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# COMPUTING THE REAL ISOLATED POINTS OF AN ALGEBRAIC HYPERSURFACE

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## ABSTRACT

Let  $\mathbf{R}$  be the field of real numbers. We consider the problem of computing the real isolated points of a real algebraic set in  $\mathbf{R}^n$  given as the vanishing set of a polynomial system. This problem plays an important role for studying rigidity properties of mechanism in material designs. In this paper, we design an algorithm which solves this problem. It is based on the computations of critical points as well as roadmaps for answering connectivity queries in real algebraic sets. This leads to a probabilistic algorithm of complexity  $(nd)^{O(n \log(n))}$  for computing the real isolated points of real algebraic hypersurfaces of degree  $d$ . It allows us to solve in practice instances which are out of reach of the state-of-the-art.

**Keywords** Semi-algebraic sets · Critical point method · Real algebraic geometry · Auxetics · Rigidity

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## 1 Introduction

Let  $\mathbf{Q}$ ,  $\mathbf{R}$  and  $\mathbf{C}$  be respectively the fields of rational, real and complex numbers. For  $\mathbf{x} \in \mathbf{R}^n$  and  $r \in \mathbf{R}$ , we denote by  $B(\mathbf{x}, r) \subset \mathbf{R}^n$  the open ball centered at  $\mathbf{x}$  of radius  $r$ .

Let  $f \in \mathbf{Q}[x_1, \dots, x_n]$  and  $\mathcal{H} \subset \mathbf{C}^n$  be the hypersurface defined by  $f = 0$ . We aim at computing the *isolated points* of  $\mathcal{H} \cap \mathbf{R}^n$ , i.e. the set of points  $\mathbf{x} \in \mathcal{H} \cap \mathbf{R}^n$  s.t. for some positive  $r$ ,  $B(\mathbf{x}, r) \cap \mathcal{H} = \{\mathbf{x}\}$ . We shall denote this set of isolated real points by  $\mathcal{I}(\mathcal{H})$ .

**Motivation** We consider here a particular instance of the more general problem of computing the isolated points of a *semi-algebraic* set. Such problems arise naturally and frequently in the design of rigid mechanism in material design. Those are modeled canonically with semi-algebraic constraints, and isolated points to the semi-algebraic set under consideration are related to mobility/rigidity properties of the mechanism. A particular example is the study of *auxetic* materials, i.e., materials that shrink in all directions under compression. These materials appear in nature (first discovered in [20]) e.g., in foams, bones or propylene; see

e.g. [32], and have various potential applications. They are an active field of research, not only on the practical side, e.g., [16, 11], but also with respect to mathematical foundations; see e.g. [5, 6]. On the constructive side, these materials are closely related to *tensegrity frameworks*, e.g., [21, 8], which can possess various sorts of rigidity properties.

Hence, we aim to provide a practical algorithm for computing these real isolated points in the particular case of real traces of complex hypersurfaces first. This simplification allows us to significantly improve the state-of-the-art complexity for this problem and to establish a new algorithmic framework for such computations.

**State-of-the-art** As far as we know, there is no established algorithm dedicated to the problem under consideration here. However, effective real algebraic geometry provides subroutines from which such a computation could be done. Let  $\mathcal{H}$  be a hypersurface defined by  $f = 0$  with  $f \in \mathbf{Q}[x_1, \dots, x_n]$  of degree  $d$ .

A first approach would be to compute a cylindrical algebraic decomposition adapted to  $\mathcal{H} \cap \mathbf{R}^n$  [7]. It partitions  $\mathcal{H} \cap \mathbf{R}^n$  into connected *cells*, i.e. subsets which are homeomorphic to  $]0, 1[^i$  for some  $1 \leq i \leq n$ . Next, one needs to identify cells which correspond to isolated points using adjacency information (see e.g. [1]). Such a procedure is at least doubly exponential in  $n$  and polynomial in  $d$ .

A better alternative is to encode real isolated points with quantified formula over the reals. Using e.g. [2, Algorithm 14.21], one can compute isolated points of  $\mathcal{H} \cap \mathbf{R}^n$  in time  $d^{O(n^2)}$ . Note also that [31] allows to compute isolated points in time  $d^{O(n^3)}$ .

A third alternative (suggested by the reviewers) is to use [2, Algorithm 12.16] to compute sample points in each connected component of  $\mathcal{H} \cap \mathbf{R}^n$  and then decide whether spheres, centered at these points, of infinitesimal radius, meet  $\mathcal{H} \cap \mathbf{R}^n$ . Note that these points are encoded with parametrizations of degree  $d^{O(n)}$  (their coordinates are evaluations of polynomials at the roots of a univariate polynomial with infinitesimal coefficients). Applying [2, Alg. 12.16] on this last real root decision problem would lead to a complexity  $d^{O(n^2)}$  since the input polynomials would have degree  $d^{O(n)}$ . Another approach would be to run [2, Alg. 12.16] modulo the algebraic extension used to define the sample points. That would lead to a complexity  $d^{O(n)}$  but this research direction requires modifications of [2, Alg. 12.16] since it assumes the input coefficients to lie in an *integral domain*, which is not satisfied in our case. Besides, we report on practical experiments showing that using [2, Alg. 12.16] to compute *only* sample points in  $\mathcal{H} \cap \mathbf{R}^n$  does not allow us to solve instances of moderate size.

The topological nature of our problem is related to connectedness. Computing isolated points of  $\mathcal{H} \cap \mathbf{R}^n$  is equivalent to computing those connected components of  $\mathcal{H} \cap \mathbf{R}^n$  which are reduced to a single point (see Lemma 1). Hence, one considers computing *roadmaps*: these are algebraic curves contained in  $\mathcal{H}$  which have a non-empty and connected intersection with all connected components of the real set under study. Once such a roadmap is computed, it suffices to compute the isolated points of a semi-algebraic curve in  $\mathbf{R}^n$ . This latter step is not trivial; as many of the algorithms computing roadmaps output either curve segments (see e.g., [4]) or algebraic curves (see e.g., [28]). Such curves are encoded through *rational parametrizations*, i.e., as the Zariski closure of the projection of the  $(x_1, \dots, x_n)$ -space of the solution set to

$$w(t, s) = 0, x_i = v_i(t, s) / \frac{\partial w}{\partial t}(t, s), \quad 1 \leq i \leq n$$

where  $w \in \mathbf{Q}[t, s]$  is square-free and monic in  $t$  and  $s$  and the  $v_i$ 's lie in  $\mathbf{Q}[t, s]$  (see e.g., [28]). As far as we know, there is no published algorithm for computing isolated points from such an encoding.

Computing roadmaps started with Canny's (probabilistic) algorithm running in time  $d^{O(n^2)}$  on real algebraic sets. Later on, [27] introduced new types of connectivity results enabling more freedom in the design of roadmap algorithms. This led to [27, 4] for computing roadmaps in time  $(nd)^{O(n^{1.5})}$ . More recently, [3], still using these new types of connectivity results, provide a roadmap algorithm running in time  $d^{O(n \log^2 n)} n^{O(n \log^3 n)}$  for general real algebraic sets (at the cost of introducing a number of infinitesimals). This is improved in [28], for smooth bounded real algebraic sets, with a probabilistic algorithm running in time  $O((nd)^{12n \log_2 n})$ . These results makes plausible to obtain a full algorithm running in time  $(nd)^{O(n \log n)}$  to compute the isolated points of  $\mathcal{H} \cap \mathbf{R}^n$ .

**Main result** We provide a probabilistic algorithm which takes as input  $f$  and computes the set of real isolated points  $\mathcal{Z}(\mathcal{H})$  of  $\mathcal{H} \cap \mathbf{R}^n$ . A few remarks on the output data-structure are in order. Any finite algebraic set  $Z \subset \mathbf{C}^n$  defined over  $\mathbf{Q}$  can be represented as the projection on the  $(x_1, \dots, x_n)$ -space of the solution set to

$$w(t) = 0, x_i = v_i(t), \quad 1 \leq i \leq n$$

where  $w \in \mathbf{Q}[t]$  is square-free and the  $v_i$ 's lie in  $\mathbf{Q}[t]$ . The sequence of polynomials  $(w, v_1, \dots, v_n)$  is called a *zero-dimensional parametrization*; such a representation goes back to Kronecker [19]. Such representations (and their variants with denominators) are widely used in computer algebra (see e.g. [12, 13, 14]). For a zero-dimensional parametrization  $\Omega$ ,  $Z(\Omega) \subset \mathbf{C}^n$  denotes the finite set represented by  $\Omega$ . Observe that considering additionally isolating boxes, one can encode  $Z(\Omega) \cap \mathbf{R}^n$ . Our main result is as follows.

**Theorem 1.** *Let  $f \in \mathbf{Q}[x_1, \dots, x_n]$  of degree  $d$  and  $\mathcal{H} \subset \mathbf{C}^n$  be the algebraic set defined by  $f = 0$ . There exists a probabilistic algorithm which, on an input  $f$  of degree  $d$ , computes a zero-dimensional parametrization  $\mathfrak{P}$  and isolating boxes which encode  $\mathcal{Z}(\mathcal{H})$  using  $(nd)^{O(n \log(n))}$  arithmetic operations in  $\mathbf{Q}$ .*

In Section 5, we report on practical experiments showing that it already allows us to solve non-trivial problems which are actually out of reach of [2, Alg. 12.16] to compute sample points in  $\mathcal{H} \cap \mathbf{R}^n$  only. We sketch now the geometric ingredients which allow

us to obtain such an algorithm. Assume that  $f$  is non-negative over  $\mathbf{R}^n$  (if this is not the case, just replace it by its square) and let  $\mathbf{x} \in \mathcal{Z}(\mathcal{H})$ . Since  $\mathbf{x}$  is isolated and  $f$  is non-negative over  $\mathbf{R}^n$ , the intuition is that for  $e > 0$  and small enough, the real solution set to  $f = e$  looks like a ball around  $\mathbf{x}$ , hence a bounded and closed connected component  $C_{\mathbf{x}}$ . Then the restriction of every projection on the  $x_i$ -axis to the algebraic set  $\mathcal{H}_e \subset \mathbf{C}^n$  defined by  $f = e$  intersects  $C_{\mathbf{x}}$ . When  $e$  tends to 0, these critical points in  $C_{\mathbf{x}}$  “tend to  $\mathbf{x}$ ”. This first process allows us to compute a superset of candidate points in  $\mathcal{H} \cap \mathbf{R}^n$  containing  $\mathcal{Z}(\mathcal{H})$ . Of course, one would like that this superset is finite and this will be the case up to some generic linear change of coordinates, using e.g. [25].

