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# ANALYSIS OF THE FESHBACH–SCHUR METHOD FOR THE PLANEWAVE DISCRETIZATIONS OF SCHRÖDINGER OPERATORS

GENEVIÈVE DUSSON, ISRAEL MICHAEL SIGAL, AND BENJAMIN STAMM

ABSTRACT. In this article, we propose a new numerical method and its analysis to solve eigenvalue problems for self-adjoint Schrödinger operators, by combining the Feshbach–Schur perturbation theory with planewave discretization. In order to analyze the method, we establish an abstract framework of Feshbach–Schur perturbation theory with minimal regularity assumptions on the potential that is then applied to the setting of the new planewave discretization method. Finally, we present some numerical results that underline the theoretical findings.

## 1. INTRODUCTION

In this article, we address the problem of the computation of eigenvalues of self-adjoint Schrödinger operators (quantum Hamiltonians)  $\mathcal{H} = -\Delta + V$ . Our main result is a reduction of this infinite dimensional problem to a finite-dimensional one in a fully controlled way. To this end we use the Feshbach–Schur (FS) method which originated in works of I. Schur on the Dirichlet problem in planar domains and H. Feshbach, on resonances in the nuclear physics, and was then developed independently in numerical analysis, computational quantum chemistry and mathematical physics, see [12, 14].

Unlike the standard applications of this method (see e.g. a series of papers [17, 18] by Löwdin on bounds on eigenvalues of a given Hamiltonian), we use it not as a fixed scheme but rather, following [2], as a map - called the Feshbach–Schur map (FSM) - from one problem to another, simpler one, involving fewer degrees of freedom. We base our analysis on the isospectrality property of this map discovered in [2] recalled in Theorem 3 below. We call this approach the *FSM method*.

We combine this approach with planewave discretizations which are widely used in numerical methods in electronic structure calculation, especially for condensed matter simulations and in materials science. Electronic structure calculation is indeed one of the problems we have in our sight. And one particular very useful aspect of planewaves is that they are eigenfunctions of the Laplace operator, which is the main part of the Hamiltonian  $\mathcal{H} = -\Delta + V$  that needs to be diagonalized in order to determine the electronic structure of the system.

Limiting the computational cost of finding the eigenstates of the Hamiltonian operator has been a key issue in electronic structure calculation, and is currently of interest, due to the ever growing size of the considered systems that is matched to the available computational resources. For example, different perturbation methods have been proposed, such as [4, 19], traditionally to introduce more physical details e.g. many-particle interactions in a given approximation. More recently, a post-processing strategy has been proposed by some authors for planewave discretizations for non-linear eigenvalue problems [8, 9, 7, 11], which considers the exact solution as a perturbation of the discrete (using the planewave basis) approximation. This is in spirit not so far from so-called two-grid methods, where a first problem is solved on a coarse basis, i.e. in a small discretization space, and a small problem is solved on a fine basis. In the case of eigenvalue

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problems, two-grid methods have been proposed e.g. in [24] in the case of a linear eigenvalue problem. A two-grid method has also been proposed for nonlinear eigenvalue problems of a Gross–Pitaevskii type in [5].

In this article, we extend the FSM-method to establish finite-dimensional approximations to solve the Hamiltonian eigenvalue problem with controlled errors on the eigenvalues and eigenvectors. To be a little more concrete, we define a new problem in a coarse space  $\mathbf{X}_M \subset \mathbf{X}$  yielding the exact eigenvalue one would obtain when computing it in the infinite-dimensional space  $\mathbf{X}$ . Indeed, our contribution follows a new Ansatz based on the question: Can we find a discrete Hamiltonian acting on the finite-dimensional space  $\mathbf{X}_M$  that has the exact eigenvalue  $\lambda_*$  of the original Hamiltonian (acting on  $\mathbf{X}$ ) as eigenvalue? It turns that the answer is yes, but that the discrete Hamiltonian depends itself on  $\lambda_*$ , through the FS-map, leading to an eigenvalue problem in  $\mathbf{X}_M$  that is nonlinear in the spectral parameter. Not surprisingly, the map cannot be computed exactly but only be approximated through a fast decaying series, that is truncated based on a parameter  $K$ , and which requires computations in a larger space  $\mathbf{X}_N$  with  $\mathbf{X}_M \subset \mathbf{X}_N \subset \mathbf{X}$ .

In this work we quantify the error introduced due to the discretization parameters  $K, N, M$  and show that the eigenvalue and eigenfunction errors are bounded by two terms: i) a term with algebraic decay with respect to the truncation parameter  $K$ , and ii) a term with a regularity-dependent convergence rate in  $N$ . We also quantify the explicit dependency of the error in terms of the parameter  $M$  defining the discrete space  $\mathbf{X}_M$  and the potential, including its regularity. Our analysis reveals that the algebraic decay rate with respect to  $K$  increases with increasing  $M$ .

Our method uses an adapted version of perturbation theory based on a slightly more regular notion of relative form-boundedness, as stated in Assumption 1, developed as an abstract theory in Section 3, which thus only requires little regularity of the potential including cases which are not covered by the standard analysis of planewave discretizations. We also illustrate our approach by computing eigenvalues of several 1D Schrödinger operators.

This article is organized as follows. In Section 2 we present the problem and numerical method that is used to find approximations thereof, as well as the main approximation result of the article and the error bounds on the eigenvalues. Section 3 provides the above-mentioned abstract framework of Feshbach–Schur perturbation theory based on the regularized version of form-boundedness whereas Section 4 contains some technical results needed to prove the main result which follows in Section 5. Finally, we present in Section 6 some numerical results to illustrate the convergence as well as the error bounds, and we conclude with some perspectives in Section 7.

## 2. SET-UP AND RESULTS

**2.1. Problem statement.** In order to simplify the notation, we consider a cubic lattice  $\mathcal{R} = L\mathbb{Z}^d$  ( $L > 0$ ,  $d = 1, 2, 3$ ), but all our arguments straightforwardly apply to the general case of any Bravais lattice. In this paper we are interested in the spectral theory of the self-adjoint Schrödinger operators (quantum Hamiltonians)

$$\mathcal{H} := -\Delta + V,$$

with reasonably regular,  $\mathcal{R}$ -periodic potentials  $V$ , acting on the Hilbert space

$$\mathbb{L}_{\text{per}}^2 := \left\{ u \in L_{\text{loc}}^2(\mathbb{R}^d) \mid u \text{ is } \mathcal{R}\text{-periodic} \right\},$$

endowed with the scalar product  $\langle u, v \rangle = \int_{\Omega} u(\mathbf{r})v(\mathbf{r})d\mathbf{r}$  and the induced norm  $\|\cdot\|$ , where  $\Omega = [0, L]^d$  is the chosen fundamental cell of the lattice  $\mathcal{R} = L\mathbb{Z}^d$ .

Specifically, we would like to solve the eigenvalue problem

$$\mathcal{H}\varphi = \lambda\varphi, \quad (1)$$

in a space  $X \subset H_{\text{per}}^1$ . Here,  $H_{\text{per}}^1$  is the Sobolev space of index 1 of periodic functions on  $\Omega$ , which is defined in precise terms later on and equation (1) is considered in the weak sense.

To this end we use the Feshbach–Schur method to reduce the problem to a finite dimensional one. To simplify the exposition, we will assume that the eigenvalue of interest  $\lambda_*$  is isolated, which is true for the smallest eigenvalue under fairly general assumptions of  $V$ . We denote by  $\|\cdot\|$  the operator norm on  $\mathcal{L}(L_{\text{per}}^2)$ , the space of bounded linear operators on  $L_{\text{per}}^2$ . To formulate our condition on the potential  $V$ , we introduce the following norm measuring its regularity

$$\|V\|_r := \|(-\Delta + 1)^{-1/2+r/2}V(-\Delta + 1)^{-1/2+r/2}\|,$$

where the operator  $(-\Delta + 1)^s$  is defined by the Fourier transform (cf. Appendix A). In what follows, we thus assume that the potential  $V$  satisfies the following condition.

**Assumption 1.** *The potential  $V$  is real,  $\mathcal{R}$ -periodic and satisfies*

$$\|V\|_r < \infty \quad \text{for some } r > 0.$$

Assumption 1 implies that  $V$  is  $\Delta$ -form bounded [10, 21], which corresponds to  $r = 0$ . The latter, weaker property implies that  $\mathcal{H}$  (a) is self-adjoint; (b) is bounded below and (c) has purely discrete spectrum (see e.g. [10, 21, 22, 15]). Moreover, potentials  $V$  belonging to the Sobolev spaces,  $H_{\text{per}}^s := (-\Delta + 1)^{s/2}L_{\text{per}}^2$  satisfy this assumption as shown in Appendix A, Lemma 13 for  $r \leq s + 1$  and  $r < 1 + \frac{s}{2} - \frac{d}{4}$ . In terms of Sobolev spaces, Assumption 1 states that  $V$ , as an operator, maps  $H_{\text{per}}^{1-r}$  into  $H_{\text{per}}^{-1+r}$ .

**2.2. Approach.** In our approach, we reduce the exact infinite dimensional eigenvalue problem to a finite dimensional one in a controlled way for fairly irregular potentials. Of course, we have to pay a price for this, which is that at one point we solve a one-dimensional fixed point problem that can be equivalently seen as a non-linear eigenvalue problem. A key ingredient of our method is the finite dimensional space and the corresponding orthogonal projection onto which we map the original problem to obtain a reduced, finite-dimensional one.

Let  $X_M$  denote the subspace of  $L_{\text{per}}^2$  spanned by the eigenfunctions of  $-\Delta$  on  $\mathcal{R}$ , thus planewaves, with eigenvalues smaller than  $\rho_M$  where

$$\rho_M := \left(\frac{2\pi M}{L}\right)^2.$$

Let  $P_M$  be the  $L_{\text{per}}^2$ -orthogonal projection onto  $X_M$  and  $P_M^\perp = 1 - P_M$ . We consider the Galerkin approximation of the linear Hamiltonian  $\mathcal{H} := -\Delta + V$ ,

$$\mathcal{H}_M := P_M(-\Delta + V)P_M.$$

We now introduce the projections  $\varphi_M = P_M\varphi$  and  $\varphi_M^\perp = P_M^\perp\varphi$  and project the exact eigenvalue problem (1) onto the subspace  $X_M$  and its complement  $X_M^\perp$  to obtain

$$P_M(\mathcal{H}_M - \lambda)\varphi_M = -P_M V\varphi_M^\perp, \quad (2)$$

$$P_M^\perp(\mathcal{H}_M^\perp - \lambda)\varphi_M^\perp = -P_M^\perp V\varphi_M, \quad (3)$$

where  $\mathcal{H}_M^\perp := P_M^\perp\mathcal{H}P_M^\perp$ . Next, in Appendix A, we prove the following

**Lemma 1.** *Let Assumption 1 hold and define  $\kappa_M := \rho_M - (\rho_M + 1)\rho_M^{-r}\|V\|_r$ . Then*

$$\mathcal{H}_M^\perp \geq \kappa_M \quad \text{on } \text{Ran } P_M^\perp. \quad (4)$$

Thus for  $\lambda < \kappa_M$ , the operator  $\mathcal{H}_M^\perp - \lambda$  is invertible and we can solve (3) for  $\varphi_M^\perp$  and thus  $\varphi_M^\perp = -(\mathcal{H}_M^\perp - \lambda)^{-1} \mathbf{P}_M^\perp V \varphi_M$ . Substituting the result into (2), we obtain the non-linear eigenvalue problem

$$(\mathcal{H}_M + U_M(\lambda)) \varphi_M = \lambda \varphi_M, \quad (5)$$

where we introduced the *effective interaction*  $U_M(\lambda) : \mathbf{X}_M \rightarrow \mathbf{X}_M$ , or a Schur complement,

$$U_M(\lambda) := -\mathbf{P}_M V \mathbf{P}_M^\perp (\mathcal{H}_M^\perp - \lambda)^{-1} \mathbf{P}_M^\perp V \mathbf{P}_M. \quad (6)$$

We then have the following proposition, which is proved in Appendix A.

**Proposition 1.** *For each  $\lambda$  such that  $\lambda < \kappa_M$ ,  $U_M(\lambda)$  is a well-defined operator as a product of three maps:  $\mathbf{P}_M, V$  and  $\mathbf{P}_M^\perp (\mathcal{H}_M^\perp - \lambda)^{-1} \mathbf{P}_M^\perp$  between various but matching Sobolev spaces.*

Now, we construct a completely computable approximation of the eigenvalue problem (5), with the operators involved being sums of products of finite matrices. Namely, we expand the resolvent  $(\mathcal{H}_M^\perp - \lambda)|_{\text{Ran } \mathbf{P}_M^\perp}^{-1} = (-\Delta + V_M^\perp - \lambda)|_{\text{Ran } \mathbf{P}_M^\perp}^{-1}$  in (6) in the formal Neumann series in  $V_M^\perp$ , then truncate this series at  $K \in \mathbb{N}$  and replace the projections  $\mathbf{P}_M^\perp := \mathbf{1} - \mathbf{P}_M$  by  $\mathbf{P}_M^N := \mathbf{P}_N - \mathbf{P}_M$ , with  $N > M$ . Introducing the notation

$$\mathbf{G}_M^N(\lambda) := (-\Delta - \lambda)|_{\text{Ran } \mathbf{P}_M^N}^{-1} \quad (7)$$

and  $V_M^N := \mathbf{P}_M^N V \mathbf{P}_M^N$ , we obtain the following truncated effective interaction

$$U_\sigma(\lambda) := -\mathbf{P}_M V \mathbf{P}_M^N R_\sigma(\lambda) \mathbf{P}_M^N V \mathbf{P}_M, \quad (8)$$

where  $\sigma := (N, M, K)$  and  $R_\sigma(\lambda) := \sum_{k=0}^K (-1)^k [\mathbf{G}_M^N(\lambda) V_M^N]^k \mathbf{G}_M^N(\lambda)$ . Since all the operators involved in (8) are finite matrices, this family is well-defined and computable. Now, we define  $\mathcal{H}_\sigma(\lambda) := \mathcal{H}_M + U_\sigma(\lambda)$  on  $\mathbf{X}_M$  and consider the eigenvalue problem: find an eigenvalue  $\lambda_{\sigma i}$  and the corresponding eigenfunctions  $\varphi_{\sigma i} \in \mathbf{X}_M$  such that

$$\mathcal{H}_\sigma(\lambda_{\sigma i}) \varphi_{\sigma i} = \lambda_{\sigma i} \varphi_{\sigma i}. \quad (9)$$

Next, we define the approximate ‘lifting’ operator whose origin will be become clear in the next section:

$$Q_\sigma(\lambda) := \mathbf{1} - R_\sigma(\lambda) \mathbf{P}_M^N V \mathbf{P}_M. \quad (10)$$

**2.3. Main results.** Within this manuscript, we denote by  $\lesssim$  upper bounds involving constants that do not depend on the parameters  $\sigma = (N, M, K), \alpha, r, \|V\|_r$ . Then, we have the following result.

