Robust Formation Control Under State Constraints of Multi-Agent Systems in Systems

T. V. Pham, N. Messai, D.H Nguyen, N. Manamanni

Version of Record: https://www.sciencedirect.com/science/article/pii/S0167691120300700
Manuscript_84df77255e70ae83f4088df5e422620e

Robust Formation Control Under State Constraints of Multi-Agent Systems in Clustered Networks

Van Thiem Pham[a,1,*], Nadhir Messai[a,1], Dinh Hoa Nguyen[b,1], Noureddine Manamanni[a,1],

[a] *University of Reims Champagne Ardenne, CReSTIC EA 3804, 51097 Reims, France*
[b] *WPI-I2CNER and IMI, Kyushu University, Fukuoka 819-0395, Japan*

Abstract

This paper studies the formation control problem in clustered network systems composing of linear agents that are subjected to state constraints. In each cluster, there exists an agent called a leader who can communicate with other leaders outside of its cluster at some specific discrete instants. Moreover, the continuous-time communication structure in each cluster is represented by a fixed and undirected graph. A robust formation control protocol is proposed to deal with the hybrid communication described above and the constraints on states of agents. It is next shown that the hybrid robust formation control design for clustered multi-agent networks can be indirectly solved through the robust stabilization design of an equivalent system obtained by matrix theory and algebraic graph theory. Then, a robust controller is designed for the initial clustered network system in terms of linear matrix inequalities. Finally, a formation design for unmanned aerial vehicles is carried out and simulated to illustrate the effectiveness of the proposed hybrid formation control design method.

Keywords: Clustered network, state constraints, hybrid communication, formation control, linear matrix inequalities (LMIs).

1. Introduction

Multi-agent systems (MASs) and their cooperative control problems have been extensively investigated in the past two decades, motivated by many practical applications such as unmanned aerial vehicles (UAVs), wireless sensor networks, autonomous underwater vehicles (AUVs), transportation networks, satellite networks, etc. Formation is one of the most interesting problems and intensively investigated issue in MASs, because of both several theoretical challenges and a wide range of potential applications [1, 2, 3, 4].

Currently, consensus-based control approaches are widely employed in solving the formation problem of MASs. The first-order MASs was studied in [5]. It was proved that they could achieve formation if the directed interaction topology contains a directed spanning tree. Extensions of consensus-based formation algorithms to second-order with linear dynamics under the fixed or the time-varying interaction graphs were also investigated in [6]. In [7], a formation control law based on local measurement of relative-positions was proposed for first-order MASs. In another research [8], based on consensus approaches, necessary and sufficient conditions were derived to deal with the time-varying formations for second-order UAV swarm systems. Another direction is to study the formation of second-order MASs with time delays [9]. Accordingly, the sufficient conditions were proposed for MASs to achieve desired stationary and moving formations. Next, the formation tracking control problems [10], which uses neighboring relative state, and position information for second-order MASs with time-varying delays, were considered. Moreover, the formation tracking problems were also investigated in [11], where the agents' local coordinate systems are applied such that the centroid of the controlled formation tracks a given trajectory. Recently, some new studies concern the influence of environments on wireless communication [12, 13]. This leads to the degradation of the performance of wireless communication, which causes the interaction topology among agents cannot hold for a long time.

Other significant and realistic issues have been encountered such as the constraints on the agent's inputs, states, or relative states because of the physical limitations of agents. This includes, for example, the formation of vehicles with limited speeds and limited working space, smart buildings energy control with constraints on temperature and humidity in specific ranges and so on. Recently, some studies have considered the cooperative control of MASs under the constraints on agent's inputs, states, or relative states [14, 15, 16, 17, 18, 19]. In [14], a constrained consensus algorithm and distributed optimization problems were proposed, where agents with state constraints are investigated and they are required to lie in individual closed convex sets. In another work, [15] studied a consensus problem of simple integrator MASs under input constraints. Following this research line, a distributed consensus of second-order MASs with nonconvex input constraints was addressed in [16]. It is shown that the input constrained consensus achieved if the graph has a directed spanning tree. Another direction to

*Corresponding author
Email addresses: van-thiem.pham@etudiant.univ-reims.fr (Van Thiem Pham), nadhir.messai@univ-reims.fr (Nadhir Messai), hoa.nd@i2cner.kyushu-u.ac.jp (Dinh Hoa Nguyen), noureddine.manamanni@univ-reims.fr (Noureddine Manamanni)

deal with input and state constraints, discarded consensus algorithms are employed [17]. Moreover, in order to achieve the global consensus in the presence of agents' inputs, states, or relative state constraints [18, 19], the MASs is reformulated in form of a network of Lure systems. However, the formation problem under state constraints in the clustered network is still an open problem.

