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# Complete Markets with Bankruptcy Risk and Pecuniary Default Penalties

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## Abstract

For an infinite horizon economy with complete contingent markets and bankruptcy risk, like the one studied by Araujo and Sandroni (1999) and Araujo, da Silva and Faro (2016), we show that an equilibrium may fail to exist even if agents' beliefs are homogeneous. In order to discourage agents from making promises that they know in advance they will not be able to keep, default penalties must be harsh enough. The minimum level of penalty compatible with equilibrium depends on the agents' distribution of beliefs and utility functions. When beliefs are asymptotically homogeneous, it is possible to find a uniform lower bound for the severity of the penalty. When beliefs are asymptotically singular, it is still possible to find default penalties compatible with equilibrium but they must be stochastic and unbounded in the long run. We also show how these positive results depend crucially on the interpretation of default penalties. In particular, if we consider explicit economic punishments, similar to those in Kehoe and Levine (1993), then an equilibrium never exists, even if agents' beliefs are homogeneous.

## 1 Introduction

Consider a stochastic, finite horizon economy with a complete set of contingent contracts. It is well known that a necessary condition for the existence of competitive equilibrium is that agents' beliefs are equivalent in the sense that agents agree on

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the set of events with positive probability. An important question to ask is whether equivalence of beliefs is a relevant property for the existence of equilibrium in infinite horizon economies. This is because, when time extends to infinite, this property implies a rather strong uniform perception of uncertainty, namely that agents have homogeneous posterior beliefs on a set of paths that all assign full measure.

We propose to consider the simple model of an infinite horizon stochastic economy with one consumption good and a complete set of contingent markets at the initial period. Agents are assumed to be expected utility maximizers. In the standard Arrow–Debreu paradigm when bankruptcy and default are not allowed, Riedel (2003) proved that agents need not agree completely about the probabilities of finite-time events to ensure existence of a competitive equilibrium. When bankruptcy is permitted but there is a penalty for it, Araujo and Sandroni (1999) proved that if agents’ posterior beliefs do not become eventually equivalent then an equilibrium without strategic bankruptcy does not exist. In particular, if information about uncertainty is generated by a sequence of independent, identically distributed random variables, then their result states that a necessary condition for equilibrium existence is the homogeneity of beliefs. This analysis was recently extended by Araujo, da Silva and Faro (2016) to economies with smooth ambiguity-averse agents. They show that if agents’ posterior belief reductions given by their “average probabilistic beliefs” do not become homogeneous then a competitive equilibrium without strategic bankruptcy does not exist.

The convergence results in Araujo and Sandroni (1999) and Araujo, da Silva and Faro (2016) hinge on the possibility of bankruptcy as well as on the specific default punishment considered by these authors: if agent  $i$  defaults one unit of consumption good at period  $t$  contingent to an event  $A$ , then the (discounted) penalty perceived by this agent at the initial period is  $\beta^t P^i(A) \lambda^i$  where  $\beta$  is a discount factor,  $P^i$  is agent  $i$ ’s beliefs and  $\lambda^i$  is the unitary default penalty. If the beliefs of two agents  $(i, k)$  do not eventually converge, then it is possible to find an event  $A$  with an arbitrary small probability for agent  $i$  but with probability arbitrarily close to 1 for agent  $k$ . The price at the initial period of the claim contingent to this event is bounded away from zero because of agent’s  $k$  beliefs while the cost (in terms of utility) for agent  $i$  to default contingent to this event is arbitrarily small. Consequently, agent  $i$  chooses to short-sell an arbitrarily large amount of the claim contingent to the event  $A$  while agent  $k$  is willing to accommodate agent’s  $i$  position. Since bankruptcy should occur only along

out-of-equilibrium paths, non-convergent beliefs and equilibrium are not compatible.

Araujo and Sandroni (1999) analyze a necessary condition on beliefs for existence of an equilibrium but do not provide sufficient conditions. It turns out that even if agents are expected utility maximizers with homogeneous beliefs, existence is not automatic. Agents may still have incentives to risk to be bankrupt provided that the default penalty is not too harsh. We show that a necessary condition for existence of equilibrium is that, in the long run, default penalties are larger than marginal utilities. Araujo, da Silva and Faro (2016) also provide sufficient conditions for equilibrium existence when agents are smooth ambiguity-averse. However, how harsh default penalties should be depends on the distribution of agents' characteristics.