**Theorem 1.** *Let Assumption 1 hold, let  $\lambda_\star$  be an isolated eigenvalue of  $\mathcal{H}$  of finite multiplicity  $m$ , let  $\gamma_0$  denote the gap of  $\lambda_\star$  to the rest of the spectrum of  $\mathcal{H}$ .*

*Then, there exists  $\alpha > 0$  and  $M_0 \in \mathbb{N}$  such that for  $M \geq M_0$ , problem (9) has  $m$  solutions  $(\varphi_{\sigma i}, \lambda_{\sigma i}) \in \mathbf{X}_M \times [\lambda_\star - \frac{\gamma_0}{2}, \lambda_\star + \frac{\gamma_0}{2}]$ . We denote  $\lambda_\circ = \lambda_\star + \gamma_0 + \alpha$ . For each  $\varphi_{\sigma i}$ , there exists  $\varphi_i \in \mathbf{X}$ , an eigenfunction of  $\mathcal{H}$  associated to  $\lambda_\star$ , such that  $(\varphi_{\sigma i}, \lambda_{\sigma i})$  approximates  $(\varphi_i, \lambda_\star)$  in the following sense:*

$$|\lambda_\star - \lambda_{\sigma i}| \lesssim (\lambda_\star + \alpha) \frac{\|V\|_r^2}{\alpha^r} \varepsilon(\sigma, r, V), \quad (11)$$

$$\|\varphi_i - Q_\sigma(\lambda_{\sigma i}) \varphi_{\sigma i}\| \lesssim \|V\|_r \left[ 1 + \frac{\lambda_\circ}{\gamma_0} \frac{\|V\|_r}{\alpha^r} \right] \varepsilon(\sigma, r, V), \quad (12)$$

where

$$\varepsilon(\sigma, r, V) := \rho_N^{-r} + \rho_M^{-r} [4\rho_M^{-r} \|V\|_r]^{K+1}.$$

This Theorem is subject to several remarks.

**Remark 1.** Note that  $\varepsilon$  is equivalent to

$$\varepsilon(\sigma, r, V) \approx N^{-2r} + M^{-2r} \left[ 4 \left( \frac{2\pi}{L} \right)^{2r} M^{-2r} \|V\|_r \right]^{K+1},$$

where the equivalence constants do not depend on the parameters  $\sigma = (N, M, K), r, \alpha, V$ .

**Remark 2.** In some cases, for instance in multi-scale problems, one might be only interested in the coarse-scale solution, i.e. the best-approximation in the coarse space  $\mathbf{X}_M$  given by  $\mathbf{P}_M \varphi_i$ . In such cases, a useful byproduct of the proof of Theorem 1 is the following estimate

$$\|(-\Delta + 1)^s (\mathbf{P}_M \varphi_i - \varphi_{\sigma i})\| \lesssim \frac{\lambda_\circ}{\gamma_0} \frac{\|V\|_r^2}{\alpha^r} \rho_M^s \varepsilon(\sigma, r, V), \quad (13)$$

for any  $s \geq 0$ , which thus compares the eigenfunctions in the space  $\mathbf{X}_M$ .

**Remark 3.** Note that convergence of the eigenvalues and the eigenfunctions can be achieved by taking the limit  $K, N \rightarrow \infty$  for fixed  $M \geq M_0$ . For practical purposes, the idea is to set  $N$  large enough so that the error is dominated by the error introduced in  $K < +\infty$ .

Further, note that the eigenvalue and eigenvector errors have the same rate of convergence with respect to  $K$ . However, the error in the eigenvector depends on the gap  $\gamma_0$  while the error in the eigenvalue does not.

The estimate with respect to  $N$  in Theorem 1 is not sharp in all cases, in particular for sufficiently regular potentials  $V$ . Nonetheless, our analysis has the merit of presenting the convergence result in one combined analysis based on perturbative techniques which also holds for low regularities of the potential where standard *a priori* approximation results of the variational approximation result do not hold (note that an estimate of the variational problem can be obtained by setting  $K \rightarrow \infty$  or  $M = N$ ). Moreover, for potentials with very low regularities, the upcoming numerical results indicate that the convergence with respect to  $N$  of our analysis is sharp, at least in the presented numerical study.

Note that we can adapt the result whenever *a priori* approximation results are available by employing the triangle inequality. Indeed, if the potential  $V$  belongs to the Sobolev space  $V \in \mathbf{H}_{\text{per}}^s$ , with  $s > d/2$ , we resort to *a priori* results in a first place to obtain a sharp bound with respect to  $N$ , see e.g., [1, 6, 3], and also [20] for a certain class of discontinuous potentials in  $H^{1/2-\varepsilon}$  for all  $\varepsilon > 0$ , in two dimensions.

More precisely, we consider  $\mathcal{H}$  acting on  $\mathbf{X}_N$  directly, i.e. substituting  $\mathcal{H}$  by  $\mathcal{H}_N := \mathbf{P}_N \mathcal{H} \mathbf{P}_N$  and using  $\mathbf{X} = \mathbf{X}_N$  with variational solution  $(\varphi_N, \lambda_N)$ , assuming a simple eigenvalue for simplicity. It is important to note that problem (9) remains unchanged and thus, the result of Theorem 1 holds with

$$\tilde{\varepsilon}(\sigma, r, V) := \rho_M^{-r} \left[ 4 \rho_M^{-r} \|V\|_r \right]^{K+1},$$

but where the exact solution is substituted by  $(\varphi_N, \lambda_N)$ . Proceeding then by the triangle inequality yields

$$\begin{aligned} |\lambda_\star - \lambda_{\sigma i}| &\leq |\lambda_\star - \lambda_N| + |\lambda_N - \lambda_{\sigma i}|, \\ \|\varphi_i - Q_\sigma(\lambda_{\sigma i})\varphi_{\sigma i}\| &\leq \|\varphi_i - \varphi_N\| + \|\varphi_N - Q_\sigma(\lambda_{\sigma i})\varphi_{\sigma i}\|. \end{aligned}$$

Combining then the aforementioned *a priori* estimates from [1, 6, 3] for the first terms of the right hand sides with Theorem 1 for the latter parts yields the following corollary.

**Corollary 2.** *Under the conditions of Theorem 1 and if  $V \in \mathbf{H}_{\text{per}}^s$ ,  $s > d/2$ , then*

$$\begin{aligned} |\lambda_\star - \lambda_{\sigma i}| &\leq C \left( N^{-(2s+2)} + \tilde{\varepsilon}(\sigma, r, V) \right), \\ \|\varphi_i - Q_\sigma(\lambda_{\sigma i})\varphi_{\sigma i}\| &\leq C \left( N^{-(s+2)} + \tilde{\varepsilon}(\sigma, r, V) \right), \end{aligned}$$

for some constant  $C > 0$  independent on  $\sigma = (N, M, K)$ .

**2.4. Theoretical background.** Let us shed the attention to the theoretical foundation on the eigenvalue formulation in form of (5), instead of (1), which is based on the following results, originally presented in [14, Theorem 11.1].

**Theorem 3.** *Let  $H$  be an operator on a Hilbert space and  $P$  and  $P^\perp$ , a pair of projections such that  $P + P^\perp = 1$ . Assume  $H^\perp := P^\perp H P^\perp$  is invertible on  $\text{Ran } P^\perp$  and the expression*

$$F_P(H) := P(H - H R^\perp H)P, \quad (14)$$

where  $R^\perp := P^\perp (H^\perp)^{-1} P^\perp$ , defines a bounded operator. Then  $F_P$ , considered as a map on the space of operators, is isospectral in the following sense:

- (a)  $\lambda \in \sigma(H) \iff 0 \in \sigma(F_P(H - \lambda))$ ;
- (b)  $H\psi = \lambda\psi \iff F_P(H - \lambda)\varphi = 0$ ;
- (c)  $\dim \text{Null}(H - \lambda) = \dim \text{Null}F_P(H - \lambda)$ .

Moreover,  $\psi$  and  $\varphi$  in (b) are related as  $\varphi = P\psi$  and  $\psi = Q_P(\lambda)\varphi$ , where

$$Q_P(\lambda) := P - P^\perp (H^\perp - \lambda)^{-1} P^\perp H P.$$

Finally, if  $H$  is self-adjoint, then so is  $F_P(H)$ .

The map  $F_P$  on the space of operators, is called the *Feshbach-Schur map*. The relation  $\psi = Q_P(\lambda)\varphi$  allows us to reconstruct the full eigenfunction from the projected one. By statement (a), we have

**Corollary 4.** *Let  $\nu_i(\lambda)$  denote the  $i$ -th eigenvalue of the operator  $F_P(H - \lambda) + \lambda \mathbf{1}$  for each  $\lambda$  in an interval  $I \subset \mathbb{R}$ . Then the eigenvalues of  $H$  in  $I$  are in one-to-one correspondence with the solutions of the equations*

$$\nu_i(\lambda) = \lambda.$$

In the current setting of planewave approximations,  $P = \mathbf{P}_M$ ,  $P^\perp = \mathbf{P}_M^\perp$ , Proposition 1 implies that the results of Theorem 3 apply for each choice of  $M \in \mathbb{N}$  and yield

$$F_{\mathbf{P}_M}(\mathcal{H} - \lambda) = \mathcal{H}_M(\lambda) - \lambda \mathbf{P}_M, \quad (15)$$

where we introduced the notation

$$\mathcal{H}_M(\lambda) := \mathcal{H}_M + U_M(\lambda). \quad (16)$$

Note that  $\mathcal{H}_M(\lambda)$  is exactly the operator entering (5). Thus, we have the following.

**Corollary 5.** *Let  $\lambda \in \mathbb{C}$  with  $\text{Re } \lambda < \kappa_M$ . Then*

- (a)  $\mathcal{H}\psi = \lambda\psi \iff (\mathcal{H}_M(\lambda) - \lambda)\varphi_M = 0$ ;
- (b)  $\dim \text{Null}(\mathcal{H} - \lambda) = \dim \text{Null}(\mathcal{H}_M(\lambda) - \lambda)$ .
- (c)  $\psi$  and  $\varphi$  in (a) are related as  $\varphi_M = \mathbf{P}_M\psi$  and  $\psi = \mathbf{Q}_M(\lambda)\varphi_M$ , where

$$\mathbf{Q}_M(\lambda) = \mathbf{1} - (\mathcal{H}_M^\perp - \lambda)^{-1} \mathbf{P}_M^\perp V \mathbf{P}_M. \quad (17)$$

*i.e. the corresponding eigenfunction can be reconstructed from  $\varphi_M$  by an explicit linear map.*

This result shows that the original infinite-dimensional spectral problem (1) is equivalent to the finite dimensional spectral problem (5) which is nonlinear in the spectral parameter  $\lambda$ . We now state a few properties of the effective interaction  $U_M(\lambda)$ , in order to characterize the solutions of the fixed-point problems  $\nu_i(\lambda) = \lambda$ .

**Proposition 2.** *For  $\lambda \in \mathbb{R}$  such that  $\lambda < \kappa_M$ ,  $U_M(\lambda)$  is (i) non-positive, (ii) monotonically decreasing with  $\lambda$ , (iii) vanishing as  $\lambda \rightarrow -\infty$ . For  $\lambda \in \mathbb{C}$  such that  $\text{Re } \lambda < \kappa_M$ ,  $U_M(\lambda)$  is (iv) complex analytic in  $\lambda$  and (v) symmetric.*

*Proof.* Properties (i)-(iv) follow directly from definition (6) and Lemma 1 above. For the last one, we use that  $\mathcal{H}$  is self-adjoint.  $\square$

**Proposition 3.** *Denote by  $\nu_{Mi}(\lambda)$  the  $i$ -th eigenvalue of  $\mathcal{H}_M(\lambda)$ . Then the equation  $\nu_{Mi}(\lambda) = \lambda$  has a unique solution on the interval  $(-\infty, \kappa_M)$ .*

*Proof.* Since, for  $\lambda < \kappa_M$ ,  $U_M(\lambda)$  is symmetric, the operator  $\mathcal{H}_M(\lambda)$  defined by (16) is (a) self-adjoint, (b) monotonically decreasing with  $\lambda$ , (c) converging to  $\mathcal{H}_M$  as  $\lambda \rightarrow -\infty$ , (d) is complex analytic in  $\lambda$  for  $\text{Re } \lambda < \kappa_M$ . We deduce from (b) that the functions  $\nu_{Mi}$  are decreasing on  $(-\infty, \kappa_M)$  and thus, if the  $i$ -th eigenvalue of  $\mathcal{H}$  is less than  $\kappa_M$ , the equation  $\nu_{Mi}(\lambda) = \lambda$  has a unique solution.  $\square$

Note also that  $\lim_{\lambda \rightarrow -\infty} \nu_{Mi}(\lambda)$  is the  $i$ -th eigenvalue of  $\mathcal{H}_M$  which is larger than the  $i$ -th eigenvalue of  $\mathcal{H}$  due to the variational principle.

These considerations motivate the numerical strategies to compute solutions to (9) in the following section.

**2.5. Numerical strategy.** In order to find solutions to the non-linear eigenvalue problem (9), we propose two strategies:

*Strategy 1:* For a fixed index  $i = 1, \dots, M$ , consider the sequence of iterates  $\lambda_\sigma^{(k)}$  obtained by

$$\lambda_\sigma^{(k)} : \text{ is the } i\text{-th eigenvalue of } \mathcal{H}_\sigma(\lambda_\sigma^{(k-1)}). \quad (18)$$

We thus introduce the notation  $\nu_{\sigma i}(\lambda)$  denoting the  $i$ -th eigenvalue (counting multiplicities) of the Hamiltonian  $\mathcal{H}_\sigma(\lambda)$  and thus have  $\lambda_\sigma^{(k)} = \nu_{\sigma i}(\lambda_\sigma^{(k-1)})$ . The limit value  $\lambda_\sigma := \lim_{k \rightarrow \infty} \lambda_\sigma^{(k)}$  then satisfies  $\lambda_\sigma = \nu_{\sigma i}(\lambda_\sigma)$  and thus (9).

*Strategy 2:* For a given target value  $\lambda_t \in \mathbb{R}$ , consider the sequence of iterates  $\lambda_\sigma^{(k)}$  obtained by

$$\lambda_\sigma^{(k)} : \text{ is the eigenvalue of } \mathcal{H}_\sigma(\lambda_\sigma^{(k-1)}) \text{ closest to } \lambda_t. \quad (19)$$

We thus introduce the notation  $\nu_{\sigma t}(\lambda)$  denoting the eigenvalue of the Hamiltonian  $\mathcal{H}_\sigma(\lambda)$  closest to  $\lambda_t$  and thus have  $\lambda_\sigma^{(k)} = \nu_{\sigma t}(\lambda_\sigma^{(k-1)})$ . The limit value  $\lambda_\sigma := \lim_{k \rightarrow \infty} \lambda_\sigma^{(k)}$  then satisfies  $\lambda_\sigma = \nu_{\sigma t}(\lambda_\sigma)$  and thus (9).

In both cases, as outlined in the upcoming Remark 7, convergence of the fixed-point maps (18) and (19) can be guaranteed under some conditions and for  $M$  large enough.

### 3. PERTURBATION ESTIMATES

In this article, we often deal with the following the eigenvalue perturbation problem: Given an operator  $H$  on a Hilbert space  $X$  of the form

$$H = H_0 + W, \quad (20)$$

where  $H_0$  is an operator with some isolated eigenvalues and  $W$  is small in an appropriate norm, show that  $H$  has eigenvalues near those of  $H_0$  and estimate these eigenvalues and the



corresponding eigenvectors. We therefore start by presenting an abstract theory which will be applied to our concrete problem in the following sections.

Specifically, we assume that  $H$  and  $H_0$  are self-adjoint and bounded from below and that  $W$  is  $\alpha$ -form-bounded w.r.t. of  $H_0$ , in the sense that for  $\alpha \in \mathbb{R}$  such that  $H_0 + \alpha$  is a positive operator ( $H_0 + \alpha > 0$ ), we have

$$\|W\|_{H_0, \alpha} := \|(H_0 + \alpha)^{-1/2}W(H_0 + \alpha)^{-1/2}\| < \infty, \quad (21)$$

where  $(H_0 + \alpha)^{-s}$ ,  $s > 0$ , is defined either by the spectral theory or by the explicit formula

$$(H_0 + \alpha)^{-s} := c_\alpha \int_0^\infty (H_0 + \alpha + \omega)^{-1} d\omega / \omega^s,$$

where  $c_\alpha := [\int_0^\infty (\alpha + \omega)^{-1} d\omega / \omega^s]^{-1}$ . This notion is equivalent to that of the relative form-boundedness, but it gives an important quantification of the latter.

We also note here that, by a known result about relatively form-bounded operators (see e.g. [21, 15]), if  $H_0$  is a self-adjoint, bounded below operator on  $X$  and  $W$  is symmetric and  $\alpha$ -form-bounded w.r.t. of  $H_0$ , then  $H = H_0 + W$  is self-adjoint.