On the other hand, it has been noticed that interaction among agents in the aforementioned networks is either continuous-time or discrete-time. However, due to either energy constraints occurring in long-time interactions or communication constraints, agents can only impulsively exchange information with their neighbors or be subjected to abrupt changes at specific instants [20, 21, 22, 23]. This leads to a hybrid interaction that combines both continuous and discrete interactions among agents. In [20], a sufficient result has been derived for the impulsive consensus of first-order MASs, where the graphs of continuous-time and impulsive-time topologies contain a spanning tree. Following this research line, there are several types of research [21, 22], which have dealt with the consensus problem of the second-order MASs under an impulsive control strategy. Moreover, inspired by the results in [24, 25], the necessary condition of consensus on graph connections among agents may require. This is investigated in [23], where the first-order MASs with hybrid delay consensus protocols are described in the form of impulsive systems. In other directions on consensus problem under hybrid communication, the network is partitioned into several groups or clusters [26, 27]. The works in [26] proposed a quasi-periodically reset strategy and provided some LMI conditions to guarantee the globally uniformly exponential consensus for MASs, where the intra-cluster communication structures are represented by directed and strongly connected graphs. The researches in [27] investigated the sufficient conditions for event-triggered consensus. Moreover, both of state and output consensus problem in the clustered network of generic linear MASs are respectively investigated in [28, 29] and [30]. However, in most of the above, constraints on the states of agents are not considered.

As a practical example, let us analyze two groups of multi-UAV. To support the cooperative flying, each UAV is equipped with a wireless device to communicate with other UAVs in the same group and on-board sensors to measure or estimate its absolute position and speed. Formation problem naturally appears when two groups are executing the same mission, e.g., scanning an area or protecting civil. Because of sensors' limited ranges and limited speeds of UAVs, the measurements of UAVs' positions and velocities are subject to saturation constraints. Moreover, due to the short-range communications of wireless technologies, it is impossible for each UAV to leave its group and continuously communicate with other UAVs in other groups. Therefore, it is more practical for the collaboration of these groups if each group has one leading UAV to communicate with other groups' leading UAVs through a cellular network at some specific discrete moments. As a result, the information exchange in these networks is not only continuous but also discrete.

Motivated by both theoretical and practical issues mentioned above, this paper investigates the state formation control problem under state constraints in clustered MASs where agents have generic linear dynamics. Our approach covers broader systems and scenarios than those in the existing studies [26, 27]. Next, a robust formation protocol, which deals with the continuous-time communication inside clusters and discrete-time information exchange between clusters, is introduced. Compared with the previous results [28, 29], the protocol is more practical and complicated. It is then shown that the considered robust formation control problem can be indirectly solved by studying the robust stability of an equivalent system by matrix theory and algebraic graph theory. In comparison with the one in [26, 28, 29], our approach shows the important role of communication between leaders at some specific discrete instants, represented by the stochastic matrix. Accordingly, a sufficient condition will be derived in terms of LMIs for the robust distributed formation of clustered networks of generic linear agents under state constraints and hybrid communications.

The rest of the paper is organized as follows. Section II is dedicated to problem formulation. In Section III, the robust formation problem of MASs under state constraints in a hybrid clustered network is investigated. A possible application of the proposed control law to the UAVs formation flying is illustrated in Section IV. Finally, conclusions are provided in Section V.

Notations and symbols. $\mathbb{N}$ denotes the sets of non-negative integers. Real, and non-negative real numbers are denoted by $\mathbb{R}$, and $\mathbb{R}^+$, respectively. $\mathbf{0}_N$ denotes the zero matrices with appropriate dimensions. $\mathbf{1}_N$ denotes the $(N \times 1)$ column vectors whose elements are all ones. A matrix is called SIA if it is stochastic, irreducible, and aperiodic. On the other hand, $diag\{\}$ represents diagonal or block-diagonal matrices, and $sym(A)$ denotes $A + A^T$ for any real matrix A. Lastly, $\circ$ and $\otimes$ stands for the Hadamard and Kronecker product.

2. Problem Formulation

2.1. Graph Theory

An undirected graph $\mathcal{G}$ consists of a pair $(\mathcal{V}, \mathcal{E})$, where the vertex set $\mathcal{V} = \{v_1, \cdots, v_N\}$ denotes the set of nodes and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents the interactions. The adjacency matrix is defined as $\mathcal{A} = [a_{(ij)}]$, with $a_{(ij)} > 0$ if $(v_j, v_i) \in \mathcal{E}$, and $a_{(ij)} = 0$ otherwise. The Laplacian matrix $\mathcal{L} = [L_{(ij)}] \in \mathbb{R}^{N \times N}$ is defined as $L_{(ii)} = \sum_{j \neq i} a_{(ij)}; L_{(ij)} = -a_{(ij)}$.