The economic relevance of these results depends on the interpretation of default penalties since the question at issue is: are the “real-life” default penalties severe enough to prevent strategic bankruptcy? According to Dubey, Geanakoplos and Shubik (2005), default penalties might be interpreted as the consequence in terms of utility of *extra-economic* punishments such as prison terms or pangs of conscience. Zame (1993) proposed to interpret default penalties as the consequences of a *non-modeled economic punishment*, such as, reputation losses and exclusion from credit markets or garnishing of future income. An infinite horizon model is the appropriate environment to model explicit economic punishments. The natural question is then: what are the explicit economic punishments that may lead to *indirect and endogenous* default penalties that are harsh enough to preclude strategic bankruptcy? In this paper, we propose to study the most favorable case. We impose the most severe economic punishment by assuming that when an agent defaults on a contract, she has both her assets and private endowments seized, and this is for ever. The striking result is that even under such a harsh default punishment, an equilibrium never exists, independently of the heterogeneity of agents' beliefs.

## 2 The model

We consider a stochastic infinite horizon economy with complete contingent markets and bankruptcy risk.

## 2.1 Uncertainty

Time is discrete and continues forever, i.e.,  $t \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ . Exogenous uncertainty is represented by the realization of some sequence of random variables taking values in a finite set  $S$ . We denote by  $\Omega$  the set of all possible overall histories of states, i.e.,  $\Omega = S^{\mathbb{N}}$  where  $\mathbb{N} = \{1, 2, \dots\}$ . For every  $t \geq 1$ , we let

$$X_t : \Omega \longrightarrow S \quad (2.1)$$

be the  $t$ -th projection from the space of sequences  $\Omega$  to  $S$ , i.e.,  $X_t((s_\tau)_{\tau \geq 1}) := s_t$ . The history up to time  $t \geq 1$  is given by the algebra  $\mathcal{F}_t$  on  $\Omega$  defined by

$$\mathcal{F}_t = \sigma(X_1, \dots, X_t). \quad (2.2)$$

There is no uncertainty at the initial date and we denote by  $\mathcal{F}_0$  the trivial algebra on  $\Omega$ . We denote by  $\mathcal{F}_\infty$  the  $\sigma$ -algebra of the overall history defined by

$$\mathcal{F}_\infty = \sigma\{X_t : t \in \mathbb{N}\} = \sigma\{\mathcal{F}_t : t \in \mathbb{N}\}. \quad (2.3)$$

Given an arbitrary set  $Z$ , we denote by  $\mathbb{A}(Z)$  the space of  $Z$ -valued processes  $(a_t)_{t \geq 0}$  adapted to the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{N}_0}$ . Since  $a_t$  is  $\mathcal{F}_t$ -measurable, the term  $a_t(\omega)$  only depends on  $s^t := (s_1, \dots, s_t) = (X_1(\omega), \dots, X_t(\omega))$ . In particular,  $a_t$  only takes finitely many values. We slightly abuse notations and write  $a_t(s^t)$  or  $a(s^t)$  for  $a_t(\omega)$ .

When  $Z$  is a subset of  $\mathbb{R}$ , we let  $\mathbb{A}^\infty(Z)$  be the subset of bounded processes in  $\mathbb{A}(Z)$ , i.e., for any  $a = (a_t)_{t \geq 0} \in \mathbb{A}(Z)$ , we have

$$a \in \mathbb{A}^\infty(Z) \iff \sup\{|a_t(s^t)| : t \in \mathbb{N}_0 \text{ and } s^t \in S^t\} < \infty.$$

We let  $\mathbb{A}^1(Z)$  be the subset of summable processes in  $\mathbb{A}(Z)$ , i.e., for any  $a = (a_t)_{t \geq 0} \in \mathbb{A}(Z)$ , we have

$$a \in \mathbb{A}^1(Z) \iff \sum_{t \geq 0} \sum_{s^t \in S^t} |a_t(s^t)| < \infty$$

with the convention that  $S^0 := \{s^0\}$  and  $a_0(s^0) := a_0$ .

Fix an arbitrary probability measure  $P$  on  $(\Omega, \mathcal{F}_\infty)$ . For each period  $t \geq 1$ , we denote by  $P_t$  the marginal probability on the space  $S^t$  of possible states of nature at

period  $t$  defined by

$$\forall s^t = (s_1, \dots, s_t) \in S^t, \quad P_t(s^t) = P\{(X_1, \dots, X_t) = s^t\}.^1 \quad (2.4)$$

Given an overall history of states  $\omega = (s_1, \dots, s_t, \dots) \in \Omega$ , the probability  $P_t(s_1, \dots, s_t)$  is also denoted by  $P_t(\omega)$ . The probability over possible shocks at date  $t+1$  conditional to current history  $s^t$  is denoted by  $P_{t+1}(\cdot|s^t)$  and defined by

$$P_{t+1}(s|s^t) := \frac{P_{t+1}(s^t, s)}{P_t(s^t)}$$

when well-defined.