All in all, at this stage we have “candidate points” that may lie in  $\mathcal{Z}(\mathcal{H})$ . Writing a quantified formula to decide if there exists a ball around these points which does not meet  $\mathcal{H} \cap \mathbf{R}^n$  raises complexity issues (those points are encoded by zero-dimensional parametrizations of degree  $d^{O(n)}$ , given as input to a decision procedure).

Hence we need new ingredients. Note that our “candidate points” lie on “curves of critical points” which are obtained by letting  $e$  vary in the polynomial systems defining the aforementioned critical points. Assume now that  $\mathcal{H} \cap \mathbf{R}^n$  is bounded, hence contained in a ball  $B$ . Then, for  $e'$  small enough, the real algebraic set defined by  $f = 0$  is “approximated” by the union of the connected components of the real set defined by  $f = e'$  which are contained in  $B$ . Besides, these “curves of critical points”, that we just mentioned, hit these connected components when one fixes  $e'$ . We actually prove that two distinct points of our set of “candidate points” are connected through these “curves of critical points” and those connected components defined by  $f = e'$  in  $B$  if and only if they do not lie in  $\mathcal{Z}(\mathcal{H})$ . Hence, we use computations of roadmaps of the real set defined by  $f = e'$  to answer those connectivity queries. Then, advanced algorithms for roadmaps and polynomial system solving allows us to achieve the announced complexity bound.

Many details are hidden in this description. In particular, we use infinitesimal deformations and techniques of semi-algebraic geometry. While infinitesimals are needed for proofs, they may be difficult to use in practice. On the algorithmic side, we go further exploiting the geometry of the problem to avoid using infinitesimals.

**Structure of the paper** In Section 2, we study the geometry of our problem and prove a series of auxiliary results (in particular Proposition 7, which coins the theoretical ingredient we need). Section 3 is devoted to describe the algorithm. Section 4 is devoted to the complexity analysis and Section 5 reports on the practical performances of our algorithm.

*Acknowledgments.* We thank the reviewers for their helpful comments.

## 2 The geometry of the problem

### 2.1 Candidates for isolated points

As above, let  $f \in \mathbf{Q}[x_1, \dots, x_n]$  and  $\mathcal{H} \subset \mathbf{C}^n$  be the hypersurface defined by  $f = 0$ . Let  $\mathbf{f}$  be a subset of  $\mathbf{C}[x_1, \dots, x_n]$ , we denote by  $V(\mathbf{f})$  the simultaneous vanishing locus in  $\mathbf{C}^n$  of  $\mathbf{f}$ .

**Lemma 1.** *The set  $\mathcal{Z}(\mathcal{H})$  is the (finite) union of the semi-algebraically connected components of  $\mathcal{H} \cap \mathbf{R}^n$  which are a singleton.*

*Proof.* Recall that real algebraic sets have a finite number of semi-algebraically connected components [2, Theorem 5.21]. Let  $\mathcal{C}$  be a semi-algebraically connected component of  $\mathcal{H} \cap \mathbf{R}^n$ .

Assume that  $\mathcal{C}$  is not a singleton and take  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathcal{C}$  with  $\mathbf{x} \neq \mathbf{y}$ . Then, there exists a semi-algebraic continuous map  $\gamma : [0, 1] \rightarrow \mathcal{C}$  s.t.  $\gamma(0) = \mathbf{x}$  and  $\gamma(1) = \mathbf{y}$ ; besides, since  $\mathbf{x} \neq \mathbf{y}$ , there exist  $t \in (0, 1)$  such that  $\gamma(t) \neq \mathbf{x}$ . By continuity of  $\gamma$  and the norm function, any ball  $B$  centered at  $\mathbf{x}$  contains a point  $\gamma(t) \neq \mathbf{x}$ .

Now assume that  $\mathcal{C} = \{\mathbf{x}\}$ . Observe that  $\mathcal{H} \cap \mathbf{R}^n - \{\mathbf{x}\}$  is closed (since semi-algebraically connected components of real algebraic sets are closed). Since  $\mathcal{H} \cap \mathbf{R}^n$  is bounded, we deduce that  $\mathcal{H} \cap \mathbf{R}^n - \{\mathbf{x}\}$  is closed and bounded. Then, the map  $\mathbf{y} \rightarrow \|\mathbf{y} - \mathbf{x}\|^2$  reaches a minimum over  $\mathcal{H} \cap \mathbf{R}^n - \{\mathbf{x}\}$ . Let  $e$  be this minimum value. We deduce that any ball centered at  $\mathbf{x}$  of radius less than  $e$  does not meet  $\mathcal{H} \cap \mathbf{R}^n - \{\mathbf{x}\}$ .  $\square$

To compute those connected components of  $\mathcal{H} \cap \mathbf{R}^n$  which are singletons, we use classical objects of optimization and Morse theory which are mainly *polar varieties*. Let  $\mathbf{K}$  be an algebraically closed field, let  $\phi \in \mathbf{K}[x_1, \dots, x_n]$  which defines the polynomial mapping  $(x_1, \dots, x_n) \mapsto \phi(x_1, \dots, x_n)$  and  $V \subset \mathbf{K}^n$  be a smooth equidimensional algebraic set. We denote by  $W(\phi, V)$  the set of critical points of the restriction of  $\phi$  to  $V$ . If  $c$  is the co-dimension of  $V$  and  $(g_1, \dots, g_s)$  generates the vanishing ideal associated to  $V$ , then  $W(\phi, V)$  is the subset of  $V$  at which the Jacobian matrix associated to  $(g_1, \dots, g_s, \phi)$  has rank less than or equal to  $c$  (see e.g., [28, Subsection 3.1]).

In particular, the case where  $\phi$  is replaced by the canonical projection on the  $i$ -th coordinate

$$\pi_i : (x_1, \dots, x_n) \mapsto x_i,$$

is excessively used throughout our paper.

In our context, we do not assume that  $\mathcal{H}$  is smooth. Hence, to exploit strong topological properties of polar varieties, we retrieve a smooth situation using deformation techniques. We consider an infinitesimal  $\varepsilon$ , i.e., a transcendental element over  $\mathbf{R}$  such that  $0 < \varepsilon < r$  for any positive element  $r \in \mathbf{R}$ , and the field of Puiseux series over  $\mathbf{R}$ , denoted by

$$\mathbf{R}(\varepsilon) = \left\{ \sum_{i \geq i_0} a_i \varepsilon^{i/q} \mid i \in \mathbb{N}, i_0 \in \mathbb{Z}, q \in \mathbb{N} - \{0\}, a_i \in \mathbf{R} \right\}.$$

Recall that  $\mathbf{R}\langle\varepsilon\rangle$  is a real closed field [2, Theorem 2.91]. One defines  $\mathbf{C}\langle\varepsilon\rangle$  as for  $\mathbf{R}\langle\varepsilon\rangle$  but taking the coefficients of the series in  $\mathbf{C}$ . Recall that  $\mathbf{C}\langle\varepsilon\rangle$  is an algebraic closure of  $\mathbf{R}\langle\varepsilon\rangle$  [2, Theorem 2.17]. Consider  $\sigma = \sum_{i \geq i_0} a_i \varepsilon^{i/q} \in \mathbf{R}\langle\varepsilon\rangle$  with  $a_{i_0} \neq 0$ . Then,  $a_{i_0}$  is called the *valuation* of  $\sigma$ . When  $i_0 \geq 0$ ,  $\sigma$  is said to be *bounded over  $\mathbf{R}$*  and the set of bounded elements of  $\mathbf{R}\langle\varepsilon\rangle$  is denoted by  $\mathbf{R}\langle\varepsilon\rangle_b$ . One defines the function  $\lim_\varepsilon : \mathbf{R}\langle\varepsilon\rangle_b \rightarrow \mathbf{R}$  that maps  $\sigma$  to  $a_0$  (which is 0 when  $i_0 > 0$ ) and writes  $\lim_\varepsilon \sigma = a_0$ ; note that  $\lim_\varepsilon$  is a ring homomorphism from  $\mathbf{R}\langle\varepsilon\rangle_b$  to  $\mathbf{R}$ . All these definitions extend to  $\mathbf{R}\langle\varepsilon\rangle^n$  componentwise. For a semi-algebraic set  $\mathcal{S} \subset \mathbf{R}\langle\varepsilon\rangle^n$ , we naturally define the limit of  $\mathcal{S}$  as  $\lim_\varepsilon \mathcal{S} = \{\lim_\varepsilon \mathbf{x} \mid \mathbf{x} \in \mathcal{S} \text{ and } \mathbf{x} \text{ is bounded over } \mathbf{R}\}$ .

Let  $\mathcal{S} \subset \mathbf{R}^n$  be a semi-algebraic set defined by a semi-algebraic formula  $\Phi$ . We denote by  $\text{ext}(\mathcal{S}, \mathbf{R}\langle\varepsilon\rangle)$  the semi-algebraic set of points which are solutions of  $\Phi$  in  $\mathbf{R}\langle\varepsilon\rangle^n$ . We refer to [2, Chap. 2] for more details on infinitesimals and real Puiseux series.

By e.g., [22, Lemma 3.5],  $\mathcal{H}_\varepsilon$  and  $\mathcal{H}_{-\varepsilon}$  respectively defined by  $f = \varepsilon$  and  $f = -\varepsilon$  are two disjoint smooth algebraic sets in  $\mathbf{C}\langle\varepsilon\rangle^n$ .