In the sequel, we consider that the network $\mathcal{G}$ is subdivided into m undirected subnetworks $\mathcal{C}_\tau, \forall \tau \in \{1, \cdots, m\}$ represented by the graphs $\mathcal{G}_1, \cdots, \mathcal{G}_m$ such that $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1), \cdots, \mathcal{G}_m = (\mathcal{V}_m, \mathcal{E}_m)$, where $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \ldots \cup \mathcal{V}_m$ and $V_\tau \cap V_g = \varnothing$ for all $\tau, g = 1, \cdots m, \tau \neq g$ and $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cdots \cup \mathcal{E}_m$. The communication graph of each subnetwork $\mathcal{G}_\tau$ is represented by a Laplacian matrix $\mathcal{L}_\tau$. Each cluster has a specific agent called the leader, and denoted in the following by $l_\tau \in \mathcal{V}_\tau, \forall \tau \in \{1, \cdots, m\}$. The remaining agents are called followers and are denoted by f_h. The set of leaders will be denoted by $\mathcal{I} = \{l_1, \cdots, l_m\}$. At particular time instant $t_k, k \in \mathbb{N}, t_k \geq 0$ of a time sequence $\{t_k\}$ that satisfies $t_1 < t_2 < \cdots, lim_{t_k \to \infty} t_k = \infty$, the leaders interact following a predefined interaction map $\mathcal{E}_l \subset \mathcal{I} \times \mathcal{I}$. The

leaders communication graph $\mathcal{G}_l = (\mathcal{I}, \mathcal{E}_l)$ is also supposed to be a undirected graph. Finally, without loss of generality, each leader is considered as the first agent of its cluster [30].

$$C_\tau = \{l_\tau, f_{m_{\tau-1}+2}, \cdots, f_{m_\tau}\}, \quad \forall \tau \in \{1, \cdots, m\}, \tag{1}$$

where $m_0 = 0, m_m = N$ and the cardinality of C_τ is given by $\|C_\tau\| = n_\tau = m_\tau - m_{\tau-1}, \forall \tau \geq 1$.

2.2. Network Dynamics

We consider a group of N linear identical agents that interact in m clusters. The dynamics of each agent i is described by

$$\dot{x}_i = Ax_i + Bu_i, \tag{2}$$

where $x_i = [x_{i,1}, \cdots, x_{i,n}]^T \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^p$ is the control input; $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{q\times n}$.

According to (1), we concede that each agent has a vector state denoted by $x_{l_\tau} = [x_{l_{\tau,1}}, \cdots, x_{l_{\tau,n}}]^T \in \mathbb{R}^n$ for the leaders l_τ and $x_{f_h} = [x_{f_{h,1}}, \cdots, x_{f_{h,n}}]^T \in \mathbb{R}^n$ for the followers $f_h, \forall \tau \neq h = 1, \cdots, N$. The desired state formation is denoted by $R = [r_1, r_2, \cdots, r_N] \in \mathbb{R}^{Nn}$, where $r_i \in \mathbb{R}^n, \forall i \in \mathcal{V}$ is a formation variable of agent i. Therefore, the desired state formation R is meaning that agents in a network need to achieve a reference coordinate frame corresponding to their positions and velocities.

However, in practice, the measurement part may have bounded nonlinearities or saturation constraints due to sensor limitations as well as physical limitations. Moreover, different agents inside each cluster may have different interactive capabilities. Therefore, the upper and lower bounds of saturation constraints may also be different. This leads to the heterogeneous asymmetric saturation levels.

Because only the leaders of these clusters can communicate together at some reset times t_k, a formation protocol for a clustered network with state constraints based on local information is given by

$$u_i = K \sum_{j=1}^{N} a_{(ij)} \left(\phi_j(z_j(t)) - \phi_i(z_i(t))\right) + Qr_i, t \in (t_k, t_{k+1}) \tag{3}$$

where $z_i(t) = x_i(t) - r_i$ denote the state formation variables of agent i, and $K, Q \in \mathbb{R}^{p\times n}$ denote the protocol gain matrices, respectively. The general continous functions $\phi_i(z_i) : \mathbb{R}^n \to \mathbb{R}^n, i = 1, \cdots, N$ satisfying the following sector-bounded conditions

$$(\phi_i(z_i(t)) - \Upsilon_1 z_i(t)) \circ (\phi_i(z_i(t)) - \Upsilon_2 z_i(t)) \leq 0 \ \forall z_i(t) \in \mathbb{R}^n, \tag{4}$$

where

$$\Upsilon_1 = diag\{\upsilon_{k,1}\}_{k=1,\cdots,n} \in \mathbb{R}^n,$$
$$\Upsilon_2 = diag\{\upsilon_{k,2}\}_{k=1,\cdots,n} \in \mathbb{R}^n$$

are matrices composed of known sector slopes $\upsilon_{k,1} < \upsilon_{k,2}$. An example of ϕ_i is the saturation function $s_i(\bullet)$ defined by

$$s_i(z_i(t)) \triangleq \begin{cases} \underline{s}_i : z_i(t) \leq \underline{s}_i, \\ z_i(t) : \underline{s}_i < z_i(t) < \overline{s}_i, \\ \overline{s}_i : z_i(t) \geq \overline{s}_i \end{cases} \tag{5}$$

where $\underline{s}_i \leq 0 \leq \overline{s}_i, \underline{s}_i < \overline{s}_i$ are known constants which are called saturation levels.