For every  $t \geq 1$  and every history of states  $\sigma \in S^t$ , we denote by  $\mathbf{1}_{t,\sigma}$  the adapted process defined by

$$\forall \tau \neq t, \quad [\mathbf{1}_{t,\sigma}]_\tau = 0 \quad \text{and} \quad \forall \xi \in S^t, \quad [\mathbf{1}_{t,\sigma}]_t(\xi) = \begin{cases} 1 & \text{if } \xi = \sigma \\ 0 & \text{if } \xi \neq \sigma. \end{cases} \quad (2.5)$$

We denote by  $\mathbf{1}_0$  the adapted process defined by  $[\mathbf{1}_0]_0 = 1$  and  $[\mathbf{1}_0]_t = 0$  for all  $t \geq 1$ .

## 2.2 Endowments and Preferences

There is a single consumption good and a finite set  $I$  of agents. Agent  $i$ 's endowments at period  $t$  are given by an  $\mathcal{F}_t$ -measurable function

$$e_t^i : \Omega \longrightarrow (0, +\infty). \quad (2.6)$$

The process of aggregate endowments is denoted by  $e = (e_t)_{t \geq 0}$  where  $e_t = \sum_{i \in I} e_t^i$ .

**Assumption 2.1.** For each agent  $i$ , the process  $e^i$  of endowments is adapted, uniformly bounded from above and takes strictly positive values. Moreover, the process of aggregate endowments is uniformly bounded away from 0. In other words,

$$\forall i \in I, \quad e^i \in \mathbb{A}((0, \infty)), \quad (2.7)$$

and

$$\exists \bar{e} > 0, \quad \exists \underline{e} > 0, \quad \forall t \geq 0, \quad \underline{e} \leq e_t \leq \bar{e}. \quad (2.8)$$

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<sup>1</sup>We slightly abuse notation since we should write  $P_t\{s^t\}$  instead of  $P_t(s^t)$ .

A consumption plan is an adapted process  $c = (c_t)_{t \geq 0}$  with value in  $[0, \infty)$ , i.e.,  $c \in \mathbb{A}(\mathbb{R}_+)$ . To define agents' preference relations over consumption plans, we introduce the following notations. Denote by  $\mathcal{L}_+^0(\mathcal{F}_\infty)$  the space of non-negative and  $\mathcal{F}_\infty$ -measurable functions. Each agent  $i$  ranks consumption plans according to the following function from  $\mathbb{A}(\mathbb{R}_+)$  to  $[0, \infty]$

$$W^i \left( \sum_{t \geq 0} \beta^t u^i(c_t) \right)$$

where  $W^i : \mathcal{L}_+^0(\mathcal{F}) \rightarrow [0, \infty]$  is an expectation operator,  $u^i : \mathbb{R}_+ \rightarrow \mathbb{R}$  is agent  $i$ 's utility function and  $\beta \in (0, 1)$  is a time-preference discount factor.<sup>2</sup>

**Assumption 2.2.** For each agent  $i$ , the function  $u^i : [0, \infty) \rightarrow \mathbb{R}_+$  is continuous, concave, strictly increasing, satisfies  $u^i(0) = 0$  and the Inada condition at the origin.<sup>3</sup>

**Assumption 2.3.** The function  $W^i : \mathcal{L}_+^0(\mathcal{F}_\infty) \rightarrow [0, \infty]$  satisfies the following properties:

- (a) the domain of  $W^i$  contains bounded random variables;<sup>4</sup>
- (b) for every  $z \in \mathbb{R}$ , we have  $W^i(z\mathbf{1}_\Omega) = z$ ;
- (c) it is continuous with respect to the sup-norm on the subset of bounded random variables, i.e.,

$$\lim_{n \rightarrow \infty} \|w_n - w\|_\infty = 0 \implies \lim_{n \rightarrow \infty} W^i(w_n) = W^i(w);<sup>5</sup>$$

- (d) it is strictly increasing in the sense that for every  $\varepsilon > 0$ ,

$$W^i(w) < \infty \quad \text{and} \quad w' \geq w + \varepsilon \mathbf{1}_\Omega \implies W^i(w') > W^i(w).$$

Observe that under the above properties, the function  $W^i$  is also increasing in the sense that  $W^i(w') \geq W^i(w)$  for any  $w' \geq w$ .

