Statistical dynamics of a hard sphere gas: fluctuating Boltzmann equation and large deviations

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Abstract. — We present a mathematical theory of dynamical fluctuations for the hard sphere gas in the Boltzmann-Grad limit. We prove that: (1) fluctuations of the empirical measure from the solution of the Boltzmann equation, scaled with the square root of the average number of particles, converge to a Gaussian process driven by the fluctuating Boltzmann equation, as predicted in [**42**]; (2) large deviations are exponentially small in the average number of particles and are characterized, under regularity assumptions, by a large deviation functional as previously obtained in [**38**] in a context of stochastic processes. The results are valid away from thermal equilibrium, but only for short times. Our strategy is based on uniform a priori bounds on the cumulant generating function, characterizing the fine structure of the small correlations.

CONTENTS

CHAPTER 1

INTRODUCTION

This paper is devoted to a detailed analysis of the correlations arising, at low density, in a deterministic particle system obeying Newton's laws. In this chapter we start by defining our model precisely, and recalling the fundamental result of Lanford on the short-time validity of the Boltzmann equation. After that, we state our main results, Theorem 2 and Theorem 3 below, regarding small fluctuations and large deviations of the empirical measure, respectively. Finally, the last section of this introduction describes the essential features of the proof, the organization of the paper, and presents some open problems.

1.1. The hard-sphere model

We consider a system of $N \geq 0$ spheres of diameter $\varepsilon > 0$ in the d-dimensional torus $\mathbb{T}^{dN}$ with $d \geq 2$. The positions $(\mathbf{x}_1^\varepsilon, \dots, \mathbf{x}_N^\varepsilon) \in \mathbb{T}^{dN}$ and velocities $(\mathbf{v}_1^\varepsilon, \dots, \mathbf{v}_N^\varepsilon) \in \mathbb{R}^{dN}$ of the particles satisfy Newton's laws

$$\frac{d\mathbf{x}_i^\varepsilon}{dt} = \mathbf{v}_i^\varepsilon \,, \quad \frac{d\mathbf{v}_i^\varepsilon}{dt} = 0 \quad \text{as long as } \; |\mathbf{x}_i^\varepsilon(t) - \mathbf{x}_j^\varepsilon(t)| > \varepsilon \quad \text{for } \; 1 \leq i \neq j \leq N \,, \tag{1.1.1}$$

with specular reflection at collisions

$$\left.\begin{aligned} (\mathbf{v}_i^\varepsilon)' &:= \mathbf{v}_i^\varepsilon - \frac{1}{\varepsilon^2}(\mathbf{v}_i^\varepsilon - \mathbf{v}_j^\varepsilon)\cdot(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon)\,(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon) \\ (\mathbf{v}_j^\varepsilon)' &:= \mathbf{v}_j^\varepsilon + \frac{1}{\varepsilon^2}(\mathbf{v}_i^\varepsilon - \mathbf{v}_j^\varepsilon)\cdot(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon)\,(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon) \end{aligned}\right\} \quad \text{if } |\mathbf{x}_i^\varepsilon(t) - \mathbf{x}_j^\varepsilon(t)| = \varepsilon \,. \tag{1.1.2}$$

Observe that these boundary conditions do not cover all possible situations, as for instance triple collisions are excluded. Nevertheless the hard-sphere flow generated by (1.1.1)-(1.1.2) (free transport of N spheres of diameter ε, plus instantaneous reflection

$$(\mathbf{v}_i^\varepsilon, \mathbf{v}_j^\varepsilon) \to \Big((\mathbf{v}_i^\varepsilon)', (\mathbf{v}_j^\varepsilon)'\Big)$$

at contact) is well defined on a full measure subset of $\mathcal{D}_N^\varepsilon$ (see [**1**], or [**17**] for instance) where $\mathcal{D}_N^\varepsilon$ is the canonical phase space

$$\mathcal{D}_N^\varepsilon := \big\{ Z_N \in \mathbb{D}^N \,/\, \forall i \neq j \,, \quad |x_i - x_j| > \varepsilon \big\} \,.$$

We have denoted $Z_N := (X_N, V_N) \in (\mathbb{T}^d \times \mathbb{R}^d)^N$ the positions and velocities in the phase space $\mathbb{D}^N := (\mathbb{T}^d \times \mathbb{R}^d)^N$ with $X_N := (x_1, \dots, x_N) \in \mathbb{T}^{dN}$ and $V_N := (v_1, \dots, v_N) \in \mathbb{R}^{dN}$. We set $Z_N = (z_1, \dots, z_N)$ with $z_i = (x_i, v_i)$.

The probability density W_N^ε of finding N hard spheres of diameter ε at configuration Z_N at time t is governed by the Liouville equation in the $2dN$-dimensional phase space

$$\partial_t W_N^\varepsilon + V_N \cdot \nabla_{X_N} W_N^\varepsilon = 0 \quad \text{on} \quad \mathcal{D}_N^\varepsilon \,, \tag{1.1.3}$$

with specular reflection on the boundary. If we denote

$$\partial\mathcal{D}_N^{\varepsilon\pm}(i,j) := \Big\{ Z_N \in \mathbb{D}^N \,/\, |x_i - x_j| = \varepsilon \,, \quad \pm(v_i - v_j)\cdot(x_i - x_j) > 0 \\ \text{and} \quad \forall k,\ell \in [1,N]^2 \setminus \{i,j\},\ k \neq \ell \,,\ |x_k - x_\ell| > \varepsilon \Big\} \,,$$

then

$$\forall Z_N \in \partial\mathcal{D}_N^{\varepsilon+}(i,j)\,, i \neq j \,, \quad W_N^\varepsilon(t,Z_N) := W_N^\varepsilon(t, Z_N^{'i,j}) \,, \tag{1.1.4}$$

where $Z_N^{'i,j}$ differs from Z_N only by $(v_i, v_j) \to (v_i', v_j')$, given by (1.1.2).

The canonical formalism consists in fixing the number N of particles, and in studying the probability density W_N^ε of particles in the state Z_N at time t, as well as its marginals. The main drawback of this formalism is that fixing the number of particles creates spurious correlations (see e.g. [**16, 35**]). We are rather going to define a particular class of distributions on the grand canonical phase space

$$\mathcal{D}^\varepsilon := \bigcup_{N\geq 0} \mathcal{D}_N^\varepsilon \,,$$

where the particle number is not fixed but given by a modified Poisson law (actually $\mathcal{D}_N^\varepsilon = \emptyset$ for large N). For notational convenience, we work with functions extended to zero over $\mathbb{D}^N \setminus \overline{\mathcal{D}_N^\varepsilon}$. Given a probability distribution $f^0 : \mathbb{D} \to \mathbb{R}$ satisfying

$$|f^0(x,v)| + |\nabla_x f^0(x,v)| \ \leq C_0 \, \exp\Big(-\frac{\beta_0}{2}|v|^2 \Big) \,, \quad C_0 > 0 \,,\ \beta_0 > 0 \,, \tag{1.1.5}$$

the initial probability density is defined on the configurations $(N, Z_N) \in \mathbb{D}^{\mathbb{N}}$ as

$$\frac{1}{N!} W_N^{\varepsilon 0}(Z_N) := \frac{1}{\mathcal{Z}^\varepsilon} \frac{\mu_\varepsilon^N}{N!} \prod_{i=1}^N f^0(z_i)\, \mathbf{1}_{\mathcal{D}_N^\varepsilon}(Z_N) \tag{1.1.6}$$

where $\mu_\varepsilon > 0$ and the normalization constant $\mathcal{Z}^\varepsilon$ is given by

$$\mathcal{Z}^\varepsilon := 1 + \sum_{N\geq 1} \frac{\mu_\varepsilon^N}{N!} \int_{\mathbb{D}^N} dZ_N \prod_{i=1}^N f^0(z_i)\, \mathbf{1}_{\mathcal{D}_N^\varepsilon}(Z_N) \,.$$

Here and below, $\mathbf{1}_A$ will be the characteristic function of the set A. We will also use the symbol $\mathbf{1}_{\text{“}*\text{”}}$ for the characteristic function of the set defined by condition “$*$”.

Note that in the chosen probability measure, particles are “exchangeable”, in the sense that $W_N^{\varepsilon 0}$ is invariant by permutation of the particle labels in its argument. Moreover, the choice (1.1.6) for the initial data is the one guaranteeing the “maximal factorization”, in the sense that particles would be i.i.d. were it not for the indicator function (‘hard-sphere exclusion’).

Our fundamental random variable is the time-zero configuration, consisting of the initial positions and velocities of all the particles of the gas. We will denote $\mathcal{N}$ the total number of particles (as a random variable) and $\mathbf{Z}_\mathcal{N}^{\varepsilon 0} = \big(\mathbf{z}_i^{\varepsilon 0}\big)_{i=1,\dots,\mathcal{N}}$ the initial particle configuration. The particle dynamics

$$t \mapsto \mathbf{Z}_\mathcal{N}^\varepsilon(t) = (\mathbf{z}_i^\varepsilon(t))_{i=1,\dots,\mathcal{N}} \tag{1.1.7}$$

is then given by the hard-sphere flow solving (1.1.1)-(1.1.2) with random initial data $\mathbf{Z}_\mathcal{N}^{\varepsilon 0}$ (well defined with probability 1). The probability of an event X with respect to the measure (1.1.6) will be denoted $\mathbb{P}_\varepsilon(X)$, and the corresponding expectation symbol will be denoted $\mathbb{E}_\varepsilon$. Notice that particles are

identified by their label, running from 1 to $\mathcal{N}$. We shall mostly deal with expectations of observables of type $\mathbb{E}_\varepsilon\big(\sum_{i=1}^{\mathcal{N}}\dots\big)$. Unless differently specified, we always imply that $\mathbb{E}_\varepsilon\big(\sum_i\dots\big)=\mathbb{E}_\varepsilon\big(\sum_{i=1}^{\mathcal{N}}\dots\big)$.

The average total number of particles $\mathcal{N}$ is fixed in such a way that

$$\lim_{\varepsilon\to 0}\mathbb{E}_\varepsilon\left(\mathcal{N}\right)\varepsilon^{d-1}=1\;. \tag{1.1.8}$$

The limit (1.1.8) ensures that the *Boltzmann-Grad scaling* holds, i.e. that the inverse mean free path is of order 1 [**19**]. Thus from now on we will set

$$\mu_\varepsilon=\varepsilon^{-(d-1)}\;.$$

Let us define the rescaled initial n-particle correlation function

$$F_n^{\varepsilon 0}(Z_n):=\mu_\varepsilon^{-n}\sum_{p=0}^{\infty}\frac{1}{p!}\int_{\mathbb{D}^p}dz_{n+1}\dots dz_{n+p}\,W_{n+p}^{\varepsilon 0}(Z_{n+p})\;.$$

We say that the initial measure admits correlation functions when the series in the right-hand side is convergent, together with the series in the inverse formula

$$W_n^{\varepsilon 0}(Z_n)=\mu_\varepsilon^n\sum_{p=0}^{\infty}\frac{(-\mu_\varepsilon)^p}{p!}\int_{\mathbb{D}^p}dz_{n+1}\dots dz_{n+p}\,F_{n+p}^{\varepsilon 0}(Z_{n+p}).$$

In this case, the set of functions $\big(F_n^{\varepsilon 0}\big)_{n\geq 1}$ describes all the properties of the system.

For any symmetric test function $h_n:\mathbb{D}^n\to\mathbb{R}$, the following holds :

$$\begin{aligned}\mathbb{E}_\varepsilon\Big(\sum_{\substack{i_1,\dots,i_n\\ i_j\neq i_k,j\neq k}}h_n\big(\mathbf{z}_{i_1}^{\varepsilon 0},\dots,\mathbf{z}_{i_n}^{\varepsilon 0}\big)\Big)&=\mathbb{E}_\varepsilon\Big(\delta_{\mathcal{N}\geq n}\frac{\mathcal{N}!}{(\mathcal{N}-n)!}h_n\big(\mathbf{z}_1^{\varepsilon 0},\dots,\mathbf{z}_n^{\varepsilon 0}\big)\Big)\\ &=\sum_{p=n}^{\infty}\int_{\mathbb{D}^p}dZ_p\,\frac{W_p^{\varepsilon 0}(Z_p)}{p!}\,\frac{p!}{(p-n)!}\,h_n\big(Z_n\big)\\ &=\mu_\varepsilon^n\int_{\mathbb{D}^n}dZ_n\,F_n^{\varepsilon 0}(Z_n)\,h_n(Z_n)\;.\end{aligned} \tag{1.1.9}$$

Starting from the initial distribution $W_N^{\varepsilon 0}$, the density $W_N^\varepsilon(t)$ evolves on $\mathcal{D}_N^\varepsilon$ according to the Liouville equation (1.1.3) with specular boundary reflection (1.1.4). At time $t\geq 0$, the (rescaled) n-particle correlation function is defined as

$$F_n^\varepsilon(t,Z_n):=\mu_\varepsilon^{-n}\sum_{p=0}^{\infty}\frac{1}{p!}\int_{\mathbb{D}^p}dz_{n+1}\dots dz_{n+p}\,W_{n+p}^\varepsilon(t,Z_{n+p}) \tag{1.1.10}$$

and, as in (1.1.9), we get

$$\mathbb{E}_\varepsilon\Big(\sum_{\substack{i_1,\dots,i_n\\ i_j\neq i_k,j\neq k}}h_n\big(\mathbf{z}_{i_1}^\varepsilon(t),\dots,\mathbf{z}_{i_n}^\varepsilon(t)\big)\Big)=\mu_\varepsilon^n\int_{\mathbb{D}^n}dZ_n\,F_n^\varepsilon(t,Z_n)\,h_n\big(Z_n\big)\;, \tag{1.1.11}$$

where we used the notation (1.1.7). Notice that $F_n^\varepsilon(t,Z_n)=0$ for $Z_n\in\mathbb{D}^n\setminus\overline{\mathcal{D}_n^\varepsilon}$. In the following we shall denote the empirical measure

$$\pi_t^\varepsilon:=\frac{1}{\mu_\varepsilon}\sum_{i=1}^{\mathcal{N}}\delta_{\mathbf{z}_i^\varepsilon(t)}\;. \tag{1.1.12}$$

Tested on a (one-particle) function $h : \mathbb{D} \to \mathbb{R}$, it reads

$$\pi_t^\varepsilon(h) = \frac{1}{\mu_\varepsilon} \sum_{i=1}^{\mathcal{N}} h\left(\mathbf{z}_i^\varepsilon(t)\right) \ . \tag{1.1.13}$$

By definition, F_1^ε describes the average behavior of (exchangeable) particles :

$$\mathbb{E}_\varepsilon\big(\pi_t^\varepsilon(h)\big) = \int_{\mathbb{D}} F_1^\varepsilon(t,z)\, h(z)\, dz\,. \tag{1.1.14}$$

1.2. Lanford's theorem : a law of large numbers

In the Boltzmann-Grad limit $\mu_\varepsilon \to \infty$, the average behavior is governed by the Boltzmann equation :

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = \displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \Big(f(t,x,w')f(t,x,v') - f(t,x,w)f(t,x,v)\Big)\big((v-w)\cdot\omega\big)_+ \, d\omega\, dw\,, \\ f(0,x,v) = f^0(x,v) \end{cases} \tag{1.2.1}$$

where the precollisional velocities (v', w') are defined by the scattering law

$$v' := v - \big((v-w)\cdot\omega\big)\,\omega\,, \qquad w' := w + \big((v-w)\cdot\omega\big)\,\omega\,. \tag{1.2.2}$$

More precisely, the convergence is described by Lanford's theorem [**28**] (in the canonical setting — for the grand-canonical setting see [**27**], where the case of smooth compactly supported potentials is also addressed), which we state here in the case of the initial measure (1.1.6).

***Theorem 1* (Lanford [28]).** — *Consider a system of hard spheres initially distributed according to the grand canonical measure* (1.1.6) *with f^0 satisfying the estimates* (1.1.5)*. Then, in the Boltzmann-Grad limit $\mu_\varepsilon \to \infty$, the rescaled one-particle density $F_1^\varepsilon(t)$ converges uniformly on compact sets to the solution $f(t)$ of the Boltzmann equation* (1.2.1) *on a time interval $[0, T_0]$ (which depends only on f^0 through C_0, β_0). Furthermore for each n, the rescaled n-particle correlation function $F_n^\varepsilon(t)$ converges almost everywhere in $\mathbb{D}^n$ to $f^{\otimes n}(t)$ on the same time interval.*

We refer to [**22, 44, 11, 17, 14, 6, 35**] for details on this result and subsequent developments.

The propagation of chaos derived in Theorem 1 implies in particular that the empirical measure concentrates on the solution of Boltzmann equation. Indeed, computing the variance for any test function h, we get that

$$\begin{aligned} &\mathbb{E}_\varepsilon\Big(\big(\pi_t^\varepsilon(h) - \int F_1^\varepsilon(t,z)\, h(z)\, dz\big)^2\Big) \\ &= \mathbb{E}_\varepsilon\Big(\frac{1}{\mu_\varepsilon^2}\sum_{i=1}^{\mathcal{N}} h^2\big(\mathbf{z}_i^\varepsilon(t)\big) + \frac{1}{\mu_\varepsilon^2}\sum_{i\neq j} h\big(\mathbf{z}_i^\varepsilon(t)\big)h\big(\mathbf{z}_j^\varepsilon(t)\big)\Big) - \Big(\int F_1^\varepsilon(t,z)\, h(z)\, dz\Big)^2 \\ &= \frac{1}{\mu_\varepsilon}\int F_1^\varepsilon(t,z)\, h^2(z)\, dz + \int F_2^\varepsilon(t,Z_2)\, h(z_1)h(z_2)\, dZ_2 - \Big(\int F_1^\varepsilon(t,z)\, h(z)\, dz\Big)^2 \xrightarrow[\mu_\varepsilon\to\infty]{} 0\,, \end{aligned} \tag{1.2.3}$$

where the convergence to 0 follows from the fact that F_2^ε converges to $f^{\otimes 2}$ and F_1^ε to f almost everywhere. This computation can be interpreted as a law of large numbers and we have that, for all $\delta > 0$, and smooth h,

$$\mathbb{P}_\varepsilon\left(\Big|\pi_t^\varepsilon(h) - \int_{\mathbb{D}} f(t,z)h(z)dz\Big| > \delta\right) \xrightarrow[\mu_\varepsilon\to\infty]{} 0\,. \tag{1.2.4}$$

Remark 1.2.1. — *The restriction to the time interval* $[0, T_0]$ *in the statement of Theorem* 1 *is probably of technical nature: it originates from a Cauchy-Kowalevski argument in the Banach space of measurable sequences* $F = (F_n)_{n\geq 1}$ *with* $F_n : \mathbb{D}^n \to \mathbb{R}$*, endowed with norm* $\sup_{n\geq 1} \sup_{\mathbb{D}^n} \left(|F_n| e^{\alpha n + \frac{\beta}{2}|V_n|^2} \right)$ *for suitable* $\alpha, \beta \in \mathbb{R}$*.*

1.3. The fluctuating Boltzmann equation

Describing the fluctuations around the Boltzmann equation is a way to capture part of the information which has been lost in the limit $\varepsilon \to 0$.

As in the classical central limit theorem, we expect these fluctuations to be of order $1/\sqrt{\mu_\varepsilon}$, which is the typical size of the remaining correlations. We therefore define the fluctuation field ζ^ε as follows: for any test function $h : \mathbb{D} \to \mathbb{R}$

$$\zeta_t^\varepsilon(h) := \sqrt{\mu_\varepsilon} \left(\pi_t^\varepsilon(h) - \int F_1^\varepsilon(t,z)\, h(z)\, dz \right) . \tag{1.3.1}$$

Initially the empirical measure starts close to the density profile f^0 and ζ_0^ε converges in law towards a Gaussian white noise ζ_0 with covariance

$$\mathbb{E}\big(\zeta_0(h_1)\, \zeta_0(h_2)\big) = \int h_1(z)\, h_2(z)\, f^0(z)\, dz\, .$$

In this paper we prove that in the limit $\mu_\varepsilon \to \infty$, starting from "almost independent" hard spheres, ζ^ε converges to a Gaussian process, solving formally

$$d\zeta_t = \mathcal{L}_t\, \zeta_t\, dt + d\eta_t\, , \tag{1.3.2}$$

where $\mathcal{L}_t$ is the *linearized Boltzmann operator* around the solution $f(t)$ of the Boltzmann equation (1.2.1)

$$\begin{aligned} \mathcal{L}_t\, h(x,v) := -v \cdot \nabla_x h(x,v) + \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} dw\, d\omega \big((v-w)\cdot\omega\big)_+ \\ \times \big(f(t,x,w')h(x,v') + f(t,x,v')h(x,w') - f(t,x,v)h(x,w) - f(t,x,w)h(x,v) \big)\, . \end{aligned} \tag{1.3.3}$$

The noise $d\eta_t(x,v)$ is Gaussian, with zero mean and covariance

$$\begin{aligned} \mathbb{E}\left(\int dt_1\, dz_1 h_1(z_1) \eta_{t_1}(z_1) \int dt_2\, dz_2\, h_2(z_2) \eta_{t_2}(z_2) \right) \\ = \frac{1}{2} \int dt d\mu(z_1, z_2, \omega) f(t,z_1)\, f(t,z_2) \Delta h_1\, \Delta h_2 \end{aligned} \tag{1.3.4}$$

denoting

$$d\mu(z_1, z_2, \omega) := \delta_{x_1 - x_2} \big((v_1 - v_2)\cdot\omega \big)_+ d\omega\, dv_1\, dv_2 dx_1 \tag{1.3.5}$$

and defining for any h

$$\Delta h(z_1, z_2, \omega) := h(z_1') + h(z_2') - h(z_1) - h(z_2)\, ,$$

where $z_i' := (x_i, v_i')$ with notation (1.2.2) for the velocities obtained after scattering. We postpone the precise definition of a weak solution to (1.3.2) to Section 6.1.2.

Our result is the following.

Theorem 2. — *Consider a system of hard spheres initially distributed according to the grand canonical measure* (1.1.6) *where* f^0 *is a smooth function satisfying* (1.1.5). *Then in the Boltzmann-Grad limit* $\mu_\varepsilon \to \infty$, *the fluctuation field* $(\zeta_t^\varepsilon)_{t\geq 0}$ *converges in law to a Gaussian process solving* (1.3.2) *in a weak sense on a time interval* $[0, T^\star]$.

The convergence towards the limiting process (1.3.2) was conjectured by Spohn in [**43**] and the non-equilibrium covariance of the process at two different times was computed in [**42**], see also [**44**]. The noise emerges after averaging the deterministic microscopic dynamics. It is white in time and space, but correlated in velocities so that momentum and energy are conserved.

At equilibrium the convergence of a discrete-velocity version of the same process at equilibrium was derived rigorously in [**37**], starting from a dynamics with stochastic collisions (see also [**25, 24, 30**] for fluctuations in space-homogeneous models).

The physical aspects of the fluctuations for the rarefied gas have been thoroughly investigated in [**16, 42, 43**]. We also refer to [**8**], where we gave an outline of our results and strategy. Here we would like to recall only a few important features.

1) *The noise in* (1.3.2) *originates from recollisions.*

It is a very general fact that, when the macroscopic equation is dissipative, the dynamical equation for the fluctuations contains a term of noise. In the case under study, "recollisions" are a class of mechanical events giving a negligible contribution to the limit $\pi_t^\varepsilon \to f(t)$ (see (1.2.4)) – for example, two particles colliding twice with each other in a finite time. The proof of Theorem 2 provides a further insight on the relation between collisions and noise. Following [**42**], we represent the dynamics in terms of a special class of trajectories, for which one can classify precisely the recollisions responsible for the term $d\eta_t$; see Section 1.5 for further explanations. For the moment we just remind the reader that there is no a priori contradiction between the dynamics being deterministic, and the appearance of noise from collisions in the singular limit. Indeed when ε goes to zero, the deflection angles are no longer deterministic (as in the probabilistic interpretation of the Boltzmann equation). The randomness, which is entirely coded on the initial data of the hard sphere system, is transferred to the dynamics in the limit.

2) *Equilibrium fluctuations can be deduced by the fluctuation-dissipation theorem.*

As a particular case, we obtain the result at thermal equilibrium $f^0 = M$, where M is Maxwellian with inverse temperature β. The stochastic process (1.3.2) boils down to a generalized Ornstein-Uhlenbeck process. The noise term compensates the dissipation induced by the linearized Boltzmann operator, and the covariance of the noise (1.3.4) can be predicted heuristically by using the invariant measure [**44**].

3) *Away from equilibrium, the fluctuating equations keep the same structure.*

The most direct way to to guess (1.3.2)-(1.3.4) is starting from the equilibrium prediction (previous point) and *assuming* that $M = M(v)$ can be substituted with $f = f(t, x, v)$. This heuristics is known as "extended local equilibrium" assumption, in the context of fluctuating hydrodynamics. It is based on the remark that the noise is white in space and time, and therefore only the local (in (x, t)) features of the gas should be relevant. If the system has a "local equilibrium", this is enough to determine the equations. This procedure gives the right result also for our gas at low density (even if $f = f(t, x, v)$ is not locally Maxwellian). The reason is that a form of local equilibrium is still true; namely, around

a little cube centered in x at time t, the hard sphere system is described by a Poisson measure with constant density $\int f(t,x,v)dv$ [**44**].

4) *Away from equilibrium, fluctuations exhibit long range correlations.*

The covariance of the fluctuation field (at equal times and) at different points x_1, x_2 is not zero when $|x_1 - x_2|$ is of order one (and decays slowly with $|x_1 - x_2|$). This is typical of non equilibrium fluctuations [**16**]. In the hard sphere gas at low density, it is again related to recollisions, and the proof of Theorem 2 will provide an explicit formula quantifying this effect.

1.4. Large deviations

While typical fluctuations are of order $O(\mu_\varepsilon^{-1/2})$, they may sometimes happen to be large, leading to a dynamics which is different from the Boltzmann equation. A classical problem is to evaluate the probability of such an atypical event, namely that the empirical measure remains close to a probability density $\varphi \neq f$ during a time interval $[0, T^\star]$. The following explicit formula for the large deviation functional was obtained by Rezakhanlou [**38**] in the case of a one-dimensional stochastic dynamics mimicking the hard-sphere dynamics, and then conjectured for the three-dimensional deterministic hard-sphere dynamics by Bouchet [**9**]:

$$\widehat{\mathcal{F}}(t,\varphi) := \widehat{\mathcal{F}}(0,\varphi_0) + \sup_p \left\{ \int_0^t ds \left[\int_{\mathbb{T}^d} dx \int_{\mathbb{R}^d} dv\; p(s,x,v)\, D_s\varphi(s,x,v) - \mathcal{H}\big(\varphi(s), p(s)\big) \right] \right\}, \tag{1.4.1}$$

where the supremum is taken over bounded measurable functions p, and the Hamiltonian is given by

$$\mathcal{H}(\varphi,p) := \frac{1}{2} \int d\mu(z_1,z_2,\omega)\varphi(z_1)\varphi(z_2)\big(\exp\big(\Delta p\big) - 1\big)\,. \tag{1.4.2}$$

We have denoted D_t the transport operator

$$D_t\varphi(t,z) := \partial_t\varphi(t,z) + v\cdot\nabla_x\varphi(t,z)\;, \tag{1.4.3}$$

and finally

$$\widehat{\mathcal{F}}(0,\varphi_0) := \int_{\mathbb{D}} dz\;\left(\varphi_0 \log\left(\frac{\varphi_0}{f^0}\right) - \varphi_0 + f^0\right) \tag{1.4.4}$$

with $\varphi_0 = \varphi|_{t=0}$, is the large deviation rate for the empirical measure at time zero. $\widehat{\mathcal{F}}(0)$ can be obtained by a standard procedure, modifying the measure (1.1.6) in such a way to make the (atypical) profile φ_0 typical. Similarly, to obtain the collisional term H in $\widehat{\mathcal{F}}(t,\varphi)$, one would like to understand the mechanism leading to an atypical path $\varphi = \varphi(t)$ at positive times. A serious difficulty then arises, due to the deterministic dynamics. Ideally, one should conceive a way of tilting the initial measure in order to observe a given trajectory. Whether such an efficient bias exists, we do not know. But we shall proceed in a different way, inspecting somehow the dynamics at *all* scales in ε. This strategy, which will be informally described in the next section, leads to Theorem 3. The remarkable feature of this result is that the large deviation behaviour of the mechanical dynamics is also ruled by the large deviation functional of the stochastic process.

Denote by $\mathcal{M}$ the set of probability measures on $\mathbb{D}$ (with the topology of weak convergence) and by $D([0,T^\star],\mathcal{M})$ the Skorokhod space (see [**4**] page 121).

Theorem 3. — *Consider a system of hard spheres initially distributed according to the grand canonical measure* (1.1.6) *where* f^0 *satisfies* (1.1.5). *There exists a time* $T^\star$ *and a functional* $\mathcal{F} = \mathcal{F}(T^\star, \cdot)$ *such that, in the Boltzmann-Grad limit* $\mu_\varepsilon \to \infty$, *the empirical measure satisfies the following large deviation estimates :*

— *For any compact set* $\mathbf{F} \subset D([0, T^\star], \mathcal{M})$,

$$\limsup_{\mu_\varepsilon \to \infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon \left(\pi^\varepsilon \in \mathbf{F}\right) \leq - \inf_{\varphi \in \mathbf{F}} \mathcal{F}(T^\star, \varphi) \; ; \tag{1.4.5}$$

— *For any open set* $\mathbf{O} \subset D([0, T^\star], \mathcal{M})$,

$$\liminf_{\mu_\varepsilon \to \infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon \left(\pi^\varepsilon \in \mathbf{O}\right) \geq - \inf_{\varphi \in \mathbf{O} \cap \mathcal{R}} \mathcal{F}(T^\star, \varphi) \; , \tag{1.4.6}$$

where $\mathcal{R}$ *is a non trivial subset of* $D([0, T^\star], \mathcal{M})$.

Moreover there exists a non trivial subset of $\mathcal{R}$, *and a time* $T \leq T^\star$, *such that the functionals* $\mathcal{F}(T, \cdot)$ *and* $\hat{\mathcal{F}}(T, \cdot)$ *coincide on* $\mathcal{R}$.

The functional $\mathcal{F}$ is determined by the solution of a variational problem (see (7.0.3) below) and the set $\mathcal{R}$ is chosen such that the extremum of this variational principle is attained in a class of sufficiently small and regular functions: see (7.1.5).

For an extensive formal discussion on large deviations in the Boltzmann gas, we refer to [**9**].

1.5. Strategy of the proofs

In this section we provide an overview of the paper and describe, informally, the core of our argument leading to Theorems 2 and 3.

We should start recalling the basic features of the proof of Theorem 1. For a deterministic dynamics of interacting particles, so far there has been only one way to access the law of large numbers rigorously. The strategy is based on the 'hierarchy of moments' corresponding to the family of correlation functions $(F_n^\varepsilon)_{n \geq 1}$, Eq. (1.1.10). The main role of F_n^ε is to project the measure on finite groups of particles (groups of cardinality n), out of the total $\mathcal{N}$. The term 'hierarchy' refers to the set of linear BBGKY equations satisfied by this collection of functions (which will be written in Section 3.1), where the equation for F_n^ε has a source term depending on F_{n+1}^ε. This hierarchy is completely equivalent to the Liouville equation (1.1.3) for the family $(W_N^\varepsilon)_{N \geq 0}$, as it contains exactly the same amount of information. However as $\mathcal{N} \sim \mu_\varepsilon$ in the Boltzmann-Grad limit (1.1.8), one should make sense of a Liouville density depending on infinitely many variables, and the BBGKY hierarchy becomes the natural convenient way to grasp the relevant information. Lanford succeeded to show that the explicit solution $F_n^\varepsilon(t)$ of the BBGKY, obtained by iteration of the Duhamel formula, converges to a product $f^{\otimes n}(t)$ (propagation of chaos), where f is the solution of the Boltzmann equation (1.2.1).

The hierarchy of moments has two important limitations. The first one is the restriction on its time of validity, which comes from too many terms in the iteration: we are indeed unable to take advantage of cancellations between gain and loss terms. The second one is a drastic loss of information. We shall not give here a precise notion of 'information'. We limit ourselves to stressing that $(F_n^\varepsilon)_{n \geq 1}$ is suited to the description of typical events. In the limit, everything is encoded in f, no matter how large n.

Moreover, the Boltzmann equation produces some entropy along the dynamics: at least formally, f satisfies

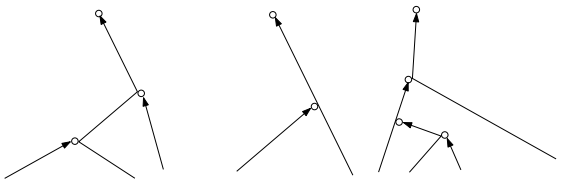
$$\partial_t\big(-\int f\log f\,dv\big)+\nabla_x\cdot\big(-\int f\log f\,v\,dv\big)\geq 0\,,$$

which is in contrast with the time-reversible hard-sphere dynamics. Our main purpose here is to overcome this second limitation (for short times) and to perform the Boltzmann-Grad limit in such a way as to keep most of the information lost in Theorem 1. In particular, the limiting functional (1.4.1) coincides with the large deviations functional of a genuine reversible Markov process, in agreement with the microscopic reversibility [**9**]. We face a significant difficulty: on the one hand, we know that *averaging* is important in order to go from Newton's equation to Boltzmann's equation; on the other hand, we want to keep track of some of the microscopic structure.

To this end, we need to go beyond the BBGKY hierarchy and turn to a more powerful representation of the dynamics. We shall replace the family $(F_n^\varepsilon)_{n\geq 1}$ (or $(W_N^\varepsilon)_{N\geq 0}$) with a third, equivalent, family of functions $(f_n^\varepsilon)_{n\geq 1}$, called (rescaled) *cumulants*[1]. Their role is to grasp information on the dynamics on finer and finer scales. Loosely speaking, f_n^ε will collect events where n particles are "completely connected" by a chain of interactions. We shall say that the n particles form a *cluster*. Since a collision between two given particles is typically of order μ_ε^{-1}, a "complete connection" would account for events of probability of order $\mu_\varepsilon^{-(n-1)}$. We therefore end up with a hierarchy of rare events, which we need to control at all orders to obtain Theorem 3. At variance with $(F_n^\varepsilon)_{n\geq 1}$, even *after* the limit $\mu_\varepsilon\to\infty$ is taken, the rescaled cumulant f_n^ε cannot be trivially obtained from the cumulant f_{n-1}^ε. Each step entails extra information, and events of increasing complexity, and decreasing probability.

The cumulants, which are a standard probabilistic tool, will be investigated here in the dynamical, non-equilibrium context. Their precise definition and basic properties are discussed in Chapter 2.

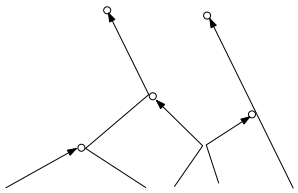
The introduction of cumulants will not entitle us to avoid the BBGKY hierarchy entirely. Unfortunately, the equations for $(f_n^\varepsilon)_{n\geq 1}$ are difficult to handle. But the moment-to-cumulant relation $(F_n^\varepsilon)_{n\geq 1}\to(f_n^\varepsilon)_{n\geq 1}$ is a bijection and, in order to construct $f_n^\varepsilon(t)$, we can still resort to the same solution representation of [**28**] for the correlation functions $(F_n^\varepsilon(t))_{n\geq 1}$. This formula is an expansion over *collision trees*, meaning that it has a geometrical representation as a sum over binary tree graphs, with vertices accounting for collisions. The formula will be presented in Chapter 3 (and generalized from the finite-dimensional case to the case of functionals over trajectories, which is needed to deal with space-time processes). For the moment, let us give an idea of the structure of this tree expansion. The Duhamel iterated solution for $F_n^\varepsilon(t)$ has a peculiar characteristic flow: n hard spheres (of diameter ε) at time t flow backwards, and collide (among themselves or) with a certain number of external particles, which are added at random times and at random collision configurations. The following picture is an example of such flow (say, $n=3$):



1. Cumulant type expansions within the framework of kinetic theory appear in [**5, 35, 29, 18**]

The net effect resembles a binary tree graph. The real graph is just a way to record which pairs of particles collided, and in which order.

It is important to notice that different subtrees are unlikely to interact: since the hard spheres are small and the trajectories involve finitely many particles, two subtrees will encounter each other with small probability. This is a rather pragmatic point of view on the propagation of chaos, and the reason why $F_n^\varepsilon(t)$ is close to a tensor product (if it is so at time zero) in the classical Lanford argument. Observe that, in this simple argument, we are giving a notion of dynamical *correlation* which is purely geometrical. Actually we will use this idea over and over. Two particles are correlated if their generated subtrees are *connected*, as represented for instance in the following picture:



which is an event of 'size' μ_ε^{-1} (the volume of a tube of diameter ε and length 1). In Chapter 4, we will give precise definitions of correlation (connection) based on geometrical constraints. It will be the elementary brick to characterize $f_n^\varepsilon(t)$ explicitly in terms of the initial data. The formula for $f_n^\varepsilon(t)$ (Section 4.4) will be supported on characteristic flows with n particles connected, through their generated subtrees (hence of expected size $\mu_\varepsilon^{-(n-1)}$). In other words, while F_n^ε projects the measure on *arbitrary* groups of particles of size n, the improvement of f_n^ε consists in restricting to *completely connected* clusters of the same size.

With this naive picture in mind, let us briefly comment again on information, and irreversibility. One nice feature of the geometric analysis of recollisions is that it reflects the transition from a time-reversible to a time-irreversible model. In [**7**] we identified, and quantified, the microscopic singular sets where F_n^ε does *not* converge. These sets are not invariant by time-reversal (they have a direction always pointing to the past, and not to the future). Looking at $F_n^\varepsilon(t)$, we lose track of what happens in these small sets. This implies, in particular, that Theorem 1 cannot be used to come back from time $t > 0$ to the initial state at time zero. The cumulants describe what happens on all the small singular sets, therefore providing the information missing to recover the reversibility.

At the end of Chapter 4, we give a uniform estimate on these cumulants (Theorem 4), which is the main advance of this paper. This L^1-bound is sharp in ε and n (n-factorial bound), roughly stating that the unscaled cumulant decays as $\mu_\varepsilon^{-(n-1)} n^{n-2}$. This estimate is intuitively simple. We have given a geometric notion of correlation as a *link* between two collision trees. Based on this notion, we can draw a random graph telling us which particles are correlated and which particles are not (each collision tree being one vertex of the graph). Since the cumulant describes n completely correlated particles, there will be at least $n-1$ edges, each one of small 'volume' μ_ε^{-1}. Of course there may be more than $n-1$ connections (the random graph has cycles), but these are hopefully unlikely as they produce extra smallness in ε. If we ignore all of them, we are left with minimally connected graphs, whose total number is n^{n-2} by Cayley's formula. Thanks to the good dependence in n of these uniform bounds, we can actually *sum up* all the family of cumulants into an analytic series, referred to as 'cumulant generating function'.

The second central result of this paper, stated in Chapter 5 (Theorem 5), is the characterization of the rescaled cumulants in the Boltzmann-Grad limit, with minimally connected graphs. Using this minimality property, we actually derive a Hamilton-Jacobi equation for the limiting cumulant generating function. Wellposedness and uniqueness for this equation can be achieved by abstract methods, based on analyticity. All the information of the microscopic mechanical model is actually encoded in this Hamilton-Jacobi equation which, in particular, allows us to characterize the large deviation functional. which is our ultimate point of arrival. From this Hamilton-Jacobi equation, we can also obtain differential equations for the limiting family of cumulants $(f_n)_{n\geq 1}$. These equations, which we may call *Boltzmann cumulant hierarchy*, have a remarkable structure and have been written first in [**16**].

The rest of the paper is devoted to the proofs of our main results.

Chapter 6 proves Theorem 2. Here, the uniform bounds of Theorem 4 are considerably better than what is required, and the proof amounts to looking at a characteristic function living on larger scales. The more technical part of the proof concerns the tightness of the process for which we adapt a Garsia-Rodemich-Rumsey's inequality on the modulus of continuity, to the case of a discontinuous process.

In Chapter 7 we prove Theorem 3. Our purpose is to show that the functional obtained in Chapter 5 is dual, through the Legendre transform, to a large deviation rate function. In the absence of global convexity, we will not succeed in proving a full large deviation principle. However, restricting to a class of regular profiles, the variational problem is uniquely solved and the rate functional can be identified with the one predicted in the physical literature, based on the analogy with stochastic dynamics.

Finally, Chapters 8 and 9 are devoted to the proof of Theorems 4 and 5, respectively. We encounter here a combinatorial issue. The number of terms in the formula for $f_n^\varepsilon(t)$ grows, at first sight, badly with n, and cancellations need to be exploited to obtain a factorial growth. At this point, cluster expansion methods ([**39**]) enter the game (summarized in Chapter 2), applied to the collision trees. The decay $\mu_\varepsilon^{-(n-1)}$ follows instead from a geometric analysis on hard-sphere trajectories with $n-1$ connecting constraints, in the spirit of previous work [**5, 7, 35**].

1.6. Remarks, and open problems

We conclude with a few remarks on our results.

- To simplify our proof, we assumed that the initial datum is a quasi-product measure, with the minimal amount of correlations (only the mutual exclusion between hard spheres is taken into account). This assumption is useful to isolate the dynamical part of the problem in the clearest way. More general initial states could be dealt with along the same lines ([**43, 35**]). However the cumulant expansions would contain more terms, describing the deterministic (linearized) transport of initial correlations.
- Similarly, fixing only the average number of particles (instead of the exact number of particles) allows to avoid spurious correlations. We therefore work in a grand canonical setting, as is customary in statistical physics when dealing with fluctuations. Notice that fixing $\mathcal{N} = N$ produces a long range term of order $1/N$ in the covariance of the fluctuation field. Note also

that the cluster expansion method, which is crucial in our analysis, is developed (with few exceptions, see [**36**] for instance) in a grand canonical framework [**33**].

— Our results could be established in the whole space $\mathbb{R}^d$, or in a parallelepiped box with periodic or reflecting boundary conditions. Different domains might be also covered, at the expense of complications in the geometrical estimates of recollisions (see [**15**] for instance).

— We do not deal with the original BBGKY hierarchy of equations, which was written for smooth potentials, but always restrict to the hard-sphere system. It is plausible that our results could be extended to smooth, compactly supported potentials as considered in [**17, 34**] (see [**2**] for a fast decaying case), but the proof would be considerably more involved.

— At thermal equilibrium, we expect Theorem 2 to be true globally in time: see [**5**] for a first step in this direction.

PART I

DYNAMICAL CUMULANTS

CHAPTER 2

COMBINATORICS ON CONNECTED CLUSTERS

This preliminary chapter consists in presenting a few notions (well-known in statistical mechanics) that will be essential in our analysis. We present in particular cumulants, and their link with exponential moments as well as with cluster expansions. We conclude the chapter with some combinatorial identities that will be useful throughout this work.

2.1. Generating functionals and cumulants

Let $h : \mathbb{D} \to \mathbb{R}$ be a bounded continuous function. We shall use the notation

$$\left\langle F_n^\varepsilon(t), h^{\otimes n} \right\rangle = \int_{\mathbb{D}^n} dZ_n \, F_n^\varepsilon(t, Z_n) h(z_1) \dots h(z_n) \, , \tag{2.1.1}$$

and

$$\mathcal{P}_n^s = \text{set of partitions of } \{1, \dots, n\} \text{ into } s \text{ parts} \, ,$$

with

$$\sigma \in \mathcal{P}_n^s \Longrightarrow \sigma = \{\sigma_1, \dots, \sigma_s\} \, , \quad |\sigma_i| = \kappa_i \, , \quad \sum_{i=1}^s \kappa_i = n \, .$$

The moment generating functional of the empirical measure (1.1.13), namely $\mathbb{E}_\varepsilon\Big(\exp\big(\pi_t^\varepsilon(h)\big)\Big)$ is related to the rescaled correlation functions (1.1.10) by the following remark. We recall that

$$\mathbb{E}_\varepsilon\Big(\exp\big(\pi_t^\varepsilon(h)\big)\Big) = \mathbb{E}_\varepsilon\left[\exp\big(\frac{1}{\mu_\varepsilon} \sum_{i=1}^{\mathcal{N}} h\big(\mathbf{z}_i^\varepsilon(t)\big)\big)\right] \, . \tag{2.1.2}$$

Proposition 2.1.1. — *We have that*

$$\mathbb{E}_\varepsilon\Big(\exp\big(\pi_t^\varepsilon(h)\big)\Big) = 1 + \sum_{n=1}^\infty \frac{\mu_\varepsilon^n}{n!} \left\langle F_n^\varepsilon(t), \Big(e^{h/\mu_\varepsilon} - 1\Big)^{\otimes n} \right\rangle \tag{2.1.3}$$

if the series is absolutely convergent.

Proof. — Starting from (2.1.2), one has

$$\sum_{k\geq 1}\frac{1}{k!}\,\mathbb{E}_\varepsilon\Big(\big(\pi_t^\varepsilon(h)\big)^k\Big)=\sum_{k\geq 1}\frac{1}{k!}\sum_{n=1}^{k}\sum_{\sigma\in\mathcal{P}_k^n}\mu_\varepsilon^{-k}\,\mathbb{E}_\varepsilon\Big(\sum_{\substack{i_1,\dots,i_n\\ i_j\neq i_\ell, j\neq\ell}} h\left(\mathbf{z}_{i_1}^\varepsilon(t)\right)^{\kappa_1}\dots h\left(\mathbf{z}_{i_n}^\varepsilon(t)\right)^{\kappa_n}\Big)$$

$$=\sum_{k\geq 1}\frac{1}{k!}\sum_{n=1}^{k}\sum_{\sigma\in\mathcal{P}_k^n}\mu_\varepsilon^{-k}\,\mu_\varepsilon^n\int_{\mathbb{D}^n}dZ_n\,F_n^\varepsilon(t,Z_n)h(z_1)^{\kappa_1}\dots h(z_n)^{\kappa_n}$$

where in the last equality we used (1.1.11). On the other hand for fixed n

$$\sum_{k\geq n}\frac{\mu_\varepsilon^{-k}}{k!}\sum_{\sigma\in\mathcal{P}_k^n}\prod_{i=1}^{n}h(z_i)^{\kappa_i}=\sum_{k\geq n}\frac{\mu_\varepsilon^{-k}}{k!\,n!}\sum_{\substack{\kappa_1\cdots\kappa_n\geq 1\\ \sum\kappa_i=k}}\binom{k}{\kappa_1}\binom{k-\kappa_1}{\kappa_2}\cdots\binom{k-\kappa_1-\cdots-\kappa_{n-2}}{\kappa_{n-1}}\prod_{i=1}^{n}h(z_i)^{\kappa_i}$$

$$=\frac{1}{n!}\prod_{i=1}^{n}\sum_{\kappa_i\geq 1}\frac{h(z_i)^{\kappa_i}}{\mu_\varepsilon^{\kappa_i}\kappa_i!}=\frac{1}{n!}\prod_{i=1}^{n}\Big(e^{h(z_i)/\mu_\varepsilon}-1\Big)\ .$$

Therefore

$$\mathbb{E}_\varepsilon\left(\exp\left(\pi_t^\varepsilon(h)\right)\right)=1+\sum_{n\geq 1}\mu_\varepsilon^n\int_{\mathbb{D}^n}dZ_n\,F_n^\varepsilon(t,Z_n)\frac{1}{n!}\prod_{i=1}^{n}\Big(e^{h(z_i)/\mu_\varepsilon}-1\Big)\ ,$$

which proves the proposition. □

The moment generating functional is just a compact representation of the information coded in the family $(F_n^\varepsilon(t))_{n\geq 1}$. After the Boltzmann-Grad limit $\mu_\varepsilon\to\infty$, the right-hand side of (2.1.3) reduces to $\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\int f(t)h\Big)^n=\exp\Big(\int f(t)h\Big)$, i.e. to the solution of the Boltzmann equation.

As discussed in the introduction, our purpose is to keep a much larger amount of information. To this end, we study the cumulant generating functional which is, by Cramér's theorem, an obvious candidate to reach atypical profiles [**46**]. Namely, we pass to the logarithm and rescale as follows:

$$\Lambda_t^\varepsilon(e^h):=\frac{1}{\mu_\varepsilon}\log\mathbb{E}_\varepsilon\left(\exp\left(\mu_\varepsilon\,\pi_t^\varepsilon(h)\right)\right)=\frac{1}{\mu_\varepsilon}\log\mathbb{E}_\varepsilon\Big(\exp\big(\sum_{i=1}^{\mathcal{N}}h\big(\mathbf{z}_i^\varepsilon(t)\big)\big)\Big)\ . \tag{2.1.4}$$

The first task is to look for a proposition analogous to the previous one. In doing so, the following definition emerges naturally, where we use the notation:

$$G_{\sigma_j}:=G_{|\sigma_j|}(Z_{\sigma_j})\,,\quad G_\sigma:=\prod_{j=1}^{|\sigma|}G_{\sigma_j} \tag{2.1.5}$$

for $\sigma=\{\sigma_1,\dots,\sigma_s\}\in\mathcal{P}_n^s$.

Definition 2.1.2 **(Cumulants)**. — *Let $(G_n)_{n\geq 1}$ be a family of distributions of n variables invariant by permutation of the labels of the variables. The* cumulants *associated with $(G_n)_{n\geq 1}$ form the family $(g_n)_{n\geq 1}$ defined, for all $n\geq 1$, by*

$$g_n:=\sum_{s=1}^{n}\sum_{\sigma\in\mathcal{P}_n^s}(-1)^{s-1}(s-1)!\,G_\sigma\ .$$

We then have the following result, which is well-known in the theory of point processes (see [**12**]).

Proposition 2.1.3. — *Let* $\left(\mu_\varepsilon^{-(n-1)} f_n^\varepsilon\right)_{n\geq 1}$ *be the family of cumulants associated with* $(F_n^\varepsilon)_{n\geq 1}$. *We have*

$$\Lambda_t^\varepsilon(e^h) = \sum_{n=1}^{\infty} \frac{1}{n!} \left\langle f_n^\varepsilon(t), \left(e^h-1\right)^{\otimes n} \right\rangle ,$$

if the series is absolutely convergent.

Proof. — Applying Proposition 2.1.1 to h in place of h/μ_ε, expanding the logarithm in a series and using Definition 2.1.2, we get

$$\begin{aligned}
\frac{1}{\mu_\varepsilon} \log \mathbb{E}_\varepsilon \left(\exp\left(\mu_\varepsilon\, \pi_t^\varepsilon(h)\right)\right) &= \frac{1}{\mu_\varepsilon} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \prod_{\ell=1}^{n} \left[\sum_{p_\ell} \frac{\mu_\varepsilon^{p_\ell}}{p_\ell!} \left\langle F_{p_\ell}^\varepsilon(t), (e^h-1)^{\otimes p_\ell} \right\rangle \right] \\
&= \frac{1}{\mu_\varepsilon} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{p_1,\dots,p_n} \frac{\mu_\varepsilon^{p_1+\cdots+p_n}}{p_1!\dots p_n!} \prod_{\ell=1}^{n} \left\langle F_{p_\ell}^\varepsilon(t), (e^h-1)^{\otimes p_\ell} \right\rangle \\
&= \sum_{p=1}^{\infty} \mu_\varepsilon^{p-1} \sum_{n=1}^{p} \sum_{\sigma\in\mathcal{P}_p^n} (-1)^{n-1}(n-1)! \prod_{\ell=1}^{n} \left\langle F_{p_\ell}^\varepsilon(t), (e^h-1)^{\otimes p_\ell} \right\rangle \\
&= \sum_{p=1}^{\infty} \frac{1}{p!} \left\langle f_p^\varepsilon(t), \left(e^h-1\right)^{\otimes p} \right\rangle ,
\end{aligned}$$

which proves the result. □

Note that cumulants measure departure from chaos in the sense that they vanish identically at order $n \geq 2$ in the case of i.i.d. random variables.

2.2. Inversion formula for cumulants

In this paragraph we prove that the cumulants (g_n) associated with a family (G_n) in the sense of Definition 2.1.2, encode all the correlations, meaning that G_n can be reconstructed from $(g_k)_{k\leq n}$ for all $n \geq 1$. More precisely, the following inversion formula holds.

Proposition 2.2.1. — *Let* $(G_n)_{n\geq 1}$ *be a family of distributions and* $(g_n)_{n\geq 1}$ *its cumulants in the sense of Definition* 2.1.2. *Then for each* $n \geq 1$, *the distribution* G_n *can be recovered from the cumulants* $(g_k)_{k\leq n}$ *by the formula*

$$\forall n \geq 1\,, \qquad G_n = \sum_{s=1}^{n} \sum_{\sigma\in\mathcal{P}_n^s} g_\sigma\,. \tag{2.2.1}$$

Proof. — Clearly $G_1 = g_1$. Suppose that the inverse formula (2.2.1) holds up to the level $n-1$ and let us check (2.2.1) at level n, i.e. that

$$G_n(Z_n) = g_n(Z_n) + \sum_{s=2}^{n} \sum_{\sigma\in\mathcal{P}_n^s} g_\sigma\,. \tag{2.2.2}$$

Replacing the cumulants g_{σ_j} by their definition, we get

$$\mathbb{A}_n := \sum_{s=2}^{n} \sum_{\sigma\in\mathcal{P}_n^s} g_\sigma = \sum_{s=2}^{n} \sum_{\sigma\in\mathcal{P}_n^s} \prod_{j=1}^{s} \Big(\sum_{k_j=1}^{\sigma_j} \sum_{\kappa_j\in\mathcal{P}_{\sigma_j}^{k_j}} (-1)^{k_j-1}(k_j-1)!\, G_{\kappa_j} \Big)\,.$$

denoting by $\mathcal{P}_V^k$ the set of partitions of V in k parts .

Using the Fubini Theorem, we can index the sum by the partitions with $r := \sum_{j=1}^{s} k_j$ sets and obtain

$$\mathbb{A}_n = \sum_{r=2}^{n} \sum_{\rho \in \mathcal{P}_n^r} G_\rho \Big(\sum_{s=2}^{r} \sum_{\omega \in \mathcal{P}_r^s} (-1)^{r-s} \prod_{i=1}^{s} (|\omega_i| - 1)! \Big) . \tag{2.2.3}$$

Note that the partition σ in the definition of $\mathbb{A}_n$ can be recovered as

$$\forall i \leq s \,, \qquad \sigma_i = \bigcup_{j \in \omega_i} \rho_j \,.$$

Using the combinatorial identity

$$\sum_{k=1}^{n} \sum_{\sigma \in \mathcal{P}_n^k} (-1)^k \prod_{i=1}^{k} (|\sigma_i| - 1)! = 0$$

(see Lemma 2.5.1 below for a proof), we find that

$$\sum_{s=2}^{r} \sum_{\omega \in \mathcal{P}_r^s} (-1)^{r-s} \prod_{i=1}^{s} (|\omega_i| - 1)! = -(-1)^{r-1} (r-1)! \,,$$

hence it follows that

$$\mathbb{A}_n = -\sum_{r=2}^{n} \sum_{\rho \in \mathcal{P}_n^r} G_\rho (-1)^{r-1} (r-1)! = -g_n(Z_n) + G_n(Z_n) \,,$$

where the last equality follows from the definition of g_n. This completes the proof of Proposition 2.2.1. □

2.3. Clusters and the tree inequality

We now prove that the cumulant of order n is supported on clusters (connected groups) of cardinality n. We shall consider an abstract situation based on a "disconnection" condition, the definition of which may change according to the context.

Definition 2.3.1. — *A* connection *is a commutative binary relation $\sim$ on a set V:*

$$x \sim y \,, \quad x, y \in V \,.$$

The (commutative) complementary relation, called disconnection*, is denoted $\not\sim$, that is $x \not\sim y$ if and only if $x \sim y$ is false.*

Consider the indicator function that n elements $\{\eta_1, \ldots, \eta_n\}$ are disconnected

$$\Phi_n\big(\eta_1, \ldots, \eta_n\big) := \prod_{1 \leq i \neq j \leq n} \mathbf{1}_{\eta_i \not\sim \eta_j} \,.$$

For $n = 1$, we set $\Phi_1\big(\eta_1\big) \equiv 1$.

The following proposition shows that the cumulant of order n of Φ_n is supported on clusters of length n, meaning configurations $(\eta_1, \ldots, \eta_n)$ in which all elements are linked by a chain of connected elements. Before stating the proposition let us recall some classical terminology on graphs. This definition, as well as Proposition 2.3.3 and its proof, are taken from [**23**].

Definition 2.3.2. — *Let V be a set of vertices and $E \subset \big\{\{v,w\},\ v,w \in V\,,\ v \neq w\big\}$ a set of edges. The pair $G = (V,E)$ is called a* graph *(undirected, no self-edge, no multiple edge). Given a graph G we denote by $E(G)$ the set of all edges in G. The graph is said* connected *if for all $v,w \in V$, $v \neq w$, there exist $v_0 = v, v_1, v_2, \dots, v_n = w$ such that $\{v_{i-1}, v_i\} \in E$ for all $i = 1, \dots, n$.*

We denote by $\mathcal{C}_V$ the set of connected graphs with V as vertices, and by $\mathcal{C}_n$ the set of connected graphs with n vertices when $V = \{1, \dots, n\}$. A minimally connected*, or* tree graph*, is a connected graph with $n-1$ edges. We denote by $\mathcal{T}_V$ the set of minimally connected graphs with V as vertices, and by $\mathcal{T}_n$ the set of minimally connected graphs with n vertices when $V = \{1, \dots, n\}$.*

Finally, the union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

The following result was originally derived by Penrose [**32**].