**Lemma 2.** *For any  $\mathbf{x}$  lying in a bounded connected component of  $\mathcal{H} \cap \mathbf{R}^n$ , there exists a point  $\mathbf{x}_\varepsilon \in (\mathcal{H}_\varepsilon \cup \mathcal{H}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle_b^n$  such that  $\lim_\varepsilon \mathbf{x}_\varepsilon = \mathbf{x}$ . For such a point  $\mathbf{x}_\varepsilon$ , let  $\mathcal{C}_\varepsilon$  be the connected component of  $(\mathcal{H}_\varepsilon \cup \mathcal{H}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^n$  containing  $\mathbf{x}_\varepsilon$ . Then,  $\mathcal{C}_\varepsilon$  is bounded over  $\mathbf{R}$ .*

*Proof.* See [22, Lemma 3.6] for the first claim. The second part can be deduced following the proof of [2, Proposition 12.51].  $\square$

**Proposition 3.** *Assume that  $\mathcal{Z}(\mathcal{H})$  is not empty and let  $\mathbf{x} \in \mathcal{Z}(\mathcal{H})$ . There exists a semi-algebraically connected component  $\mathcal{C}_\varepsilon$  that is bounded over  $\mathbf{R}$  of  $(\mathcal{H}_\varepsilon \cup \mathcal{H}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^n$  such that  $\lim_\varepsilon \mathcal{C}_\varepsilon = \{\mathbf{x}\}$ .*

*Consequently, for  $1 \leq i \leq n$ , there exists an  $\mathbf{x}_\varepsilon \in (W(\pi_i, \mathcal{H}_\varepsilon) \cup W(\pi_i, \mathcal{H}_{-\varepsilon})) \cap \mathcal{C}_\varepsilon$  such that  $\lim_\varepsilon \mathbf{x}_\varepsilon = \mathbf{x}$ . Hence we have that*

$$\mathcal{Z}(\mathcal{H}) \subset \bigcap_{i=1}^n \lim_\varepsilon ((W(\pi_i, \mathcal{H}_\varepsilon) \cup W(\pi_i, \mathcal{H}_{-\varepsilon})) \cap \mathbf{R}\langle\varepsilon\rangle_b^n).$$

*Proof.* By Lemma 2, there exists  $\mathbf{x}_\varepsilon \in (\mathcal{H}_\varepsilon \cup \mathcal{H}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^n$  such that  $\lim_\varepsilon \mathbf{x}_\varepsilon = \mathbf{x}$ . Assume that  $\mathbf{x}_\varepsilon \in \mathcal{H}_\varepsilon$  and let  $\mathcal{C}_\varepsilon$  be the connected component of  $\mathcal{H}_\varepsilon \cap \mathbf{R}\langle\varepsilon\rangle^n$  containing  $\mathbf{x}_\varepsilon$ . Again, by Lemma 2,  $\mathcal{C}_\varepsilon$  is bounded over  $\mathbf{R}$ . We prove that  $\lim_\varepsilon \mathcal{C}_\varepsilon = \{\mathbf{x}\}$  by contradiction. The case  $\mathbf{x}_\varepsilon \in \mathcal{H}_{-\varepsilon}$  is done similarly.

Assume that there exists a point  $\mathbf{y}_\varepsilon \in \mathcal{C}_\varepsilon$  such that  $\lim_\varepsilon \mathbf{y}_\varepsilon = \mathbf{y}$  and  $\mathbf{y} \neq \mathbf{x}$ . Since  $\mathcal{C}_\varepsilon$  is semi-algebraically connected, there exists a semi-algebraically continuous function  $\gamma : \text{ext}([0, 1], \mathbf{R}\langle\varepsilon\rangle) \rightarrow \mathcal{C}_\varepsilon$  such that  $\gamma(0) = \mathbf{x}_\varepsilon$  and  $\gamma(1) = \mathbf{y}_\varepsilon$ . By [2, Proposition 12.49],  $\lim_\varepsilon \text{Im}(\gamma)$  is connected and contains  $\mathbf{x}$  and  $\mathbf{y}$ . As  $\lim_\varepsilon$  is a ring homomorphism,  $f(\lim_\varepsilon \gamma(t)) = \lim_\varepsilon f(\gamma(t)) = 0$ , so  $\lim_\varepsilon \text{Im}(\gamma)$  is contained in  $\mathcal{H} \cap \mathbf{R}^n$ . This contradicts the isolatedness of  $\mathbf{x}$ , then we conclude that  $\lim_\varepsilon \mathcal{C}_\varepsilon = \{\mathbf{x}\}$ .

Since  $\mathcal{C}_\varepsilon$  is a semi-algebraically connected component of the real algebraic set  $\mathcal{H}_\varepsilon \cap \mathbf{R}\langle\varepsilon\rangle^n$ , it is closed. Also,  $\mathcal{C}_\varepsilon$  is bounded over  $\mathbf{R}$ . Hence, for any  $1 \leq i \leq n$ , the projection  $\pi_i$  reaches its extrema over  $\mathcal{C}_\varepsilon$  [2, Proposition 7.6], which implies that  $\mathcal{C}_\varepsilon \cap W(\pi_i, \mathcal{H}_\varepsilon)$  is non-empty. Take  $\mathbf{x}_\varepsilon \in W(\pi_i, \mathcal{H}_\varepsilon) \cap \mathcal{C}_\varepsilon$ , then  $\mathbf{x}_\varepsilon$  is bounded over  $\mathbf{R}$  and its limit is  $\mathbf{x}$ . Thus,  $\mathcal{Z}(\mathcal{H}) \subset \lim_\varepsilon (W(\pi_i, \mathcal{H}_\varepsilon) \cap \mathbf{R}\langle\varepsilon\rangle_b^n)$  for any  $1 \leq i \leq n$ , which implies  $\mathcal{Z}(\mathcal{H}) \subset \bigcap_{i=1}^n \lim_\varepsilon (W(\pi_i, \mathcal{H}_\varepsilon) \cap \mathbf{R}\langle\varepsilon\rangle_b^n)$ .  $\square$

## 2.2 Simplification

We introduce in this subsection a method to reduce our problem to the case where  $\mathcal{H} \cap \mathbf{R}^n$  is bounded for all  $\mathbf{x} \in \mathbf{R}^n$ . Such assumptions are required to prove the results in Subsection 2.3. Our technique is inspired by [2, Section 12.6]. The idea is to associate to the possibly unbounded algebraic set  $\mathcal{H} \cap \mathbf{R}^n$  a bounded real algebraic set whose isolated points are strongly related to  $\mathcal{Z}(\mathcal{H})$ . The construction of such an algebraic set is as follows.

Let  $x_{n+1}$  be a new variable and  $0 < \rho \in \mathbf{R}$  such that  $\rho$  is greater than the Euclidean norm  $\|\cdot\|$  of every isolated point of  $\mathcal{H} \cap \mathbf{R}^n$ . Note that such a  $\rho$  can be obtained from a finite set of points containing the the isolated points of  $\mathcal{H} \cap \mathbf{R}^n$ . We explain in Subsection 3.2 how to compute such a finite set.

We consider the algebraic set  $\mathcal{V}$  defined by the system

$$f = 0, \quad x_1^2 + \dots + x_n^2 + x_{n+1}^2 - \rho^2 = 0.$$

Let  $\pi_{\mathbf{x}}$  be the projection  $(x_1, \dots, x_n, x_{n+1}) \mapsto (x_1, \dots, x_n)$ .

The real counterpart of  $\mathcal{V}$  is the intersection of  $\mathcal{H}$  lifted to  $\mathbf{R}^{n+1}$  with the sphere of center  $\mathbf{0}$  and radius  $\rho$ . Therefore,  $\mathcal{V}$  is a bounded real algebraic set in  $\mathbf{R}^{n+1}$ . Moreover, the restriction of  $\pi_{\mathbf{x}}$  to  $\mathcal{V} \cap \mathbf{R}^{n+1}$  is exactly  $\mathcal{H} \cap B(\mathbf{0}, \rho)$ . By the definition of  $\rho$ , this image contains all the real isolated points of  $\mathcal{H}$ . Lemma 4 below relates  $\mathcal{Z}(\mathcal{H})$  to the isolated points of  $\mathcal{V} \cap \mathbf{R}^{n+1}$ .

**Lemma 4.** *Let  $\mathcal{V}$  and  $\pi_{\mathbf{x}}$  as above. We denote by  $\mathcal{Z}(\mathcal{V}) \subset \mathbf{R}^{n+1}$  the set of real isolated points of  $\mathcal{V}$  with non-zero  $x_{n+1}$  coordinate. Then,  $\pi_{\mathbf{x}}(\mathcal{Z}(\mathcal{V})) = \mathcal{Z}(\mathcal{H})$ .*

*Proof.* Note that  $\pi_{\mathbf{x}}(\mathcal{V} \cap \mathbf{R}^{n+1}) = (\mathcal{H} \cap \mathbf{R}^n) \cap B(\mathbf{0}, \rho)$ . We consider a real isolated point  $\mathbf{x}' = (\alpha_1, \dots, \alpha_n, \alpha_{n+1})$  of  $\mathcal{V}$  with  $\alpha_{n+1} \neq 0$  and  $\mathbf{x} = \pi_{\mathbf{x}}(\mathbf{x}') = (\alpha_1, \dots, \alpha_n)$ . Assume by contradiction that  $\mathbf{x} \notin \mathcal{Z}(\mathcal{H})$ , we will prove that  $\mathbf{x}' \notin \mathcal{Z}(\mathcal{V})$ , i.e., for any  $r > 0$ , there exists  $\mathbf{y}' = (\beta_1, \dots, \beta_n, \beta_{n+1}) \in \mathcal{V} \cap \mathbf{R}^{n+1}$  such that  $\|\mathbf{y}' - \mathbf{x}'\| < r$ . Since  $\mathbf{x}$  is not isolated, there exists a point  $\mathbf{y} \neq \mathbf{x}$  such that  $\|\mathbf{y} - \mathbf{x}\| < \frac{r}{1+2\rho/|\alpha_{n+1}|}$ . Let  $\mathbf{y}' \in \pi_{\mathbf{x}}^{-1}(\mathbf{y})$  such that  $\alpha_{n+1}\beta_{n+1} \geq 0$ . We have that  $\|\mathbf{x}\|^2 + \alpha_{n+1}^2 = \|\mathbf{y}\|^2 + \beta_{n+1}^2 = \rho^2$ . Now we estimate

$$\|\mathbf{y}\|^2 - \|\mathbf{x}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|) \cdot \|\mathbf{y}\| - \|\mathbf{x}\|^2 \leq 2\rho \cdot \|\mathbf{y} - \mathbf{x}\|,$$

$$|\alpha_{n+1} - \beta_{n+1}| \leq \frac{|\alpha_{n+1}^2 - \beta_{n+1}^2|}{|\alpha_{n+1}|} = \frac{\|\mathbf{y}\|^2 - \|\mathbf{x}\|^2}{|\alpha_{n+1}|} \leq \frac{2\rho \cdot \|\mathbf{y} - \mathbf{x}\|}{|\alpha_{n+1}|}.$$

Finally,

$$\|\mathbf{y}' - \mathbf{x}'\| \leq \|\mathbf{y} - \mathbf{x}\| + |\alpha_{n+1} - \alpha_{n+1}| \leq \left(1 + \frac{2\rho}{|\alpha_{n+1}|}\right) \|\mathbf{y} - \mathbf{x}\| < r.$$

So,  $\mathbf{x}'$  is not isolated in  $\mathcal{V} \cap \mathbf{R}^{n+1}$ . This contradiction implies that  $\pi_{\mathbf{x}}(\mathcal{Z}(\mathcal{V})) \subset \mathcal{Z}(\mathcal{H})$ .