We start with a general result on the eigenvalue difference.

**Proposition 4.** *Let  $H_0$  be a self-adjoint bounded below operator on  $X$  and  $W$  symmetric and  $\alpha$ -form-bounded w.r.t. of  $H_0$ , and let  $H = H_0 + W$ . Let  $\alpha \in \mathbb{R}$  be such that  $H_0 + \alpha > 0$ . Then  $H$  the eigenvalues of  $H$  and  $H_0$  satisfy the estimates*

$$|\nu_i(H) - \nu_i(H_0)| \leq (\nu_i(H_0) + \alpha) \|W\|_{H_0, \alpha}, \quad (22)$$

where  $\nu_i(A)$  denotes the  $i$ -th eigenvalue of the operator  $A$ .

*Proof.* Let  $u \in X$  be arbitrary and define  $v = (H_0 + \alpha)^{1/2}u$  noting that  $(H_0 + \alpha) > 0$ . Then,

$$\langle u, Hu \rangle = \langle u, H_0u \rangle + \langle v, (H_0 + \alpha)^{-1/2}W(H_0 + \alpha)^{-1/2}v \rangle.$$

Note that

$$\langle v, (H_0 + \alpha)^{-1/2}W(H_0 + \alpha)^{-1/2}v \rangle \leq \|W\|_{H_0, \alpha} \langle v, v \rangle = \|W\|_{H_0, \alpha} \langle u, (H_0 + \alpha)u \rangle,$$

and therefore

$$\begin{aligned} \langle u, H_0u \rangle (1 - \|W\|_{H_0, \alpha}) - \alpha \|u\|^2 \|W\|_{H_0, \alpha} &\leq \langle u, Hu \rangle \\ &\leq \langle u, H_0u \rangle (1 + \|W\|_{H_0, \alpha}) + \alpha \|u\|^2 \|W\|_{H_0, \alpha}. \end{aligned}$$

Using the min-max principle (Courant–Fisher), there holds

$$\nu_i(H_0) (1 - \|W\|_{H_0, \alpha}) - \alpha \|W\|_{H_0, \alpha} \leq \nu_i(H) \leq \nu_i(H_0) (1 + \|W\|_{H_0, \alpha}) + \alpha \|W\|_{H_0, \alpha},$$

which leads to the result.  $\square$

Let us now assume that  $\lambda_0$  is an isolated eigenvalue of  $H_0$  of finite multiplicity  $m$  and let  $P_0$  be the orthogonal projection onto the span of the the eigenfunctions of  $H_0$  corresponding to the eigenvalue  $\lambda_0$ , and let  $P_0^\perp := \mathbf{1} - P_0$ . We further introduce  $H_{0, -\lambda} := H_0^\perp - \lambda P_0^\perp$  and thus  $H_{0, \alpha} = H_0^\perp + \alpha P_0^\perp$ .

Let  $\gamma_0$  denote the gap of  $\lambda_0$  to its closest eigenvalue in the remaining spectrum of  $H_0$  and we introduce the spectral interval  $I_0 = [\lambda_0 - \frac{1}{2}\gamma_0, \lambda_0 + \frac{1}{2}\gamma_0]$ .

Our next result gives estimates on the difference of eigenvectors of  $H$  and  $H_0$ , as well as on the difference of their corresponding eigenvalues. For standard approaches to the spectral perturbation theory, see [23, 16, 22, 15].

**Theorem 6.** *Let  $H_0$  be a self-adjoint bounded below operator on  $X$ , with the eigenvalue  $\lambda_0$  as above, and  $W$  symmetric and  $\alpha$ -form-bounded w.r.t.  $H_0$ , and let  $H = H_0 + W$ . Let  $\alpha \in \mathbb{R}$  be such that  $H_0 + \alpha > 0$ . If  $\|W\|_{H_0, \alpha} \leq \frac{1}{2} \frac{\gamma_0}{\lambda_0 + \alpha}$ , then the self-adjoint operator  $H$  has exactly  $m$  eigenvalues (counting the multiplicities), denoted by  $\mu_i$ , in the interval  $I_0 = [\lambda_0 - \frac{1}{2}\gamma_0, \lambda_0 + \frac{1}{2}\gamma_0]$  which satisfy*

$$|\mu_i - \lambda_0| \leq (\lambda_0 + \alpha) \|W\|_{H_0, \alpha} \leq \frac{1}{2} \gamma_0. \quad (23)$$

Further, if  $\|W\|_{H_0, \alpha} \leq \frac{1}{4} \frac{\gamma_0}{\lambda_0}$ , then any normalized eigenfunction,  $\psi_i$ , of  $H$  for the eigenvalue  $\mu_i$  satisfies the estimates

$$\|H_{0, \alpha}^{1/2}(\psi_{0i} - \psi_i)\| \leq 4 \frac{\lambda_0}{\gamma_0} (\lambda_0 + \alpha)^{1/2} \|W\|_{H_0, \alpha}, \quad (24)$$

$$\|\psi_{0i} - \psi_i\| \leq 4 \frac{\lambda_0}{\gamma_0} \|W\|_{H_0, \alpha}, \quad (25)$$

where  $\lambda_0 = \lambda_0 + \alpha + \gamma_0$  and  $\psi_{0i}$  is an appropriate eigenfunction of  $H_0$  corresponding to the eigenvalue  $\lambda_0$ , namely  $\psi_{0i} := P_0 \psi_i$ .

**Remark 4.** *We note that similar estimates can be obtained for normalized eigenfunctions  $\tilde{\psi}_{0i} := P_0 \psi_i / \|P_0 \psi_i\|$  with an additional factor 2 using the estimate*

$$\|\tilde{\psi}_{0i} - \psi_i\| \leq |1 - \|\psi_{0i}\|| + \|\psi_{0i} - \psi_i\| \leq \left| \|\psi_i\| - \|\psi_{0i}\| \right| + \|\psi_{0i} - \psi_i\| \leq 2 \|\psi_{0i} - \psi_i\|.$$

We first develop the following preliminary results.

**Lemma 2.** *Let  $\alpha \in \mathbb{R}$  be such that  $H_0 + \alpha > 0$  and  $\lambda_0 := \lambda_0 + \gamma_0 + \alpha$ . Then, for all  $\lambda \in I_0^c := \{z \in \mathbb{C} : \operatorname{Re} z \in I_0\}$ , there holds  $|\mathbf{1}P_0^\perp - (\lambda + \alpha)H_{0, \alpha}^{-1}P_0^\perp| \geq \frac{\gamma_0}{2\lambda_0}$ .*

*Proof.* The eigenvalues of  $|\mathbf{1}P_0^\perp - (\lambda + \alpha)H_{0, \alpha}^{-1}P_0^\perp|$  on  $\operatorname{Ran} P_0^\perp$  are

$$\left| 1 - \frac{\lambda + \alpha}{\lambda_{0i} + \alpha} \right| = \left| \frac{\lambda_{0i} - \lambda}{\lambda_{0i} + \alpha} \right|,$$

where  $\lambda_{0i}$  denotes the eigenvalues of  $H_0$  and the index  $i$  runs over all eigenvalues except  $i$  such that  $\lambda_{0i} = \lambda_0$ . For  $\lambda \in I_0^c$ , we write  $\lambda = \lambda_r + i\lambda_i$ , with  $\lambda_r \in I_0$ ,  $\lambda_i \in \mathbb{R}$ . Since for any  $x \in \mathbb{R}$ ,  $|x - \lambda| \geq |x - \lambda_r|$  we have thus to study the function

$$f(x) = \left| \frac{x - \lambda}{x + \alpha} \right|, \quad x \in K_\alpha := [-\alpha, +\infty) \setminus (\lambda_0 - \gamma_0, \lambda_0 + \gamma_0),$$

for  $\lambda \in I_0$  in order to lower bound the eigenvalues. Since

$$f'(x) = \frac{x - \lambda}{|x - \lambda|} \cdot \frac{\alpha + \lambda}{(x + \alpha)^2},$$

if  $\alpha + \lambda \leq 0$ , there holds  $f'(x) < 0$  for  $x > -\alpha$  so that

$$\min_{x \in K_\alpha} f(x) \geq 1.$$

If  $\alpha + \lambda > 0$ , there holds

$$f'(x) < 0 \text{ for } x < \lambda, \quad f'(x) > 0 \text{ for } x > \lambda,$$

and thus, for  $\lambda \in I_0$ ,

$$\min_{x \in K_\alpha} f(x) = \min(f(\lambda_0 - \gamma_0), f(\lambda_0 + \gamma_0)) = \min\left(\frac{|\lambda_0 - \gamma_0 - \lambda|}{\lambda_0 - \gamma_0 + \alpha}, \frac{|\lambda_0 + \gamma_0 - \lambda|}{\lambda_0 + \gamma_0 + \alpha}\right) \geq \frac{1}{2} \frac{\gamma_0}{\lambda_0},$$

yielding the result.  $\square$

Denote  $H^\perp := P_0^\perp H P_0^\perp|_{\text{Ran } P_0^\perp}$  and  $R^\perp(\lambda) := P_0^\perp (H^\perp - \lambda)^{-1} P_0^\perp$ . We have

**Lemma 3.** *Let  $\alpha \in \mathbb{R}$  be such that  $H_0 + \alpha > 0$  and  $\lambda_\circ := \lambda_0 + \gamma_0 + \alpha$ . Let  $I_0^c := \{z \in \mathbb{C} : \text{Re } z \in I_0\}$  and assume  $\|W\|_{H_{0,\alpha}} \leq \frac{1}{4} \frac{\gamma_0}{\lambda_\circ}$ . Then, for  $\lambda \in I_0^c$ , the following statements hold*

- (a) *The operator  $H^\perp - \lambda$  is invertible on  $\text{Ran } P_0^\perp$ ;*
- (b) *The inverse  $R^\perp(\lambda) := P_0^\perp (H^\perp - \lambda)^{-1} P_0^\perp$  defines a bounded, analytic operator-family;*
- (c) *The expression*

$$U(\lambda) := -P_0 H R^\perp(\lambda) H P_0 \quad (26)$$

*defines a finite-rank, analytic operator-family and bounded as*

$$\|U(\lambda)\|_{H_{0,\alpha}} \leq 4 [\lambda_\circ / \gamma_0] \|P_0 W P_0^\perp\|_{H_{0,\alpha}}^2 \leq 4 [\lambda_\circ / \gamma_0] \|W\|_{H_{0,\alpha}}^2. \quad (27)$$

*Further,  $U(\lambda)$  is symmetric for any  $\lambda \in I_0$ .*

*Proof.* (a) Since  $H^\perp$  is self-adjoint, the operator  $H^\perp - \lambda$  is invertible for any  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . For  $\lambda \in I_0$ , we argue as follows. With the notation  $A^\perp := P_0^\perp A P_0^\perp|_{\text{Ran } P_0^\perp}$ , we write

$$H^\perp = H_0^\perp + W^\perp.$$

Now, we write

$$H^\perp - \lambda P_0^\perp = H_{0,\alpha}^{1/2} [\mathbf{1} P_0^\perp - (\lambda + \alpha) H_{0,\alpha}^{-1} + K_\lambda] H_{0,\alpha}^{1/2}, \quad (28)$$

with  $K_\lambda = H_{0,\alpha}^{-1/2} W^\perp H_{0,\alpha}^{-1/2}$ . Lemma 2 yields that  $|\mathbf{1} P_0^\perp - (\lambda + \alpha) H_{0,\alpha}^{-1} P_0^\perp| \geq \frac{\gamma_0}{2\lambda_\circ}$  and thus, the operator  $\mathbf{1} - (\lambda + \alpha) H_{0,\alpha}^{-1} + K_\lambda$  is invertible as soon as  $\|K_\lambda\| < \frac{\gamma_0}{2\lambda_\circ}$ , which is in particular the case if  $\|W\|_{H_{0,\alpha}} \leq \frac{1}{4} \frac{\gamma_0}{\lambda_\circ}$ . Then, we also have  $|\mathbf{1} - (\lambda + \alpha) H_{0,\alpha}^{-1} + K_\lambda| \geq \frac{1}{4} \frac{\gamma_0}{\lambda_\circ}$ . Hence the operator  $H^\perp - \lambda$  is a product of three invertible operators and therefore is invertible itself on  $\text{Ran } P_0^\perp$ .

For (b), since  $H^\perp - \lambda$  is invertible on  $\text{Ran } P_0^\perp$ , the interval  $I_0$  is contained in the resolvent set,  $\rho(H^\perp|_{\text{Ran } P_0^\perp})$ , of  $H^\perp|_{\text{Ran } P_0^\perp}$  and therefore, since  $H^\perp|_{\text{Ran } P_0^\perp}$  is self-adjoint,  $I_0^c \subset \rho(H^\perp|_{\text{Ran } P_0^\perp})$ . Since

$$R^\perp(\lambda) := P_0^\perp (H^\perp - \lambda)^{-1} P_0^\perp \quad (29)$$

is the resolvent of the operator  $H^\perp - \lambda$  restricted to  $\text{Ran } P_0^\perp$ , it is analytic on its resolvent set and in particular on  $I_0^c$ .

To prove statement (c), we note that the operators  $R^\perp(\lambda), H P_0$  and  $P_0 H = (H P_0)^*$  are bounded and  $R^\perp(\lambda)$  is symmetric for  $\lambda \in I_0$ . Hence so is  $U(\lambda)$ . The analyticity of  $U(\lambda)$  follows from the analyticity of  $R^\perp(\lambda)$ . It is clear that  $U(\lambda)$  is of finite rank due to its definition.

Finally, to prove estimate (27), we first show that the operator  $R^\perp(\lambda) := P_0^\perp (H^\perp - \lambda)^{-1} P_0^\perp$ ,  $\lambda \in I_0^c$ , satisfies

$$\|H_{0,\alpha}^{1/2} R^\perp(\lambda) H_{0,\alpha}^{1/2}\| \leq 4 \frac{\lambda_\circ}{\gamma_0}. \quad (30)$$

To this end, we invert (28) on  $\text{Ran } P_0^\perp$  and use that  $|\mathbf{1} - (\lambda + \alpha) H_{0,\alpha}^{-1} + K_\lambda| \geq \frac{1}{4} \frac{\gamma_0}{\lambda_\circ}$  to obtain (30) for  $\lambda \in I_0^c$ .

Finally, we prove inequality (27). Since  $P_0 H_0 = H_0 P_0$  and  $P_0 P_0^\perp = 0$ , we have

$$P_0 H P_0^\perp = P_0 W P_0^\perp, \quad P_0^\perp H P_0 = P_0^\perp W P_0.$$

These relations and definition (26) yield

$$U(\lambda) = -P_0 W R^\perp(\lambda) W P_0. \quad (31)$$

Combining (30) and (31), we obtain (27).  $\square$

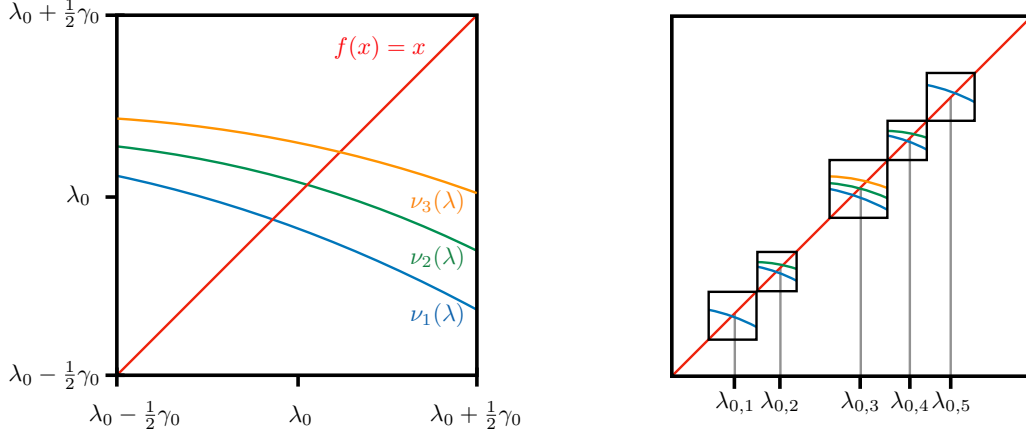


FIGURE 1. (Left) Schematic illustration of the eigenvalues  $\nu_i(\lambda)$  of  $H(\lambda)$  in the neighborhood of  $\lambda_0$  for the case of  $m = 3$ . (Right) Illustration of the spectrum of  $H_0$  consisting of five eigenvalues  $\lambda_{0,1} \dots, \lambda_{0,5}$  of multiplicity  $m_{0,1} = 1, m_{0,2} = 2, m_{0,3} = 4, m_{0,4} = 2, m_{0,5} = 1$  and the corresponding situation when zooming in close to  $\lambda_0 = \lambda_{0,i}$ .