The exchanged information between leader's states at the reset time t_k can be described by

$$\Delta z_{l_i}(t_k) = \sum_{j=1}^{m} a_{l(ij)}(z_{l_j}(t_k) - z_{l_i}(t_k)), \quad t = t_k, \tag{6}$$

where $\Delta z_{l_i}(t_k) = z_{l_i}(t_k^+) - z_{l_i}(t_k^-)$, where $z_{l_i}(t_k^+)$ and $z_{l_i}(t_k^-)$ represent the right and left limit of z_{l_i} at t_k, respectively. Without loss of generality, we assume that $z_{l_i}(t_k^-) = z_{l_i}(t_k)$. Furthermore, $a_{l(i,j)}$ is the (i, j)th entry of the weighted adjacent matrix $\mathcal{A}_l = [a_{l(ij)}]$, and m denotes the set of leaders in graph $\mathcal{G}_l$. The Laplacian matrix $\mathcal{L}_l = [L_{l(ij)}] \in \mathbb{R}^{m\times m}$ is defined as $L_{l(ii)} = \sum_{j\neq i=1}^{m} a_{l(ij)}; L_{l(ij)} = -a_{l(ij)}$.

Then the collective dynamics of system (2) under the consensus protocol (3) and the interaction between leaders (6) can be rewritten as

$$\begin{aligned} &\dot{z}_i(t) = Az_i(t) - BK \sum_{j=1}^{N} L_{(ij)} w_j(t) + (A + BQ)r_i, t \in (t_k, t_{k+1}), \\ &\Delta z_{l_i}(t_k) = -\sum_{j=1}^{m} L_{l(ij)} z_{l_j}(t_k), \quad t = t_k, \\ &w_i(t) = \phi_i(z_i(t)). \end{aligned} \tag{7}$$

It can be seen that the evolution of the dynamic system described by (7) is influenced by the events that happen at the reset moments t_k. Thus, the evolution of the whole system can be viewed as a hybrid system that evolves as follows: during interval time (t_{k-1}, t_k), the interaction among nodes in each cluster C_τ is related only to the graph $\mathcal{G}_\tau$. Moreover, at each instant time t_k, the leaders update their states instantaneously according to the topology of $\mathcal{G}_l$, and thus, there exit jump phenomena in the states' leader, while the states' followers in a cluster are recalculated based on the new information's leader. The objective of the above collaboration is to achieve a specific global objective namely state formation defined as follows.

Definition 1. *The MASs (2) is said to be achieve state formation anticipated by R if there is vector $h(t) \in \mathbb{R}^n$ such that*

$$\lim_{t\to\infty}(x_i(t) - r_i - h(t)) = 0, \tag{8}$$

for any given bounded initial condition.

For the sake of clarity, an example, as depicted in Fig. 1, is given to illustrate this kind of problem. There are 6 agents divided into two clusters, red and black, that have to realize a hexagon formation. Each agent receives only the state information of its neighbors in the same subnetwork (the same color). If there is no communication between subnetworks (for example, agents 2 and 5 or agents 3 and 6), then the 6-agent network cannot achieve the desired formation. Therefore, in order to ensure the task of 6 agents, at some discrete-time instants, a communication between one red and one black agent (called leader 1 and leader 2, respectively) is activated.

Next, the following assumptions are utilized.