It remains to prove that  $\mathcal{Z}(\mathcal{H}) \subset \pi_{\mathbf{x}}(\mathcal{Z}(\mathcal{V}))$ . For any real isolated point  $\mathbf{x} \in \mathcal{Z}(\mathcal{H})$ , we consider a ball  $B(\mathbf{x}, r') \subset B(\mathbf{0}, \rho) \subset \mathbf{R}^n$  such that  $B(\mathbf{x}, r') \cap \mathcal{H} = \{\mathbf{x}\}$ . We have that  $\pi_{\mathbf{x}}^{-1}(B(\mathbf{x}, r')) \cap \mathcal{V} \cap \mathbf{R}^{n+1}$  is equal to  $\pi_{\mathbf{x}}^{-1}(\mathbf{x}) \cap \mathcal{V} \cap \mathbf{R}^{n+1}$ , which is finite. So, all the points in  $\pi_{\mathbf{x}}^{-1}(B(\mathbf{x}, r')) \cap \mathcal{V} \cap \mathbf{R}^{n+1}$  are isolated. Since  $\mathcal{Z}(\mathcal{H}) \subset B(\mathbf{0}, \rho)$ , we deduce that  $\mathcal{Z}(\mathcal{H})$  is contained in  $\pi_{\mathbf{x}}(\mathcal{Z}(\mathcal{V}))$ .

Thus, we conclude that  $\pi_{\mathbf{x}}(\mathcal{Z}(\mathcal{V})) = \mathcal{Z}(\mathcal{H})$ .  $\square$

Note that the condition  $x_{n+1} \neq 0$  is crucial. For a connected component  $\mathcal{C}$  of  $\mathcal{H} \cap \mathbf{R}^n$  that is not a singleton, its intersection with the closed ball  $\overline{B(\mathbf{0}, \rho)}$  can have an isolated point on the boundary of the ball, which corresponds to an isolated point of  $\mathcal{V} \cap \mathbf{R}^{n+1}$ . This situation depends on the choice of  $\rho$  and can be easily detected by checking the vanishing of the coordinate  $x_{n+1}$ .

### 2.3 Identification of isolated points

By Proposition 3, the real points of  $\bigcap_{i=1}^n \lim_{\varepsilon} W(\pi_i, \mathcal{H}_{\varepsilon})$  are potential isolated points of  $\mathcal{H} \cap \mathbf{R}^n$ . We study now how to identify, among those candidates, which points are truly isolated.

We use the same  $g = x_1^2 + \dots + x_{n+1}^2 - \rho^2$  and  $\mathcal{V} = V(f, g) \subset \mathbf{C}^{n+1}$  as in Subsection 2.2. Let  $\mathcal{V}_{\varepsilon} = V(f - \varepsilon, g)$  and  $\mathcal{V}_{-\varepsilon} = V(f + \varepsilon, g)$ , note that they are both algebraic subsets of  $\mathbf{C}\langle\varepsilon\rangle^{n+1}$ .

**Lemma 5.** *Let  $\mathbf{x} \in \mathcal{V} \cap \mathbf{R}^{n+1}$  such that its  $x_{n+1}$  coordinate is non-zero. Then,  $\mathbf{x}$  is not an isolated point of  $\mathcal{V} \cap \mathbf{R}^{n+1}$  if and only if there exists a semi-algebraically connected component  $\mathcal{C}_{\varepsilon}$  of  $(\mathcal{V}_{\varepsilon} \cup \mathcal{V}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^{n+1}$ , bounded over  $\mathbf{R}$ , such that  $\{\mathbf{x}\} \subsetneq \lim_{\varepsilon} \mathcal{C}_{\varepsilon}$ .*

*Proof.* Let  $\mathbf{x} = (\alpha_1, \dots, \alpha_{n+1}) \in \mathcal{V} \cap \mathbf{R}^{n+1}$  such that  $\alpha_{n+1} \neq 0$ . As  $f(\alpha_1, \dots, \alpha_n) = 0$ , by Lemma 2, there exists a point  $\mathbf{x}_{\varepsilon} = (\beta_1, \dots, \beta_{n+1}) \in \mathbf{R}\langle\varepsilon\rangle^{n+1}$  such that  $(\beta_1, \dots, \beta_n) \in (\mathcal{H}_{\varepsilon} \cup \mathcal{H}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^n$  and  $\lim_{\varepsilon} (\beta_1, \dots, \beta_n) = (\alpha_1, \dots, \alpha_n)$ . Since  $\alpha_{n+1} \neq 0$ , we can choose  $\beta_{n+1}$  such that  $g(\mathbf{x}_{\varepsilon}) = 0$ . Therefore, for any  $\mathbf{x}$  as above, there exists  $\mathbf{x}_{\varepsilon} \in (\mathcal{V}_{\varepsilon} \cup \mathcal{V}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^{n+1}$  such that  $\lim_{\varepsilon} \mathbf{x}_{\varepsilon} = \mathbf{x}$ .

Since  $(\mathcal{V}_{\varepsilon} \cup \mathcal{V}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^{n+1}$  lies on the sphere (in  $\mathbf{R}\langle\varepsilon\rangle^{n+1}$ ) defined by  $g = 0$ , every connected component of  $(\mathcal{V}_{\varepsilon} \cup \mathcal{V}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^{n+1}$  is bounded over  $\mathbf{R}$ . Hence, the points of  $\mathcal{V} \cap \mathbf{R}^{n+1}$  whose  $x_{n+1}$  coordinates are not zero are contained in  $\lim_{\varepsilon} (\mathcal{V}_{\varepsilon} \cup \mathcal{V}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^{n+1}$ .

Let  $\mathbf{x}$  be a non-isolated point of  $\mathcal{V} \cap \mathbf{R}^{n+1}$  whose  $x_{n+1}$ -coordinate is not zero. We assume by contradiction that for any semi-algebraically connected component  $\mathcal{C}_{\varepsilon}$  of  $(\mathcal{V}_{\varepsilon} \cup \mathcal{V}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^{n+1}$  (which is bounded over  $\mathbf{R}$  by above), then it happens that either  $\lim_{\varepsilon} \mathcal{C}_{\varepsilon} = \{\mathbf{x}\}$  or  $\mathbf{x} \notin \lim_{\varepsilon} \mathcal{C}_{\varepsilon}$ .

Since  $(\mathcal{V}_{\varepsilon} \cup \mathcal{V}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^{n+1}$  has finitely many connected components, the number of connected components of the second type is also finite. Since  $\mathcal{V} \cap \mathbf{R}^{n+1}$  is not a singleton (by the existence of  $\mathbf{x}$ ), the connected components of the second type exist. So, we enumerate them as  $\mathcal{C}_1, \dots, \mathcal{C}_k$  and  $\mathbf{x} \notin \lim_{\varepsilon} \mathcal{C}_j$  for  $1 \leq j \leq k$ .

As  $\mathbf{x}$  is not isolated in  $\mathcal{V} \cap \mathbf{R}^{n+1}$  with non-zero  $x_{n+1}$  coordinate by assumption, there exists a sequence of points  $(\mathbf{x}_i)_{i \geq 0}$  in  $\mathcal{V} \cap \mathbf{R}^{n+1}$  of non-zero  $x_{n+1}$  coordinates that converges to  $\mathbf{x}$ . Since there are finitely many  $\mathcal{C}_i$ , there exists an index  $j$  such that  $\lim_{\varepsilon} \mathcal{C}_j$  contains a sub-sequence of  $(\mathbf{x}_i)_{i \geq 0}$ . By Proposition 12.49 [BPR], the limit of the semi-algebraically connected component  $\mathcal{C}_j$  (which is bounded over  $\mathbf{R}$ ) is a closed and connected semi-algebraic set. It follows that  $\mathbf{x} \in \lim_{\varepsilon} \mathcal{C}_j$ , which is a contradiction. Therefore, there exists a semi-algebraically connected component of  $(\mathcal{V}_{\varepsilon} \cup \mathcal{V}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^{n+1}$ , bounded over  $\mathbf{R}$ , such that  $\{\mathbf{x}\} \subsetneq \lim_{\varepsilon} \mathcal{C}_{\varepsilon}$ .

It remains to prove the reverse implication. Assume that  $\{\mathbf{x}\} \subsetneq \lim_{\varepsilon} \mathcal{C}_{\varepsilon}$  for some semi-algebraically connected component  $\mathcal{C}_{\varepsilon}$  of  $(\mathcal{V}_{\varepsilon} \cup \mathcal{V}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^{n+1}$  that is bounded over  $\mathbf{R}$ . As  $\lim_{\varepsilon} \mathcal{C}_{\varepsilon}$  is connected, we finish the proof.  $\square$

**Lemma 6.** *Let  $\mathbf{x} \in \mathcal{V} \cap \mathbf{R}^{n+1}$  whose  $x_{n+1}$  coordinate is non-zero. Assume that  $\mathbf{x}$  is not an isolated point of  $\mathcal{V} \cap \mathbf{R}^{n+1}$ . For any semi-algebraically connected component  $\mathcal{C}_{\varepsilon}$  of  $(\mathcal{V}_{\varepsilon} \cup \mathcal{V}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^{n+1}$ , bounded over  $\mathbf{R}$ , such that  $\{\mathbf{x}\} \subsetneq \lim_{\varepsilon} \mathcal{C}_{\varepsilon}$ , there exists  $1 \leq i \leq n$  such that  $\mathcal{C}_{\varepsilon} \cap (W(\pi_i, \mathcal{V}_{\varepsilon}) \cup W(\pi_i, \mathcal{V}_{-\varepsilon}))$  contains a point  $\mathbf{x}'_{\varepsilon}$  which satisfies  $\lim_{\varepsilon} \mathbf{x}'_{\varepsilon} \neq \mathbf{x}$ .*

*Proof.* Let  $\mathcal{C}_{\varepsilon}$  be semi-algebraically connected component of  $(\mathcal{V}_{\varepsilon} \cup \mathcal{V}_{-\varepsilon}) \cap \mathbf{R}\langle\varepsilon\rangle^{n+1}$ , bounded over  $\mathbf{R}$ , such that  $\{\mathbf{x}\} \subsetneq \lim_{\varepsilon} \mathcal{C}_{\varepsilon}$ . Lemma 5 ensures the existence of such a connected component  $\mathcal{C}_{\varepsilon}$ .

Now let  $\mathbf{x}_{\varepsilon}$  and  $\mathbf{y}_{\varepsilon}$  be two points contained in  $\mathcal{C}_{\varepsilon}$  such that  $\lim_{\varepsilon} \mathbf{x}_{\varepsilon} = \mathbf{x}$ ,  $\lim_{\varepsilon} \mathbf{y}_{\varepsilon} = \mathbf{y}$  and  $\mathbf{x} \neq \mathbf{y}$ . Let  $\mathbf{x} = (\alpha_1, \dots, \alpha_{n+1})$  and  $\mathbf{y} = (\beta_1, \dots, \beta_{n+1})$ . Since  $\mathbf{x} \neq \mathbf{y}$ , there exists  $1 \leq i \leq n+1$  such that  $\alpha_i \neq \beta_i$ . Note that if  $(\alpha_1, \dots, \alpha_n) = (\beta_1, \dots, \beta_n)$  for any  $\mathbf{y} \in \lim_{\varepsilon} \mathcal{C}_{\varepsilon}$ , then  $\lim_{\varepsilon} \mathcal{C}_{\varepsilon}$  contains at most two points (by the constraint  $g = 0$ ). However, since  $\lim_{\varepsilon} \mathcal{C}_{\varepsilon}$  is connected and contains at least two points, it must be an infinite set. So, we can choose  $\mathbf{y}$  such that have that  $1 \leq i \leq n$ .