**Remark 5.** *In addition, we have the estimate*

$$\|U(\lambda)\| \leq 4 \frac{\lambda_0^2}{\gamma_0} \|P_0 W P_0^\perp\|_{H_0, \alpha}^2. \quad (32)$$

Indeed, since  $H_0 P_0 = \lambda_0 P_0$ , we have  $\|(H_0 + \alpha)^{1/2} P_0\|^2 = \lambda_0 + \alpha$ , which implies the estimate

$$\|P_0 A P_0\| = (\lambda_0 + \alpha) \|P_0 A P_0\|_{H_0, \alpha}, \quad (33)$$

which, together with estimate (27), yields (32).

Hence, under the conditions of Lemma 3 and for  $\lambda \in I_0$  the following Hamiltonian is well defined

$$H(\lambda) := P_0 H P_0 + U(\lambda). \quad (34)$$

Note that  $P_0 H P_0 = \lambda_0 P_0$ . Lemma 3 above implies

**Corollary 7.** *The operator family  $H(\lambda)$  is (i) self-adjoint for  $\lambda \in I_0$  and (ii) complex analytic in  $\lambda \in I_0^c$ .*

In what follows, we label the eigenvalue families  $\nu_i(\lambda)$ ,  $i = 1, \dots, m$ , of  $H(\lambda)$  in the order of their increase and so that

$$\nu_1(\lambda) \leq \dots \leq \nu_m(\lambda). \quad (35)$$

Note that the eigenvalue branches  $\nu_i(\lambda)$  can also be of higher multiplicity. On a subinterval  $I_i \subset I_0$ , we say that the branch  $\nu_i(\lambda)$  is isolated on  $I_i$  if each other branch  $\nu_j(\lambda)$ , with  $\lambda \in I_i$ , either i) coincides with  $\nu_i(\lambda)$  or ii) satisfies

$$\min_{\lambda \in I_i} |\nu_i(\lambda) - \nu_j(\lambda)| \geq \gamma_i > 0. \quad (36)$$

Further, we have the following result.

**Proposition 5.** *Let  $\alpha \in \mathbb{R}$  be such that  $H_0 + \alpha > 0$  and let  $I_i \subset I_0$  be such that the branch  $\nu_i(\lambda)$  is isolated on  $I_i$ . For  $\lambda \in I_i$ , (i) the eigenvalues  $\nu_i(\lambda)$  of  $H(\lambda)$  are continuously differentiable; (ii) the derivative  $\nu'_i(\lambda)$  is non-positive; (iii) the solutions to the equations  $\nu_i(\lambda) = \lambda$  are unique if  $\lambda \in I_i$ ; (iv) if  $\|W\|_{H_0, \alpha} \leq \frac{1}{4} \frac{\gamma_0}{\lambda_0}$ , the derivatives  $\nu'_i(\lambda)$ ,  $\lambda \in I'_0 := [\lambda_0 - \frac{1}{4}\gamma_0, \lambda_0 + \frac{1}{4}\gamma_0] \cap I_i$ , are bounded as*

$$|\nu'_i(\lambda)| \leq \frac{8}{\pi} \frac{(\lambda_0 + \alpha)\lambda_0}{\gamma_0^2} \|P_0 W P_0^\perp\|_{H_0, \alpha}^2.$$

where  $\lambda_0 := \lambda_0 + \gamma_0 + \alpha$ .

*Proof.* Proof of (i) of a simple eigenvalue  $\lambda_0$ , i.e.,  $m = 1$ . In such a case,  $P_0$  is a rank-one projector on the space spanned by the eigenvector  $\varphi_0$  of  $H_0$  corresponding to the eigenvalue  $\lambda_0$  and therefore Eq. (34) implies that  $H(\lambda) = \nu_1(\lambda)P_0$ , with

$$\nu_1(\lambda) := \langle \varphi_0, H(\lambda)\varphi_0 \rangle. \quad (37)$$

This and Corollary 7 show that the eigenvalue  $\nu_1(\lambda)$  is analytic.

We now prove (i) in the general case. First, we claim the following well-known formula

$$\nu'_i(\lambda) = \langle \chi_i(\lambda), U'(\lambda)\chi_i(\lambda) \rangle, \quad (38)$$

for  $\lambda \in I_i$ , where  $\chi_i(\lambda)$  are well-chosen normalized eigenfunction of  $H(\lambda)$  corresponding to the eigenvalue  $\nu_i(\lambda)$ , namely that they are differentiable in  $\lambda$ . To this end, we observe that for each  $\mu \in I_i$ , we can find a local neighborhood  $I_\mu \subset I_i$  of  $\mu$  such that

$$\bigcup_{j \neq i} \{\nu_j(\lambda) \mid \lambda \in I_\mu\} \cap \{\nu_i(\lambda) \mid \lambda \in I_\mu\} = \emptyset, \quad (39)$$

due to the isolated branch property, i.e.,  $\gamma_i > 0$  in (36). Second, since  $H(\lambda)$  is self-adjoint for  $\lambda \in I_0$  and analytic (say, in the resolvent sense) in  $\lambda \in I_0^c$ , the Riesz projection, corresponding to the eigenvalue  $\nu_i(\lambda)$ :

$$P_i(\lambda) := \frac{1}{2\pi i} \oint_{\Gamma_i(\mu)} (H(\lambda) - z)^{-1} dz, \quad (40)$$

where  $\Gamma_i(\mu)$  is a closed curve in the resolvent set of  $H(\lambda)$  surrounding the eigenvalue branch  $\{\nu_i(\lambda) : \lambda \in I_\mu\}$ , is also self-adjoint and analytic in  $\lambda \in I_\mu^c$  and therefore in  $\lambda \in I_0^c$  (see [22, 15]), condition (39) guarantees that we can choose such a closed curve which contains no other points of  $\sigma(H(\mu))$  on  $I_\mu$ , and that, combining all neighborhoods of  $\mu$  for  $\mu \in I_i$ , there holds that  $P_i(\lambda)$  is analytic in  $I_i$ . From [22, Theorem XII.12], there exists an analytic family of unitary operators  $V_i(\lambda)$  such that  $P_i(\lambda) = V_i(\lambda)P_i(\lambda_0)[V_i(\lambda)]^{-1}$ ,  $\lambda_0$  being possibly replaced by some arbitrary  $\mu \in I_i$  if  $\lambda_0$  does not belong to  $I_i$ . We then define

$$\chi_i(\lambda) = V_i(\lambda)\psi_{0i},$$

where  $\psi_{0i}$  is an eigenvector of  $H_0$  corresponding to the eigenvalue  $\lambda_0$ . Since  $V_i(\lambda)$  is analytic,  $\chi_i(\lambda)$  is also analytic in  $I_i$ , so in particular differentiable, and one can easily check that  $\chi_i(\lambda)$  is of norm 1 and that  $P_i(\lambda)\chi_i(\lambda) = \chi_i(\lambda)$ , which guarantees that  $\chi_i(\lambda)$  is a normalized eigenfunction of  $H(\lambda)$ . Now, we use that

$$\begin{aligned} \langle \chi'_i(\lambda), H(\lambda)\chi_i(\lambda) \rangle + \langle \chi_i(\lambda), H(\lambda)\chi'_i(\lambda) \rangle &= \nu_i(\lambda)(\langle \chi'_i(\lambda), \chi_i(\lambda) \rangle + \langle \chi_i(\lambda), \chi'_i(\lambda) \rangle) \\ &= \langle \chi_i(\lambda), \chi_i(\lambda) \rangle' = 0 \end{aligned}$$

to obtain  $\nu'_i(\lambda) = \langle \chi_i(\lambda), H'(\lambda)\chi_i(\lambda) \rangle$ , which gives (38). The differentiability of  $\chi_i(\lambda)$  and the analyticity of  $H(\lambda)$  then implies the differentiability of  $\nu_i$  in each neighborhood of  $\lambda$ .

In order to prove (ii), note that  $U'(\lambda) \leq 0$ , as follows by the explicit formula

$$U'(\lambda) := -PW P_0^\perp (H^\perp - \lambda)^{-2} P_0^\perp W P \leq 0. \quad (41)$$

Hence,  $\nu'_i(\lambda) < 0$  by (38). The monotonicity of  $\nu_i(\lambda)$  also implies the well-posedness of the equations  $\nu_i(\lambda) = \lambda$  under the condition that  $\lambda \in I_i$ , thus statement (iii).

We now aim to prove (iv). Starting from (38), we estimate  $\nu'_i(\lambda)$  with

$$|\nu'_i(\lambda)| \leq \|(H_0 + \alpha)^{1/2} P_0\|^2 \|U'(\lambda)\|_{H_0, \alpha}. \quad (42)$$

The first factor on the right hand side is exactly known as

$$\|(H_0 + \alpha)^{1/2} P_0\|^2 = (\lambda_0 + \alpha). \quad (43)$$

To investigate the second factor on the r.h.s. of (42), we use the analyticity  $U(\lambda)$  and the estimate (27). Indeed, by the Cauchy integral formula, we have

$$\|U'(\lambda)\|_{H_0, \alpha} \leq \frac{1}{2\pi R} \sup_{\substack{\mu \in \mathbb{C}, \\ |\mu - \lambda| = R}} \|U(\mu)\|_{H_0, \alpha},$$

where  $R$  is such that  $\{\mu \in \mathbb{C} : |\mu - \lambda| \leq R\} \subset I_0^c$ . Taking  $R = \frac{1}{4}\gamma_0$  gives, under the conditions of Lemma 3, the estimate

$$\|U'(\lambda)\|_{H_0, \alpha} \leq \frac{8}{\pi} \frac{\lambda_0}{\gamma_0^2} \|P_0 W P_0^\perp\|_{H_0, \alpha}^2. \quad (44)$$

Combining equations (42), (43) and (44) shows (iv).  $\square$

**Corollary 8.** *Let  $\alpha \in \mathbb{R}$  be such that  $H_0 + \alpha > 0$  and let  $I_i \subset I_0$  be such that the branch  $\nu_i(\lambda)$  is isolated on  $I_i$ . Under the condition that*

$$\frac{8}{\pi} \frac{(\lambda_0 + \alpha)\lambda_0}{\gamma_0^2} \|P_0 W P_0^\perp\|_{H_0, \alpha}^2 < 1,$$

*and that the unique solution  $\lambda$  of  $\nu_i(\lambda) = \lambda$  satisfies  $\lambda \in I_0'$ , the fixed-point iteration  $\lambda^{(k+1)} = \nu_i(\lambda^{(k)})$  converges to  $\lambda$  for initial values in  $I_0'$ .*

Now, we proceed directly to the proof of Theorem 6.

*Proof of Theorem 6.* For the estimate on the eigenvalues, we first remark that applying Proposition 4 to the  $m$  eigenvalues corresponding to  $\lambda_0$  for  $H_0$  provides the first inequality in (23). The second one follows immediately from the condition  $\|W\|_{H_0, \alpha} \leq \frac{1}{2} \frac{\gamma_0}{\lambda_0 + \alpha}$  and thus  $\mu_i \in I_0$ .

The fact that the operator  $H$  has exactly  $m$  eigenvalues (counting the multiplicities) in  $I_0$  follows from Corollary 4 and Proposition 5(iii) and the fact that  $H(\lambda)$  is a  $m \times m$  symmetric matrix.

For the estimates on the eigenfunctions, recall from Theorem 3 that  $Q_0(\mu_i)\psi_{0i} = \psi_i$ , where  $\mu_i = \nu_i(H)$  and the operator  $Q_0(\lambda)$  is given by

$$Q_0(\lambda) := \mathbf{1} - R^\perp(\lambda) P_0^\perp W P_0, \quad (45)$$

with  $R^\perp(\lambda)$  defined in (29). This yields

$$\psi_{0i} - \psi_i = \psi_{0i} - Q_0(\mu_i)\psi_{0i} = R^\perp(\mu_i) P_0^\perp W P_0 \psi_{0i}. \quad (46)$$

Then, for  $\gamma \in \{0, 1/2\}$ ,

$$\begin{aligned} \|H_{0, \alpha}^\gamma(\psi_{0i} - \psi_i)\| &\leq \|H_{0, \alpha}^\gamma R^\perp(\mu_i) P_0^\perp W P_0\| \\ &\leq \|H_{0, \alpha}^\gamma H_{0, \alpha}^{-1/2} P_0^\perp\| \|H_{0, \alpha}^{1/2} R^\perp(\mu_i) H_{0, \alpha}^{1/2}\| \|P_0^\perp W P_0\|_{H_0, \alpha} \|P_0 H_{0, \alpha}^{1/2}\|. \end{aligned}$$

In the previous expression, we can use (30) to estimate  $\|H_{0, \alpha}^{1/2} R^\perp(\mu_i) H_{0, \alpha}^{1/2}\|$ . Then, we note that

$$\|P_0^\perp W P_0\|_{H_0, \alpha} \leq \|W\|_{H_0, \alpha},$$

as well as

$$\|P_0 H_{0,\alpha}^{1/2}\| = (\lambda_0 + \alpha)^{1/2}.$$

Finally, in the case  $\gamma = 0$

$$\|H_{0,\alpha}^{-1/2} P_0^\perp\| \leq (\lambda_0 + \alpha)^{-1/2},$$

and for  $\gamma = 1/2$ ,

$$\|H_{0,\alpha}^\gamma H_{0,\alpha}^{-1/2} P_0^\perp\| \leq 1.$$

Combining the four bounds leads to (24) and (25).  $\square$

**Remark 6.** Note that by Theorem 3, any solution  $\mu_i$  to the equation  $\nu_i(\mu_i) = \mu_i$ , for  $i = 1, \dots, m$  and where the  $\mu_i$  are in ascending order, is an eigenvalue of  $H$ . Under the condition

$$\|W\|_{H_0,\alpha} \leq \frac{1}{2} \frac{\gamma_0}{\lambda_0 + \alpha}, \quad (47)$$

Theorem 6 guarantees that the eigenvalues  $\lambda_i$  satisfy  $|\mu_i - \lambda_0| \leq \frac{\gamma_0}{2}$  and thus  $\mu_i \in I_0$ .

On the contrary, the eigenvalues  $\mu_i$  are the only  $m$  eigenvalues of  $H$  belonging to  $I_0$  if a similar condition as (47), but for the next larger eigenvalue of  $H_0$  than  $\lambda_0$  holds whereas such a condition is automatically satisfied by (47) for the preceding eigenvalue of  $H_0$ .

#### 4. PRELIMINARY RESULTS

We now derive a few preliminary results that will be useful for proving Theorem 1. For the following proofs, we define the following quantities:  $h_\lambda := -\Delta - \lambda$ ,  $V_M^\perp = P_M^\perp V P_M^\perp$ ,  $V_M^N = P_M^N V P_M^N$ .