Proposition 2.3.3. — *The cumulant of Φ_n defined as in Definition* 2.1.2 *is equal to*

$$\varphi_n\big(\eta_1, \dots, \eta_n\big) = \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in E(G)} \big(-\mathbf{1}_{\eta_i \sim \eta_j}\big)\,. \tag{2.3.1}$$

Furthermore, one has the following "tree inequality"

$$\big|\varphi_n\big(\eta_1, \dots, \eta_n\big)\big| \leq \sum_{T \in \mathcal{T}_n} \prod_{\{i,j\} \in E(T)} \mathbf{1}_{\eta_i \sim \eta_j}\,. \tag{2.3.2}$$

Proof. — The first step is to check the representation formula (2.3.1) for the cumulant φ_n. The starting point is the definition of Φ_n

$$\Phi_n\big(\eta_1, \dots, \eta_n\big) = \prod_{1 \leq i \neq j \leq n} \big(1 - \mathbf{1}_{\{\eta_i \sim \eta_j\}}\big) = \sum_{G} \prod_{\{i,j\} \in E(G)} \big(-\mathbf{1}_{\eta_i \sim \eta_j}\big)\,,$$

where the sum over G runs over all graphs with n vertices. We then decompose these graphs in connected components and obtain that

$$\Phi_n\big(\eta_1, \dots, \eta_n\big) = \sum_{s=1}^{n} \sum_{\sigma \in \mathcal{P}_n^s} \prod_{k=1}^{s} \left(\sum_{G_k \in \mathcal{C}_{\sigma_k}} \prod_{\{i,j\} \in E(G_k)} \big(-\mathbf{1}_{\eta_i \sim \eta_j}\big) \right)\,.$$

By identification with the formula (2.2.1), we therefore deduce that

$$\varphi_n\big(\eta_1, \dots, \eta_n\big) = \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in E(G)} \big(-\mathbf{1}_{\eta_i \sim \eta_j}\big)\,.$$

The second step is to compare connected graphs and trees, defining a tree partition scheme, i.e. a map $\pi : \mathcal{C}_n \to \mathcal{T}_n$ such that for any $T \in \mathcal{T}_n$, there is a graph $R(T) \in \mathcal{C}_n$ satisfying

$$\pi^{-1}(\{T\}) = \big\{G \in \mathcal{C}_n \,/\, E(T) \subset E(G) \subset E(R(T))\big\}\,.$$

Penrose's partition scheme is obtained in the following way. Given a graph G, we define its image T iteratively starting from the root 1

- the first generation of T consists of all i such that $\{1, i\} \in G$; these vertices are labeled in increasing order $t_{1,1}, \dots, t_{1,r_1}$.
- the ℓ-th generation consists of all i which are not already in the tree, and such that $\{t_{\ell-1,j}, i\}$ belongs to $E(G)$ for some $j \in \{1, \dots, r_{\ell-1}\}$; these vertices are labeled in increasing order of $j = 1, \dots, r_{\ell-1}$, then increasing order of i.

The procedure ends obviously with a unique tree $T \in \mathcal{T}_n$. In order to characterize $R(T)$, we then have to investigate which edges of G have been discarded. Denote by $d(i)$ the graph distance of the vertex i to the root (which is just its generation). Let $\{i,j\} \in E(G)\backslash E(T)$ and assume without loss of generality that $d(i) \leq d(j)$. By construction $d(j) \leq d(i)+1$. Furthermore, if $d(j) = d(i)+1$, the parent i' of j in the tree is such that $i' < i$. Therefore $E(G) \setminus E(T)$ is a subset of the set $E'(T)$ consisting of edges within a generation ($d(i) = d(j)$), and of edges towards a younger uncle ($d(j) = d(i)+1$ and $i' < i$). Conversely, we can check that any graph satisfying $E(T) \subset G \subset E(T) \cup E'(T)$ belongs to $\pi^{-1}(\{T\})$. We therefore define $R(T)$ as the graph with edges $E(T) \cup E'(T)$.

The last step is to exploit the non trivial cancellations between graphs associated with the same tree. There holds, with the above notation,

$$\begin{aligned}\sum_{G\in\mathcal{C}_n}\prod_{\{i,j\}\in E(G)}(-\mathbf{1}_{\eta_i\sim\eta_j}) &= \sum_{T\in\mathcal{T}_n}\sum_{G\in\pi^{-1}(T)}\prod_{\{i,j\}\in E(G)}(-\mathbf{1}_{\eta_i\sim\eta_j})\\ &= \sum_{T\in\mathcal{T}_n}\left(\prod_{\{i,j\}\in E(T)}(-\mathbf{1}_{\eta_i\sim\eta_j})\right)\left(\sum_{E'\subset E'(T)}\prod_{\{i,j\}\in E'}(-\mathbf{1}_{\eta_i\sim\eta_j})\right)\\ &= \sum_{T\in\mathcal{T}_n}\left(\prod_{\{i,j\}\in E(T)}(-\mathbf{1}_{\eta_i\sim\eta_j})\right)\left(\prod_{\{i,j\}\in E'(T)}(1-\mathbf{1}_{\eta_i\sim\eta_j})\right).\end{aligned}$$

The conclusion follows from the fact that $(1-\mathbf{1}_{\eta_i\sim\eta_j}) \in [0,1]$. The proposition is proved. □

2.4. Number of minimally connected graphs

The following classical result will be used in Chapter 8.

Lemma 2.4.1. — *The cardinality of the set of minimally connected graphs on n vertices with degrees (number of edges per vertex) of the vertices $1,\dots,n$ fixed respectively at the values $d_1,\dots,d_n$ is*

$$\left|\left\{T\in\mathcal{T}_n \;\middle|\; d_1(T)=d_1,\dots,d_n(T)=d_n\right\}\right| = \frac{(n-2)!}{\prod_{i=1}^n (d_i-1)!}\,. \tag{2.4.1}$$

Proof. — We notice preliminarily that this implies Cayley's formula $|\mathcal{T}_n| = n^{n-2}$. Indeed the graph is minimal, so there are exactly $n-1$ edges hence (each edge has two vertices) the sum of the degrees has to be equal to $2n-2$. Thus

$$|\mathcal{T}_n| = \sum_{\substack{d_1,\dots,d_n\\ 1\leq d_i\leq n-1\\ \sum_i d_i = 2(n-1)}} \frac{(n-2)!}{\prod_{i=1}^n (d_i-1)!} = \sum_{\substack{d_1,\dots,d_n\\ 0\leq d_i\leq n-2\\ \sum_i d_i = n-2}} \frac{(n-2)!}{\prod_{i=1}^n d_i!} = \left(\sum_{i=1}^n 1\right)^{n-2}.$$

The lemma can be proved by induction. For $n=2$ the result is trivial, so we suppose to have proved it for the set $\mathcal{T}_n^{d_1,\dots,d_n} := \{T\in\mathcal{T}_n \mid d_1(T)=d_1,\dots,d_n(T)=d_n\}$, for arbitrary $d_1,\dots,d_n$, and consider the set $\mathcal{T}_{n+1}^{d_1,\dots,d_{n+1}} := \{T\in\mathcal{T}_{n+1} \mid d_1(T)=d_1,\dots,d_{n+1}(T)=d_{n+1}\}$. Since there is always at least one vertex of degree 1, we can assume without loss of generality that $d_{n+1}=1$. Notice that, if the vertex $n+1$ is linked to the vertex j, then necessarily $d_j \geq 2$. We therefore compute the number of

minimally connected graphs on n vertices with degrees $d_1, \dots, d_{j-1}, d_j - 1, d_{j+1}, \dots, d_n$, and sum then over j (all the ways to attach the vertex $n+1$ of degree 1). This leads to

$$|\mathcal{T}_{n+1}^{d_1,\dots,d_{n+1}}| = \sum_{j=1}^{n} \frac{(n-2)!}{(d_j-2)! \prod_{i\neq j}(d_i-1)!}\,,$$

hence

$$|\mathcal{T}_{n+1}^{d_1,\dots,d_{n+1}}| = \frac{(n-2)!}{\prod_{i=1}^{n+1}(d_i-1)!} \sum_{j=1}^{n+1}(d_j-1) = \frac{(n-1)!}{\prod_{i=1}^{n}(d_i-1)!}$$

having used again $\sum_{j=1}^{n+1} d_j = 2(n+1-1)$. □

2.5. Combinatorial identities

In the previous paragraphs and later in this work the following combinatorial identities are used.

Lemma 2.5.1. — *For $n \geq 2$ there holds*

$$\sum_{k=1}^{n} \sum_{\sigma\in\mathcal{P}_n^k} (-1)^k (k-1)! = 0\,, \tag{2.5.1}$$

$$\sum_{k=1}^{n} \sum_{\sigma\in\mathcal{P}_n^k} (-1)^k \prod_{i=1}^{k} (|\sigma_i|-1)! = 0\,. \tag{2.5.2}$$

Proof. — From the Taylor series of $x \mapsto \log\big(\exp(x)\big)$, we deduce that

$$\forall n \geq 2, \qquad \sum_{k=1}^{n} \sum_{\ell_1+\dots+\ell_k=n} \frac{(-1)^k}{k} \frac{1}{\ell_1!\dots\ell_k!} = 0\,.$$

The number of partitions of $\{1,\dots,n\}$ into k sets with cardinals $\ell_1,\dots,\ell_k$ is given by

$$\sharp\mathcal{P}_n^k(\ell_1,\dots,\ell_k) = \frac{1}{k!}\binom{n}{\ell_1}\binom{n-\ell_1}{\ell_2}\cdots\binom{n-\ell_1-\dots-\ell_{k-1}}{\ell_k} = \frac{1}{k!}\,\frac{n!}{\ell_1!\,\dots\,\ell_k!}\,, \tag{2.5.3}$$

where the factor $k!$ arises to take into account the fact that the sets of the partition are not ordered. Combining (2.5.3) and the previous identity, we get

$$\begin{aligned} 0 &= \sum_{k=1}^{n} \sum_{\ell_1+\dots+\ell_k=n} \frac{(-1)^k}{k} \frac{1}{\ell_1!\dots\ell_k!} = \sum_{k=1}^{n} \frac{(-1)^k}{k} \sum_{\ell_1+\dots+\ell_k=n} \frac{k!}{n!} \sharp\mathcal{P}_n^k(\ell_1,\dots,\ell_k) \\ &= \frac{1}{n!} \sum_{k=1}^{n} (-1)^k (k-1)! \sharp\mathcal{P}_n^k \end{aligned}$$

and this completes the first identity (2.5.1).

From the Taylor series of $x \mapsto \exp\big(\log(1+x)\big)$, we deduce that

$$\forall n \geq 2, \qquad \sum_{k=1}^{n} \frac{1}{k!} \sum_{\ell_1+\dots+\ell_k=n} \frac{(-1)^k}{\ell_1\dots\ell_k} = 0\,. \tag{2.5.4}$$

Combining (2.5.3) and the previous identity, we get

$$0 = \sum_{k=1}^{n} \frac{1}{k!} \sum_{\ell_1+\cdots+\ell_k=n} \frac{(-1)^k}{\ell_1 \dots \ell_k} = \frac{1}{n!} \sum_{k=1}^{n} \sum_{\sigma \in \mathcal{P}_n^k} (-1)^k \prod_{i=1}^{k} (|\sigma_i| - 1)!$$

and this completes the second identity (2.5.2).

The lemma is proved. □

CHAPTER 3

TREE EXPANSIONS OF THE HARD-SPHERE DYNAMICS

Here and in the next chapter, we explain how the combinatorial methods presented in the previous chapter can be applied to study the dynamical correlations of hard spheres. The first steps in this direction are to define a suitable family describing the correlations of order n, and then to obtain a graphical representation of this family which will be helpful to identify the clustering structure.

3.1. Space correlation functions

For the sake of simplicity, we start by describing correlations in phase space. Recall that the n-particle correlation function $F_n^\varepsilon \equiv F_n^\varepsilon(t, Z_n)$ defined by (1.1.10) counts how many groups of n particles are, in average, in a given configuration Z_n at time t: see Eq. (1.1.11).

Let us now discuss the time evolution of the correlation functions: by integration of the Liouville equation (1.1.3), we get that the family $(F_n^\varepsilon)_{n\geq 1}$ satisfies the so-called BBGKY hierarchy (going back to [**10**]) :

$$\partial_t F_n^\varepsilon + V_n \cdot \nabla_{X_n} F_n^\varepsilon = C_{n,n+1}^\varepsilon F_{n+1}^\varepsilon \quad \text{in} \quad \mathcal{D}_n^\varepsilon \tag{3.1.1}$$

with specular boundary reflection

$$\forall Z_n \in \partial\mathcal{D}_n^{\varepsilon +}(i,j)\,, \quad F_n^\varepsilon(t, Z_n) := F_n^\varepsilon(t, Z_n^{'i,j})\,, \tag{3.1.2}$$

where $Z_N^{'i,j}$ differs from Z_N only by (1.1.2). The collision operator in the right-hand side of (3.1.1) comes from the boundary terms in Green's formula (using the reflection condition to rewrite the gain part in terms of pre-collisional velocities):

$$C_{n,n+1}^\varepsilon F_{n+1}^\varepsilon := \sum_{i=1}^n C_{n,n+1}^{i,\varepsilon} F_{n+1}^\varepsilon$$

with

$$\begin{aligned}(C_{n,n+1}^{i,\varepsilon} F_{n+1}^\varepsilon)(Z_n) := &\int F_{n+1}^\varepsilon(Z_n^{\langle i\rangle}, x_i, v_i', x_i + \varepsilon\omega, w')\big((w - v_i)\cdot\omega\big)_+ \, d\omega dw \\ &- \int F_{n+1}^\varepsilon(Z_n, x_i + \varepsilon\omega, w)\big((w - v_i)\cdot\omega\big)_- \, d\omega dw\,,\end{aligned} \tag{3.1.3}$$

where (v_i', w') is recovered from (v_i, w) through the scattering laws (1.1.2), and with the notation

$$Z_n^{\langle i\rangle} := (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)\,. \tag{3.1.4}$$

Note that the collision operator is defined as a trace, and thus some regularity on F^ε_{n+1} is required to make sense of this operator. The classical way of dealing with this issue (see for instance [**17, 41**]) is to consider the integrated form of the equation, obtained by Duhamel's formula

$$F^\varepsilon_n(t) = S^\varepsilon_n(t)F^{\varepsilon 0}_n + \int_0^t S^\varepsilon_n(t-t_1)C^\varepsilon_{n,n+1}F^\varepsilon_{n+1}(t_1)dt_1\,,$$

denoting by S^ε_n the group associated with free transport in $\mathcal{D}^\varepsilon_n$ with specular reflection on the boundary $\partial\mathcal{D}^\varepsilon_n$.

Iterating Duhamel's formula, we can express the solution as a sum of operators acting on the initial data :

$$F^\varepsilon_n(t) = \sum_{m\geq 0} Q^\varepsilon_{n,n+m}(t)F^{\varepsilon 0}_{n+m}\,, \tag{3.1.5}$$

where we have defined for $t>0$

$$\begin{aligned} Q^\varepsilon_{n,n+m}(t)F^{\varepsilon 0}_{n+m} := \int_0^t\int_0^{t_1}\dots\int_0^{t_{m-1}} & S^\varepsilon_n(t-t_1)C^\varepsilon_{n,n+1}S^\varepsilon_{n+1}(t_1-t_2)C^\varepsilon_{n+1,n+2} \\ & \dots S^\varepsilon_{n+m}(t_m)F^{\varepsilon 0}_{n+m}\,dt_m\dots dt_1 \end{aligned} \tag{3.1.6}$$

and $Q^\varepsilon_{n,n}(t)F^{\varepsilon 0}_n := S^\varepsilon_n(t)F^{\varepsilon 0}_n$, $Q^\varepsilon_{n,n+m}(0)F^{\varepsilon 0}_{n+m} := \delta_{m,0}F^{\varepsilon 0}_{n+m}$.

3.2. Geometrical representation with collision trees

The usual way to study the Duhamel series (3.1.5) is to introduce "pseudo-dynamics" describing the action of the operator $Q^\varepsilon_{n,n+m}$. In the following, particles will be denoted by two different types of labels: either integers i or labels $i*$ (this difference will correspond to the fact that particles labeled with an integer i will be added to the pseudo-dynamics through the Duhamel formula as time goes backwards, while those labeled by $i*$ are already present at time t). The configuration of the particle labeled $i*$ will be denoted indifferently $z^*_i = (x^*_i, v^*_i)$ or $z_{i*} = (x_{i*}, v_{i*})$.

Definition 3.2.1 **(Collision trees).** — *Given $n\geq 1\,, m\geq 0$, an (ordered)* collision tree $a\in\mathcal{A}_{n,m}$ *is a family $(a_i)_{1\leq i\leq m}$ with $a_i\in\{1,\dots,i-1\}\cup\{1*,\dots,n*\}$.*

Note that $|\mathcal{A}_{n,m}| = n(n+1)\dots(n+m-1)$.

Given a collision tree $a\in\mathcal{A}_{n,m}$, we define pseudo-dynamics starting from a configuration $Z^*_n = (x^*_i, v^*_i)_{1\leq i\leq n}$ in the n-particle phase space at time t as follows.

Definition 3.2.2 **(Pseudo-trajectory).** — *Given $Z^*_n\in\mathcal{D}^\varepsilon_n$, $m\in\mathbb{N}$ and $a\in\mathcal{A}_{n,m}$, we consider a collection of times, angles and velocities $(T_m,\Omega_m,V_m) := (t_i,\omega_i,v_i)_{1\leq i\leq m}$ satisfying the constraint*

$$0\leq t_m<\dots<t_1\leq t=t_0\,.$$

We define recursively pseudo-trajectories *as follows:*

— *in between the collision times t_i and t_{i+1} the particles follow the $(n+i)$-particle (backward) hard-sphere flow;*

— *at time t_i^+, particle i is adjoined to particle a_i at position $x_{a_i}+\varepsilon\omega_i$ and with velocity v_i, provided it remains at a distance larger than ε from all the other particles. If $(v_i - v_{a_i}(t_i^+))\cdot\omega_i > 0$, velocities at time t_i^- are given by the scattering laws*

$$\begin{aligned} v_{a_i}(t_i^-) &:= v_{a_i}(t_i^+) - \big((v_{a_i}(t_i^+) - v_i)\cdot\omega_i\big)\,\omega_i\,, \\ v_i(t_i^-) &:= v_i + \big((v_{a_i}(t_i^+) - v_i)\cdot\omega_i\big)\,\omega_i\,. \end{aligned} \tag{3.2.1}$$

We denote by $\Psi^\varepsilon_{n,m} = \Psi^\varepsilon_{n,m}(t)$ (we shall sometimes omit to emphasize the number of created particles and simply denote generically by Ψ^ε_n) the so constructed pseudo-trajectory, and by $Z_{n,m}(\tau) = \big(Z^*_n(\tau), Z_m(\tau)\big)$ the coordinates of the particles in the pseudo-trajectory at time $\tau \le t_m$. It depends on the parameters $a, Z^*_n, T_m, \Omega_m, V_m$, and t. We also define $\mathcal{G}^\varepsilon_m(a, Z^*_n)$ the set of parameters (T_m, Ω_m, V_m) such that the pseudo-trajectory exists up to time 0, meaning in particular that on adjunction of a new particle, its distance to the others remains larger than ε. For $m = 0$, there is no adjoined particle and the pseudo-trajectory $\Psi^\varepsilon_{n,0}(\tau) = Z_{n,0}(\emptyset, Z^*_n, \tau)$ for $\tau \in (0,t)$ is the n-particle (backward) hard-sphere flow.

For a given time $t > 0$, the sample path pseudo-trajectory of the n ($*-$labeled) particles is denoted by $Z^*_n([0,t])$.

Remark 3.2.3. — *We stress the difference in notation: "$z_i(\tau)$" in the above definition denotes the configuration of particle i in the pseudo-trajectory while the real, $\mathcal{N}$-particle hard-sphere flow is denoted $\mathbf{Z}^\varepsilon_{\mathcal{N}}(\tau)$ as in* (1.1.7).

With these notations, the representation formula (3.1.5) for the n-particle correlation function can be rewritten as

$$F^\varepsilon_n(t, Z^*_n) = \sum_{m\ge 0}\sum_{a\in\mathcal{A}_{n,m}} \int_{\mathcal{G}^\varepsilon_m(a,Z^*_n)} dT_m d\Omega_m dV_m \Big(\prod_{i=1}^m \big(v_i - v_{a_i}(t_i)\big)\cdot\omega_i\Big) F^{\varepsilon 0}_{n+m}\big(\Psi^{\varepsilon 0}_{n,m}\big)\,, \tag{3.2.2}$$

where

$$dT_m := dt_1 \dots dt_m\, \mathbf{1}_{0\le t_m\le\dots\le t_1\le t}\,,$$

we have denoted by $(F^{\varepsilon 0}_n)_{n\ge 1}$ the initial rescaled correlation function, and $\Psi^{\varepsilon 0}_{n,m}$ is the configuration at time 0 associated with the pseudo-trajectory $\Psi^\varepsilon_{n,m}$. Note that the variables ω_i are integrated over spheres and the scalar products take positive and negative values (corresponding to the positive and negative parts of the collision operators). Equivalently, we can introduce decorated trees $(a, s_1, \dots, s_m)$ with signs $s_i = \pm$ specifying the collision hemispheres: denoting by $\mathcal{A}^\pm_{n,m}$ the set of all such trees, we can write Eq. (3.2.2) as

$$F^\varepsilon_n(t, Z^*_n) = \sum_{m\ge 0}\sum_{a\in\mathcal{A}^\pm_{n,m}} \int_{\mathcal{G}^\varepsilon_m(a,Z^*_n)} dT_m d\Omega_m dV_m \Big(\prod_{i=1}^m s_i\,\big(\big(v_i - v_{a_i}(t_i)\big)\cdot\omega_i\big)_+\Big) F^{\varepsilon 0}_{n+m}\big(\Psi^{\varepsilon 0}_{n,m}\big)\,, \tag{3.2.3}$$

where the pseudo-trajectory is defined as before, with the scattering (3.2.1) applied in the case $s_i = +$ and the creation at position $x_i + s_i\varepsilon\omega_i$.

3.3. Averaging over trajectories

To describe dynamical correlations. More precisely, we are going to follow the particle trajectories. As noted in Remark 3.2.3, pseudo-trajectories provide a geometric representation of the iterated Duhamel series (3.1.5), but they are not physical trajectories of the particle system. Nevertheless, the probability

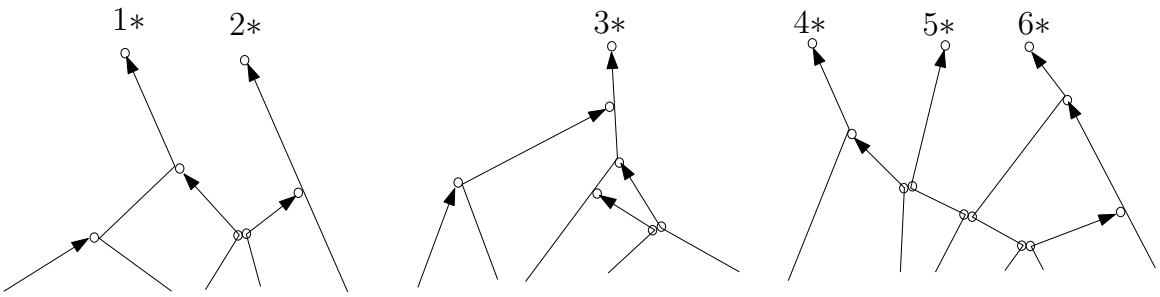


FIGURE 1. An example of pseudo-trajectory with $n = 6$, $m = 10$. In this symbolic picture, time is thought of as flowing upwards (at the top we have a configuration Z_6^*, at the bottom $\Psi_{6,10}^{\varepsilon 0}$). The little circles represent hard spheres of diameter ε. Notice that several collisions are possible between the adjunction times T_m. For simplicity, the hard spheres have been drawn only at their first time of existence (going backwards), and at collisions between adjunction times.

on the trajectories of n particles can be derived from the Duhamel series, as we are going to explain now.

For a given time $t > 0$, the sample path of n particles labeled i_1 to i_n, among the $\mathcal{N}$ hard spheres, is denoted $(\mathbf{z}_{i_1}^\varepsilon([0,t]), \dots, \mathbf{z}_{i_n}^\varepsilon([0,t]))$. In the case when $i_j = j$ for all $1 \le j \le n$ we denote that sample path by $\mathbf{Z}_n^\varepsilon([0,t])$. As $\mathbf{Z}_n^\varepsilon$ has jumps in velocity, it is convenient to work in the space $D_n([0,t])$ of functions that are right-continuous with left limits in $\mathbb{D}^n$. This space is endowed with the Skorokhod topology. In the case when $n = 1$ we denote it simply $D([0,t])$.

Let H_n be a bounded measurable function on $D_n([0,t])$ (the assumption on boundedness will be relaxed later). We define

$$F_{n,[0,t]}^\varepsilon(H_n) := \int dZ_n^* \sum_{m\ge 0} \sum_{a\in\mathcal{A}_{n,m}^\pm} \int_{\mathcal{G}_m^\varepsilon(a,Z_n^*)} dT_m d\Omega_m dV_m \times H_n\big(Z_n^*([0,t])\big)\Big(\prod_{i=1}^m s_i\left(\big(v_i - v_{a_i}(t_i)\big)\cdot\omega_i\right)_+\Big) F_{n+m}^{\varepsilon 0}\big(\Psi_{n,m}^{\varepsilon 0}\big)\,. \tag{3.3.1}$$

This formula generalizes the representation introduced in Section 3.2 in the sense that, in the case when $H_n(Z_n^*([0,t])) = h_n(Z_n^*(t))$, we obtain

$$F_{n,[0,t]}^\varepsilon(H_n) = \int F_n^\varepsilon(t, Z_n^*) h_n(Z_n^*) dZ_n^*\,.$$

More generally, in analogy with (1.1.11), Eq. (3.3.1) gives the average (under the initial probability measure) of the function H_n as stated in the next proposition.

Proposition 3.3.1. — *Let H_n be a symmetric bounded measurable function on $D_n([0,t])$. Then*

$$\mathbb{E}_\varepsilon\Big(\sum_{\substack{i_1,\dots,i_n\\ i_j\neq i_k, j\neq k}} H_n\big(\mathbf{z}_{i_1}^\varepsilon([0,t]),\dots,\mathbf{z}_{i_n}^\varepsilon([0,t])\big)\Big) = \mu_\varepsilon^n F_{n,[0,t]}^\varepsilon(H_n)\,. \tag{3.3.2}$$

Proof. — To establish (3.3.2), we first look at the case of a discrete sampling of trajectories

$$H_n(\mathbf{Z}_n^\varepsilon([0,t])) = \prod_{i=1}^p h_n^{(i)}(\mathbf{Z}_n^\varepsilon(\theta_i))$$

for some decreasing sequence of times $\Theta = (\theta_i)_{1\le i\le p}$ in $[0,t]$, and some family of bounded continuous functions $\left(h_n^{(i)}\right)_{1\le i\le p}$ with $h_n^{(i)} : \mathbb{D}^n \to \mathbb{R}$.

First step. To take into account the discrete sampling H_n, we proceed recursively and define for any $\tau \in [0,t]$

$$H_{n,\tau}(\mathbf{Z}_n^\varepsilon([0,t])) := \left(\prod_{\theta_i\le\tau} h_n^{(i)}(\mathbf{Z}_n^\varepsilon(\theta_i))\right)\left(\prod_{\theta_j>\tau} h_n^{(j)}(\mathbf{Z}_n^\varepsilon(\tau))\right).$$

In particular, for $\tau \le \theta_p \le \cdots \le \theta_1$, the function $H_{n,\tau}$ depends only on the density at time τ so that

$$\mathbb{E}_\varepsilon\Big(\sum_{\substack{i_1,\dots,i_n\\ i_j\neq i_k, j\neq k}} H_{n,\tau}\big(\mathbf{z}_{i_1}^\varepsilon([0,t]),\dots,\mathbf{z}_{i_n}^\varepsilon([0,t])\big)\Big) = \mu_\varepsilon^n \int F_n^\varepsilon(\tau, Z_n^*)\prod_{j=1}^p h_n^{(j)}(Z_n^*)dZ_n^*\,.$$

We then define the biased distribution

$$\tilde F_n^\varepsilon(\tau, Z_n^*) := F_n^\varepsilon(\tau, Z_n^*)\prod_{j=1}^p h_n^{(j)}(Z_n^*) \quad \text{for} \quad \tau\in[0,\theta_p]$$

and then extend this biased correlation function $\tilde F_n^\varepsilon(\tau, Z_n^*)$ on $[0,t]$ so that

$$\mathbb{E}_\varepsilon\Big(\sum_{\substack{i_1,\dots,i_n\\ i_j\neq i_k, j\neq k}} H_{n,\tau}\big(\mathbf{z}_{i_1}^\varepsilon([0,t]),\dots,\mathbf{z}_{i_n}^\varepsilon([0,t])\big)\Big) = \mu_\varepsilon^n \int \tilde F_n^\varepsilon(\tau, Z_n^*)dZ_n^*\,.$$

In order to characterize $\tilde F_n^\varepsilon(\tau, Z_n^*)$, we have to iterate the Duhamel formula (3.1.5) in time slices $[\theta_{i+1},\theta_i]$ as in the proof of Proposition 2.4 of [**6**] (see also [**3, 5**]). On $[\theta_{i+1},\theta_i]$, $\tilde F_n^\varepsilon(\tau, Z_n^*)$ is the product of the weight $\prod_{j\le i} h_n^{(j)}(Z_n^*)$ by a correlation function which satisfies the BBGKY hierarchy. Therefore the expansion (3.1.5) can be applied to describe its evolution in $[\theta_{i+1},\theta_i]$. We obtain by iteration on i that

$$\begin{aligned}\tilde F_n^\varepsilon(t) &= \sum_{k_1\ge 0} Q_{n,n+k_1}^\varepsilon(t-\theta_1)\tilde F_{n+k_1}^\varepsilon(\theta_1)\\ &= \sum_{k_1+\cdots+k_{p+1}\ge 0} Q_{n,n+k_1}^\varepsilon(t-\theta_1)h_n^{(1)}(Z_n^*(\theta_1))Q_{n+k_1,n+k_1+k_2}^\varepsilon(\theta_1-\theta_2)\\ &\qquad \dots h_n^{(p)}(Z_n^*(\theta_p))Q_{n+k_1+\cdots+k_p,n+k_1+\cdots+k_{p+1}}^\varepsilon(\theta_p)F_{n+k_1+\cdots+k_{p+1}}^{\varepsilon 0}\,,\end{aligned} \tag{3.3.3}$$

which leads to (3.3.2) for discrete samplings.

Second step. More generally any function H_n on $(\mathbb{D}^n)^p$ can be approximated in terms of products of functions on $\mathbb{D}^n$, thus (3.3.3) leads to

$$\begin{aligned}\mathbb{E}_\varepsilon\Big(\sum_{\substack{i_1,\dots,i_n\\ i_j\neq i_k, j\neq k}} H_n\big(\mathbf{z}_{i_1}^\varepsilon([0,t]),\dots,\mathbf{z}_{i_n}^\varepsilon([0,t])\big)\Big) &= \mu_\varepsilon^n \sum_{k_1+\cdots+k_{p+1}\ge 0} Q_{n,n+k_1}^\varepsilon(t-\theta_1)Q_{n+k_1,n+k_1+k_2}^\varepsilon(\theta_1-\theta_2)\\ &\qquad \dots Q_{n+k_1+\cdots+k_p,n+k_1+\cdots+k_{p+1}}^\varepsilon(\theta_p)H_n(Z_n^*(\theta_1),\dots,Z_n^*(\theta_p))F_{n+k_1+\cdots+k_{p+1}}^{\varepsilon 0}\end{aligned}$$

where the Duhamel series is weighted by the n-particle pseudo-trajectories at times $\theta_1,\dots,\theta_p$.

Third step. For any $0 \le \theta_p < \cdots < \theta_1 < t$, we denote by $\pi_{\theta_1,\dots,\theta_p}$ the projection from $D_n([0,t])$ to $(\mathbb{D}^n)^p$

$$\pi_{\theta_1,\dots,\theta_p}(Z_n([0,t])) = (Z_n(\theta_1),\dots,Z_n(\theta_p))\,. \tag{3.3.4}$$

The σ-field of Borel sets for the Skorokhod topology can be generated by the sets of the form $\pi^{-1}_{\theta_1,\dots,\theta_p} A$ with A a subset of $(\mathbb{D}^n)^p$ (see Theorem 12.5 in [**4**], page 134). This completes the proof of Proposition 3.3.1. □

To simplify notation, we are going to denote by Ψ^ε_n the pseudo-trajectory during the whole time interval $[0,t]$, which is encoded by its starting points Z^*_n and the evolution parameters (a, T_m, Ω_m, V_m). Similarly we use the compressed notation $\mathbf{1}_{\mathcal{G}^\varepsilon}$ for the constraint that the parameters (T_m, Ω_m, V_m) should be in $\mathcal{G}^\varepsilon_m(a, Z^*_n)$ as in Definition 3.2.2. The parameters (a, T_m, Ω_m, V_m) are distributed according to the measure

$$d\mu(\Psi^\varepsilon_n) := \sum_m \sum_{a\in\mathcal{A}^\pm_{n,m}} dT_m d\Omega_m dV_m \mathbf{1}_{\mathcal{G}^\varepsilon}(\Psi^\varepsilon_n) \prod_{k=1}^m \Big(s_k \big(\big(v_k - v_{a_k}(t_k)\big)\cdot\omega_k \big)_+ \Big) . \tag{3.3.5}$$

The weight coming from the function H_n will be denoted by

$$\mathcal{H}\big(\Psi^\varepsilon_n\big) := H_n\big(Z^*_n([0,t])\big) . \tag{3.3.6}$$

Formula (3.3.1) can be rewritten

$$F^\varepsilon_{n,[0,t]}(H_n) = \int dZ^*_n \int d\mu(\Psi^\varepsilon_n)\ \mathcal{H}\big(\Psi^\varepsilon_n\big)\ F^{\varepsilon 0}\big(\Psi^{\varepsilon 0}_n\big) , \tag{3.3.7}$$

and $F^{\varepsilon 0}\big(\Psi^{\varepsilon 0}_n\big)$ stands for the initial data evaluated on the configuration at time 0 of the pseudo-trajectory (containing $n+m$ particles).

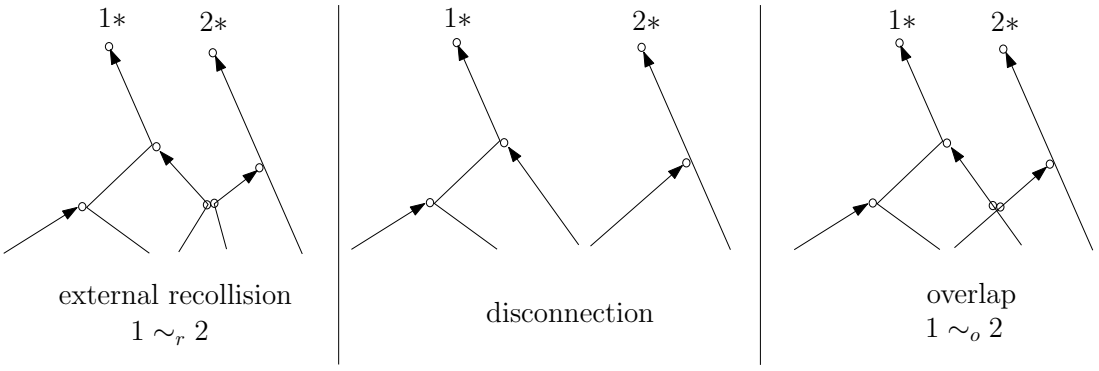
The series expansion (3.3.7) is absolutely convergent, uniformly in ε, for times smaller than some $T_0 > 0$: this determines the time restriction in Theorem 1. More precisely, T_0 is defined by the following condition:

$$\forall t\in[0,T_0] , \qquad \sup_{n\geq 1} \left[\sup_{\mathbb{D}^n} \int d|\mu|(\Psi^\varepsilon_n)\ F^{\varepsilon 0}\big(\Psi^{\varepsilon 0}_n\big)\, e^{\frac{\beta_0}{4}|V^*_n|^2} \right]^{\frac{1}{n}} < +\infty . \tag{3.3.8}$$

CHAPTER 4

CUMULANTS FOR THE HARD-SPHERE DYNAMICS

To understand the structure of dynamical correlations, we are going to describe how the collision trees introduced in the previous chapter (which are the elementary dynamical objects) can be grouped into clusters. We shall identify three different types of correlations (treated in Section 4.1, 4.2, 4.3 respectively). Our starting point will be Formula (3.3.7). We will also need the notation $\Psi_n^\varepsilon = \Psi_{\{1,\dots,n\}}^\varepsilon$, where a pseudo-trajectory is labeled by the ensemble of its roots. Notice that the two collision trees in $\Psi_{\{1,2\}}^\varepsilon$ do *not* scatter if and only if $\Psi_{\{1\}}^\varepsilon$ and $\Psi_{\{2\}}^\varepsilon$ keep a mutual distance larger than ε. Therefore we shall write the non-scattering condition as the complement of an *overlap* condition, meaning that $\Psi_{\{1\}}^\varepsilon$ and $\Psi_{\{2\}}^\varepsilon$ reach a mutual distance smaller than ε (without scattering with each other). The scattering, disconnection and overlap situations are represented in the following picture (recall also Figure 1), together with some nomenclature which is made precise below.



4.1. External recollisions

A pseudo-trajectory Ψ_n^ε is made of n collision trees starting from the roots Z_n^*. These elementary collision trees will be called *subtrees*, and will be indexed by the label of their root. The parameters (a, T_m, Ω_m, V_m) associated with each collision tree are independent, and can be separated into n subsets.

The corresponding pseudotrajectories $\Psi_{\{1\}}^\varepsilon, \dots \Psi_{\{n\}}^\varepsilon$ evolve independently until two particles belonging to different trees collide, in which case the corresponding two trees get correlated. The next definition introduces the notion of recollision and distinguishes whether the recolliding particles are in the same tree or not.

***Definition 4.1.1* (External/internal recollisions).** — *A recollision occurs when two pre-existing particles in a pseudo-trajectory scatter. A recollision between two particles will be called an external recollision if the two particles involved are in different subtrees. A recollision between two particles will be called an internal recollision if the two particles involved are in the same subtree.*

Let us now decompose the integral (3.3.7) depending on whether subtrees are correlated or not. Recall Definitions 2.3.1 and 2.3.2.

Notation 4.1.2. — *We denote by*

$$\{j\} \sim_r \{j'\}$$

the condition: "there exists an external recollision between particles in the subtrees indexed by j and j'". Given $\lambda \subset \{1,\dots,n\}$, we denote by $\mathbb{M}_\lambda$ the indicator function that any two elements of λ are connected by a chain of external recollisions. In other words

$$\mathbb{M}_\lambda = 1 \quad \Longleftrightarrow \quad \exists\, G \in \mathcal{C}_\lambda\,, \qquad \prod_{\{j,j'\}\in E(G)} \mathbf{1}_{\{j\}\sim_r\{j'\}} = 1\,. \tag{4.1.1}$$

Notice that $\mathbb{M}_\lambda$ depends only on Ψ^ε_λ. We set $\mathbb{M}_\lambda = 1$ when $|\lambda| = 1$. We extend $\mathbb{M}_\lambda$ to zero outside $\mathcal{G}^\varepsilon(Z^*_\lambda)$. We therefore have the partition of unity

$$\mathbf{1}_{\mathcal{G}^\varepsilon}\big(\Psi^\varepsilon_n\big) = \sum_{\ell=1}^{n} \sum_{\lambda\in\mathcal{P}^\ell_n} \left(\prod_{i=1}^{\ell} \mathbb{M}_{\lambda_i}\, \mathbf{1}_{\mathcal{G}^\varepsilon}\big(\Psi^\varepsilon_{\lambda_i}\big) \right) \Phi_\ell\,(\lambda_1,\dots,\lambda_\ell) \tag{4.1.2}$$

where $\Phi_1 = 1$, and Φ_ℓ for $\ell > 1$ is the indicator function that the subtrees indexed by $\lambda_1,\dots,\lambda_\ell$ keep mutual distance larger than ε. Φ_ℓ is defined on $\cup_i \mathcal{G}^\varepsilon(Z^*_{\lambda_i})$.

Using the notation (3.3.7), we can partition the pseudo-trajectories in terms of the external recollisions

$$F^\varepsilon_{n,[0,t]}(H^{\otimes n}) = \int dZ^*_n \sum_{\ell=1}^{n} \sum_{\lambda\in\mathcal{P}^\ell_n} \int d\mu(\Psi^\varepsilon_n)\mathcal{H}\big(\Psi^\varepsilon_n\big) \Big(\prod_{i=1}^{\ell} \mathbb{M}_{\lambda_i} \Big)\, \Phi_\ell\big(\lambda_1,\dots,\lambda_\ell\big) F^{\varepsilon 0}\big(\Psi^{\varepsilon 0}_n\big)\,.$$

There is no external recollision between the subtrees indexed by $\lambda_1,\dots,\lambda_\ell$, so the pseudo-trajectories are defined independently; in particular, assuming from now on that

$$H_n = H^{\otimes n}$$

with H a measurable function on the space of trajectories $D([0,t])$, the cross-sections, the weights and the constraint imposed by $\mathcal{G}^\varepsilon$ factorize

$$\Phi_\ell\big(\lambda_1,\dots,\lambda_\ell\big)\mathcal{H}\big(\Psi^\varepsilon_n\big)d\mu\big(\Psi^\varepsilon_n\big) = \Phi_\ell\big(\lambda_1,\dots,\lambda_\ell\big)\Big(\prod_{i=1}^{\ell}\mathcal{H}\big(\Psi^\varepsilon_{\lambda_i}\big)d\mu\big(\Psi^\varepsilon_{\lambda_i}\big)\Big)$$

and we get

$$F^\varepsilon_{n,[0,t]}(H^{\otimes n}) = \int dZ^*_n \sum_{\ell=1}^{n} \sum_{\lambda\in\mathcal{P}^\ell_n} \int \Big(\prod_{i=1}^{\ell} d\mu\big(\Psi^\varepsilon_{\lambda_i}\big)\mathcal{H}\big(\Psi^\varepsilon_{\lambda_i}\big)\mathbb{M}_{\lambda_i} \Big)\, \Phi_\ell\big(\lambda_1,\dots,\lambda_\ell\big) F^{\varepsilon 0}\big(\Psi^{\varepsilon 0}_n\big)\,. \tag{4.1.3}$$

The function Φ_ℓ forbids any overlap between different forests in (4.1.3). In particular, notice that Φ_ℓ is equal to zero if $|x^*_i - x^*_j| < \varepsilon$ for some $i \neq j$ (compatibly with the definition of $F^\varepsilon_{n,[0,t]}$).

Although the subtrees $\Psi^\varepsilon_{\lambda_1},\dots,\Psi^\varepsilon_{\lambda_\ell}$ in the above formula have no external recollisions, they are not yet fully independent as their parameters are constrained precisely by the fact that no external recollision should occur. Thus we are going to decompose further the collision integral.

4.2. Overlaps

In order to identify all possible correlations, we now introduce a cumulant expansion of the constraint Φ_ℓ encoding the fact that no external recollision should occur between the different λ_i.

***Definition 4.2.1* (Overlap).** — *An overlap occurs between two subtrees if two pseudo-particles, one in each subtree, find themselves at a distance less than ε one from the other for some $\tau \in [0,t]$.*

Notation 4.2.2. — *We denote by*

$$\lambda_i \sim_o \lambda_j$$

the relation: "there exists an overlap between two subtrees belonging to λ_i and λ_j respectively", and we denote $\lambda_i \not\sim_o \lambda_j$ the complementary relation. Therefore

$$\Phi_\ell(\lambda_1,\dots,\lambda_\ell) = \prod_{1\le i\neq j\le \ell} \mathbf{1}_{\lambda_i \not\sim_o \lambda_j}\,. \tag{4.2.1}$$

The inversion formula (2.2.1) implies that

$$\Phi_\ell(\lambda_1,\dots,\lambda_\ell) = \sum_{r=1}^{\ell} \sum_{\rho\in\mathcal{P}_\ell^r} \varphi_\rho\,,$$

denoting

$$\varphi_\rho := \prod_{j=1}^{r} \varphi_{\rho_j}\,.$$

The cumulants associated with the partition $\{\lambda_1,\dots,\lambda_\ell\}$ are defined for any subset ρ_j of $\{1,\dots,\ell\}$ as

$$\varphi_{\rho_j} = \sum_{u=1}^{|\rho_j|} \sum_{\omega\in\mathcal{P}^u_{\rho_j}} (-1)^{u-1}(u-1)!\,\Phi_\omega\,, \tag{4.2.2}$$

where ω is a partition of ρ_j, and recalling the notation

$$\Phi_\omega = \prod_{i=1}^{u} \Phi_{\omega_i}\,, \quad \Phi_{\omega_i} = \Phi_{|\omega_i|}(\lambda_k; k\in\omega_i)\,.$$

Note that as stated in Proposition 2.3.3, the function φ_ρ is supported on clusters formed by overlapping collision trees, i.e.

$$\varphi_{\rho_j} = \sum_{G\in\mathcal{C}_{\rho_j}} \prod_{\{i_1,i_2\}\in E(G)} (-\mathbf{1}_{\lambda_{i_1}\sim_o\lambda_{i_2}})\,. \tag{4.2.3}$$

For the time being let us return to (4.1.3), which can thus be further decomposed as

$$F^\varepsilon_{n,[0,t]}(H^{\otimes n}) = \int dZ_n^* \sum_{\ell=1}^{n} \sum_{\lambda\in\mathcal{P}_n^\ell} \sum_{r=1}^{\ell} \sum_{\rho\in\mathcal{P}_\ell^r} \int \Big(\prod_{i=1}^{\ell} d\mu(\Psi^\varepsilon_{\lambda_i})\mathcal{H}(\Psi^\varepsilon_{\lambda_i})\mathbb{M}_{\lambda_i}\Big)\,\varphi_\rho F^{\varepsilon 0}(\Psi_n^{\varepsilon 0})\,. \tag{4.2.4}$$

By abuse of notation, the partition ρ can be also interpreted as a partition of $\{1,\dots,n\}$

$$\forall j \le |\rho|\,, \qquad \rho_j = \bigcup_{i\in\rho_j} \lambda_i\,, \tag{4.2.5}$$

coarser than the partition λ. The relative coarseness (4.2.5) will be denoted by

$$\lambda \hookrightarrow \rho\,.$$

4.3. Initial clusters

In (4.2.4), the pseudo-trajectory is evaluated at time 0 on the initial distribution $F^{\varepsilon 0}\big(\Psi_n^{\varepsilon 0}\big)$. Thus the pseudo-trajectories $\{\Psi^{\varepsilon}_{\rho_j}\}_{j\le r}$ remain correlated by the initial data, so we are finally going to decompose the initial measure in terms of cumulants.

Given $\rho=\{\rho_1,\dots,\rho_r\}$ a partition of $\{1,\dots,n\}$ into r subsets, we define the cumulants of the initial data associated with ρ as follows. For any subset $\tilde\sigma$ of $\{1,\dots,r\}$, we set

$$f^{\varepsilon 0}_{\tilde\sigma} := \sum_{u=1}^{|\tilde\sigma|}\sum_{\omega\in\mathcal{P}^u_{\tilde\sigma}} (-1)^{u-1}(u-1)!\, F^{\varepsilon 0}_{\omega}\,, \tag{4.3.1}$$

where ω is a partition of $\tilde\sigma$, and denoting as previously

$$F^{\varepsilon 0}_{\omega}=\prod_{i=1}^{u} F^{\varepsilon 0}_{\omega_i}\,,\quad F^{\varepsilon 0}_{\omega_i}=F^{\varepsilon 0}(\Psi^{\varepsilon 0}_{\rho_j}; j\in\omega_i)\,.$$

We recall that $\Psi^{\varepsilon 0}_{\rho_j}$ represents the pseudo-trajectories rooted in $Z^*_{\rho_j}$ computed at time 0. They involve m_j new particles, so there are $|\rho_j|+m_j$ particles at play at time 0, with of course $\sum_{j=1}^r(|\rho_j|+m_j)=n+\sum_{j=1}^r m_j=n+m$. We stress that the cumulant decomposition depends on ρ (in the same way as (4.2.2) was depending on λ).

Given $\rho=\{\rho_1,\dots,\rho_r\}$, the initial data can thus be decomposed as

$$F^{\varepsilon 0}\big(\Psi_n^{\varepsilon 0}\big)=\sum_{s=1}^{r}\sum_{\sigma\in\mathcal{P}^s_r} f^{\varepsilon 0}_{\sigma}\,,\quad\text{with}\quad f^{\varepsilon 0}_{\sigma}=\prod_{i=1}^{s} f^{\varepsilon 0}_{\sigma_i}\,.$$

By abuse of notation as above in (4.2.5), the partition σ can be also interpreted as a partition of $\{1,\dots,n\}$

$$\forall i\le|\sigma|\,,\qquad \sigma_i=\bigcup_{j\in\sigma_i}\rho_j\,,$$

coarser than the partition ρ. Hence there holds $\rho\hookrightarrow\sigma$.

We finally get

$$F^{\varepsilon}_{n,[0,t]}(H^{\otimes n})=\int dZ^*_n\sum_{\ell=1}^{n}\sum_{\lambda\in\mathcal{P}^\ell_n}\sum_{r=1}^{\ell}\sum_{\rho\in\mathcal{P}^r_\ell}\sum_{s=1}^{r}\sum_{\sigma\in\mathcal{P}^s_r}\int\Big(\prod_{i=1}^{\ell}d\mu\big(\Psi^{\varepsilon}_{\lambda_i}\big)\mathcal{H}\big(\Psi^{\varepsilon}_{\lambda_i}\big)\mathbb{M}_{\lambda_i}\Big)\,\varphi_\rho\, f^{\varepsilon 0}_{\sigma}\,.$$

The n subtrees generated by Z^*_n have been decomposed into nested partitions $\lambda\hookrightarrow\rho\hookrightarrow\sigma$ (see Figure 2).

Thus we can write

$$F^{\varepsilon}_{n,[0,t]}(H^{\otimes n})=\int dZ^*_n\sum_{\substack{\lambda,\rho,\sigma\\ \lambda\hookrightarrow\rho\hookrightarrow\sigma}}\int\Big(\prod_{i=1}^{\ell}d\mu\big(\Psi^{\varepsilon}_{\lambda_i}\big)\mathcal{H}\big(\Psi^{\varepsilon}_{\lambda_i}\big)\mathbb{M}_{\lambda_i}\Big)\,\varphi_\rho\, f^{\varepsilon 0}_{\sigma}\,. \tag{4.3.2}$$

The order of the sums can be exchanged, starting from the coarser partition σ: we obtain

$$F^{\varepsilon}_{n,[0,t]}(H^{\otimes n})=\int dZ^*_n\sum_{s=1}^{n}\sum_{\sigma\in\mathcal{P}^s_n}\prod_{j=1}^{s}\sum_{\substack{\lambda,\rho\\ \lambda\hookrightarrow\rho\hookrightarrow\sigma_j}}\int\Big(\prod_{i=1}^{\ell}d\mu\big(\Psi^{\varepsilon}_{\lambda_i}\big)\mathcal{H}\big(\Psi^{\varepsilon}_{\lambda_i}\big)\mathbb{M}_{\lambda_i}\Big)\,\varphi_\rho f^{\varepsilon 0}_{\sigma_j} \tag{4.3.3}$$

where the generic variables λ,ρ denote now nested partitions of the subset σ_j.

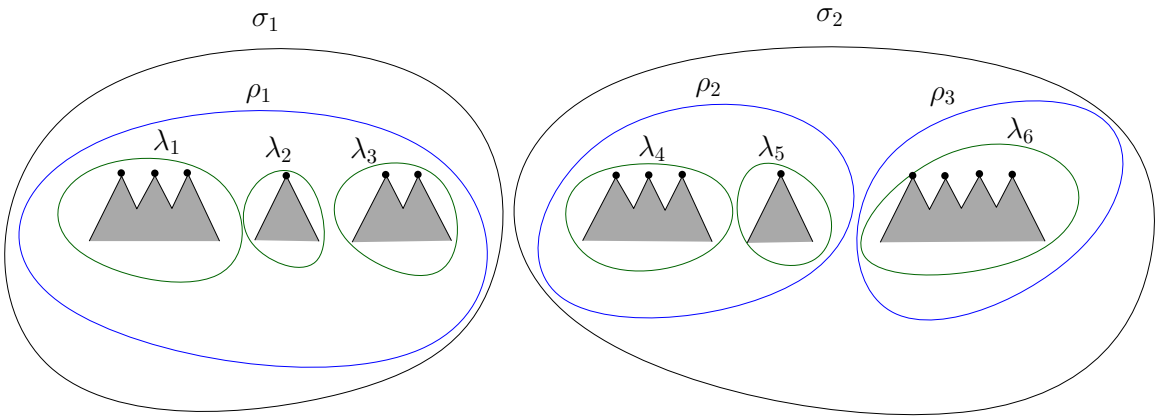


FIGURE 2. The figure illustrates the nested decomposition $\lambda \hookrightarrow \rho \hookrightarrow \sigma$ in (4.3.2). The configuration Z_n^* at time t is represented by $n = 14$ black dots. Collision trees, depicted by grey triangles, are created from each dots and all the trees with labels in a subset λ_i interact via external recollisions, forming connected clusters (grey mountains). These trees are then regrouped in coarser partitions ρ and σ in order to evaluate the corresponding cumulants. Green clusters λ are called forests, blue clusters ρ are called jungles, and black clusters σ are called initial clusters.

4.4. Dynamical cumulants

Using the inversion formula (2.2.1), the cumulant of order n is defined as the term in (4.3.3) such that σ has only 1 element, i.e. $\sigma = \{1, \dots, n\}$. We therefore define the (scaled) cumulant, recalling notation (4.3.1),

(4.4.1)
$$f^{\varepsilon}_{n,[0,t]}(H^{\otimes n}) = \int dZ_n^* \mu_\varepsilon^{n-1} \sum_{\ell=1}^{n} \sum_{\lambda \in \mathcal{P}_n^\ell} \sum_{r=1}^{\ell} \sum_{\rho \in \mathcal{P}_\ell^r} \int \Big(\prod_{i=1}^{\ell} d\mu(\Psi^{\varepsilon}_{\lambda_i}) \mathcal{H}(\Psi^{\varepsilon}_{\lambda_i}) \mathbb{\Delta}_{\lambda_i} \Big) \, \varphi_\rho \, f^{\varepsilon 0}_{\{1,\dots,r\}}(\Psi^{\varepsilon 0}_{\rho_1}, \dots, \Psi^{\varepsilon 0}_{\rho_r}) \, .$$

In the simple case $n = 2$, the above formula reads

$$\begin{aligned} f^{\varepsilon}_{2,[0,t]}(H^{\otimes 2}) = \int dZ_2^* \, \mu_\varepsilon \Big\{ & \int d\mu(\Psi^{\varepsilon}_{\{1,2\}}) \, \mathbf{1}_{\{1\} \sim_r \{2\}} \, \mathcal{H}(\Psi^{\varepsilon}_{\{1,2\}}) F^{\varepsilon 0}(\Psi^{\varepsilon 0}_{\{1,2\}}) \\ & - \int \prod_{i=1}^{2} \Big[d\mu(\Psi^{\varepsilon}_{\{i\}}) \, \mathcal{H}(\Psi^{\varepsilon}_{\{i\}}) \Big] \mathbf{1}_{\{1\} \sim_o \{2\}} F^{\varepsilon 0} \left(\Psi^{\varepsilon 0}_{\{1\}}, \Psi^{\varepsilon 0}_{\{2\}} \right) \\ & + \int \prod_{i=1}^{2} \Big[d\mu(\Psi^{\varepsilon}_{\{i\}}) \, \mathcal{H}(\Psi^{\varepsilon}_{\{i\}}) \Big] \left(F^{\varepsilon 0} \left(\Psi^{\varepsilon 0}_{\{1\}}, \Psi^{\varepsilon 0}_{\{2\}} \right) - F^{\varepsilon 0} \left(\Psi^{\varepsilon 0}_{\{1\}} \right) F^{\varepsilon 0} \left(\Psi^{\varepsilon 0}_{\{2\}} \right) \right) \Big\}, \end{aligned}$$

where we used (4.1.1), (4.2.3) and (4.3.1). The three lines on the right hand side represent the three possible correlation mechanisms between particles 1^* and 2^* (i.e. between the subtrees 1 and 2): respectively the (clustering) recollision, the (clustering) overlap and the correlation of initial data.