As  $\mathcal{C}_{\varepsilon}$  is closed in  $\mathbf{R}\langle\varepsilon\rangle^{n+1}$  (as a connected component of an algebraic set) and bounded over  $\mathbf{R}$  by definition, its projection on the  $x_i$  coordinate is a closed interval  $[a, b] \subset \mathbf{R}\langle\varepsilon\rangle$  (see [2, Theorem 3.23]), which is bounded over  $\mathbf{R}$  (because  $\mathcal{C}_{\varepsilon}$  is). Also, since  $[a, b]$  is closed, there exist  $\mathbf{x}'_a$  and  $\mathbf{x}'_b$  in  $\mathbf{R}\langle\varepsilon\rangle^{n+1}$  such that  $\mathbf{x}'_a \in \pi_i^{-1}(a) \cap \mathcal{C}_{\varepsilon} \cap (W(\pi_i, \mathcal{V}_{\varepsilon}) \cup W(\pi_i, \mathcal{V}_{-\varepsilon}))$  and

$\mathbf{x}'_b \in \pi_i^{-1}(b) \cap \mathcal{C}_\varepsilon \cap (W(\pi_i, \mathcal{V}_\varepsilon) \cup W(\pi_i, \mathcal{V}_{-\varepsilon}))$ . Since  $\alpha_i \neq \beta_i$  both lying in  $\mathbf{R}$ ,  $\{\alpha_i, \beta_i\} \subset [\lim_\varepsilon a, \lim_\varepsilon b]$  implies that  $\lim_\varepsilon a \neq \lim_\varepsilon b$ . It follows that  $\lim_\varepsilon \mathbf{x}'_a \neq \lim_\varepsilon \mathbf{x}'_b$ . Thus, at least one point among  $\lim_\varepsilon \mathbf{x}'_a$  and  $\lim_\varepsilon \mathbf{x}'_b$  does not coincide with  $\mathbf{x}$ . Hence, there exists a point  $\mathbf{x}'_\varepsilon$  in  $\mathcal{C}_\varepsilon \cap (W(\pi_i, \mathcal{V}_\varepsilon) \cup W(\pi_i, \mathcal{V}_{-\varepsilon}))$  such that  $\lim_\varepsilon \mathbf{x}'_\varepsilon \neq \mathbf{x}$ .  $\square$

We can easily deduce from Lemma 5 and Lemma 6 the following proposition, which is the main ingredient of our algorithm.

**Proposition 7.** *Let  $\mathbf{x} \in \bigcap_{i=1}^n \lim_\varepsilon W(\pi_i, \mathcal{V}_\varepsilon) \cup W(\pi_i, \mathcal{V}_{-\varepsilon})$  whose  $x_{n+1}$  coordinate is non-zero. Then,  $\mathbf{x}$  is not an isolated point of  $\mathcal{V} \cap \mathbf{R}^{n+1}$  if and only if there exist  $1 \leq i \leq n$  and a connected component  $\mathcal{C}_\varepsilon$  of  $\mathcal{V}_\varepsilon \cap \mathbf{R}(\varepsilon)^{n+1}$ , which is bounded over  $\mathbf{R}$ , such that  $\mathcal{C}_\varepsilon \cap W(\pi_i, \mathcal{H}_\varepsilon)$  contains  $\mathbf{x}_\varepsilon, \mathbf{x}'_\varepsilon$  satisfying  $\mathbf{x} = \lim_\varepsilon \mathbf{x}_\varepsilon \neq \lim_\varepsilon \mathbf{x}'_\varepsilon$ .*

## 3 Algorithm

### 3.1 General description

The algorithm takes as input a polynomial  $f \in \mathbf{R}[x_1, \dots, x_n]$ .

The first step consists in computing a parametrization  $\mathfrak{P}$  encoding a finite set of points which contains  $\mathcal{Z}(\mathcal{H})$ . Let  $\mathcal{H}_\varepsilon$  and  $\mathcal{H}_{-\varepsilon}$  be the algebraic subsets of  $\mathbf{C}(\varepsilon)^n$  respectively defined by  $f = \varepsilon$  and  $f = -\varepsilon$ . By Proposition 3, the set  $\bigcap_{i=1}^n \lim_\varepsilon W(\pi_i, \mathcal{H}_\varepsilon) \cup W(\pi_i, \mathcal{H}_{-\varepsilon})$  contains the real isolated points of  $\mathcal{H}$ . To ensure that this set is finite, we use *generically chosen* linear change of coordinates.

Given a matrix  $\mathbf{A} \in GL_n(\mathbf{Q})$ , a polynomial  $p \in \mathbf{Q}[x_1, \dots, x_n]$  and an algebraic set  $\mathcal{S} \subset \mathbf{C}^n$ , we denote by  $p^{\mathbf{A}}$  the polynomial  $p(\mathbf{A} \cdot \mathbf{x})$  obtained by applying the change of variables  $\mathbf{A}$  to  $p$  and  $\mathcal{S}^{\mathbf{A}} = \{\mathbf{A}^{-1} \cdot \mathbf{x} \mid \mathbf{x} \in \mathcal{S}\}$ . Then, we have that  $V(p)^{\mathbf{A}} = V(p^{\mathbf{A}})$ .

In [25], it is proved that, with  $\mathbf{A}$  outside a prescribed proper Zariski closed subset of  $GL_n(\mathbf{Q})$ ,  $W(\pi_i, \mathcal{H}_\varepsilon^{\mathbf{A}}) \cup W(\pi_i, \mathcal{H}_{-\varepsilon}^{\mathbf{A}})$  is finite for  $1 \leq i \leq n$ . Additionally, since  $\mathbf{A}$  is assumed to be generically chosen, [24] shows that the ideal  $\langle \ell \cdot \frac{\partial f^{\mathbf{A}}}{\partial x_i} - 1, \frac{\partial f^{\mathbf{A}}}{\partial x_j} \text{ for all } j \neq i \rangle$  defines either an empty set or a one-equidimensional algebraic set, where  $\ell$  is a new variable. Those extra assumptions are required in our subroutine `Candidates` (see the next subsection). Note that, for any matrix  $\mathbf{A}$ , the real isolated points of  $\mathcal{H}^{\mathbf{A}}$  is the image of  $\mathcal{Z}(\mathcal{H})$  by the linear mapping associated to  $\mathbf{A}^{-1}$ . Thus, in practice, we will choose randomly a  $\mathbf{A} \in GL_{n+1}(\mathbf{Q})$ , compute the real isolated points of  $\mathcal{H}^{\mathbf{A}}$ , and then go back to  $\mathcal{Z}(\mathcal{H})$  by applying the change of coordinates induced by  $\mathbf{A}^{-1}$ . This random choice of  $\mathbf{A}$  makes our algorithm probabilistic.

The next step consists of identifying those of the candidates which are isolated in  $\mathcal{H}^{\mathbf{A}} \cap \mathbf{R}^n$ ; this step relies on Proposition 7. To reduce our problem to the context where Proposition 7 can be applied, we use Lemma 4. One needs to compute  $\rho \in \mathbf{R}$ , such that  $\rho$  is larger than the maximum norm of the real isolated points we want to compute. This value of  $\rho$  can be easily obtained by isolating the real roots of the zero-dimensional parametrization encoding the candidates. Further, we call `GetNormBound` a subroutine which takes as input  $\mathfrak{P}$  and returns  $\rho$  as we just sketched. We let  $g = x_1^2 + \dots + x_n^2 + x_{n+1}^2 - \rho^2$ . By Lemma 4,  $\mathcal{Z}(\mathcal{H})$  is the projection of the set of real isolated points of the algebraic set  $\mathcal{V}$  defined by  $f = g = 0$  at which  $x_{n+1} \neq 0$ . Let  $\mathcal{X}$  be the set of points of  $\mathcal{V}$  projecting to the candidates encoded by  $\mathfrak{P}$ .

Proposition 7 would lead us to compute  $W(\pi_i, \mathcal{V}_\varepsilon^{\mathbf{A}}) \cup W(\pi_i, \mathcal{V}_{-\varepsilon}^{\mathbf{A}})$  as well as a roadmap of  $\mathcal{V}_\varepsilon^{\mathbf{A}} \cup \mathcal{V}_{-\varepsilon}^{\mathbf{A}}$ . As explained in the introduction, this induces computations over the ground field  $\mathbf{R}(\varepsilon)$  which we want to avoid. We bypass this computational difficulty as follows. We compute a roadmap  $\mathcal{R}_e$  for  $\mathcal{V}_\varepsilon^{\mathbf{A}} \cup \mathcal{V}_{-\varepsilon}^{\mathbf{A}} \cap \mathbf{R}^{n+1}$  (defined by  $\{f^{\mathbf{A}} = e, g = 0\}$  and  $\{f^{\mathbf{A}} = -e, g = 0\}$  respectively) for  $e$  small enough (see Subsection 3.3) and define a semi-algebraic curve  $\mathcal{K}$  containing  $\mathcal{X}$  such that  $\mathbf{x} \in \mathcal{X}$  is isolated in  $\mathcal{V}^{\mathbf{A}} \cap \mathbf{R}^{n+1}$  if and only if it is not connected to any other  $\mathbf{x}' \in \mathcal{X}$  by  $\mathcal{K}$ . We call `Isolated` the subroutine that takes as input  $\mathfrak{P}$ ,  $f^{\mathbf{A}}$  and  $g$  and returns  $\mathfrak{P}$  with isolating boxes  $B$  of the real points of defined by  $\mathfrak{P}$  which are isolated in  $\mathcal{V}^{\mathbf{A}} \cap \mathbf{R}^{n+1}$ .

Once the real isolated points of  $\mathcal{V}^{\mathbf{A}}$  is computed, we remove the boxes corresponding to points at which  $x_{n+1} = 0$  and project the remaining points on the  $(x_1, \dots, x_n)$ -space to obtain the isolated points of  $\mathcal{H}^{\mathbf{A}}$ . This whole step uses a subroutine which we call `Remove` (see [28, Appendix J]). Finally, we reverse the change of variable by applying  $\mathbf{A}^{-1}$  to get  $\mathcal{Z}(\mathcal{H})$ .

We summarize our discussion in Algorithm 1 below.