**Lemma 4.** For  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda < \frac{1}{2}\rho_M$  and  $\rho_M \geq 1$ , the following bounds hold

$$\|h_\lambda^{-1/2} V_M^N h_\lambda^{-1/2}\| \leq \|h_\lambda^{-1/2} V_M^\perp h_\lambda^{-1/2}\| \leq 4\rho_M^{-r} \|V\|_r. \quad (48)$$

*Proof.* First, we note that

$$\|h_\lambda^{-1/2} V_M^N h_\lambda^{-1/2}\| = \|P_N h_\lambda^{-1/2} V_M^\perp h_\lambda^{-1/2} P_N\| \leq \|h_\lambda^{-1/2} V_M^\perp h_\lambda^{-1/2}\|.$$

Then, we estimate for any  $s \geq 0$ , using the assumption  $\operatorname{Re} \lambda \leq \frac{1}{2}\rho_M$

$$\|h_{-1}^s h_\lambda^{-s} P_M^\perp\| \leq \frac{|\rho_M + 1|^s}{|\rho_M - \lambda|^s} \leq 2^s (1 + \rho_M^{-1})^s \leq 4^s, \quad (49)$$

This implies in particular that  $\|h_\lambda^{-1/2} V_M^\perp h_\lambda^{-1/2}\| \leq 4 \|V_M^\perp\|_0$ . The result follows noting that  $\|V_M^\perp\|_0 \leq \rho_M^{-r} \|V\|_r$ .  $\square$

**Lemma 5.** For  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda \leq \min(\frac{1}{2}\rho_M, \kappa_M)$  and  $\rho_M \geq 1$ , the following bound holds

$$\|U_\sigma(\lambda)\|_r \leq 4\rho_M^{-r} \|V\|_r^2 \sum_{k=0}^K [4\rho_M^{-r} \|V\|_r]^k. \quad (50)$$

Moreover, if  $4\rho_M^{-r} \|V\|_r < 1$ ,

$$\|U_\sigma(\lambda)\|_r \leq \rho_M^{-r} \frac{4 \|V\|_r^2}{1 - 4\rho_M^{-r} \|V\|_r}, \quad (51)$$

and in particular

$$\|U_M(\lambda)\|_r \leq \rho_M^{-r} \frac{4 \|V\|_r^2}{1 - 4\rho_M^{-r} \|V\|_r}. \quad (52)$$

*Proof.* By definition (8), we can write  $U_\sigma(\lambda)$  which is well-defined for  $\lambda < \kappa_M$  as

$$U_\sigma(\lambda) = - \sum_{k=0}^K \mathbf{P}_M V \mathbf{P}_M^N h_\lambda^{-1/2} \left[ -h_\lambda^{-1/2} V_M^N h_\lambda^{-1/2} \right]^k h_\lambda^{-1/2} \mathbf{P}_M^N V \mathbf{P}_M. \quad (53)$$

Using estimate (48), there holds

$$\|U_\sigma(\lambda)\|_r \leq \sum_{k=0}^K [4\rho_M^{-r} \|V\|_r]^k \|h_{-1}^{-1/2+r/2} \mathbf{P}_M V \mathbf{P}_M^N h_\lambda^{-1/2} \| \|h_\lambda^{-1/2} \mathbf{P}_M^N V \mathbf{P}_M h_{-1}^{-1/2+r/2}\|.$$

and by (49),  $\|h_\lambda^{-1/2} \mathbf{P}_M^N V \mathbf{P}_M h_{-1}^{-1/2+r/2}\| \leq 2\rho_M^{-r/2} \|V\|_r$ , so that we obtain (50). The bound (51) is easily obtained from (50) and taking  $K, N = \infty$  in (51), we arrive at (52).  $\square$

**Lemma 6.** *For  $\lambda < \frac{1}{2}\kappa_M$ ,  $\rho_M \geq 1$  and if  $4\rho_M^{-r} \|V\|_r < 1$ , the following bounds hold*

$$\|U'_M(\lambda)\|_r \leq \frac{1}{\pi(\kappa_M - 2\lambda)} \rho_M^{-r} \frac{4 \|V\|_r^2}{1 - 4\rho_M^{-r} \|V\|_r}. \quad (54)$$

*Proof.* Since from Proposition 2(iv),  $U_M(\lambda)$  is complex analytic in  $\lambda$  for  $\text{Re } \lambda < \kappa_M$ , by the Cauchy integral formula, we have

$$\|U'_M(\lambda)\|_r \leq \frac{1}{2\pi R_M(\lambda)} \sup_{\substack{\mu \in \mathbb{C}, \\ |\mu - \lambda| = R_M(\lambda)}} \|U_M(\mu)\|_r,$$

with  $R_M(\lambda) = \frac{1}{2}\kappa_M - \lambda > 0$ . Using (52) and noting that  $\frac{1}{2}\kappa_M \leq \frac{1}{2}\rho_M$ , we obtain (54).  $\square$

**Lemma 7.** *For  $\lambda < \min(\kappa_M, \frac{1}{2}\rho_M)$ ,  $\rho_N \geq \rho_M > 1$  and if  $4\rho_M^{-r} \|V\|_r < 1$  and  $4\rho_N^{-r} \|V\|_r + \frac{16\rho_M^{-2r} \|V\|_r^2}{1 - 4\rho_M^{-r} \|V\|_r} < 1$ , the following bound holds*

$$\begin{aligned} & \|U_M(\lambda) - U_\sigma(\lambda)\|_r \\ & \leq \frac{4\rho_N^{-r} \|V\|_r^2}{1 - 4\rho_N^{-r} \|V\|_r - \frac{16\rho_M^{-2r} \|V\|_r^2}{1 - 4\rho_M^{-r} \|V\|_r}} \left[ 1 + \frac{4\rho_M^{-r} \|V\|_r}{1 - 4\rho_M^{-r} \|V\|_r} \right]^2 + \frac{4\rho_M^{-r} \|V\|_r^2}{1 - 4\rho_M^{-r} \|V\|_r} [4\rho_M^{-r} \|V\|_r]^{K+1}. \end{aligned} \quad (55)$$

*Proof.* We first write  $U_M(\lambda) - U_\sigma(\lambda)$  as

$$U_M(\lambda) - U_\sigma(\lambda) = (U_M(\lambda) - U_{MN}(\lambda)) + (U_{MN}(\lambda) - U_\sigma(\lambda)), \quad (56)$$

where, using the notation  $\mathcal{H}_M^N = \mathbf{P}_M^N \mathcal{H} \mathbf{P}_M^N$

$$\begin{aligned} U_{MN}(\lambda) & := -\mathbf{P}_M V \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N V \mathbf{P}_M \\ & = -\mathbf{P}_M V \mathbf{P}_M^N (-\Delta + V_M^N - \lambda)^{-1} \mathbf{P}_M^N V \mathbf{P}_M. \end{aligned}$$

Since  $U_M(\lambda) = -\mathbf{P}_M V \mathbf{P}_M^\perp (\mathcal{H}_M^\perp - \lambda)^{-1} \mathbf{P}_M^\perp V \mathbf{P}_M$ , with  $\mathcal{H}_M^\perp = \mathbf{P}_M^\perp \mathcal{H} \mathbf{P}_M^\perp$ , the first term  $U_M(\lambda) - U_{MN}(\lambda)$  is estimated as follows.

Denoting by  $\mathcal{H}_N^\perp = \mathbf{P}_N^\perp \mathcal{H} \mathbf{P}_N^\perp$ , and the Schur complement

$$A = \mathbf{P}_N^\perp \left( (\mathcal{H}_N^\perp - \lambda \mathbf{P}_N^\perp) - \mathbf{P}_N^\perp V \mathbf{P}_M^N (\mathcal{H}_M^\perp - \lambda)^{-1} \mathbf{P}_M^N V \mathbf{P}_N^\perp \right)^{-1} \mathbf{P}_N^\perp,$$

there holds, using a block matrix inversion

$$\begin{aligned} \mathbf{P}_M^\perp (\mathcal{H}_M^\perp - \lambda)^{-1} \mathbf{P}_M^\perp & = \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N + \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N V A V \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N \\ & \quad - \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N V A - A V \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N + A. \end{aligned} \quad (57)$$



Therefore,  $U_M(\lambda) - U_{MN}(\lambda)$  can be decomposed into four terms as

$$\begin{aligned} U_M(\lambda) - U_{MN}(\lambda) &= -\mathbf{P}_M V \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N V A V \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N V \mathbf{P}_M, \\ &\quad + \mathbf{P}_M V \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N V A V \mathbf{P}_M, \\ &\quad + \mathbf{P}_M V A V \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N V \mathbf{P}_M, \\ &\quad - \mathbf{P}_M V A V \mathbf{P}_M. \end{aligned}$$

Then, the  $r$ -norm can be estimated as

$$\begin{aligned} \|U_M(\lambda) - U_{MN}(\lambda)\|_r &\leq \|V\|_r^2 \left[ \|h_{-1}^{1/2-r/2} \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N V A V \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N h_{-1}^{1/2-r/2}\| \right. \\ &\quad + \|h_{-1}^{1/2-r/2} \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N V A h_{-1}^{1/2-r/2}\| \\ &\quad + \|h_{-1}^{1/2-r/2} A V \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N h_{-1}^{1/2-r/2}\| \\ &\quad \left. + \|h_{-1}^{1/2-r/2} A h_{-1}^{1/2-r/2}\| \right]. \end{aligned}$$

Introducing appropriate  $h_{-1}^{1/2-r/2}$  and  $h_{-1}^{-1/2+r/2}$  terms, we obtain

$$\begin{aligned} \|U_M(\lambda) - U_{MN}(\lambda)\|_r &\leq \|V\|_r^2 \left[ \|h_{-1}^{1/2-r/2} \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N h_{-1}^{1/2-r/2}\|^2 \|V\|_r^2 \|h_{-1}^{1/2-r/2} A h_{-1}^{1/2-r/2}\| \right. \\ &\quad + 2 \|h_{-1}^{1/2-r/2} \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N h_{-1}^{1/2-r/2}\| \|V\|_r \|h_{-1}^{1/2-r/2} A h_{-1}^{1/2-r/2}\| \\ &\quad \left. + \|h_{-1}^{1/2-r/2} A h_{-1}^{1/2-r/2}\| \right] \\ &\leq \|h_{-1}^{1/2-r/2} A h_{-1}^{1/2-r/2}\| \|V\|_r^2 \\ &\quad \times \left[ 1 + \|h_{-1}^{1/2-r/2} \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N h_{-1}^{1/2-r/2}\| \|V\|_r \right]^2. \end{aligned}$$

We are therefore left with the estimation of  $\|h_{-1}^{1/2-r/2} \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N h_{-1}^{1/2-r/2}\|$  and  $\|h_{-1}^{1/2-r/2} A h_{-1}^{1/2-r/2}\|$ . First, noting from (48) that

$$\|h_{-1}^{1/2} \mathbf{P}_M^N (\mathcal{H}_M^{\perp} - \lambda)^{-1} \mathbf{P}_M^N h_{-1}^{1/2}\| \leq \|(I + h_{-1}^{-1/2} V_M^{\perp} h_{-1}^{-1/2})^{-1}\| \leq \frac{1}{1 - 4\rho_M^{-r} \|V\|_r}, \quad (58)$$

and using that

$$\|h_{-1}^{1/2-r/2} h_{-1}^{-1/2} \mathbf{P}_M^N\| \leq 2\rho_M^{-r/2}, \quad (59)$$

we obtain

$$\|h_{-1}^{1/2-r/2} \mathbf{P}_M^N (\mathcal{H}_M^N - \lambda)^{-1} \mathbf{P}_M^N h_{-1}^{1/2-r/2}\| \leq \frac{4\rho_M^{-r}}{1 - 4\rho_M^{-r} \|V\|_r}. \quad (60)$$

Second,

$$\|h_{-1}^{1/2-r/2} A h_{-1}^{1/2-r/2}\| = \|h_{-1}^{1/2-r/2} \mathbf{P}_N^{\perp} \left[ (\mathcal{H}_N^{\perp} - \lambda \mathbf{P}_N^{\perp}) - \mathbf{P}_N^{\perp} V \mathbf{P}_M^N (\mathcal{H}_M^{\perp} - \lambda)^{-1} \mathbf{P}_M^N V \mathbf{P}_N^{\perp} \right]^{-1} \mathbf{P}_N^{\perp} h_{-1}^{1/2-r/2}\|.$$

Noting that

$$\|h_{-1}^{1/2-r/2} h_{-1}^{-1/2} \mathbf{P}_N^{\perp}\| \leq 2\rho_N^{-r/2},$$

there holds

$$\|h_{-1}^{1/2-r/2} A h_{-1}^{1/2-r/2}\| \leq 4\rho_N^{-r} \|h_{-1}^{1/2} \mathbf{P}_N^{\perp} \left[ (\mathcal{H}_N^{\perp} - \lambda \mathbf{P}_N^{\perp}) - \mathbf{P}_N^{\perp} V \mathbf{P}_M^N (\mathcal{H}_M^{\perp} - \lambda)^{-1} \mathbf{P}_M^N V \mathbf{P}_N^{\perp} \right]^{-1} \mathbf{P}_N^{\perp} h_{-1}^{1/2}\|.$$

Factorizing  $h_{-1}$ , we deduce

$$\|h_{-1}^{1/2-r/2} A h_{-1}^{1/2-r/2}\| \leq 4\rho_N^{-r} \left\| \left[ I + h_{-1}^{-1/2} V_N^{\perp} h_{-1}^{-1/2} - h_{-1}^{-1/2} \mathbf{P}_N^{\perp} V \mathbf{P}_M^N (\mathcal{H}_M^{\perp} - \lambda)^{-1} \mathbf{P}_M^N V \mathbf{P}_N^{\perp} h_{-1}^{-1/2} \right]^{-1} \right\|.$$

From (48) with  $N$  in place of  $M$ , we obtain

$$\|h_\lambda^{-1/2} V_N^\perp h_\lambda^{-1/2}\| \leq 4\rho_N^{-r} \|V\|_r. \quad (61)$$

Moreover,

$$\|h_\lambda^{-1/2} P_N^\perp V P_M^N (\mathcal{H}_M^\perp - \lambda)^{-1} P_M^N V P_N^\perp h_\lambda^{-1/2}\| \leq \|h_\lambda^{-1/2} P_N^\perp V P_M^N h_\lambda^{-1/2}\|^2 \|h_\lambda^{1/2} P_M^N (\mathcal{H}_M^\perp - \lambda)^{-1} P_M^N h_\lambda^{1/2}\|,$$

which, from (48) and (58), leads to

$$\|h_\lambda^{-1/2} P_N^\perp V P_M^N (\mathcal{H}_M^\perp - \lambda)^{-1} P_M^N V P_N^\perp h_\lambda^{-1/2}\| \leq \frac{16\rho_M^{-2r} \|V\|_r^2}{1 - 4\rho_M^{-r} \|V\|_r}.$$

Combining this last line with (61), we obtain the bound

$$\|h_{-1}^{1/2-r/2} A h_{-1}^{1/2-r/2}\| \leq 4\rho_N^{-r} \frac{1}{1 - 4\rho_N^{-r} \|V\|_r - \frac{16\rho_M^{-2r} \|V\|_r^2}{1 - 4\rho_M^{-r} \|V\|_r}}.$$

This leads to the following bound for the difference  $U_M(\lambda) - U_{MN}(\lambda)$

$$\|U_M(\lambda) - U_{MN}(\lambda)\|_r \leq \rho_N^{-r} \frac{4\|V\|_r^2}{1 - 4\rho_N^{-r} \|V\|_r - \frac{16\rho_M^{-2r} \|V\|_r^2}{1 - 4\rho_M^{-r} \|V\|_r}} \left[ 1 + \frac{4\rho_M^{-r} \|V\|_r}{1 - 4\rho_M^{-r} \|V\|_r} \right]^2. \quad (62)$$

For the second term on the right handside of (56), we write  $U_{MN}(\lambda) - U_\sigma(\lambda)$  as

$$U_{MN}(\lambda) - U_\sigma(\lambda) = - \sum_{k=K+1}^{\infty} P_M V P_M^N h_\lambda^{-1/2} \left[ -h_\lambda^{-1/2} V_M^N h_\lambda^{-1/2} \right]^k h_\lambda^{-1/2} P_M^N V P_M. \quad (63)$$

Using (48) and (49), we obtain

$$\begin{aligned} \|U_{MN}(\lambda) - U_\sigma(\lambda)\|_r &\leq \|V\|_r^2 \|h_{-1}^{1/2-r/2} h_\lambda^{-1/2} P_M^\perp\|^2 \sum_{k=K+1}^{\infty} \|h_\lambda^{-1/2} V_M^N h_\lambda^{-1/2}\|^k \\ &\leq 4\|V\|_r^2 \rho_M^{-r} \sum_{k=K+1}^{\infty} [4\rho_M^{-r} \|V\|_r]^k, \end{aligned}$$

from which we deduce that

$$\|U_{MN}(\lambda) - U_\sigma(\lambda)\|_r \leq \frac{4\|V\|_r^2}{1 - 4\rho_M^{-r} \|V\|_r} \rho_M^{-r} [4\rho_M^{-r} \|V\|_r]^{K+1}. \quad (64)$$

Combining (62) and (64), we obtain (55).  $\square$

**Lemma 8.** For  $\lambda < \mu < \min(\kappa_M, \frac{1}{2}\rho_M)$ ,  $\rho_M \geq 1$  and if  $4\rho_M^{-r} \|V\|_r < 1$ , there holds

$$\|U_M(\mu) - U_M(\lambda)\|_r \leq |\mu - \lambda| \frac{\rho_M^{-r}}{\pi(\kappa_M - 2\lambda)} \frac{4\|V\|_r^2}{1 - 4\rho_M^{-r} \|V\|_r} \quad (65)$$

*Proof of Lemma 8.* Without loss of generality, we assume that  $\lambda < \mu$ . Writing

$$U_M(\mu) - U_M(\lambda) = \int_\lambda^\mu U_M'(s) ds,$$

yields

$$\|U_M(\mu) - U_M(\lambda)\|_r \leq |\mu - \lambda| \max_{s \in [\lambda, \mu]} \|U_M'(s)\|_r.$$

We conclude by applying (54) of Lemma 6.  $\square$

## 5. PROOF OF THE MAIN RESULTS

The goal of this section is to provide the proof for Theorem 1.