**Example 2.1.** The standard example is the expected utility model considered by Araujo and Sandroni (1999) where

$$\forall w \in \mathcal{L}_+^0(\mathcal{F}_\infty), \quad W^i(w) = \mathbb{E}_{P^i}(w)$$

<sup>2</sup>We assume that  $\beta$  is homogeneous to simplify the presentation.

<sup>3</sup>In the sense that  $\lim_{x \rightarrow 0} (u^i(x) - u^i(0))/x = \infty$ .

<sup>4</sup>The domain of  $W^i$  is the set of random variables  $w \in \mathcal{L}_+^0(\mathcal{F}_\infty)$  satisfying  $W^i(w) \in \mathbb{R}$ .

<sup>5</sup>For every  $w \in \mathcal{L}^0(\mathbb{R})$ , we let  $\|w\|_\infty := \sup\{|w(\omega)| : \omega \in \Omega\}$ .

for some probability measure  $P^i$  on  $\mathcal{F}_\infty$  representing agent  $i$ 's subjective beliefs.

**Example 2.2.** One may also follow Araujo, da Silva and Faro (2016) and consider the expectation operator associated to the smooth ambiguity averse model of Klibanoff, Marinacci and Mukerji (2005) (see also Klibanoff, Marinacci and Mukerji (2009)):

$$\forall w \in \mathcal{L}_+^0(\mathcal{F}_\infty), \quad W^i(w) = (\phi^i)^{-1} \left( \int_{\Delta(\Omega)} \phi^i(\mathbb{E}_P(w)) \mu^i(dP) \right)$$

where  $\Delta(\Omega)$  is the space of probability measures on  $(\Omega, \mathcal{F}_\infty)$ ,  $\mu^i$  is a probability measure on  $\Delta(\Omega)$  and  $\phi^i : \mathbb{R} \rightarrow \mathbb{R}$  is strictly increasing and concave.

### 2.3 Investment Strategies

Trade occurs only at the initial period  $t = 0$ . There exists a complete market of claims contingent on future histories of states of nature. Contingent to each possible date- $t$  history  $s^t \in S^t$ , each agent chooses a claim  $\theta_t(s^t)$  of units of consumption at unitary price  $q_t(s^t)$  (in units of the consumption good at  $t = 0$ ). Short-sales are allowed and a negative claim means a promise to deliver units of the consumption good. As usual we make the following abuse of notation:

$$\forall (t, \omega) \in \mathbb{N} \times \Omega, \quad \theta_t(\omega) = \theta_t(X_1(\omega), \dots, X_t(\omega)). \quad (2.9)$$

By convention, we pose  $\theta_0 = 0$ . Observe that  $\theta = (\theta_t)_{t \geq 0}$  belongs to subspace  $\mathbb{A}_0(\mathbb{R})$  of all adapted processes  $a = (a_t)_{t \geq 0} \in \mathbb{A}$  satisfying  $a_0 = 0$ . A vector in  $\mathbb{A}_0(\mathbb{R})$  is called an investment strategy (or a portfolio).

Since we want to formally define the cost of a portfolio  $\theta \in \mathbb{A}_0(\mathbb{R})$ , we restrict investment strategies to satisfy the following property

$$q(\theta) := \lim_{\tau \rightarrow \infty} \sum_{t=1}^{\tau} q_t(s^t) \theta_t(s^t) \in \mathbb{R}. \quad (2.10)$$

We denote by  $\Theta$  the subset of all ‘‘admissible’’ portfolios  $\theta \in \mathbb{A}_0(\mathbb{R})$  satisfying the restriction (2.10).

*Remark 2.1.* In Araujo, da Silva and Faro (2016), agents are restricted to choose portfolios in the set  $\mathbb{A}^\infty(\mathbb{R})$  of bounded processes. Since they also assume that asset prices  $q = (q_t)_{t \geq 0}$  are summable, then (2.10) is automatically satisfied.



### 3 Competitive Equilibrium with Risk of Bankruptcy

At the initial period  $t = 0$ , each agent  $i$  chooses a consumption  $c_0^i \geq 0$  and an investment strategy  $\theta^i \in \Theta$  satisfying the budget restriction

$$c_0^i + q(\theta^i) \leq e_0^i. \quad (3.1)$$

At period  $t$ , agent  $i$  has an endowment  $e_t^i$  and, either receives the payment  $\theta_t^i$  of his investment if  $\theta_t^i \geq 0$ , or has to honor his debt by paying  $-\theta_t^i$  units of the consumption good if  $\theta_t^i < 0$ . Bankruptcy is allowed in the sense that agent  $i$  is not restricted at  $t = 0$  to choose an investment strategy  $\theta^i$  satisfying the solvency constraint

$$\forall t \geq 1, \quad \theta_t^i + e_t^i \geq 0. \quad (3.2)$$

If agent  $i$  is solvent at event  $s^t$ , i.e.,  $\theta_t^i + e_t^i \geq 0$  then he consumes  $\theta_t^i + e_t^i$  units of the good and gets the utility  $u^i(\theta_t^i + e_t^i)$ . If agent  $i$  is not solvent, i.e.,  $\theta_t^i + e_t^i < 0$  then he goes bankrupt.