### 3.2 Computation of candidates

Further, we let  $\mathcal{H}_\varepsilon^{\mathbf{A}}$  (resp.  $\mathcal{H}_{-\varepsilon}^{\mathbf{A}}$ ) be the algebraic set associated to  $f^{\mathbf{A}} = \varepsilon$  (resp.  $f^{\mathbf{A}} = -\varepsilon$ ). To avoid to overload notation, we omit the change of variables  $\mathbf{A}$  as upper script. Let  $\ell$  be a new variable. For  $1 \leq i \leq n$ ,  $I_i$  denotes the ideal of  $\mathbf{Q}[\ell, x_1, \dots, x_n]$  generated by the set of polynomials  $\{\ell \cdot \frac{\partial f}{\partial x_i} - 1, \frac{\partial f}{\partial x_j} \text{ for all } j \neq i\}$ .

Following the discussion in Subsection 3.1, the algebraic set associated to  $I_i$  is either empty or one-equidimensional and  $W(\pi_i, \mathcal{H}_\varepsilon) \cup W(\pi_i, \mathcal{H}_{-\varepsilon})$  is finite. Hence, [24, Theorem 1] shows that the algebraic set associated to the ideal  $\langle f \rangle + (I_i \cap \mathbf{Q}[x_1, \dots, x_n])$  is zero-dimensional and contains  $\lim_\varepsilon W(\pi_i, \mathcal{H}_\varepsilon) \cup W(\pi_i, \mathcal{H}_{-\varepsilon})$ .

In our problem, the intersection of  $\lim_\varepsilon W(\pi_i, \mathcal{H}_\varepsilon) \cup W(\pi_i, \mathcal{H}_{-\varepsilon})$  is needed rather than each limit itself. Hence, we use the inclusion

$$\bigcap_{i=1}^n \lim_\varepsilon W(\pi_i, \mathcal{H}_\varepsilon) \cup W(\pi_i, \mathcal{H}_{-\varepsilon}) \subset V(\langle f \rangle + \sum_{i=1}^n I_i \cap \mathbf{Q}[x_1, \dots, x_n]).$$

---

**Algorithm 1: IsolatedPoints**

---

**Input:** A polynomial  $f \in \mathbf{Q}[x_1, \dots, x_n]$ **Output:** A zero-dimensional parametrization  $\mathfrak{P}$  such that  $\mathcal{Z}(\mathcal{H}) \subset Z(\mathfrak{P})$  and a set of boxes isolating  $\mathcal{Z}(\mathcal{H})$ 

```
1 A chosen randomly in  $GL_{n+1}(\mathbf{Q})$ 
2  $\mathfrak{P} \leftarrow \text{Candidates}(f^A)$ 
3  $\rho \leftarrow \text{GetNormBound}(\mathfrak{P})$ 
4  $g \leftarrow x_1^2 + \dots + x_n^2 + x_{n+1}^2 - \rho^2$ 
5  $\mathfrak{P}, B \leftarrow \text{Isolated}(\mathfrak{P}, f^A, g)$ 
6  $\mathfrak{P}, B \leftarrow \text{Removes}(\mathfrak{P}, B, x_{n+1})$ 
7  $\mathfrak{P}, B \leftarrow \mathfrak{P}^{A^{-1}}, B^{A^{-1}}$ 
8 return  $(\mathfrak{P}, B)$ 
```

---

We can compute the algebraic set on the right-hand side as follows:

1. For each  $1 \leq i \leq n$ , compute a set  $G_i$  of generators of the ideal  $I_i \cap \mathbf{Q}[x_1, \dots, x_n]$ .
2. Compute a zero-dimensional parametrization  $\mathfrak{P}$  of the set of polynomials  $\{f\} \cup G_1 \cup \dots \cup G_n$ .

Such computations mimic those in [24]. The complexity of this algorithm of course depends on the algebraic elimination procedure we use. For the complexity analysis in Section 4, we employ the geometric resolution [14]. It basically consists in computing a one-dimensional parametrization of the curve defined by  $I_i$  and next computes a zero-dimensional parametrization of the finite set obtained by intersecting this curve with the hypersurface defined by  $f = 0$ . We call `ParametricCurve` a subroutine that, taking the polynomial  $f$  and  $1 \leq i \leq n$ , computes a one-dimensional parametrization  $\mathfrak{G}_i$  of the curve defined above. Also, let `IntersectCurve` be a subroutine that, given a one-dimensional rational parametrization  $\mathfrak{G}_i$  and  $f$ , outputs a zero-dimensional parametrization  $\mathfrak{P}_i$  of their intersection. Finally, we use a subroutine `Intersection` that, from the parametrizations  $\mathfrak{P}_i$ 's, computes a zero-dimensional parametrization of  $\bigcap_{i=1}^n Z(\mathfrak{P}_i)$ .

---

**Algorithm 2: Algorithm Candidates**

---

**Input:** The polynomial  $f \in \mathbf{Q}[x_1, \dots, x_n]$ **Output:** A zero-dimensional parametrization  $\mathfrak{P}$ 

```
1 for  $1 \leq i \leq n$  do
2    $\mathfrak{G}_i \leftarrow \text{ParametricCurve}(g, i)$ 
3    $\mathfrak{P}_i \leftarrow \text{IntersectCurve}(\mathfrak{G}_i, g)$ 
4 end
5  $\mathfrak{P} \leftarrow \text{Intersection}(\mathfrak{P}_1, \dots, \mathfrak{P}_n)$ 
6 return  $\mathfrak{P}$ 
```

---

### 3.3 Description of Isolated

This subsection is devoted to the subroutine `Isolated` that identifies isolated points of  $\mathcal{H}^A \cap \mathbf{R}^n$  among the candidates  $Z(\mathfrak{P}) \cap \mathbf{R}^n$  computed in the previous subsection. We keep using  $f$  to address  $f^A$ . Let  $\mathcal{P}$  be the set  $\{\mathbf{x} = (x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \mid (x_1, \dots, x_n) \in Z(\mathfrak{P}), g(\mathbf{x}) = 0, x_{n+1} \neq 0\}$ .

For  $e \in \mathbf{R}$ ,  $\mathcal{V}_e \subset \mathbf{R}^{n+1}$  denotes the algebraic set defined by  $f = e$  and  $g = 0$ . We follow the idea mentioned in the end of Subsection 3.1, that is to replace the infinitesimal  $\varepsilon$  by a sufficiently small  $e \in \mathbf{R}$  then adapt the results of Subsection 2.3 to  $\mathcal{V}_e$ .

By definition,  $\mathcal{V}_e \cap \mathbf{R}^{n+1}$  is bounded for any  $e \in \mathbf{R}$ . Let  $t$  be a new variable,  $\pi_x : (\mathbf{x}, t) \mapsto \mathbf{x}$  and  $\pi_t : (\mathbf{x}, t) \mapsto t$ . For a semi-algebraic set  $\mathcal{S} \subset \mathbf{R}^{n+1} \times \mathbf{R}$  in the coordinate  $(\mathbf{x}, t)$  and a subset  $\mathcal{I}$  of  $\mathbf{R}$ , the notation  $\mathcal{S}_{\mathcal{I}}$  stands for the fiber  $\pi_t^{-1}(\mathcal{I}) \cap \mathcal{S}$ . Let  $\mathcal{V}_t = \{(\mathbf{x}, t) \in \mathbf{R}^{n+1} \times \mathbf{R} \mid f(\mathbf{x}) = t, g(\mathbf{x}) = 0\}$ . Note that  $\mathcal{V}_t$  is smooth. Recall that the set of critical values of the restriction of  $\pi_t$  to  $\mathcal{V}_t$  is finite by the algebraic Sard's theorem (see e.g., [28, Proposition B.2]).

Since for  $e \in \mathbf{R}$ , the set  $\mathcal{V}_e \cap \mathbf{R}^{n+1}$  is compact, restriction of  $\pi_t$  to  $\mathcal{V}_t$  is proper. Then, by Thom's isotopy lemma [9],  $\pi_t$  realizes a locally trivial fibration over any open connected subset of  $\mathbf{R}$  which does not intersect the set of critical values of the restriction of  $\pi_t$  to  $\mathcal{V}_t$ . Let  $\eta \in \mathbf{R}$  such that the open set  $] - \eta, 0[ \cup ] 0, \eta[$  does not contain any critical value of the restriction of  $\pi_t$  to the algebraic set  $\mathcal{V}_t$ . Hence,  $\mathcal{V}_e$  is nonsingular for  $e \in ] - \eta, 0[ \cup ] 0, \eta[$ ,  $(\mathcal{V}_e \cap \mathbf{R}^{n+1}) \times (] - \eta, 0[ \cup ] 0, \eta[)$  is diffeomorphic to  $\mathcal{V}_{t, ] - \eta, 0[ \cup ] 0, \eta[}$ .

We need to mention that  $W(\pi_i, \mathcal{H}_e)$  corresponds to the critical points of  $\pi_i$  restricted to  $\mathcal{V}_e$  with non-zero  $x_{n+1}$  coordinate. Further, we use  $W(\pi_i, \mathcal{V}_e)$  to address those latter critical points.



Now, for  $1 \leq i \leq n$ , we define  $\mathcal{W}_i$  as the closure of

$$\left\{ (\mathbf{x}, t) \in \mathbf{R}^{n+2} \mid \frac{\partial f}{\partial x_i}(\mathbf{x}) \neq 0, \frac{\partial f}{\partial x_j}(\mathbf{x}) = 0 \text{ for } j \neq i, x_{n+1} \neq 0 \right\} \cap \mathcal{V}_t.$$

Since  $\mathbf{A}$  is assumed to be generically chosen,  $\mathcal{W}_i$  is either empty or one-equidimensional (because  $\langle \ell \cdot \frac{\partial f}{\partial x_i} - 1, \frac{\partial f}{\partial x_j} \forall j \neq i \rangle$  either defines an empty set or a one-equidimensional algebraic set by [24]). This implies that the set of singular points of  $\mathcal{W}_i$  is finite.

By [18], the set of non-properness of the restriction of  $\pi_t$  to  $\mathcal{W}_i$  is finite (this is the set of points  $y$  such that for any closed interval  $U$  containing  $y$ ,  $\pi_t^{-1}(U) \cap \mathcal{W}_i$  is not bounded). Using again [18], the restriction of  $\pi_t$  to  $\mathcal{W}_i$  realizes a locally trivial fibration over any connected open subset which does not meet the union of the images by  $\pi_t$  of the singular points of  $\mathcal{W}_i$ , the set of non-properness, and the set of critical values of the restriction of  $\pi_t$  to  $\mathcal{W}_i$ . We let  $\eta_i'$  be the minimum of the absolute values of the points in this union.

We choose now  $0 < e_0 < \min(\eta, \eta_1', \dots, \eta_n')$ . We call `SpecializationValue` a subroutine that takes as input  $f$  and  $g$  and returns such a rational number  $e_0$ . Note that `SpecializationValue` is easily obtained from elimination algorithms solving polynomial systems (from which we can compute critical values) and from [26] to compute the set of non-properness of some map.