**5.1. Proof of Theorem 1.** We first prove the following technical lemmas which will be useful later.

**Lemma 9.** *For  $H, W$  such that  $H = -\Delta + W$ , there holds for  $\alpha \geq (2\|W\|_r)^{1/r}$  such that  $H + \alpha > 0$*

$$\|A\|_{H,\alpha} \leq 2\|A\|_{-\Delta,\alpha} \leq 2\alpha^{-r}\|A\|_r.$$

*Proof.* First, denoting again  $h_{-\alpha} := -\Delta + \alpha$ , note that

$$\|A\|_{H,\alpha} \leq \|A\|_{-\Delta,\alpha} \|h_{-\alpha}^{1/2}(H + \alpha)^{-1/2}\|^2.$$

Let  $v$  be arbitrary and define  $u = (H + \alpha)^{-1/2}v$ . Note that

$$\|h_{-\alpha}^{1/2}u\|^2 = \langle u, h_{-\alpha}u \rangle = \langle u, (H + \alpha - W)u \rangle = \|v\|^2 - \langle u, Wu \rangle.$$

Further, there holds

$$\langle u, Wu \rangle \leq \|W\|_{-\Delta,\alpha} \|h_{-\alpha}^{1/2}u\|^2,$$

and using the inequality  $\|W\|_{-\Delta,\alpha} \leq \alpha^{-r}\|W\|_r$  we obtain that  $\|W\|_{-\Delta,\alpha} \leq \frac{1}{2}$  for all  $\alpha \geq (2\|W\|_r)^{1/r}$ , and that  $\|h_{-\alpha}^{1/2}(H + \alpha)^{-1/2}\|^2 \leq 2$  yielding the first inequality.

The second inequality results from using, once again, the inequality  $\|A\|_{-\Delta,\alpha} \leq \alpha^{-r}\|A\|_r$ .  $\square$

Before starting the proof of Theorem 1, we analyze the relation of the spectra of  $\mathcal{H}$  and  $\mathcal{H}_M(\lambda)$ .

**Lemma 10.** *Let  $\lambda_i < \lambda_j$  be the  $i$ -th resp.  $j$ -th eigenvalue of  $\mathcal{H}$  and let  $\nu_{Mi}(\lambda)$  denote the  $i$ -th eigenvalue of the operator  $\mathcal{H}_M(\lambda)$ . Then, we have the following  $M$ -independent lower bound*

$$\lambda_j - \lambda_i \leq \min(\nu_{Mj}(\lambda_i) - \nu_{Mi}(\lambda_i), \nu_{Mj}(\lambda_j) - \nu_{Mi}(\lambda_j)). \quad (66)$$

*Proof.* Since  $\lambda_i = \nu_{Mi}(\lambda_i)$  and  $\lambda_j = \nu_{Mj}(\lambda_j)$ , we have  $\lambda_j - \lambda_i = \nu_{Mj}(\lambda_j) - \nu_{Mi}(\lambda_i)$ . By definition (16) and Proposition 2, the family  $\mathcal{H}_M(\lambda)$  is monotonically decreasing with  $\lambda$ . Hence  $\nu_{Mi}(\lambda_j) \leq \nu_N^i(\lambda_i)$  and  $\nu_{Mj}(\lambda_j) \leq \nu_{Mj}(\lambda_i)$ , so that  $\lambda_j - \lambda_i = \nu_{Mj}(\lambda_j) - \nu_{Mi}(\lambda_i)$  implies (66).  $\square$

**Corollary 9.** *Let  $\lambda_i$  be an isolated eigenvalue of  $H$ . Then, the gap  $\gamma_{Mi}$  of  $\lambda_i$  to the rest of the spectrum of  $\mathcal{H}_M(\lambda_i)$  is bounded below by the gap  $\gamma_0$  of  $\lambda_i$  to the rest of the spectrum of  $H$ .*

**Lemma 11.** *Let be  $\alpha > 0$  such that*

$$\mathcal{H}_M(\lambda_\star) + \alpha > 0, \quad \text{and} \quad \alpha > (2\|U_M(\lambda_\star) - U_\sigma(\lambda_{\sigma i})\|_r)^{1/r}, \quad (67)$$

*$M$  large enough such that  $\lambda_\star, \lambda_{\sigma i} < \min(\kappa_M, \frac{1}{2}\rho_M)$ ,  $\rho_M \geq 1$  and  $4\rho_M^{-r}\|V\|_r \leq \frac{1}{3}$ , and  $N \geq M$ .*

*Then, there holds*

$$\|U_M(\lambda_\star) - U_\sigma(\lambda_{\sigma i})\|_{\mathcal{H}_M(\lambda_\star),\alpha} \leq |\lambda_{\sigma i} - \lambda_\star| \frac{8\alpha^{-r}\rho_M^{-r}\|V\|_r^2}{\pi(\kappa_M - \min(\lambda_\star, \lambda_{\sigma i}))} + 36 \frac{\|V\|_r^2}{\alpha^r} \varepsilon(\sigma, r, V) \quad (68)$$

*with, using the notation  $\sigma = (N, M, K)$ ,*

$$\varepsilon(\sigma, r, V) := \rho_N^{-r} + \rho_M^{-r} [4\rho_M^{-r}\|V\|_r]^{K+1}.$$

*Proof.* Applying Lemma 9 to the present case, we obtain

$$\|U_M(\lambda_\star) - U_\sigma(\lambda_{\sigma i})\|_{\mathcal{H}_M(\lambda_\star),\alpha} \leq 2\alpha^{-r}\|U_M(\lambda_\star) - U_\sigma(\lambda_{\sigma i})\|_r.$$

Employing the triangle inequality, there holds

$$\|U_M(\lambda_\star) - U_\sigma(\lambda_{\sigma i})\|_r \leq \|U_M(\lambda_\star) - U_M(\lambda_{\sigma i})\|_r + \|U_M(\lambda_{\sigma i}) - U_\sigma(\lambda_{\sigma i})\|_r. \quad (69)$$

For the first term of the right handside, Lemma 8 yields

$$\|U_M(\lambda_{\sigma i}) - U_M(\lambda_\star)\|_r \leq |\lambda_{\sigma i} - \lambda_\star| \frac{8\rho_M^{-r} \|V\|_r^2}{\pi(\kappa_M - 2\min(\lambda_\star, \lambda_{\sigma i}))}.$$

Finally, we use Lemma 7 in order to bound the second term of (69), we conclude that

$$\|U_M(\lambda_{\sigma i}) - U_\sigma(\lambda_{\sigma i})\|_r \leq \|V\|_r^2 \left[ 18\rho_N^{-r} + 6\rho_M^{-r} [4\rho_M^{-r} \|V\|_r]^{K+1} \right].$$

Combining all estimates concludes the proof.  $\square$

Finally, we are now ready to prove Theorem 1.

*Proof of Theorem 1.* Note that  $\lambda_\star = \nu_{Mi}(\lambda_\star)$ ,  $\lambda_{\sigma i} = \nu_{\sigma i}(\lambda_{\sigma i})$  where  $\nu_{Mi}(\lambda)$  and  $\nu_{\sigma i}(\lambda)$  represents the  $i$ -th eigenvalue of  $\mathcal{H}_M(\lambda)$  and  $\mathcal{H}_\sigma(\lambda)$  respectively.

We choose  $\alpha > 0$  in order to satisfy the condition (67) of Lemma 11. By the expression (68) of Lemma 11 we can choose  $\widetilde{M}_0 \in \mathbb{N}$  such that

$$\|U_M(\lambda_\star) - U_\sigma(\lambda_{\sigma i})\|_{\mathcal{H}_M(\lambda_\star), \alpha} \leq \frac{1}{4} \frac{\gamma_0}{\lambda_\circ},$$

for any  $M \geq \widetilde{M}_0$  and where  $\gamma_0$  denotes the gap of  $\lambda_\star$  to the rest of the spectrum of  $\mathcal{H}$ , which is a lower bound of  $\gamma_{Mi}$  by Corollary 9, and  $\lambda_\circ = \lambda_\star + \gamma_0 + \alpha$ . In consequence, we can apply Theorem 6 with

$$H_0 = \mathcal{H}_M(\lambda_\star), \quad W = U_\sigma(\lambda_{\sigma i}) - U_M(\lambda_\star), \quad \lambda_0 = \lambda_\star, \quad (70)$$

and thus  $H = \mathcal{H}_\sigma(\lambda_{\sigma i})$ , yielding

$$|\lambda_\star - \lambda_{\sigma i}| = |\nu_{Mi}(\lambda_\star) - \nu_{\sigma i}(\lambda_{\sigma i})| \leq (\lambda_\star + \alpha) \|U_M(\lambda_\star) - U_\sigma(\lambda_{\sigma i})\|_{\mathcal{H}_M(\lambda_\star), \alpha} \quad (71)$$

$$\leq (\lambda_\star + \alpha) \left[ |\lambda_{\sigma i} - \lambda_\star| \frac{16\alpha^{-r} \rho_M^{-r} \|V\|_r^2}{\pi(\kappa_M - 2\min(\lambda_\star, \lambda_{\sigma i}))} + 54 \frac{\|V\|_r^2}{\alpha^r} \varepsilon(\sigma, r, V) \right] \quad (72)$$

It exists a  $M_0 \in \mathbb{N}$ , with  $M_0 \geq \widetilde{M}_0$ , such that

$$4\rho_M^{-r} \|V\|_r \leq \frac{1}{3} \quad \text{and} \quad (\lambda_\star + \alpha) \frac{16\alpha^{-r} \rho_M^{-r} \|V\|_r^2}{\pi(\kappa_M - 2\min(\lambda_\star, \lambda_{\sigma i}))} \leq \frac{1}{2}, \quad (73)$$

for all  $M \geq M_0$  and thus

$$|\lambda_\star - \lambda_{\sigma i}| \lesssim (\lambda_\star + \alpha) \frac{\|V\|_r^2}{\alpha^r} \varepsilon(\sigma, r, V). \quad (74)$$

A similar development for the eigenfunctions based on the estimates (24)–(25) of Theorem 6 can be applied. Indeed, we denote by  $\varphi_{\sigma i}$  the  $i$ -th normalized eigenfunction of  $\mathcal{H}_\sigma(\lambda_{\sigma i})$ , thus, in terms of the notation used in Theorem 6, we have  $\psi_i = \varphi_{\sigma i}$ . Then, the corresponding eigenfunction in the span of all eigenfunctions corresponding to  $\lambda_\star$  is given by  $\varphi_{Mi} = \mathbf{P}_0 \varphi_{\sigma i}$ , where  $\mathbf{P}_0$  is the projector onto this span of all eigenfunctions of  $\mathcal{H}_M(\lambda_\star)$  corresponding to  $\lambda_\star$ . Thus,  $\varphi_{Mi}$  is an eigenfunction of  $\mathcal{H}_M(\lambda_\star)$  associated to the eigenvalue  $\lambda_\star$ . Taking, again, Corollary 9 into account yields

$$\|\varphi_{Mi} - \varphi_{\sigma i}\| \leq 4 \frac{\lambda_\circ}{\gamma_0} \|U_M(\lambda_\star) - U_\sigma(\lambda_{\sigma i})\|_{\mathcal{H}_M(\lambda_\star), \alpha}, \quad (75)$$

Using the bounds of  $\|U_M(\lambda_\star) - U_\sigma(\lambda_{\sigma i})\|_{\mathcal{H}_M(\lambda_\star), \alpha}$  expression (68) of Lemma 11 and combining with (74) yields the auxiliary result (see (13)): for  $s \geq 0$  and for all  $M \geq M_0$ ,

$$\|(-\Delta + 1)^s (\varphi_{Mi} - \varphi_{\sigma i})\| \lesssim \rho_M^s \|\varphi_{Mi} - \varphi_{\sigma i}\| \lesssim \frac{\lambda_\circ}{\gamma_0} \frac{\|V\|_r^2}{\alpha^r} \rho_M^s \varepsilon(\sigma, r, V). \quad (76)$$

Now we define the eigenfunction  $\varphi_i := \mathbf{Q}_M(\lambda_\star)\varphi_{Mi}$  where  $\mathbf{Q}_M$  is defined by (17). Following Corollary 5,  $\varphi_i$  is an eigenfunction of  $\mathcal{H}$  associated to the eigenvalue  $\lambda_\star$ . Note that

$$\varphi_i - Q_\sigma(\lambda_{\sigma i})\varphi_{\sigma i} = \varphi_{Mi} - \varphi_{\sigma i} - [S_M(\lambda_\star)\varphi_{Mi} - S_\sigma(\lambda_{\sigma i})\varphi_{\sigma i}],$$

with  $S_M(\lambda) := \mathbf{P}_M^\perp(\mathcal{H}_M^\perp - \lambda)^{-1}\mathbf{P}_M^\perp V \mathbf{P}_M$ ,  $S_\sigma(\lambda) := R_\sigma(\lambda)\mathbf{P}_M^N V \mathbf{P}_M$ . Applying the triangle inequality several times yields

$$\|\varphi_i - Q_\sigma(\lambda_{\sigma i})\varphi_{\sigma i}\| \leq (1 + I_1)\|\varphi_{Mi} - \varphi_{\sigma i}\| + I_2 + I_3, \quad (77)$$

with

$$I_1 = \|S_M(\lambda_\star)\|, \quad I_2 = \|S_M(\lambda_\star) - S_M(\lambda_{\sigma i})\|, \quad I_3 = \|S_M(\lambda_{\sigma i}) - S_\sigma(\lambda_{\sigma i})\|.$$

For  $I_1$ , proceeding as in the proof of Lemma 5, writing

$$S_M(\lambda) = \sum_{k=0}^{\infty} \mathbf{P}_M^\perp h_\lambda^{-1/2} \left[ -h_\lambda^{-1/2} V_M^\perp h_\lambda^{-1/2} \right]^k h_\lambda^{-1/2} \mathbf{P}_M^\perp V \mathbf{P}_M.$$

and under the assumptions  $\lambda \leq \min(\frac{1}{2}\rho_M, \kappa_M)$ ,  $\rho_M \geq 2$ ,  $4\rho_M^{-r}\|V\|_r \leq \frac{1}{2}$ , using the estimates of Lemma 4  $\|\mathbf{P}_M^\perp h_\lambda^{-1/2}\| \leq \sqrt{2}\rho_M^{-1/2}$ ,  $\|\mathbf{P}_M h_{s-1}^s\| \leq (\rho_M + 1)^s \leq (3\rho_M/2)^s$  for  $s \geq 0$ , one obtains

$$\|S_M(\lambda)\| \leq 2 \|\mathbf{P}_M^\perp h_\lambda^{-1/2}\| \|\mathbf{P}_M^\perp h_\lambda^{-1/2} \mathbf{P}_M^\perp V \mathbf{P}_M h_{-1}^{-1/2+r/2}\| \|\mathbf{P}_M h_{-1}^{1/2-r/2}\| \quad (78)$$

$$\leq 2\sqrt{2}(3/2)^{1/2-r/2} \rho_M^{-1/2} \rho_M^{-r/2} \|V\|_r \rho_M^{1/2-r/2} \leq 12\rho_M^{-r} \|V\|_r. \quad (79)$$

For  $I_2$ , we proceed as in Lemma 8 based on the results of Lemma 6 but substituting  $U_M(\lambda)$  by  $S_M(\lambda)$  in order to obtain

$$I_2 \leq \frac{|\lambda_\star - \lambda_{\sigma i}|}{\pi(\kappa_M - 2\min(\lambda_\star, \lambda_{\sigma i}))} 12\rho_M^{-r} \|V\|_r, \quad (80)$$

based upon the estimate  $\|S_M(\lambda)\| \leq 12\rho_M^{-r} \|V\|_r$  from (78)–(79).