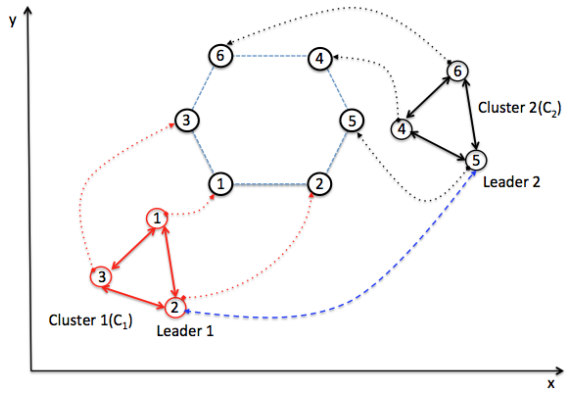

Figure 1: The scheme of formation problem

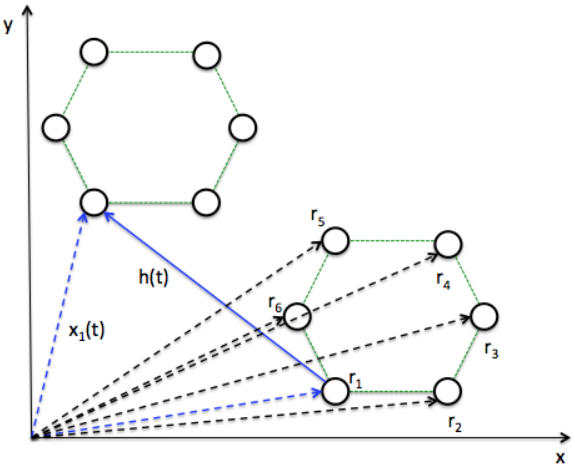

Figure 2: State formation of six agents moving in the XY plane

A1. The graphs $\mathcal{G}_\tau$ and $\mathcal{G}_l$ are undirected and connected.

A2. $a_{l(ij)} > 0, \sum_{j\neq i=1}^{m} a_{l(ij)} < 1$.

A3. The matrix pair (A, B) is stabilizable.

Remark 1. *Assumption **A1** is needed to guarantee that the Laplacian matrix $\mathcal{L}_\tau$ of $\mathcal{G}_\tau, \forall \tau \in \{1, \cdots, m\}$ satisfies the following proprieties $\mathcal{L}_\tau \mathbf{1}_{N_\tau} = 0, r_\tau^T \mathcal{L}_\tau = 0$ and $r_v^T \mathbf{1}_{N_\tau} = 1$, where $\mathbf{1}_{N_\tau}$, and $r_\tau^T = \frac{1}{N_\tau}\mathbf{1}_{N_\tau}$ are the right and left eigenvectors of $\mathcal{L}_\tau$ associated with zero eigenvalue, respectively. Assumption **A2** ensures the matrix $P_l = I - \mathcal{L}_l$ is a stochastic matrix with positive diagonal elements (more information is shown in (9)). Moreover, assumption **A3** is for the existence of a controller.*

Definition 2. *[24] Let F_p a nonnegative and square matrix whose row sums are all equal 1 (i.e., $F_p\mathbf{1} = \mathbf{1}$). Then, matrix F_p is called row stochastic.*

Lemma 1. *[25] Let Γ be a compact set consisting of $n \times n$ SIA matrices with the property that for any nonnegative integer k and any $B_1, \cdots, B_k \in \Gamma$ (repetitions permitted), $\prod_{i=1}^{k} B_i$ is SIA. Then, given any infinite sequence $B_1, B_2, \cdots$ (repetitions permitted), of matrices Γ, there exits a column vector c^T such that $\lim_{k\to\infty} \prod_{i=1}^{k} B_i = \mathbf{1}c^T$.*

Lemma 2. *[25] If $B = [b_{ij}]_{n\times n}$ is a stochastic matrix with positive diagonal elements, and the graph associated with B has a spanning tree, then B is SIA.*

Hereafter, the time index t is omitted in expressions of x_i, h and other variables just for conciseness of mathematical representations. Now, the considered problem in the current research is stated as follows.

□ **Hybrid Robust Formation Control Problem:** Consider a group of N agents with their auxiliary systems defined in (7), and suppose that assumptions **A1**–**A3** hold. Design the hybrid robust distributed formation control (3) such that the collective dynamics of the clustered MAS (7) satisfies *Definition 1*.

3. Robust Distributed Formation Design

In this section, we propose a solution for the above hybrid robust formation control problem with fixed topologies $\mathcal{G}_\tau$ and $\mathcal{G}_l$ satisfying assumption **A1**. The proposed design is composed of two steps. First, by employing results from matrix theory and algebraic graph theory, we show that the considered problem can be indirectly solved by the robust stability of an equivalent system. Then, the robust stability design of the equivalent system is derived in terms of LMIs.

3.1. Prerequisites

From the Geršgorin theorem [31], we know that $\lambda_m \leq 2d_{max}(\mathcal{G}_l)$, where λ_m is the largest eigenvalue of the Lalapcian of the graph $\mathcal{G}_l$, and $d_{max}(\mathcal{G}_l)$ is the maximum out-degree of the nodes of $\mathcal{G}_l$, where

$$\deg_{out}(v_{li}) = \sum_{i=1}^{m} a_{l(ij)}$$

Therefore, we can get $0 < \lambda_m < 2$. Let us introduce

$$\begin{aligned} P_{l(ij)} &= -L_{l(ij)} = a_{l(ij)} > 0, \\ P_{l(ii)} &= 1 - L_{l(ii)} = 1 - \sum_{j\neq i=1}^{m} a_{l(ij)} > 0, \end{aligned} \tag{9}$$

then $\sum_{j=1}^{m} P_{l(ij)} = 1$, and $P_l = I - \mathcal{L}_l$ is a row stochastic matrix with positive diagonal elements, and according to **A1** and **A2**, it has an eigenvalue $\lambda_1 = 1$ with algebraic multiplicity equal to one, and all the other eigenvalues satisfy $0 < |\lambda_i| < 1, i = 2, \cdots, m$.