We should now specify the punishment for default.

#### 3.1 Default Penalties

Following Zame (1993) and Dubey, Geanakoplos and Shubik (2005), Araujo and Sandroni (1999) and Araujo, da Silva and Faro (2016) assume that in case of bankruptcy, an agent does not consume and suffers a penalty  $\lambda_t^i[\theta_t^i + e_t^i]^-$  proportional to his default where  $\lambda_t^i(\omega) \geq 0$  is the unitary default penalty at period  $t$  contingent to the history  $\omega$ . There is no other punishment. In particular, there is no garnishment of assets or future income. Under this assumption, the indirect utility associated to an investment strategy  $\theta^i \in \Theta$  is given by

$$V^i(\theta^i) := W^i \left( u^i(e_0^i - q(\theta^i)) + \sum_{t=1}^{\infty} \beta^t v^i(e_t^i + \theta_t^i) \right)$$

where  $v^i$  is the felicity function defined on  $\mathbb{R}$  by

$$\forall z \in \mathbb{R}, \quad v^i(z) = \begin{cases} u^i(z) & \text{if } z \geq 0 \\ \lambda^i z & \text{if } z < 0. \end{cases} \quad (3.3)$$

Following Araujo and Sandroni (1999) and Araujo, da Silva and Faro (2016), we introduce the following concept of competitive equilibrium with risk of bankruptcy and default penalties.

**Definition 3.1.** A competitive equilibrium with risk of bankruptcy and default penalties is a family  $(q, (\theta^i)_{i \in I})$  of asset prices  $q = (q_t)_{t \geq 1}$  and an allocation  $(\theta^i)_{i \in I}$  of investment strategies  $\theta^i \in \Theta$  such

(a) the investment strategy is optimal, i.e.,

$$\theta^i \in \operatorname{argmax} \{V^i(\theta) : \theta \in \Theta \text{ and } e_0^i - q(\theta) \geq 0\}, \quad (3.4)$$

(b) asset markets clear at  $t = 0$ , i.e.,

$$\sum_{i \in I} \theta^i = 0, \quad (3.5)$$

(c) for every period  $t \geq 1$ , there is no bankruptcy, i.e.,

$$\forall i \in I, \quad \theta_t^i + e_t^i \geq 0. \quad (3.6)$$

*Remark 3.1.* When bankruptcy is allowed, agents have the option to take the risk to be bankrupt on some states. Araujo and Sandroni (1999) and Araujo, da Silva and Faro (2016) look for equilibrium solutions for which agents optimally opt to be solvent at any contingency. This is in sharp contrast with equilibrium concepts analyzed in Zame (1993) and Dubey, Geanakoplos and Shubik (2005) where default may occur along the equilibrium path. Naturally, default penalties should be large enough to discourage agents from making promises that they know in advance they will not be able to honor.

## 3.2 Implicit Assumptions

Two important assumptions are implicitly imposed for the validity of the model and the equilibrium concept. The first one concerns the ex-ante information available to agents. In sharp contrast with Kehoe and Levine (1993) (see also Alvarez and Jermann (2000)), it is assumed that the information about agents' endowments and utility functions is not common knowledge at  $t = 0$ . Indeed, if the market knows at  $t = 0$  that agent  $i$ 's

endowment at period  $t$  contingent to the history  $s^t \in S^t$  is  $e_t^i(s^t)$ , then agent  $i$  should be constrained to choose an affordable claim  $\theta_t^i(s^t) \geq -e_t^i(s^t)$ , as it is the case in the standard Arrow–Debreu paradigm (see for instance Arrow and Debreu (1954), Dana (1993) and Riedel (2003)).