Finally for  $I_3$ , we proceed as in the proof of Lemma 7. Indeed, we apply, once again, the triangle inequality to obtain  $I_3 \leq I_{3,1} + I_{3,2}$  with

$$I_{3,1} = \|S_M(\lambda_{\sigma i}) - S_{MN}(\lambda_{\sigma i})\| \quad I_{3,2} = \|S_{MN}(\lambda_{\sigma i}) - S_\sigma(\lambda_{\sigma i})\|.$$

with  $S_{MN}(\lambda) := \mathbf{P}_M^N(\mathcal{H}_M^N - \lambda)^{-1}\mathbf{P}_M^N V \mathbf{P}_M$ . To estimate  $I_{3,1}$ , we first note that

$$\|S_M(\lambda) - S_{MN}(\lambda)\| \leq \|h_{-1}^{-1/2+r/2} \mathbf{P}_M^N\| \|h_{-1}^{1/2-r/2} \left[ \mathbf{P}_M^\perp(\mathcal{H}_M^\perp - \lambda)^{-1}\mathbf{P}_M^\perp - \mathbf{P}_M^N(\mathcal{H}_M^N - \lambda)^{-1}\mathbf{P}_M^N \right] h_{-1}^{1/2-r/2}\| \|h_{-1}^{-1/2+r/2} V \mathbf{P}_M\|$$

Using that  $\|h_{-1}^{-1/2+r/2} \mathbf{P}_M^N\| \leq \rho_M^{-1/2+r/2}$  and  $\|h_{-1}^{-1/2+r/2} V \mathbf{P}_M\| \leq \|V\|_r \rho_M^{1/2-r/2}$ , we obtain

$$\|S_M(\lambda) - S_{MN}(\lambda)\| \leq \|V\|_r \|h_{-1}^{1/2-r/2} \left[ \mathbf{P}_M^\perp(\mathcal{H}_M^\perp - \lambda)^{-1}\mathbf{P}_M^\perp - \mathbf{P}_M^N(\mathcal{H}_M^N - \lambda)^{-1}\mathbf{P}_M^N \right] h_{-1}^{1/2-r/2}\|.$$

Adapting the proof of (62), and noting that  $4\rho_M^{-r}\|V\|_r - \frac{16\rho_M^{-2r}\|V\|_r^2}{1-4\rho_M^{-r}\|V\|_r} < 1$ , and  $4\rho_M^{-r}\|V\|_r < 1$ , we obtain

$$I_{3,1} \lesssim \rho_M^{-r} \|V\|_r. \quad (81)$$

Finally, for  $I_{3,2}$ , we proceed as in the derivation of (64), starting from a similar ansatz than (63)

$$S_{MN}(\lambda) - S_\sigma(\lambda) = \sum_{k=K+1}^{\infty} \mathbf{P}_M^N h_\lambda^{-1/2} \left[ -h_\lambda^{-1/2} V_M^N h_\lambda^{-1/2} \right]^k h_\lambda^{-1/2} \mathbf{P}_M^N V \mathbf{P}_M,$$

yielding, together with  $\|\mathbf{P}_M^N h_\lambda^{-1/2}\| \lesssim \rho_M^{-1/2}$  and  $\|h_\lambda^{-1/2} \mathbf{P}_M^N V \mathbf{P}_M\| \lesssim \rho_M^{1/2-r} \|V\|_r$ ,

$$I_{3,2} = \|S_{MN}(\lambda_{\sigma i}) - S_\sigma(\lambda_{\sigma i})\| \lesssim \rho_M^{-r} \|V\|_r [4\rho_M^{-r} \|V\|_r]^{K+1}. \quad (82)$$

Starting from (77) and combining (78)–(79), (80), (81), (82) with the estimates (76) (with  $s = 0$ ), (74) and the bound (73) concludes the proof.  $\square$

**Remark 7.** *Having now understood how to apply Theorem 6 in this context, i.e. using (70), and seen the abstract theory of Section 3 we can now make a statement about the convergence of the fixed-point iteration schemes (18) and (19): Following Corollary 8, we note that if  $\nu_{\sigma i}(\lambda)$  parametrizes an isolated branch on some interval  $I_i$  containing  $\lambda_{\sigma i}$ ,  $M \geq M_0$  and, using (71)–(72) and (74), if  $\varepsilon(\sigma, r, V)$  is small enough, then, the fixed-point iterations converge for any starting point in  $I_i$ . Unfortunately, it is difficult to assess whether the isolated branch property holds in practical application and this remains an abstract result.*

## 6. NUMERICAL RESULTS

In this section, we test the theoretical estimates developed in this article. We consider a one-dimensional test case with  $\Omega = (0, 1)$  and a potential  $V_t$  given by its Fourier coefficients:

$$\widehat{V}_0 = -10, \quad \widehat{V}_n = -\frac{5}{|n|^t},$$

so that  $V_t \in \mathbf{H}_{\text{per}}^{t-\frac{1}{2}-\varepsilon}$  for any  $\varepsilon > 0$ . We then consider the two different values  $t = 1$  or  $t = 0$ , see Figure 2 for a graphical illustration for the case  $t = 1$ , so that including the embedding expressed under the constraints (86) yields that  $\|V\|_r < \infty$  for  $r = 1$  if  $t = 1$  and for all  $r < 1/2$  if  $t = 0$ .

Note that, to the best of our knowledge, the classical convergence analysis does not cover such low regularities of the potentials. While the standard analysis can probably be extended to potentials in  $L^2$  and thus covering the case  $t = 1$  (although we are not aware of any published analysis in this case), it certainly does not hold without further developments for  $t = 0$ .

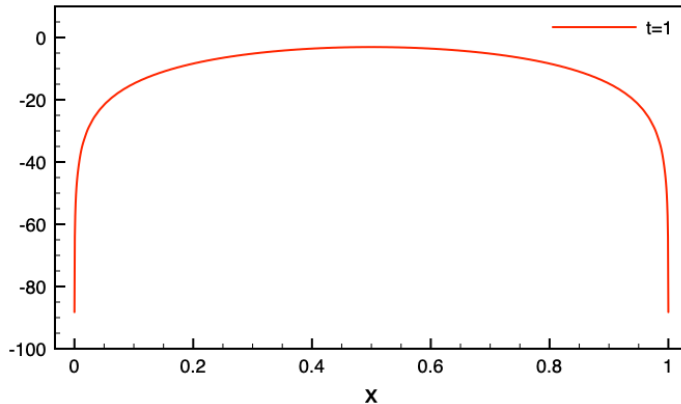
In Figure 3, we illustrate the convergence of the discrete solutions with respect to  $K$  for different values of  $M$  and  $t = 1$  and  $t = 0$ , and for fixed  $N = 500$ . The error in the eigenvalue and eigenvector are defined as

$$\text{err}_{\text{val}} = |\lambda_\star - \lambda_{\sigma i}| \quad \text{and} \quad \text{err}_{\text{vec}} = \|\varphi_i - Q_\sigma(\lambda_{\sigma i})\varphi_{\sigma i}\|,$$

where  $(\lambda_\star, \varphi_{M i})$ ,  $(\lambda_{\sigma i}, \varphi_{\sigma i})$  are the  $i$ -th solution to (5) and (9) respectively, using the computational Strategy 1 defined by (18) targeting the smallest eigenvalue. The “exact” solution is obtained by computing the variational approximation for  $N_e = 1000$ .

We observe that, in agreement with the theory for small enough  $K, M$ , the convergence rate with respect to  $K$  for different value of  $M$  is the same for the eigenvalue and eigenvector error and that the convergence rate improves with increased values of  $M$ . In this example, we observe that the condition  $M \geq M_0$  is not restrictive and convergence can be observed for all values of  $M$ . It is also noted, in particular for the higher values of  $M$  (but still very moderate), that the number on the truncation order  $K$  can be kept very low to achieve a good accuracy. This, in turn, means that the number of computations on the fine grid (essentially matrix-vector products involving the fine grid for each matrix-vector product involving the coarse Hamiltonian  $\mathcal{H}_\sigma(\lambda)$ ) can be kept to a minimum.

A similar behaviour is reported in Figure 5 for the third eigenvalue using the potential  $V_{t=1}$  and again  $N = 500$ .

FIGURE 2. Illustration of the potential  $V_{t=1}$ .

$M = 1$		$M = 2$		$M = 3$		$M = 4$		$M = 5$		$M = 6$	
$K$	# scf	$K$	# scf	$K$	# scf	$K$	# scf	$K$	# scf	$K$	# scf
1	6	1	5	1	4	1	4	1	4	1	4
2	6	2	5	2	4	2	4	2	4	2	4
3	6	3	5	3	4	3	4	3	4	3	4
4	6	4	5	4	4	4	4	4	4	4	4

TABLE 1. The number of SCF-like iterations of the solution strategy (18) to reach an increment of  $10^{-12}$  in the eigenvalue for the approximation of the first eigenvalue with  $V_{t=1}$  using  $N = 500$ .

The number of required SCF-iterations (18) to converge to an increment in the eigenvalue smaller than  $10^{-12}$ , thus very tight, is stable and very moderate over all test cases as reported in Tables 1 and 2 (the case of  $t = 0$  behaves similarly and is not reported here).

Finally, Figure 4 illustrates the error in the eigenvalue and eigenvector with respect to  $N$  for different values of  $K$  and  $M$  for the first eigenvalue with  $V_{t=1}$  and  $V_{t=0}$ . We observe two regimes. First, for small values of  $N$ , the error is limited by the small size of the fine grid  $X_N$ , i.e. by a small  $N$ , and decreases with increasing  $N \in [25, 500]$ . Second, when  $N$  is large enough, the error due to the moderate values of  $M$ ,  $K$  is dominating and the error stagnates. As  $M$  or  $K$  increase, the transition of the two regimes moves to lower accuracy. This agrees well with the theoretical result presented in Theorem 1. For the potential  $V_{t=1}$ , we observe that the convergence rate in  $N$  (for  $M, K$  large enough) is roughly 3 for the eigenvalue error and 2.5 for the eigenvector error, which are the rates predicted by the standard analysis as outlined in Corollary 2. For the less regular case  $V_{t=0}$ , and thus  $\|V\|_r < \infty$  for any  $r < 1/2$ , we observe a rate of roughly 1 in both cases, eigenvalue and eigenvector  $L^2$ -error, which is exactly as predicted by Theorem 1.

## 7. CONCLUSION AND PERSPECTIVES

In this paper, we have proposed a new numerical method based on the Feshbach-Schur map in combination with planewave discretizations for linear Schrödinger eigenvalue problems. The method does not rely on the variational principle but reformulates the infinite-dimensional problem as an equivalent problem, non-linear in the spectral parameter, on a finite dimensional grid whose unknowns are the exact eigenvalue and the best-approximation of the exact eigenfunctions on the given grid. Such a problem can then be approximated by evaluating the Feshbach-Schur

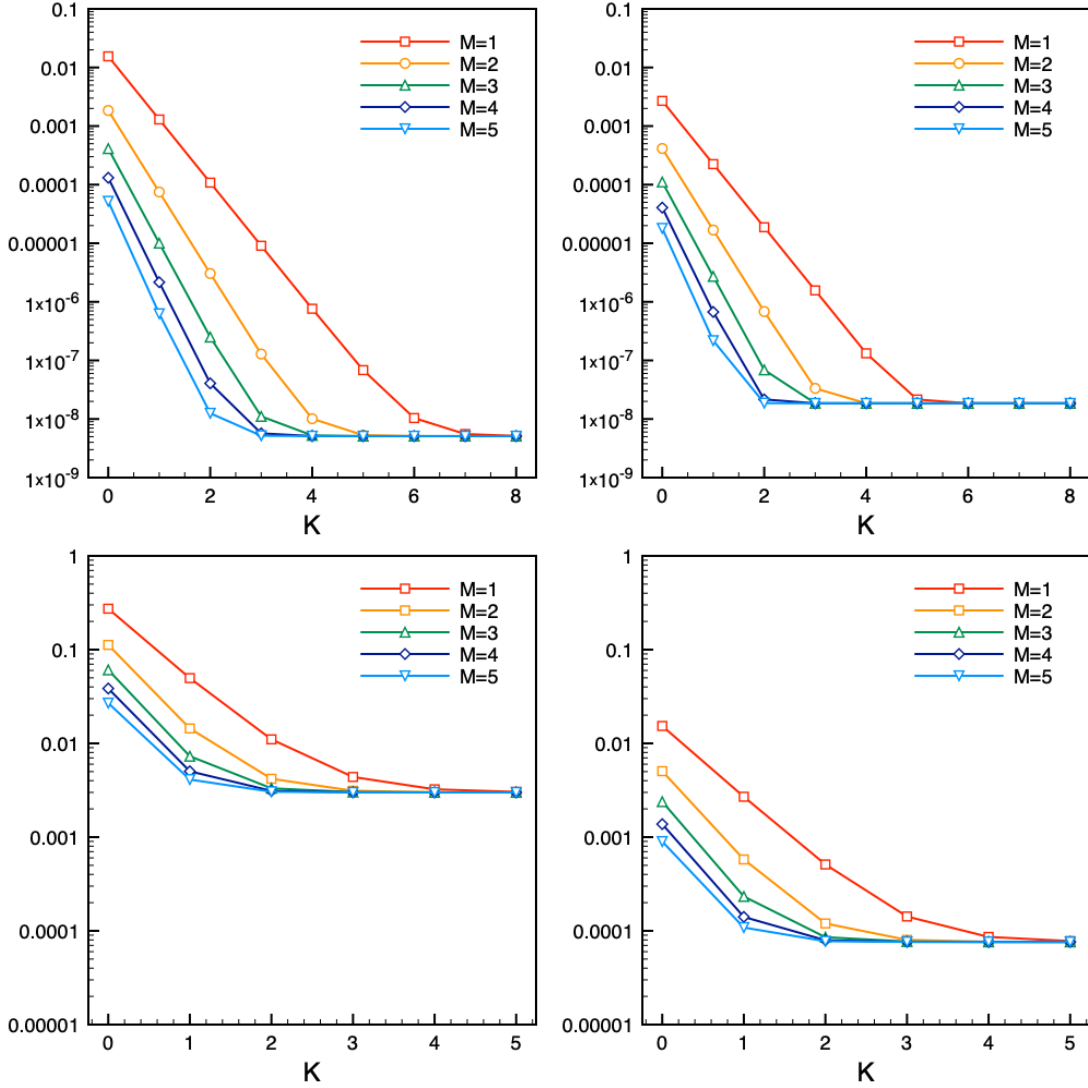


FIGURE 3. The convergence of the eigenvalue error  $\text{err}_{\text{val}}$  (left) and the eigenvector error  $\text{err}_{\text{vec}}$  (right) corresponding to the first eigenvalue with respect to  $K$  for different values of  $M$  and for fixed  $N = 500$ . The top row corresponds to a potential  $V_{t=1}$  whereas the bottom row corresponds to  $V_{t=0}$ .

map on a second finer grid. The substantial contribution of this paper is an analysis in order to provide error estimates of the proposed method in all discretization parameters.