With  $e_0$  as above, we denote  $\mathcal{I} = ] - e_0, 0[ \cup ] 0, e_0[$ . Let  $\mathcal{W}_{i,\mathcal{I}}$  is semi-algebraically diffeomorphic to  $\mathcal{W}_{i,e} \times \mathcal{I}$  for every  $e \in \mathcal{I}$ . As  $\mathcal{V}_e$  is nonsingular, the critical locus  $W(\pi_i, \mathcal{V}_e)$  is guaranteed to be finite by the genericity of the change of variables  $\mathbf{A}$  (hence  $\mathcal{W}_{i,e}$  is) and that  $W(\pi_i, \mathcal{V}_e) \cap \mathbf{R}^{n+1}$  coincides with  $\pi_{\mathbf{x}}(\mathcal{W}_{i,e})$ . Thus, the above diffeomorphism implies that, for any connected component  $\mathcal{C}$  of  $\mathcal{W}_{i,\mathcal{I}}$ ,  $\mathcal{C}$  is diffeomorphic to an open interval in  $\mathbf{R}$ . Moreover, if  $\mathcal{C}$  is bounded, then  $\overline{\mathcal{C}} \setminus \mathcal{C}$  contains exactly two points which satisfy respectively  $f = 0$  and  $f^2 = e_0^2$ . We now consider

$$\mathcal{L}_i = \left\{ \mathbf{x} \in \mathbf{R}^{n+1} \mid 0 < f < e_0, g = 0, \frac{\partial f}{\partial x_j} = 0 \text{ for } j \neq i, x_{n+1} \neq 0 \right\}.$$

It is the intersection of the Zariski closure  $\mathcal{K}_i$  of the solution set to  $\left\{ \frac{\partial f}{\partial x_i} \neq 0, \frac{\partial f}{\partial x_j} = 0 \text{ for } j \neq i, x_{n+1} \neq 0 \right\}$  with the semi-algebraic set defined by  $0 < f < e_0$ . Note that  $\mathcal{K}_i$  is either empty or one-equidimensional. As  $\mathcal{V}_e$  is nonsingular for  $e \in \mathcal{I}$ ,  $\mathcal{L}_i$  and  $\mathcal{L}_j$  are disjoint for  $i \neq j$ . Since the restriction of  $\pi_{\mathbf{x}}$  to  $\mathcal{V}_t$  is an isomorphism between the algebraic sets  $\mathcal{V}_t$  and  $\mathbf{R}^{n+1}$  with the inverse map  $\mathbf{x} \mapsto (\mathbf{x}, f(\mathbf{x}))$ , the properties of  $\mathcal{W}_{\mathcal{I}}$  mentioned above are transferred to its image  $\mathcal{L}_i$  by the projection  $\pi_{\mathbf{x}}$ .

Further, we consider a subroutine `ParametricCurve` which takes as input  $f$  and  $i \in [1, n]$  and returns a rational parametrization  $\mathfrak{R}_i$  of  $\mathcal{K}_i$ . Also, let `Union` be a subroutine that takes a family of rational parametrizations  $\mathfrak{R}_1, \dots, \mathfrak{R}_n$  to compute a rational parametrization encoding the union of the algebraic curves defined by the  $\mathfrak{R}_i$ 's. We denote by  $\mathfrak{R}$  the output of `Union`; it encodes  $\mathcal{K} = \bigcup_{i=1}^n \mathcal{K}_i$ . We refer to [28, Appendix J.2] for these two subroutines.

Lemma 8 below establishes a *well-defined* notion of limit for a point  $\mathbf{x}_e \in W(\pi_i, \mathcal{V}_e) \cup W(\pi_i, \mathcal{V}_{-e})$  when  $e$  tends to 0.

**Lemma 8.** *Let  $e_0$  and  $\mathcal{L}_i$  be as above. For  $e \in ]0, e_0[$  and  $\mathbf{x}_e \in (W(\pi_i, \mathcal{V}_e) \cup W(\pi_i, \mathcal{V}_{-e})) \cap \mathbf{R}^{n+1}$ , there exists a (unique) connected component  $\mathcal{C}$  of  $\mathcal{L}_i$  containing  $\mathbf{x}_e$ . If  $\mathcal{C}$  is bounded, let  $\mathbf{x}$  be the only point in  $\overline{\mathcal{C}}$  satisfying  $f(\mathbf{x}) = 0$ , then  $\mathbf{x} \in \lim_{\varepsilon} (W(\pi_i, \mathcal{V}_\varepsilon) \cup W(\pi_i, \mathcal{V}_{-\varepsilon})) \cap \mathbf{R}(\varepsilon)^{n+1}$ . Thus, we set  $\lim_0 \mathbf{x}_e = \mathbf{x}$ .*

*Moreover, the extension  $\text{ext}(\mathcal{C}, \mathbf{R}(\varepsilon))$  contains exactly one point  $\mathbf{x}_\varepsilon$  such that  $f(\mathbf{x}_\varepsilon)^2 = \varepsilon^2$  and  $\lim_{\varepsilon} \mathbf{x}_\varepsilon = \mathbf{x}$ .*

*Proof.* Since  $\mathbf{x}_e \in (W(\pi_i, \mathcal{V}_e) \cup W(\pi_i, \mathcal{V}_{-e})) \cap \mathbf{R}^{n+1}$  and  $0 < e < e_0$ , we have  $\mathbf{x}_e \in \mathcal{L}_i$ , the existence of  $\mathcal{C}$  follows naturally. Let  $\mathbf{x}$  be the unique point of  $\overline{\mathcal{C}}$  satisfying  $f = 0$ . Then, the notion  $\lim_0$  is well-defined. From the proof of [2, Theorem 12.43], we have that

$$\lim_{\varepsilon} (W(\pi_i, \mathcal{V}_\varepsilon) \cup W(\pi_i, \mathcal{V}_{-\varepsilon})) \cap \mathbf{R}(\varepsilon)^{n+1} = \pi_{\mathbf{x}}(\overline{W_{(0,+\infty)}} \cap V(t)).$$

As  $\pi_{\mathbf{x}}(\overline{W_{(0,+\infty)}} \cap V(t))$  is the set of points corresponding to  $f = 0$  of  $\mathcal{L}_i$ , we deduce that  $\mathbf{x} \in \lim_{\varepsilon} (W(\pi_i, \mathcal{V}_\varepsilon) \cup W(\pi_i, \mathcal{V}_{-\varepsilon})) \cap \mathbf{R}(\varepsilon)^{n+1}$ .

Since the extension  $\text{ext}(\mathcal{C}, \mathbf{R}(\varepsilon))$  is a connected component of  $\text{ext}(\mathcal{L}_i, \mathbf{R}(\varepsilon))$  and homeomorphic to an open interval in  $\mathbf{R}(\varepsilon)$ , there exists  $\mathbf{x}_\varepsilon \in \text{ext}(\mathcal{C}, \mathbf{R}(\varepsilon))$  such that  $f(\mathbf{x}_\varepsilon)^2 = \varepsilon^2$ . Moreover, since  $0 = \lim_{\varepsilon} f(\mathbf{x}_\varepsilon)^2 = f(\lim_{\varepsilon} \mathbf{x}_\varepsilon)^2$  and  $\mathbf{x}$  is the only point in  $\overline{\mathcal{C}}$  satisfying  $f = 0$ , we conclude that  $\lim_{\varepsilon} \mathbf{x}_\varepsilon = \mathbf{x}$ .  $\square$

Now, let  $\mathcal{R}_e$  be a roadmap associated to the algebraic set  $\mathcal{V}_e \cup \mathcal{V}_{-e}$ , i.e.  $\mathcal{R}_e$  is contained in  $(\mathcal{V}_e \cup \mathcal{V}_{-e}) \cap \mathbf{R}^{n+1}$ , of at most dimension one and has non-empty intersection with every connected component of  $(\mathcal{V}_e \cup \mathcal{V}_{-e}) \cap \mathbf{R}^{n+1}$ . We also require that  $\mathcal{R}_e$  contains  $\bigcup_{i=1}^n (W(\pi_i, \mathcal{V}_e) \cup W(\pi_i, \mathcal{V}_{-e})) \cap \mathbf{R}^{n+1}$ . The proposition below is the key to describe `Isolated`.

**Proposition 9.** *Given  $e \in ]0, e_0[$  and  $\mathcal{I} = ] - e_0, 0[ \cup ] 0, e_0[$  as above. Let  $\mathcal{L} = \bigcup_{i=1}^n \mathcal{L}_i$  and  $\mathbf{x} \in \mathcal{P}$ . Then  $\mathbf{x}$  is not isolated in  $\mathcal{V} \cap \mathbf{R}^{n+1}$  if and only if there exists  $\mathbf{x}' \in \mathcal{P}$  such that  $\mathbf{x}$  and  $\mathbf{x}'$  are connected in  $\mathcal{P} \cup \mathcal{L} \cup \mathcal{R}_e$ .*

*Proof.* Assume first that  $\mathbf{x}$  is not isolated. By Proposition 7, there exists  $1 \leq i \leq n$  and a connected component  $\mathcal{C}_\varepsilon$  of  $(\mathcal{V}_\varepsilon \cup \mathcal{V}_{-\varepsilon}) \cap \mathbf{R}(\varepsilon)^{n+1}$ , which is bounded over  $\mathbf{R}$ , such that  $\mathcal{C}_\varepsilon \cap (W(\pi_i, \mathcal{V}_\varepsilon) \cup W(\pi_i, \mathcal{V}_{-\varepsilon}))$  contains  $\mathbf{x}_\varepsilon$  and  $\mathbf{x}'_\varepsilon$  satisfying  $\mathbf{x} = \lim_{\varepsilon} \mathbf{x}_\varepsilon \neq \lim_{\varepsilon} \mathbf{x}'_\varepsilon$ . By the choice of  $e_0$ , there exist a diffeomorphism  $\theta : \mathcal{V}_{t,\mathcal{I}} \rightarrow \mathcal{V}_e \times \mathcal{I}$  such that  $\theta(\mathcal{W}_{i,\mathcal{I}}) = \theta(\mathcal{W}_{i,e}) \times \mathcal{I}$ . Using [2, Exercise 3.2],  $\text{ext}(\theta, \mathbf{R}(\varepsilon))$  is a diffeomorphism between:

$$\text{ext}(\mathcal{V}_{t,\mathcal{I}}, \mathbf{R}(\varepsilon)) \cong \text{ext}(\mathcal{V}_e, \mathbf{R}(\varepsilon)) \times \text{ext}(\mathcal{I}, \mathbf{R}(\varepsilon)),$$

$$\text{ext}(\mathcal{W}_{i,\mathcal{I}}, \mathbf{R}(\varepsilon)) \cong \text{ext}(\mathcal{W}_{i,e}, \mathbf{R}(\varepsilon)) \times \text{ext}(\mathcal{I}, \mathbf{R}(\varepsilon)).$$