More generally, looking at Eq. (4.4.1), we are going to check that $f^{\varepsilon}_{n,[0,t]}(H^{\otimes n})$ is a cluster of order n.

— We start with n trees which are grouped into ℓ forests in the partition λ. In each forest λ_i we shall identify $|\lambda_i| - 1$ "clustering recollisions". These recollisions give rise to $\sum_{i=1}^{\ell} (|\lambda_i| - 1) = n - \ell$ constraints.

— The ℓ forests are then grouped into r jungles ρ and in each jungle ρ_i, we shall identify $|\rho_i|-1$ "clustering overlaps". These give rise to $\sum_{i=1}^{r}(|\rho_i|-1)=\ell-r$ constraints.
— The r elements of ρ are then coupled by the initial cluster, and this gives rise to $r-1$ constraints.

By construction $n-1=\sum_{i=1}^{r}(|\rho_i|-1)+\sum_{i=1}^{\ell}(|\lambda_i|-1)+r-1$. The dynamical decomposition (4.4.1) implies therefore that the cumulant of order n is associated with pseudo-trajectories with $n-1$ clustering constraints, and we expect that each of these $n-1$ clustering constraints will provide a small factor of order $1/\mu_\varepsilon$. To quantify rigorously this smallness, we need to identify $n-1$ "independent" degrees of freedom. For clustering overlaps this will be an easy task. Clustering recollisions will require more attention, as they introduce a strong dependence between different trees.

Let us now analyze Eq. (4.4.1) in more detail. The decomposition can be interpreted in terms of a graph in which the edges represent all possible correlations (between points in a tree, between trees in a forest and between forests in a jungle). In these correlations, some play a special role as they specify minimally connected subgraphs in jungles or forests: this is made precise in the two following important notions.

Let us start with the easier case of overlaps in a jungle. The following definition assigns a minimally connected graph (cf. Definition 2.3.2) on the set of forests grouped into a given jungle.

***Definition 4.4.1* (Clustering overlaps).** — *Given a jungle $\rho_i=\{\lambda_{j_1},\dots,\lambda_{j_{|\rho_i|}}\}$ and a pseudo-trajectory $\Psi^{\varepsilon}_{\rho_i}$, we call "clustering overlaps" a set of $|\rho_i|-1$ overlaps*

$$(\lambda_{j_1}\sim_o\lambda_{j_1'}),\dots,(\lambda_{j_{|\rho_i|-1}}\sim_o\lambda_{j'_{|\rho_i|-1}}) \tag{4.4.2}$$

such that

$$\Big\{\{\lambda_{j_1},\lambda_{j_1'}\},\dots,\{\lambda_{j_{|\rho_i|-1}},\lambda_{j'_{|\rho_i|-1}}\}\Big\}=E(T_{\rho_i})$$

where T_{ρ_i} is a minimally connected graph on ρ_i. Given a pseudo-trajectory $\Psi^{\varepsilon}_{\rho_i}$ with clustering overlaps, we define $|\rho_i|-1$ overlap times as follows: the k-th overlap time is

$$\tau_{\mathrm{ov},k}:=\sup\Big\{\tau\geq 0\;\Big|\min_{\substack{q\text{ in }\Psi^{\varepsilon}_{\lambda_{j_k}}\\ q'\text{ in }\Psi^{\varepsilon}_{\lambda_{j_k'}}}}|x_{q'}(\tau)-x_q(\tau)|<\varepsilon\Big\}\,. \tag{4.4.3}$$

Each one of the $|\rho_i|-1$ overlaps is a strong geometrical constraint which will be used in Part III to gain a small factor $1/\mu_\varepsilon$. More precisely, in Chapter 8 we assign to each forest λ_{j_k} a root $z^*_{\lambda_{j_k}}$ (chosen among the roots of $\Psi^{\varepsilon}_{\lambda_{j_k}}$). Then, it will be possible to "move rigidly" the whole pseudo-trajectory $\Psi^{\varepsilon}_{\lambda_{j_k}}$, acting just on $x^*_{\lambda_{j_k}}$. It follows that one easily translates the condition of "clustering overlap" into $|\rho_i|-1$ independent constraints on the relative positions of the roots. In fact remember that the pseudo-trajectories $\Psi^{\varepsilon}_{\lambda_{j_k}},\Psi^{\varepsilon}_{\lambda_{j_k'}}$ do not interact with each other by construction. Therefore $\lambda_{j_k}\sim_o\lambda_{j_k'}$ means that the two pseudo-trajectories meet at some time $\tau_{\mathrm{ov},k}>0$ and, immediately after (going backwards), they cross each other freely. Which corresponds to a small measure set in the variable $x^*_{\lambda_{j_k'}}-x^*_{\lambda_{j_k}}$.

Contrary to overlaps, recollisions are unfortunately not independent one from the other. For this reason, the study of recollisions of trees in a forest needs more care. In this case the main idea is based on fixing the order of the recollision times. Then we can identify an ordered sequence of relative positions (between trees) which do not affect the previous recollisions. One by one and following the ordering, such degrees of freedom are shown to belong to a small measure set. The precise identification of degrees of freedom will be explained in Section 8.1 and is based in the following notion.

***Definition 4.4.2* (Clustering recollisions).** — *Given a forest $\lambda_i = \{i_1, \dots, i_{|\lambda_i|}\}$ and a pseudo-trajectory $\Psi^{\varepsilon}_{\lambda_i}$, we call "clustering recollisions" the set of recollisions identified by the following iterative procedure.*

- The first clustering recollision is the first external recollision in $\Psi^{\varepsilon}_{\lambda_i}$ (going backward in time); we rename the recolliding trees j_1, j_1' and the recollision time $\tau_{\mathrm{rec},1}$.

- The k-th clustering recollision is the first external recollision in $\Psi^{\varepsilon}_{\lambda_i}$ (going backward in time) such that, calling j_k, j_k' the recolliding trees, $\{\{j_1, j_1'\}, \dots, \{j_k, j_k'\}\} = E\left(G^{(k)}\right)$ where $G^{(k)}$ is a graph with no cycles (and no multiple edges). We denote the recollision time $\tau_{\mathrm{rec,k}}$.

In particular,

$$\tau_{\mathrm{rec},1} \geq \cdots \geq \tau_{\mathrm{rec},|\lambda_i|-1} \quad \text{and} \quad \left\{\{j_1, j_1'\}, \dots, \{j_{|\lambda_i|-1}, j_{|\lambda_i|-1}'\}\right\} = E(T_{\lambda_i}) \tag{4.4.4}$$

where T_{λ_i} is a minimally connected graph on λ_i.

If q, q' are the particles realizing the recollision, we define the corresponding recollision vector by

$$\omega_{\mathrm{rec,k}} := \frac{x_{q'}(\tau_{\mathrm{rec,k}}) - x_q(\tau_{\mathrm{rec,k}})}{\varepsilon} \,. \tag{4.4.5}$$

The important difference between Definition 4.4.2 and Definition 4.4.1 is that we have given an order to the recollision times in Eq. (4.4.4) (which is missing in Eq. (4.4.3)).

From now on, in order to distinguish, at the level of graphs, between clustering recollisions and clustering overlaps, we shall decorate edges as follows.

***Definition 4.4.3* (Edge sign).** — *An edge has sign + if it represents a clustering recollision. An edge has sign − if it represents a clustering overlap.*

Collecting together clustering recollisions and clustering overlaps, we obtain r minimally connected clusters, one for each jungle. In particular, we can construct a graph $G_{\lambda,\rho}$ made of r minimally connected components. To each $e \in E(G_{\lambda,\rho})$, we associate a sign (+ for a recollision and − for an overlap), and a clustering time τ_e^{clust}.

Our main results describing the structure of dynamical correlations will be proved in the third part of this paper. The major breakthrough in this work is to remark that one can obtain uniform bounds for the cumulant of order n for all n, with a controlled growth, as stated in the next theorem.

Theorem 4. — *Consider the system of hard spheres under the initial measure* (1.1.6)*, with f^0 satisfying* (1.1.5)*. Let $H : D([0,\infty[) \mapsto \mathbb{R}$ be a continuous function such that*

$$|H^{\otimes n}(Z_n([0,t]))| \leq \exp\left(\alpha_0 n + \frac{\beta_0}{4} \sup_{s\in[0,t]} |V_n(s)|^2\right) \tag{4.4.6}$$

for some $\alpha_0 \in \mathbb{R}$. Define the scaled cumulant $f^{\varepsilon}_{n,[0,t]}(H^{\otimes n})$ by (4.4.1)*, with the notation* (3.3.5)*. Then there exists a positive constant C and a time $T^\star = T^\star(C_0, \beta_0)$ such that the following uniform a priori bound holds for any $t \leq T^\star$:*

$$|f^{\varepsilon}_{n,[0,t]}(H^{\otimes n})| \leq (Ce^{\alpha_0})^n \big(t + \varepsilon\big)^{n-1} n! \,.$$

In particular setting $H = e^h - 1$ and up to restricting $T^\star = T^\star(\alpha_0, C_0, \beta_0)$, the series defining the cumulant generating function is absolutely convergent on $[0, T^\star]$:

$$\Lambda^\varepsilon_{[0,t]}(e^h) := \frac{1}{\mu_\varepsilon} \log \mathbb{E}_\varepsilon \left(\exp\Big(\sum_{i=1}^{\mathcal{N}} h\big(\mathbf{z}^\varepsilon_i([0,t])\big)\Big)\right) = \sum_{n=1}^{\infty} \frac{1}{n!} f^\varepsilon_{n,[0,t]}\big((e^h-1)^{\otimes n}\big)\,. \tag{4.4.7}$$

Note that (4.4.7) follows easily from the uniform bounds on the rescaled cumulants, recalling Proposition 2.1.3.

CHAPTER 5

CHARACTERIZATION OF THE LIMITING CUMULANTS

Due to the uniform bounds obtained in Theorem 4, for all n there is a limit $f_{n,[0,t]}(H^{\otimes n})$ for $f^{\varepsilon}_{n,[0,t]}(H^{\otimes n})$ as $\mu_\varepsilon \to \infty$. Our goal in this chapter is first to obtain a formula for $f_{n,[0,t]}(H^{\otimes n})$ similar to (4.4.1), with a precise definition of the limiting pseudo-trajectories (see Theorem 5 in Section 5.1 below): the main feature of those pseudo-trajectories is that they correspond to minimally connected collision graphs. In Section 5.2 we derive a formula for the limiting cumulant generating function (Theorem 6) which enables us in Section 5.3 to deduce that this function satisfies a Hamilton-Jacobi equation. The fact that the limiting graphs have no cycles is crucial to the derivation of the equation. The well-posedness of the Hamilton-Jacobi equation is investigated in Section 5.4. It provides finally a rather direct access to the dynamical equations satisfied by the limiting cumulants in Section 5.5.

5.1. Limiting pseudo-trajectories and graphical representation of limiting cumulants

In this section we characterize the limiting cumulants $f_{n,[0,t]}(H^{\otimes n})$ by their integral representation. This means that we have to specify both the admissible graphs, the limiting pseudo-trajectories and the limiting measure.

We first describe the formal limit of (4.4.1). To do this, we start by giving a definition of pseudo-trajectories associated with cumulants for fixed ε. Recall that the cumulant $f^{\varepsilon}_{n,[0,t]}(H^{\otimes n})$ of order n corresponds to graphs of size n which are completely connected, either by recollisions, or by overlaps, or by initial correlations. The clusterings coming from the initial data, being smaller by a factor $O(\varepsilon)$, will not contribute to the limit, and they will be disregarded in this section. The clusterings associated with recollisions and overlaps can be expressed as an additional condition for pseudo-trajectories to be admissible.

Definition 5.1.1 **(Cumulant pseudo-trajectories)**. — *Let $m \geq 0$. The cumulant pseudo-trajectory $\Psi^{\varepsilon}_{n,m}$ associated with the minimally connected graph $T \in \mathcal{T}^{\pm}_n$ decorated with edge signs $\left(s^{\rm clust}_e\right)_{e\in E(T)}$, and collision tree $a \in \mathcal{A}^{\pm}_{n,m}$ is obtained by fixing Z^*_n and a collection of m ordered creation times T_m, and parameters (Ω_m, V_m).*

At each creation time t_k a new particle, labeled k, is adjoined at position $x_{a_k}(t_k) + s_k\varepsilon\omega_k$ and with velocity v_k.

- *if $s_k > 0$ then the velocities v_k and v_{a_k} are changed to $v_k(t_k^-)$ and $v_{a_k}(t_k^-)$ according to the laws* (3.2.1),

— *then all particles are transported (backwards) in* $\mathcal{D}^{\varepsilon}_{n+k}$.

The cumulant pseudo-trajectory is admissible, if the following holds: for all edges $e = \{j, j'\} \in E(T)$, *there exists a pair of particles* q_e *and* q'_e, *respectively in the subtrees* j *and* j', *producing a clustering recollision if* $s_e^{\rm clust} = +$ *or a clustering overlap if* $s_e^{\rm clust} = -$, *at time* $\tau_e^{\rm clust}$ *(respectively* $\tau_e^{\rm rec}$, $\tau_e^{\rm ov}$*). We say that* $\{q_e, q'_e\}$ *is a representative of the edge* e, *and we denote this by* $\{q_e, q'_e\} \approx e$. *We also denote by* $\Theta_{n-1}^{\rm clust}$ *the collection of clustering times.*

Now let us introduce the limiting cumulant pseudo-trajectories and measure. Since we have established a uniform convergence of the series expansion (with respect to m), it is enough to look at a fixed m and a fixed tree $a \in \mathcal{A}^{\pm}_{n,m}$. We prove in Chapter 9 that there are two situations :

— either the parameters $(Z_n^*, T_m, \Omega_m, V_m)$ are such that there is an additional (non clustering) recollision in $\Psi^{\varepsilon}_{n,m}$, but this may happen only for a vanishing set of integration parameters which does not contribute to the limit integral;

— or the parameters $(Z_n^*, T_m, \Omega_m, V_m)$ are such that there is no additional (non clustering) recollision in $\Psi^{\varepsilon}_{n,m}$, and we can prove that $\Psi^{\varepsilon}_{n,m}$ converges to a limit pseudo-trajectory where the collisions, recollisions and overlaps become pointwise, and the scattering angles decouple from the dynamics and become random parameters.

Thus *all* the external recollisions and overlaps in the limiting pseudo-trajectory (corresponding to the constraints $\mathbb{\Delta}_{\lambda_i}$ and φ_ρ in (4.4.1)) are represented by a minimally connected graph (with positive edges connecting vertices of the same forest, negative edges connecting vertices of different forests), as explained before Theorem 4.

The clustering constraints provide $n-1$ singular conditions on the roots $(z_i^*)_{1\le i\le n}$ of the trees, so only one root is free. We set this root to be z_n^*. This will reflect on the limiting measure. For fixed $\varepsilon > 0$ the clustering condition associated with the edge $e = \{i, j\}$ takes the form

$$\omega_e^{\rm clust} := \frac{x_{q_e}(\tau_e^{\rm clust}) - x_{q'_e}(\tau_e^{\rm clust})}{\varepsilon} \in \mathbb{S}^{d-1}\,.$$

Notice that, according to the definitions given in Section 4.4, the clustering recollision constraint implies the existence of a clustering vector $\omega_e^{\rm clust} = \omega_e^{\rm rec}$. This is also true for clustering overlaps ($\omega_e^{\rm clust} = \omega_e^{\rm ov}$), after neglecting a set of parameters whose contribution will be shown to vanish, as $\varepsilon \to 0$, in Section 9.2.2 below.

Given (x_i^*, v_i^*) and v_j^* as well as collision parameters (a, T_m, Ω_m, V_m), since the trajectories are piecewise affine one can perform the local change of variables

$$x_j^* \in \mathbb{T}^d \mapsto (\tau_e^{\rm clust}, \omega_e^{\rm clust}) \in (0, t) \times \mathbb{S}^{d-1} \tag{5.1.1}$$

with Jacobian $\mu_\varepsilon^{-1}\big((v_{q_e}(\tau_e^{\rm clust+}) - v_{q'_e}(\tau_e^{\rm clust+}))\cdot\omega_e^{\rm clust}\big)_+$. This provides the identification of measures

$$\mu_\varepsilon dx_i^* dv_i^* dx_j^* dv_j^* = dx_i^* dv_i^* dv_j^* d\tau_e^{\rm clust} d\omega_e^{\rm clust}\big((v_{q_e}(\tau_e^{\rm clust}) - v_{q'_e}(_e^{\rm clust}))\cdot\omega_e^{\rm clust}\big)_+\,. \tag{5.1.2}$$

We shall explain in Section 8.1 how to identify a good sequence of roots to perform this change of variables iteratively. We can therefore define the limiting singular measure for each tree $a \in \mathcal{A}^{\pm}_{n,m}$, and each minimally connected graph $T \in \mathcal{T}_n^{\pm}$

$$\begin{aligned} d\mu_{{\rm sing},T,a}\left(\Psi^{\varepsilon}_{n,m}\right) := {} & dT_m d\Omega_m dV_m dx_n^* dV_n^* d\Theta_{n-1}^{\rm clust} d\Omega_{n-1}^{\rm clust} \prod_{i=1}^{m} s_i\big((v_i - v_{a_i}(t_i))\cdot\omega_i\big)_+ \\ & \times \prod_{e\in E(T)} \sum_{\{q_e,q'_e\}\approx e} s_e^{\rm clust}\big((v_{q_e}(\tau_e^{\rm clust}) - v_{q'_e}(\tau_e^{\rm clust}))\cdot\omega_e^{\rm clust}\big)_+\,, \end{aligned} \tag{5.1.3}$$

where $\Omega_{n-1}^{\mathrm{clust}}$ denotes the collection of clustering vectors $\big(\omega_e^{\mathrm{clust}}\big)_{e\in E(T)}$.

***Definition 5.1.2* (Limiting cumulant pseudo-trajectories).** — *Let $m \geq 0$. The limiting cumulant pseudo-trajectories $\Psi_{n,m}$ associated with the minimally connected graph $T \in \mathcal{T}_n^{\pm}$ and tree $a \in \mathcal{A}_{n,m}^{\pm}$ are obtained by fixing x_n^* and V_n^*,*

- *for each $e \in E(T)$, a representative $\{q_e, q'_e\} \approx e$,*
- *a collection of m ordered creation times T_m, and parameters (Ω_m, V_m);*
- *a collection of clustering times $(\tau_e^{\mathrm{clust}})_{e\in E(T)}$ and clustering angles $(\omega_e^{\mathrm{clust}})_{e\in E(T)}$.*

At each creation time t_k, a new particle, labeled k, is adjoined at position $x_{a_k}(t_k)$ and with velocity v_k:

- *if $s_k = +$, then the velocities v_k and v_{a_k} are changed to $v_k(t_k^-)$ and $v_{a_k}(t_k^-)$ according to the laws* (3.2.1),
- *then all particles follow the backward free flow until the next creation or clustering time.*

For $\Psi_{n,m}$ to be admissible, at each time τ_e^{clust} the particles q_e and q'_e have to be at the same position:

- *if $s_e = +$, then the velocities v_{q_e} and $v_{q'_e}$ are changed according to the scattering rule, with scattering vector $\omega_e^{\mathrm{clust}}$.*
- *then all particles follow the backward free flow until the next creation or clustering time.*

Thanks to our assumption on the initial data (quasi-product measure), we will show that $f^{\varepsilon 0}_{\{1,\dots,r\}}$ in (4.4.1) becomes a pure product as $\varepsilon \to 0$.

Equipped with these notations, we can now state the result that will be proved in Chapter 9.

Theorem 5. — *Under the assumptions of Theorem* 4, *for all $t \leq T^\star$, $f^{\varepsilon}_{n,[0,t]}(H^{\otimes n})$ converges when $\mu_\varepsilon \to \infty$ to $f_{n,[0,t]}(H^{\otimes n})$ given by*

$$f_{n,[0,t]}(H^{\otimes n}) = \sum_{T\in\mathcal{T}_n^{\pm}} \sum_{m} \sum_{a\in\mathcal{A}_{n,m}^{\pm}} \int d\mu_{\mathrm{sing},T,a}(\Psi_{n,m})\, \mathcal{H}\big(\Psi_{n,m}\big)\, \big(f^0\big)^{\otimes m+n}\,(\Psi^0_{n,m})\,. \tag{5.1.4}$$

In particular by Theorem 4 *there holds*

$$|f_{n,[0,t]}(H^{\otimes n})| \leq C^n t^{n-1} n!\,.$$

5.2. Limiting cumulant generating function

Thanks to Theorem 4 we know that the limiting cumulant generating function has the form

$$\Lambda_{[0,t]}(\gamma) = \sum_{n=1}^{\infty} \frac{1}{n!} f_{n,[0,t]}\big((\gamma-1)^{\otimes n}\big)\,. \tag{5.2.1}$$

The following result provides a graphical expansion of $\Lambda_{[0,t]}(\gamma)$.

Theorem 6. — *Under the assumptions of Theorem* 4, *the limiting cumulant generating function $\Lambda_{[0,t]}$ satisfies*

$$\Lambda_{[0,t]}(\gamma) + 1 = \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{\tilde{T}\in\mathcal{T}_K^{\pm}} \int d\mu_{\mathrm{sing},\tilde{T}}(\Psi_{K,0}) \gamma^{\otimes K}(\Psi_{K,0}) f^{0\otimes K}(\Psi^0_{K,0})\,, \tag{5.2.2}$$

where

$$d\mu_{\mathrm{sing},\tilde{T}} := dx_K^* dV_K \prod_{e=\{q,q'\}\in E(\tilde{T})} s_e\big((v_q(\tau_e) - v_{q'}(\tau_e))\cdot\omega_e\big)_+ d\tau_e d\omega_e\,. \tag{5.2.3}$$

Furthermore the series is absolutely convergent for $t \in [0, T^\star]$:

$$\int d|\mu_{\mathrm{sing},\tilde{T}}(\Psi_{K,0})|\,|\gamma^{\otimes K}(\Psi_{K,0})|\,f^{0\otimes K}(\Psi^0_{K,0}) \leq \big(Ct\big)^{K-1}\,. \tag{5.2.4}$$

Proof. — By definition,

$$\Lambda_{[0,t]}\left(\gamma\right) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{T\in\mathcal{T}_n^{\pm}} \sum_{m} \sum_{a\in\mathcal{A}^{\pm}_{n,m}} \int d\mu_{\mathrm{sing},T,a}(\Psi_{n,m})(\gamma-1)^{\otimes n}\left(f^0\right)^{\otimes(m+n)}\,.$$

Note that the trajectories of particles $i \in \{1,\dots,m\}$ can be extended on the whole interval $[0,t]$ just by transporting i without collision on $[t_i, t]$: this is actually the only way to have a set of $m+n$ pseudotrajectories which is minimally connected (any additional collision would add a non clustering constraint, or require adding new particles). It can therefore be identified to some $\Psi_{m+n,0}$.

Let us now fix $K = n+m$ and symmetrize over all arguments :

$$\begin{aligned}\Lambda_{[0,t]}\left(\gamma\right) &= \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{n=1}^{K} \frac{K!}{n!(K-n)!}(K-n)! \sum_{T\in\mathcal{T}_n^{\pm}} \sum_{a\in\mathcal{A}^{\pm}_{n,K-n}} \int d\mu_{\mathrm{sing},T,a}(\Psi_{n,K-n})(\gamma-1)^{\otimes n}\left(f^0\right)^{\otimes K}\\ &= \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{n=1}^{K} \sum_{|\eta|=n}^{\eta} \sum_{(\eta^c)^\prec} \sum_{T\in\mathcal{T}_\eta^{\pm}} \sum_{a\in\mathcal{A}^{\pm}_{\eta,(\eta^c)^\prec}} \int d\mu_{\mathrm{sing},T,a}(\Psi_{\eta,(\eta^c)^\prec})(\gamma-1)^{\otimes \eta}\left(f^0\right)^{\otimes K}\end{aligned}$$

where η stands for a subset of $\{1^*,\dots,n^*,1,\dots,K-n\}$ with cardinal n; η^c denotes its complement and $(\eta^c)^\prec$ indicates that we have chosen an order on the set η^c. We denote by $\mathcal{A}^{\pm}_{\eta,(\eta^c)^\prec}$ the set of signed trees with roots η and added particles with prescribed order in $(\eta^c)^\prec$.

Note that the combinatorics of collisions a and recollisions T (together with the choice of the representatives $\{q_e, q'_e\}_{e\in E(T)}$) can be described by a single minimally connected graph $\tilde{T} \in \mathcal{T}^{\pm}_K$. In order to apply Fubini's theorem, we then need to understand the mapping

$$(a, T, \{q_e, q'_e\}_{e\in E(T)}) \mapsto (\tilde{T}, \eta)\,.$$

It is easy to see that this mapping is injective but not surjective. Given a pseudo-trajectory $\Psi_{K,0}$ compatible with $\tilde{T}$ and a set η of cardinality n, we reconstruct $(a, T, \{q_e, q'_e\}_{e\in E(T)})$ as follows. We color in red the n particles belonging to η at time t, and in blue the $K-n$ other particles. Then we follow the dynamics backward. At each clustering, we apply the following rule

- if the clustering involves one red particle and one blue particle, then it corresponds to a collision in the Duhamel pseudo-trajectory. The corresponding edge of $\tilde{T}$ will be described by a. We then change the color of the blue particle to red.
- if the clustering involves two red particles, then it corresponds to a recollision in the Duhamel pseudo-trajectory. The corresponding edge of $\tilde{T}$ is therefore an edge $e \in E(T)$ and the two colliding particles determine the representative $\{q_e, q'_e\}$.
- if the clustering involves two blue particles, then the pseudo-trajectory is not admissible for $(\tilde{T}, \eta)$, as it is not associated to any $(a, T, \{q_e, q'_e\}_{e\in E(T)})$.

However the contribution of the non admissible pseudo-trajectories $\Psi_{K,0}$ to

$$\sum_{\tilde{T}\in\mathcal{T}^{\pm}_K} \int d\mu_{\mathrm{sing},\tilde{T}}(\Psi_{K,0})\gamma^{\otimes K}(\Psi_{K,0}) f^{0\otimes K}(\Psi^0_{K,0})$$

is exactly zero, as the overlap and recollision terms associated to the first clustering between two blue particles (i.e. the $\pm$ signs of the corresponding edge) exactly compensate.

We therefore conclude that

$$\Lambda_{[0,t]}(\gamma) = \sum_{K\geq 1}\frac{1}{K!}\sum_{\tilde{T}\in\mathcal{T}_K^{\pm}}\int d\mu_{\mathrm{sing},\tilde{T}}(\Psi_{K,0})\left(f^0\right)^{\otimes K}\sum_{n=1}^{K}\sum_{\eta\in\mathcal{P}_K^n}(\gamma-1)^{\otimes\eta}$$
$$= \sum_{K\geq 1}\frac{1}{K!}\sum_{\tilde{T}\in\mathcal{T}_K^{\pm}}\int d\mu_{\mathrm{sing},\tilde{T}}(\Psi_{K,0})\gamma^{\otimes K}\left(f^0\right)^{\otimes K} - 1$$

which is exactly (5.2.2). Note that the compensation mechanism described above does not work for $n = 0$ and $K = 1$, which is the reason for the -1 in the final formula.

The bound (5.2.4) comes from the definition of $\mu_{\mathrm{sing},\tilde{T}}$ together with the estimates used in the proof of Theorem 4 to control the collision cross-sections. □

5.3. Hamilton-Jacobi equations

For our purpose, it will be convenient to consider test functions on the trajectories which write as

$$e^{h(z([0,t]))} = \gamma\big(z(t)\big)\exp\Big(-\int_0^t\phi\big(s,z(s)\big)ds\Big)\,, \tag{5.3.1}$$

where $\phi : [0,t]\times\mathbb{D}\to\mathbb{C}$ and $\gamma : \mathbb{D}\to\mathbb{C}$ are two functions. We choose complex-valued functions here as we shall be using properties on analytic functionals of γ later in this chapter. All the results obtained so far can easily be adapted to this more general setting. To stress the dependence on ϕ and γ, we introduce a specific notation for the corresponding exponential moment (5.2.1)

$$\mathcal{J}(t,\phi,\gamma) := \Lambda_{[0,t]}(\gamma e^{-\int_0^t\phi})\,. \tag{5.3.2}$$

For $t\in[0,T^*]$, $\alpha\geq 0$ and $\beta > 0$, we define the functional space

$$\mathcal{B}_{\alpha,\beta,t} := \Big\{(\phi,\gamma)\in C^0([0,t]\times\mathbb{D};\mathbb{C})\times C^0(\mathbb{D};\mathbb{C})\ /\ |\gamma(z)|\leq e^{(1-\frac{t}{2T^\star})(\alpha+\frac{\beta}{4}|v|^2)},$$
$$\sup_{s\in[0,t]}|\phi(s,z)|\leq\frac{1}{2T^\star}(\alpha+\frac{\beta}{4}|v|^2)\Big\}\,. \tag{5.3.3}$$

We can now state the main result of this chapter, which shows that $\mathcal{J}$ satisfies a Hamilton-Jacobi equation.

Theorem 7. — *For all $(\phi,\gamma^\star)\in\mathcal{B}_{\alpha_0,\beta_0,T^\star}$, define γ by*

$$D_t\gamma_t - \phi_t\gamma_t = 0,\quad \gamma_{T^\star} = \gamma^\star\,. \tag{5.3.4}$$

Then the functional $\mathcal{J}(t,\phi,\gamma_t)$ satisfies the following Hamilton-Jacobi equation on $[0,T^\star]$:

$$\partial_t\mathcal{J}(t,\phi,\gamma_t) = \frac{1}{2}\int\frac{\partial\mathcal{J}}{\partial\gamma}(t,\phi,\gamma_t)(z_1)\frac{\partial\mathcal{J}}{\partial\gamma}(t,\phi,\gamma_t)(z_2)\Big(\gamma_t(z_1')\gamma_t(z_2')-\gamma_t(z_1)\gamma_t(z_2)\Big)d\mu(z_1,z_2,\omega)\,, \tag{5.3.5}$$

where we used the notation (1.3.5)

$$d\mu(z_1,z_2,\omega) := \delta_{x_1-x_2}((v_1-v_2)\cdot\omega)_+ d\omega dv_1 dv_2 dx_1\,.$$

Remark 5.3.1. — *Note that, if $(\phi,\gamma_t)\in\mathcal{B}_{\alpha,\beta,t}$ and γ solves on $[0,t]$*

$$D_s\gamma_s - \phi_s\gamma_s = 0\,,\quad \gamma_{|t} = \gamma_t\,,$$

then $(\phi,\gamma_s)\in\mathcal{B}_{\alpha,\beta,s}$ for all $s\in[0,t]$.

Proof. — We start by choosing $(\phi,\gamma^\star)\in\mathcal{B}_{\alpha_0,\beta_0,T^\star}$ smooth and with compact support (in v), set $t<T^\star$ and compute the limit as δ goes to zero of the rate of change

$$\Delta_\delta\mathcal{J}(t,\phi,\gamma):=\frac{1}{\delta}\Big(\mathcal{J}(t+\delta,\phi,\gamma_{t+\delta})-\mathcal{J}(t,\phi,\gamma_t)\Big)\,.$$

The following remark is crucial for the computation of $\Delta_\delta\mathcal{J}(t,\phi,\gamma)$, and its limit as δ goes to 0 : for any $t'\geq t$,

$$\mathcal{J}(t,\phi,\gamma)=\Lambda_{[0,t']}(\gamma e^{-\int_0^t\phi})\,.$$

This is easily seen by going back to the definition (4.4.7) of $\Lambda^\varepsilon_{[0,t]}$ and taking the limit $\varepsilon\to 0$. Therefore thanks to (5.2.2)

$$\Delta_\delta\mathcal{J}(t,\phi,\gamma)=\sum_K\frac{1}{K!}\sum_{\tilde T\in\mathcal{T}_K^\pm}\int d\mu_{\mathrm{sing},\tilde T}(\Psi_{K,0})\mathcal{H}_\delta(\Psi_{K,0})\left(f^0\right)^{\otimes K}(\Psi^0_{K,0})$$

with

$$\mathcal{H}_\delta(\Psi_{K,0}):=\frac{1}{\delta}\Big[\prod_{i=1}^K\gamma_{t+\delta}\left(z_i(t+\delta)\right)\,e^{-\int_0^{t+\delta}\phi(s,z_i(s))ds}-\prod_{i=1}^K\gamma_t\big(z_i(t)\big)\,e^{-\int_0^t\phi(s,z_i(s))ds}\Big]\tag{5.3.6}$$

and $\Psi_{K,0}$ are trajectories on $[0,t+\delta]$ (having exactly $K-1$ connections prescribed by $\tilde T$).

We claim that

$$\mathcal{H}_\delta=\mathcal{H}_1+\mathcal{H}_{\delta,2}+\mathcal{H}_{\delta,3}\tag{5.3.7}$$

where

$$\mathcal{H}_1\big(\Psi_{K,0}\big):=\sum_{i=1}^K\big(\partial_t\gamma_t-\gamma_t\phi_t\big)\big(z_i(t)\big)\,e^{-\int_0^t\phi(s,z_i(s))ds}\prod_{j\neq i}\gamma_t\big(z_j(t)\big)\,e^{-\int_0^t\phi(s,z_j(s))ds}\,,$$

$$\mathcal{H}_{\delta,2}\big(\Psi_{K,0}\big):=\frac{1}{\delta}\Big[\prod_{i=1}^K\gamma_t\left(z_i(t+\delta)\right)\,e^{-\int_0^t\phi(s,z_i(s))ds}-\prod_{i=1}^K\gamma_t\big(z_i(t)\big)\,e^{-\int_0^t\phi(s,z_i(s))ds}\Big]\,,$$

while

$$\lim_{\delta\to0}\sum_K\frac{1}{K!}\sum_{\tilde T\in\mathcal{T}_K^\pm}\int d\mu_{\mathrm{sing},\tilde T}(\Psi_{K,0})\mathcal{H}_{\delta,3}(\Psi_{K,0})\left(f^0\right)^{\otimes K}(\Psi^0_{K,0})=0\,.\tag{5.3.8}$$

In the following we identify the contributions of $\mathcal{H}_1$ and $\mathcal{H}_{\delta,2}$, which lead to the Hamilton-Jacobi equation, and prove that the remainder term $\mathcal{H}_{\delta,3}$ goes to zero. We shall use the fact that the function $\gamma\mapsto\mathcal{J}(t,\phi,\gamma)$ is an analytic function of γ since the series is converging uniformly. Thus the derivative with respect to γ in the direction Υ is given by a term-wise derivation of the series (5.2.2):

$$\begin{aligned}\int_{\mathbb{D}}dz\frac{\partial\mathcal{J}(t,\phi,\gamma)}{\partial\gamma}(z)\Upsilon(z)=\sum_K\frac{1}{K!}\sum_{\tilde T\in\mathcal{T}_K^\pm}\sum_{i=1}^K\int d\mu_{\mathrm{sing},\tilde T}(\Psi_{K,0})\big(\Upsilon(z_i(t))e^{-\int_0^t\phi(z_i(s),s)}\big)\\ \times\prod_{j\neq i}(\gamma(z_j(t))e^{-\int_0^t\phi(z_j(s),s)})\left(f^0\right)^{\otimes K}(\Psi^0_{K,0})\end{aligned}\tag{5.3.9}$$

Contribution of $\mathcal{H}_1$: Let us set

$$\Delta_1\mathcal{J}(t,\phi,\gamma):=\sum_K\frac{1}{K!}\sum_{\tilde T\in\mathcal{T}_K^\pm}\int d\mu_{\mathrm{sing},\tilde T}(\Psi_{K,0})\mathcal{H}_1(\Psi_{K,0})\left(f^0\right)^{\otimes K}(\Psi^0_{K,0})\,.$$

Thanks to (5.3.9) there holds directly

$$\Delta_1 \mathcal{J}(t,\phi,\gamma) = \int_{\mathbb{D}} dz \frac{\partial \mathcal{J}(t,\phi,\gamma)}{\partial \gamma}(z) \left(\partial_t \gamma_t - \gamma_t \phi_t\right)(z)\,. \tag{5.3.10}$$

Contribution of $\mathcal{H}_{\delta,2}$: Let us set

$$\Delta_{\delta,2}\mathcal{J}(t,\phi,\gamma) := \sum_K \frac{1}{K!} \sum_{\tilde{T}\in\mathcal{T}_K^{\pm}} \int d\mu_{\mathrm{sing},\tilde{T}}(\Psi_{K,0}) \mathcal{H}_{\delta,2}(\Psi_{K,0}) \left(f^0\right)^{\otimes K}(\Psi^0_{K,0})\,.$$

If there is no collision on $[t,t+\delta]$, then $z_i(t+\delta) - z_i(t) = (0,\delta v_i(t))$ so

$$\mathcal{H}_{\delta,2}\big(\Psi_{K,0}\big) = \sum_{i=1}^K v_i(t)\cdot\nabla_x \gamma_t(z_i(t)) e^{-\int_0^t \phi(s,z_i(s))ds} \times \prod_{j\neq i} \Big((\gamma(z_j(t)) e^{-\int_0^t \phi(z_j(s),s)} \Big) + O(\delta)\,.$$

If there is one collision at $\tau\in[t,t+\delta]$, say between j_1 and j_2, we have an additional contribution

$$\frac{1}{\delta}\Big(\gamma_t(z_{j_1}(\tau^+))\gamma_t(z_{j_2}(\tau^+)) - \gamma_t(z_{j_1}(\tau^-))\gamma_t(z_{j_2}(\tau^-))\Big) e^{-\int_0^t \phi(s,z_{j_1}(s))ds} e^{-\int_0^t \phi(s,z_{j_2}(s))ds} \\ \times \prod_{i\neq j_1,j_2} \Big(\gamma_t\big(z_i(t)\big)\, e^{-\int_0^t \phi(s,z_i(s))ds}\Big)\,,$$

but of course it imposes a strong constraint on $\mu_{\mathrm{sing},\tilde{T}}$ as $\tau\in[t,t+\delta]$:

$$d\mu_{\mathrm{sing},\tilde{T}} = dx_K dV_K\, s_{\{j_1,j_2\}}\big((v_{j_1}(\tau) - v_{j_2}(\tau))\cdot\omega\big)_+ \mathbf{1}_{\tau\in[t,t+\delta]} d\tau d\omega \\ \times \prod_{e=\{q,q'\}\in E(\tilde{T})\setminus\{j_1,j_2\}} s_e\big((v_q(\tau_e) - v_{q'}(\tau_e))\cdot\omega_e\big)_+ d\tau_e d\omega_e\,.$$

Having at least two collisions in $[t,t+\delta]$ provides a contribution of order $O(\delta)$ since the two collision times have to be in $[t,t+\delta]$, so we can neglect this term.

Now, since $\tilde{T}$ is a minimally connected graph, removing the edge $\{j_1,j_2\}$ splits it into two (minimally connected) graphs :

$$\sum_K \frac{1}{K!} \sum_{\tilde{T}\in\mathcal{T}_K^{\pm}} \sum_{\{j_1,j_2\}\in E(\tilde{T})} = \frac{1}{2} \sum_{K_1,K_2} \frac{1}{K_1!K_2!} \sum_{\tilde{T}_1\in\mathcal{T}_{K_1}^{\pm}} \sum_{\tilde{T}_2\in\mathcal{T}_{K_2}^{\pm}} \sum_{j_1=1}^{K_1} \sum_{j_2=1}^{K_2}\,.$$

We therefore end up with the following identity

$$\Delta_{\delta,2}\mathcal{J}(t,\phi,\gamma) = \Delta^{\mathrm{T}}_{\delta,2}\mathcal{J}(t,\phi,\gamma) + \Delta^{\mathrm{C}}_{\delta,2}\mathcal{J}(t,\phi,\gamma) + O(\delta)$$

with

$$\Delta^{\mathrm{T}}_{\delta,2}\mathcal{J}(t,\phi,\gamma) := \sum_K \frac{1}{K!} \sum_{i=1}^K \sum_{\tilde{T}\in\mathcal{T}_K^{\pm}} \int d\mu_{\mathrm{sing},\tilde{T}}(\Psi_{K,0}) v_i(t)\cdot\nabla_x\gamma_t(z_i(t)) e^{-\int_0^t \phi(s,z_i(s))ds} \\ \times \prod_{j\neq i} \Big(\gamma(z_j(t) e^{-\int_0^t \phi(z_j(s),s)}\Big) \left(f^0\right)^{\otimes K}(\Psi^0_{K,0})$$

and

$$\Delta^{\mathrm{C}}_{\delta,2}\mathcal{J}(t,\phi,\gamma) := \frac{1}{2}\sum_{K_1,K_2}\frac{1}{K_1!K_2!}\sum_{\tilde T_1\in\mathcal{T}^{\pm}_{K_1}}\sum_{\tilde T_2\in\mathcal{T}^{\pm}_{K_2}}\sum_{j_1=1}^{K_1}\sum_{j_2=1}^{K_2}\int d\mu_{\mathrm{sing},\tilde T_1}(\Psi_{K_1,0})d\mu_{\mathrm{sing},\tilde T_2}(\Psi_{K_2,0})$$
$$\times\big(f^0\big)^{\otimes(K_1+K_2)}\,(\Psi^0_{K_1,0}\otimes\Psi^0_{K_2,0})\prod_{i\neq j_1,j_2}\Big(\gamma_t\big(z_i(t)\big)\,e^{-\int_0^t\phi(s,z_i(s))ds}\Big)\,e^{-\int_0^t\phi(s,z_{j_1}(s))ds}e^{-\int_0^t\phi(s,z_{j_2}(s))ds}$$
$$\times\Big(\gamma_t(z_{j_1}(\tau^+))\gamma_t(z_{j_2}(\tau^+))-\gamma_t(z_{j_1}(\tau^-))\gamma_t(z_{j_1}(\tau^-))\Big)\delta_{x_{j_1}(\tau)-x_{j_2}(\tau)}\big((v_{j_1}(\tau)-v_{j_2}(\tau))\cdot\omega\big)_+\frac{1}{\delta}\mathbf{1}_{\tau\in[t,t+\delta]}d\tau d\omega\,.$$

Putting together all those contributions and recalling (5.3.9) gives rise to

$$\begin{aligned}\lim_{\delta\to0}\Delta_{\delta,2}\mathcal{J}(t,\phi,\gamma) &= \int_{\mathbb{D}}dz\frac{\partial\mathcal{J}(t,\phi,\gamma)}{\partial\gamma}(z)\,\big(v\cdot\nabla_x\gamma_t\big)(z)\\ &+\frac{1}{2}\int\frac{\partial\mathcal{J}}{\partial\gamma}(t,\phi,\gamma_t)(z_1)\frac{\partial\mathcal{J}}{\partial\gamma}(t,\phi,\gamma_t)(z_2)\Big(\gamma_t(z_1')\gamma_t(z_2')-\gamma_t(z_1)\gamma_t(z_2)\Big)d\mu(z_1,z_2,\omega)\,.\end{aligned}\tag{5.3.11}$$

Notice that it is very important that the graphs are minimal in this computation: if that had not been the case, the above splitting in $\Delta^{\mathrm{C}}_{\delta,2}$ would not have been possible and would have given rise to second order derivatives in $\mathcal{J}$.

Computation of $\mathcal{H}_{\delta,3}$: By definition,

$$\mathcal{H}_\delta\big(\Psi_{K,0}\big)-\mathcal{H}_{\delta,2}\big(\Psi_{K,0}\big)=\frac{1}{\delta}\Big[\prod_{i=1}^K\Big(\gamma_{t+\delta}\left(z_i(t+\delta)\right)\,e^{-\int_0^{t+\delta}\phi(s,z_i(s))ds}\Big)-\prod_{i=1}^K\Big(\gamma_t\left(z_i(t+\delta)\right)\,e^{-\int_0^t\phi(s,z_i(s))ds}\Big)\Big]\,.$$

Then we decompose, for any $1\le i\le K$,

$$\begin{aligned}&\gamma_{t+\delta}\left(z_i(t+\delta)\right)\,e^{-\int_0^{t+\delta}\phi(s,z_i(s))ds}-\gamma_t\left(z_i(t+\delta)\right)\,e^{-\int_0^t\phi(s,z_i(s))ds}\\ &=\big(\gamma_{t+\delta}-\gamma_t\big)\left(z_i(t+\delta)\right)\,e^{-\int_0^t\phi(s,z_i(s))ds}+\gamma_{t+\delta}\left(z_i(t+\delta)\right)\Big(e^{-\int_t^{t+\delta}\phi(s,z_i(s))ds}-1\Big)e^{-\int_0^t\phi(s,z_i(s))ds}\,.\end{aligned}$$

Since ϕ and γ are smooth and have compact support, there holds

$$\begin{aligned}\big(\gamma_{t+\delta}-\gamma_t\big)\left(z_i(t+\delta)\right) &= \delta\,\partial_t\gamma_t\left(z_i(t)\right)+o(\delta)\\ &\quad+\big(\gamma_{t+\delta}-\gamma_t\big)\left(z_i(t+\delta)\right)-\big(\gamma_{t+\delta}-\gamma_t\big)\left(z_i(t)\right)\,.\end{aligned}$$

Similarly

$$\begin{aligned}&\gamma_{t+\delta}\left(z_i(t+\delta)\right)\Big(e^{-\int_t^{t+\delta}\phi(s,z_i(s))ds}-1\Big)\\ &\quad=-\gamma_t\big(z_i(t)\big)\int_t^{t+\delta}\phi(s,z_i(s))ds+o(\delta)-\Big(\gamma_t\big(z_i(t+\delta)\big)-\gamma_t\big(z_i(t)\big)\Big)\int_t^{t+\delta}\phi(s,z_i(s))ds\\ &\quad=-\delta\gamma_t\big(z_i(t)\big)\phi(t,z_i(t))+o(\delta)-\Big(\gamma_t\big(z_i(t+\delta)\big)-\gamma_t\big(z_i(t)\big)\Big)\int_t^{t+\delta}\phi(s,z_i(s))ds\\ &\qquad+\gamma_t\big(z_i(t)\big)\Big(\int_t^{t+\delta}\phi(s,z_i(t))ds-\int_t^{t+\delta}\phi(s,z_i(s))ds\Big)\,.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{\delta}\Big(\gamma_{t+\delta}\left(z_i(t+\delta)\right)\,e^{-\int_0^{t+\delta}\phi(s,z_i(s))ds}-\gamma_t\left(z_i(t+\delta)\right)\,e^{-\int_0^t\phi(s,z_i(s))ds}\Big)\\ &\quad=\Big(\partial_t\gamma_t\left(z_i(t)\right)-\gamma_t\big(z_i(t)\big)\phi(t,z_i(t))+o(1)+\mathcal{E}\big(z_i([t,t+\delta])\big)\Big)e^{-\int_0^t\phi(s,z_i(s))},\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}\big(z_i([t,t+\delta])\big) := \frac{1}{\delta}\Big[\big(\gamma_{t+\delta}-\gamma_t\big)\big(z_i(t+\delta)\big)-\big(\gamma_{t+\delta}-\gamma_t\big)\big(z_i(t)\big)\Big]\\ -\frac{1}{\delta}\Big(\gamma_t\big(z_i(t+\delta)\big)-\gamma_t\big(z_i(t)\big)\Big)\int_t^{t+\delta}\phi(s,z_i(s))ds\\ +\frac{1}{\delta}\gamma_t\big(z_i(t)\big)\int_t^{t+\delta}\Big(\phi(s,z_i(t))-\phi(s,z_i(s))\Big)ds\end{aligned}$$

satisfies

$$\Big|\mathcal{E}\big(z_i([t,t+\delta])\big)\Big| \le C\sup_{s\in[t,t+\delta]}|z_i(t)-z_i(s)|\,.$$

Finally we have

$$\begin{aligned}\mathcal{H}_\delta\big(\Psi_{K,0}\big)-\mathcal{H}_{\delta,2}\big(\Psi_{K,0}\big)=\sum_{i=1}^K\Big(\partial_t\gamma_t\left(z_i(t)\right)-\gamma_t\big(z_i(t)\big)\phi(t,z_i(t))+o(1)+\mathcal{E}\big(z_i([t,t+\delta])\big)\Big)e^{-\int_0^t\phi(s,z_i(s))}\\ \times\prod_{j\neq i}\Big(\gamma_t(z_j(t))\,e^{-\int_0^t\phi(s,z_j(s))ds}\Big)+O(K^2\delta)\end{aligned}$$

so by definition of $\mathcal{H}_1$, the remainder $\mathcal{H}_{\delta,3}$ is of the form

$$\sum_{i=1}^K\Big(o(1)+\mathcal{E}\big(z_i([t,t+\delta])\big)\Big)e^{-\int_0^t\phi(s,z_i(s))}\prod_{j\neq i}\Big(\gamma_t(z_j(t))\,e^{-\int_0^t\phi(s,z_j(s))ds}\Big)+O(K^2\delta)\,.$$

Summing over K and using (5.2.4) to get

$$\sum_{K=1}^\infty\frac{1}{K!}K^2\sum_{\tilde T\in\mathcal{T}_K^\pm}\int d|\mu_{\mathrm{sing},\tilde T}(\Psi_{K,0})|\exp\Big(\alpha_0K+\frac{\beta_0}{4}|V_K|^2\Big)\left(f^0\right)^{\otimes K}(\Psi^0_{K,0})<+\infty$$

we obtain that the contribution of the terms of order $o(1)$ converges to 0 when δ goes to 0. The arguments used to control $\mathcal{H}_{\delta,2}$ show that the error term generated by $\mathcal{E}$ is $O(\delta)$.

Conclusion: The above analysis gives rise to

$$\begin{aligned}\partial_t\mathcal{J}(t,\phi,\gamma_t)&=\lim_{\delta\to0}\Delta_\delta\mathcal{J}(t,\phi,\gamma_t)\\&=\Delta_1\mathcal{J}(t,\phi,\gamma_t)+\lim_{\delta\to0}\Delta_{\delta,2}\mathcal{J}(t,\phi,\gamma_t)\end{aligned}$$

and putting together (5.3.10) and (5.3.11) we find that

$$\begin{aligned}\partial_t\mathcal{J}(t,\phi,\gamma_t)=\int\frac{\partial\mathcal{J}(t,\phi,\gamma)}{\partial\gamma}(z)\left(\partial_t\gamma_t+v\cdot\nabla_x\gamma_t-\gamma_t\phi_t\right)(z)\,dz\\ +\frac{1}{2}\int\frac{\partial\mathcal{J}}{\partial\gamma}(t,\phi,\gamma_t)(z_1)\frac{\partial\mathcal{J}}{\partial\gamma}(t,\phi,\gamma_t)(z_2)\Big(\gamma_t(z_1')\gamma_t(z_2')-\gamma_t(z_1)\gamma_t(z_2)\Big)d\mu(z_1,z_2,\omega)\,.\end{aligned}\tag{5.3.12}$$

The transport equation (5.3.4) on γ_t yields finally

$$\partial_t\mathcal{J}(t,\phi,\gamma_t)=\frac{1}{2}\int\frac{\partial\mathcal{J}}{\partial\gamma}(t,\phi,\gamma_t)(z_1)\frac{\partial\mathcal{J}}{\partial\gamma}(t,\phi,\gamma_t)(z_2)\Big(\gamma_t(z_1')\gamma_t(z_2')-\gamma_t(z_1)\gamma_t(z_2)\Big)d\mu(z_1,z_2,\omega)\,.$$

The theorem is proved in the case when Φ and γ are smooth and compactly supported in v. In the case when (Φ,γ^*) belongs to $\mathcal{B}_{\alpha_0,\beta_0,T^\star}$, we use an approximation procedure and take limits in the mild form of the Hamilton-Jacobi equation: this turns out to be possible thanks to the stability results of the next section : see in particular Proposition 5.4.1. □

5.4. Stability of the Hamilton-Jacobi equation

Making sense of the Hamilton-Jacobi equation (5.3.5) requires some regularity in the space variable on the functional derivatives in order to define the Dirac distribution $\delta_{x_1-x_2}$, as well as some integrability with respect to the velocity variable. We prove here some properties of the functional $\mathcal{J}$. We use the notation

$$\|\mathcal{J}(t)\|_{\alpha,\beta} := \sup_{(\phi,\gamma)\in\mathcal{B}_{\alpha,\beta,t}} \left|\mathcal{J}(t,\phi,\gamma)\right|. \tag{5.4.1}$$

Proposition 5.4.1. — *The limiting cumulant generating function $\gamma \mapsto \mathcal{J}(t,\phi,\gamma)$ introduced in (5.3.2) is an analytic function of γ, on $\mathcal{B}_{\alpha_0,\beta_0,t}$. In particular, the following estimates hold: for any $\alpha' \in]\alpha,\alpha_0]$, $\beta' \in]\beta,\beta_0]$ and all $(\phi,\gamma) \in \mathcal{B}_{\alpha,\beta,t}$, the derivative $\dfrac{\partial\mathcal{J}(t,\phi,\gamma)}{\partial\gamma}$ at γ satisfies the following loss continuity estimate:*

$$\left\|\frac{\partial\mathcal{J}(t,\phi,\gamma)}{\partial\gamma}\right\|_{\mathcal{M}\left((1+|v|)\exp\left((1-\frac{t}{2T^\star})(\alpha+\frac{\beta}{4}|v|^2)\right)dxdv\right)} \leq C\left(\frac{1}{\alpha'-\alpha}+\frac{1}{\beta'-\beta}\right)\|\mathcal{J}(t)\|_{\alpha',\beta'}. \tag{5.4.2}$$

Moreover, the derivative $\dfrac{\partial\mathcal{J}(t,\phi,\gamma)}{\partial\gamma}$ is a continuous function on $\mathbb{D}$ and if $(\phi,\gamma) \in \mathcal{B}_{\alpha_0,\beta_0,T^\star}$,

$$\left\|\frac{\partial\mathcal{J}(t,\phi,\gamma)}{\partial\gamma}(1+|v|)\exp(\frac{\beta_0}{4}|v|^2)\right\|_{L^\infty(\mathbb{D})} \leq C_0. \tag{5.4.3}$$

Proof. — Thanks to (5.3.9) we find that $\dfrac{\partial\mathcal{J}(t,\phi,\gamma)}{\partial\gamma}$ is a function on $\mathbb{D}$, for which we are going to establish properties (5.4.2) and (5.4.3).

Step 1. Proof of (5.4.2). Let (ϕ,γ) be in $\mathcal{B}_{\alpha,\beta,t}$ and let Υ be a continuous function on $\mathbb{D}$ satisfying

$$|\Upsilon(x,v)| \leq (1+|v|)\exp\left((1-\frac{t}{2T^\star})(\alpha+\frac{\beta}{4}|v|^2)\right).$$

It is easy to check that for a suitable choice of $\lambda > 0$, the couple $(\phi,\gamma+\lambda e^{i\theta}\Upsilon)$ belongs to $\mathcal{B}_{\alpha',\beta',t}$. Indeed it suffices to notice that

$$\begin{aligned}\left|\gamma+\lambda e^{i\theta}\Upsilon\right| &< \left(1+\lambda(1+|v|)\right)\exp\left((1-\frac{t}{2T^\star})(\alpha+\frac{\beta}{4}|v|^2)\right)\\ &\leq \exp\left((1-\frac{t}{2T^\star})(\alpha+\frac{\beta}{4}|v|^2)+2\lambda+\frac{\lambda}{2}|v|^2\right)\\ &\leq \exp\left((1-\frac{t}{2T^\star})(\alpha'+\frac{\beta'}{4}|v|^2)\right),\end{aligned}$$

provided that $\lambda \leq \min\left(\dfrac{\alpha'-\alpha}{4}, \dfrac{\beta'-\beta}{4}\right)$. Then by analyticity, choosing $\lambda = \min\left(\dfrac{\alpha'-\alpha}{4}, \dfrac{\beta'-\beta}{4}\right)$, the derivative can be estimated by a contour integral

$$\int_{\mathbb{D}} dz \frac{\partial\mathcal{J}(t,\phi,\gamma)}{\partial\gamma}(z)\ \Upsilon(z) = \frac{1}{2\pi\lambda}\int_0^{2\pi}\mathcal{J}(t,\phi,\gamma+\lambda e^{i\theta}\Upsilon)e^{-i\theta}d\theta,$$

and we conclude that for all (ϕ,γ) in $\mathcal{B}_{\alpha,\beta,t}$,

$$\left\|\frac{\partial\mathcal{J}(t,\phi,\gamma)}{\partial\gamma}\right\|_{\mathcal{M}\left((1+|v|)\exp\left((1-\frac{t}{2T^\star})(\alpha+\frac{\beta}{4}|v|^2)\right)\right)} \leq C\left(\frac{1}{\alpha'-\alpha}+\frac{1}{\beta'-\beta}\right)\|\mathcal{J}(t)\|_{\alpha',\beta'}.$$

This completes (5.4.2).

Step 2. Proof of (5.4.3). For the second estimate, we use the series expansion (5.3.9). The singular measure $\mu_{\mathrm{sing},\tilde{T}}$ is invariant under global translations, and since Υ depends only on one variable in $\mathbb{D}$, (5.3.9) still makes sense if $\exp(-\frac{\beta_0}{4}|v|^2)\Upsilon$ is only a measure. Up to changing the parameter of the weights, we get the result.