For this, we developed in Section 3 a version of perturbation theory that relies on the notion of form-boundedness with increased regularity, as stated by Assumption 1.

Having established the method and its analysis, its full benefits shall be further analyzed in future. At the present stage, it is worth to mention that, for the considered one-dimensional problem, the contraction in the perturbation is rather small and the non-linear iteration converge rapidly. Also, in view of more sophisticated non-linear eigenvalue problems, the artificial extra non-linearity does not seem to be much of a burden.

The future developments include the extension of Section 3 to a more general family of operators, including non-symmetric perturbations of self-adjoint operators as well as extending



$M = 1$		$M = 2$		$M = 3$		$M = 4$		$M = 5$		$M = 6$	
$K$	# scf	$K$	# scf	$K$	# scf	$K$	# scf	$K$	# scf	$K$	# scf
1	6	1	4	1	4	1	4	1	3	1	3
2	6	2	4	2	4	2	4	2	3	2	3
3	6	3	4	3	4	3	4	3	3	3	3
4	6	4	4	4	4	4	4	4	3	4	3

TABLE 2. The number of SCF-like iterations of the solution strategy (18) to reach an increment of  $10^{-12}$  in the eigenvalue for the approximation of the third eigenvalue with  $V_{t=1}$  using  $N = 500$ .

the numerical method and its analysis to cluster of eigenvalues using a density-matrix based formulation.

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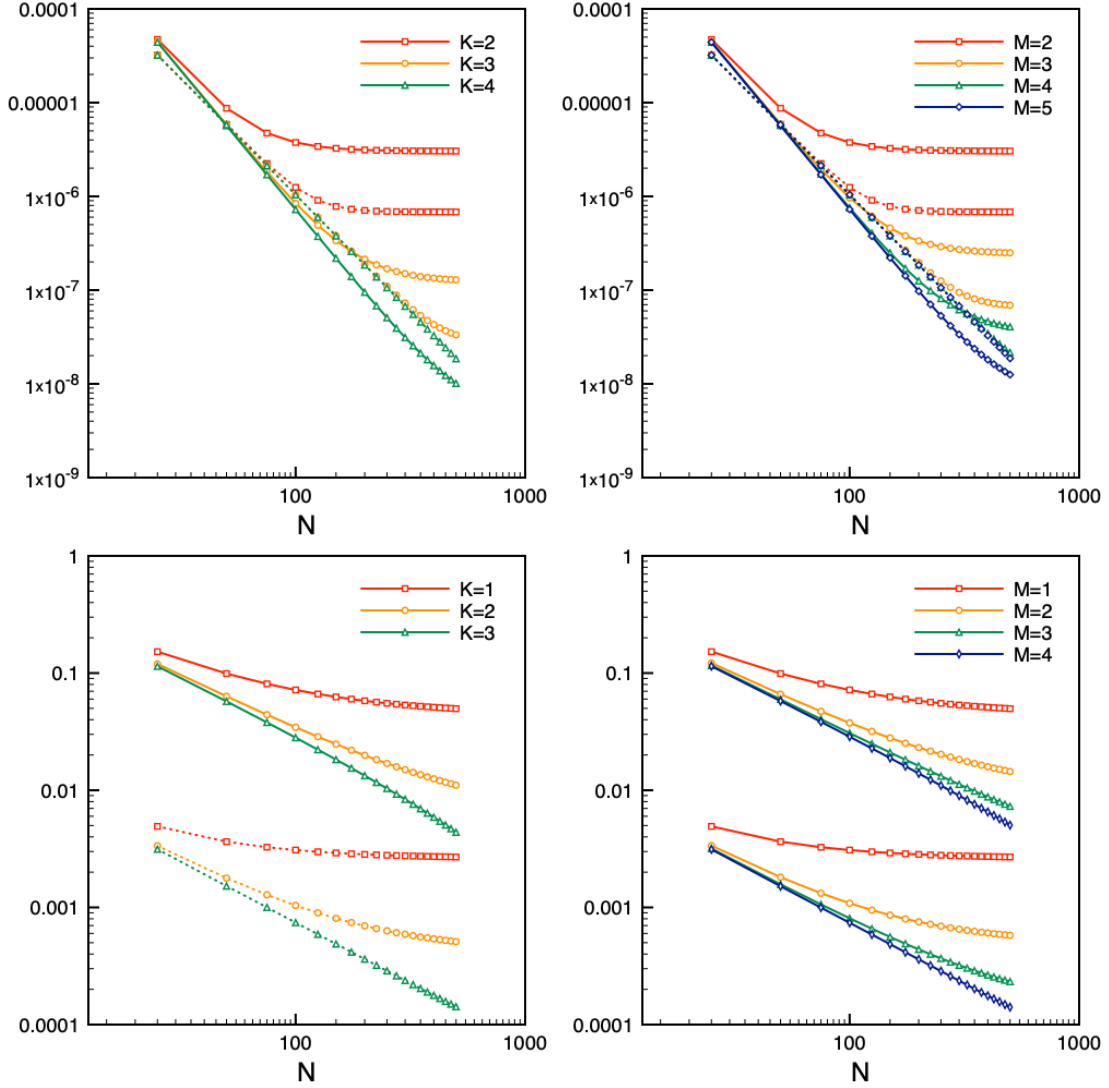


FIGURE 4. The convergence of the eigenvalue error  $\text{err}_{\text{val}}$  (straight lines) and the eigenvector error  $\text{err}_{\text{vec}}$  (dotted lines) corresponding to the first eigenvalue with respect to  $N$  for different values of  $K$  (left) with  $M = 2$  (top) and  $M = 1$  (bottom) and different values of  $M$  (right) with  $K = 2$  (top) and  $K = 1$  (bottom) for the potential  $V_{t=1}$  (top) and  $V_{t=0}$  (bottom).

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#### APPENDIX A. TECHNICAL RESULTS AND PROOFS

We present proofs of some technical statements made in the previous sections and some additional technical results. In what follows, we denote  $V_N^\perp := P_N^\perp V P_N^\perp$ .

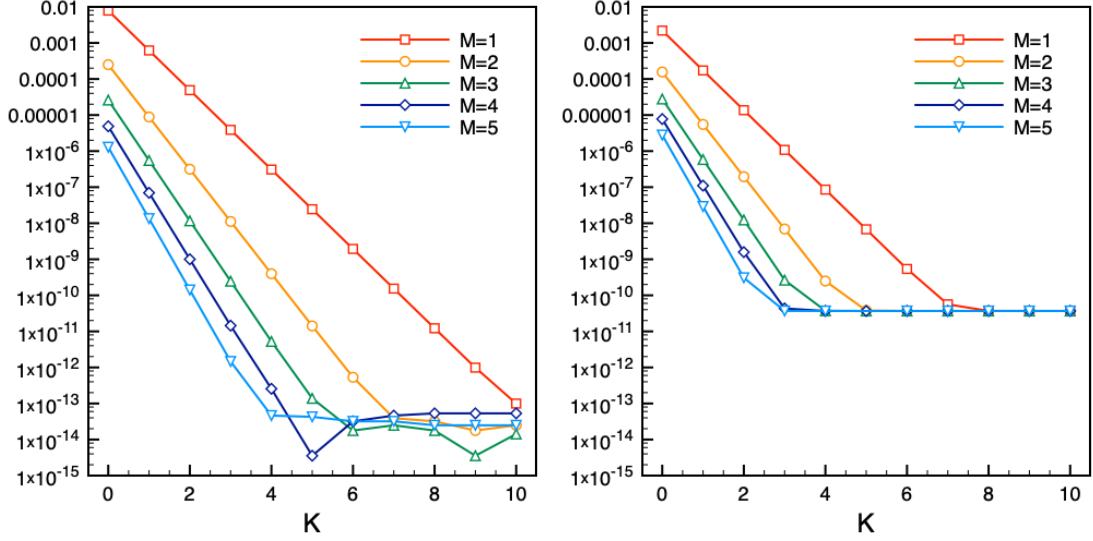


FIGURE 5. The convergence of the eigenvalue error  $\text{err}_{\text{val}}$  (left) and the eigenvector error  $\text{err}_{\text{vec}}$  (right) corresponding to the third eigenvalue with respect to  $K$  for different values of  $M$  and for fixed  $N = 500$  and with  $V_{t=1}$ .

**Lemma 12.** *Under Assumption 1, (i) the norm  $\|V\|_{0,\alpha} := \|(-\Delta + \alpha)^{-1/2}V(-\Delta + \alpha)^{-1/2}\|$  is well-defined for  $\alpha > 0$  and  $\|V\|_{0,\alpha} \rightarrow 0$  when  $\alpha \rightarrow +\infty$  and (ii)  $\mathcal{H}$  is bounded below as*

$$\mathcal{H} \geq [1 - \|V\|_{0,\alpha}](-\Delta + \alpha) - \alpha, \quad (83)$$

*Proof.* The first statement is obvious and the second one follows from the identity

$$\mathcal{H} = (-\Delta + \alpha)^{1/2}[\mathbf{1} + (-\Delta + \alpha)^{-1/2}V(-\Delta + \alpha)^{-1/2}](-\Delta + \alpha)^{1/2} - \alpha \quad (84)$$

and the fact that the expression in square braces is bounded below by  $1 - \|V\|_{0,\alpha}$ .  $\square$

*Proof of Lemma 1.* Proceeding as in (83), we obtain for any  $\alpha \in \mathbb{R}$  large enough so that  $\|V_M^\perp\|_{0,\alpha} < 1$

$$-\Delta + V_M^\perp \geq [1 - \|V_M^\perp\|_{0,\alpha}](-\Delta + \alpha) - \alpha.$$

Using that  $-\Delta \geq (2\pi M/L)^2 = \rho_M$  on  $\text{Ran } P_M^\perp$  and the estimate

$$\begin{aligned} \|V_M^\perp\|_{0,\alpha} &\leq \|(1 - \Delta)^{1/2}(-\Delta + \alpha)^{1/2}P_M^\perp\|^2 \|(1 - \Delta)^{-1/2}P_M^\perp V P_M^\perp (1 - \Delta)^{-1/2}\| \\ &\leq \frac{\rho_M + 1}{\rho_M + \alpha} \rho_M^{-r} \|V\|_r, \end{aligned}$$

we deduce

$$\begin{aligned} -\Delta + V_M^\perp &\geq \left(1 - \frac{\rho_M + 1}{\rho_M + \alpha} \rho_M^{-r} \|V\|_r\right) (\rho_M + \alpha) - \alpha, \\ &\geq \rho_M + \alpha - (\rho_M + 1) \rho_M^{-r} \|V\|_r - \alpha, \\ &\geq \rho_M - (\rho_M + 1) \rho_M^{-r} \|V\|_r, \end{aligned}$$

from which we obtain the result.  $\square$

Now, we introduce the Sobolev spaces  $\mathbf{H}_{\text{per}}^s$  of periodic real functions resp. distributions which can be characterized in a simple way using Fourier series: for  $s \in \mathbb{R}$ , we have

$$\mathbf{H}_{\text{per}}^s := \left\{ v = \sum_{\mathbf{k} \in \mathcal{R}^*} \widehat{v}_{\mathbf{k}} e_{\mathbf{k}} \mid \forall \mathbf{k} \in \mathcal{R}^* : \widehat{v}_{-\mathbf{k}} = \overline{\widehat{v}_{\mathbf{k}}}, \quad \|v\|_{\mathbf{H}_{\text{per}}^s} < \infty \right\}, \quad (85)$$

where the norm  $\|v\|_{\mathbf{H}_{\text{per}}^s}$  is given by the  $\mathbf{H}_{\text{per}}^s$  inner product is defined by

$$\forall u, v \in \mathbf{H}_{\text{per}}^s, \quad (u, v)_{\mathbf{H}_{\text{per}}^s} := \sum_{\mathbf{k} \in \mathcal{R}^*} (1 + |\mathbf{k}|^2)^s \overline{u_{\mathbf{k}}} v_{\mathbf{k}}.$$

**Lemma 13.** *If  $V \in \mathbf{H}_{\text{per}}^s$  for some  $s \in \mathbb{R}$ , then  $V$  satisfies Assumption 1 for all  $r \geq 0$  satisfying  $s \geq r - 1$  and  $s > -2(1 - r) + d/2$  resp.*

$$r \leq s + 1 \quad \text{and} \quad r < 1 + \frac{s}{2} - \frac{d}{4}, \quad (86)$$

where  $d$  is the spatial dimension. Moreover,

$$\|V\|_r \leq C_{r,s} \|V\|_{\mathbf{H}_{\text{per}}^s}.$$

*Proof.* From [13, Theorem 1.4.4.2], for any  $s_1, s_2 \geq t$ , such that  $s_1 + s_2 > t + d/2$ , there exists  $C_{s_1, s_2, t} > 0$  such that for all  $u \in \mathbf{H}_{\text{per}}^{s_1}, v \in \mathbf{H}_{\text{per}}^{s_2}$ , then  $uv \in \mathbf{H}_{\text{per}}^t$  and

$$\|uv\|_{\mathbf{H}_{\text{per}}^t} \leq C_{s_1, s_2, t} \|u\|_{\mathbf{H}_{\text{per}}^{s_1}} \|v\|_{\mathbf{H}_{\text{per}}^{s_2}}.$$

Hence if  $V \in \mathbf{H}_{\text{per}}^s$ ,  $s > -2(1 - r) + d/2$ , and  $s \geq r - 1$ , then there exists  $C_{r,s}$  such that

$$\|V\|_r = \sup_{\psi \in \mathbf{H}_{\text{per}}^{1-r}} \frac{\|V\psi\|_{\mathbf{H}_{\text{per}}^{-1+r}}}{\|\psi\|_{\mathbf{H}_{\text{per}}^{1-r}}} \leq C_{r,s} \|V\|_{\mathbf{H}_{\text{per}}^s},$$

which implies the estimate in the lemma.  $\square$

*Proof of Proposition 1.* For each  $\lambda$  such that  $\lambda < \kappa_M$ ,  $U_M(\lambda)$  can be written as the product of five bounded operators:  $-\mathbf{P}_M(-\Delta + 1)^{1/2-r/2}$ ,  $(-\Delta + 1)^{-1/2+r/2}V(-\Delta + 1)^{-1/2+r/2}$ ,  $(-\Delta + 1)^{1/2-r/2}\mathbf{P}_M^\perp(\mathcal{H}_M^\perp - \lambda)^{-1}\mathbf{P}_M^\perp(-\Delta + 1)^{1/2-r/2}$ ,  $(-\Delta + 1)^{-1/2+r/2}V(-\Delta + 1)^{-1/2+r/2}$ ,  $(-\Delta + 1)^{1/2-r/2}\mathbf{P}_M$ . Therefore,  $U_M(\lambda)$  is a well-defined operator.  $\square$

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