As  $\pi_{\mathbf{x}}$  is an isomorphism from  $\mathcal{V}_t$  to  $\mathbf{R}^{n+1}$ , there exists a (unique) bounded connected component  $\mathcal{C}_e$  of  $\mathcal{V}_e \cap \mathbf{R}^{n+1}$  s.t.  $\mathcal{C}_e$  is diffeomorphic to  $\text{ext}(\mathcal{C}_e, \mathbf{R}(\varepsilon))$ . Moreover, let  $L$  and  $L'$  be the connected components of  $\text{ext}(\mathcal{L}_i, \mathbf{R}(\varepsilon))$  containing  $\mathbf{x}_\varepsilon$  and  $\mathbf{x}'_\varepsilon$  respectively and  $\mathbf{x}_e$  and  $\mathbf{x}'_e$  ( $\in \text{ext}(\mathcal{C}_e, \mathbf{R}(\varepsilon))$ ) be the intersections of  $\text{ext}(\mathcal{C}_e, \mathbf{R}(\varepsilon))$  with  $L$  and  $L'$  respectively. Then,  $\lim_\varepsilon \mathbf{x}_\varepsilon$  ( $\lim_\varepsilon L$ ) connects  $\lim_\varepsilon \mathbf{x}_e$  ( $\lim_\varepsilon \mathbf{x}'_e$ ) to  $\mathbf{x}$  ( $\mathbf{x}'$ ). As  $\lim_\varepsilon \mathbf{x}_e$  and  $\lim_\varepsilon \mathbf{x}'_e$  are connected in  $\mathcal{C}_e$ , we conclude that  $\mathbf{x}$  and  $\mathbf{x}'$  are also connected in  $\mathcal{P} \cup \mathcal{L} \cup \mathcal{R}_e$ . The reverse implication is immediate using the above techniques  $\square$

From Lemma 8 and Proposition 9, any  $e$  lying in the interval  $]0, e_0[$  defined above can be used to replace the infinitesimal  $\varepsilon$ . So, we simply take  $e = e_0/2$ . For  $1 \leq i \leq n$ , we use a subroutine `ZeroDimSolve` which takes as input  $\left\{ f - e_0/2, g, \frac{\partial f}{\partial x_j} \text{ for all } j \neq i \right\}$  to compute a zero-dimensional parametrization  $\Omega_i$  such that  $W(\pi_i, \mathcal{V}_e) = \{\mathbf{x} \in Z(\Omega_i) | x_{n+1} \neq 0\}$ .

To use Proposition 9, we need to compute  $\mathcal{R}_{e_0/2}$ , which we refer to the algorithm `Roadmap` in [28]. This algorithm allows us to compute roadmaps for smooth and bounded real algebraic sets, which is indeed the case of  $(\mathcal{V}_{e_0/2} \cup \mathcal{V}_{-e_0/2}) \cap \mathbf{R}^{n+1}$ . First, we call (another) `Union` that, on the zero-dimensional parametrizations  $\Omega_i$ , it computes a zero-dimensional parametrization  $\Omega$  encoding  $\cup_{i=1}^n Z(\Omega_i)$ . Given the polynomials  $f, g$ , the value  $e_0/2$  and the parametrization  $\Omega$ , a combination of `Union` and `Roadmap` returns a one-dimensional parametrization  $\mathfrak{R}$  representing  $\mathcal{R}_{e_0/2}$ .

Deciding connectivity over  $\mathcal{P} \cup \mathcal{L} \cup \mathcal{R}_e$  is done as follows. We use `Union` to compute a rational parametrization  $\mathfrak{S}$  encoding  $\mathcal{K} \cup \mathcal{R}_e$ . Then, with input  $\mathfrak{S}, \mathfrak{P}, x_{n+1} \neq 0$  and the inequalities  $0 < f < e_0$ , we use Newton Puiseux expansions and cylindrical algebraic decomposition (see [10, 30]) following [27], taking advantage of the fact that polynomials involved in rational parametrizations of algebraic curves are bivariate. We denote by `ConnectivityQuery` the subroutine that takes those inputs and returns  $\mathfrak{B}$  and isolating boxes of the points defined by  $\mathfrak{B}$  which are not connected to other points of  $\mathfrak{B}$ .

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### Algorithm 3: `Isolated`

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**Input:** The polynomials  $f^A \in \mathcal{Q}[x_1, \dots, x_n]$  and  $g \in \mathcal{Q}[x_1, \dots, x_{n+1}]$  and the zero-dimensional parametrization  $\mathfrak{P}$ .

**Output:**  $\mathfrak{B}$  with isolating boxes of the isolated points of  $\mathcal{V}^A \cap \mathbf{R}^{n+1}$

- 1  $e_0 \leftarrow \text{SpecializationValue}(f^A, g)$
  - 2 **for**  $1 \leq i \leq n$  **do**
  - 3      $\Omega_i \leftarrow \text{ZeroDimSolve} \left( \left\{ f^A - e_0/2, g, \frac{\partial f^A}{\partial x_j} \text{ for all } j \neq i \right\} \right)$
  - 4      $\mathfrak{R}_i \leftarrow \text{ParametricCurve}(f^A, i)$
  - 5 **end**
  - 6  $\mathfrak{R} \leftarrow \text{Union}(\mathfrak{R}_1, \dots, \mathfrak{R}_n)$
  - 7  $\Omega \leftarrow \text{Union}(\Omega_1, \dots, \Omega_n)$
  - 8  $\mathfrak{R} \leftarrow \text{Union}(\text{RoadMap}(f^A - e_0/2, g, \Omega), \text{RoadMap}(f^A + e_0/2, g, \Omega))$
  - 9  $\mathfrak{S} \leftarrow \text{Union}(\mathfrak{R}, \mathfrak{P})$
  - 10  $B \leftarrow \text{ConnectivityQuery}(\mathfrak{S}, \mathfrak{P}, x_{n+1} \neq 0, 0 < f^A < e_0)$
  - 11 **return**  $(\mathfrak{B}, B)$
- 

## 4 Complexity analysis

All complexity results are given in the number of arithmetic operations in  $\mathcal{Q}$ . Hereafter, we assume that a generic enough matrix  $A$  is found from a random choice. In order to end the proof of Theorem 1, we now estimate the arithmetic runtime of the calls to `Candidates` and `Isolated`.

**Complexity of Algorithm 2** Since  $W(\pi_i, \mathcal{H}_\varepsilon^A)$  is the finite algebraic set associated to  $\left\langle f^A - e, \frac{\partial f^A}{\partial x_j} \text{ for all } j \neq i \right\rangle$ , its degree is bounded by  $d(d-1)^n$  [17]. Consequently, the degree of the output zero-dimensional parametrization lies in  $d^{O(n)}$ . Using [24, Theorem 6] (which is based on the geometric resolution algorithm in [14]), it is computed within  $d^{O(n)}$  arithmetic operations in  $\mathcal{Q}$ . The last step which takes intersections is done using the algorithm in [28, Appendix J.1]; it does not change the asymptotic complexity.

We have seen that `GetNormBound` reduces to isolate the real roots of a zero-dimensional parametrization of degree  $d^{O(n)}$ . This can be done within  $d^{O(n)}$  operations by Uspensky's algorithm [23].

**Complexity of Algorithm 3** Each call to `SpecializationValue` reduces to computing critical values of  $\pi_i$  of a smooth algebraic set defined by polynomials of degree  $\leq d$ . This is done using  $(nd)^{O(n)}$  arithmetic operations in  $\mathcal{Q}$  (see [15]). Using [14] for

ZeroDimSolve and [29] for ParametricCurve does not increase the overall complexity. The loop is performed  $n$  times ; hence the complexity lies in  $(nd)^{O(n)}$ . All output zero-dimensional parametrizations have degree bounded by  $d^{O(n)}$ . Running Union on these parametrizations does not increase the asymptotic complexity. One gets then parametrizations of degree bounded by  $nd^{O(n)}$ . Finally, using [28] for Roadmap uses  $(nd)^{O(n \log(n))}$  arithmetic operations in  $\mathcal{Q}$  and outputs a rational parametrization of degree lying in  $(nd)^{O(n \log(n))}$ . The call to ConnectivityQuery, done as explained in [27] is polynomial in the degree of the roadmap.

The final steps which consist in calling Removes and undoing the change of variables does not change the asymptotic complexity.

Summing up altogether the above complexity estimates, one obtains an algorithm using  $(nd)^{O(n \log(n))}$  arithmetics operations in  $\mathcal{Q}$  at most. This ends the proof of Theorem 1.

## 5 Experimental results

We report on practical performances of our algorithm. Computations were done on an Intel(R) Xeon(R) CPU E3-1505M v6 @ 3.00GHz with 32GB of RAM. We take sums of squares of  $n$  random dense quadrics in  $n$  variables (with a non-empty intersection over  $\mathbf{R}$ ) ; we obtain *dense quartics* defining a finite set of points. Timings are given in seconds (s.), minutes (m.), hours (h.) and days (d.).

We used Faugère’s FGB library for computing Gröbner bases in order to perform algebraic elimination in Algorithms 1, 2 and 3. We also used our C implementation for bivariate polynomial system solving (based on resultant computations) which we need to analyze connectivity queries in roadmaps. Timings for Algorithm 2 are given in the column CAND below. Timings for the computation of the roadmaps are given in the column RMP and timings for the analysis of connectivity queries are given in the column QRI below.

Roadmaps are obtained as the union of critical loci of some maps in slices of the input variety [28]. We report on the highest degree of these critical loci in the column SRMP. The column SQRI reports on the maximum degree of the bivariate zero-dimensional system we need to study to analyze connectivity queries on the roadmap.

None of the examples we considered could be tackled using the implementations of Cylindrical Algebraic Decomposition algorithms in Maple and Mathematica.

We also implemented [2, Alg. 12.16] using the FLINT C library with evaluation/interpolation techniques instead to tackle coefficients involving infinitesimals. This algorithm only computes sample points per connected components. *That implementation was not able to compute sample points of the input quartics for any of our examples.* We then report in the column [BPR] on the degree of the zero-dimensional system which is expected to be solved by [2, BPR]. This is to be compared with columns SRMP and SQRI.

$n$	CAND	RMP	QRI	total	SRMP	SQRI	[BPR]
4	2 s.	15 s.	33 s.	50 s.	36	359	7290
5	< 10 min.	1h.	7h.	8 h.	108	4644	65 610
6	< 12h	2 d.	18 d.	20 d.	308	47952	590 490

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