Moreover, as mentioned above the network is subdivided into m undirected subnetworks. Then, $\mathcal{L} \in \mathbb{R}^{N\times N}$ stands for the Laplacian matrix associated with the graph $\mathcal{G}$, which has a diagonal form

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{L}_m \end{bmatrix}. \tag{10}$$

Now, some algebraic properties of $\mathcal{L}$ are presented in the following *Proposition*.

Proposition 1. *Let us consider a network of m clusters satisfying assumption **A1**, with the Laplacian $\mathcal{L} \in \mathbb{R}^{N\times N}$, then rank($\mathcal{L}$) = $N - m$ and $\mathcal{L}$ has m eigenvalues at zero and all the other $N - m$ eigenvalues of the Laplacian $\mathcal{L} \in \mathbb{R}^{N\times N}$ are positive.*

Proof. See in [29]. □

Next, it follows (9), the system (7) can be written in a form of the overall network dynamics

$$\begin{aligned} &\dot{z} = (I_N \otimes A)z - (\mathcal{L} \otimes BK)w + (I_N \otimes \mathbb{A})R, \ t \in (t_k, t_{k+1}) \\ &z_l(t_k^+) = (P_l \otimes I_n)z_l(t_k), \quad t = t_k \\ &w = \Phi(z), \end{aligned} \tag{11}$$

where $P_l = I - \mathcal{L}_l, \mathbb{A} = A + BQ, \Phi(z) = [\phi_1^T(z_1), \cdots, \phi_N^T(z_N)]^T$ and

$$\begin{aligned} &z = [z_{l_1}^T, z_{f_2}^T, \cdots, z_{f_{m_1}}^T, \cdots, z_{l_m}^T, \cdots, z_{f_{m_m}}^T = z_{f_N}^T]^T \in \mathbb{R}^{Nn}, \\ &z_l = [z_{l_1}^T, \cdots, z_{l_m}^T]^T \in \mathbb{R}^{mn}. \end{aligned}$$

containing respectively the states of agents and leader's states.

Let us introduce the extended stochastic matrix P_e as follows

$$P_e = \mathcal{M}^T \begin{bmatrix} P_l & 0 \\ 0 & I_{N-m} \end{bmatrix} \mathcal{M} \in \mathbb{R}^{N\times N}, \tag{12}$$

where $\mathcal{M}$ is a permutation matrix.

Then, the second equation in (11) can be expressed by

$$z(t_k^+) = (P_e \otimes I_n)z(t_k), \quad t = t_k.$$

In the following, let $U \in \mathbb{R}^{N\times N}$ be an orthogonal matrix, and employing *Proposition 1*, we obtain

$$\begin{aligned} &U^{-1}\mathcal{L}U = \begin{bmatrix} 0_m & 0 \\ 0 & \Gamma \end{bmatrix} = \Lambda \in \mathbb{R}^{(N)\times(N)}, \\ &\Gamma = diag\{\gamma_{m+1}, \cdots, \gamma_N\} \in \mathbb{R}^{(N-m)\times(N-m)}. \end{aligned} \tag{13}$$

Finally, let us also introduce the new variable

$$\psi = (U^{-1} \otimes I_n)z. \tag{14}$$

It follows the variable ψ in (14), we now formulate our statement as the following

$$\begin{aligned} &\dot{\psi} = (I_N \otimes A)\psi - (\Lambda U^{-1} \otimes BK)w + \mathcal{H}R, t \in (t_k, t_{k+1}) \\ &\psi(t_k^+) = (P_\psi \otimes I_n)\psi(t_k), \ \ t = t_k, \\ &z = (U \otimes I_n)\psi, \\ &w = \Phi(z), \end{aligned} \tag{15}$$

where $P_\psi = U^{-1}P_eU$ and $\mathcal{H} = (U^{-1} \otimes \mathbb{A})$.

In the next part of this paper, thanks to results from matrix theory and algebraic graph theory, we show that the robust formation control problem of MASs in clustered network (11) is indirectly solved by considering the robust stability of the system (15).