Proposition 5.4.1 is proved. □

A natural question at this stage is to know whether the Hamilton-Jacobi equation provides a complete characterization of the limiting cumulant generating function $\mathcal{J}(t,\phi,\gamma)$. In other words, we need to prove that the Hamilton-Jacobi equation (5.3.5) has a unique solution (at least in a good class of functionals). Note that the existence of a solution is not an issue here since we already know that the Hamilton-Jacobi equation has a solution (by construction of the limit). To prove uniqueness we use an analyticity-type argument taken from [**26**]; such analytic techniques will also be used later, in Chapter 6 — see the statement of Theorem 8). The important point in the proof is the use of the norm

$$\mathcal{N}(\mathcal{J}) := \sup_{\substack{\rho<1\\ t<T(1-\rho)}} \|\mathcal{J}(t)\|_{\alpha_0\rho,\beta_0\rho}\Big(1-\frac{t}{T(1-\rho)}\Big)$$

for some well chosen T.

Proposition 5.4.2. — *There exists $T\in]0,T^\star]$ such that the Hamilton-Jacobi equation* (5.3.5) *has locally a unique (analytic) solution $\mathcal{J}$, in the class of functionals which satisfy the a priori estimates* (5.4.2) *and* (5.4.3).

Proof. — Assume that there are two different solutions $\mathcal{J}$ and $\mathcal{J}'$ of (5.3.5) with same initial data, analytic with respect to γ, and satisfying the a priori estimates (5.4.2)-(5.4.3). In particular, their difference is analytic with respect to γ and satisfies an estimate similar to (5.4.2). Then we write

$$\begin{aligned}\mathcal{J}(t,\phi,\gamma)-\mathcal{J}'(t,\phi,\gamma) = \frac12\int_0^t\int \frac{\partial\big(\mathcal{J}(s,\phi,\gamma)-\mathcal{J}'(s,\phi,\gamma)\big)}{\partial\gamma}(z_1)\frac{\partial\big(\mathcal{J}(s,\phi,\gamma)+\mathcal{J}'(s,\phi,\gamma)\big)}{\partial\gamma}(z_2)\\ \times\Big(\gamma_s(z_1')\gamma_s(z_2') \ -\gamma_s(z_1)\gamma_s(z_2)\big)\Big)d\mu(z_1,z_2,\omega)\end{aligned}$$

from which we deduce that for any $(\phi,\gamma)\in\mathcal{B}_{\alpha,\beta,t}$ and any α',β' with $0<\beta<\beta'\le\beta_0$, $0<\alpha<\alpha'\le\alpha_0$

$$\begin{aligned}\Big|\big(\mathcal{J}(t,\phi,\gamma_t)-\mathcal{J}'(t,\phi,\gamma_t)\big)\Big| &\le C\int_0^t ds\Big\|\frac{\partial\big(\mathcal{J}(s,\phi,\gamma_s)-\mathcal{J}'(s,\phi,\gamma_s)\big)}{\partial\gamma}\Big\|_{\mathcal{M}\Big((1+|v|)\exp\left((1-\frac{s}{2T^\star})(\alpha+\frac{\beta}{4}|v|^2)\right)dxdv\Big)}\\ &\quad\times\Big\|\frac{\partial\big(\mathcal{J}(s,\phi,\gamma_s)+\mathcal{J}'(s,\phi,\gamma_s)\big)}{\partial\gamma}(1+|v|)\exp(\frac{\beta_0}{4}|v|^2)\Big\|_{C^0(\mathbb{D})}\\ &\le C\int_0^t ds\left(\frac{1}{\alpha'-\alpha}+\frac{1}{\beta'-\beta}\right)\|\mathcal{J}(s)-\mathcal{J}'(s)\|_{\alpha',\beta'}\end{aligned}$$

where C is a generic constant depending only on α_0,β_0.

Taking the supremum on all couples $(\phi,\gamma_t)\in\mathcal{B}_{\alpha,\beta,t}$, we obtain that

$$\Big\|\mathcal{J}(t)-\mathcal{J}'(t)\Big\|_{\alpha,\beta}\le C\int_0^t ds\left(\frac{1}{\alpha'-\alpha}+\frac{1}{\beta'-\beta}\right)\Big\|\mathcal{J}(s)-\mathcal{J}'(s)\Big\|_{\alpha',\beta'}.$$

We finally introduce the weight in time: we set $\beta = \rho\beta_0$ and $\alpha = \rho\alpha_0$, and we let β' and α' depend on s in the following way: $\beta' = \widetilde{\rho}(s)\beta_0$ and $\alpha' = \widetilde{\rho}(s)\alpha_0$ with

$$\widetilde{\rho}(s) := \frac{1}{2}\Big(1+\rho-\frac{s}{T}\Big)$$

with T to be chosen small enough. Then

$$\widetilde{\rho}-\rho = \frac{1}{2}\Big(1-\rho-\frac{s}{T}\Big) \quad \text{and} \quad 1-\widetilde{\rho} = \frac{1}{2}\Big(1-\rho+\frac{s}{T}\Big)\,,$$

and we get, for a constant C depending only on α_0 and β_0,

$$\begin{aligned}
\Big\|\mathcal{J}(t)-\mathcal{J}'(t)\Big\|_{\alpha,\beta}\Big(1-\frac{t}{T(1-\rho)}\Big) &\le C\mathcal{N}(\mathcal{J}-\mathcal{J}')\Big(1-\frac{t}{T(1-\rho)}\Big)\int_0^t\Big(\frac{1}{\widetilde{\rho}-\rho}\Big)\Big(1-\frac{s}{T(1-\widetilde{\rho})}\Big)^{-1}ds\\
&\le 2C\mathcal{N}(\mathcal{J}-\mathcal{J}')\Big(1-\frac{t}{T(1-\rho)}\Big)\int_0^t\Big(\frac{1}{1-\rho-\frac{s}{T}}\Big)\Big(\frac{1-\rho+\frac{s}{T}}{1-\rho-\frac{s}{T}}\Big)ds\\
&\le 4C\mathcal{N}(\mathcal{J}-\mathcal{J}')\Big(1-\rho-\frac{t}{T}\Big)\int_0^t\frac{ds}{\Big(1-\rho-\frac{s}{T}\Big)^2}\\
&\le 4CT\mathcal{N}(\mathcal{J}-\mathcal{J}')\,.
\end{aligned}$$

For T sufficiently small, we obtain that the constant $4CT$ is strictly less than 1, which implies finally that $\mathcal{N}(\mathcal{J}-\mathcal{J}')=0$. Proposition 5.4.2 is proved. □

5.5. Dynamical equations for the limiting cumulant densities

The Hamilton-Jacobi equation (5.3.5) enables us to deduce dynamical equations for the limiting cumulants. More precisely we consider now the case when the weight acts only at the final time ($\Phi \equiv 0$ and $\gamma \equiv \gamma_t$) and study the limiting cumulant densities $(f_n(t))_{n\ge1}$ defined (by abuse of notation since they are singular measures)

$$f_{n,[0,t]}\big((\gamma-1)^{\otimes n}\big) = \int dZ_n f_n(t,Z_n)(\gamma-1)^{\otimes n}(Z_n) \quad \text{with} \quad \gamma(z([0,t])) = \gamma(z(t))\,. \tag{5.5.4}$$

Note that γ is no longer assumed to satisfy a transport equation. Setting

$$\tilde{\mathcal{J}}(t,\gamma) := \Lambda_{[0,t]}(\gamma)\,,$$

we find from (5.3.12) that

$$\partial_t\tilde{\mathcal{J}}(t,\gamma) - \Big\langle\frac{\partial\tilde{\mathcal{J}}}{\partial\gamma}(t,\gamma), v\cdot\nabla_x\gamma\Big\rangle = \frac{1}{2}\int\frac{\partial\tilde{\mathcal{J}}}{\partial\gamma}(t,\gamma)(z_1)\frac{\partial\tilde{\mathcal{J}}}{\partial\gamma}(t,\gamma)(z_2)\Big(\gamma(z_1')\gamma(z_2')-\gamma(z_1)\gamma(z_2)\Big)d\mu(z_1,z_2,\omega)\,.$$

Recalling (5.3.9), there holds on the one hand

$$\begin{aligned}
\Big\langle\frac{\partial\tilde{\mathcal{J}}}{\partial\gamma}(t,\gamma), v\cdot\nabla_x\gamma\Big\rangle &= \sum_{n\ge1}\sum_{i=1}^n\frac{1}{n!}\int dZ_n f_n(t,Z_n)(v_i\cdot\nabla_{x_i}\gamma)(z_i)(\gamma-1)^{\otimes(n-1)}(Z_n^{\langle i\rangle})\\
&= -\sum_{n\ge1}\frac{1}{n!}\int dZ_n V_n\cdot\nabla_{X_n}f_n(t,Z_n)(\gamma-1)^{\otimes n}(Z_n)\,.
\end{aligned}$$

On the other hand

$$\frac{1}{2}\int \frac{\partial\tilde{\mathcal{J}}}{\partial\gamma}(t,\gamma)(z_1)\frac{\partial\tilde{\mathcal{J}}}{\partial\gamma}(t,\gamma)(z_2)\Big(\gamma(z_1')\gamma(z_2')-\gamma(z_1)\gamma(z_2)\Big)d\mu(z_1,\bar{z}_2,\omega)$$
$$=\frac{1}{2}\sum_{n_1\geq 1}\sum_{i_1=1}^{n_1}\sum_{n_2\geq 1}\sum_{i_2=1}^{n_2}\frac{1}{n_1!}\frac{1}{n_2!}\int dZ_{n_1}^{\langle i_1\rangle}d\bar{Z}_{n_2}^{\langle i_2\rangle}d\mu(z_{i_1},\bar{z}_{i_2},\omega)f_{n_1}(t,Z_{n_1})f_{n_2}(t,\bar{Z}_{n_2})$$
$$\times\big(\gamma(z_{i_1}')\gamma(\bar{z}_{i_2}')-\gamma(z_{i_1})\gamma(\bar{z}_{i_2})\big)(\gamma-1)^{\otimes(n_1-1)}(Z_{n_1}^{\langle i_1\rangle})(\gamma-1)^{\otimes(n_2-1)}(\bar{Z}_{n_2}^{\langle i_2\rangle})\,.$$

Now let us decompose

$$\gamma(z_{i_1}')\gamma(\bar{z}_{i_2}')-\gamma(z_{i_1})\gamma(\bar{z}_{i_2})=(\gamma(z_{i_1}')-1)(\gamma(\bar{z}_{i_2}')-1)-(\gamma(z_{i_1})-1)(\gamma(\bar{z}_{i_2})-1)$$
$$+(\gamma(z_{i_1}')-1)-(\gamma(z_{i_1})-1)+(\gamma(\bar{z}_{i_2}')-1)-(\gamma(\bar{z}_{i_2})-1)\,.$$

The first contribution can be understood as a recollision term, while the second corresponds to a collision term. Indeed there holds

$$\mathcal{R}:=\frac{1}{2}\sum_{n_1\geq 1}\sum_{i_1=1}^{n_1}\sum_{n_2\geq 1}\sum_{i_2=1}^{n_2}\frac{1}{n_1!}\frac{1}{n_2!}\int dZ_{n_1}^{\langle i_1\rangle}d\bar{Z}_{n_2}^{\langle i_2\rangle}d\mu(z_{i_1},\bar{z}_{i_2},\omega)f_{n_1}(t,Z_{n_1})f_{n_2}(t,\bar{Z}_{n_2})$$
$$\times(\gamma-1)^{\otimes(n_1-1)}(Z_{n_1}^{\langle i_1\rangle})(\gamma-1)^{\otimes(n_2-1)}(\bar{Z}_{n_2}^{\langle i_2\rangle})\big((\gamma(z_{i_1}')-1)(\gamma(\bar{z}_{i_2}')-1)-(\gamma(z_{i_1})-1)(\gamma(\bar{z}_{i_2})-1)\big)$$
$$=\frac{1}{2}\sum_{n\geq 2}\frac{1}{n!}\sum_{\substack{n_1+n_2=n\\ n_1,n_2\geq 1}}\binom{n}{n_1}\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\int dZ_{n_1}^{\langle 1\rangle}d\bar{Z}_{n_2}^{\langle 2\rangle}d\mu(z_1,\bar{z}_2,\omega)(\gamma-1)^{\otimes(n_1-1)}(Z_{n_1}^{\langle 1\rangle})(\gamma-1)^{\otimes(n_2-1)}(\bar{Z}_{n_2}^{\langle 2\rangle})$$
$$\times\big((\gamma(z_1')-1)(\gamma(\bar{z}_2')-1)-(\gamma(z_1)-1)(\gamma(\bar{z}_2)-1)\big)f_{n_1}(t,Z_{n_1})f_{n_2}(t,\bar{Z}_{n_2})\,.$$

The change of variables $(v_{i_1},\bar{v}_{i_2},\omega)\mapsto(v_{i_1}',\bar{v}_{i_2}',\omega)$ gives

$$\mathcal{R}=\frac{1}{2}\sum_{n\geq 2}\frac{1}{n!}\sum_{\substack{n_1+n_2=n\\ n_1,n_2\geq 1}}\binom{n}{n_1}\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\int dZ_{n_1}^{\langle 1\rangle}d\bar{Z}_{n_2}^{\langle 2\rangle}d\mu(z_1,\bar{z}_2,\omega)(\gamma-1)^{\otimes(n_1-1)}(Z_{n_1}^{\langle 1\rangle})(\gamma-1)^{\otimes(n_2-1)}(\bar{Z}_{n_2}^{\langle 2\rangle})$$
$$\times(\gamma(z_1)-1)(\gamma(\bar{z}_2)-1)\Big(f_{n_1}(t,Z_{n_1}^{'i_1,i_2})f_{n_2}(t,\bar{Z}_{n_2}^{'i_1,i_2})-f_{n_1}(t,Z_{n_1})f_{n_2}(t,\bar{Z}_{n_2})\Big)$$

where $Z_{n_1}^{'i_1,i_2},\bar{Z}_{n_2}^{'i_1,i_2}$ differ from $Z_{n_1},\bar{Z}_{n_2}$ only by $z_{i_1}',\bar{z}_{i_2}'$. Finally defining

$$R^{i,j}(f_{|\eta_i|},f_{|\eta_j|})(Z_n):=\int\Big(f_{|\eta_i|}(Z_{\eta_i}^{'i,j})f_{|\eta_j|}(Z_{\eta_j}^{'i,j})-f_{|\eta_i|}(Z_{\eta_i})f_{|\eta_j|}(Z_{\eta_j})\Big)d\mu_{z_i,z_j}(\omega)$$

with

$$d\mu_{z_i,z_j}(\omega):=\delta_{x_i-x_j}\big((v_i-v_j)\cdot\omega\big)_+d\omega \tag{5.5.5}$$

we can write

$$\mathcal{R}=\sum_{n=2}^{\infty}\frac{1}{n!}\sum_{i\neq j}\sum_{\eta\in\mathcal{S}_n^{i,j}}\int R^{i,j}(f_{|\eta_i|},f_{|\eta_j|})(\gamma-1)^{\otimes n}\,dZ_n\,,$$

denoting by $\mathcal{S}_n^{i,j}$ the set of all partitions of $\{1,\dots,n\}$ in two parts η_i and η_j separating i and j. Similarly

$$\mathcal{C}:=\sum_{n_1\geq 1}\sum_{i_1=1}^{n_1}\sum_{n_2\geq 1}\sum_{i_2=1}^{n_2}\frac{1}{n_1!}\frac{1}{n_2!}\int dZ_{n_1}^{\langle i_1\rangle}d\bar{Z}_{n_2}^{\langle i_2\rangle}d\mu(z_{i_1},\bar{z}_{i_2},\omega)f_{n_1}(t,Z_{n_1})f_{n_2}(t,\bar{Z}_{n_2})$$
$$\times(\gamma-1)^{\otimes(n_1-1)}(Z_{n_1}^{\langle i_1\rangle})(\gamma-1)^{\otimes(n_2-1)}(\bar{Z}_{n_2}^{\langle i_2\rangle})\big(\gamma(z_{i_1}')-1)-(\gamma(z_{i_1})-1)\big)$$
$$=\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i=1}^{n}\sum_{\eta\in\mathcal{S}_{n+1}^{i,n+1}}\int C^{i,n+1}(f_{|\eta_i|},f_{|\eta_{n+1}|})(\gamma-1)^{\otimes n}\,dZ_n\,,$$

with

$$C^{i,n+1}(f_{|\eta_i|}, f_{|\eta_{n+1}|})(Z_n) := \int \Big(f_{|\eta_i|}(Z'_{\eta_i}) f_{|\eta_{n+1}|}(Z'_{\eta_{n+1}}) - f_{|\eta_i|}(Z_{\eta_i}) f_{|\eta_{n+1}|}(Z_{\eta_{n+1}}) \Big) d\mu_{z_i}(z_{n+1}, \omega)$$

denoting by $\mathcal{S}_{n+1}^{i,n+1}$ the set of all partitions of $\{1, \dots, n+1\}$ in two parts separating i and $n+1$ and with

$$d\mu_{z_i}(z_{n+1}, \omega) := \delta_{x_i - x_{n+1}} \big((v_{n+1} - v_i) \cdot \omega \big)_+ d\omega dv_{n+1} \,. \tag{5.5.6}$$

Putting all those contributions together and identifying the factor of $(\gamma - 1)^{\otimes n}$ provides the following equation

$$\partial_t f_n + V_n \cdot \nabla_{X_n} f_n = \sum_{i=1}^{n} \sum_{\eta \in \mathcal{S}_{n+1}^{i,n+1}} C^{i,n+1}(f_{|\eta_i|}, f_{|\eta_{n+1}|}) + \sum_{i \neq j} \sum_{\eta \in \mathcal{S}_n^{i,j}} R^{i,j}(f_{|\eta_i|}, f_{|\eta_j|}) \,. \tag{5.5.7}$$

In particular the equation for f_1 is of course the Boltzmann equation

$$\partial_t f_1 + v_1 \cdot \nabla_x f_1 = C^{1,2}(f_1, f_1) \tag{5.5.8}$$

while the equation for second cumulant density is

$$\partial_t f_2 + V_2 \cdot \nabla_{X_2} f_2 = L_{f_1}(f_2) + R^{1,2}(f_1, f_1) \tag{5.5.9}$$

with

$$L_{f_1}(f_2)(Z_2) := \sum_{i=1}^{2} \int \Big(f_2(z_j, z'_i) f_1(z'_3) + f_1(z'_i) f_2(z_j, z'_3) - f_2(z_i, z_j) f_1(z_3) - f_1(z_i) f_2(z_3, z_j) \Big) d\mu_{z_i}(z_3, \omega) \,.$$

PART II

FLUCTUATIONS AROUND THE BOLTZMANN DYNAMICS

CHAPTER 6

FLUCTUATING BOLTZMANN EQUATION

The goal of this chapter is to study the limit of the fluctuation field $(\zeta_t^\varepsilon)_{t\leq T^\star}$ introduced in (1.3.1), i.e. defined for any smooth test function φ as

$$\zeta_t^\varepsilon\big(\varphi\big) := \frac{1}{\sqrt{\mu_\varepsilon}}\Big(\sum_{i=1}^{\mathcal{N}} \varphi\big(\mathbf{z}_i^\varepsilon(t)\big) - \mu_\varepsilon \int F_1^\varepsilon(t,z)\,\varphi\big(z\big)\,dz\Big)\,.$$

In this chapter, we prove Theorem 2, namely that, in the Boltzmann-Grad limit, ζ_t^ε converges to a stochastic process solving (in a sense we make precise below) on $[0,T^\star]$

$$d\hat{\zeta}_t = \mathcal{L}_t\,\hat{\zeta}_t\,dt + d\eta_t\,. \tag{6.0.1}$$

We recall that f is the solution of the Boltzmann equation on $[0,T^\star]$, that the linearized Boltzmann operator is defined as $\mathcal{L}_t := -v\cdot\nabla_x + \mathbf{L}_t$ with the collision part

$$\mathbf{L}_t\,\varphi(z_1) := \int d\mu_{z_1}(z_2,\omega)\Big(f(t,z_2')\varphi(z_1') + f(t,z_1')\varphi(z_2') - f(t,z_2)\varphi(z_1) - f(t,z_1)\varphi(z_2)\Big)\,, \tag{6.0.2}$$

and that $d\eta_t(x,v)$ is a Gaussian noise with zero mean and covariance

$$\mathbf{Cov}_t(\varphi,\psi) := \frac{1}{2}\int d\mu(z_1,z_2,\omega)\,f(t,z_1)\,f(t,z_2)\,\Delta\psi\Delta\varphi\,, \tag{6.0.3}$$

where the scattering measure is defined as in (1.3.5) and (5.5.6)

$$d\mu_{z_1}(z_2,\omega) = \delta_{x_1-x_2}\big((v_1-v_2)\cdot\omega\big)_+ d\omega dv_2, \quad d\mu(z_1,z_2,\omega) = \delta_{x_2-x_1}\left((v_1-v_2)\cdot\omega\right)_+ d\omega\,dx_1dv_1dv_2\,, \tag{6.0.4}$$

and we recall the notation

$$\Delta\psi(z_1,z_2,\omega) = \psi(z_1') + \psi(z_2') - \psi(z_1) - \psi(z_2)\,. \tag{6.0.5}$$

In order to obtain the convergence of the fluctuation field, we shall proceed in two steps, establishing first the convergence of the characteristic function in Section 6.2, and then some tightness in Section 6.3. Section 6.1 is devoted to explaining in what sense solutions to (6.0.1) are to be understood.

6.1. Weak solutions for the limit process

In this section we provide a notion of weak solution to (6.0.1).

Denote by $\mathcal{U}(t,s)$ the semigroup associated with $\mathcal{L}_\tau$ between times $s<t$, meaning that

$$\partial_t \mathcal{U}(t,s)\varphi - \mathcal{L}_t \mathcal{U}(t,s)\varphi = 0\,, \qquad \mathcal{U}(s,s)\varphi = \varphi\,,$$

and

$$\partial_s \mathcal{U}(t,s)\varphi + \mathcal{U}(t,s)\mathcal{L}_s\varphi = 0\,, \qquad \mathcal{U}(t,t)\varphi = \varphi\,.$$

By definition, $\psi_s := \mathcal{U}^*(t,s)\varphi$ satisfies the backward equation

$$\partial_s \psi_s + \mathcal{L}_s^* \psi_s = 0\,, \qquad \psi_t = \varphi\,, \tag{6.1.1}$$

where we recall that $\mathcal{L}_s^* = v\cdot\nabla_x + \mathbf{L}_s^*$ with

$$\mathbf{L}_s^*\,\psi(z_1) := \int d\mu_{z_1}(z_2,\omega) f(s,z_2)\,\Delta\psi(z_1,z_2,\omega). \tag{6.1.2}$$

Formally, a solution of the limit process (6.0.1) satisfies for any test function φ

$$\hat\zeta_t(\varphi) = \zeta_0(\mathcal{U}^*(t,0)\varphi) + \int_0^t d\eta_s(\mathcal{U}^*(t,s)\varphi)\,.$$

For any $t\geq s$ and test functions φ,ψ, the covariance is given by

$$\begin{aligned}
\mathbb{E}\big(\hat\zeta_t(\psi)\hat\zeta_s(\varphi)\big) &= \mathbb{E}\Big(\zeta_0\big(\mathcal{U}^*(t,0)\psi\big)\;\zeta_0\big(\mathcal{U}^*(s,0)\varphi\big)\Big) + \mathbb{E}\left(\int_0^t\int_0^s d\eta_u\, d\eta_{u'}\big(\mathcal{U}^*(t,u)\psi\big)\big(\mathcal{U}^*(s,u')\varphi\big)\right)\\
&\quad + \mathbb{E}\left(\zeta_0\big(\mathcal{U}^*(t,0)\psi\big)\;\int_0^s d\eta_{u'}\big(\mathcal{U}^*(s,u')\varphi\big)\right) + \mathbb{E}\left(\zeta_0\big(\mathcal{U}^*(s,0)\varphi\big)\;\int_0^t d\eta_u\big(\mathcal{U}^*(t,u)\psi\big)\right)\\
&= \mathbb{E}\Big(\zeta_0\big(\mathcal{U}^*(t,0)\psi\big)\;\zeta_0\big(\mathcal{U}^*(s,0)\varphi\big)\Big) + \int_0^s du\;\mathbf{Cov}_u\left(\mathcal{U}^*(t,u)\psi,\mathcal{U}^*(s,u)\varphi\right)\,.
\end{aligned} \tag{6.1.3}$$

Let us first describe briefly the equilibrium case (when $f^0=M$ is a Maxwellian). Denote by $\mathcal{U}_{\rm eq}(t,s)$ the semigroup associated with $\mathcal{L}_{\rm eq} := -v\cdot\nabla_x + \mathbf{L}_{\rm eq}$, where $\mathbf{L}_{\rm eq}$ is the (autonomous) linearized operator around M, between times $s<t$. For solutions of the generalized Ornstein-Uhlenbeck equation

$$d\hat\zeta_t = \mathcal{L}_{\rm eq}\,\hat\zeta_t\,dt + d\eta_t \tag{6.1.4}$$

the expression of the covariance (6.1.3) simplifies by stationarity of the process at equilibrium

$$\mathbb{E}\big(\hat\zeta_t(\psi)\hat\zeta_s(\varphi)\big) = \mathbb{E}\big(\hat\zeta_{t-s}(\psi)\hat\zeta_0(\varphi)\big) = \mathbb{E}\Big(\hat\zeta_0\big(\mathcal{U}_{\rm eq}^*(t-s,0)\psi\big)\;\hat\zeta_0\big(\varphi\big)\Big).$$

Notice that at equilibrium the second term in Formula (6.1.3) is well defined as

$$\int_0^t du\,\mathbf{Cov}\big(\mathcal{U}_{\rm eq}^*(t,u)\varphi,\mathcal{U}_{\rm eq}^*(t,u)\varphi\big) < +\infty$$

is satisfied for any $\varphi\in L^2(Mdvdx)$. Indeed using the symmetry of the equilibrium measure $M(z_1')M(z_2') = M(z_1)M(z_2)$ and denoting by $\mathcal{U}_{\rm eq}^*$ the adjoint of $\mathcal{U}_{\rm eq}$ in $L^2(\mathbb{D})$, one gets

$$\begin{aligned}
\int_0^t du\,\mathbf{Cov}\big(\mathcal{U}_{\rm eq}^*(t,u)\varphi,\mathcal{U}_{\rm eq}^*(t,u)\varphi\big) &= -2\int_0^t du\int \mathcal{U}_{\rm eq}^*(t,u)\varphi M\mathbf{L}_{\rm eq}^*\,\mathcal{U}_{\rm eq}^*(t,u)\varphi\\
&= -2\int_0^t du\int \mathcal{U}_{\rm eq}^*(t,u)\varphi M(-\partial_u - v\cdot\nabla_x)\,\mathcal{U}_{\rm eq}^*(t,u)\varphi\\
&= \int M|\varphi|^2 - \int M\,|\mathcal{U}_{\rm eq}^*(t,0)\varphi|^2\,.
\end{aligned}$$

Note that this means that the fluctuations exactly compensate the dissipation. Moreover since the operator $\mathcal{U}_{\rm eq}^*$ is a semigroup of self-adjoint contractions on $L^2(Mdvdx)$, the method of [**21**] implies that one can construct a martingale solution to (6.1.4).

The situation is very different in the non equilibrium case. First, the computation above does not provide any control on the variance, since a similar integration by parts gives rise to additional terms coming from the equation $(\partial_u + v\cdot\nabla_x)f_u = C^{1,2}(f_u,f_u)$. Moreover the linearized operator $\mathcal{L}_t$ is non autonomous, non self-adjoint, and the corresponding semigroup is not a contraction. It is therefore unclear how to define the noise $d\eta_t$, nor a solution to (6.0.1) as in [**21**]. We shall thus proceed differently, by defining a function space in which the semi-group $\mathcal{U}^*(t,s)$ associated with $\mathcal{L}^*_\tau$ is well-defined, and for which we can define the covariance (6.1.3) of the stochastic process. This will enable us to give a notion of weak solution to (6.0.1).

6.1.1. Functional setting. — For any $\beta>0$, we introduce the weighted L^2 space

$$L^2_\beta := \left\{\varphi=\varphi(x,v),\qquad \int_{\mathbb{D}} \exp\big(-\frac{\beta}{2}|v|^2\big)\,\varphi^2(x,v)dxdv<+\infty\right\} \tag{6.1.5}$$

and the associate norm

$$\|\varphi\|_\beta := \left(\int_{\mathbb{D}} \exp\big(-\frac{\beta}{2}|v|^2\big)\,\varphi^2(x,v)dxdv\right)^{\frac12}.$$

We are going to establish estimates on the semigroup $\mathcal{U}^*(t,s)$ in order to define the covariance of the limiting stochastic process, which will be denoted, for $s\le t$, by

$$\begin{aligned}\forall\varphi,\psi\in L^2_{\beta_0},\quad \hat{\mathcal{C}}(s,t,\varphi,\psi) := &\int dz f^0(z)\,\big(\mathcal{U}^*(t,0)\psi\big)(z)\,\big(\mathcal{U}^*(s,0)\varphi\big)(z)\\ &+\int_0^s du\,\mathbf{Cov}_u\left(\big(\mathcal{U}^*(t,u)\psi\big),\big(\mathcal{U}^*(s,u)\varphi\big)\right).\end{aligned} \tag{6.1.6}$$

As a corollary of Lanford's proof, the solution f to the Boltzmann equation has been built on the time interval $[0,T^\star]$ by a Cauchy-Kowalewski fixed point argument [**45**] : it belongs to the functional space

$$\left\{g=g(t,x,v)\,,\quad \sup_{t,x,v}\exp\Big((\alpha_0+\frac{\beta_0}{2}|v|^2)(1-\frac{t}{2T^\star})\Big)g(t,x,v)<+\infty\right\} \tag{6.1.7}$$

where the variations in the coefficients α and β have been introduced to compensate the loss in the continuity estimates for the collision operator.

Proposition 6.1.1. — *There is a time $T\in(0,T^\star]$ such that for any φ in $L^2_{\beta_0/2}$, $\psi_s := \mathcal{U}^*(t,s)\varphi$ belongs to $L^2_{3\beta_0/4}$ for any $s\le t\le T$.*

Proof of Proposition 6.1.1. — Denoting by S_t the transport operator in $\mathbb{D}$, we get from (6.1.1)-(6.1.2) that

$$\psi_s = S_{s-t}\varphi + \int_s^t S_{s-\sigma}\mathbf{L}^*_\sigma\psi_\sigma\,d\sigma\,. \tag{6.1.8}$$

Using the uniform bound (6.1.7), it is easy to see that for any function ϕ and any $\frac{\beta_0}{2}<\beta'<\beta$, there is a constant C such that

$$\|\mathbf{L}^*_s\,\phi\|_\beta \le \frac{C}{\beta-\beta'}\|\phi\|_{\beta'}\,, \tag{6.1.9}$$

the loss coming from the collision cross-section in (6.1.2). On the other hand, the transport S_s preserves the spaces L^2_β. For $T\le T^\star$, we introduce the functional space

$$\mathbf{X} = \left\{\psi(s,x,v):[0,T]\times\mathbb{D}\mapsto\mathbb{R}\,,\quad \forall s\in[0,T],\quad \psi_s\in L^2_{\beta_0(3-\frac{s}{T})/4}\right\}. \tag{6.1.10}$$

We then obtain that $\psi = \mathcal{U}^*(t,s)\varphi \in \mathbf{X}$, for T small enough, using the following reformulation of the Cauchy-Kowalewski theorem.

***Theorem 8* ([26, 31]).** — *Let $(X_\rho)_{\rho>0}$ be a decreasing sequence of Banach spaces. Consider the equation*

$$u(t) = u_0(t) + \int_0^t F\big(s, u(s)\big) ds \tag{6.1.11}$$

where

— *$F(\cdot, 0) = 0$, and F is continuous from $[0,T] \times B_R(X_{\rho'})$ to X_ρ for all $\rho' > \rho$. Moreover there is a constant C such that for all $u, v \in B_R(X_{\rho'})$,*

$$\|F(t,u) - F(t,v)\|_\rho \leq \frac{C\|u-v\|_{\rho'}}{\rho' - \rho}, \qquad \rho < \rho' < \rho_0. \tag{6.1.12}$$

— *u_0 is continuous from $[0,T]$ to X_ρ and there are constants R_0, ρ_0 and η such that $\rho_0 - t/\eta > 0$ and*

$$\forall t \in [0,T]\,, \quad \|u_0(t)\|_\rho \leq R_0 \quad \text{for } \rho < \rho_0 \quad \text{and} \quad t < \eta(\rho_0 - \rho).$$

Then there exists a constant $\eta' \leq \eta$ such that (6.1.11) *has a unique solution on the time interval $[0, \eta'\rho_0[$ satisfying*

$$\sup_{\substack{\rho<\rho_0 \\ 0\leq t<\eta'(\rho_0-\rho)}} \|u(t)\|_\rho \Big(1 - \frac{t}{\eta'(\rho_0 - \rho)}\Big) < +\infty\,.$$

We stress the fact that (6.1.8) defines a backward evolution, instead Theorem 8 is stated for a forward evolution as it will be more convenient for later use. Notice that the spaces L^2_β in (6.1.5) are increasing, this explains the different order of the parameters $\beta' < \beta$ in (6.1.9) and $\rho' > \rho$ in (6.1.12).

Note that this procedure provides a solution on $[0,T]$ for some $T \leq T^\star$ (a careful look at the constant in the loss estimate (6.1.9) would show that in fact $T = T^\star$ but we shall not pursue this matter here). □

By (6.0.3), (6.1.7) and Proposition 6.1.1, for any φ and $\psi \in L^2_{\beta_0/2}$ there holds

$$\forall s \leq t \leq T\,, \qquad \int_0^s du\, \mathbf{Cov}_u\Big(\big(\mathcal{U}^*(t,u)\psi\big), \big(\mathcal{U}^*(s,u)\varphi\big)\Big) < +\infty\,. \tag{6.1.13}$$

6.1.2. Covariance of the limit stochastic process. — In the following, a weak solution of the fluctuating Boltzmann equation (6.0.1) is defined as a probability measure on the space $D\big([0,T^\star], \mathcal{D}'(\mathbb{D})\big)$ whose marginals have Gaussian law with covariance $\hat{\mathcal{C}}$ satisfying, for any φ, ψ in $\mathcal{D}(\mathbb{D})$,

$$s \leq t \leq T^\star, \qquad \begin{cases} \partial_t \hat{\mathcal{C}}(s,t,\varphi,\psi) = \hat{\mathcal{C}}(s,t,\varphi,\mathcal{L}_t^*\psi), \\ \partial_t \hat{\mathcal{C}}(t,t,\varphi,\psi) = \hat{\mathcal{C}}(t,t,\varphi,\mathcal{L}_t^*\psi) + \hat{\mathcal{C}}(t,t,\mathcal{L}_t^*\varphi,\psi) + \mathbf{Cov}_t(\psi,\varphi)\,, \\ \hat{\mathcal{C}}(0,0,\varphi,\psi) = \displaystyle\int dz \varphi(z)\psi(z) f^0(z)\,. \end{cases} \tag{6.1.14}$$

The following result characterises fully the covariance on a restricted time interval.

Lemma 6.1.2. — *The covariance $\hat{\mathcal{C}}$ defined in* (6.1.6) *is the unique solution in the sense of distributions of the dynamical equations* (6.1.14) *on $[0,T]$ with T as in Proposition 6.1.1.*

Proof. — The fact that (6.1.6) is the Duhamel formulation of (6.1.14) is an easy computation: for $t > s$, the time derivative gives

$$\partial_t \hat{\mathcal{C}}(s,t,\varphi,\psi) = \mathbb{E}\left(\zeta_0\big(\mathcal{U}^*(t,0)\mathcal{L}_t^*\psi\big)\ \zeta_0\big(\mathcal{U}^*(s,0)\varphi\big)\right) + \int_0^s du\ \mathbf{Cov}_u\left(\big(\mathcal{U}^*(t,u)\mathcal{L}_t^*\psi\big), \big(\mathcal{U}^*(s,u)\varphi\big)\right)$$

$$= \hat{\mathcal{C}}(s,t,\varphi,\mathcal{L}_t^*\psi)\,. \tag{6.1.15}$$

For $s = t$, the time derivative is

$$\partial_t \hat{\mathcal{C}}(t,t,\varphi,\psi) = \hat{\mathcal{C}}(t,t,\varphi,\mathcal{L}_t^*\psi) + \hat{\mathcal{C}}(t,t,\mathcal{L}_t^*\varphi,\psi) + \mathbf{Cov}_t(\psi,\varphi)\,. \tag{6.1.16}$$

At time $t = s = 0$, the covariance (6.1.6) is given by $\hat{\mathcal{C}}(0,0,\varphi,\psi) = \int dz\varphi(z)\psi(z)f^0(z)$, so that (6.1.14) holds.

To prove the uniqueness for (6.1.14), we consider $\delta\mathcal{C} = \hat{\mathcal{C}}_1 - \hat{\mathcal{C}}_2$ the difference between two solutions with initial data under the general form

$$\delta\mathcal{C}(0,0,\varphi,\psi) = \int dz\varphi(z)\psi(z)\,\delta f^0(z)\,.$$

Then $\delta\mathcal{C}$ satisfies the linear evolution

$$\partial_t \delta\mathcal{C}(s,t,\varphi,\psi) = \delta\mathcal{C}(s,t,\varphi,\mathcal{L}_t^*\psi)\,,$$
$$\partial_t \delta\mathcal{C}(t,t,\varphi,\psi) = \delta\mathcal{C}(t,t,\varphi,\mathcal{L}_t^*\psi) + \delta\mathcal{C}(t,t,\mathcal{L}_t^*\varphi,\psi)\,.$$

Uniqueness follows by writing the equation in integral form as in (6.1.6) and choosing $\delta f^0 \equiv 0$. □

In the next section, we are going to identify the limiting measure of the fluctuation field as a Gaussian measure with covariance given by (6.1.6): this will be a consequence of the fact that the covariance will be shown to satisfy (6.1.14) and of the uniqueness property provided in Lemma 6.1.2.

6.2. Convergence of the characteristic function

We are going to prove the convergence of time marginals of the process ζ^ε. Let $\theta_1, \dots, \theta_\ell$ be a collection of times in $[0, T^\star]$. Given a collection of smooth bounded test functions $\{\varphi_j\}_{j\le\ell}$, we consider the discrete sampling $H\big(z([0,T^\star])\big) = \sum_{j=1}^{\ell} \varphi_j\big(z(\theta_j)\big)$. Let us define

$$\big\langle\!\big\langle \zeta^\varepsilon, H \big\rangle\!\big\rangle := \frac{1}{\sqrt{\mu_\varepsilon}} \sum_{j=1}^{\ell} \left[\sum_{i=1}^{\mathcal{N}} \varphi_j\big(\mathbf{z}_i^\varepsilon(\theta_j)\big) - \mu_\varepsilon \int F_1^\varepsilon(\theta_j, z)\,\varphi_j\big(z\big)\,dz\right]. \tag{6.2.1}$$

Proposition 6.2.1. — *The characteristic function* $\mathbb{E}_\varepsilon\left(\exp\left(\mathbf{i}\big\langle\!\big\langle \zeta^\varepsilon, H\big\rangle\!\big\rangle\right)\right)$ *converges to the characteristic function of a Gaussian process, the covariance of which solves the dynamical equations* (6.1.14).

By Lemma 6.1.2 the covariance is given by (6.1.6) on $[0,T]$. The covariance of the density field out of equilibrium was first computed in [**42**]. A discussion on the result of [**42**] is postponed to Section 6.5 at the end of this chapter.

Proof. — **Step 1.** *Convergence to a Gaussian process.*

The characteristic function can be rewritten in terms of the empirical measure

$$\mathbb{E}_\varepsilon\left(\exp\left(\mathbf{i}\langle\!\langle \zeta^\varepsilon, H\rangle\!\rangle\right)\right) = \mathbb{E}_\varepsilon\Big(\exp\left(\mathbf{i}\sqrt{\mu_\varepsilon}\langle\!\langle \pi^\varepsilon, H\rangle\!\rangle\right)\Big)\exp\left(-\mathbf{i}\sqrt{\mu_\varepsilon}\sum_{j=1}^{\ell}\int F_1^\varepsilon(\theta_j,z)\,\varphi_j(z)\,dz\right). \tag{6.2.2}$$

Thanks to Proposition 2.1.3, we get

$$\log\mathbb{E}_\varepsilon\left(\exp\left(\mathbf{i}\langle\!\langle \zeta^\varepsilon, H\rangle\!\rangle\right)\right) = \mu_\varepsilon\sum_{n=1}^{\infty}\frac{1}{n!}f^\varepsilon_{n,[0,t]}\left(\left(e^{\frac{\mathbf{i}H}{\sqrt{\mu_\varepsilon}}}-1\right)^{\otimes n}\right) - \mathbf{i}\sqrt{\mu_\varepsilon}\sum_{j=1}^{\ell}\int F_1^\varepsilon(\theta_j,z)\varphi_j(z)\,dz\,.$$

As H is bounded, the series converges uniformly for any μ_ε large enough. At leading order, only the terms $n=1$ and $n=2$ will be relevant in the limit since by Theorem 9

$$\left|f^\varepsilon_{n,[0,t]}\left(\left(e^{\frac{\mathbf{i}H}{\sqrt{\mu_\varepsilon}}}-1\right)^{\otimes n}\right)\right| \le \left(\frac{C\|H\|_\infty}{\sqrt{\mu_\varepsilon}}\right)^n n!\,.$$

Expanding the exponential with respect to μ_ε, we notice that the term of order $\sqrt{\mu_\varepsilon}$ cancels so

$$\log\mathbb{E}_\varepsilon\left(\exp\left(\mathbf{i}\langle\!\langle \zeta^\varepsilon, H\rangle\!\rangle\right)\right) = -\frac{1}{2}f^\varepsilon_{1,[0,t]}\left(H^2\right) - \frac{1}{2}f^\varepsilon_{2,[0,t]}\left(H^{\otimes 2}\right) + O\left(\frac{\|H\|^3_\infty}{\sqrt{\mu_\varepsilon}}\right).$$

As the cumulants $f^\varepsilon_{1,[0,t]}\left(H^2\right), f^\varepsilon_{2,[0,t]}\left(H^{\otimes 2}\right)$ converge (see Theorem 5), the characteristic function has a limit

$$\lim_{\mu_\varepsilon\to\infty}\mathbb{E}_\varepsilon\left(\exp\left(\mathbf{i}\langle\!\langle \zeta^\varepsilon, H\rangle\!\rangle\right)\right) = \exp\left(-\frac{1}{2}\sum_{i,j\le\ell}\mathcal{C}(\theta_i,\theta_j,\varphi_i,\varphi_j)\right),$$

where the limiting covariance reads for any test functions φ and ψ

$$\mathcal{C}(s,t,\varphi,\psi) = f_{1,[0,t]}\big(\psi(z(s))\varphi(z(t))\big) + f_{2,[0,t]}\big(\psi(z(s)),\varphi(z(t))\big) \tag{6.2.3}$$

denoting abusively by $f_{2,[0,t]}\left(\psi,\varphi\right)$ the bilinear symmetric form obtained by polarization

$$f_{2,[0,t]}\left(\psi,\varphi\right) := \frac{1}{2}\Big(f_{2,[0,t]}\left((\psi+\varphi)^{\otimes 2}\right) - f_{2,[0,t]}\left(\psi^{\otimes 2}\right) - f_{2,[0,t]}\left(\varphi^{\otimes 2}\right)\Big).$$

Step 2. *Identification of the limiting covariance* (6.2.3). Let us prove that the limiting covariance solves (6.1.14) for all test functions φ and ψ. At time $t=s=0$, the covariance is

$$\mathcal{C}(0,0,\varphi,\psi) = \int dz\varphi(z)\psi(z)f^0(z)dz\,.$$

Let us recall the dynamical equations (5.5.8), (5.5.9) satisfied by the first two limiting cumulants

$$\partial_t f_1 + v_1\cdot\nabla_{x_1}f_1 = C^{1,2}(f_1,f_1), \qquad \partial_t f_2 + V_2\cdot\nabla_{X_2}f_2 = L_{f_1}f_2 + R^{1,2}(f_1,f_1).$$

For $t=s$, taking the time derivative in (6.2.3) gives

$$\partial_t\mathcal{C}(t,t,\varphi,\psi) = \Big\langle\partial_t f_1(t),\psi\varphi\big(z_1\big)\Big\rangle + \Big\langle\partial_t f_2(t),\psi\big(z_1\big)\varphi\big(z_2\big)\Big\rangle = \mathbb{A}_1+\mathbb{A}_2\,,$$

with the decomposition

$$\begin{aligned}
\mathbb{A}_1 &:= \Big\langle f_1(t), v_1\cdot\nabla_{x_1}\psi\varphi(z_1)\Big\rangle + \Big\langle f_2(t), v_1\cdot\nabla_{x_1}\psi(z_1)\varphi(z_2)\Big\rangle + \Big\langle f_2(t), \psi(z_1)\ v_2\cdot\nabla_{x_2}\varphi(z_2)\Big\rangle \\
&\quad + \int d\mu(z_1,z_3,\omega)dz_2\ f_1(t,z_3)f_2(t,z_1,z_2)\Big(\psi(z_1')+\psi(z_3')-\psi(z_1)-\psi(z_3)\Big)\varphi(z_2) \\
&\quad + \int dz_1 d\mu(z_2,z_3,\omega)\ f_1(t,z_3)f_2(t,z_1,z_2)\psi(z_1)\Big(\varphi(z_2')+\varphi(z_3')-\varphi(z_2)-\varphi(z_3)\Big) \\
&\quad + \Big\langle f_1(t), \psi\ \mathbf{L}_t^*\varphi(z_1)\Big\rangle + \Big\langle f_1(t), \varphi\ \mathbf{L}_t^*\psi(z_1)\Big\rangle \\
&= \Big\langle f_1(t), v_1\cdot\nabla_{x_1}\psi\varphi(z_1)\Big\rangle + \Big\langle f_2(t), v_1\cdot\nabla_{x_1}\psi(z_1)\varphi(z_2)\Big\rangle + \Big\langle f_2(t), \psi(z_1)\ v_2\cdot\nabla_{x_2}\varphi(z_2)\Big\rangle \\
&\quad + \Big\langle f_1(t), \psi\ \mathbf{L}_t^*\varphi + \varphi\ \mathbf{L}_t^*\psi\Big\rangle + \Big\langle f_2(t), \psi\ \mathbf{L}_t^*\varphi + \varphi\mathbf{L}_t^*\psi\Big\rangle, \\
\mathbb{A}_2 &:= \int d\mu(z_1,z_2,\omega)\ f_1(t,z_1)f_1(t,z_2)\Big[\Big(\psi\varphi(z_1')-\psi\varphi(z_1)\Big)+\Big(\psi(z_1')\varphi(z_2')-\psi(z_1)\varphi(z_2)\Big)\Big] \\
&\quad - \Big\langle f_1(t), \psi\ \mathbf{L}_t^*\varphi(z_1)\Big\rangle - \Big\langle f_1(t), \varphi\ \mathbf{L}_t^*\psi(z_1)\Big\rangle .
\end{aligned}$$

To show that $\mathcal{C}(t,t,\varphi,\psi)$ satisfies (6.1.16), it remains now to identify $\mathbb{A}_1$ with the linearized part and $\mathbb{A}_2$ with the covariance term. Note that the derivatives of f_1 and f_2 are both contributing to $\mathbb{A}_1$ and $\mathbb{A}_2$. Furthermore the last term involving $\mathbf{L}_t^*$ in $\mathbb{A}_2$ has been added to $\mathbb{A}_1$ and removed from $\mathbb{A}_2$ in order to identify the covariance. From (6.1.2), one gets that

$$\mathbb{A}_1 = \mathcal{C}(t,t,\varphi,\mathcal{L}_t^*\psi) + \mathcal{C}(t,t,\mathcal{L}_t^*\varphi,\psi)\,.$$

Using again (6.1.2), we deduce that

$$\begin{aligned}
\mathbb{A}_2 &= \int d\mu(z_1,z_2,\omega)\ f_1(t,z_1)f_1(t,z_2)\Big[\psi(z_1')\big(\varphi(z_1')+\varphi(z_2')\big)-\big(\psi\varphi(z_1)+\psi(z_1)\varphi(z_2)\big) \\
&\qquad - \varphi(z_1)\Big(\psi(z_1')+\psi(z_2')-\psi(z_1)-\psi(z_2)\Big)-\psi(z_1)\Big(\varphi(z_1')+\varphi(z_2')-\varphi(z_1)-\varphi(z_2)\Big)\Big] \\
&= \frac{1}{2}\int d\mu(z_1,z_2,\omega)\ f_1(t,z_1)f_1(t,z_2) \\
&\qquad \Big(\psi(z_1')+\psi(z_2')-\psi(z_1)-\psi(z_2)\Big)\Big(\varphi(z_1')+\varphi(z_2')-\varphi(z_1)-\varphi(z_2)\Big) = \mathbf{Cov}_t(\psi,\varphi)\,.
\end{aligned}$$

We have therefore recovered that $\mathcal{C}(t,t,\varphi,\psi)$ satisfies (6.1.14).

In the same way, one can check that for $t>s$, Equation (6.1.14) holds for $\mathcal{C}(s,t,\varphi,\psi)$. □

6.3. Tightness and proof of Theorem 2

In this section we prove a tightness property for the law of the process ζ_t^ε. It turns out that this is made possible by considering test functions in a space with more regularity than $L^2_{\beta_0}$. In order to construct a convenient function space let us consider a Fourier-Hermite basis of $\mathbb{D}$: let $\{\tilde e_{j_1}(x)\}_{j_1\in\mathbb{Z}^d}$ be the Fourier basis of $\mathbb{T}^d$ and $\{e_{j_2}(v)\}_{j_2\in\mathbb{N}^d}$ be the Hermite basis of $L^2(\mathbb{R}^d)$ constituted of the eigenmodes of the harmonic oscillator $-\Delta_v+|v|^2$. This provides a basis $\big\{h_j(z)=\tilde e_{j_1}(x)e_{j_2}(v)\big\}_{j=(j_1,j_2)}$ of Lipschitz functions on $\mathbb{D}$, exponentially decaying in v, such that for all $j=(j_1,j_2)$

$$\|h_j\|_\infty \le c\,,\qquad \|\nabla h_j\|_\infty = \|\nabla_v h_j\|_\infty + \|\nabla_x h_j\|_\infty < c(1+|j|),\qquad \|v\cdot\nabla_x h_j\|_\infty < c(1+|j|)^{\frac{3}{2}}\,, \tag{6.3.1}$$

with $|j| := |j_1| + |j_2|$ and for some constant c (see [**20**]). Then we define for any real number $k \in \mathbb{R}$ the Sobolev-type space $\mathcal{H}_k(\mathbb{D})$ by the norm

$$\|\varphi\|_k^2 := \sum_{j=(j_1,j_2)} (1+|j|^2)^k \left(\int_{\mathbb{D}} dz\, \varphi(z) h_j(z) \right)^2 . \tag{6.3.2}$$

Following [**4**] (Theorem 13.2 page 139), the tightness of the law of the process in $D\big([0,T^\star], \mathcal{H}_{-k}(\mathbb{D})\big)$ (for some large positive k) is a consequence of the following proposition.

Proposition 6.3.1. — *There is $k > 0$ large enough such that*

$$\forall \delta' > 0, \qquad \lim_{\delta\to 0} \lim_{\mu_\varepsilon \to \infty} \mathbb{P}_\varepsilon \Big(\sup_{\substack{|s-t|\le\delta \\ s,t\in[0,T^\star]}} \big\| \zeta_t^\varepsilon - \zeta_s^\varepsilon \big\|_{-k} \ge \delta' \Big) = 0, \tag{6.3.3}$$

$$\lim_{A\to\infty} \lim_{\mu_\varepsilon \to \infty} \mathbb{P}_\varepsilon \Big(\sup_{t\in[0,T^\star]} \big\| \zeta_t^\varepsilon \big\|_{-k} \ge A \Big) = 0 \,. \tag{6.3.4}$$

The tightness property above combined with the identification of the time marginals in Proposition 6.2.1 implies the convergence, on $[0,T^\star]$, to a weak solution of the fluctuating Boltzmann equation (6.0.1) (in the sense given in Section 6.1.2). This completes the proof of Theorem 2.

The proof of Proposition 6.3.1 relies on the following modified version of the Garsia, Rodemich, Rumsey inequality [**46**] which will be used to control the modulus of continuity (its derivation is postponed to Section 6.4).

Proposition 6.3.2. — *Choose two functions $\Psi(u) = u^4$ and $p(u) = u^{\gamma/4}$ with γ belonging to $]2,3[$. Let $\varphi : [0,T^\star] \to \mathbb{R}$ be a given function and define for $\alpha > 0$*

$$B_\alpha := \int_0^{T^\star} \int_0^{T^\star} dsdt\; \Psi\left(\frac{|\varphi_t - \varphi_s|}{p(|t-s|)} \right) \mathbf{1}_{|t-s|>\alpha} \,. \tag{6.3.5}$$

The modulus of continuity of φ is controlled by

$$\sup_{\substack{0\le s,t\le T^\star \\ |t-s|\le\delta}} \big|\varphi_t - \varphi_s\big| \le 2 \sup_{\substack{0\le s,t\le T^\star \\ |t-s|\le 2\alpha}} \big|\varphi_t - \varphi_s\big| \;+\; 8\sqrt{2} B_\alpha^{1/4} \delta^{\frac{\gamma}{4}-\frac{1}{2}} \,. \tag{6.3.6}$$

In the standard Garsia, Rodemich, Rumsey inequality, (6.3.5) is assumed to hold with $\alpha = 0$ leading to a stronger conclusion as φ is then proved to be Hölder continuous. The cut-off $\alpha > 0$ allows us to consider functions φ which may be discontinuous.

Proof of Proposition 6.3.1. — At time 0, all the moments of ζ_0^ε are bounded, so (6.3.4) can be deduced from the control of the initial fluctuations and the bound (6.3.3) on the modulus of continuity. Thus it is enough to prove (6.3.3), i.e. to show that

$$\forall \delta' > 0\,, \qquad \lim_{\delta\to 0} \lim_{\mu_\varepsilon \to \infty} \mathbb{P}_\varepsilon \left(\sup_{\substack{|s-t|\le\delta \\ s,t\in[0,T^\star]}} \sum_j \frac{1}{(1+|j|^2)^k} \big|\zeta_t^\varepsilon(h_j) - \zeta_s^\varepsilon(h_j)\big|^2 \ge \delta' \right) = 0 \,, \tag{6.3.7}$$

where $\{h_j(z)\}_{j=(j_1,j_2)}$ is the family of test functions introduced above.

We are going to apply Proposition 6.3.2 to the functions $t \mapsto \zeta_t^\varepsilon(h_j)$. In order to do so, the short time fluctuations have first to be controlled. This will be achieved thanks to the following lemma.

Lemma 6.3.3. — *The time scale cut-off will be denoted by $\alpha_\varepsilon = \mu_\varepsilon^{-7/3}$. For the basis of functions introduced in* (6.3.1)*, there is $k > 0$ large enough so that*

$$\forall \delta' > 0\,, \qquad \lim_{\mu_\varepsilon \to \infty} \mathbb{P}_\varepsilon \left(\sum_j \frac{1}{(1+|j|^2)^k} \sup_{\substack{|s-t|\le 2\alpha_\varepsilon \\ s,t\in[0,T^\star]}} \left|\zeta_t^\varepsilon(h_j) - \zeta_s^\varepsilon(h_j)\right|^2 \ge \delta' \right) = 0\,. \tag{6.3.8}$$

Then, to control the fluctuations on time scales of order δ, it will be enough to rely on averaged estimates of the following type.

Lemma 6.3.4. — *For any function h, there exists a constant C depending on $\|h\|_\infty$ such that for any $\varepsilon > 0$ and $s, t \in [0, T^\star]$*

$$\mathbb{E}_\varepsilon \left(\left(\zeta_t^\varepsilon(h) - \zeta_s^\varepsilon(h)\right)^4 \right) \le C(\|\nabla h\|_{L^\infty} + 1) \left(|t-s|^2 + \frac{1}{\mu_\varepsilon}|t-s| \right). \tag{6.3.9}$$

We postpone the proofs of the two previous statements and conclude first the proof of (6.3.7).