The second assumption concerns ex-post enforcement. In Dubey and Geanakoplos (1989), Zame (1993) and Dubey, Geanakoplos and Shubik (2005), each agent chooses whether to repay his debt or to bear the penalty for defaulting. In particular an agent may decide not to meet at all his liabilities even if he could afford them. In Kehoe and Levine (1993) and Alvarez and Jermann (2000) it is assumed that assets can be seized but endowments are non-recourse. In Geanakoplos (1997), Geanakoplos and Zame (2002), Araujo, Páscoa and Torres-Martinez (2002) and Kubler and Schmedders (2003), the only enforcement mechanism is the seizure of collateral bundles. In the model of Araujo and Sandroni (1999) and Araujo, da Silva and Faro (2016), it is implicitly assumed that in case of bankruptcy, endowments can be seized but assets cannot.

### 3.3 Default Penalties Need to Be Large Enough

Since agents have the option to go bankrupt but optimally decides not to do so, default penalties must be sufficiently large, even if agents are expected utility maximizers with homogeneous beliefs. We show below that default penalties have to be larger than equilibrium marginal utilities.

**Proposition 3.1.** Assume that there exists a probability measure  $P \in \Delta(\Omega)$  with full support such that  $W^i(w) = \mathbb{E}_P(w)$ .<sup>6</sup> If there is competitive equilibrium with risk of bankruptcy and default penalties, then, in the long run and for every agent, default penalties must be larger than marginal utilities. More precisely, we must have

$$\forall i \in I, \quad \forall \omega \in \Omega, \quad \limsup_{t \rightarrow \infty} \{ \nabla u^i(c_t^i(\omega)) - \lambda_t^i(\omega) \} \leq 0 \quad (3.7)$$

where  $c_t^i := e^i + \theta_t^i$  is the equilibrium consumption.

*Proof of Proposition 3.1.* Consider a competitive equilibrium  $(q, (\theta^i)_{i \in I})$ . Let  $c_t^i := e^i + \theta_t^i$  and recall that  $c_t^i \geq 0$ . Since  $u^i$  satisfies Inada's condition, we have that  $c_t^i > 0$ . In particular, we can define the marginal utility  $\nabla u^i(c_t^i(\omega))$  for every  $t \geq 0$ .

<sup>6</sup> $P$  has full support means that  $P_t(s^t) > 0$  for every date- $t$  event  $s^t \in S^t$ .

Euler equations associated to the optimal choice of  $\theta^i$  imply that

$$\forall t \geq 1, \quad \forall \omega \in \Omega, \quad q_t(\omega) = \beta^t P_t(\omega) \frac{\nabla u^i(c_t^i(\omega))}{\nabla u^i(c_0^i)}.$$

Observe that we always have  $c_t^i(\omega) \leq \bar{e}$  and  $c_0^i \leq \bar{e}$ . Moreover, since the allocation  $(c^i)_{i \in I}$  is feasible, there exists  $i_1 \in I$  such that  $c_0^{i_1} \geq \underline{e}$  and for each  $t \geq 1$  and history of states  $\sigma \in S^t$ , there exists  $i_2 \in I$  such that  $c_t^{i_2}(\omega) \geq \underline{e}$ . Since utility functions are concave, we get

$$0 < m := \min_{k \in I} \left\{ \frac{\nabla u^k(\bar{e})}{\nabla u^k(\underline{e})} \right\} \leq \frac{\nabla u^i(c_t^i(\omega))}{\nabla u^i(c_0^i)} \leq \max_{k \in I} \left\{ \frac{\nabla u^k(\underline{e})}{\nabla u^k(\bar{e})} \right\} := M < \infty.$$

Assume by way of contradiction that there exists an agent  $i \in I$  and an history of states  $\omega \in \Omega$  such that

$$\limsup_{t \rightarrow \infty} \{ \nabla u^i(c_t^i(\omega)) - \lambda_t^i(\omega) \} > 0.$$

Fix arbitrarily  $\alpha > 0$  and consider the investment strategy  $\eta^i(\alpha) \in \mathbb{A}_0(\mathbb{R})$  defined by

$$\eta^i(\alpha) := \theta^i - (\alpha + \theta_t^i(\omega)) \mathbf{1}_{t,\omega}.$$

With the portfolio  $\eta^i(\alpha)$ , agent  $i$  is bankrupt at date  $t$  contingent to state  $\omega$ . The indirect utility is then

$$V^i(\eta^i(\alpha)) = V^i(\theta^i) + [u^i(c_0^i + f_t^i(\alpha)) - u^i(c_0^i)] - [\alpha \lambda_t^i(\omega) + u^i(c_t^i(\omega))] \beta^t P_t(\omega)$$

where

$$f_t^i(\alpha) = \alpha \beta^t P_t(\omega) \frac{\nabla u^i(c_t^i(\omega))}{\nabla u^i(c_0^i)}.$$