3.2. Formation Analysis in Clustered Network

In order to simplify the presentation of the next results let us partition the matrices U^{-1}, U into

$$U^{-1} = [U_3^T \ \ U_4^T]^T, U = [U_1 \ \ U_2], \tag{16}$$

where $U_3^T \in \mathbb{R}^{m\times N}, U_4^T \in \mathbb{R}^{(N-m)\times N}$ and $U_1 \in \mathbb{R}^{N\times m}$, $U_2 \in \mathbb{R}^{N\times(N-m)}$.

$$U_1 = \begin{bmatrix} \mathbf{1}_{N_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{1}_{N_m} \end{bmatrix}, U_3^T = \begin{bmatrix} r_1^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_m^T \end{bmatrix}, \tag{17}$$

which satisfies $\mathcal{L}U_1 = 0_{N\times n}, U_3^T\mathcal{L} = 0_{m\times N}$. This allows to decompose (14) into two parts :

$$\psi_1 = (U_3^T \otimes I_n)z, \quad \psi_2 = (U_4^T \otimes I_n)z, \tag{18}$$

where $\psi_1 \in \mathbb{R}^{mn}$ and $\psi_2 \in \mathbb{R}^{Nn-mn}$. Now we are able to introduce the first main results of this paper.

Theorem 1. *Consider the overall network dynamics system (15) satisfying **A1–A3** the hybrid robust formation control problem is solved if the following formation feasibility condition holds*

$$(A + BQ)(r_i - r_j) = 0, \quad \forall i, j = 1, \cdots, N. \tag{19}$$

and

$$\lim_{t\to\infty} \psi_2 \to 0 \tag{20}$$

for any given bounded initial conditions.

Proof. If the condition (19) holds, then one has that

$$[\mathcal{L} \otimes (A + BQ)]R = 0. \tag{21}$$

Pre-multiplying both sides of (21) with $(U^{-1} \otimes I_n)$ yields

$$[\Lambda U^{-1} \otimes (A + BQ)]R = 0. \tag{22}$$

Then pre-multiplying both the sides of (22) with

$$\begin{bmatrix} 0_m & 0 \\ 0 & \Gamma^{-1} \end{bmatrix} \otimes I_n$$

gives us $[U^{-1} \otimes (A + BQ)]R = 0$, which is equivalent to $\mathcal{H}R = 0$. Therefore, the system (15) leads to the following system

$$\begin{aligned} &\dot{\psi} = (I_N \otimes A)\psi - (\Lambda U^{-1} \otimes BK)w, t \in (t_k, t_{k+1}) \\ &\psi(t_k^+) = (P_\psi \otimes I_n)\psi(t_k), \ \ t = t_k, \\ &z = (U \otimes I_n)\psi, \\ &w = \Phi(z). \end{aligned} \tag{23}$$

In the following, we firstly show that ψ_1 reaches a constant value, which depends on the dynamics of agents, the graph of each cluster, the interaction between leaders, and the initial conditions. Second, based on the analysis of the first step, the hybrid robust formation control problem, satisfying *Definition 1*, is solved.

First, by employing (13), (18) and (23), the dynamics of ψ_1 can be represented as:

$$\begin{aligned}\dot{\psi}_1 &= (I_m \otimes A)\psi_1, \\ \psi_1(t_k^+) &= (U_3^T P_e U_1 \otimes I_n)\psi_1(t_k) + (U_3^T P_e U_2 \otimes I_n)\psi_2(t_k).\end{aligned} \tag{24}$$

Then, the solution of (24) with initial condition $\psi_1(t_0) = \psi_{10}$ can be obtained by

$$\psi_1 = e^{(I_m \otimes A)(t-t_k)}\psi_1(t_k^+) = (I_m \otimes e^{A(t-t_k)})\psi_1(t_k^+), \tag{25}$$

and if $\lim_{t\to\infty} \psi_2 \to 0$, then $\psi_1(t_k^+)$ can be expressed as:

$$\begin{aligned}\psi_1(t_k^+) &= \lim_{k\to\infty} \prod_{i=1}^{k} (U_3^T P_e U_1 \otimes I_n) e^{(I_m \otimes A)(t_i - t_{i-1})}\psi_1(t_0) \\ &= \lim_{k\to\infty} (U_3^T P_e U_1)^k \otimes e^{A(t_k - t_0)}\psi_1(t_0).\end{aligned} \tag{26}$$

In the following, by using results from *Lemma 1* and *Lemma 2*, we prove that matrix $U_3^T P_e U_1 \in \mathbb{R}^{m\times m}$ is a row stochastic matrix with positive diagonal elements.