Notice that Lemma 6.3.4 implies that the random variable associated with any function h_j satisfying (6.3.1)

$$B_{\alpha_\varepsilon}(h_j) := \int_0^{T^\star} \int_0^{T^\star} ds\, dt \frac{\left|\zeta_t^\varepsilon(h_j) - \zeta_s^\varepsilon(h_j)\right|^4}{|t-s|^\gamma} \mathbf{1}_{|t-s|>\alpha_\varepsilon} \tag{6.3.10}$$

has finite expectation

$$\mathbb{E}_\varepsilon\big(B_{\alpha_\varepsilon}(h_j)\big) \le C(1+|j|) \int_0^{T^\star} \int_0^{T^\star} dsdt \left(|t-s|^{2-\gamma} + \frac{1}{\mu_\varepsilon}|t-s|^{1-\gamma} \mathbf{1}_{|t-s|>\alpha_\varepsilon} \right). \tag{6.3.11}$$

Setting now $\gamma = 7/3$, we get an upper bound uniform with respect to ε for $\alpha_\varepsilon = \mu_\varepsilon^{-7/3}$

$$\mathbb{E}_\varepsilon\big(B_{\alpha_\varepsilon}(h_j)\big) \le C(1+|j|)^{\frac{3}{2}} \left(1 + \frac{\alpha_\varepsilon^{2-\gamma}}{\mu_\varepsilon} \right) \le C'(1+|j|). \tag{6.3.12}$$

From Proposition 6.3.2, a large modulus of continuity of $t \mapsto \zeta_t^\varepsilon(h_j)$ induces a deviation of the random variable $B_{\alpha_\varepsilon}(h_j)$. This implies that on average

$$\begin{aligned} &\mathbb{P}_\varepsilon\Big(\sup_{\substack{|s-t|\le\delta \\ s,t\in[0,T^\star]}} \sum_j \frac{1}{(1+|j|^2)^k} \left|\zeta_t^\varepsilon(h_j) - \zeta_s^\varepsilon(h_j)\right|^2 \ge \delta' \Big) \\ &\le \mathbb{P}_\varepsilon\Big(\sum_j \frac{1}{(1+|j|^2)^k} \sup_{\substack{|s-t|\le 2\alpha_\varepsilon \\ s,t\in[0,T^\star]}} \left|\zeta_t^\varepsilon(h_j) - \zeta_s^\varepsilon(h_j)\right|^2 \ge \frac{\delta'}{16} \Big) + \mathbb{P}_\varepsilon\Big(\sum_j \frac{\sqrt{B_{\alpha_\varepsilon}(h_j)}}{(1+|j|^2)^k} \ge \frac{\delta'}{2^9\, \delta^{\frac{\gamma}{2}-1}} \Big)\,. \end{aligned} \tag{6.3.13}$$

The first term in (6.3.13) tends to 0 by Lemma 6.3.3 and the second one can be estimated by the Markov inequality and by the upper bound (6.3.12)

$$\mathbb{P}_\varepsilon\Big(\sum_j \frac{\sqrt{B_{\alpha_\varepsilon}(h_j)}}{(1+|j|^2)^k} \ge \frac{\delta'}{2^8\, \delta^{\frac{\gamma}{2}-1}} \Big) \le C_1 \frac{\delta^{\gamma-2}}{\delta'^2} \sum_j \frac{1}{(1+|j|^2)^k} \mathbb{E}_\varepsilon\big(B_{\alpha_\varepsilon}(h_j)\big) \le \frac{C_2}{\delta'^2}\, \delta^{\gamma-2}\,,$$

for some constants C_1, C_2 and k large enough. As $\gamma = 7/3$, the limit (6.3.7) holds. Proposition 6.3.1 is proved. □

6.3.1. Averaged time continuity. — We prove now Lemma 6.3.4. Denoting

$$H(z([0,t])) := h(z(t)) - h(z(s))\,,$$

the moments can be recovered by taking derivatives of the exponential moments

$$\mathbb{E}_\varepsilon\left(\left(\zeta_t^\varepsilon(h)-\zeta_s^\varepsilon(h)\right)^4\right) = \left(\frac{\partial^4}{\partial\lambda^4}\mathbb{E}_\varepsilon\left(\exp\left(\mathbf{i}\lambda\langle\!\langle\zeta^\varepsilon,H\rangle\!\rangle\right)\right)\right)_{|\lambda=0}. \tag{6.3.14}$$

We recall from Proposition 2.1.3 that

$$\log\mathbb{E}_\varepsilon\left(\exp\left(\mathbf{i}\lambda\langle\!\langle\zeta^\varepsilon,H\rangle\!\rangle\right)\right) = \mu_\varepsilon\sum_{n=1}^\infty\frac{1}{n!}f^\varepsilon_{n,[0,t]}\left(\left(e^{\frac{\mathbf{i}\lambda H}{\sqrt{\mu_\varepsilon}}}-1\right)^{\otimes n}\right) - \sqrt{\mu_\varepsilon}\,\mathbf{i}\,\lambda F_1^\varepsilon(H) = O(\lambda^2).$$

Thus expanding the exponential moment at the 4th order leads to

$$\begin{aligned}\mathbb{E}_\varepsilon\left(\exp\left(\mathbf{i}\lambda\langle\!\langle\zeta^\varepsilon,H\rangle\!\rangle\right)\right) =& 1+\mu_\varepsilon\sum_{n=1}^\infty\frac{1}{n!}f^\varepsilon_{n,[0,t]}\left(\left(e^{\frac{\mathbf{i}\lambda H}{\sqrt{\mu_\varepsilon}}}-1\right)^{\otimes n}\right) - \sqrt{\mu_\varepsilon}\mathbf{i}\lambda F_1^\varepsilon(H)\\ &+\frac{\lambda^4}{2}\left(\frac{1}{2}f^\varepsilon_{1,[0,t]}\left(H^2\right)+\frac{1}{2}f^\varepsilon_{2,[0,t]}\left((H)^{\otimes 2}\right)\right)^2+o(\lambda^4)\,.\end{aligned}$$

The fourth moment can be recovered by taking the 4th derivative with respect to λ

$$\begin{aligned}\mathbb{E}\left(\left(\zeta_t^\varepsilon(h)-\zeta_s^\varepsilon(h)\right)^4\right) =& 3\left(f^\varepsilon_{1,[0,t]}\left(H^2\right)+f^\varepsilon_{2,[0,t]}\left(H^{\otimes 2}\right)\right)^2\\ &+\frac{1}{\mu_\varepsilon}\sum_{n=1}^4\sum_{\kappa_1+\cdots+\kappa_n=4}C_\kappa\; f^\varepsilon_{n,[0,t]}(H^{\kappa_1},\ldots,H^{\kappa_n})\end{aligned} \tag{6.3.15}$$

denoting abusively by $f^\varepsilon_{n,[0,t]}$ the n-linear form obtained by polarization. A refinement of Theorem 4 stated in (8.2.1) combined with (6.3.15) leads to

$$\mathbb{E}\left(\left(\zeta_t^\varepsilon(h)-\zeta_s^\varepsilon(h)\right)^4\right) \le C(\|\nabla h\|_\infty+1)\;|t-s|\left(|t-s|+\frac{1}{\mu_\varepsilon}\right), \tag{6.3.16}$$

where C depends only on $\|h\|_\infty$. This concludes the proof of Lemma 6.3.4.

6.3.2. Control of small time fluctuations. — We are now going to prove Lemma 6.3.3 by localizing the estimates into short time intervals. For this divide $[0,T]$ into overlapping intervals $I_i := [i\alpha_\varepsilon,(i+2)\alpha_\varepsilon]$ of size $2\alpha_\varepsilon$. Define also the set of trajectories such that at least two distinct collisions occur in the particle system during the time interval I_i

$$\mathcal{A}_i := \Big\{\text{At least two collisions occur in the Newtonian dynamics } \{\mathbf{z}_\ell^\varepsilon(t)\}_{\ell\le\mathcal{N}} \text{ during } I_i\Big\}. \tag{6.3.17}$$

We are going to show that the probability of $\mathcal{A}=\cup_i\mathcal{A}_i$ vanishes in the limit

$$\lim_{\varepsilon\to 0}\mathbb{P}_\varepsilon(\mathcal{A})=0. \tag{6.3.18}$$

Assuming the validity of (6.3.18) for the moment, let us first conclude the proof of Lemma 6.3.3 by restricting to the event $\mathcal{A}^c$. By construction for any trajectory in $\mathcal{A}^c$, there is at most one collision during each time interval I_i. Then, except for at most 2 particles, the particles move in straight lines

as their velocities remain unchanged and it is enough to track the variations of the test functions with respect to the positions. Thus, for any t, s in I_i and a smooth function h_j, we get

$$\begin{aligned}\sqrt{\mu_\varepsilon}\left(\zeta_t^\varepsilon(h_j)-\zeta_s^\varepsilon(h_j)\right)&=\sum_{\ell=1}^{\mathcal{N}}\left(h_j\big(\mathbf{z}_\ell^\varepsilon(t)\big)-h_j\big(\mathbf{z}_\ell^\varepsilon(s)\big)\right)-\mu_\varepsilon\int dz\big(F_1^\varepsilon(t,z)-F_1^\varepsilon(s,z)\big)h_j(z)\\&=\sum_{\ell=1}^{\mathcal{N}}\int_s^t du\,\mathbf{v}_\ell^\varepsilon(u)\cdot\nabla h_j\big(\mathbf{z}_\ell^\varepsilon(u)\big)-\mu_\varepsilon\int dz\big(F_1^\varepsilon(t,z)-F_1^\varepsilon(s,z)\big)h_j(z)+O(\|h_j\|_\infty)\,,\end{aligned}$$

where the error occurs from the fact that at most two particles may have collided in the time interval $[s,t]\subset I_i$. Using the Duhamel formula, the particle density (at fixed ε) can be also estimated by the free transport up to small corrections which may occur from the collision operator $C_{1,2}^\varepsilon F_2^\varepsilon$

$$\mu_\varepsilon\int dz\big(F_1^\varepsilon(t,z)-F_1^\varepsilon(s,z)\big)h_j(z)=\mu_\varepsilon\int_s^t du\int dz F_1^\varepsilon(u,z)\,v\cdot\nabla h_j(z)+\mu_\varepsilon\alpha_\varepsilon O(\|h_j\|_\infty)\,.$$

Recall that $\mu_\varepsilon\alpha_\varepsilon\to 0$ when μ_ε tends to infinity. Setting $\bar h_j(z):=v\cdot\nabla h_j(z)$, the time difference can be rewritten for any trajectory in $\mathcal{A}^c$ as a time integral

$$\begin{aligned}\zeta_t^\varepsilon(h_j)-\zeta_s^\varepsilon(h_j)&=\frac{1}{\sqrt{\mu_\varepsilon}}\int_s^t du\left(\langle\pi_u^\varepsilon,\bar h_j\rangle-\mu_\varepsilon\int F_1^\varepsilon(u,z)\bar h_j(z)dz\right)+\frac{1}{\sqrt{\mu_\varepsilon}}O(\|h_j\|_\infty)\\&=\int_s^t du\,\zeta_u^\varepsilon(\bar h_j)+\frac{1}{\sqrt{\mu_\varepsilon}}O(\|h_j\|_\infty)\,.\end{aligned}\tag{6.3.19}$$

Thanks to (6.3.19), we get

$$\begin{aligned}U:=\mathbb{P}_\varepsilon&\left(\mathcal{A}^c\bigcap\left\{\sum_j\frac{1}{(1+|j|^2)^k}\sup_{\substack{|s-t|\le 2\alpha_\varepsilon\\ s,t\in[0,T^\star]}}\left|\zeta_t^\varepsilon(h_j)-\zeta_s^\varepsilon(h_j)\right|^2\ge\delta'\right\}\right)\\&\le\mathbb{P}_\varepsilon\left(\mathcal{A}^c\bigcap\left\{\sum_j\frac{1}{(1+|j|^2)^k}\sup_{i\le\frac{T^\star}{\alpha_\varepsilon}}\sup_{s,t\in I_i}\left|\zeta_t^\varepsilon(h_j)-\zeta_s^\varepsilon(h_j)\right|^2\ge\delta'\right\}\right)\\&\le\mathbb{P}_\varepsilon\left(\mathcal{A}^c\bigcap\left\{\sum_j\frac{1}{(1+|j|^2)^k}\sup_{i\le\frac{T^\star}{\alpha_\varepsilon}}\sup_{s,t\in I_i}\Big|\int_s^t du\,\zeta_u^\varepsilon(\bar h_j)\Big|^2\ge\frac{\delta'}{2}\right\}\right),\end{aligned}$$

where the error term in (6.3.19) was controlled by choosing k large enough and ε small enough so that $\frac{1}{\sqrt{\mu_\varepsilon}}\ll\delta'/2$. At this stage, the constraint $\mathcal{A}^c$ can be dropped and by the Bienaymé-Tchebichev inequality there holds

$$\begin{aligned}U&\le\sum_j\frac{1}{\delta'(1+|j|^2)^k}\mathbb{E}_\varepsilon\left(\sup_{i\le\frac{T^\star}{\alpha_\varepsilon}}\sup_{s,t\in I_i}\Big|\int_s^t du\,\zeta_u^\varepsilon(\bar h_j)\Big|^2\right)\\&\le\sum_{i=1}^{\frac{T^\star}{\alpha_\varepsilon}}\sum_j\frac{1}{\delta'(1+|j|^2)^k}\mathbb{E}_\varepsilon\left(\sup_{s,t\in I_i}\Big|\int_s^t du\,\zeta_u^\varepsilon(\bar h_j)\Big|^2\right).\end{aligned}\tag{6.3.20}$$

Using the Cauchy-Schwarz inequality and then the fact that t, s belong to $I_i=[i\alpha_\varepsilon,(i+1)\alpha_\varepsilon]$, we get

$$\begin{aligned}\mathbb{E}_\varepsilon\left(\sup_{s,t\in I_i}\Big|\int_s^t du\,\zeta_u^\varepsilon(\bar h_j)\Big|^2\right)&\le\mathbb{E}_\varepsilon\left(\sup_{s,t\in I_i}|t-s|\int_s^t du\,|\zeta_u^\varepsilon(\bar h_j)|^2\right)\\&\le\alpha_\varepsilon\int_{i\alpha_\varepsilon}^{(i+1)\alpha_\varepsilon}du\,\mathbb{E}_\varepsilon\Big(\zeta_u^\varepsilon\big(\bar h_j\big)^2\Big)\le c\,\alpha_\varepsilon^2(1+|j|)^3.\end{aligned}\tag{6.3.21}$$

In the last inequality, an argument similar argument to (6.3.16) leads to the control of the second moment of $\zeta_u^\varepsilon(\bar{h}_j)$ by $\|\bar{h}_j\|_\infty^2 \leq c(1+|j|)^3$ as $\bar{h}_j = v\cdot\nabla_x h_j$ (see (6.3.1)).

Combining (6.3.20) and (6.3.21), we deduce that for k large enough

$$U \leq \sum_{i=1}^{\frac{T^\star}{\alpha_\varepsilon}} \sum_j \frac{c\,\alpha_\varepsilon^2(1+|j|)^3}{\delta'(1+|j|^2)^k} \leq \frac{C}{\delta'}\alpha_\varepsilon \xrightarrow{\varepsilon\to 0} 0. \tag{6.3.22}$$

Thus to complete the proof of Lemma 6.3.3, it remains only to show (6.3.18), i.e. that the probability concentrates on $\mathcal{A}$. To the estimate the probability of the set $\mathcal{A}_i$ introduced in (6.3.17), we distinguish two cases :

— A particle has at least two collisions during I_i. This event will be denoted by $\mathcal{A}_i^1$ if the corresponding particle has label 1, and can be separated into two subcases: either particle 1 encounters two different particles during I_i, or it encounters the same one due to space periodicity.

— Two collisions occur involving different particles. This event will be denoted by $\mathcal{A}_i^{1,2}$ if the corresponding particles are 1 and 2.

The occurence of two collisions in a time interval of length α_ε has a probability which can be estimated by using Proposition 3.3.1 with $n = 1, 2$, which allows to reduce to an estimate on pseudo-trajectories thanks to the Duhamel formula: noticing that the space-periodic situation leads to an exponentially small contribution, since it forces the velocity of the colliding particles to be of order $1/\alpha_\varepsilon$, we find

$$\mathbb{P}_\varepsilon(\mathcal{A}_i) \leq \mu_\varepsilon \mathbb{P}_\varepsilon\left(\mathcal{A}_i^1\right) + \mu_\varepsilon^2 \mathbb{P}_\varepsilon\left(\mathcal{A}_i^{1,2}\right) \leq C\big(\mu_\varepsilon + \mu_\varepsilon^2\big)\alpha_\varepsilon^2 \leq C\alpha_\varepsilon \mu_\varepsilon^{-1/3}, \tag{6.3.23}$$

where we used that $\alpha_\varepsilon = \mu_\varepsilon^{-7/3}$. Summing over the $\frac{T^\star}{\alpha_\varepsilon}$ time intervals, we deduce that $\mathbb{P}_\varepsilon(\mathcal{A}) \leq CT^\star\mu_\varepsilon^{-1/3}$. Thus the probability of $\mathcal{A}$ vanishes as ε tends to 0. This completes the proof of (6.3.18) and thus of Lemma 6.3.3.

6.4. The modified Garsia, Rodemich, Rumsey inequality

Proposition 6.3.2 is a slight adaptation of [**46**]. For simplicity we suppose that $T^\star = 1$ and set

$$B_\alpha := \int_0^1 \int_0^1 dsdt\ \Psi\left(\frac{|\varphi_t - \varphi_s|}{p(|t-s|)}\right)\mathbf{1}_{|t-s|>\alpha}\,. \tag{6.4.1}$$

Step 1:
We are first going to show that there exists $w, w' \in [0, 2\alpha]$ such that

$$\big|\varphi_{1-w'} - \varphi_w\big| \leq 8\int_0^1 \Psi^{-1}\left(\frac{4B_\alpha}{u^2}\right)dp(u) \leq 8\sqrt{2}B_\alpha^{1/4}\int_0^1 \frac{d(u^{\frac{\gamma}{4}})}{\sqrt{u}} \leq cB_\alpha^{1/4}\,. \tag{6.4.2}$$

Define

$$B_\alpha(t) = \int_0^1 ds\ \Psi\left(\frac{\varphi_t - \varphi_s}{p(|t-s|)}\right)\mathbf{1}_{|t-s|>\alpha} \quad \text{with} \quad B_\alpha = \int_0^1 dtB_\alpha(t). \tag{6.4.3}$$

There is $t_0 \in (0,1)$ such that $B_\alpha(t_0) \leq B_\alpha$. Suppose that $t_0 > 2\alpha$, then we are going to prove that there is $w \in [0, 2\alpha]$ such that

$$\big|\varphi_w - \varphi_{t_0}\big| \leq 4\int_\alpha^1 \Psi^{-1}\left(\frac{4B_\alpha}{u^2}\right)dp(u). \tag{6.4.4}$$

If $t_0 < 1 - 2\alpha$, we can show the reverse inequality

$$\left|\varphi_{1-w'} - \varphi_{t_0}\right| \leq 4 \int_\alpha^1 \Psi^{-1}\left(\frac{4B_\alpha}{u^2}\right) dp(u).$$

Combining both inequalities, will be enough to complete (6.4.2).

Let us assume that $t_0 > 2\alpha$, we are going to build a sequence $\{t_n, u_n\}_n$

$$t_0 > u_1 > t_1 > u_2 > \dots$$

such that $t_{n-1} > 2\alpha$ and u_n is defined by

$$p(u_n) = \frac{1}{2} p(t_{n-1}), \quad \text{i.e.} \quad u_n = \frac{1}{2^{4/\gamma}} t_{n-1}. \tag{6.4.5}$$

The sequence will be stopped as soon as $t_n < 2\alpha$.

Initially $t_0 > 2\alpha$ and u_1 is defined by (6.4.5). Suppose that the sequence has been built up to t_{n-1}. By construction

$$t_{n-1} - u_n = \left(1 - \frac{1}{2^{4/\gamma}}\right) t_{n-1} > \alpha \quad \text{since} \quad t_{n-1} > 2\alpha.$$

Thus

$$\int_0^{u_n} ds\ \Psi\left(\frac{|\varphi_{t_{n-1}} - \varphi_s|}{p(|t_{n-1} - s|)}\right) = \int_0^{u_n} ds\ \Psi\left(\frac{|\varphi_{t_{n-1}} - \varphi_s|}{p(|t_{n-1} - s|)}\right) \mathbf{1}_{|t_{n-1}-s|>\alpha} \leq B_\alpha(t_{n-1}).$$

Furthermore

$$\int_0^{u_n} dt B_\alpha(t) \leq B_\alpha,$$

thus there is $t_n \in [0, u_n]$ such that

$$B_\alpha(t_n) \leq \frac{2B_\alpha}{u_n} \quad \text{and} \quad \Psi\left(\frac{|\varphi_{t_{n-1}} - \varphi_{t_n}|}{p(|t_{n-1} - t_n|)}\right) \leq \frac{2B_\alpha(t_{n-1})}{u_n} \leq \frac{4B_\alpha}{u_{n-1}\, u_n} \leq \frac{4B_\alpha}{u_n^2}.$$

We deduce that

$$|\varphi_{t_{n-1}} - \varphi_{t_n}| \leq \Psi^{-1}\left(\frac{4B_\alpha}{u_n^2}\right) p(|t_{n-1} - t_n|) \leq \Psi^{-1}\left(\frac{4B_\alpha}{u_n^2}\right) p(t_{n-1}).$$

Suppose that $t_n > 2\alpha$ then using that

$$u_n > t_n \Rightarrow p(u_n) > p(t_n) = 2p(u_{n+1}),$$

we get

$$p(t_{n-1}) = 2p(u_n) = 4\big(p(u_n) - p(u_n)/2\big) \leq 4\big(p(u_n) - p(u_{n+1})\big)$$

and also

$$|\varphi_{t_{n-1}} - \varphi_{t_n}| \leq 4\Psi^{-1}\left(\frac{4B_\alpha}{u_n^2}\right)\big(p(u_n) - p(u_{n+1})\big) \leq 4\int_{u_{n+1}}^{u_n} \Psi^{-1}\left(\frac{4B_\alpha}{u^2}\right) dp(u). \tag{6.4.6}$$

We then iterate the procedure to define t_{n+1}.

If $t_n < 2\alpha$, we set $w = t_n$ and we stop the procedure at step n with the inequality

$$|\varphi_{t_{n-1}} - \varphi_w| = |\varphi_{t_{n-1}} - \varphi_{t_n}| \leq 4 \int_0^{u_n} \Psi^{-1}\left(\frac{4B_\alpha}{u^2}\right) dp(u), \tag{6.4.7}$$

where we used that

$$p(t_{n-1}) = 2p(u_n) \leq 4\big(p(u_n) - p(0)\big).$$

Summing the previous inequalities of the form (6.4.6), we deduce (6.4.4) from

$$|\varphi_{t_0} - \varphi_w| \leq \sum_{i=1}^{n} |\varphi_{t_{i-1}} - \varphi_{t_i}| \leq 4 \int_0^{u_1} \Psi^{-1}\left(\frac{4B_\alpha}{u^2}\right) dp(u). \tag{6.4.8}$$

This completes the proof of (6.4.2).

Step 2: proof of (6.3.6)**.**
We are going to proceed by a change of variables. Given $x < y$ such that $y - x > 4\alpha$, we set $p_{y-x}(u) = p((y-x)u)$ and $\psi_t = \varphi(x + (y-x)t)$

$$\begin{aligned} B^{(\psi)}_{\frac{\alpha}{y-x}} &= \int_0^1 \int_0^1 ds dt\ \Psi\left(\frac{|\varphi_t - \varphi_s|}{p_{y-x}(|t-s|)}\right) \mathbf{1}_{\{|t-s| > \frac{\alpha}{|y-x|}\}} \\ &= \frac{1}{|y-x|^2} \int_x^y \int_x^y ds' dt'\ \Psi\left(\frac{|\psi_{t'} - \psi_{s'}|}{p(|t'-s'|)}\right) \mathbf{1}_{\{|t'-s'|>\alpha\}} \leq \frac{B_\alpha}{|y-x|^2}\,. \end{aligned}$$

Applying (6.4.2) to the function ψ, there exists $w, w' \in [0, 2\alpha]$ such that

$$\left|\psi_{1-\frac{w'}{y-x}} - \psi_{\frac{w}{y-x}}\right| \leq 8 \int_0^1 \Psi^{-1}\left(\frac{4B^{(\psi)}_{\frac{\alpha}{y-x}}}{u^2}\right) dp_{y-x}(u) \leq 8 \int_0^1 \Psi^{-1}\left(\frac{4B_\alpha}{|y-x|^2 u^2}\right) dp_{y-x}(u).$$

Changing again variables, we get

$$\left|\varphi_{y-w'} - \varphi_{x+w}\right| \leq 8\,(y-x)^{\frac{\gamma}{4}-\frac{1}{2}} \int_0^1 \Psi^{-1}\left(\frac{4B_\alpha}{u^2}\right) dp(u) \leq 8\sqrt{2} B_\alpha^{1/4}\,(y-x)^{\frac{\gamma}{4}-\frac{1}{2}}.$$

By bounding $\left|\varphi_y - \varphi_{y-w'}\right|$ and $\left|\varphi_{x+w} - \varphi_y\right|$ by the supremum of the local fluctuations in a time interval less than 2α, we conclude to (6.3.6). The proposition is proved. □

6.5. Spohn's formula for the covariance

For the sake of completeness, we are going to show that the covariance $\hat{\mathcal{C}}$ of the Ornstein-Uhlenbeck process computed in (6.1.6) coincides with the formula obtained by Spohn in [**42**] and recalled below in (6.5.1). Formula (6.5.1) is striking as the recollision operator $R^{1,2}$ emphasises the contribution to the covariance of the recollisions in the microscopic dynamics.

Proposition 6.5.1. — *Recall that $\mathcal{U}(t,s)$ stands for the semi-group associated with the time dependent operator $\mathcal{L}_\tau$ for τ between times $s < t$. Given two times $t \geq s$, there holds*

$$\begin{aligned} \hat{\mathcal{C}}(s,t,\varphi,\psi) &= \int dz\, \mathcal{U}^*(t,s)\psi(z)\ \varphi(z)\, f(s,z) \\ &\quad + \int_0^t d\tau \int dx dv dw\, R^{1,2}\left(f(\tau), f(\tau)\right)(x,v,w)\ \left(\mathcal{U}^*(t,\tau)\psi\right)(x,v)\ \left(\mathcal{U}^*(s,\tau)\varphi\right)(x,w)\,, \end{aligned} \tag{6.5.1}$$

where the recollision operator $R^{1,2}$ is defined as in (5.5.7)

$$R^{1,2}(g,g)(z_1,z_2) := \int \Big(g(z_1')g(z_2') - g(z_1)g(z_2)\Big) d\mu_{z_1,z_2}(\omega). \tag{6.5.2}$$

Proof. — The covariance at time $t = s = 0$ is indeed given by

$$\mathbb{E}\left(\zeta_0(\varphi)\zeta_0(\psi)\right) = \int dz \varphi(z) f^0 \psi(z) = \int dz \varphi(z)\psi(z) f(0,z)\,.$$

We will simply derive (6.5.1) when $s = t$ and the case $s < t$ can be easily deduced. The covariance $\mathbf{Cov}_t$ introduced in (6.0.3) can be rewritten in terms of the operator Σ_t

$$\Sigma_t\psi(z_1) := -\int d\mu_{z_1}(z_2,\omega)\,\Big[f(t,z_1)f(t,z_2) + f(t,z_1')f(t,z_2')\Big]\Delta\psi\,, \tag{6.5.3}$$

with the notation $d\mu_{z_1}$ as in (5.5.6) and $\Delta\psi$ as in (6.0.5). Indeed, one can check that for any functions φ, ψ, the covariance can be recovered as follows

$$\begin{aligned}\int \varphi\Sigma_t\psi(z_1)dz_1 &= -\frac{1}{2}\int d\mu(z_1,z_2,\omega)\Big[f(t,z_1)f(t,z_2) + f(t,z_1')f(t,z_2')\Big]\Delta\psi(\varphi(z_1)+\varphi(z_2))\\ &= \frac{1}{2}\int d\mu(z_1,z_2,\omega)f(t,z_1)f(t,z_2)(\Delta\psi)(\Delta\varphi) = \mathbf{Cov}_t(\varphi,\psi)\,.\end{aligned}$$

The covariance $\hat{\mathcal{C}}$ of the Ornstein-Uhlenbeck process computed in (6.1.6) reads

$$\hat{\mathcal{C}}(t,t,\varphi,\psi) = \int dz_1\,\mathcal{U}^*(t,0)\psi(z_1)\,f^0\,\mathcal{U}^*(t,0)\varphi(z_1) + \int_0^t du\int dz_1\,\varphi(z_1)\,\big[\mathcal{U}(t,u)\,\Sigma_u\,\mathcal{U}^*(t,u)\psi\big](z_1)\,. \tag{6.5.4}$$

The following identity is the key to identify (6.5.4) and (6.5.1)

$$\Sigma_t\varphi(z_1) = -\Big(f_t\mathcal{L}_t^* + \mathcal{L}_t f_t\Big)\varphi(z_1) + \partial_t f(t,z_1)\,\varphi(z_1) + \int dz_2\,R^{1,2}\big(f(t),f(t)\big)(z_1,z_2)\varphi(z_2). \tag{6.5.5}$$

Let us postpone for a while the proof of this identity and complete first the proof of (6.5.1).

Replacing the expression (6.5.5) of Σ_u in the second line of (6.5.4) and recalling that $\mathcal{U}(t,t)\varphi = \varphi$, we get that

$$\begin{aligned}&\int_0^t du\int dz_1\,\varphi(z_1)\,\big[\mathcal{U}(t,u)\,\Sigma_u\,\mathcal{U}^*(t,u)\psi\big](z_1)\\ &\qquad= \int_0^t du\int dz_1\,\varphi(z_1)\,\Big[\mathcal{U}(t,u)\,\Big(-\big(\mathcal{L}_u f_u + f_u\mathcal{L}_u^*\big) + \partial_u f(u)\Big)\,\mathcal{U}^*(t,u)\psi\Big](z_1)\\ &\qquad\quad+\int_0^t du\int dz_1dz_2\,\mathcal{U}^*(t,u)\varphi(z_1)\,R^{1,2}\big(f(u),f(u)\big)(z_1,z_2)\,\mathcal{U}^*(t,u)\psi(z_2)\,.\end{aligned}$$

Noticing that the time derivative is given by

$$\partial_u\Big[\mathcal{U}(t,u)\,f_u\,\mathcal{U}^*(t,u)\Big] = \mathcal{U}(t,u)\,\Big(-\big(\mathcal{L}_u f_u + f_u\mathcal{L}_u^*\big) + \partial_u f(u)\Big)\,\mathcal{U}^*(t,u)\,,$$

we conclude that

$$\begin{aligned}\int_0^t du\int dz_1\,\varphi(z_1)\,\big[\mathcal{U}(t,u)\,\Sigma_u\,\mathcal{U}^*(t,u)\psi\big](z_1) &= \int dz_1\,\Big(\varphi(z_1)\,f_t\psi(z_1) - \varphi(z_1)\,\mathcal{U}(t,0)f^0\,\mathcal{U}^*(t,0)\psi(z_1)\Big)\\ &\quad+\int_0^t du\int dz_1dz_2\,\mathcal{U}^*(t,u)\varphi(z_1)\,R^{1,2}\big(f(u),f(u)\big)(z_1,z_2)\,\mathcal{U}^*(t,u)\psi(z_2).\end{aligned}$$

Finally the covariance (6.5.4) reads

$$\hat{\mathcal{C}}(t,t,\varphi,\psi) = \int dz\varphi(z)\,f_t\psi(z) + \int_0^t du\int dz_1dz_2\,\mathcal{U}^*(t,u)\varphi(z_1)\,R^{1,2}\big(f(u),f(u)\big)(z_1,z_2)\,\mathcal{U}^*(t,u)\psi(z_2).$$

This completes the proof of Proposition 6.5.1.

It remains then to establish the identity (6.5.5). Let us write the decomposition $\Sigma_t = \Sigma_t^+ + \Sigma_t^-$ with

$$\Sigma_t^+\psi(z_1) := -\int d\mu_{z_1}(z_2,\omega)f(t,z_1')f(t,z_2')\Delta\psi, \qquad \Sigma_t^-\psi(z_1) := -\int d\mu_{z_1}(z_2,\omega)f(t,z_1)f(t,z_2)\Delta\psi.$$

Recall that $\mathcal{L}_T^*$ was computed in (6.1.2). We get

$$f(t)\mathcal{L}_t^*\varphi(z_1) = f(t)\; v_1\cdot\nabla\varphi(z_1) + \int d\mu_{z_1}(z_2,\omega) f(t,z_1)f(t,z_2)\Delta\varphi = f(t)\; v_1\cdot\nabla\varphi(z_1) - \Sigma_t^-\varphi(z_1)\,.$$

and

$$\begin{aligned}\mathcal{L}_t f(t)\varphi(z_1) &= -v_1\cdot\nabla[f(t)\varphi](z_1) + \int d\mu_{z_1}(z_2,\omega)\Big(f(t,z_1')f(t,z_2')\big(\varphi(z_1')+\varphi(z_2')\big)\\ &\qquad\qquad - f(t,z_1)f(t,z_2)\big(\varphi(z_2)+\varphi(z_1)\big)\Big)\\ &= -v_1\cdot\nabla[f(t)\varphi](z_1) + \int d\mu_{z_1}(z_2,\omega)\Big(f(t,z_1')f(t,z_2')\Delta\varphi\\ &\qquad + \big[f(t,z_1')f(t,z_2') - f(t,z_1)f(t,z_2)\big]\big(\varphi(z_1)+\varphi(z_2)\big)\Big)\\ &= -v_1\cdot\nabla[f(t)\varphi](z_1) - \Sigma_t^+\varphi(z_1) + \int dz_2\, R^{1,2}\big(f(t),f(t)\big)(z_1,z_2)\big(\varphi(z_1)+\varphi(z_2)\big)\,,\end{aligned}$$

where we used the notation (6.5.2). As a consequence, we get that

$$f(t)\mathcal{L}_t^*\varphi(z_1) + \mathcal{L}_t f(t)\varphi(z_1) = -\varphi\; v_1\cdot\nabla f(t,z_1) - \Sigma_t\varphi(z_1) + \int dz_2 R^{1,2}\big(f(t),f(t)\big)(z_1,z_2)\big(\varphi(z_1)+\varphi(z_2)\big)\,.$$

As f solves the Boltzmann equation, we have

$$\partial_t f(t,z_1) = -v_1\cdot\nabla f(t,z_1) + \int dz_2 R^{1,2}\big(f(t),f(t)\big)(z_1,z_2)\,.$$

This leads to further simplifications as

$$f(t)\mathcal{L}_t^*\varphi(z_1) + \mathcal{L}_t f(t)\varphi(z_2) = \varphi\;\partial_t f(t,z_1) - \Sigma_t\varphi(z_1) + \int dz_2\, R^{1,2}\big(f(t),f(t)\big)(z_1,z_2)\varphi(z_2),$$

thus (6.5.5) holds. □

CHAPTER 7

LARGE DEVIATIONS

This chapter is devoted to the study of large deviations, and to the proof of Theorem 3. We are going to evaluate the probability of an atypical event, namely that the empirical measure remains close to a probability density φ (which is different from the solution to the Boltzmann equation f) during the time interval $[0, T^\star]$.

The strategy we will use to evaluate this probability is rather indirect as we cannot describe the bias we have to impose on the initial data to observe such a trajectory φ : changes in the collision process (both on the rate and on the cross section) depend indeed in a very intricate way on the microscopic realization of the initial data. We will therefore proceed in a completely different way, using a kind of duality argument. The idea is to compute (with exponential accuracy) the average of functionals $H\big(z([0,t])\big)$ of the trajectories for a large class of test functionals H, and then to deduce the weight of a trajectory φ using a minimizing argument.

The duality on $\mathbb{D}$ (resp. $[0,T]\times\mathbb{D}$) will be denoted, as in (2.1.1)-(6.2.1) by $\langle\cdot,\cdot\rangle$ (resp. $\langle\!\langle\cdot,\cdot\rangle\!\rangle$)

$$\langle\varphi,\psi\rangle := \int_{\mathbb{D}} dz\ \varphi(z)\ \psi(z)\,, \qquad \langle\!\langle\varphi,\psi\rangle\!\rangle := \int_0^T dt \int_{\mathbb{D}} dz\ \varphi(t,z)\ \psi(t,z)\,.$$

Using notation (1.4.3), (5.3.2) and (5.3.3), define the set of test functions

$$\mathbb{B} = \big\{ g \in C^1([0,T^\star]\times\mathbb{D},\mathbb{R}) \,/\, \big(D_t g, \exp(g(T^\star))\big) \in \mathcal{B}_{\alpha_0,\beta_0,T^\star} \big\} \tag{7.0.1}$$

and set for any g in $\mathbb{B}$ and $t \le T^\star$

$$\mathcal{I}(t,g) := \mathcal{J}\big(t, Dg, \exp(g(t))\big)\,. \tag{7.0.2}$$

For a restricted class of functions φ, we prove in Section 7.1 that the large deviation functional on the time interval $[0,t]$ is given by the variational principle

$$\mathcal{F}(t,\varphi) := \sup_{g\in\mathbb{B}} \Big\{ -\langle\!\langle\varphi, Dg\rangle\!\rangle + \langle\varphi_t, g_t\rangle - \mathcal{I}(t,g) \Big\}\,. \tag{7.0.3}$$

This functional can be identified with the functional $\widehat{\mathcal{F}}(t)$ predicted by [**38**] and [**9**] (see Section 7.2).

7.1. Large deviation asymptotics

We first start by proving upper and lower large deviation bounds in a topology weaker than the Skorohod topology. We are going to consider the weak topology on $D([0,T^\star],\mathcal{M}(\mathbb{D}))$ generated by

open sets of the form below, for any $\nu \in D([0,T^\star],\mathcal{M}(\mathbb{D}))$ and for test functions g in $\mathbb{B}$ and $\delta>0$:

$$\mathbf{O}_{\delta,g}(\nu) := \Big\{ \nu' \in D([0,T^\star],\mathcal{M}(\mathbb{D}))\,,\ \Big| \big(\langle\!\langle \nu', Dg\rangle\!\rangle - \langle \nu'_{T^\star}, g_{T^\star}\rangle\big) - \big(\langle\!\langle \nu, Dg\rangle\!\rangle - \langle \nu_{T^\star}, g_{T^\star}\rangle\big)\Big| < \delta/2 \Big\}. \tag{7.1.1}$$

7.1.1. Upper bound. — We are going to prove the large deviation upper bound (1.4.5) for any compact set $\mathbf{F}$ in the weak topology

$$\limsup_{\mu_\varepsilon\to\infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon\left(\pi^\varepsilon \in \mathbf{F}\right) \le - \inf_{\varphi\in\mathbf{F}} \mathcal{F}(T^\star,\varphi)\,. \tag{7.1.2}$$

To prove (7.1.2), we consider a compact set $\mathbf{F}$ of $D([0,T^\star],\mathcal{M}(\mathbb{D}))$. We are first going to show that for any density φ in $\mathbf{F}$ and $\delta>0$, there exists $g\in\mathbb{B}$ and an open set $\mathbf{O}_{\delta,g}(\varphi)$ of φ such that

$$\limsup_{\mu_\varepsilon\to\infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon\left(\pi^\varepsilon \in \mathbf{O}_{\delta,g}(\varphi)\right) \le -\mathcal{F}_{[0,T^\star]}(\varphi) + \delta\,. \tag{7.1.3}$$

Then by compactness, for any $\delta>0$, a finite covering of $\mathbf{F}\subset \cup_{i\le K}\mathbf{O}_{\delta,g_i}(\varphi_i)$ can be extracted so that

$$\limsup_{\mu_\varepsilon\to\infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon\left(\pi^\varepsilon \in \mathbf{F}\right) \le - \inf_{i\le K} \mathcal{F}(T^\star,\varphi_i) + \delta \le - \inf_{\varphi\in\mathbf{F}} \mathcal{F}(T^\star,\varphi) + \delta\,.$$

Letting $\delta\to 0$, we recover the upper bound (7.1.2).

We turn now to the derivation of (7.1.3). For any density φ in $\mathbf{F}$, we know from (7.0.3) that there exists $g\in\mathbb{B}$ such that

$$\mathcal{F}(T^\star,\varphi) \le -\langle\!\langle \varphi, Dg\rangle\!\rangle + \langle \varphi_{T^\star}, g_{T^\star}\rangle - \mathcal{I}(T^\star,g) + \delta/2\,.$$

This leads to the upper bound

$$\begin{aligned}\mathbb{P}_\varepsilon\left(\pi^\varepsilon \in \mathbf{O}_{\delta,g}(\varphi)\right) &\le \exp\Big(\mu_\varepsilon\frac{\delta}{2} + \mu_\varepsilon\langle\!\langle \varphi, Dg\rangle\!\rangle - \mu_\varepsilon\langle \varphi_{T^\star}, g_{T^\star}\rangle\Big)\ \mathbb{E}_\varepsilon\Big(\exp\Big(-\mu_\varepsilon\langle\!\langle \pi^\varepsilon, Dg\rangle\!\rangle + \mu_\varepsilon\langle \pi^\varepsilon_{T^\star}, g_{T^\star}\rangle\Big)\Big)\\ &\le \exp\Big(\mu_\varepsilon\frac{\delta}{2} + \mu_\varepsilon\langle\!\langle \varphi, Dg\rangle\!\rangle - \mu_\varepsilon\langle \varphi_{T^\star}, g_{T^\star}\rangle + \mu_\varepsilon\mathcal{I}^\varepsilon(T^\star,g)\Big)\,,\end{aligned}$$

with

$$\mathcal{I}^\varepsilon(t,g) := \mathcal{J}^\varepsilon\big(t, Dg, \exp(g(t))\big) = \Lambda^\varepsilon_{[0,t]}(e^{g-\int_0^t Dg})\,.$$

Passing to the limit thanks to the upper bounds provided by Theorem 4, this completes (7.1.3)

$$\limsup_{\mu_\varepsilon\to\infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon\Big(\pi^\varepsilon \in \mathbf{O}_{\delta,g}(\varphi)\Big) \le \mathcal{I}(T^\star,g) + \langle\!\langle \varphi, Dg\rangle\!\rangle - \langle \varphi_{T^\star}, g_{T^\star}\rangle + \delta/2 \le -\mathcal{F}(T^\star,\varphi) + \delta\,.$$

7.1.2. Lower bound. — We are going to prove the large deviation lower bound (1.4.6) for any open set $\mathbf{O}$ in the weak topology

$$\liminf_{\mu_\varepsilon\to\infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon\left(\pi^\varepsilon \in \mathbf{O}\right) \ge - \inf_{\varphi\in\mathbf{O}\cap\mathcal{R}} \mathcal{F}(T^\star,\varphi)\,, \tag{7.1.4}$$

where the restricted set $\mathcal{R}$ of trajectories is the set of densities φ such that the supremum in (7.0.3) is reached for some $g\in\mathbb{B}$

$$\mathcal{R} := \Big\{ \varphi\in C^1([0,T^\star]\times\mathbb{D})\,,\ \exists g\in\mathbb{B} \text{ such that } \mathcal{F}(T^\star,\varphi) = \langle \varphi_{T^\star}, g_{T^\star}\rangle - \langle\!\langle \varphi, Dg\rangle\!\rangle - \mathcal{I}(T^\star,g)\Big\}\,. \tag{7.1.5}$$

Let us fix $\varphi \in \mathbf{O} \cap \mathcal{R}$ and denote by g the associated test function as in (7.1.5). There exists a collection of test functions $g^{(1)}, \dots, g^{(\ell)}$ in $\mathbb{B}$ such that the following open neighborhood of φ

$$\begin{aligned} \mathbf{O}_{\delta,\{g^{(i)}\}}(\varphi) := \Big\{ & \nu \in D([0,T^\star], \mathcal{M}(\mathbb{D}))\,,\ \forall i \le \ell\,, \\ & \Big| \big\langle\!\big\langle \nu, Dg^{(i)} \big\rangle\!\big\rangle - \langle \nu_{T^\star}, g^{(i)}_{T^\star} \rangle - \big(\big\langle\!\big\langle \varphi, Dg^{(i)} \big\rangle\!\big\rangle - \langle \varphi_{T^\star}, g^{(i)}_{T^\star} \rangle \big) \Big| < \delta \Big\} \end{aligned} \tag{7.1.6}$$

is included in $\mathbf{O}$ for any $\delta > 0$ small enough. We impose also that g is one of the test functions $g^{(1)}, \dots, g^{(\ell)}$. To complete the lower bound

$$\liminf_{\mu_\varepsilon \to \infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon \left(\pi^\varepsilon \in \mathbf{O} \right) \ge - \mathcal{F}(T^\star, \varphi)\,,$$

it is enough to show that

$$\liminf_{\delta \to 0} \liminf_{\mu_\varepsilon \to \infty} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon \left(\pi^\varepsilon \in \mathbf{O}_{\delta,\{g^{(i)}\}}(\varphi) \right) \ge - \mathcal{F}(T^\star, \varphi)\,. \tag{7.1.7}$$

We start by tilting the measure

$$\begin{aligned} & \mathbb{P}_\varepsilon \left(\mathbf{O}_{\delta,\{g^{(i)}\}}(\varphi) \right) \\ & \ge \exp\Big(- \delta\mu_\varepsilon + \mu_\varepsilon \big\langle\!\big\langle \varphi, Dg \big\rangle\!\big\rangle - \mu_\varepsilon \langle \varphi_{T^\star}, g_{T^\star} \rangle \Big)\, \mathbb{E}_\varepsilon \Big(\exp\Big(- \mu_\varepsilon \big\langle\!\big\langle \pi^\varepsilon, Dg \big\rangle\!\big\rangle + \mu_\varepsilon \langle \pi^\varepsilon_{T^\star}, g_{T^\star} \rangle \Big)\, \mathbf{1}_{\mathbf{O}_{\delta,\{g^{(i)}\}}(\varphi)} \Big) \\ & \ge \exp\Big(-\delta\mu_\varepsilon + \mu_\varepsilon \mathcal{I}^\varepsilon_{[0,T^\star]}(g) + \mu_\varepsilon \big\langle\!\big\langle \varphi, Dg \big\rangle\!\big\rangle - \mu_\varepsilon \langle \varphi_{T^\star}, g_{T^\star} \rangle \Big)\, \mathbb{E}_{\varepsilon,g} \Big(\mathbf{1}_{\mathbf{O}_{\delta,\{g^{(i)}\}}(\varphi)} \Big)\,, \end{aligned}$$

where we defined the tilted measure for any function Ψ on the particle trajectories as

$$\mathbb{E}_{\varepsilon,g} \left(\Psi(\pi^\varepsilon) \right) := \exp\Big(-\mu_\varepsilon \mathcal{I}^\varepsilon_{[0,T^\star]}(g) \Big)\, \mathbb{E}_\varepsilon \Big(\exp\Big(- \mu_\varepsilon \big\langle\!\big\langle \pi^\varepsilon, Dg \big\rangle\!\big\rangle + \mu_\varepsilon \langle \pi^\varepsilon_{T^\star}, g_{T^\star} \rangle \Big)\, \Psi(\pi^\varepsilon) \Big)\,.$$

If we can show that the trajectory φ is typical under the tilted measure

$$\forall \delta > 0\,, \qquad \lim_{\mu_\varepsilon \to \infty} \mathbb{P}_{\varepsilon,g} \left(\pi^\varepsilon \in \mathbf{O}_{\delta,\{g^{(i)}\}}(\varphi) \right) = 1\,, \tag{7.1.8}$$

this will complete the proof of (7.1.7).

Let $\tilde{g}$ be one of the functions $g^{(1)}, \dots, g^{(\ell)}$ used to define the weak neighborhood $\mathbf{O}_{\delta,\{g^{(i)}\}}(\varphi)$. Choose $u \in \mathbb{C}$ in a neigborhood of 0 so that the function below is analytic

$$u \in \mathbb{C} \mapsto \mathcal{I}(T^\star, u\tilde{g} + g) = \lim_{\mu_\varepsilon \to \infty} \mathcal{I}^\varepsilon(T^\star, u\tilde{g} + g)\,.$$

As a consequence the derivative and the limit as $\mu_\varepsilon \to \infty$ commute, so that taking the derivative at $u = 0$, we get

$$- \left\langle\!\!\left\langle \frac{\partial \mathcal{I}(T^\star)}{\partial Dg}(g), D\tilde{g} \right\rangle\!\!\right\rangle + \left\langle \frac{\partial \mathcal{I}(T^\star)}{\partial g_{T^\star}}(g), \tilde{g}_{T^\star} \right\rangle = \lim_{\mu_\varepsilon \to \infty} \mathbb{E}_{\varepsilon,g} \Big(- \big\langle\!\big\langle \pi^\varepsilon, D\tilde{g} \big\rangle\!\big\rangle + \langle \pi^\varepsilon_{T^\star}, \tilde{g}_{T^\star} \rangle \Big)\,.$$

Note that in the above equation, the functional derivative is taken over both coordinates of the functional $\mathcal{I}(T^\star, g) = \mathcal{J}(T^\star, Dg, g_t)$. As the supremum in (7.0.3) is reached at g, we deduce from (7.1.5) that

$$- \left\langle\!\!\left\langle \frac{\partial \mathcal{I}(T^\star)}{\partial Dg}(g), D\tilde{g} \right\rangle\!\!\right\rangle + \left\langle \frac{\partial \mathcal{I}(T^\star)}{\partial g_{T^\star}}(g), \tilde{g}_{T^\star} \right\rangle = \langle \varphi_{T^\star}, \tilde{g}_{T^\star} \rangle - \big\langle\!\big\langle \varphi, D\tilde{g} \big\rangle\!\big\rangle\,. \tag{7.1.9}$$

This allows us to characterize the mean under the tilted measure

$$\lim_{\mu_\varepsilon \to \infty} \mathbb{E}_{\varepsilon,g} \left(\langle \pi^\varepsilon_{T^\star}, \tilde{g}_{T^\star} \rangle - \big\langle\!\big\langle \pi^\varepsilon, D\tilde{g} \big\rangle\!\big\rangle \right) = \langle \varphi_{T^\star}, \tilde{g}_{T^\star} \rangle - \big\langle\!\big\langle \varphi, D\tilde{g} \big\rangle\!\big\rangle\,. \tag{7.1.10}$$

Taking twice the derivative, we obtain

$$\lim_{\mu_\varepsilon\to\infty}\mu_\varepsilon\mathbb{E}_{\varepsilon,g}\left(\left[\Big(\langle\pi^\varepsilon_{T^\star},\tilde g_{T^\star}\rangle-\langle\!\langle\pi^\varepsilon,D\tilde g\rangle\!\rangle\Big)-\mathbb{E}_{\varepsilon,g}\Big(\langle\pi^\varepsilon_{T^\star},\tilde g_{T^\star}\rangle-\langle\!\langle\pi^\varepsilon,D\tilde g\rangle\!\rangle\Big)\right]^2\right)<\infty\,.$$

Combined with (7.1.10), this implies that the empirical measure concentrates to φ in a weak sense

$$\lim_{\mu_\varepsilon\to\infty}\mathbb{E}_{\varepsilon,g}\left(\left[\Big(\langle\pi^\varepsilon_{T^\star},\tilde g_{T^\star}\rangle-\langle\!\langle\pi^\varepsilon,D\tilde g\rangle\!\rangle\Big)-\Big(\langle\varphi_{T^\star},\tilde g_{T^\star}\rangle-\langle\!\langle\varphi,D\tilde g\rangle\!\rangle\Big)\right]^2\right)=0\,.$$

In particular, this holds for any test functions $g^{(1)},\dots,g^{(\ell)}$ defining the neighborhood $\mathbf{O}_{\delta,\{g^{(i)}\}}(\varphi)$ in (7.1.6). This completes (7.1.8).

7.1.3. Uniform continuity in time. — In this paragraph we strenghten the Large Deviations Principle derived in the previous section, and show that is holds in the Skorohod topoloyy. It is well known (see Corollary 4.2.6 of [**13**]) that large deviation estimates can be derived in a strong topology from a coarser topology by proving a tightness property in this strong topology. The following proposition shows that the sample paths concentrate on equicontinuous trajectories in $[0,T^\star]$, which is a kind of tightness property.

Let $(h_j)_{j\geq0}$ denote the basis of Fourier-Hermite functions (as in (6.3.1)). We define a distance on the set of probability measures $\mathcal{M}(\mathbb{D})$ by

$$d(\mu,\nu):=\sum_j 2^{-j}\left|\int dz\;h_j(z)\big(d\mu(z)-d\nu(z)\big)\right|\,. \tag{7.1.11}$$

Proposition 7.1.1. — *The modulus of continuity is controlled by*

$$\forall\delta'>0,\qquad \lim_{\delta\to0}\lim_{\varepsilon\to0}\frac{1}{\mu_\varepsilon}\log\mathbb{P}_\varepsilon\left(\sup_{\substack{|t-s|\leq\delta\\ t,s\in[0,T^\star]}}d(\pi^\varepsilon_t,\pi^\varepsilon_s)>\delta'\right)=-\infty\,. \tag{7.1.12}$$

Before proving Proposition 7.1.1, let us first show that it implies a large deviation estimate in the Skorohod space of trajectories $D([0,T^\star],\mathcal{M}(\mathbb{D}))$ (for a definition see Section 12 in [**4**]). First of all notice that the upper bound holds as the closed sets for the Skorohod topology are also closed for the weak topology. We consider now an open set $\mathcal{O}$ for the strong topology and φ a trajectory in $\mathcal{O}\cap\mathcal{R}$, recalling $\mathcal{R}$ is defined in (7.1.5). We would like to apply the same proof as in Section 7.1.2 and to reduce the estimates to sample paths in a weak open set of the form (7.1.6). We proceed in several steps. First note that there exists $\delta>0$ such that

$$\left\{\nu\,,\ \sup_{t\leq T^\star}d(\nu_t,\varphi_t)<2\delta\right\}\subset\mathcal{O}\,.$$

Since φ belongs to $\mathcal{R}$, the density φ is smooth in time. Choosing a time step $\gamma>0$ small enough, we can restrict to computing the distance at discrete times

$$\left\{\nu\,,\ \sup_{\substack{i\in\mathbb{N}\\ i\gamma\leq T^\star}}d(\nu_{i\gamma},\varphi_{i\gamma})<\delta\right\}\bigcap\left\{\nu\,,\ \sup_{|t-s|\leq\gamma}d(\nu_t,\nu_s)<\frac{\delta}{2}\right\}\subset\mathcal{O}\,.$$

Since φ is regular in time and we consider only $T^\star/\gamma$ times, the set above can be approximated by a set of the form $\mathbf{O}_\delta(\varphi)$ as in (7.1.6). As a consequence we have shown that there is an open set $\mathbf{O}_\delta(\varphi)$

such that

$$\mathbb{P}_\varepsilon\left(\pi^\varepsilon \in \mathcal{O}\right) \geq \mathbb{P}_\varepsilon\left(\pi^\varepsilon \in \mathbf{O}_\delta(\varphi) \bigcap \left\{ \sup_{|t-s|\leq\gamma} d(\pi^\varepsilon_t, \pi^\varepsilon_s) < \delta \right\}\right)$$
$$\geq \mathbb{P}_\varepsilon\left(\pi^\varepsilon \in \mathbf{O}_\delta(\varphi)\right) - \mathbb{P}_\varepsilon\left(\left\{ \sup_{|t-s|\leq\gamma} d(\pi^\varepsilon_t, \pi^\varepsilon_s) > \delta \right\}\right).$$

By Proposition 7.1.1 the last term can be made arbitrarily small for γ small. Thus the proof of the lower bound reduces now to the one of weak open sets as in Section 7.1.2.