We then get

$$V^i(\eta^i(\alpha)) - V^i(\theta^i) \geq \nabla u^i(c_0^i + f_t^i(\alpha)) f_t^i(\alpha) - [\alpha \lambda_t^i(\omega) + u^i(\bar{e})] \beta^t P_t(\omega).$$

Now fix arbitrarily  $\gamma > 0$  and choose  $\alpha > 0$  such that

$$\alpha \beta^t P_t(\omega) = \gamma.$$

It follows that

$$V^i(\eta^i(\alpha)) - V^i(\theta^i) \geq \nabla u^i(c_0^i + \gamma M) \gamma \frac{\nabla u^i(c_t^i(\omega))}{\nabla u^i(c_0^i)} - [\gamma \lambda_t^i(\omega) + u^i(\bar{e}) \beta^t P_t(\omega)]$$

implying that

$$\limsup_{t \rightarrow \infty} \frac{V^i(\eta^i(\alpha)) - V^i(\theta^i)}{\gamma} \geq [\nabla u^i(c_0^i + \gamma M) - \nabla u^i(c_0^i)] m + \limsup_{t \rightarrow \infty} [\nabla u^i(c_t^i(\omega)) - \lambda_t^i(\omega)].$$

Choosing  $t$  large enough and  $\gamma > 0$  small enough, we get that  $V^i(d^i(\alpha)) > V^i(c^i)$ . This contradicts the optimality of  $\theta^i$ .  $\square$

Proposition 3.1 provides a necessary condition relating unitary default penalties which are primitives of the model and marginal utilities at equilibrium which are endogenous variables. However, we can easily provide sufficient conditions for non-existence, only in terms of primitives: if there exists an every overall history of states  $\omega \in \Omega$  such that either

$$\forall i \in I, \quad \liminf_{t \rightarrow \infty} \lambda_t^i(\omega) < \nabla u^i(\underline{e}) \quad (3.8)$$

or

$$\exists i \in I, \quad \liminf_{t \rightarrow \infty} \lambda_t^i(\omega) < \nabla u^i(\bar{e}) \quad (3.9)$$

then there does not exist a competitive equilibrium without default.

## 4 Explicit Economic Punishment

We have seen that default penalties should be large enough to ensure existence of a competitive equilibrium even when agents are expected utility maximizers with homogeneous beliefs. Araujo, da Silva and Faro (2016) show that when agents are smooth ambiguity averse, then under a strong compatibility condition on second order beliefs, it is possible to identify large enough default penalties for which there exists a competitive equilibrium.

Since the level of default penalties matters, we should investigate what exactly are these default penalties. According to Dubey, Geanakoplos and Shubik (2005), default

penalties might be interpreted as the consequence in terms of utility of *extra-economic* punishment such as prison terms or pangs of conscience. Default penalties may also be interpreted as the consequence in terms of utility of a *non-modeled economic punishment* such as reputation losses and exclusion from credit markets or garnishing of future income. We quote hereafter Zame (1993):

In the real world, such default may entail many and varied consequences: creditors might be able to seize assets and be awarded judgments against future earnings, defaulters might be barred from future credit markets, and so on. Rather than attempt to model such institutional details, I assume that the only consequences of default are penalties assessed against the defaulters, and these penalties are assessed directly in terms of utility.

Since existence of a competitive equilibrium depends on the level of the default penalty, the natural question is then: what are the explicit economic punishments that lead to *endogenously determined* default penalties that are large enough to preclude strategic bankruptcy? Is there a sufficiently severe economic punishment such that, independently of the distribution of agents' characteristics, strategic bankruptcy is precluded?

To answer this question, we consider the most severe economic punishment. As in Araujo and Sandroni (1999) and Araujo, da Silva and Faro (2016), we assume that upon default, current endowment is seized. In addition, in the line of Kehoe and Levine (1993) and Alvarez and Jermann (2000), we assume that assets are also seized. Moreover, not only current endowment is seized, but all future endowments are seized. This means that if agent  $i$  chooses an investment strategy  $\theta^i$  such that for some history  $\omega$  and date  $t$ , we have  $\theta_t^i(\omega) > -e_t^i(\omega)$ , then for every  $\tau \geq t$  the agent consumption is  $c_\tau^i(\omega) = 0$ . Since we assumed that  $u^i(0) = 0$ , the indirect utility of an investment strategy  $\theta$  is now defined by

$$V^i(\theta) := W^i \left( u^i(e_0^i - q(\theta)) + \sum_{t=1}^{\tau^i(\theta)} \beta^t u^i(e_t^i + \theta_t) \right) \quad (4.1)$$

where the stopping time  $\tau^i(\theta) : \Omega \mapsto \mathbb{N}$  is defined by

$$\tau^i(\theta)(\omega) = \inf\{t \geq 1 : \theta_t(\omega) + e_t^i(\omega) < 0\} \quad (4.2)$$

with the convention that  $\inf \emptyset = \infty$  and  $\sum_{t=1}^0 = 0$ .

Even when the most severe explicit economic punishment is imposed, a competitive equilibrium without default never exists.

**Theorem 4.1.** *Assume that the default punishment is complete garnishment of current and future income (assets and endowments). Then, a competitive equilibrium with risk of bankruptcy does not exist.*

*Proof.* Assume, by way of contradiction, that there exists a competitive equilibrium  $(q, (\theta^i)_{i \in I})$  with risk of bankruptcy. Fix an agent  $i$  and a date  $\xi \geq 1$ . Consider the alternative investment strategy  $\eta^{i,\xi}$  defined by

$$\eta^{i,\xi} := \begin{cases} \theta_t^i, & \text{if } t \neq \xi + 1 \\ \theta_{\xi+1}^i - \alpha_{\xi+1}, & \text{otherwise,} \end{cases}$$

where  $\alpha_{\xi+1} > 0$  is such that

$$\sum_{s^{\xi+1} \in S^{\xi+1}} q_{\xi+1}(s^{\xi+1}) \alpha_{\xi+1} = 1.$$

In the new strategy  $\eta^{i,\xi}$ , debt is increased uniformly at all events at date  $\xi + 1$  such that the corresponding amount borrowed at date 0 is exactly 1. Recall that in a competitive equilibrium with risk of bankruptcy, the equilibrium portfolio  $\theta^i$  is such that agent  $i$  is never bankrupt, i.e.,

$$e_t^i + \theta^i \geq 0, \quad \forall t \geq 1.$$

When  $\xi$  tends to infinite, the unit price  $q_{\xi+1}(s^{\xi+1})$  may converge to zero. Therefore, the new debt  $\alpha_{\xi+1}$  may tend to infinite. This implies that we may have  $e_{\xi+1}^i(\omega) + \eta_{\xi+1}^{i,\xi}(\omega) < 0$  for some history  $\omega$ . Therefore, we get the following lower bound of the indirect utility associated to the new investment strategy

$$V^i(\eta^{i,\xi}) \geq W^i \left( u^i(e_0^i - q(\theta) + 1) + \sum_{t=1}^{\xi} \beta^t u^i(e_t^i + \theta_t^i) \right). \quad (4.3)$$

Comparing the two indirect utilities, we get

$$V^i(\eta^{i,\xi}) - V^i(\theta^i) \geq W^i \left( \underbrace{u^i(e_0^i - q(\theta) + 1) + \sum_{t=1}^{\xi} \beta^t u^i(e_t^i + \theta_t^i)}_{=: \tilde{U}^{i,\xi}} \right) - W^i \left( \underbrace{u^i(e_0^i - q(\theta)) + \sum_{t=1}^{\infty} \beta^t u^i(e_t^i + \theta_t^i)}_{=: U^i} \right).$$

Observe that

$$\tilde{U}^{i,\xi} = U^i + \delta_0^i \mathbf{1}_\Omega - \sum_{t=\xi+1}^{\infty} \beta^t u^i(e_t^i + \theta_t^i) \quad \text{with} \quad \delta_0^i := u^i(e_0^i + q(\theta^i) + 1) - u^i(e_0^i + q(\theta^i)).$$

Since markets clear at equilibrium, we have  $e_t^i + \theta_t^i \leq \bar{e}$  for every  $t \geq 1$ . This implies that

$$0 \leq \sum_{t \geq \xi+1}^{\infty} \beta^t u^i(e_t^i + \theta_t^i) \leq \frac{\beta^{\xi+1}}{1-\beta} u(\bar{e}).$$

We then get that

$$\lim_{\xi \rightarrow \infty} \left\| \tilde{U}^{i,\xi} - (U^i + \delta_0^i \mathbf{1}_\Omega) \right\|_\infty = 0.$$

By sup-norm continuity of  $W^i$ , we deduce that

$$\lim_{\xi \rightarrow \infty} V^i(\eta^{i,\xi}) - V^i(\theta^i) = W^i(U^i + \delta_0^i \mathbf{1}_\Omega) - W^i(U^i)$$

which is strictly positive since  $W^i$  is strictly increasing. This contradicts the optimality of  $\theta^i$ .  $\square$

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