Proof of Proposition 7.1.1. — As the test functions used for defining the distance $d(\pi^\varepsilon_t, \pi^\varepsilon_s)$ in (7.1.11) are bounded, it is enough to consider a finite number of test functions. Indeed, for any δ' there is $K = K(\delta')$ such that

$$d(\mu,\nu) > \delta' \quad \Rightarrow \quad \sum_{|j|\leq K} 2^{-j} \left| \int dz\; h_j(z)\big(d\mu(z) - d\nu(z)\big) \right| > \frac{\delta'}{2}\cdot$$

By the union bound, we can then reduce (7.1.12) to controlling a single test function h

$$\forall \delta' > 0\,, \qquad \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon\left(\sup_{|t-s|\leq\delta} \big|\langle \pi^\varepsilon_t, h\rangle - \langle \pi^\varepsilon_s, h\rangle\big| > \delta' \right) = -\infty \tag{7.1.13}$$

where t, s are restricted to $[0, T^\star]$. Next, we localize the constraint on the time interval $[0, T^\star]$ to smaller time intervals

$$\mathbb{P}_\varepsilon\left(\sup_{|t-s|\leq\delta} \big|\langle \pi^\varepsilon_t, h\rangle - \langle \pi^\varepsilon_s, h\rangle\big| > \delta' \right) \leq \sum_{i=2}^{T^\star/\delta} \mathbb{P}_\varepsilon\left(\sup_{t,s\in[(i-2)\delta, i\delta]} \big|\langle \pi^\varepsilon_t, h\rangle - \langle \pi^\varepsilon_s, h\rangle\big| > \delta' \right). \tag{7.1.14}$$

By assumption (1.1.5), the initial density f^0 is bounded, up to a multiplicative constant C_0, by the Maxwellian M_β (uniformly distributed in x). By modifying the weights $W^{\varepsilon 0}_N$ in (1.1.6), we deduce that the probability of any event $\mathcal{A}$ under $\mathbb{P}_\varepsilon$ can be bounded from above in terms of the probability $\tilde{\mathbb{P}}_\varepsilon$ with initial density M_β (its expectation is denoted by $\tilde{\mathbb{E}}_\varepsilon$)

$$\mathbb{P}_\varepsilon(\mathcal{A}) \leq \frac{\tilde{\mathcal{Z}}^\varepsilon}{\mathcal{Z}^\varepsilon} \tilde{\mathbb{E}}_\varepsilon(C_0^{\mathcal{N}}\, 1_{\mathcal{A}}) \leq \frac{\tilde{\mathcal{Z}}^\varepsilon}{\mathcal{Z}^\varepsilon} \tilde{\mathbb{E}}_\varepsilon(C_0^{2\mathcal{N}})^{\frac12}\; \tilde{\mathbb{E}}_\varepsilon(1_{\mathcal{A}})^{\frac12} \leq \exp(C\mu_\varepsilon)\; \tilde{\mathbb{P}}_\varepsilon(\mathcal{A})^{\frac12},$$

for some constant C and $\tilde{\mathcal{Z}}^\varepsilon$ stands for the partition function of this new density. Using the fact that the probability $\tilde{\mathbb{P}}_\varepsilon$ is time invariant, we can reduce the estimate of the events in (7.1.14) to a single time interval. Thus (7.1.13) will follow if one can show that

$$\forall \delta' > 0\,, \qquad \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{\mu_\varepsilon} \log \tilde{\mathbb{P}}_\varepsilon\left(\sup_{t,s\in[0,2\delta]} \big|\langle \pi^\varepsilon_t, h\rangle - \langle \pi^\varepsilon_s, h\rangle\big| > \delta' \right) = -\infty. \tag{7.1.15}$$

By the Markov inequality and using the notation $L_\delta = \log|\log\delta|$, we get

(7.1.16)

$$\tilde{\mathbb{P}}_\varepsilon\left(\sup_{t,s\in[0,2\delta]} \big|\langle \pi^\varepsilon_t, h\rangle - \langle \pi^\varepsilon_s, h\rangle\big| > \delta' \right) \leq e^{-\delta'\, L_\delta\, \mu_\varepsilon} \tilde{\mathbb{E}}_\varepsilon\Big(\exp\Big(\sup_{t,s\in[0,2\delta]} L_\delta \Big| \sum_{i=1}^{\mathcal{N}} h\big(\mathbf{z}^\varepsilon_i(t)\big) - h\big(\mathbf{z}^\varepsilon_i(s)\big)\Big|\Big)\Big)$$
$$\leq e^{-\delta'\, L_\delta\, \mu_\varepsilon} \tilde{\mathbb{E}}_\varepsilon\Big(\exp\Big(\sum_{i=1}^{\mathcal{N}} \sup_{t,s\in[0,2\delta]} L_\delta \big|h\big(\mathbf{z}^\varepsilon_i(t)\big) - h\big(\mathbf{z}^\varepsilon_i(s)\big)\big|\Big)\Big).$$

The last inequality is very crude, but it is enough for the large deviation asymptotics and it allows us to reduce to a sum of functions depending only on the trajectory of each particle via

$$\tilde{h}\big(z([0,2\delta])\big) := \sup_{t,s\in[0,2\delta]} L_\delta \left| h\big(z(t)\big) - h\big(z(s)\big)\right|.$$

Thanks to Proposition 2.1.3, the last expectation in (7.1.16) can be rewritten in terms of the cumulants

$$\frac{1}{\mu_\varepsilon}\log\tilde{\mathbb{E}}_\varepsilon\left(\exp\Big(\sum_{i=1}^{\mathcal{N}}\tilde{h}\big(\mathbf{z}_i^\varepsilon([0,2\delta])\big)\Big)\right) = \sum_{n=1}^{\infty}\frac{1}{n!}\Big|\tilde{f}^\varepsilon_{n,[0,2\delta]}\Big(\big(\exp(\tilde{h})-1\big)^{\otimes n}\Big)\Big|, \tag{7.1.17}$$

where $\tilde{f}^\varepsilon_n$ stands for the dynamical cumulant under the new distribution.

For $n \geq 2$, the statement 1 of Theorem 9 page 95 can be applied

$$\Big|\tilde{f}^\varepsilon_{n,[0,2\delta]}\Big(\big(\exp(\tilde{h})-1\big)^{\otimes n}\Big)\Big| \leq n!\big(C(2\delta+\varepsilon)\big)^{n-1}\,|\log\delta|^{2n\|h\|_\infty},$$

with $L_\delta = \log|\log\delta|$. The term $n=1$ is controlled thanks to the statement 3 of Theorem 9

$$\Big|\tilde{f}^\varepsilon_{1,[0,2\delta]}\big(\exp(\tilde{h})-1\big)\Big| \leq \delta\left(\|\nabla h\|_\infty L_\delta + 1\right)e^{L_\delta\|h\|_\infty} \leq \delta\left(\|v\cdot\nabla_x h\|_\infty L_\delta + 1\right)|\log\delta|^{\|h\|_\infty}.$$

Thus (7.1.17) converges to 0 as $\varepsilon\to 0$, then δ tends to 0. Furthermore L_δ diverges to ∞ as δ vanishes, one deduces from (7.1.16) that (7.1.15) holds for any $\delta' > 0$. This completes the proof of (7.1.13) and therefore of Proposition 7.1.1. □

7.2. Identification of the large deviation functionals $\mathcal{F} = \widehat{\mathcal{F}}$

In this section, we are going to identify, for some time $T>0$, the functional $\mathcal{F}(T)$ obtained from the mechanical particle system in (7.0.3) with the large deviation functional (1.4.1) derived from stochastic collision processes

$$\begin{aligned}\widehat{\mathcal{F}}(T,\varphi) &= \widehat{\mathcal{F}}(0,\varphi_0) + \sup_p\left\{\int_0^T ds\int_{\mathbb{D}} dz\, p(s,z)\,D\varphi(s,z) - \mathcal{H}\big(\varphi(s),p(s)\big)\right\}\\ &= \widehat{\mathcal{F}}(0,\varphi_0) + \sup_p\left\{\big\langle\!\big\langle p, D\varphi\big\rangle\!\big\rangle - \int_0^T ds\,\mathcal{H}\big(\varphi(s),p(s)\big)\right\},\end{aligned} \tag{7.2.1}$$

where the supremum is taken over measurable functions p with at most a quadratic growth in v (which is the natural extension of Equation (1.14) in [**38**] to the case of unbounded velocities), and the Hamiltonian is given by

$$\mathcal{H}(\varphi,p) := \frac{1}{2}\int d\mu(z_1,z_2,\omega)\varphi(z_1)\varphi(z_2)\big(\exp\big(\Delta p\big)-1\big), \tag{7.2.2}$$

with $d\mu$ as in (6.0.4). Recall that $\widehat{\mathcal{F}}(0,\cdot)$ stands for the large deviation functional on the initial data

$$\widehat{\mathcal{F}}(0,\varphi_0) = \int dz\,\left(\varphi_0\log\left(\frac{\varphi_0}{f^0}\right) - \varphi_0 + f^0\right). \tag{7.2.3}$$

Note that at equilibrium, a derivation of large deviations by using cluster expansion can be found in [**40**] for a larger range of densities.

Given $T\in]0,T^\star]$, define $\mathcal{R}$ as in (7.1.5) on the time interval $[0,T]$, and denote by $\hat{\mathcal{R}}$ the set of densities φ such that the supremum in (7.2.1) is reached for some function $p\in\mathbb{B}$. We have the following identification.

Proposition 7.2.1. — *There exists $T \in]0, T^\star]$ such that for any positive function φ in $\mathcal{R} \cap \hat{\mathcal{R}}$, there holds $\mathcal{F}(T, \varphi) = \widehat{\mathcal{F}}(T, \varphi)$.*

We briefly explain the strategy of the proof. The main step is to identify the functional $\mathcal{I}(t)$ defined in (7.0.2) for any t in $[0, T]$ with the Legendre transform $\widehat{\mathcal{I}}$ of the large deviation functional $\widehat{\mathcal{F}}(t)$ defined in (7.2.1). Indeed, for any (real) test function $g \in \mathbb{B}$, we expect that $\widehat{\mathcal{I}}(t, g)$ coincides with the solution of the following variational problem :

$$
\begin{aligned}
&\sup_{\varphi} \Big\{ \langle g_t, \varphi_t \rangle - \big\langle\!\big\langle Dg, \varphi \big\rangle\!\big\rangle - \widehat{\mathcal{F}}(t, \varphi) \Big\} \\
&\qquad = \sup_{\varphi} \inf_{p} \left\{ \langle g_t, \varphi_t \rangle - \big\langle\!\big\langle Dg, \varphi \big\rangle\!\big\rangle - \big\langle\!\big\langle p, D\varphi \big\rangle\!\big\rangle + \int_0^t ds\, \mathcal{H}\big(\varphi(s), p(s)\big) - \widehat{\mathcal{F}}(0, \varphi_0) \right\}, \\
&\qquad = \sup_{\varphi} \inf_{p} \left\{ \langle g_t - p_t, \varphi_t \rangle + \langle p_0, \varphi_0 \rangle - \widehat{\mathcal{F}}(0, \varphi_0) + \big\langle\!\big\langle Dp - Dg, \varphi \big\rangle\!\big\rangle + \int_0^t ds\, \mathcal{H}\big(\varphi(s), p(s)\big) \right\},
\end{aligned}
\tag{7.2.4}
$$

where the supremum is taken over positive trajectories $\varphi \in D([0, T^\star], \mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d))$, and $\widehat{\mathcal{F}}(t)$ was replaced by its variational expression (7.2.1) in the second equation.

Using the variational principle (7.2.4), we are going to construct a functional $\widehat{\mathcal{I}}(t, g)$ and show that it coincides with $\mathcal{I}(t, g)$. This identification will rely on the fact that both $\mathcal{I}(t, g)$ and $\widehat{\mathcal{I}}(t, g)$ satisfy the Hamilton-Jacobi equation as derived in Theorem 7. The difficulty is that the uniqueness result stated in Proposition 5.4.2 holds in the setting of functionals defined on complex valued functions g. The main issue in Section 7.2.1 is therefore to provide a definition of $\widehat{\mathcal{I}}$ corresponding to the variational problem (7.2.4), and which can be extended to complex valued functions.

In Section 7.2.2, we then deduce from this first step that the functionals $\mathcal{F}(T, \varphi)$ and $\widehat{\mathcal{F}}(T, \varphi)$ coincide for positive $\varphi \in \mathcal{R} \cap \hat{\mathcal{R}}$.

7.2.1. A variational characterization of the functional $\mathcal{I}$. — Our starting point here is the variational principle (7.2.4) which we rewrite formally in terms of the functions $\psi_s = \varphi_s \exp(-p_s)$ and $\eta_s = \exp(p_s)$ for $s \in [0, t]$. Setting $\gamma = \exp(g_t)$, and $D_s g = \phi_s$, the last expression in (7.2.4) becomes

$$
\begin{aligned}
\sup_{\psi} \inf_{\eta} \Big\{ &\langle \log \gamma - \log \eta_t, \psi_t \eta_t \rangle + \langle \log \eta_0, \psi_0 \eta_0 \rangle - \widehat{\mathcal{F}}(0, \psi_0 \eta_0) - \big\langle\!\big\langle \phi, \psi\eta \big\rangle\!\big\rangle + \big\langle\!\big\langle D\eta, \psi \big\rangle\!\big\rangle \\
&+ \frac{1}{2} \int_0^t ds \int d\mu(z_1, z_2, \omega) \psi_s(z_1) \psi_s(z_2) \Big(\eta_s(z_1') \eta_s(z_2') - \eta_s(z_1) \eta_s(z_2) \Big) \Big\}.
\end{aligned}
\tag{7.2.5}
$$

The Euler-Lagrange equations associated with this variational problem are given for any $s \in [0, t]$ by

$$
\begin{aligned}
D_s \psi &= -\psi_s \phi_s + \int d\mu_{z_1}(z_2, \omega)\, \eta_s(z_2) \Big(\psi_s(z_1') \psi_s(z_2') - \psi_s(z_1) \psi_s(z_2) \Big) \quad \text{with} \quad \psi_0 = f^0, \\
D_s \eta &= \eta_s \phi_s - \int d\mu_{z_1}(z_2, \omega)\, \psi_s(z_2) \Big(\eta_s(z_1') \eta_s(z_2') - \eta_s(z_1) \eta_s(z_2) \Big) \quad \text{with} \quad \eta_t = \gamma\,,
\end{aligned}
\tag{7.2.6}
$$

recalling notation (5.5.5).

We stress the fact that the evolution of η is constrained by a final time condition.

Plugging these solutions in (7.2.5), we expect that the variational principle (7.2.5) is formally equivalent to

$$\begin{aligned}\widehat{\mathcal{J}}(t,\phi,\gamma) =&\langle f^0,(\eta_0-1)\rangle - \big\langle\!\big\langle \phi,\psi\eta\big\rangle\!\big\rangle + \big\langle\!\big\langle D\eta,\psi\big\rangle\!\big\rangle \\ &+\frac{1}{2}\int_0^t ds\int d\mu(z_1,z_2,\omega)\psi_s(z_1)\psi_s(z_2)\Big(\eta_s(z_1')\eta_s(z_2')-\eta_s(z_1)\eta_s(z_2)\Big),\end{aligned} \tag{7.2.7}$$

where we used that $\log\gamma = \log\eta_t$ and the explicit form (7.2.3) of $\widehat{\mathcal{F}}(0)$ to simplify the term at the initial time.

Since $\gamma = \exp(g_t)$ and $D_s g = \phi_s$, we could have simply defined $\widehat{\mathcal{I}}(t,g) = \widehat{\mathcal{J}}(t,\phi,\gamma)$ as our goal is to identify $\widehat{\mathcal{I}}(t,g)$ and $\mathcal{I}(t,g)$ defined in (7.0.2). However, this identification will rely on the uniqueness of the Hamilton-Jacobi derived in Proposition 5.4.2 and this requires to consider the general functional $\widehat{\mathcal{J}}(t,\phi,\gamma)$ with complex functions $(\phi,\gamma)\in\mathcal{B}_{\alpha,\beta,t}$.

Remark 7.2.2. — *Note that the computations leading to the definition of $\widehat{\mathcal{I}}$ are formal, and involve quantities which make sense only if ψ and η are positive functions (for instance $\log\eta_t$). However the final formula* (7.2.7) *is well defined for any complex functions $(\phi,\gamma)\in\mathcal{B}_{\alpha,\beta,t}$.*

For any $\beta\in\mathbb{R}$, define the norm

$$\|\Upsilon\|_\beta := \sup_{x,v}\Big(\exp\Big(-\frac{\beta}{4}|v|^2\Big)\ |\Upsilon(x,v)|\Big), \tag{7.2.8}$$

and denote by L^∞_β the corresponding functional space. The next lemma provides conditions on (ϕ,γ) to control the solutions of (7.2.6) and $\widehat{\mathcal{J}}(t,\phi,\gamma)$.

Lemma 7.2.3. — *There exists a time $T\in]0,T^\star]$ such that for any $(\phi,\gamma)\in\mathcal{B}_{\alpha_0,\beta_0,T}$, there is a unique solution to the system of equations* (7.2.6) *on $[0,T]$ with $\psi\in L^\infty_{-3\beta_0/2}$ and $\eta\in L^\infty_{5\beta_0/4}$. For any $t\in[0,T]$, the functional $\widehat{\mathcal{J}}(t,\phi,\gamma)$ depends analytically on γ and there holds*

$$\frac{\partial\widehat{\mathcal{J}}}{\partial\gamma}(t,\phi,\gamma)=\psi_t\,. \tag{7.2.9}$$

Furthermore estimates (5.4.2) *and* (5.4.3) *hold for $\widehat{\mathcal{J}}$.*

Proof. — We start by rewriting (7.2.6) in a mild form, denoting S_t the transport operator in $\mathbb{D}$

$$\begin{aligned}\psi(s) &= S_s f^0 + \int_0^s S_{s-\sigma}F_1(\phi_\sigma,\eta_\sigma,\psi_\sigma)d\sigma\,,\\ \eta(s) &= S_{s-t}\gamma - \int_s^t S_{s-\sigma}F_2(\phi_\sigma,\eta_\sigma,\psi_\sigma)d\sigma\,,\end{aligned} \tag{7.2.10}$$

with

$$\begin{aligned}F_1(\phi,\eta,\psi) &= -\psi\,\phi + \int d\mu_{z_1}(z_2,\omega)\,\eta(z_2)\Big(\psi(z_1')\psi(z_2')-\psi(z_1)\psi(z_2)\Big)\,,\\ F_2(\phi,\eta,\psi) &= \eta\,\phi - \int d\mu_{z_1}(z_2,\omega)\,\psi(z_2)\Big(\eta(z_1')\eta(z_2')-\eta(z_1)\eta(z_2)\Big)\,.\end{aligned}$$

Note that, since this is a coupled system and η satisfies a backward equation, this is not exactly the standard formulation to apply a Cauchy-Kowalewski argument. Nevertheless, this is still the right form to apply a fixed point argument provided that we find suitable functional spaces to encode the loss

continuity estimates on F_1 and F_2. Using the fact that $(\phi, \gamma) \in \mathcal{B}_{\alpha,\beta_0,T}$, we indeed have in particular that

$$|\phi(x,v)| \leq C(1+|v|^2) \quad \text{and} \quad \gamma \in L^\infty_{\beta_0}\,.$$

Recall moreover that f^0 belongs to $L^\infty_{-2\beta_0}$, so let us define

$$\bar{C} := 2\left(\|\gamma\|_{\beta_0} + \|f^0\|_{-2\beta_0}\right)\,.$$

Now there are constants C_1 and C_2 such that for any $\beta_1 > \beta_1' \geq 5\beta_0/4 > 0$ and $\beta_2 < \beta_2' \leq 3\beta_0/2$ there holds

$$\|F_1(\phi,\eta,\psi)\|_{-\beta_1'} \leq \frac{C_1\beta_0}{\beta_1-\beta_1'}\|\psi\|_{-\beta_1}\Big(1+\|\psi\|_{-\beta_1}\|\eta\|_{5\beta_0/4}\Big)\,, \tag{7.2.11}$$

$$\|F_2(\phi,\eta,\psi)\|_{\beta_2'} \leq \frac{C_2\beta_0}{\beta_2'-\beta_2}\|\eta\|_{\beta_2}\Big(1+\|\psi\|_{-3\beta_0/2}\|\eta\|_{\beta_2}\Big)\,. \tag{7.2.12}$$

By Theorem 8 in Chapter 6, we infer from (7.2.11) that as long as $\sup_{t\in[0,T]}\|\eta(t)\|_{5\beta_0/4} \leq \bar{C}$, there is $\bar{C}_1$ such that

$$\sup_{\substack{\rho<1,\\ t<\bar{C}_1(1-\rho)}} \|\psi(t)\|_{-2\beta_0\rho}\big(1-\frac{t}{\bar{C}_1(1-\rho)}\big) \leq \bar{C}\,.$$

In the same way, provided that $\sup_{t\in[0,T]}\|\psi(t)\|_{-3\beta_0/2} \leq \bar{C}$, (7.2.12) provides using the backward equation on η, that

$$\sup_{\substack{\rho<1,\\ s<\bar{C}_2(1-\rho)}} \|\eta(T-s)\|_{\beta_0(2-\rho)}\big(1-\frac{s}{\bar{C}_2(1-\rho)}\big) \leq \bar{C}\,.$$

Therefore, choosing $T < \frac{1}{4}\min(\bar{C}_1,\bar{C}_2)$, and applying a fixed point argument, we find that there exists a unique solution (ψ,η) to (7.2.10), satisfying

$$\sup_{t\in[0,T]}\|\eta(t)\|_{5\beta_0/4} \leq \bar{C}\,, \quad \sup_{t\in[0,T]}\|\psi(t)\|_{-3\beta_0/2} \leq \bar{C}\,.$$

For convenience we can assume that $T \leq T^\star$.

We turn now to the proof of (7.2.9). Since the solution ψ,η to the Euler-Lagrange equations is obtained as a fixed point of a contracting (polynomial) map depending linearly on γ (see (7.2.10)), it is straightforward to check that ψ,η depends analytically on γ (for instance using the iterated Duhamel series expansion). Using the symmetry

$$\int d\mu(z_1,z_2,\omega)\psi_s(z_1)\psi_s(z_2)\Big(\eta_s(z_1')\eta_s(z_2') - \eta_s(z_1)\eta_s(z_2)\Big)$$
$$= \int d\mu(z_1,z_2,\omega)\eta_s(z_1)\eta_s(z_2)\Big(\psi_s(z_1')\psi_s(z_2') - \psi_s(z_1)\psi_s(z_2)\Big)$$

one gets

(7.2.13)

$$\begin{aligned}\langle\frac{\partial\widehat{\mathcal{J}}}{\partial\gamma}(t,\phi,\gamma),\delta\gamma\rangle &= \langle f^0,\delta\eta_0\rangle - \big\langle\!\big\langle \phi,\delta\psi\,\eta+\psi\,\delta\eta\big\rangle\!\big\rangle + \big\langle\!\big\langle D\delta\eta,\psi\big\rangle\!\big\rangle + \big\langle\!\big\langle D\eta,\delta\psi\big\rangle\!\big\rangle\\ &+\frac{1}{2}\int_0^t ds\int d\mu(z_1,z_2,\omega)\Big(\delta\psi_s(z_1)\psi_s(z_2)+\psi_s(z_1)\delta\psi_s(z_2)\Big)\Big(\eta_s(z_1')\eta_s(z_2')-\eta_s(z_1)\eta_s(z_2)\Big)\\ &+\frac{1}{2}\int_0^t ds\int d\mu(z_1,z_2,\omega)\Big(\delta\eta_s(z_1)\eta_s(z_2)+\eta_s(z_1)\delta\eta_s(z_2)\Big)\Big(\psi_s(z_1')\psi_s(z_2')-\psi_s(z_1)\psi_s(z_2)\Big),\end{aligned}$$

where $\delta\psi, \delta\eta$ stand for the variations of the solutions of (7.2.6) when γ changes. Recall that $\psi_0 = f^0$ and $\delta\eta_t = \delta\gamma$, so that

$$\langle\!\langle D\delta\eta, \psi\rangle\!\rangle = -\langle\!\langle \delta\eta, D\psi\rangle\!\rangle + \langle \delta\gamma, \psi_t\rangle - \langle f^0, \delta\eta_0\rangle. \tag{7.2.14}$$

Thus (7.2.13) simplifies

$$\begin{aligned}\langle \frac{\partial\widehat{\mathcal{J}}}{\partial\gamma}(t,\phi,\gamma), \delta\gamma\rangle = &-\langle\!\langle \phi, \delta\psi\,\eta + \psi\,\delta\eta\rangle\!\rangle + \langle\!\langle D\eta, \delta\psi\rangle\!\rangle - \langle\!\langle \delta\eta, D\psi\rangle\!\rangle + \langle\delta\gamma, \psi_t\rangle \\ &+ \frac{1}{2}\int_0^t ds \int d\mu(z_1,z_2,\omega)\Big(\delta\psi_s(z_1)\psi_s(z_2) + \psi_s(z_1)\delta\psi_s(z_2)\Big)\Big(\eta_s(z_1')\eta_s(z_2') - \eta_s(z_1)\eta_s(z_2)\Big) \\ &+ \frac{1}{2}\int_0^t ds \int d\mu(z_1,z_2,\omega)\Big(\delta\eta_s(z_1)\eta_s(z_2) + \eta_s(z_1)\delta\eta_s(z_2)\Big)\Big(\psi_s(z_1')\psi_s(z_2') - \psi_s(z_1)\psi_s(z_2)\Big) \\ &= \langle\delta\gamma, \psi_t\rangle,\end{aligned}$$

where the last equality follows from the Euler-Lagrange equations (7.2.6). Lemma 7.2.3 is proved. □

Proposition 7.2.4. — *Let T be as in Proposition* 5.4.2 *and Lemma* 7.2.3*, then the functional $\widehat{\mathcal{J}}$ introduced in* (7.2.7) *satisfies the Hamilton-Jacobi equation* (5.3.5) *in the time interval* $[0,T]$

$$\partial_t\widehat{\mathcal{J}}(t,\phi,\gamma_t) = \frac{1}{2}\int \frac{\partial\widehat{\mathcal{J}}}{\partial\gamma}(t,\phi,\gamma_t)(z_1)\frac{\partial\widehat{\mathcal{J}}}{\partial\gamma}(t,\phi,\gamma_t)(z_2)\Big(\gamma_t(z_1')\gamma_t(z_2') - \gamma_t(z_1)\gamma_t(z_2)\Big)d\mu(z_1,z_2,\omega)\,, \tag{7.2.15}$$

for all $(\phi,\gamma_T) \in \mathcal{B}_{\alpha_0,\beta_0,T}$ with $\gamma \in C^0\big([0,T]\times\mathbb{D};\mathbb{C}\big)$ defined by $D_t\gamma_t - \phi_t\gamma_t = 0$ for $t \le T$ as in (5.3.4)*. By uniqueness of the Hamilton-Jacobi equation (Proposition* 5.4.2*), this implies that $\mathcal{J}(t,\phi,\gamma_t) = \widehat{\mathcal{J}}(t,\phi,\gamma_t)$ for all $t \le T$ and $(\phi,\gamma_t) \in \mathcal{B}_{\alpha_0,\beta_0,t}$.*

Proof of Proposition 7.2.4. — We split the proof in two parts.

Step 1 : Derivation of the Hamilton-Jacobi equation (7.2.15)**.**

Taking the time derivative of (7.2.7), we get two types of terms, those coming from the explicit dependence in t (appearing in the bounds of the integrals), and those coming from the variations of the solutions of (7.2.6) when the time interval changes from $[0,t]$ to $[0,t+dt]$. The same computations as in (7.2.13)-(7.2.14) show that the second contribution is $\langle\delta\eta_t, \psi_t\rangle$, so that

$$\begin{aligned}\partial_t\widehat{\mathcal{J}}(t,\phi,\gamma) = &\langle\delta\eta_t, \psi_t\rangle - \langle\phi_t, \psi_t\eta_t\rangle + \langle D\eta_t, \psi_t\rangle \\ &+ \frac{1}{2}\int d\mu(z_1,z_2,\omega)\psi_t(z_1)\psi_t(z_2)\Big(\eta_t(z_1')\eta_t(z_2') - \eta_t(z_1)\eta_t(z_2)\Big).\end{aligned} \tag{7.2.16}$$

Formula (7.2.16) simplifies thanks to the Euler-Lagrange equations (7.2.6)

$$\partial_t\widehat{\mathcal{J}}(t,\phi,\gamma) = \langle\delta\eta_t, \psi_t\rangle - \frac{1}{2}\int d\mu(z_1,z_2,\omega)\psi_t(z_1)\psi_t(z_2)\Big(\eta_t(z_1')\eta_t(z_2') - \eta_t(z_1)\eta_t(z_2)\Big).$$

By construction, the variation of the solutions at time t is

$$\delta\eta_t = \partial_t\gamma - \partial_t\eta = D_t\gamma - D_t\eta\,,$$

as the boundary condition implies that $\eta_t = \gamma$. Since $\partial_t\gamma = \phi_t\gamma_t$, we find

$$\delta\eta_t = D_t\gamma - D_t\eta = \int dv_2 d\omega\,\delta_{x_1=x_2}\left((v_1-v_2)\cdot\omega\right)_+\psi_t(z_2)\Big(\eta_t(z_1')\eta_t(z_2') - \eta_t(z_1)\eta_t(z_2)\Big).$$

As consequence, we recover the Hamilton-Jacobi equation (7.2.15).

Step 2 : Identifying the functionals $\widehat{\mathcal{J}}(t)$ and $\mathcal{J}(t)$.

At time 0, one can check that both functionals are equal. Indeed one has both $\psi_0 = f^0$ and $\eta_0 = \gamma$, from which we deduce that

$$\widehat{\mathcal{J}}(0,\phi,\gamma) = \langle f^0, (\gamma - 1)\rangle = \int dz f^0(z)\big(\gamma(z) - 1\big).$$

This coincides with $\mathcal{J}(0,\phi,\gamma)$ which can be obtained from a non-interacting gas of particles under the grand canonical measure.

By Lemma 7.2.3, the functional $\widehat{\mathcal{J}}(t,\phi,\gamma)$ depends analytically on γ. Both functionals satisfy the same Hamilton-Jacobi equation which has a unique solution (in the class of analytic functionals) by Proposition 5.4.2. This completes the claim that $\mathcal{J}(t,\phi,\gamma) = \widehat{\mathcal{J}}(t,\phi,\gamma)$ for $t \le T$. □

7.2.2. Identification of the large deviation functional. — We now turn to the proof of Proposition 7.2.1 and fix a positive density φ in $\mathcal{R} \cap \hat{\mathcal{R}}$.

By analogy with the definition (7.0.2) of $\mathcal{I}(t)$, we set for any $g \in \mathbb{B}$

$$\widehat{\mathcal{I}}(t,g) := \widehat{\mathcal{J}}\big(t, Dg, \exp(g(t))\big). \tag{7.2.17}$$

We start with a preliminary result.

Lemma 7.2.5. — *Let T be as in Lemma 7.2.3, then for any g in $\mathbb{B}$ and $t \le T$,*
— *the functions ψ, η in* (7.2.6) *associated with $(\phi,\gamma) = \big(Dg, \exp(g(t))\big)$ are both positive functions;*
— *there holds*

$$\frac{\partial \widehat{\mathcal{I}}(t,g)}{\partial g_0} = \psi_0\eta_0\,, \qquad \frac{\partial \widehat{\mathcal{I}}(t,g)}{\partial g} = D(\psi\eta). \tag{7.2.18}$$

Proof. — The first property is proved by rewriting (7.2.6) in the form

$$D_s\psi + \psi_s\Big(\phi_s + K_1(\psi,\eta)\Big) = \int dv_2 d\omega\, \delta_{x_1=x_2}\, \big((v_1 - v_2)\cdot\omega\big)_+ \eta_s(z_2)\psi_s(z_1')\psi_s(z_2')$$
$$\text{with } \psi_0 = f^0,$$
$$D_s\eta + \eta_s\Big(-\phi_s + K_2(\psi,\eta)\Big) = -\int dv_2 d\omega\, \delta_{x_1=x_2}\, \big((v_1 - v_2)\cdot\omega\big)_+ \psi_s(z_2)\eta_s(z_1')\eta_s(z_2')$$
$$\text{with } \eta_t = \gamma\,.$$

The first equation is a transport equation with a (nonlinear) damping term $\phi_s + K_1(\psi,\eta)$ and a source term which is nonnegative (as long as ψ,η are positive). It therefore preserves the positivity. The second equation is a backward transport equation with a damping term $-\phi_s + K_2(\psi,\eta)$ and a source term which is non positive (as long as ψ,η are positive). It also preserves the positivity. The solution (ψ,η) obtained by iteration (using the fixed point argument) is therefore positive.

Integrating by parts (7.2.7) for $\phi = Dg$, we get

$$\begin{aligned}\widehat{\mathcal{I}}(t,g) =& \big\langle\!\big\langle g, D(\psi\eta)\big\rangle\!\big\rangle - \langle g_t, \psi_t\eta_t\rangle + \langle g_0, \psi_0\eta_0\rangle + \langle f^0, (\eta_0 - 1)\rangle + \big\langle\!\big\langle D\eta, \psi\big\rangle\!\big\rangle \\ &+ \frac{1}{2}\int_0^t ds \int d\mu(z_1,z_2,\omega)\psi_s(z_1)\psi_s(z_2)\Big(\eta_s(z_1')\eta_s(z_2') - \eta_s(z_1)\eta_s(z_2)\Big).\end{aligned} \tag{7.2.19}$$

To identify the functional derivatives, we use the fact that ψ,η are the solutions of the Euler-Lagrange equation (7.2.6) for the variational problem (7.2.5). This means that the shifts in ψ or η due to a small variation of g will not affect $\widehat{\mathcal{I}}(t,g)$ at leading order. Thus (7.2.18) follows. □

We turn now to the identification of the large deviation functionals $\mathcal{F}$ and $\widehat{\mathcal{F}}$. Recall from Proposition 7.2.4 that $\mathcal{I}(T)$ and $\widehat{\mathcal{I}}(T)$ coincide on $\mathbb{B}$, thus the functional $\mathcal{F}(T)$ introduced in (7.0.3) can be rewritten as

$$\mathcal{F}(T,\varphi) = \sup_{u\in\mathbb{B}} \Big\{ \langle \varphi_T, u_T\rangle - \big\langle\!\!\big\langle \varphi, Du \big\rangle\!\!\big\rangle - \widehat{\mathcal{I}}(T,u) \Big\} = \sup_{u\in\mathbb{B}} \Big\{ \langle \varphi_0, u_0\rangle + \big\langle\!\!\big\langle D\varphi, u \big\rangle\!\!\big\rangle - \widehat{\mathcal{I}}(T,u) \Big\}.$$

First, we are going to derive an equation satisfied by φ. As φ belongs to $\mathcal{R}$, the supremum is reached for some function $g \in \mathbb{B}$

$$\mathcal{F}(T,\varphi) = \langle \varphi_0, g_0\rangle + \big\langle\!\!\big\langle D\varphi, g \big\rangle\!\!\big\rangle - \widehat{\mathcal{I}}(T,g), \tag{7.2.20}$$

which satisfies the condition

$$\frac{\partial \widehat{\mathcal{I}}(T,g)}{\partial g} = D\varphi \quad \text{and} \quad \frac{\partial \widehat{\mathcal{I}}(T,g)}{\partial g_0} = \varphi_0. \tag{7.2.21}$$

By (7.2.18), we also have that

$$\frac{\partial \widehat{\mathcal{I}}(T,g)}{\partial g} = D(\psi\eta) \quad \text{and} \quad \frac{\partial \widehat{\mathcal{I}}(T,g)}{\partial g_0} = \psi_0\eta_0,$$

from which we deduce that $\varphi = \psi\eta$ with ψ, η defined by (7.2.6). In particular, we deduce from (7.2.6)

$$D_s\varphi = D_s(\eta\psi) = \int d\mu_{z_1}(z_2,\omega) \Big(\eta_s(z_1)\eta_s(z_2)\psi_s(z_1')\psi_s(z_2') - \psi_s(z_1)\psi_s(z_2)\eta_s(z_1')\eta_s(z_2') \Big). \tag{7.2.22}$$

In the next step, we relate η to the Lagrange parameter associated with $\widehat{\mathcal{F}}$. As φ belongs to $\hat{\mathcal{R}}$, the supremum in the variational problem $\widehat{\mathcal{F}}(T,\varphi)$ is reached for a function p so that

$$\widehat{\mathcal{F}}(T,\varphi) = \widehat{\mathcal{F}}(0,\varphi_0) + \big\langle\!\!\big\langle p, D\varphi \big\rangle\!\!\big\rangle - \int_0^T ds\, \mathcal{H}\big(\varphi(s), p(s)\big), \tag{7.2.23}$$

and the corresponding Euler-Lagrange equation reads

$$D\varphi = \int dv_* d\nu \, \big((v-v_*)\cdot\nu\big)_+ \big(\varphi'\varphi_*' \exp(-\Delta p) - \varphi\varphi_* \exp(\Delta p)\big) \quad \text{with } \varphi_0 = f^0 \exp(p_0). \tag{7.2.24}$$

Since $\varphi > 0$, the Hamiltonian $\Delta p \mapsto \mathcal{H}(\varphi, p)$ introduced in (7.2.2) is strictly convex, any drifts p and $\bar{p}$ compatible with the evolution (7.2.24), satisfy $\Delta(p-\bar{p}) = 0$. Recall from Lemma 7.2.3 that η is a positive function. We then deduce from (7.2.22) that $\Delta(p_s - \log\eta_s) = 0$ at all times $s \in [0,T]$.

Finally, thanks to the identity $\Delta(p - \log\eta) = 0$, we are going to conclude that $\mathcal{F} = \widehat{\mathcal{F}}$. Using the following integration by parts in (7.2.19)

$$\big\langle\!\!\big\langle D\eta, \psi \big\rangle\!\!\big\rangle = -\big\langle\!\!\big\langle \log\eta, D(\eta\psi) \big\rangle\!\!\big\rangle + \langle \log\eta_t, \eta_t\psi_t\rangle - \langle \log\eta_0, \eta_0\psi_0\rangle,$$

and then replacing $\eta\psi$ by φ and $\log\eta$ by p, one gets

$$\begin{aligned}
\widehat{\mathcal{I}}(t,g) =& \big\langle\!\!\big\langle g - \log\eta, D\varphi \big\rangle\!\!\big\rangle + \langle \log\eta_t - g_t, \varphi_t\rangle + \langle g_0, \varphi_0\rangle + \langle f^0, (\eta_0 - 1)\rangle - \langle \log\eta_0, \varphi_0\rangle \\
&+ \frac{1}{2}\int_0^t ds \int d\mu(z_1,z_2,\omega)\psi_s(z_1)\psi_s(z_2)\Big(\eta_s(z_1')\eta_s(z_2') - \eta_s(z_1)\eta_s(z_2)\Big) \\
=& \langle g_0, \varphi_0\rangle + \big\langle\!\!\big\langle g - \log\eta, D\varphi \big\rangle\!\!\big\rangle + \int_0^T ds\, \mathcal{H}\big(\varphi_s, p_s\big) - \widehat{\mathcal{F}}(0,\varphi_0),
\end{aligned}$$

as $\log\eta_t = g_t$. In the last equality, $\widehat{\mathcal{F}}(0,\varphi_0)$ is recovered by (7.2.3) and the identity $\varphi_0 = \eta_0 f^0$. The symmetries of the equation (7.2.24) imply

$$\big\langle\!\!\big\langle \log\eta - p, D\varphi \big\rangle\!\!\big\rangle = \big\langle\!\!\big\langle \Delta\big(\log\eta - p\big), D\varphi \big\rangle\!\!\big\rangle = 0.$$

This leads to

$$\widehat{\mathcal{I}}(T,g) = \langle g_0, \varphi_0 \rangle + \langle\!\langle g - p, D\varphi \rangle\!\rangle + \int_0^T ds\, \mathcal{H}\big(\varphi_s, p_s\big) - \widehat{\mathcal{F}}(0, \varphi_0).$$

Applying (7.2.23), we conclude that

$$\widehat{\mathcal{I}}(T,g) = \langle g_0, \varphi_0 \rangle + \langle\!\langle g, D\varphi \rangle\!\rangle - \widehat{\mathcal{F}}(T, \varphi),$$

and combined with (7.2.20), we get $\mathcal{F}(T,\varphi) = \widehat{\mathcal{F}}(T,\varphi)$.

This completes the proof of Proposition 7.2.1. □

PART III

UNIFORM A PRIORI BOUNDS AND CONVERGENCE OF THE CUMULANTS

CHAPTER 8

CLUSTERING CONSTRAINTS AND CUMULANT ESTIMATES

In this chapter we consider the cumulants $f^{\varepsilon}_{n,[0,t]}(H^{\otimes n})$, whose definition (Eq. (4.4.1)) we recall:

$$f^{\varepsilon}_{n,[0,t]}(H^{\otimes n}) = \int dZ^*_n \mu_\varepsilon^{n-1} \sum_{\ell=1}^{n} \sum_{\lambda\in\mathcal{P}^\ell_n} \sum_{r=1}^{\ell} \sum_{\rho\in\mathcal{P}^r_\ell} \int \Big(\prod_{i=1}^{\ell} d\mu\big(\Psi^{\varepsilon}_{\lambda_i}\big) \mathcal{H}\big(\Psi^{\varepsilon}_{\lambda_i}\big) \mathbb{M}_{\lambda_i} \Big) \varphi_\rho \, f^{\varepsilon 0}_{\{1,\dots,r\}} \, . \tag{8.0.1}$$

We prove the upper bound stated in Theorem 4 page 35. We shall actually prove a more general statement, see Theorem 9 page 95.

The key idea behind this result is that the clustering structure of $f^{\varepsilon}_{n,[0,t]}(H^{\otimes n})$ imposes strong geometric constraints on the integration parameters $(Z^*_n, T_m, V_m, \Omega_m)$ (where we recall that m is the size of the collision tree), which imply that the integral defining $f^{\varepsilon}_{n,[0,t]}(H^{\otimes n})$ involves actually only a small measure set of parameters, of size $O(1/\mu_\varepsilon^{n-1})$. More precisely, what we prove is that:

— there are $n-1$ "independent" geometric constraints (clustering conditions) and each of them provides a small factor $O(1/\mu_\varepsilon)$;

— the integration measure (which is unbounded because of possibly large velocities in the collision cross-sections) does not induce any divergence.

Section 8.1 is devoted to characterizing the small measure set. Actually we only provide necessary conditions for the parameters $(Z^*_n, T_m, V_m, \Omega_m)$ to belong to such set (which is enough to get an upper bound). This characterization can be expressed as a succession of geometric conditions on the initial (at time t) relative positions of particles.

Section 8.2 then explains how to control the integral defining $f^{\varepsilon}_{n,[0,t]}(H^{\otimes n})$. Recall that, by (4.4.6) and by conservation of the energy,

$$|\mathcal{H}(\Psi^{\varepsilon}_n)| = |H_n\big(Z^*_n([0,t])\big)| \leq e^{\alpha_0 n + \frac{\beta_0}{4}|V^*_n(0)|^2 + \frac{\beta_0}{4}|V_m(0)|^2} \, .$$

Since the initial data satisfy a Gaussian bound

$$(f^0)^{\otimes n+m}(\Psi^{\varepsilon 0}_n) \leq C_0^{n+m} e^{-\frac{\beta_0}{2}|V^*_n(0)|^2 - \frac{\beta_0}{2}|V_m(0)|^2} \, ,$$

the growth of $|\mathcal{H}(\Psi^{\varepsilon}_n)|$ is easily controlled, so the main difficulty is to control the cross-sections

$$\mathcal{C}\big(\Psi^{\varepsilon}_n\big) := \prod_{k=1}^{m} s_k\Big(\big(v_k - v_{a_k}(t_k)\big)\cdot\omega_k\Big)_+ \tag{8.0.2}$$

present in the measure $d\mu\big(\Psi^{\varepsilon}_n\big)$. In order for this term not to create any divergence for large m, we need a symmetry argument as in the classical proof of Lanford, but intertwined here with the estimates on

the size of the small measure set. A similar procedure will be used in Section 8.1 to cure high energy singularities arising from the geometric constraints themselves.

8.1. Dynamical constraints

Let $\lambda \hookrightarrow \rho$ be a nested partition of $\{1^*, \dots, n^*\}$. We fix the velocities V_n^* at time t, as well as the collision parameters $(m, a, T_m, V_m, \Omega_m)$ of the pseudo-trajectories. We recall that $V_m = (v_1, \dots, v_m)$ where v_i is the velocity of particle i at the moment of its creation.

We denote by

$$\mathbb{V}^2 := (V_n^*)^2 + V_m^2 = \sum_{i=1}^{n} (v_i^*)^2 + \sum_{i=1}^{m} v_i^2$$

(twice) the total energy of the whole pseudo-trajectory Ψ_n^ε appearing in (8.0.1), and by $K = n+m$ its total number of particles. We also indicate by $\mathbb{V}_i^2$ (resp. $\mathbb{V}_\lambda^2$ for any $\lambda \subset \{1^*, \dots, n^*\}$) and K_i (resp. K_λ) the corresponding energy and number of particles of the collision tree with root at z_i^* (resp. Z_λ^*), that is:

$$\begin{aligned} \mathbb{V}_i^2 &= (v_i^*)^2 + \sum_{j \text{ created in } \Psi^\varepsilon_{\{i\}}} v_j^2 \,, \\ K_i &= 1 + \#\left(\text{particles created in } \Psi^\varepsilon_{\{i\}}\right) \end{aligned} \tag{8.1.1}$$

and

$$\begin{aligned} \mathbb{V}_\lambda^2 &= \sum_{i \text{ tree in } \lambda} \mathbb{V}_i^2 \,, \\ K_\lambda &= \sum_{i \text{ tree in } \lambda} K_i \,. \end{aligned} \tag{8.1.2}$$

Note that $\mathbb{V}^2 = \sum_{i=1}^n \mathbb{V}_i^2$ and $K = \sum_{i=1}^n K_i$.

In what follows, it will be important to remember the notations and definitions introduced in Chapter 4, as well as the rules of construction of pseudo-trajectories explained in Section 3.2. In particular we recall that, because of these rules, $\mathbb{V}^2/2$ is the energy at time zero of the configuration $\Psi_n^{\varepsilon 0}$, while $\mathbb{V}_i^2/2$ is not, in general, the energy of $\Psi_{\{i\}}^{\varepsilon 0}$ (because of external recollisions which can perturb the velocities of the particles inside the tree), unless $\Psi_{\{i\}}^\varepsilon$ does not recollide with the other $\Psi_{\{j\}}^\varepsilon$, $j \neq i$.

– *Clustering recollisions.* We first study the constraints associated with clustering recollisions in the pseudo-trajectory of the generic forest $\Psi_{\lambda_1}^\varepsilon$. Up to a renaming of the integration variables, we can assume that

$$\lambda_1 = \{1, \dots, \ell_1\} \,.$$

We call $z_{\lambda_1}^* := z_{\ell_1}^*$ the *root* of the forest.

By definition of $\mathbb{A}_{\lambda_1}$ and by Definition 4.4.2 of clustering recollisions, there exist $\ell_1 - 1$ clustering recollisions occurring at times $\tau_{\mathrm{rec},1} \geq \tau_{\mathrm{rec},2} \geq \cdots \geq \tau_{\mathrm{rec},\ell_1-1}$. Moreover, the corresponding chain of recolliding trees $\{j_1, j_1'\}, \dots, \{j_{\ell_1-1}, j_{\ell_1-1}'\}$ is a minimally connected graph $T \in \mathcal{T}_{\lambda_1}$, equipped with an

ordering of the edges. We shall denote by $T^{\prec}$ a minimally connected graph equipped with an ordering of edges, and by $\mathcal{T}_{\lambda_1}^{\prec}$ the set of all such graphs on λ_1. Hence we have

$$\mathbb{\Delta}_{\lambda_1} = \sum_{T^{\prec} \in \mathcal{T}_{\lambda_1}^{\prec}} \mathbb{\Delta}_{\lambda_1, T^{\prec}}$$

almost surely, where $\mathbb{\Delta}_{\lambda_1, T^{\prec}}$ is the indicator function that the clustering recollisions for the forest λ_1 are given by $T^{\prec}$. We also recall that, by definition, $\mathbb{\Delta}_{\lambda_1}$ is equal to zero whenever two particles find themselves at mutual distance strictly smaller than ε.

It will be convenient to represent the set of graphs $\mathcal{T}_{\lambda_1}^{\prec}$ in terms of sequences of merged subforests. The subforests are obtained following the dynamics of the pseudo-trajectory $\Psi_{\lambda_1}^{\varepsilon}$ backward in time, and putting together the groups of trees that recollide. An example is provided by Figure 3.

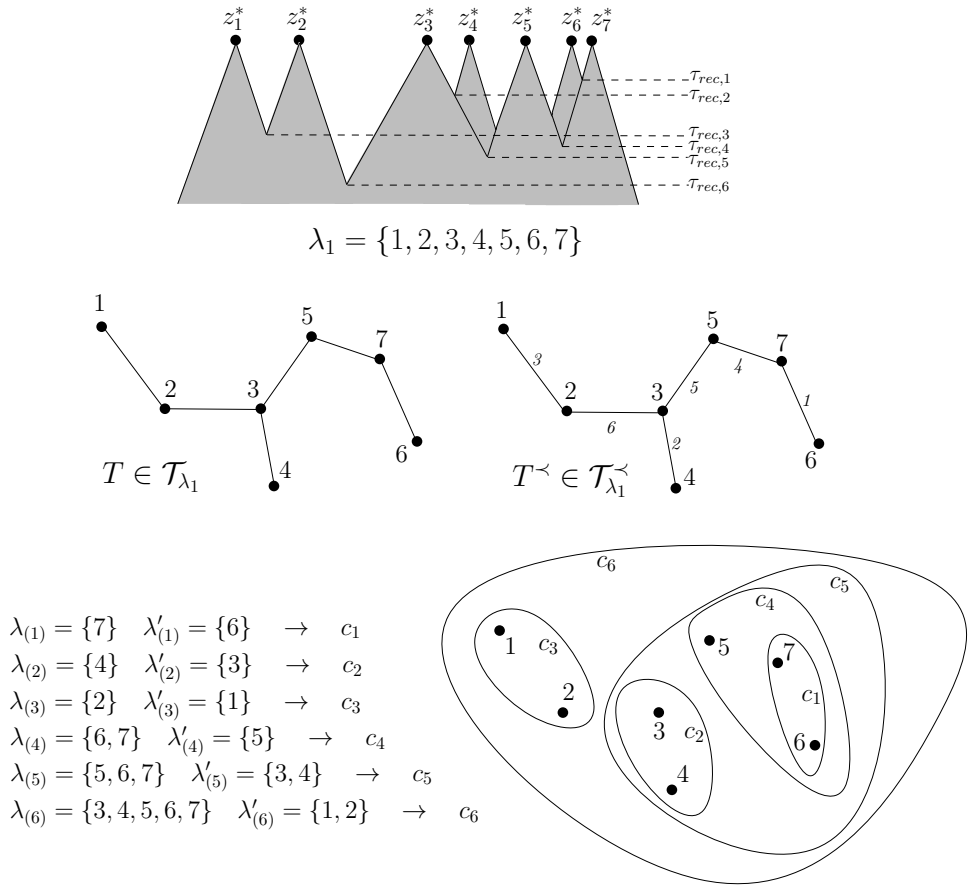


FIGURE 3. An example of pseudo-trajectory $\Psi_{\lambda_1}^{\varepsilon}$ ($\ell_1 = 7$) satisfying the constraint $\mathbb{\Delta}_{\lambda_1, T^{\prec}}$, together with its minimally connected graph T, ordered graph $T^{\prec}$, and sequence of *merged subforests* $\left(\lambda_{(k)}, \lambda'_{(k)}\right)_k$. The roots of the trees $z_i^* = (x_i^*, v_i^*)$ and the clustering recollision times appear in the picture on the top.

More precisely, we define the map

$$\mathcal{T}_{\lambda_1}^{\prec} \ni T^{\prec} \to \left(\lambda_{(k)}, \lambda'_{(k)}\right)_k$$

by the following iteration :

— start from $\lambda_1 = \{1, \dots, \ell_1\}$;
— take the first edge $\{j_1, j_1'\}$ of $T^\prec$, and set $\Big(\lambda_{(1)}, \lambda'_{(1)}\Big) = (\{j_1\}, \{j_1'\})$; these two elements are merged into a single cluster c_1; set $L_1 := c_1 \cup (\lambda_1 \setminus \{j_1, j_1'\})$;
— at step $k > 1$, take $\Big(\lambda_{(k)}, \lambda'_{(k)}\Big)$ of L_{k-1} in such a way that $j_k \in \lambda_{(k)}, j_k' \in \lambda'_{(k)}$ where $\{j_k, j_k'\}$ is the k-th edge of $T^\prec$, and merge them into a single cluster c_k; set $L_k := c_k \cup \Big(L_{k-1} \setminus \{\lambda_{(k)}, \lambda'_{(k)}\}\Big)$. We can assume without loss of generality that $\max \lambda'_{(k)} < \max \lambda_{(k)}$.

The last step is given by $\Big(\lambda_{(\ell_1-1)}, \lambda'_{(\ell_1-1)}\Big)$, which merges the two remaining clusters.

However this map is not a bijection, because the merged subforests do not specify which vertices of $j_k \in \lambda_{(k)}$ and $j_k' \in \lambda'_{(k)}$ are connected by the edge. A bijection is therefore given by

$$\mathcal{T}_{\lambda_1}^\prec \ni T^\prec \to \Big(\lambda_{(k)}, \lambda'_{(k)}, j_k \in \lambda_{(k)}, j_k' \in \lambda'_{(k)}\Big)_k \ . \tag{8.1.3}$$

We define the *root* of the subforest $\lambda_{(k)}$ by

$$z^*_{\lambda_{(k)}} := z^*_{\max \lambda_{(k)}} \ ,$$

and same definition for the root of $\lambda'_{(k)}$. We can then define

$$\hat{x}_k := x^*_{\lambda'_{(k)}} - x^*_{\lambda_{(k)}} \ , \qquad k = 1, \dots, \ell_1 - 1$$

as the relative position between the two recolliding subforests at time t. It is easy to see that, for any given root position $x^*_{\lambda_1} = x^*_{\ell_1} \in \mathbb{T}^d$, the map of translations

$$X^*_{\ell_1-1} = \big(x^*_1, \dots, x^*_{\ell_1-1}\big) \longrightarrow \hat{X}_{\ell_1-1} := (\hat{x}_1, \dots, \hat{x}_{\ell_1-1}) \tag{8.1.4}$$

is one-to-one on $\mathbb{T}^{d(\ell_1-1)}$ and such that

$$dX^*_{\ell_1-1} = d\hat{X}_{\ell_1-1} \ .$$

Thus (8.1.4) is a legitimate change of variables in (8.0.1).

Our purpose is to prove iteratively that, for $k = \ell_1 - 1, \dots, 1$, the variable $\hat{x}_k$ associated with the k-th clustering recollision has to be in a small set, the measure of which is uniformly small of size $O(1/\mu_\varepsilon)$.

We define $\Psi^\varepsilon_{\lambda_{(k)}}$ (respectively $\Psi^\varepsilon_{\lambda'_{(k)}}$) the pseudo-trajectory with starting particles $\lambda_{(k)}$ ($\lambda'_{(k)}$). Since $\tau_{\mathrm{rec,k}} \geq (\tau_{\mathrm{rec,s}})_{s>k}$, the collision trees in $\lambda_1 \setminus \Big(\lambda_{(k)} \cup \lambda'_{(k)}\Big)$ do not affect the subforests $\lambda_{(k)}, \lambda'_{(k)}$ in the time interval $(\tau_{\mathrm{rec,k}}, t)$. The clustering structure prescribed by $T^\prec$ implies that $\Psi^\varepsilon_{\lambda'_{(k)}}$ and $\Psi^\varepsilon_{\lambda_{(k)}}$, regarded as independent trajectories, reach mutual distance ε at some time $\tau_{\mathrm{rec,k}} \in (0, \tau_{\mathrm{rec,k}-1})$.

Given $(\hat{x}_s)_{s<k}$ fixed by the previous recollisions, we are going to vary $\hat{x}_k$ so that an external recollision between the subforests occurs. This corresponds to moving rigidly $\Psi^\varepsilon_{\lambda'_{(k)}}$ and $\Psi^\varepsilon_{\lambda_{(k)}}$ by acting on their relative distance $\hat{x}_k$. In fact, the recollision condition depends only on this distance.

Given a sequence of merged subforests $\Big(\lambda_{(k)}, \lambda'_{(k)}\Big)_k$ and a set of variables $(\hat{x}_s)_{s<k}$ (with $|\hat{x}_s| > \varepsilon$), the $k-$th clustering recollision condition is defined by

$$\hat{x}_k \in \mathcal{B}_k := \bigcup_{\substack{q \text{ in the subforest } \lambda_{(k)} \\ q' \text{ in the subforest } \lambda'_{(k)}}} B_{qq'} \ ,$$

with

$$B_{qq'} := \Big\{ \hat{x}_k \in \mathbb{T}^d \ \Big| \ |x_{q'}(\tau_{\mathrm{rec},k}) - x_q(\tau_{\mathrm{rec},k})| = \varepsilon \quad \text{for some } \tau_{\mathrm{rec,k}} \in (0, \tau_{\mathrm{rec},k-1}) \Big\} \ . \tag{8.1.5}$$

Here $x_q(\tau), x_{q'}(\tau)$ are the particle trajectories in the flows $\Psi^{\varepsilon}_{\lambda_{(k)}}, \Psi^{\varepsilon}_{\lambda'_{(k)}}$ (and τ is of course restricted to their existence times). In other words there exists a time $\tau_{\mathrm{rec,k}} \in (0, \tau_{\mathrm{rec,k-1}})$ and a vector $\omega_{\mathrm{rec,k}} \in \mathbb{S}^{d-1}$ such that

$$x_{q'}(\tau_{\mathrm{rec},k}) - x_q(\tau_{\mathrm{rec},k}) = \varepsilon\, \omega_{\mathrm{rec},k}\,. \tag{8.1.6}$$

The particle trajectories $x_q(\tau), x_{q'}(\tau)$ are piecewise affine (because there are almost surely a finite number of collisions and recollisions within the trees $\Psi^{\varepsilon}_{\lambda_{(k)}}, \Psi^{\varepsilon}_{\lambda'_{(k)}}$). Moreover, $(x_q(\tau) - x_{q'}(\tau)) - (x^*_{\lambda_{(k)}} - x^*_{\lambda'_{(k)}})$ does not depend on $\hat{x}_k := x^*_{\lambda'_{(k)}} - x^*_{\lambda_{(k)}}$, because all positions in the collision tree are translated rigidly. This means that $\hat{x}_k$ has to be in a tube of radius ε around the parametric curve $(x^*_{\lambda_{(k)}} - x^*_{\lambda'_{(k)}}) - (x_q(\tau) - x_{q'}(\tau))$. This tube is a union of cylinders, with two spherical caps at both ends (see Figure 4). Note however that we have to remove from this tube the sphere corresponding to the exclusion at the creation time (or at time t if q and q' exist up to time t).

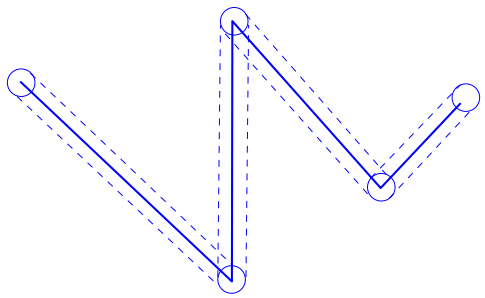


FIGURE 4. The tube $B_{qq'}$ leading to a recollision between particles q and q'. The tube has section μ_ε^{-1}.

Therefore

$$B_{qq'} = \bigcup_j B_{qq'}(\delta\tau_j)$$

for a suitable finite decomposition of $(0, \tau_{\mathrm{rec,k-1}})$ (depending on all the history). We therefore end up with the estimate (see Figure 4)

$$|B_{qq'}| \leq \frac{C}{\mu_\varepsilon} \sum_j |v_q^{(\delta\tau_j)} - v_{q'}^{(\delta\tau_j)}|\, |\delta\tau_j|$$

for some pure constant $C > 0$.

We sum now over all q, q' to obtain an estimate of the set $\mathcal{B}_k$. To exploit the conservation of energy, we exchange the sums over $\delta\tau_j$ and over q, q'. We get

$$|\mathcal{B}_k| \leq \frac{C}{\mu_\varepsilon} \sum_j |\delta\tau_j| \sum_{q,q'} |v_q^{(\delta\tau_j)} - v_{q'}^{(\delta\tau_j)}|\,.$$

Applying the Cauchy-Schwarz inequality, the sum over q, q' is bounded by

$$\sqrt{\sum_q \left(v_q^{(\delta\tau_j)}\right)^2} \sqrt{K_{\lambda_{(k)}}}\, K_{\lambda'_{(k)}} + \sqrt{\sum_{q'} \left(v_{q'}^{(\delta\tau_j)}\right)^2} \sqrt{K_{\lambda'_{(k)}}}\, K_{\lambda_{(k)}} \leq \mathbb{V}_{\lambda_{(k)}} \sqrt{K_{\lambda_{(k)}}}\, K_{\lambda'_{(k)}} + \mathbb{V}_{\lambda'_{(k)}} \sqrt{K_{\lambda'_{(k)}}}\, K_{\lambda_{(k)}}$$

where we use the notations for energy and mass of subforests introduced at the beginning of this section. In the above inequality, we have used the independence of $\Psi^{\varepsilon}_{\lambda_{(k)}}$ and $\Psi^{\varepsilon}_{\lambda'_{(k)}}$ on $[\tau_{\mathrm{rec},k}, t]$, and

bounded their energies in $\delta\tau_j$ with $\mathbb{V}_{\lambda_{(k)}}$ and $\mathbb{V}_{\lambda'_{(k)}}$ respectively (see Eq.s (8.1.1)-(8.1.2)). Therefore we infer that

$$\begin{aligned} |\mathcal{B}_k| &\leq \frac{C}{\mu_\varepsilon}\,\tau_{\mathrm{rec},k-1}\,\left(\mathbb{V}^2_{\lambda_{(k)}}+K_{\lambda_{(k)}}\right)\left(\mathbb{V}^2_{\lambda'_{(k)}}+K_{\lambda'_{(k)}}\right) \\ &= \frac{C}{\mu_\varepsilon}\,\tau_{\mathrm{rec},k-1}\sum_{\substack{j_k\in\lambda_{(k)}\\ j'_k\in\lambda'_{(k)}}}\left(\mathbb{V}^2_{j_k}+K_{j_k}\right)\left(\mathbb{V}^2_{j'_k}+K_{j'_k}\right)\,. \end{aligned} \tag{8.1.7}$$

In this way we have obtained an estimate which depends only on the energy and the number of particles enclosed in the trees $\Psi^\varepsilon_{\lambda_{(k)}}, \Psi^\varepsilon_{\lambda'_{(k)}}$.

Coming back to Equation (8.0.1) we observe that, if $\text{Δ}_{\lambda_1}=1$, then there exist merged subforests such that $\hat{x}_k\in\mathcal{B}_k$ for $k=\ell_1-1,\dots,1$. Hence, iterating the procedure leading to (8.1.7) for $k=\ell_1-1,\dots,1$, leads to an upper bound on the cost of the clustering recollisions in λ_1:

(8.1.8)

$$\begin{aligned} \int dX^*_{\ell_1-1}\,\text{Δ}_{\lambda_1}\,\mathbf{1}_{\mathcal{G}^\varepsilon}\left(\Psi^\varepsilon_{\lambda_1}\right) &\leq \sum_{\left(\lambda_{(k)},\lambda'_{(k)}\right)}\int d\hat{x}_1\mathbf{1}_{\mathcal{B}_1}\int d\hat{x}_2\dots\int d\hat{x}_{\ell_1-1}\mathbf{1}_{\mathcal{B}_{\ell_1-1}} \\ &\leq \left(\frac{C}{\mu_\varepsilon}\right)^{\ell_1-1}\int_0^t d\tau_{\mathrm{rec},1}\cdots\int_0^{\tau_{\mathrm{rec},\ell_1-2}} d\tau_{\mathrm{rec},\ell_1-1}\sum_{\left(\lambda_{(k)},\lambda'_{(k)}\right)}\sum_{\substack{j_k\in\lambda_{(k)}\\ j'_k\in\lambda'_{(k)}}}\prod_{k=1}^{\ell_1-1}\left(\mathbb{V}^2_{j_k}+K_{j_k}\right)\left(\mathbb{V}^2_{j'_k}+K_{j'_k}\right) \\ &= \left(\frac{Ct}{\mu_\varepsilon}\right)^{\ell_1-1}\frac{1}{(\ell_1-1)!}\sum_{\left(\lambda_{(k)},\lambda'_{(k)}\right)}\sum_{\substack{j_k\in\lambda_{(k)}\\ j'_k\in\lambda'_{(k)}}}\prod_{k=1}^{\ell_1-1}\left(\mathbb{V}^2_{j_k}+K_{j_k}\right)\left(\mathbb{V}^2_{j'_k}+K_{j'_k}\right)\,. \end{aligned}$$

Using the bijection (8.1.3) and compensating the $1/(\ell_1-1)!$ with the ordering of the edges in $T^\prec$, we rewrite this result as

$$\int dX^*_{\ell_1-1}\text{Δ}_{\lambda_1}\,\mathbf{1}_{\mathcal{G}^\varepsilon}\left(\Psi^\varepsilon_{\lambda_1}\right)\leq\left(\frac{Ct}{\mu_\varepsilon}\right)^{\ell_1-1}\sum_{T\in\mathcal{T}_{\lambda_1}}\prod_{\{j,j'\}\in E(T)}\left(\mathbb{V}^2_j+K_j\right)\left(\mathbb{V}^2_{j'}+K_{j'}\right)\,,$$

where $E(T)$ is the set of edges of T. Equivalently,

$$\int dX^*_{\ell_1-1}\text{Δ}_{\lambda_1}\,\mathbf{1}_{\mathcal{G}^\varepsilon}\left(\Psi^\varepsilon_{\lambda_1}\right)\leq\left(\frac{Ct}{\mu_\varepsilon}\right)^{\ell_1-1}\sum_{T\in\mathcal{T}_{\lambda_1}}\prod_{j\in\lambda_1}\left(\mathbb{V}^2_j+K_j\right)^{d_j(T)}\,, \tag{8.1.9}$$

where $d_j(T)$ is the degree of the vertex j in the graph T.

– *Clustering overlaps.* We are now going to estimate the constraints associated with clustering overlaps in the pseudo-trajectory of the generic jungle $\Psi^\varepsilon_{\rho_1}$. The argument is similar, but not identical, to the one just seen for clustering recollisions. Below we shall indicate the differences, without repeating the identical parts.

Up to a renaming of the summation variables, we can assume that

$$\rho_1=\{\lambda_1,\dots,\lambda_{r_1}\}\,.$$

We recall that each forest λ_i has a root $z^*_{\lambda_i}$, which did not play any role in the previous estimate of clustering recollisions. We call $z^*_{\rho_1}:=z^*_{\lambda_{r_1}}$ the *root* of the jungle.

By definition of φ_{ρ_1}, by Definition 4.4.1 of the clustering overlaps, there exist $r_1 - 1$ clustering overlaps, and the corresponding chain of overlapping forests $(\lambda_{j_1}, \lambda_{j'_1}), \cdots, (\lambda_{j_{r_1-1}}, \lambda_{j'_{r_1-1}})$ is a minimally connected graph $T \in \mathcal{T}_{\rho_1}$. Then, thanks to the tree inequality stated in Proposition 2.3.3,

$$|\varphi_{\rho_1}| \leq \sum_{T \in \mathcal{T}_{\rho_1}} \prod_{\{\lambda_j, \lambda_{j'}\} \in E(T)} \mathbf{1}_{\lambda_j \sim_o \lambda_{j'}} \,. \tag{8.1.10}$$

Note that, as mentioned in Section 4.4, we have more flexibility when dealing with overlaps than with recollisions, as $\left(\Psi^{\varepsilon}_{\lambda_j}\right)_{1 \leq j \leq r_1}$ are completely independent trajectories, whatever the ordering of the overlap times. We therefore have more freedom in choosing the integration variables.

To define the change of variables, we assign an ordering of the edges $E(T)$ in the following way. Consider $T \in \mathcal{T}_{\rho_1}$ as a rooted graph, with root λ_{ρ_1}. We start from the vertices of T which have the maximal depth, say $\bar{k}$ (the depth is defined as the number of edges connecting the vertex to the root). These vertices have degree 1, hence each one of the vertices identifies exactly one edge. We label these edges in such a way that they keep the same mutual order of the vertices, starting from the biggest one. We rename the ordered edges as $e_{r_1-1}, e_{r_1-2}, \ldots$. Next we prune the edges, obtaining a smaller minimally connected tree graph, on which we can repeat the labelling operation. We iterate this procedure $\bar{k}$ times, producing a complete ordering of edges $e_{r_1-1}, \ldots, e_1$.

Let us write $e_k = \{\lambda_{[k]}, \lambda'_{[k]}\}$ where $\lambda'_{[k]}$ has depth larger than $\lambda_{[k]}$. We can then define

$$\hat{x}_k := x^*_{\lambda'_{[k]}} - x^*_{\lambda_{[k]}} \,, \qquad k = 1, \ldots, r_1 - 1$$

as the relative position between the two overlapping forests at time t. As in the case of clustering recollisions, for any given root position $x^*_{\rho_1} := x^*_{\lambda_{r_1}} \in \mathbb{T}^d$, the map of translations

$$\left(x^*_{\lambda_1}, \ldots, x^*_{\lambda_{r_1-1}}\right) \longmapsto \hat{X}_{r_1-1} := (\hat{x}_1, \ldots, \hat{x}_{r_1-1}) \tag{8.1.11}$$

is one-to-one on $\mathbb{T}^{d(r_1-1)}$ and it has unit Jacobian determinant. Thus (8.1.11) is a legitimate change of variables in (8.0.1).

Given a graph $T \in \mathcal{T}_{\rho_1}$ and the corresponding sequence $\left(\lambda_{[k]}, \lambda'_{[k]}\right)_k$, the $k-$th clustering overlap condition is defined by

$$\hat{x}_k \in \tilde{\mathcal{B}}_k := \bigcup_{\substack{q \text{ in the forest } \lambda_{[k]} \\ q' \text{ in the forest } \lambda'_{[k]}}} \tilde{B}_{qq'} \,,$$

with

$$\tilde{B}_{qq'} = \left\{ \hat{x}_k \in \mathbb{T}^d \;\middle|\; \exists \tau \in [0,t] \quad \text{such that} \quad |x_q(\tau) - x_{q'}(\tau)| = \varepsilon \right\}$$

where we used (4.4.3), and $x_q(\tau), x_{q'}(\tau)$ are the particle trajectories in the flows $\Psi^{\varepsilon}_{\lambda_{[k]}}, \Psi^{\varepsilon}_{\lambda'_{[k]}}$. This set has small measure

$$|\tilde{\mathcal{B}}_k| \leq \frac{C}{\mu_\varepsilon} (t + \varepsilon) \left(\mathbb{V}^2_{\lambda_{[k]}} + K_{\lambda_{[k]}}\right) \left(\mathbb{V}^2_{\lambda'_{[k]}} + K_{\lambda'_{[k]}}\right) \tag{8.1.12}$$

for some constant $C > 0$. Notice that the correction of $O(\varepsilon)$ comes from the extremal spherical caps of the tubes in Figure 4 (since $\mathbf{1}_{\lambda_{[k]} \sim_o \lambda'_{[k]}} = 1$ inside those regions).

Remark 8.1.1. — *Note that overlaps can be classified in two types*

— *those arising at time t or involving a particle q at its creation time t_q : in this case, the distance between the overlapping particles at $\tau_{\rm ov}$ satisfies only the inequality*

$$|x_q(\tau_{\rm ov}) - x_{q'}(\tau_{\rm ov})| \leq \varepsilon\,.$$

This corresponds to one spherical end of the tube in Figure 4*;*

— *and the* regular *ones, for which the two overlapping particles are exactly at distance ε at $\tau_{\rm ov}$. We then have the same parametrization as for recollisions*

$$x_q(\tau_{\rm ov}) - x_{q'}(\tau_{\rm ov}) = \varepsilon\omega_{\rm ov}\,. \tag{8.1.13}$$

This corresponds to the tube in Figure 4 *minus the spherical end.*

The final result is thus:

$$\int dx^*_{\lambda_1}\cdots dx^*_{\lambda_{r_1-1}}|\varphi_{\rho_1}| \leq \left(\frac{C}{\mu_\varepsilon}\right)^{r_1-1}(t+\varepsilon)^{r_1-1}\sum_{T\in\mathcal{T}_{\rho_1}}\prod_{\lambda_j\in\rho_1}\left(\mathbb{V}^2_{\lambda_j}+K_{\lambda_j}\right)^{d_{\lambda_j}(T)}\,. \tag{8.1.14}$$

– *Initial clustering.* Finally, we are going to estimate the non-overlap constraints in the initial data, which are encoded in (4.3.1).

Recall that $f^{\varepsilon 0}_{\{1,\dots,r\}}(\Psi^{\varepsilon 0}_{\rho_1},\dots,\Psi^{\varepsilon 0}_{\rho_r})$ is a measure of the correlations between all the different clusters of particles $\Psi^{\varepsilon 0}_{\rho_1},\dots,\Psi^{\varepsilon 0}_{\rho_r}$ at time zero, and its definition has been adapted to reconstruct the dynamical cumulants. An estimate of this correlation is obtained by integrating over the root coordinates of the jungles $x^*_{\rho_1},\dots,x^*_{\rho_{r-1}}$, as stated in the following proposition.

We recall that $K_{\rho_i} := m_{\rho_i} + |\rho_i|$ denotes the number of particles in the configuration $\Psi^{\varepsilon,0}_{\rho_i}$ at time 0, and that $K := \sum_{i=1}^{r} K_{\rho_i} = m+n$.

Proposition 8.1.2. — *Under Assumption* (1.1.5)*, there exists $C>0$ such that, for ε small enough,*

$$\int_{\mathbb{T}^{d(r-1)}} |f^{\varepsilon 0}_{\{1,\dots,r\}}(\Psi^{\varepsilon 0}_{\rho_1},\dots,\Psi^{\varepsilon 0}_{\rho_r})|\,dx^*_{\rho_1}\dots dx^*_{\rho_{r-1}} \leq (r-2)!\,C^K\,\exp\Big(-\frac{\beta_0}{2}\mathbb{V}^2\Big)\,\varepsilon^{d(r-1)}$$

for all $\Psi^{\varepsilon 0}_{\rho_i}\in\mathcal{D}^\varepsilon_{K_{\rho_i}}$ at time 0. We have used the convention $0!=(-1)!=1$.

Recall that $f^{\varepsilon 0}_{\{1,\dots,r\}}$ is extended to $\mathbb{D}^K\setminus\mathcal{D}^\varepsilon_K$ by setting $F^{\varepsilon 0}_{\omega_i}=0$ in (4.3.1) wherever it is not defined.

The following proof is an application of known cluster expansion techniques, see e.g. [**33**] and references therein.

Proof. — Set $Z_K := (\Psi^{\varepsilon 0}_{\rho_1},\dots,\Psi^{\varepsilon 0}_{\rho_r})$ with $\Psi^{\varepsilon 0}_{\rho_i}\in\mathcal{D}^\varepsilon_{K_{\rho_i}}$ at time 0. To make notation lighter we shall omit the superscripts $^{\varepsilon 0}$ and also omit to specify the exclusion constraints inside each $\Psi^\varepsilon_{\rho_i}$ in the sequel. We define Φ_{r+p} the indicator function of the mutual exclusion between the elements of the set $\{\Psi^\varepsilon_{\rho_1},\dots,\Psi^\varepsilon_{\rho_r},\bar z_1,\dots,\bar z_p\}$ (where $\Psi^\varepsilon_{\rho_1},\dots,\Psi^\varepsilon_{\rho_r}$ form r clusters and $\bar z_1,\dots,\bar z_p$ are the configurations of p single particles):

$$\Phi_{r+p} = \prod_{h\neq h'}\mathbf{1}_{\eta_h\not\sim\eta_{h'}}\,,$$

with $(\eta_1,\dots,\eta_{r+p}) = (\Psi^\varepsilon_{\rho_1},\dots,\Psi^\varepsilon_{\rho_r},\bar z_1,\dots,\bar z_p)$ and "$\eta_h \not\sim \eta_{h'}$" meaning that the minimum distance between elements of η_h and $\eta_{h'}$ is larger than ε. So we start from

$$F_K^{\varepsilon 0}(Z_K) = \frac{(f^0)^{\otimes K}(Z_K)}{\mathcal{Z}^\varepsilon} \sum_{p\ge 0} \frac{\mu_\varepsilon^p}{p!} \int_{\mathbb{D}^p} (f^0)^{\otimes p}(\bar Z_p)\, \Phi_{r+p}(\Psi^\varepsilon_{\rho_1},\dots,\Psi^\varepsilon_{\rho_r},\bar Z_p)\, d\bar Z_p\,. \tag{8.1.15}$$

We want to expand Φ_{r+p} in order to compensate the factor $\mathcal{Z}^\varepsilon$ whose definition we recall

$$\mathcal{Z}^\varepsilon := \sum_{p\ge 0} \frac{\mu_\varepsilon^p}{p!} \int_{\mathbb{D}^p} (f^0)^{\otimes p}(\bar Z_p)\, \Phi_p(\bar Z_p)\, d\bar Z_p\,, \tag{8.1.16}$$

and to identify the elements in the decomposition

$$F_K^{\varepsilon 0}(\Psi^\varepsilon_{\rho_1},\dots,\Psi^\varepsilon_{\rho_r}) = \sum_{s=1}^r \sum_{\sigma\in\mathcal{P}_r^s} \prod_{i=1}^s f^{\varepsilon 0}_{|\sigma_i|}(\Psi^\varepsilon_{\sigma_i})\,.$$

This will enable us to compute, and estimate, $f^{\varepsilon 0}_{\{1,\dots,r\}}(\Psi^\varepsilon_{\rho_1},\dots,\Psi^\varepsilon_{\rho_r})$. To do so, we naturally develop Φ_{r+p} into s clusters (each of them corresponding to one connected graph containing at least one element of $\{\Psi^\varepsilon_{\rho_1},\dots,\Psi^\varepsilon_{\rho_r}\}$), plus a background $\bar\sigma_0$ of mutually excluding particles (necessary to reconstruct $\mathcal{Z}_\varepsilon$). Such a partition can be reconstructed isolating first the background component, and then splitting $\{\Psi^\varepsilon_{\rho_1},\dots,\Psi^\varepsilon_{\rho_r}\}$ in s parts, to which we adjoin the remaining single particles (see Figure 5).

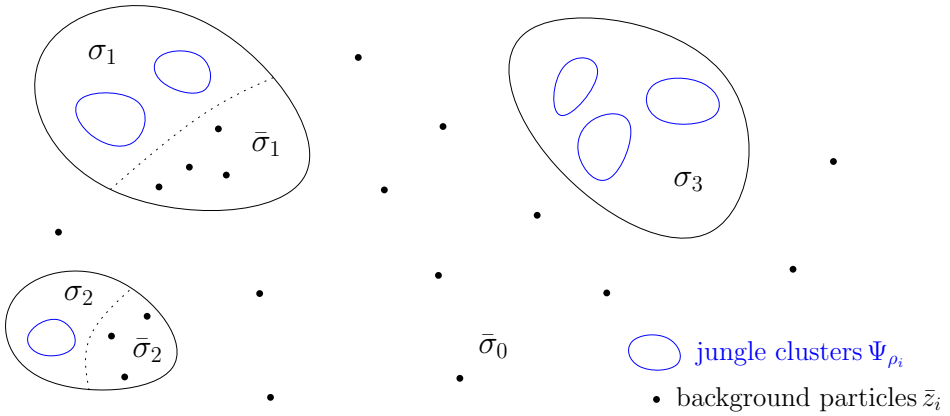


FIGURE 5. Initial configurations are decomposed in s clusters containing at least one jungle $\Psi^\varepsilon_{\rho_1},\dots,\Psi^\varepsilon_{\rho_r}$, plus a background of mutually excluding particles (for which we do not expand the exclusion condition).

This amounts to introducing truncated functions φ via the following formula:

$$\Phi_{r+p}(\Psi^\varepsilon_{\rho_1},\dots,\Psi^\varepsilon_{\rho_r},\bar Z_p) = \sum_{\bar\sigma_0\subset\{1,\dots,p\}} \Phi_{|\bar\sigma_0|}(\bar Z_{\bar\sigma_0}) \sum_{s=1}^r \sum_{\sigma\in\mathcal{P}_r^s} \sum_{\substack{\bar\sigma_1,\dots,\bar\sigma_s \\ \cup_{i=0}^s \bar\sigma_i=\{1,\dots,p\} \\ \bar\sigma_k\cap\bar\sigma_h=\emptyset, k\neq h}} \sum_{j_1,\dots,j_s} \prod_{i=1}^s \varphi(\Psi^\varepsilon_{\sigma_i}, \bar Z_{\bar\sigma_{j_i}}) \tag{8.1.17}$$

where the sum $\sum_{j_1,\dots,j_s}$ runs over the permutations of $\{1,\dots,s\}$. Note that the $\bar\sigma_i$ may be empty (in particular all $\bar\sigma_i$ are empty if $|\bar\sigma_0| = p$). By (2.3.1), we see that

$$\varphi(\Psi^\varepsilon_{\rho_1},\dots,\Psi^\varepsilon_{\rho_r},\bar Z_p) = \sum_{G\in\mathcal{C}_{r+p}} \prod_{(h,h')\in E(G)} (-\mathbf{1}_{\eta_h\sim\eta_{h'}})\,,$$

where the sum runs over the set of connected graphs with $r+p$ vertices; more generally,

$$\varphi(\Psi^\varepsilon_{\sigma_i}, \bar{Z}_{\bar{\sigma}_{j_i}}) = \sum_{G \in \mathcal{C}_{|\sigma_i|+|\bar{\sigma}_{j_i}|}} \prod_{(h,h')\in E(G)} (-\mathbf{1}_{\eta_h \sim \eta_{h'}}) \,.$$

Using the symmetry in the exchange of particle labels, we get, denoting $\bar{s}_i := |\bar{\sigma}_i|$,

$$\binom{p}{\bar{s}_1}\binom{p-\bar{s}_1}{\bar{s}_2}\cdots\binom{p-\bar{s}_1-\cdots-\bar{s}_{s-1}}{\bar{s}_s} = \frac{p!}{\bar{s}_0!\,\bar{s}_1!\ldots\bar{s}_s!}$$

choices for the repartition of the background particles, so that

$$\sum_{p\geq 0}\frac{1}{p!}\int_{\mathbb{D}^p}\Phi_{r+p}(\Psi^\varepsilon_{\rho_1},\ldots,\Psi^\varepsilon_{\rho_r},\bar{Z}_p)\,d\bar{Z}_p = \sum_{s=1}^{r}\sum_{\sigma\in\mathcal{P}^s_r}\sum_{p\geq 0}\sum_{\substack{\bar{s}_0,\ldots,\bar{s}_s\geq 0\\ \sum\bar{s}_i=p}}\int_{\mathbb{D}^p}\Phi_{\bar{s}_0}(\bar{Z}_{\bar{s}_0})\prod_{i=1}^{s}\frac{\varphi(\Psi^\varepsilon_{\sigma_i},\bar{Z}_{\bar{s}_i})}{\bar{s}_i!}d\bar{Z}_p\,.$$

Therefore, plugging (8.1.17) into (8.1.15) first and then using (8.1.16), we obtain

$$\begin{aligned}F^{\varepsilon 0}_K(Z_K) = \frac{(f^0)^{\otimes K}(Z_K)}{\mathcal{Z}^\varepsilon}\sum_{s=1}^{r}\sum_{\sigma\in\mathcal{P}^s_r}\sum_{p\geq 0}\sum_{\substack{\bar{s}_0,\ldots,\bar{s}_s\geq 0\\ \sum\bar{s}_i=p}}&\left(\frac{\mu_\varepsilon^{\bar{s}_0}}{\bar{s}_0!}\int(f^0)^{\otimes\bar{s}_0}(\bar{Z}_{\bar{s}_0})\Phi_{\bar{s}_0}(\bar{Z}_{\bar{s}_0})d\bar{Z}_{\bar{s}_0}\right)\\ &\times\prod_{i=1}^{s}\frac{\mu_\varepsilon^{\bar{s}_i}}{\bar{s}_i!}\int(f^0)^{\otimes\bar{s}_i}(\bar{Z}_{\bar{s}_i})\varphi(\Psi^\varepsilon_{\sigma_i},\bar{Z}_{\bar{s}_i})d\bar{Z}_{\bar{s}_i}\\ = (f^0)^{\otimes K}(Z_K)\sum_{s=1}^{r}\sum_{\sigma\in\mathcal{P}^s_r}\prod_{i=1}^{s}\sum_{\bar{s}_i\geq 0}&\frac{\mu_\varepsilon^{\bar{s}_i}}{\bar{s}_i!}\int(f^0)^{\otimes\bar{s}_i}(\bar{Z}_{\bar{s}_i})\varphi(\Psi^\varepsilon_{\sigma_i},\bar{Z}_{\bar{s}_i})d\bar{Z}_{\bar{s}_i}\,,\end{aligned}$$

hence finally

$$f^{\varepsilon 0}_{\{1,\ldots,r\}}(\Psi^\varepsilon_{\rho_1},\ldots,\Psi^\varepsilon_{\rho_r}) = (f^0)^{\otimes K}(Z_K)\sum_{p\geq 0}\frac{\mu_\varepsilon^p}{p!}\int(f^0)^{\otimes p}(\bar{Z}_p)\varphi(\Psi^\varepsilon_{\rho_1},\ldots,\Psi^\varepsilon_{\rho_r},\bar{Z}_p)d\bar{Z}_p\,. \tag{8.1.18}$$

Applying again Proposition 2.3.3 implies that φ is bounded by

$$|\varphi(\Psi^\varepsilon_{\rho_1},\ldots,\Psi^\varepsilon_{\rho_r},\bar{Z}_p)| \leq \sum_{T\in\mathcal{T}_{r+p}}\prod_{(h,h')\in E(T)}\mathbf{1}_{\eta_h\sim\eta_{h'}} \tag{8.1.19}$$

where $\mathcal{T}_{r+p}$ is the set of minimally connected graphs with $r+p$ vertices labelled by $\Psi^\varepsilon_{\rho_1},\ldots,\Psi^\varepsilon_{\rho_r},\bar{z}_1,\ldots,\bar{z}_p$.

By Lemma 2.4.1, the number of minimally connected graphs with specified vertex degrees $d_1,\ldots,d_{r+p}$ is given by

$$(r+p-2)!/\prod_{i=1}^{r+p}(d_i-1)!\,.$$

On the other hand, the product of indicator functions in (8.1.19) is a sequence of $r+p-1$ constraints, confining the space coordinates to balls of size ε centered at the positions of the clusters $\Psi^\varepsilon_{\rho_1},\ldots,\Psi^\varepsilon_{\rho_r},\bar{z}_1,\ldots,\bar{z}_p$. Such clusters have cardinality $K_{\rho_1},\ldots,K_{\rho_r}\geq 1$ with the constraint

$$\sum_i K_{\rho_i} = K\,.$$

We deduce that for some $C' > 0$

$$\begin{aligned}
&\int_{\mathbb{T}^{d(r-1)}} |f^{\varepsilon 0}_{\{1,\dots,r\}}(\Psi^{\varepsilon}_{\rho_1},\dots,\Psi^{\varepsilon}_{\rho_r})| dx^*_{\rho_1}\dots dx^*_{\rho_{r-1}} \\
&\quad\le C'^K \varepsilon^{d(r-1)} e^{-\frac{\beta_0}{2}\mathbb{V}^2} \sum_{p\ge 0} \frac{(r+p-2)!}{p!} (C'\varepsilon^d \mu_\varepsilon)^p \sum_{d_1,\dots,d_{r+p}\ge 1} \frac{\prod_{i=1}^r K_{\rho_i}^{d_i}}{\prod_{i=1}^{r+p}(d_i-1)!} \\
&\quad\le C'^K \varepsilon^{d(r-1)} e^{-\frac{\beta_0}{2}\mathbb{V}^2} \sum_{p\ge 0} \frac{(r+p-2)!}{p!} (C'\varepsilon^d \mu_\varepsilon)^p \, e^{2K+p} \\
&\quad\le C'^K \varepsilon^{d(r-1)} e^{-\frac{\beta_0}{2}\mathbb{V}^2} 2^{r-2}(r-2)! \sum_{p\ge 0} (C'\varepsilon^d \mu_\varepsilon)^p \, e^{2K+p} \, .
\end{aligned}$$

In the second inequality we used that

$$\prod_{i=1}^r \sum_{d_i\ge 1} \frac{K_{\rho_i}^{d_i}}{(d_i-1)!} \le \prod_{i=1}^r K_{\rho_i} e^{K_{\rho_i}} \le \prod_{i=1}^r e^{2K_{\rho_i}} = e^{2K} \, .$$

Since $C'\varepsilon^d \mu_\varepsilon$ is arbitrarily small with ε, this proves Proposition 8.1.2 with $C = 4C'e^2$. □

8.2. Decay estimate for the cumulants

In this section we shall prove the bound provided in Theorem 4 stated page 35 for $f^{\varepsilon}_{n,[0,t]}(H^{\otimes n})$, and actually the following, slightly more general statement.

Theorem 9. — *Consider the system of hard spheres under the initial measure* (1.1.6)*, with* f^0 *satisfying* (1.1.5)*. Let* $H_n : D([0,\infty[) \mapsto \mathbb{R}$ *be a continuous factorized function:*

$$H_n\big(Z_n([0,\infty[) = \prod_{i=1}^n H^{(i)}\big(z_i([0,\infty[)\big)$$

and define the scaled cumulant $f^{\varepsilon}_{n,[0,t]}(H_n)$ *by polarization of the* n *linear form* (4.4.1)*. Then there exists a positive constant* C *and a time* $T^\star = T^\star(C_0,\beta_0)$ *such that the following uniform a priori bounds hold:*

1. *If* H_n *is bounded, then on* $[0,T^\star]$

$$|f^{\varepsilon}_{n,[0,t]}(H_n)| \le n! C^n (t+\varepsilon)^{n-1} \prod_{i=1}^n \|H^{(i)}\|_\infty \, .$$

2. *If* H_n *has a controlled growth*

$$\big|H_n(Z_n([0,t]))\big| \le \exp\Big(\alpha_0 n + \frac{\beta_0}{4} \sup_{s\in[0,t]} |V_n(s)|^2\Big) \, ,$$

then on $[0,T^\star]$

$$|f^{\varepsilon}_{n,[0,t]}(H_n)| \le (Ce^{\alpha_0})^n (t+\varepsilon)^{n-1} n! \, .$$

3. *Fix* $\delta > 0$*. If* H_n *measures in addition the time regularity in the time interval* $[t-\delta,t]$*, i.e. if for some* $i \in \{1,\dots,n\}$

$$\big|H_n(Z_n([0,t]))\big| \le C_{Lip} \min\Big(\sup_{|t-s|\le\delta} |z_i(t)-z_i(s)|, 1\Big) \exp\Big(\alpha_0 n + \frac{\beta_0}{4} \sup_{s\in[0,t]} |V_n(s)|^2\Big) \, ,$$

then on $[0, T^\star]$

$$|f^{\varepsilon}_{n,[0,t]}(H_n)| \leq C_{Lip}\delta(Ce^{\alpha_0})^n(t+\varepsilon)^{n-1}n!\,. \tag{8.2.1}$$

In the previous section, we considered a nested partition $\lambda \hookrightarrow \rho \hookrightarrow \sigma$ (with $|\sigma| = 1$) of the set $\{1^*, \dots, n^*\}$. We fixed the velocities V_n^* as well as the collision parameters of the pseudo-trajectories $(m, a, T_m, V_m, \Omega_m)$. We then exhibited $n-1$ "independent" conditions on the positions X_n^* for the pseudo-trajectories to be compatible with the partitions λ, ρ. Now we shall conclude the proof of Theorem 9, by integrating successively on all the available parameters. The order of integration is pictured in Figure 6.

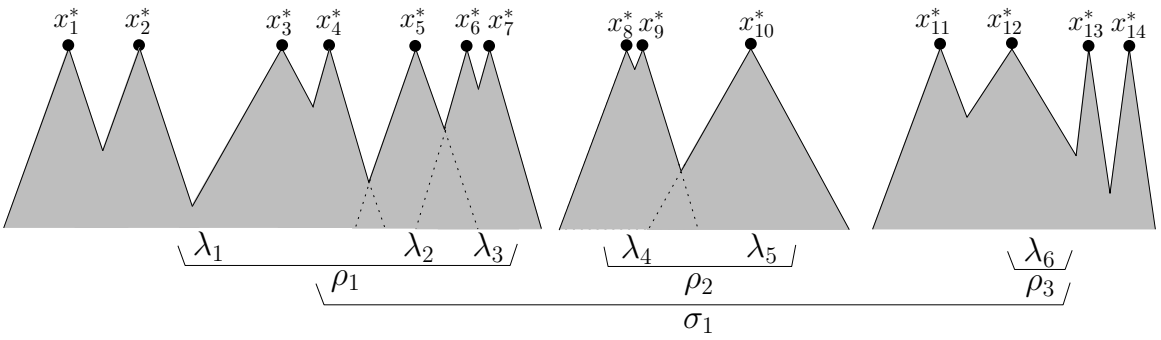


FIGURE 6. In this contribution to the cumulant of order $n = 14$, we integrate over the positions of the roots in the following order: (i) first we integrate over the initial clustering $\hat{x}_{\rho_2} = x^*_{10} - x^*_{14}$ and $\hat{x}_{\rho_1} = x^*_7 - x^*_{14}$; (ii) secondly over the clustering overlaps $\hat{x}_{\lambda_4} = x^*_9 - x^*_{10}$ and $\hat{x}_{\lambda_1} = x^*_4 - x^*_5\,, \hat{x}_{\lambda_2} = x^*_5 - x^*_7$; (iii) finally over the clustering recollisions : $\hat{x}_3^{(\lambda_1)} = x^*_2 - x^*_3$, $\hat{x}_2^{(\lambda_1)} = x^*_1 - x^*_2$, $\hat{x}_1^{(\lambda_1)} = x^*_3 - x^*_4$, $\hat{x}_1^{(\lambda_3)} = x^*_6 - x^*_7$, $\hat{x}_1^{(\lambda_4)} = x^*_8 - x^*_9$, $\hat{x}_3^{(\lambda_6)} = x^*_{13} - x^*_{14}$, $\hat{x}_2^{(\lambda_6)} = x^*_{12} - x^*_{13}$, $\hat{x}_1^{(\lambda_6)} = x^*_{11} - x^*_{12}$. Notice that the variable x^*_{14} remains free.

For the proof of the first two statements in Theorem 9, we start by controlling the weight, simply using the bounds

$$|\mathcal{H}(\Psi^{\varepsilon}_n)| \leq \prod_{i=1}^{n} \|H^{(i)}\|_\infty \quad \text{or} \quad |\mathcal{H}(\Psi^{\varepsilon}_n)| \leq e^{\alpha_0 n + \frac{\beta_0}{4}\mathbb{V}^2}\,. \tag{8.2.2}$$

Then we use that nothing depends on the root coordinates of the jungles $x^*_{\rho_1}, \dots, x^*_{\rho_{r-1}}$ inside the integrand in (8.0.1), except the initial datum $f^{\varepsilon 0}_{\{1,\dots,r\}}$. Therefore by Fubini and according to Proposition 8.1.2,

$$\int_{\mathbb{T}^{d(r-1)}} |f^{\varepsilon 0}_{\{1,\dots,r\}}(\Psi^{\varepsilon 0}_{\rho_1}, \dots, \Psi^{\varepsilon 0}_{\rho_r})| dx^*_{\rho_1} \dots dx^*_{\rho_{r-1}} \leq (r-2)!\, C^K \exp\Big(-\frac{\beta_0}{2}\mathbb{V}^2\Big)\, \varepsilon^{d(r-1)} \tag{8.2.3}$$

for some $C > 0$, uniformly with respect to all other parameters.

Next, the clustering condition on the jungles gives an extra smallness when integrating over the roots of the forests (see (8.1.14))

$$\prod_{i=1}^{r} \int |\varphi_{\rho_i}| \prod_{j=1}^{r_i - 1} dx^*_{\lambda_j} \leq \Big(\frac{C}{\mu_\varepsilon}\Big)^{\ell - r} (t+\varepsilon)^{\ell - r} \prod_{i=1}^{r} \sum_{T \in \mathcal{T}_{\rho_i}} \prod_{\lambda_j \in \rho_i} \Big(\mathbb{V}^2_{\lambda_j} + K_{\lambda_j}\Big)^{d_{\lambda_j}(T)}\,, \tag{8.2.4}$$

uniformly with respect to all other parameters, for some possibly larger constant C.

The clustering condition on the forests gives finally an extra smallness when integrating over the remaining variables $\hat{x}_k$, according to (8.1.9). Notice however that the latter inequality cannot be directly applied to (4.4.1), due to the presence of the cross section factors (8.0.2) in the measure (3.3.5).

It is then useful to combine the estimate with the sum over trees $a_{|\lambda_i}$. The argument is depicted in Figure 7. We will present the arguments for λ_1, assuming without loss of generality that $\lambda_1 = \{1, \dots, \ell_1\}$. We will denote by $\tilde{a}$ the restriction of the tree a to λ_1 with fixed total numbers of particles $K_1, \cdots, K_{\ell_1}$, and by $\tilde{a}_k$, $\mathcal{C}_k$ the tree variables and the cross section factors associated with the s_k creations occurring in the time interval $(\tau_{\mathrm{rec,k}}, \tau_{\mathrm{rec,k-1}})$ for $1 \leq k \leq \ell_1$.

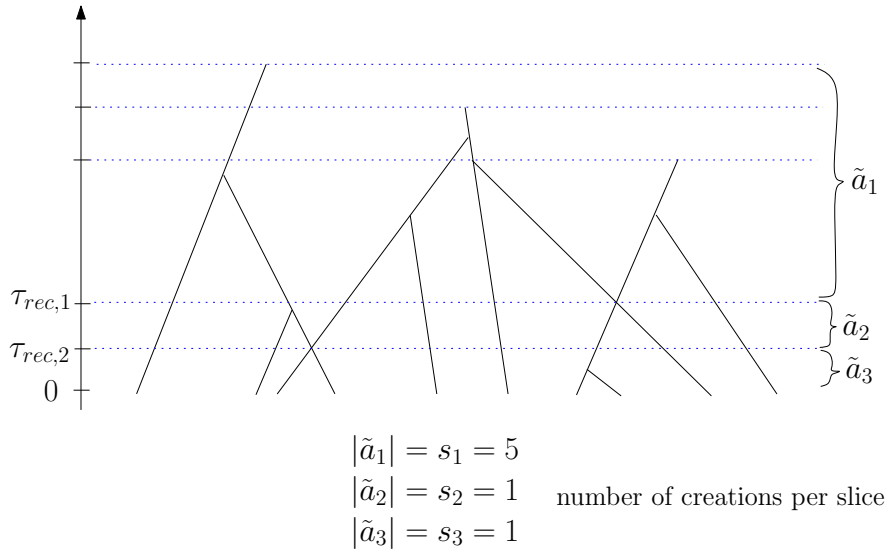


FIGURE 7. Integration over time slices.

As in the first line of (8.1.8), we have that

(8.2.5)

$$\sum_{\tilde{a}} \int dX^*_{\ell_1 - 1} \, \mathbb{\Delta}_{\lambda_1} \, \mathbf{1}_{\mathcal{G}^\varepsilon}\big(\Psi^\varepsilon_{\lambda_1}\big) |\mathcal{C}\big(\Psi^\varepsilon_{\lambda_1}\big)|$$

$$\leq \sum_{\left(\lambda_{(k)}, \lambda'_{(k)}\right)} \sum_{\tilde{a}_1} |\mathcal{C}^\varepsilon_1\left(\Psi_{\lambda_1}\right)| \int d\hat{x}_1 \mathbf{1}_{\mathcal{B}_1} \sum_{\tilde{a}_2} |\mathcal{C}_2\big(\Psi^\varepsilon_{\lambda_1}\big)| \int d\hat{x}_2 \dots \int d\hat{x}_{\ell_1 - 1} \mathbf{1}_{\mathcal{B}_{\ell_1 - 1}} \sum_{\tilde{a}_{\ell_1}} |\mathcal{C}_{\ell_1}\big(\Psi^\varepsilon_{\lambda_1}\big)| \, .$$

We can therefore apply iteratively the inequality (8.1.7) and the classical Cauchy-Schwarz argument used in Lanford's proof. Denote by

$$S_k := \sum_{i=1}^{k} s_i$$

the number of particles added before time $\tau_{\mathrm{rec,k}}$, so that

$$S_{\ell_1} = m_{\lambda_1}$$

(denoting abusively $\tau_{\mathrm{rec},\ell_1} = 0$). We get:

$$\begin{aligned}\sum_{\tilde{a}_k} \big|\mathcal{C}_k\big(\Psi_{\lambda_1}\big)\big| &\leq \prod_{s=S_{k-1}+1}^{S_k} \left(\sum_{u=1}^{s-1} |v_s - v_u(t_s)| + \sum_{u=1}^{\ell_1} |v_s - v_u^*(t_s)|\right) \\ &\leq \prod_{s=S_{k-1}+1}^{S_k} \left((\ell_1 + s - 1)|v_s| + \sum_{u=1}^{s-1} |v_u(t_s)| + \sum_{u=1}^{\ell_1} |v_u^*(t_s)|\right) \\ &\leq \prod_{s=S_{k-1}+1}^{S_k} \big((\ell_1 + m_{\lambda_1})(1 + |v_s|) + |\mathbb{V}_{\lambda_1}|^2\big)\end{aligned} \tag{8.2.6}$$

and

$$\begin{aligned}&\sum_{\tilde{a}} \int dX^*_{\ell_1-1}\, \mathbb{M}_{\lambda_1}\, \mathbf{1}_{\mathcal{G}^\varepsilon}\big(\Psi_{\lambda_1}\big) |\mathcal{C}\big(\Psi_{\lambda_1}\big)| \\ &\leq \left(\frac{C}{\mu_\varepsilon}\right)^{\ell_1 - 1} (t+\varepsilon)^{\ell_1 - 1} \sum_{T \in \mathcal{T}_{\lambda_1}} \prod_{j \in \lambda_1} \big(\mathbb{V}_j^2 + K_j\big)^{d_j(T)} \prod_{s=1}^{m_{\lambda_1}} \big((\ell_1 + m_{\lambda_1})(1 + |v_s|) + |\mathbb{V}_{\lambda_1}|^2\big)\ ,\end{aligned}$$

for some positive C.

Recall that

$$\exp\left(-\frac{\beta_0}{16m}|V|^2\right)|V|^2 \leq Cm.$$

Combining (8.2) with the bound (8.2.2) on $\mathcal{H}$, (8.2.3) and (8.2.4) leads therefore to

$$\begin{aligned}&\int \Big| \sum_a \prod_{i=1}^{\ell} \mathbb{M}_{\lambda_i}\, \mathcal{C}\big(\Psi^\varepsilon_{\lambda_i}\big)\, \mathbf{1}_{\mathcal{G}^\varepsilon}\big(\Psi^\varepsilon_{\lambda_i}\big) \mathcal{H}\big(\Psi^\varepsilon_{\lambda_i}\big) \varphi_\rho\, f^{\varepsilon 0}_{\{1,\dots,r\}}(\Psi^{\varepsilon 0}_{\rho_1}, \dots, \Psi^{\varepsilon 0}_{\rho_r})\Big|\, dX^*_n \\ &\leq (r-2)!\, C^K \exp\big(\alpha_0 n - \frac{\beta_0}{4}\mathbb{V}^2\big)\, \varepsilon^{d(r-1)} \left(\frac{C}{\mu_\varepsilon}\right)^{n-r} (t+\varepsilon)^{n-r} \\ &\qquad \times \left(\prod_{i=1}^{r} \sum_{T \in \mathcal{T}_{\rho_i}} \prod_{\lambda_j \in \rho_i} \Big(\mathbb{V}^2_{\lambda_j} + K_{\lambda_j}\Big)^{d_{\lambda_j}(T)}\right) \left(\prod_{i=1}^{\ell} \sum_{T \in \mathcal{T}_{\lambda_i}} \prod_{j \in \lambda_i} \big(\mathbb{V}_j^2 + K_j\big)^{d_j(T)}\right) \\ &\qquad \times (m+n)^m \prod_{s=1}^{m} (1 + |v_s|)\,,\end{aligned} \tag{8.2.7}$$

valid uniformly with respect to all other parameters. Here and below, we indicate by C a large enough constant, possibly depending on C_0, β_0 (but on nothing else) and changing from line to line.

The following step then consists in integrating (8.2.7) with respect to the remaining parameters (T_m, Ω_m, V_m) and V_n^* (with m fixed for the time being). Recalling the condition that $t_1 \geq t_2 \geq \cdots \geq t_m$,

we get

$$\begin{aligned}
&\int \Big| \sum_a \prod_{i=1}^{\ell} \mathbb{M}_{\lambda_i} \, \mathcal{C}\big(\Psi^{\varepsilon}_{\lambda_i}\big) \, \mathbf{1}_{\mathcal{G}^{\varepsilon}}\big(\Psi^{\varepsilon}_{\lambda_i}\big) \mathcal{H}\big(\Psi^{\varepsilon}_{\lambda_i}\big) \varphi_\rho \, f^{\varepsilon 0}_{\{1,\dots,r\}}(\Psi^{\varepsilon 0}_{\rho_1}, \dots, \Psi^{\varepsilon 0}_{\rho_r}) \, dT_m d\Omega_m dV_m \Big| dZ^*_n \\
&\leq (r-2)! \, C^K \varepsilon^{d(r-1)} \left(\frac{C}{\mu_\varepsilon}\right)^{n-r} (t+\varepsilon)^{n-r} \, \frac{(Ct)^m}{m!} (m+n)^m \\
&\times \sum_{T_1 \in \mathcal{T}_{\rho_1}} \cdots \sum_{T_r \in \mathcal{T}_{\rho_r}} \sum_{\tilde{T}_1 \in \mathcal{T}_{\lambda_1}} \cdots \sum_{\tilde{T}_\ell \in \mathcal{T}_{\lambda_\ell}} \int \exp\left(\alpha_0 n - \frac{\beta_0}{16} \mathbb{V}^2\right) \prod_{s=1}^{m} (1+|v_s|) dV_n^* dV_m \\
&\qquad \times \sup \left(\exp\big(-\frac{\beta_0}{16} \mathbb{V}^2\big) \left(\prod_{i=1}^{r} \prod_{\lambda_j \in \rho_i} \left(\mathbb{V}^2_{\lambda_j} + K_{\lambda_j} \right)^{d_{\lambda_j}(T_i)} \right) \left(\prod_{i=1}^{\ell} \prod_{j \in \lambda_i} \left(\mathbb{V}^2_j + K_j \right)^{d_j(\tilde{T}_i)} \right) \right) .
\end{aligned}$$

Using the facts that

$$\begin{aligned}
\int \exp\left(-\frac{\beta_0}{16}|w|^2\right) |w| dw \leq C \, , \\
\exp\left(-\frac{\beta_0}{16}|V|^2\right) \left(|V|^2 + K\right)^D \leq C^K \left(\frac{16D}{\beta_0}\right)^D
\end{aligned}$$

for positive K, D, we arrive at

$$\begin{aligned}
&\int \Big| \sum_a \prod_{i=1}^{\ell} \mathbb{M}_{\lambda_i} \, \mathcal{C}\big(\Psi^{\varepsilon}_{\lambda_i}\big) \, \mathbf{1}_{\mathcal{G}^{\varepsilon}}\big(\Psi^{\varepsilon}_{\lambda_i}\big) \mathcal{H}\big(\Psi^{\varepsilon}_{\lambda_i}\big) \varphi_\rho \, f^{\varepsilon 0}_{\{1,\dots,r\}}(\Psi^{\varepsilon 0}_{\rho_1}, \dots, \Psi^{\varepsilon 0}_{\rho_r}) \, dT_m d\Omega_m dV_m \Big| dZ^*_n \\
&\leq (r-2)! \left(\frac{C}{\mu_\varepsilon}\right)^{n-r} (t+\varepsilon)^{n-r} \, \varepsilon^{d(r-1)} (Ct)^m e^{n\alpha_0} \qquad (8.2.8) \\
&\qquad \times \left(\prod_{i=1}^{r} \sum_{T \in \mathcal{T}_{\rho_i}} \prod_{\lambda_j \in \rho_i} \big(d_{\lambda_j}(T)\big)^{d_{\lambda_j}(T)} \right) \left(\prod_{i=1}^{\ell} \sum_{\tilde{T} \in \mathcal{T}_{\lambda_i}} \prod_{j \in \lambda_i} \left(d_j(\tilde{T})\right)^{d_j(\tilde{T})} \right) .
\end{aligned}$$

For each forest (jungle) we ended up with a factor $\sum_{T \in \mathcal{T}_k} \prod_{i=1}^{k} \big(d_i(T)\big)^{d_i(T)}$ where k is the cardinality of the forest (jungle). Applying again Lemma 2.4.1, this number is bounded above by

$$\begin{aligned}
(k-2)! \sum_{\substack{d_1, \cdots, d_k \\ 1 \leq d_i \leq k-1 \\ \sum_i d_i = 2(k-1)}} \prod_{i=1}^{k} \frac{d_i^{d_i}}{(d_i-1)!} &\leq (k-2)! \, e^{k-2} \sum_{\substack{d_1, \cdots, d_k \\ 1 \leq d_i \leq k-1 \\ \sum_i d_i = 2(k-1)}} \prod_{i=1}^{k} d_i \\
&\leq (k-2)! \, e^{2(k-2)} \sum_{\substack{d_1, \cdots, d_k \\ 1 \leq d_i \leq k-1 \\ \sum_i d_i = 2(k-1)}} 1 \, .
\end{aligned}$$

The last sum is also bounded by C^k. Taking the sum over the number of created particles m, we arrive at

$$\begin{aligned}
&\int \left| \int \prod_{i=1}^{\ell} \Big[\mu(d\Psi^{\varepsilon}_{\lambda_i}) \mathbb{M}_{\lambda_i} \, \mathcal{C}\big(\Psi^{\varepsilon}_{\lambda_i}\big) \, \mathbf{1}_{\mathcal{G}^{\varepsilon}}\big(\Psi^{\varepsilon}_{\lambda_i}\big) \mathcal{H}\big(\Psi^{\varepsilon}_{\lambda_i}\big) \Big] \times \varphi_\rho \, f^{\varepsilon 0}_{\{1,\dots,r\}}(\Psi^{\varepsilon 0}_{\rho_1}, \dots, \Psi^{\varepsilon 0}_{\rho_r}) \right| dZ^*_n \\
&\leq (r-2)! (Ce^{\alpha_0})^n \left(\frac{(t+\varepsilon)^{n-r} \varepsilon^{r-1}}{\mu_\varepsilon^{n-1}} \right) \prod_{i=1}^{r} (r_i - 2)! \prod_{j=1}^{\ell} (\ell_j - 2)! \sum_m (Ct)^m
\end{aligned} \qquad (8.2.9)$$

valid uniformly with respect to all partitions $\lambda \hookrightarrow \rho$, and for t small enough. Finally, summing (8.2.9) over the partitions $\lambda \hookrightarrow \rho$ we find (recalling the convention $0! = (-1)! = 1$)

$$\sum_{\ell=1}^{n} \sum_{\lambda \in \mathcal{P}_n^\ell} \sum_{r=1}^{\ell} \sum_{\rho \in \mathcal{P}_\ell^r} (r-2)! \prod_{i=1}^{r} (r_i - 2)! \prod_{j=1}^{\ell} (\ell_j - 2)!$$

$$= \sum_{\ell=1}^{n} \sum_{\substack{\ell_1, \cdots, \ell_\ell \geq 1 \\ \sum_i \ell_i = n}} \sum_{r=1}^{\ell} \sum_{\substack{r_1, \cdots, r_r \geq 1 \\ \sum_i r_i = \ell}} \frac{n!}{\ell! \ell_1! \dots \ell_\ell!} \frac{\ell!}{r! r_1! \dots r_r!} (r-2)! \prod_{i=1}^{r} (r_i - 2)! \prod_{j=1}^{\ell} (\ell_j - 2)!$$

$$\leq n! \left(1 + \sum_{r \geq 2} \frac{1}{r(r-1)} \right)^{2n} .$$

This concludes the proof of the first two estimates in Theorem 9. The third statement (8.2.1) is obtained in a very similar way. If the pseudo-particle i has no collision or recollision during $[t-\delta, t]$ then

$$\sup_{|t-s| \leq \delta} |z_i(t) - z_i(s)| \leq \delta |v_i(t)| \leq \delta |V_n(t)| .$$

This is enough to gain a factor δ from the assumption on H_n.

If a collision occurs during $[t-\delta, t]$, then by localizing the time integral of this collision in Duhamel formula, one gets the additional factor δ (with a factor m corresponding to the symmetry breaking in the time integration dT_m).

Finally, it may happen that a recollision occurs during $[t-\delta, t]$. This imposes an additional constraint on the parents of the recolliding particles and the recollision time has to be integrated now in $[t-\delta, t]$. Thus an additional factor δ is also obtained (together with a factor n corresponding to the symmetry breaking in the time integration $d\Theta_{n-1}^{\text{clust}}$). This completes the proof of (8.2.1). □

Remark 8.2.1. — *Note that the sum over m in* (8.2.9) *is converging uniformly in ε, which means that the contribution of pseudo-trajectories involving a large number m of created particles can be made as small as needed. In particular, to study the convergence as $\varepsilon \to 0$, it will be enough to look at pseudo-trajectories with a controlled number $m \leq m_0$ of added particles.*

CHAPTER 9

MINIMAL TREES AND CONVERGENCE OF THE CUMULANTS

The goal of this chapter is to prove Theorem 5 p. 39, which can be restated as follows.

Theorem 10. — *Let $H_n : (D([0,+\infty[)^n \mapsto \mathbb{R}$ be a continuous factorized function $H_n(Z_n([0,t])) = \prod_{i=1}^n H^{(i)}(z_i([0,t]))$ such that*

$$\left|H_n(Z_n([0,t]))\right| \leq \exp\left(\alpha_0 n + \frac{\beta_0}{4} \sup_{s\in[0,t]} |V_n(s)|^2\right), \tag{9.0.1}$$

with β_0 defined in (1.1.5)*.*

Then the scaled cumulant $f^{\varepsilon}_{n,[0,t]}(H_n)$ converges for any $t \leq T^\star$ to the limiting cumulant introduced in (5.1.4)

$$f_{n,[0,t]}(H_n) = \sum_{T\in\mathcal{T}_n^{\pm}} \sum_{m} \sum_{a\in\mathcal{A}^{\pm}_{n,m}} \int d\mu_{\mathrm{sing},T,a}(\Psi_{n,m}) \mathcal{H}(\Psi_{n,m}) f^{0\otimes(n+m)}(\Psi^0_{n,m})\,.$$

After some preparation in Section 9.1, we present in Section 9.2 the leading order asymptotics of $f^{\varepsilon}_{n,[0,t]}(H^{\otimes n})$ by eliminating all pseudo-trajectories involving non clustering recollisions and overlaps. Section 9.3 is devoted to the conclusion of the proof, by estimating the discrepancy between the remaining pseudo-trajectories Ψ^{ε}_n and their limits Ψ_n.

9.1. Truncated cumulants

An inspection of the arguments in the previous chapter shows that initial clusterings are negligible compared to dynamical clusterings. Indeed Estimate (8.2.9) shows that the leading order term in the cumulant decomposition (4.4.1) corresponds to choosing $r = 1$: this term is indeed of order

$$C^n n!(t+\varepsilon)^{n-1}$$

while the error is smaller by one order of ε. We are therefore reduced to studying

$$\mu_\varepsilon^{n-1} \sum_{\ell=1}^{n} \sum_{\lambda\in\mathcal{P}^\ell_n} \int \Big(\prod_{i=1}^{\ell} d\mu(\Psi^{\varepsilon}_{\lambda_i}) \mathcal{H}(\Psi^{\varepsilon}_{\lambda_i}) \mathbb{\Delta}_{\lambda_i}\Big)\, \varphi_{\{1,\dots,\ell\}}\, f^{\varepsilon 0}_{\{1\}}(\Psi^{\varepsilon 0}_{\rho_1})\,.$$

We shall furthermore consider only trees of controlled size: we define, for any integer m_0,

$$f^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n}) := \mu_\varepsilon^{n-1} \sum_{\ell=1}^{n} \sum_{\lambda\in\mathcal{P}_n^\ell} \int dZ_n^* \int \prod_{i=1}^{\ell} \Big[d\mu_{m_0}(\Psi^\varepsilon_{\lambda_i}) \mathbb{\Delta}_{\lambda_i} \mathcal{H}\big(\Psi^\varepsilon_{\lambda_i}\big)\Big] \varphi_{\{1,\dots,\ell\}}\ f^{\varepsilon 0}_{\{1\}}(\Psi^{\varepsilon 0}_{\rho_1})\,, \tag{9.1.1}$$

where the measure on the pseudo-trajectories is defined as in (3.3.5) by

$$d\mu_{m_0}(\Psi^\varepsilon_{\lambda_i}) := \sum_{m_i\le m_0} \sum_{a\in\mathcal{A}^\pm_{\lambda_i,m_i}} dT_{m_i} d\Omega_{m_i} dV_{m_i}\ \mathbf{1}_{\mathcal{G}^\varepsilon}(\Psi^\varepsilon_{\lambda_i}) \prod_{k=1}^{m_i} \Big(s_k \left(\big(v_k - v_{a_k}(t_k)\big)\cdot\omega_k \right)_+ \Big).$$

Then by Remark 8.2.1, we have

$$\lim_{m_0\to\infty} \big| f^\varepsilon_{n,[0,t]}(H^{\otimes n}) - f^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n}) \big| = 0 \text{ uniformly in } \varepsilon\,. \tag{9.1.2}$$

Next let us define

$$\tilde f^\varepsilon_{n,[0,t]}(H^{\otimes n}) := \mu_\varepsilon^{n-1} \sum_{\ell=1}^{n} \sum_{\lambda\in\mathcal{P}_n^\ell} \int dZ_n^* \int \prod_{i=1}^{\ell} \Big[d\mu(\Psi^\varepsilon_{\lambda_i}) \tilde{\mathbb{\Delta}}_{\lambda_i} \mathcal{H}\big(\Psi^\varepsilon_{\lambda_i}\big)\Big] \tilde\varphi_{\{1,\dots,\ell\}}\ f^{\varepsilon 0}_{\{1\}}(\Psi^{\varepsilon 0}_{\rho_1})$$

where $\tilde{\mathbb{\Delta}}_{\lambda_i}$ is the characteristic function supported on the forests λ_i having exactly $|\lambda_i|-1$ recollisions, and $\tilde\varphi_{\{1,\dots,\ell\}}$ is supported on jungles having exactly $\ell-1$ regular overlaps, so that

— all recollisions and overlaps are clustering;

— all overlaps are regular in the sense of Remark 8.1.1.

Since $\tilde f^\varepsilon_{n,[0,t]}(H^{\otimes n})$ is defined simply as the restriction of $f^\varepsilon_{n,[0,t]}(H^{\otimes n})$ to some pseudo-trajectories (with a special choice of initial data), the same estimates as in the previous chapter show that

$$|\tilde f^\varepsilon_{n,[0,t]}(H^{\otimes n})| \le C^n n! (t+\varepsilon)^{n-1}\,.$$

Furthermore, defining its truncated counterpart

$$\tilde f^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n}) := \mu_\varepsilon^{n-1} \sum_{\ell=1}^{n} \sum_{\lambda\in\mathcal{P}_n^\ell} \int dZ_n^* \int \prod_{i=1}^{\ell} \Big[d\mu_{m_0}(\Psi^\varepsilon_{\lambda_i}) \tilde{\mathbb{\Delta}}_{\lambda_i} \mathcal{H}\big(\Psi^\varepsilon_{\lambda_i}\big)\Big] \tilde\varphi_{\{1,\dots,\ell\}}\ f^{\varepsilon 0}_{\{1\}}(\Psi^{\varepsilon 0}_{\rho_1})$$

there holds

$$\lim_{m_0\to\infty} \big| \tilde f^\varepsilon_{n,[0,t]}(H^{\otimes n}) - \tilde f^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n}) \big| = 0 \text{ uniformly in } \varepsilon\,. \tag{9.1.3}$$

The limits (9.1.2) and (9.1.3) imply that it is enough to prove that the truncated decompositions $f^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n})$ and $\tilde f^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n})$ are close: we shall indeed see in the next paragraph that non clustering recollisions or overlaps as well as non regular overlaps induce some extra smallness.

Note finally that the estimates provided in Theorem 9 show that the series $f^\varepsilon_{n,[0,t]}(H^{\otimes n})/n!$ converges uniformly in ε for $t\le T^\star$, so a termwise (in n) convergence as $\varepsilon\to 0$ is sufficient for our purposes. We therefore shall make no attempt at optimality in the dependence of the constants in n in this chapter.

9.2. Removing non clustering recollisions/overlaps and non regular overlaps

Let us now estimate $|f^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n}) - \tilde f^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n})|$. We first show how to express non clustering recollisions as additional constraints on the set of integration parameters $(Z_n^*, T_m, V_m, \Omega_m)$. The constraints may be either "independent" of the constraints described in the previous chapter, or can reinforce one

of them in an explicit way: in particular, we shall see that the size of the sets $\mathcal{B}_k$ of parameters, introduced in Section 8.1, becomes smaller by a factor $O(\varepsilon^{\frac{1}{8}})$ in the presence of a non clustering recollision (and similarly for clustering overlaps). This argument is actually very similar to the argument used to control (internal) recollisions in Lanford's proof (which focuses primarily on the expansion of the first cumulant).

In the coming section we discuss one elementary step, which is the estimate of a given non clustering event, by treating separately different geometrical cases – we shall actually only deal with non clustering recollisions, the case of overlaps being simpler. Then in Section 9.2.2 we apply the argument to provide a global estimate.

9.2.1. Additional constraint due to non clustering recollisions and overlaps. — We consider a partition λ of $\{1^*, \dots, n^*\}$ in ℓ forests $\lambda_1, \dots, \lambda_\ell$. We fix the velocities V_n^*, as well as the collision parameters (T_m, V_m, Ω_m), with $m \leq m_0 \ell$. As in Section 8.1 we denote by $\mathbb{V}^2 := (V_n^*)^2 + V_m^2$ (twice) the total energy and by $K = n + m$ the total number of particles, and by $\mathbb{V}_i^2$ and K_i the energy and number of particles of the collision tree $\Psi^\varepsilon_{\{i\}}$ with root at z_i^*.

Let us consider a pseudo-trajectory (compatible with λ) involving a non clustering recollision. We denote by $t_{\rm rec}$ the time of occurrence of the first non clustering recollision (going backwards in time) and we denote by $q, q' \in \{1^*, \dots, n^*\} \cup \{1, \dots, m\}$ the labels of the two particles involved in that recollision. By definition, they belong to the same forest, say λ_1, and we denote by $\Psi^\varepsilon_{\{i\}}$ and $\Psi^\varepsilon_{\{i'\}}$ their respective trees (note that it may happen that $i = i'$).

The recollision between q and q' imposes strong constraints on the history of these particles, especially on the first deflection of the couple q, q', moving up the forest (thus forward in time) towards the root. These constraints can be expressed by different equations depending on the recollision scenario.

Self-recollision. Let us assume that moving up the tree starting at the recollision time, the first deflection of q and q' is between q and q' themselves at time $\bar{t}$: this means that the recollision occurs due to periodicity in space.

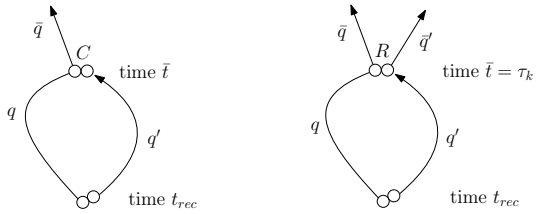


FIGURE 8. The first deflection of q and q' can be either the creation of one of them (say q), or a clustering recollision.

This has a very small cost, as described in the following proposition (with the notation of Section 8.1).

Proposition 9.2.1. — *Let q and q' be the labels of the two particles recolliding due to space periodicity, and denote by $\bar{t}$ the first time of deflection of q and q', moving up their respective trees from the recollision time. The following holds:*

— *If q is created next to q' at time $\bar t$ with collision parameters $\bar\omega$ and $\bar v$, and if $\bar v_q$ is the velocity of q at time $\bar t^+$, then denoting by $\Psi^\varepsilon_{\{i\}}$ their collision tree there holds*

$$\int \mathbf{1}_{\text{Self-recollision with creation of } q \text{ at time } \bar t} \left|\big(\bar v - \bar v_q\big)\cdot\bar\omega\right| d\bar t d\bar\omega d\bar v \le \frac{C}{\mu_\varepsilon}\mathbb{V}_i^{2d+1}(1+t)^{d+1}\,.$$

— *If $\bar t$ corresponds to the k-th clustering recollision in $\Psi^\varepsilon_{\lambda_1}$, between the trees $\Psi^\varepsilon_{\{j_k\}}$ and $\Psi^\varepsilon_{\{j'_k\}}$, then*

$$\int \mathbf{1}_{\text{Self-recollision with a clustering recollision at time } \bar t}\, d\hat x_k \le \frac{C}{\mu_\varepsilon^2}\Big((\mathbb{V}_{j_k}+\mathbb{V}_{j'_k})(1+t)\Big)^{d+1}\,.$$

Note that in the second case, the condition is expressed in terms of the root $\hat x_k$ with the notation of Section 8.1: it is not independent of the condition (8.1.5) defining $B_{qq'}$, but it reinforces it as the estimate provides a factor $1/\mu_\varepsilon^2$ instead of $1/\mu_\varepsilon$.

Geometry of the first recollision. Without loss of generality, we may assume that the first deflection moving up the tree from time $t_{\rm rec}$ involves q. We denote by $\bar t$ the time of that first deflection and by $c\neq q,q'$ the particle involved in the collision with q (see Figure 9). To simplify we denote by $\bar v_q$ the post-collisional velocity of particle q if c is created at time $\bar t$ and the post-collisional velocity of particle c if q is created at time $\bar t$. Similarly we denote by $\bar v_{q'}$ the velocity of particle q' at time $\bar t$.

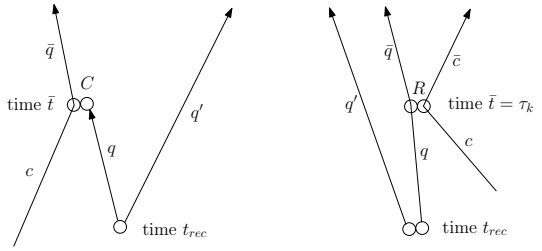


FIGURE 9. The first deflection of q can be either a collision, or a clustering recollision.

The result is the following.

Proposition 9.2.2. — *Let q and q' be the labels of the two particles involved in the first non clustering recollision. Assume that the first deflection moving up their trees from time $t_{\rm rec}$ involves q and a particle $c\neq q'$, at some time $\bar t$. Then with the above notation*

— *If $\bar t$ is the creation time of q (or c), denoting by $\bar\omega$ and $\bar v$ the corresponding collision parameters, by $\Psi^\varepsilon_{\{i\}}$ their collision tree and by $\Psi^\varepsilon_{\{i'\}}$ the collision tree of q', there holds*

$$\int \mathbf{1}_{\text{Recollision with a creation at time } \bar t} \left|(\bar v-\bar v_q(\bar t))\cdot\bar\omega\right| d\bar t d\bar\omega d\bar v \le C\big(\mathbb{V}_i+\mathbb{V}_{i'}\big)^{2d+\frac32}(1+t)^{d+\frac12}\min\left(1,\frac{\varepsilon^{1/2}}{|\bar v_q-\bar v_{q'}|}\right)\,.$$

— *If $\bar t$ corresponds to the k-th clustering recollision in the tree $\Psi^\varepsilon_{\lambda_1}$, between $\Psi^\varepsilon_{\{j_k\}}$ and $\Psi^\varepsilon_{\{j'_k\}}$, and if $\Psi^\varepsilon_{\{i'\}}$ is the collision tree of q', then*

$$\int \mathbf{1}_{\text{Recollision with a clustering recollision at time } \bar t}\, d\hat x_k \le \frac{C}{\mu_\varepsilon}\big(\mathbb{V}_{j_k}+\mathbb{V}_{j'_k}+\mathbb{V}_{i'}\big)^{\frac32}(1+t)^{\frac12}\min\left(1,\frac{\varepsilon^{1/2}}{|\bar v_q-\bar v_{q'}|}\right)\,.$$

Note that as in the periodic situation, the condition in the second case is expressed in terms of the root $\hat{x}_k$, and reinforces the condition (8.1.5) defining $B_{qq'}$ by a factor $\varepsilon^{1/2}$, up to a singularity in velocities that has to be eliminated.

The geometric analysis of these scenarios and the proof of Propositions 9.2.1 and 9.2.2 are postponed to Section 9.4. The estimates in the first case were actually already proved in [**5**], while the second one (the case of a clustering recollision) requires a slight adaptation.

Elimination of the singularity. It finally remains to eliminate the singularity $1/|\bar{v}_q - \bar{v}_{q'}|$, using the next deflection moving up the tree. Note that this singularity arises only if the first non clustering recollision is not a self-recollision, which ensures that the recolliding particles have at least two deflections before the non clustering recollision. The result is the following.

Proposition 9.2.3. — *Let q and q' be the labels of two particles with velocities v_q and $v_{q'}$, and denote by $\bar{t}$ the time of the first deflection of q or q' moving up their trees.*

— *If the deflection at $\bar{t}$ corresponds to a collision in a tree $\Psi^{\varepsilon}_{\{i\}}$ with parameters $\bar{\omega}, \bar{v}$, then*

$$\int \mathbf{1}_{\text{Recollision with a creation at time } \bar{t}} \min\left(1, \frac{\varepsilon^{1/2}}{|v_q - v_{q'}|}\right) \left|(\bar{v} - \bar{v}_q)\cdot\bar{\omega}\right| d\bar{t}\, d\bar{v}\, d\bar{\omega} \leq C t \mathbb{V}_i^{d+1} \varepsilon^{\frac{1}{8}} .$$

— *if $\bar{t}$ corresponds to the k-th clustering recollision in the tree $\Psi^{\varepsilon}_{\lambda_1}$, between $\Psi^{\varepsilon}_{\{j_k\}}$ and $\Psi^{\varepsilon}_{\{j'_k\}}$, then*

$$\int \min\left(1, \frac{\varepsilon^{1/2}}{|v_q - v_{q'}|}\right) d\hat{x}_k \leq \frac{C\varepsilon^{\frac{1}{8}}(\mathbb{V}_{j_k} + \mathbb{V}_{j'_k})t}{\mu_\varepsilon} .$$

The proposition is also proved in Section 9.4 of this chapter.

9.2.2. Removing pathological cumulant pseudo-trajectories. — We apply now the geometrical estimates of the previous section to show the following result.

Proposition 9.2.4. — *With the previous notation, for any finite m_0, there holds*

$$\lim_{\varepsilon\to 0} \left| \tilde{f}^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n}) - f^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n}) \right| = 0 .$$

Proof. — We first consider the case of pathological pseudo-trajectories involving a non regular clustering overlap. By definition (see Remark 8.1.1), this means that the corresponding $\tau_{\rm ov}$ has to be equal either to t or to the creation time of one of the overlapping particles. In other words, instead of being a union of tubes of volume $O((t+\varepsilon)/\mu_\varepsilon)$, the set $\tilde{\mathcal{B}}_k$ describing the k-th clustering overlap (see (8.1.12)) reduces to a union of balls of volume $O(\varepsilon^d)$, so that

$$|\tilde{\mathcal{B}}_k| \leq C\varepsilon^d K_{\lambda_{[k]}} K_{\lambda'_{[k]}} .$$

The non clustering condition is therefore reinforced and we gain additional smallness.

Let us now consider the case of pathological pseudo-trajectories involving some non clustering recollision/overlap. We can assume without loss of generality that the first non clustering recollision (recall that we leave the case of regular overlaps to the reader) occurs in the forest $\lambda_1 = \{1, \dots, \ell_1\}$. The compatibility condition on the jungles gives smallness when integrating over the roots of the jungles (see (8.2.4)). The compatibility condition on the forests $\lambda_2, \dots, \lambda_\ell$ is obtained by integrating (8.2.5) as in Section 8.2. We now have to combine the recollision condition with the compatibility conditions on λ_1 to obtain the desired estimate. As in the previous chapter, we denote by $\tilde{a}$ the restriction of

the tree a to λ_1, and by $\tilde{a}_k$, $\mathcal{C}_k$ the tree variables and the cross section factors associated with the s_k creations occurring in the time interval $(\tau_{\mathrm{rec,k}}, \tau_{\mathrm{rec,k-1}})$.

We start from (8.2.5), adding the recollision condition: we get

$$\sum_{\tilde{a}} \int dx^*_{\lambda_1,1} \dots dx^*_{\lambda_1,\ell_1-1} \, \mathbb{\Delta}_{\lambda_1} \, \mathbf{1}_{\mathcal{G}}\big(\Psi^{\varepsilon}_{\lambda_1}\big) |\mathcal{C}\big(\Psi^{\varepsilon}_{\lambda_1}\big)| \, \mathbf{1}_{\Psi^{\varepsilon}_{\lambda_1} \text{ has a non clustering recollision}}$$
$$\leq \sum_{\tilde{a}_1} |\mathcal{C}_1\big(\Psi^{\varepsilon}_{\lambda_1}\big)| \int d\hat{x}_1 \mathbf{1}_{\mathcal{B}_1} \sum_{\tilde{a}_2} |\mathcal{C}_2\big(\Psi^{\varepsilon}_{\lambda_1}\big)| \int d\hat{x}_2 \dots$$
$$\times \int d\hat{x}_{\ell_1-1} \mathbf{1}_{\mathcal{B}_{\ell_1-1}} \sum_{\tilde{a}_{\ell_1}} |\mathcal{C}_{\ell_1}\big(\Psi^{\varepsilon}_{\lambda_1}\big)| \, \mathbf{1}_{\Psi^{\varepsilon}_{\lambda_1} \text{ has a non clustering recollision}} \cdot$$

As shown in the previous section, the set of parameters leading to the additional recollision can be described in terms of a first deflection at a time $\bar{t}$. We then have to improve the iteration scheme of Section 8.2, on the time interval $[\tau_{\mathrm{rec},k}, \tau_{\mathrm{rec},k+1}]$ containing the time $\bar{t}$. There are two different situations depending on whether the time $\bar{t}$ corresponds to a creation, or to a clustering recollision.

If $\bar{t}$ corresponds to a creation of a particle, say c, the condition on the recollision can be expressed in terms of the collision parameters $(\bar{t}, \bar{v}, \bar{\omega}) = (t_c, v_c, \omega_c)$. We therefore have to

— use (8.2.6) to control the collision cross sections $\big|\mathcal{C}_j\big(\Psi^{\varepsilon}_{\lambda_1}\big)\big|$ for integration variables indexed by $s \in \{c+1, \dots, S_j\}$;

— use the integral with respect to $\bar{t}, \bar{\omega}, \bar{v}$ to gain a factor

$$C(1+\mathbb{V}_{n+m})^{2d+3/2}(1+t)^{d+1/2} \min\left(1, \frac{\varepsilon^{1/2}}{|\bar{v}_q - v_{q'}|}\right)$$

by Proposition 9.2.2. Note that the geometric condition for the recollision between q and q' does not depend on the parameters which have been integrated already at this stage, and to simplify from now on all velocities are bounded by $\mathbb{V}_{n+m}$;

— use (8.2.6) to control the collision cross sections $\big|\mathcal{C}_j\big(\Psi^{\varepsilon}_{\lambda_1}\big)\big|$ for $s \in \{S_{j-1}+1, \dots, c-1\}$;

— use the integral with respect to $\hat{x}_j$ to gain smallness due to the clustering recollision.

Note that, since $\bar{t}$ is dealt with separately, we shall lose a power of t as well as a factor $m \leq \ell m_0$ in the time integral. We shall also lose another factor K^2 corresponding to all possible choices of recollision pairs (q, q'): at this stage we shall not be too precise in the control of the constants in terms of n, and m_0, contrary to the previous chapter.

If $\bar{t} = \tau_{\mathrm{rec},k}$ corresponds to a clustering recollision, we use the same iteration as in Section 8.2:

— use (8.2.6) to control the collision cross sections $\big|\mathcal{C}_k\big(\Psi^{\varepsilon}_{\lambda_1}\big)\big|$;

— use the integral with respect to $\hat{x}_k$ to gain some smallness due to the clustering recollision, multiplied by the additional smallness due to the non clustering recollision.

As in the first case, we shall lose a factor K^2 corresponding to all possible choices of recollision pairs.

After this first stage, we still need to integrate the singularity with respect to velocity variables, which requires introducing the next deflection (moving up the root).

We therefore perform the same steps as above, but integrate the singularity

$$\min\left(1, \frac{\varepsilon^{1/2}}{|v_q - v_{q'}|}\right)$$

by using Proposition 9.2.3.

Remark 9.2.5. — *Note that it may happen that the two deflection times used in the process are in the same time interval* $[\tau_{\mathrm{rec},k}, \tau_{\mathrm{rec},k+1}]$*, which does not bring any additional difficulty. We just set apart the two corresponding integrals in the collision parameters if both correspond to the creation of new particles.*

Integrating with respect to the remaining variables in (T_m, Ω_m, V_m) and following the strategy described above leads to the following bound

$$\left| \int \left(\prod_{i=1}^{\ell} \mathbb{\Delta}_{\lambda_i} \, \mathcal{C}\big(\Psi^{\varepsilon}_{\lambda_i}\big) \, \mathbf{1}_{\mathcal{G}}\big(\Psi^{\varepsilon}_{\lambda_i}\big) \mathcal{H}\big(\Psi_{\lambda_i}\big) \right) \mathbf{1}_{\Psi^{\varepsilon}_{\lambda_1} \text{ has a non clustering recollision}} \quad \varphi_{\{1,\dots,\ell\}} f^{\varepsilon 0}_{\{1\}} dT_m d\Omega_m dV_m dZ^*_n \right|$$

$$\leq \ell! \varepsilon^{\frac{1}{8}} (\ell m_0)^4 C^n \left(\frac{(t+\varepsilon)}{\mu_\varepsilon} \right)^{n-1} (Ct)^m (1+t)^d \, .$$

Finally summing over $m \leq \ell m_0$ and over all possible partitions, we find

$$\forall n \geq 1, \qquad \big| f^{\varepsilon, m_0}_{n,[0,t]}(H^{\otimes n}) - \tilde{f}^{\varepsilon, m_0}_{n,[0,t]}(H^{\otimes n}) \big| \leq C^n (t+1)^{n+d-1} n! \varepsilon^{1/8} \, ,$$

where C depends on C_0, α_0, β_0 and m_0. This concludes the proof of Proposition 9.2.4. □

In the following, we shall denote by $\mathcal{B}^\varepsilon$ the set of integration parameters leading to a non clustering recollision/overlap or to a non regular overlap.

9.3. Proof of Theorem 10 : convergence of the cumulants

In order to conclude the proof of Theorem 10, we now have to compare $\tilde{f}^{\varepsilon, m_0}_{n,[0,t]}(H^{\otimes n})$ and $f_{n,[0,t]}(H^{\otimes n})$ defined in (5.1.4) as

$$f_{n,[0,t]}(H^{\otimes n}) = \sum_{T \in \mathcal{T}^{\pm}_n} \sum_m \sum_{a \in \mathcal{A}^{\pm}_{n,m}} \int d\mu_{\mathrm{sing},T,a}(\Psi_{n,m}) \mathcal{H}(\Psi_{n,m}) \left(f^0\right)^{\otimes(n+m)} (\Psi^0_{n,m}) \, .$$

The comparison will be achieved by coupling the pseudo-trajectories and this requires discarding the pathological trajectories leading to non clustering recollisions/overlaps and non regular overlaps. Thus we define the modified limiting cumulants by restricting the integration parameters to the set $\mathcal{G}^\varepsilon$, which avoids internal overlaps in collision trees of the same forest at the creation times, and by removing the set $\mathcal{B}^\varepsilon$ introduced at the end of the previous section

$$\tilde{f}^{m_0}_{n,[0,t]}(H^{\otimes n}) := \sum_{T \in \mathcal{T}^{\pm}_n} \sum_m \sum_{a \in \mathcal{A}^{\pm}_{n,m}} \int d\mu^{m_0}_{\mathrm{sing},T,a}(\Psi_{n,m}) \mathcal{H}(\Psi_{n,m}) \mathbf{1}_{\mathcal{G}^\varepsilon \setminus \mathcal{B}^\varepsilon} \left(f^0\right)^{\otimes(n+m)} (\Psi^0_{n,m}) \, ,$$

where $d\mu^{m_0}_{\mathrm{sing},T,a}$ stands for the singular measure with at most m_0 collisions in each forest. We stress the fact that $\tilde{f}^{m_0}_{n,[0,t]}(H^{\otimes n})$ depends on ε only through the sets $\mathcal{B}^\varepsilon$ and $\mathcal{G}^\varepsilon$. We are going to check that

$$\lim_{m_0 \to \infty} \lim_{\varepsilon \to 0} |f_{n,[0,t]}(H^{\otimes n}) - \tilde{f}^{m_0}_{n,[0,t]}(H^{\otimes n})| = 0 \, . \tag{9.3.1}$$

The analysis of the two previous sections may be performed for the limiting cumulants so that restricting the number of collisions to be less than m_0 in each forest and the integration parameters outside the set $\mathcal{B}^\varepsilon$ leads to a small error. The control of internal overlaps, associated with $\mathcal{G}^\varepsilon$, relies on the same geometric arguments as discussed in Section 9.2.1: indeed, in order for an overlap to arise when adding particle k at time t_k, one should already have a particle which is at distance less than 2ε from particle a_k, which is a generalized recollision situation (replacing ε by 2ε). This completes (9.3.1).

In order to compare $\tilde{f}^{m_0}_{n,[0,t]}(H^{\otimes n})$ and $\tilde{f}^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n})$, we first compare the initial measures, namely $f^{\varepsilon 0}_{\{1\}}$ with $(f^0)^{\otimes(n+m)}$. This is actually an easy matter as returning to (8.1.18) we see that the leading order term in the decomposition of $f^{\varepsilon 0}_{\{1\}}$ is F^0_{n+m}, which is well known to tensorize asymptotically as μ_ε goes to infinity (for fixed $n+m$), as stated by the following proposition.

Proposition 9.3.1 **([17])**. — *If f^0 satisfies* (1.1.5)*, there exists $C>0$ such that*

$$\forall m\,, \qquad \left| \left(F^0_m - \left(f^0\right)^{\otimes m} \right) \mathbf{1}_{\mathcal{D}^m_\varepsilon}(Z_m) \right| \leq C^m \varepsilon\, e^{-\frac{3\beta_0}{8}|V_m|^2}\,.$$

At this stage, we are left with a final discrepancy between $\tilde{f}^{m_0}_{n,[0,t]}(H^{\otimes n})$ and $\tilde{f}^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n})$ which is due to the initial data and $\mathcal{H}$ being evaluated at different configurations (namely Ψ_n and Ψ^ε_n). We then need to introduce a suitable coupling.

In Chapter 5, we used the change of variables (5.1.1) to reparametrize the limiting pseudo-trajectories in terms of x^*_n, V^*_n and $n-1$ recollision parameters (times and angles). In the same way, for fixed ε, we can use the parametrization of clustering recollisions (4.4.5) and of regular clustering overlaps (8.1.13) to reparametrize the non pathological pseudo-trajectories in terms of x^*_n, V^*_n and $n-1$ recollision parameters (times and angles). As in (5.1.3), we define the singular measure for each tree $a \in \mathcal{A}^{\pm}_{n,m}$ and each minimally connected graph $T \in \mathcal{T}^{\pm}_n$

$$\begin{aligned} d\mu^\varepsilon_{\mathrm{sing},T,a} := {} & dT_m d\Omega_m dV_m dx^*_n dV^*_n d\Theta^{\mathrm{clust}}_{n-1} d\omega^{\mathrm{clust}}_{n-1} \prod_{i=1}^m s_i\big((v_i - v_{a_j}(t_i)\cdot\omega_i\big)_+ \\ & \times \prod_{e\in E(T)} \sum_{\{q_e,q'_e\}\approx e} s^{\mathrm{clust}}_e\big((v_{q_e}(\tau^{\mathrm{clust}}_e) - v_{q'_e}(\tau^{\mathrm{clust}}_e))\cdot\omega^{\mathrm{clust}}_e\big)_+ \mathbf{1}_{\mathcal{G}^\varepsilon\setminus\mathcal{B}^\varepsilon} \end{aligned} \tag{9.3.2}$$

denoting by $\{q_e, q'_e\} \approx e$ the fact that $\{q_e, q'_e\}$ is a representative of the edge e, and by $\Theta^{\mathrm{clust}}_{n-1}$ and $\Omega^{\mathrm{clust}}_{n-1}$ the $n-1$ clustering times τ^{clust}_e and angles $\omega^{\mathrm{clust}}_e$ for $e \in E(T)$.

We can therefore couple the pseudo-trajectories Ψ_n and Ψ^ε_n by their (identical) collision and clustering parameters. The error between the two configurations Ψ^ε_n and Ψ_n is due to the fact that collisions, recollisions and overlaps become pointwise in the limit but generate a shift of size $O(\varepsilon)$ for fixed ε. We then have

$$|\Psi^\varepsilon_n(\tau) - \Psi_n(\tau)| \leq C(n+m)\,\varepsilon \quad \text{ for all } \tau\in[0,t]\,.$$

Such discrepancies concern only the positions, as the velocities remain equal in both flows.

It follows that

$$\left| \left(f^0\right)^{\otimes(n+m)}(\Psi^{\varepsilon 0}_n) - \left(f^0\right)^{\otimes(n+m)}(\Psi^0_n) \right| \leq C_{n,m_0}\varepsilon e^{-\frac{3\beta}{8}|V_{m+n}|^2}\,,$$

having used the Lipschitz continuity (1.1.5) of f^0. Using the same reasoning for $\mathcal{H}$ (assumed to be continuous), we find finally that for all n, m_0

$$\lim_{\varepsilon\to 0} |\tilde{f}^{\varepsilon,m_0}_{n,[0,t]}(H^{\otimes n}) - \tilde{f}^{m_0}_{n,[0,t]}(H^{\otimes n})| = 0\,.$$

This result, along with Proposition 9.2.4, Estimates (9.1.2), (9.1.3) and (9.3.1) proves Theorem 10. □

9.4. Analysis of the geometric conditions

In this section we prove Propositions 9.2.1 to 9.2.3. Without loss of generality, we will assume that the velocities $\mathbb{V}_j$ are all larger than 1.

Self-recollision: proof of Proposition 9.2.1. Denote by q, q' the recolliding particles. By definition of a self-recollision, their first deflection (going forward in time) involves both particles q and q'. It can be either a creation (say of q without loss of generality, in the tree $\Psi^{\varepsilon}_{\{i\}}$ of q'), or a clustering recollision between two trees (say $\Psi^{\varepsilon}_{\{j_k\}}$ and $\Psi^{\varepsilon}_{\{j'_k\}}$ in $\Psi^{\varepsilon}_{\lambda_1}$) (see Figure 8).

• If the first deflection corresponds to the creation of q, we denote by $(\bar t, \bar\omega, \bar v)$ the parameters encoding this creation. We also denote by $\bar v_q$ the velocity of q' just after that deflection in the forward dynamics, and by $\Psi^{\varepsilon}_{\{i\}}$ the collision tree of q' (and q). Denoting by v_q and $v_{q'}$ the velocities of q and q' after adjunction of q (in the backward dynamics) there holds

$$\varepsilon\bar\omega + (v_q - v_{q'})(t_{\rm rec} - \bar t) = \varepsilon\omega_{\rm rec} + \zeta \text{ with } \zeta \in \mathbb{Z}^d \setminus \{0\} \tag{9.4.1}$$

which implies that $v_q - v_{q'}$ has to belong to the intersection K_ζ of a cone of opening ε with a ball of radius $2\mathbb{V}_i$.

Note that the number of ζ's for which the sets are not empty is at most $O\big(\mathbb{V}_i^d t^d\big)$.

— If the creation of q is without scattering, then $v_q - v_{q'} = \bar v - \bar v_q$ has to belong to the union of the K_ζ's, and

$$\int \mathbf{1}_{\text{Self-recollision with creation at time } \bar t \text{ without scattering}} \big|(\bar v - \bar v_q)\cdot\bar\omega\big| d\bar t d\bar\omega d\bar v \\ \leq C\mathbb{V}_i^d t^d \sup_\zeta \int \mathbf{1}_{\bar v - \bar v_q \in K_\zeta} \big|(\bar v - \bar v_q)\cdot\bar\omega\big| d\bar t d\bar\omega d\bar v \leq C\varepsilon^{d-1}\mathbb{V}_i^d (\mathbb{V}_i t)^{d+1}\,.$$

— If the creation of q is with scattering, then $v_q - v_{q'} = \bar v - \bar v_q - 2(\bar v - \bar v_q)\cdot\bar\omega\,\bar\omega$ has to belong to the union of the K_ζ's. Equivalently $\bar v - \bar v_q$ lies in the union of the $S_{\bar\omega}K_\zeta$'s (obtained from K_ζ by symmetry with respect to $\bar\omega$), and there holds

$$\int \mathbf{1}_{\text{Self-recollision with creation at time } \bar t \text{ with scattering}} \big|(\bar v - \bar v_q)\cdot\bar\omega\big| d\bar t d\bar\omega d\bar v \\ \leq C\mathbb{V}_i^d t^d \sup_\zeta \int \mathbf{1}_{\bar v - \bar v_q \in S_{\bar\omega}K_\zeta} \big|(\bar v - \bar v_q)\cdot\bar\omega\big| d\bar t d\bar\omega d\bar v \leq C\varepsilon^{d-1}\mathbb{V}_i^d (\mathbb{V}_i t)^{d+1}\,.$$

• If the first deflection corresponds to the k-th clustering recollision between $\Psi^{\varepsilon}_{\{j_k\}}$ and $\Psi^{\varepsilon}_{\{j'_k\}}$ in the forest $\Psi^{\varepsilon}_{\lambda_1}$ for instance, in addition to the condition $\hat x_k \in B_{qq'}$ which encodes the clustering recollision (see Section 8.1), we obtain the condition

$$\begin{aligned} \varepsilon\omega_{{\rm rec},k} + (v_q - v_{q'})(t_{\rm rec} - \tau_{{\rm rec},k}) &= \varepsilon\omega_{\rm rec} + \zeta \text{ with } \zeta \in \mathbb{Z}^d \\ \text{and } v_q - v_{q'} &= \bar v_q - \bar v_{q'} - 2(\bar v_q - \bar v_{q'})\cdot\omega_{{\rm rec},k}\,\omega_{{\rm rec},k} \end{aligned} \tag{9.4.2}$$

denoting by $\bar v_q, \bar v_{q'}$ the velocities before the clustering recollision in the backwards dynamics, and by $\omega_{{\rm rec},k}$ the impact parameter at the clustering recollision. We deduce from the first relation that $v_q - v_{q'}$ has to be in a small cone K_ζ of opening ε, which implies by the second relation that $\omega_{{\rm rec},k}$ has to be in a small cone S_ζ of opening ε.

Using the change of variables (5.1.1), it follows that

$$\int \mathbf{1}_{\text{Self-recollision with clustering at time } \bar t}\, d\hat x_k \leq C\varepsilon^{d-1} t \sup_\zeta \int \mathbf{1}_{\omega_{{\rm rec},k}\in S_\zeta} \big((\bar v_q - \bar v_{q'})\cdot\omega_{{\rm rec},k}\big) d\omega_{{\rm rec},k} \\ \leq C\varepsilon^{2(d-1)} \Big(t\big(\mathbb{V}_{j_k} + \mathbb{V}_{j'_k}\big)\Big)^{d+1}\,.$$

This concludes the proof of Proposition 9.2.1. □

Non clustering recollision: proof of Proposition 9.2.2

Denote by q, q' the recolliding particles. Without loss of generality, we can assume that the first deflection (when going up the tree) involves only particle q, at some time $\bar{t}$. It can be either a creation (with or without scattering), or a clustering recollision.

• If the first deflection of q corresponds to a creation, we denote by $(\bar{t}, \bar{\omega}, \bar{v})$ the parameters encoding this creation, and by $(\bar{x}_q, \bar{v}_q)$ the position and velocity of the particle q before the creation in the backward dynamics. As explained before the statement of Proposition 9.2.2, we use that notation even if q is created at time $\bar{t}$ and c is its parent. Note that locally in time (up to the next deflection) $\bar{v}_q$ is constant, and $\bar{x}_q$ is an affine function. In the same way, denoting by $(\bar{x}_{q'}, \bar{v}_{q'})$ the position and velocity of the particle q', we have that $\bar{v}_{q'}$ is locally constant while $\bar{x}_{q'}$ is affine.

There are actually three subcases :

(a) particle q is created without scattering : $v_q = \bar{v}$;
(b) particle q is created with scattering : $v_q = \bar{v} + (\bar{v} - \bar{v}_q) \cdot \bar{\omega}\, \bar{\omega}$;
(c) another particle is created next to q, and q is scattered : $v_q = \bar{v}_q + (\bar{v} - \bar{v}_q) \cdot \bar{\omega}\, \bar{\omega}$.

The equation for the recollision states

$$\begin{aligned} \bar{x}_q(\bar{t}) + \varepsilon\bar{\omega} - \bar{x}_{q'}(\bar{t}) + (v_q - \bar{v}_{q'})(t_{\rm rec} - \bar{t}) = \varepsilon\omega_{\rm rec} + \zeta \ \text{ in cases (a)-(b)}, \\ \bar{x}_q(\bar{t}) - \bar{x}_{q'}(\bar{t}) + (v_q - \bar{v}_{q'})(t_{\rm rec} - \bar{t}) = \varepsilon\omega_{\rm rec} + \zeta \ \text{ in case (c)}\,. \end{aligned} \tag{9.4.3}$$

Let us set

$$\mathbb{V}_{i,i'} := \mathbb{V}_i + \mathbb{V}_{i'}\,.$$

We fix from now on the parameter $\zeta \in \mathbb{Z}^d \cap B_{\mathbb{V}_{i,i'}t}$ encoding the periodicity, and the estimates will be multiplied by $\mathbb{V}_{i,i'}^d t^d$ at the very end. Define

$$\begin{aligned} \delta x &:= \frac{1}{\varepsilon}(\bar{x}_{q'}(\bar{t}) - \varepsilon\bar{\omega} - \bar{x}_q(\bar{t}) + \zeta) =: \delta x_\perp + \frac{1}{\varepsilon}(\bar{v}_{q'} - \bar{v}_q)(\bar{t} - t_0) \text{ in cases (a)-(b)}\ , \\ \delta x &:= \frac{1}{\varepsilon}(\bar{x}_{q'}(\bar{t}) - \bar{x}_q(\bar{t}) + \zeta) =: \delta x_\perp + \frac{1}{\varepsilon}(\bar{v}_{q'} - \bar{v}_q)(\bar{t} - t_0) \text{ in case (c)}\,, \\ \tau_{\rm rec} &:= (t_{\rm rec} - \bar{t})/\varepsilon \quad \text{ and } \tau := (\bar{t} - t_0)/\varepsilon\,, \end{aligned}$$

where $\delta x_\perp$ is orthogonal to $\bar{v}_{q'} - \bar{v}_q$ (this constraint defines the parameter t_0). Then (9.4.3) can be rewritten

$$v_q - \bar{v}_{q'} = \frac{1}{\tau_{\rm rec}}\Big(\omega_{\rm rec} + \delta x_\perp + \tau(\bar{v}_{q'} - \bar{v}_q)\Big). \tag{9.4.4}$$

We know that $v_q - \bar{v}_{q'}$ belongs to a ball of radius $\mathbb{V}_{i,i'}$. In the case when $|\tau(\bar{v}_{q'} - \bar{v}_q)| \geq 2$, the triangular inequality gives

$$\frac{1}{2\tau_{\rm rec}}\big|\tau(\bar{v}_{q'} - \bar{v}_q)\big| \leq \frac{1}{\tau_{\rm rec}}\Big|\omega_{\rm rec} + \delta x_\perp + \tau(\bar{v}_{q'} - \bar{v}_q)\Big| = |v_q - \bar{v}_{q'}| \leq \mathbb{V}_{i,i'}$$

and we deduce that

$$\frac{1}{\tau_{\rm rec}} \leq \frac{2\mathbb{V}_{i,i'}}{|\tau||\bar{v}_{q'} - \bar{v}_q|}$$

hence $v_q - \bar{v}_{q'}$ belongs to a cylinder of main axis $\delta x_\perp + \tau(\bar{v}_{q'} - \bar{v}_q)$ and of width $2\mathbb{V}_{i,i'}/|\tau||\bar{v}_q - \bar{v}_{q'}|$. In any case, (9.4.4) forces $v_q - \bar{v}_{q'}$ to belong to a cylinder $\mathcal{R}$ of main axis $\delta x_\perp + \tau(\bar{v}_{q'} - \bar{v}_q)$ and of width $C\mathbb{V}_{i,i'} \min\Big(\frac{1}{|\tau||\bar{v}_q - \bar{v}_{q'}|}, 1\Big)$. In any dimension $d \geq 2$, the volume of this cylinder is less than $C\mathbb{V}_{i,i'}^d \min\Big(\frac{1}{|\tau||\bar{v}_q - \bar{v}_{q'}|}, 1\Big)$.

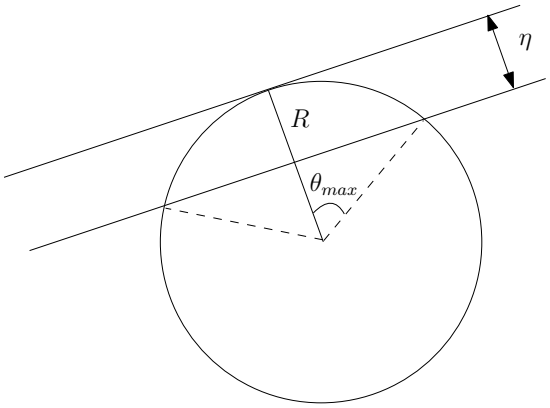
Case (a). Since $v_q = \bar{v}$, Equation (9.4.4) forces $\bar{v} - \bar{v}_{q'}$ to belong to the cylinder $\mathcal{R}$. Recall that τ is a rescaled time, with

$$|(\bar{v}_q - \bar{v}_{q'})\tau| \leq \frac{t}{\varepsilon}|\bar{v}_q - \bar{v}_{q'}| + |\delta x_\parallel| \leq \frac{C}{\varepsilon}(\mathbb{V}_{i,i'}t + 1)\,.$$

Then

$$\begin{aligned}\int_{|\bar{v}|\leq \mathbb{V}_{i,i'}} \mathbf{1}_{\bar{v}-\bar{v}_{q'}\in\mathcal{R}}\,\big|(\bar{v}-\bar{v}_q)\cdot\bar{\omega}\big|d\bar{t}d\bar{\omega}d\bar{v} &\leq C\mathbb{V}_{i,i'}^{d+1}\int_{-C(\mathbb{V}_{i,i'}t+1)/\varepsilon}^{C(\mathbb{V}_{i,i'}t+1)/\varepsilon}\min\left(\frac{1}{|u|},1\right)\varepsilon\frac{du}{|\bar{v}_q-\bar{v}_{q'}|}\\ &\leq C\mathbb{V}_{i,i'}^{d+1}\frac{\varepsilon\big(|\log(\mathbb{V}_{i,i'}t+1)|+|\log\varepsilon|\big)}{|\bar{v}_q-\bar{v}_{q'}|}\,.\end{aligned}$$

Cases (b) and (c). By definition, v_q belongs to the sphere of diameter $[\bar{v}, \bar{v}_q]$. The intersection I of this sphere and of the cylinder $\bar{v}_{q'} + \mathcal{R}$ is a union of spherical caps, and we can estimate the solid angles of these caps.



$$\theta \leq \theta_{max} \leq C\left(\tfrac{\eta}{R}\right)^{1/2}$$

FIGURE 10. Intersection of a cylinder and a sphere. The solid angle of the spherical caps is less than $C_d \min(1, (\eta/R)^{1/2})$.

A basic geometrical argument shows that $\bar{\omega}$ has therefore to be in a union of solid angles of measure less than $C\min\Big(\big(\frac{\mathbb{V}_{i,i'}}{|\tau||\bar{v}_q-\bar{v}_{q'}||\bar{v}_q-\bar{v}|}\big)^{1/2},1\Big)$. Integrating first with respect to $\bar{\omega}$, then with respect to $\bar{v}$ and $\bar{t}$, we obtain

$$\begin{aligned}\int_{|\bar{v}|\leq \mathbb{V}_{i,i'}} \mathbf{1}_{v_q\in I}\,\big|(\bar{v}-\bar{v}_q)\cdot\bar{\omega}\big|d\bar{t}d\bar{\omega}d\bar{v} &\leq \mathbb{V}_{i,i'}\int_{|\bar{v}|\leq\mathbb{V}_{i,i'}}\min\Big(\Big(\frac{1}{|\tau||\bar{v}_q-\bar{v}_{q'}|}\Big)^{1/2},1\Big)d\bar{t}d\bar{v}\\ &\leq C\mathbb{V}_{i,i'}^{d+1}\int_{-C(\mathbb{V}_{i,i'}t+1)/\varepsilon}^{C(\mathbb{V}_{i,i'}t+1)/\varepsilon}\min\Big(\frac{1}{|u|^{1/2}},1\Big)\varepsilon\frac{du}{|\bar{v}_q-\bar{v}_{q'}|}\\ &\leq C\mathbb{V}_{i,i'}^{d+\frac{3}{2}}\frac{\varepsilon^{1/2}t^{\frac{1}{2}}}{|\bar{v}_q-\bar{v}_{q'}|}\,.\end{aligned}$$

We obtain finally that

$$\int \mathbf{1}_{\text{Recollision of type (a)(b)(c)}}\big|(\bar{v}-\bar{v}_q)\cdot\bar{\omega}\big|d\bar{t}d\bar{\omega}d\bar{v} \leq C\mathbb{V}_{i,i'}^{2d+\frac{3}{2}}(1+t)^{d+\frac{1}{2}}\frac{\varepsilon^{\frac{1}{2}}}{|\bar{v}_q-\bar{v}_{q'}|}\,.$$

• If the first deflection of q corresponds to a clustering recollision. With the notation of Section 8.1 we assume the clustering recollision is the k-th recollision in $\Psi^\varepsilon_{\lambda_1}$ between the trees $\Psi^\varepsilon_{j_k}$ and $\Psi^\varepsilon_{j'_k}$,

involving particles $q \in \Psi^\varepsilon_{\{j_k\}}$ and $c \in \Psi^\varepsilon_{\{j'_k\}}$ (with $c \neq q'$) at time $\bar t = \tau_{\mathrm{rec},k}$. Then in addition to the condition

$$\hat x_k \in B_{qc}$$

which encodes the clustering recollision (see Section 8.1), we obtain the condition

$$\begin{aligned} \big(\bar x_q(\tau_{\mathrm{rec},k}) - x_{q'}(\tau_{\mathrm{rec},k})\big) + (v_q - \bar v_{q'})(t_{\mathrm{rec}} - \tau_{\mathrm{rec},k}) = \varepsilon\omega_{\mathrm{rec}} + \zeta\,, \\ \text{and } v_q = \bar v_q - (\bar v_q - \bar v_c)\cdot\omega_{\mathrm{rec},k}\,\omega_{\mathrm{rec},k} \end{aligned} \tag{9.4.5}$$

denoting by $(\bar x_q, \bar v_q)$ and $(\bar x_c, \bar v_c)$ the positions and velocities of q and c before the clustering recollision. Note that, as previously, $\bar v_q$ and $\bar v_c$ are locally constant. Defining as above

$$\delta x := \frac{1}{\varepsilon}(\bar x_q(\tau_{\mathrm{rec},k}) - x_q(\tau_{\mathrm{rec},k}) + \zeta) =: \delta x_\perp + (\bar v_{q'} - \bar v_q)(\tau_{\mathrm{rec},k} - t_0)/\varepsilon\,,$$

and the rescaled times

$$\tau_{\mathrm{rec}} := (t_{\mathrm{rec}} - \tau_{\mathrm{rec},k})/\varepsilon \quad \text{ and } \tau =: (\tau_{\mathrm{rec},k} - t_0)/\varepsilon\,,$$

we end up with the equation (9.4.4), which forces $v_q - \bar v_{q'}$ to belong to a cylinder $\mathcal{R}$ of main axis $\delta x_\perp - \tau(\bar v_q - \bar v_{q'})$ and of width $C\mathbb{V}_{j_k,j'_k,i'}\min\Big(\frac{1}{|\tau(\bar v_q - \bar v_{q'})|}, 1\Big)$, where

$$\mathbb{V}_{j_k,j'_k,i'} := \mathbb{V}_{j_k} + \mathbb{V}_{j'_k} + \mathbb{V}_{i'}$$

and $\Psi^\varepsilon_{\{i'\}}$ is the collision tree of q'. Then v_q has to be in the intersection of the sphere of diameter $[\bar v_q, \bar v_c]$ and of the cylinder $\bar v_{q'} + \mathcal{R}$. This implies that $\omega_{\mathrm{rec},k}$ has to belong to a union of spherical caps S, of solid angle less than $C\min\Big(\big(\frac{\mathbb{V}_{j_k,j'_k,i'}}{|\tau||\bar v_q - \bar v_{q'}||\bar v_q - \bar v_c|}\big)^{1/2}, 1\Big)$. Using the (local) change of variables $\hat x_k \mapsto (\tau_{\mathrm{rec},k}, \varepsilon\omega_{\mathrm{rec},k})$, it follows that

$$\begin{aligned} \int \mathbf{1}_{\text{Recollision of type (d)}} d\hat x_k &\le \frac{C}{\mu_\varepsilon}\int \mathbf{1}_{\omega_{\mathrm{rec},k}\in S}|(\bar v_q - \bar v_c)\cdot\omega_{\mathrm{rec},k}|d\omega_{\mathrm{rec},k}d\tau_{\mathrm{rec},k} \\ &\le \frac{C}{\mu_\varepsilon}\mathbb{V}^{\frac32}_{j_k,j'_k,i'}(1+t)^{\frac12}\frac{\varepsilon^{1/2}}{|\bar v_q - \bar v_{q'}|}\,. \end{aligned}$$

This concludes the proof of Proposition 9.2.2. □

Integration of the singularity in relative velocities: proof of Proposition 9.2.3

We start with the obvious estimate

$$\min\Big(1, \frac{\varepsilon^{1/2}}{|v_q - v_{q'}|}\Big) \le \varepsilon^{\frac14} + \mathbf{1}_{|v_q - v_{q'}|\le\varepsilon^{1/4}}\,. \tag{9.4.6}$$

Thus we only need to control the set of parameters leading to small relative velocities.

Without loss of generality, we shall assume that the first deflection (when going up the tree) involves particle q. It can be either a creation (with or without scattering), or a clustering recollision, say between $q \in \Psi^\varepsilon_{\{j_k\}}$ and $c \in \Psi^\varepsilon_{\{j'_k\}}$.

• If the first deflection of q corresponds to a creation, we denote by $(\bar t, \bar\omega, \bar v)$ the parameters encoding this creation, and by $(\bar x_q, \bar v_q)$ and $(\bar x_{q'}, \bar v_{q'})$ the positions and velocities of the pseudo-particles q and q' before the creation.

There are actually four subcases :

(a) particle q' is created next to particle q in the tree $\Psi^\varepsilon_{\{i\}}$: $|v_q - v_{q'}| = |\bar v - \bar v_q|$;

(b) particle q' is not deflected and particle q is created without scattering next to $\bar q$ in the tree $\Psi^\varepsilon_{\{i\}}$: $|v_q - v_{q'}| = |\bar v - \bar v_{q'}|$;

(c) particle q' is not deflected and particle q is created with scattering next to $\bar{q}$ in the tree $\Psi^{\varepsilon}_{\{i\}}$: $v_q = \bar{v} - (\bar{v} - \bar{v}_q) \cdot \bar{\omega}\, \bar{\omega}$;

(d) particle q' is not deflected, another particle is created next to q in the tree $\Psi^{\varepsilon}_{\{i\}}$, and q is scattered so $v_q = \bar{v}_q + (\bar{v} - \bar{v}_q) \cdot \bar{\omega}\, \bar{\omega}$.

In cases (a) and (b), we obtain that $\bar{v}$ has to be in a small ball of radius $\varepsilon^{1/4}$. Then,

$$\int \mathbf{1}_{\text{Small relative velocity of type (a)(b)}} \big|(\bar{v} - \bar{v}_q) \cdot \bar{\omega}\big| d\bar{t} d\bar{\omega} d\bar{v} \leq C \mathbb{V}_i t \varepsilon^{d/4} .$$

In cases (c) and (d), we obtain that v_q has to be in the intersection of a small ball of radius $\varepsilon^{1/4}$ and of the sphere of diameter $[\bar{v}, \bar{v}_q]$. This condition imposes that $\bar{\omega}$ has to be in a spherical cap of solid angle less than $\varepsilon^{\frac{1}{8}} / |\bar{v} - \bar{v}_q|^{1/2}$ (see Figure 10). We find that

$$\int \mathbf{1}_{\text{Small relative velocity of type (c)(d)}} \big|(\bar{v} - \bar{v}_q) \cdot \bar{\omega}\big| d\bar{t} d\bar{\omega} d\bar{v} \leq C \mathbb{V}_i^{d+\frac{1}{2}} t \varepsilon^{\frac{1}{8}} .$$

Combining these two estimates with (9.4.6), we get

$$\int \min\Big(1, \frac{\varepsilon^{1/2}}{|v_q - v_{q'}|}\Big) \big|(\bar{v} - \bar{v}_q) \cdot \bar{\omega}\big| d\bar{t} d\bar{\omega} d\bar{v} \leq C \mathbb{V}_i^{d+1} t \varepsilon^{\frac{1}{8}} .$$

• If the first deflection of q corresponds to the k-th clustering recollision in $\Psi^{\varepsilon}_{\lambda_1}$ between $q \in \Psi^{\varepsilon}_{\{j_k\}}$ and $c \in \Psi^{\varepsilon}_{\{j'_k\}}$ at time $\bar{t} = \tau_{\mathrm{rec},k}$, in addition to the condition $\hat{x}_k \in B_{qc}$ which encodes the clustering recollision (see Section 8.1), we obtain a condition on the velocity.

There are actually two subcases :

(e) $q' = c$ and $|v_q - v_{q'}| = |\bar{v}_q - \bar{v}_{q'}|$;

(f) q' is not deflected, and $v_q = \bar{v}_q - (\bar{v}_q - \bar{v}_c) \cdot \omega_{\mathrm{rec},k}\, \omega_{\mathrm{rec},k}$.

In case (e), there holds

$$\int \mathbf{1}_{\text{Small relative velocity of type (e)}} d\hat{x}_k \leq \frac{C}{\mu_\varepsilon} \int \mathbf{1}_{|\bar{v}_q - \bar{v}_{q'}| \leq \varepsilon^{1/4}} \big|(\bar{v}_q - \bar{v}_{q'}) \cdot \omega\big| d\omega d\tau_{\mathrm{rec},k} \leq \frac{C t \varepsilon^{\frac{1}{4}}}{\mu_\varepsilon} .$$

In case (f), we obtain that v_q has to be in the intersection of a small ball of radius $\varepsilon^{1/4}$ and of the sphere of diameter $[\bar{v}_q, \bar{v}_c]$. This condition imposes that $\omega_{\mathrm{rec},k}$ has to be in a spherical cap of solid angle less than $\varepsilon^{\frac{1}{8}} / |\bar{v}_q - \bar{v}_c|^{1/2}$ (see Figure 10). We find

$$\int \mathbf{1}_{\text{Small relative velocity of type (f)}} d\hat{x}_k \leq \frac{C}{\mu_\varepsilon} \varepsilon^{\frac{1}{8}} \int \big|\bar{v}_q - \bar{v}_c\big|^{1/2} d\tau_{\mathrm{rec},k} \leq \frac{C t \mathbb{V}_{j_k, j'_k}^{\frac{1}{2}} \varepsilon^{\frac{1}{8}}}{\mu_\varepsilon} .$$

Combining these two estimates with (9.4.6), we get

$$\int \min\Big(1, \frac{\varepsilon^{1/2}}{|v_q - v_{q'}|^{1/2}}\Big) d\hat{x}_k \leq \frac{C \mathbb{V}_{j_k, j'_k} t \varepsilon^{\frac{1}{8}}}{\mu_\varepsilon} .$$

This concludes the proof of Proposition 9.2.3. □

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