According to (9), the extended stochastic matrix P_e in (12) can be re-expressed as follows

$$P_e = \begin{bmatrix} P_{l_{11}} & 0 & \cdots & P_{l_{1m}} & 0 \\ 0 & I_{N_1-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{l_{m1}} & 0 & \cdots & P_{l_{mm}} & 0 \\ 0 & 0 & \cdots & 0 & I_{N_m-1} \end{bmatrix} \in \mathbb{R}^{N\times N}, \tag{27}$$

where

$$P_l = \begin{bmatrix} P_{l_{11}} & \cdots & P_{l_{1m}} \\ \vdots & \vdots & \vdots \\ P_{l_{m1}} & \cdots & P_{l_{mm}} \end{bmatrix} = \begin{bmatrix} P_{l_1} \\ \vdots \\ P_{l_m} \end{bmatrix} \in \mathbb{R}^{m\times m}, \tag{28}$$

and the matrix $U_1 \in \mathbb{R}^{N\times m}$ and $U_3^T \in \mathbb{R}^{m\times N}$ are given in (17). Now, the matrix $P_e U_1$ is calculated as

$$P_e U_1 = \begin{bmatrix} P_{l_{11}} \cdots & P_{l_{1m}} \\ \mathbf{1}_{N_1-1} & 0_{(N_1-1)\times(N_1-1)} \\ \vdots & \vdots \\ P_{l_{m1}} \cdots & P_{l_{mm}} \\ \mathbf{1}_{N_m-1} & 0_{(N_m-1)\times(N_m-1)} \end{bmatrix} = \begin{bmatrix} E_1 \\ \vdots \\ E_m \end{bmatrix}, \tag{29}$$

where $\forall\tau \in \{1, \cdots, m\}$, and

$$E_\tau = \begin{bmatrix} P_{l_{\tau 1}} \cdots & P_{l_{\tau m}} \\ \mathbf{1}_{N_\tau - 1} & 0_{(N_\tau-1)\times(N_\tau-1)} \end{bmatrix} \in \mathbb{R}^{N_\tau \times m}. \tag{30}$$

Then, the matrix $U_3^T P_e U_1$ is determined as follows

$$U_3^T P_e U_1 = \begin{bmatrix} r_1^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_m^T \end{bmatrix} \begin{bmatrix} E_1 \\ \vdots \\ E_m \end{bmatrix} = \begin{bmatrix} r_1^T E_1 \\ \vdots \\ r_m^T E_m \end{bmatrix}, \tag{31}$$

where $r_\tau^T = [r_{\tau_1}, \cdots, r_{\tau_{N_\tau}}] \in \mathbb{R}^{1\times N_\tau}$, $\forall\tau \in \{1, \cdots, m\}$, and

$$r_\tau^T E_\tau = [r_{\tau_1} P_{l_{\tau 1}} + r_{\tau_2} + \cdots + r_{\tau_{N_\tau}}, r_{\tau_1} P_{l_{\tau 2}}, \cdots, r_{\tau_1} P_{l_{\tau m}}].$$

The sum of the row matrix $r_\tau^T E_\tau$ is calculated by

$$\sum_{k=1}^{N_\tau} r_\tau^T E_\tau = r_{\tau_1} P_{l_{\tau 1}} + \sum_{k=2}^{N_\tau} r_{\tau_k} + r_{\tau_1} \sum_{k=2}^{N_\tau} P_{l_{\tau k}}. \tag{32}$$

According to **A1, A2** and (9), $P_{l_{\tau 1}} = 1 - \sum_{k=2}^{m} P_{l_{\tau k}}$, then

$$\sum_{k=1}^{N_\tau} r_\tau^T E_\tau = \sum_{k=1}^{N_\tau} r_{\tau_k} = 1, \tag{33}$$

and $P_{l_{\tau k}} > 0$, then

$$\begin{aligned} r_{\tau_1} P_{l_{\tau 1}} + r_{\tau_2} + \cdots + r_{\tau_{N_\tau}} > 0, \\ r_{\tau_1} P_{l_{\tau 2}} > 0, \cdots, \\ r_{\tau_1} P_{l_{\tau m}} > 0. \end{aligned} \tag{34}$$

Subsequently, by employing (33) and (34), and according to *Definition 2*, we see that the matrix $U_3^T P_e U_1 \in \mathbb{R}^{m\times m}$ is a row stochastic matrix with positive diagonal elements.

Furthermore, by employing (9), and $\forall i, \tau \in \{1, \cdots, m\}$ Eq. (34) becomes

$$\begin{aligned} 1 + r_{\tau_1}(P_{l_{\tau 1}} - 1) = 1 - r_{\tau_1} \sum_{i\neq j=1}^{m} a_{l(ij)}, \\ r_{\tau_1} P_{l_{\tau 2}} = r_{\tau_1} a_{l_{i2}}, \\ \cdots \\ r_{\tau_1} P_{l_{\tau m}} = r_{\tau_1} a_{l_{im}}. \end{aligned} \tag{35}$$

then the $(i, j)^{th}$ entry of $U_3^T P_e U_1$ is $r_{\tau_1} a_{l(ij)}$, which implies that the graph $\mathcal{G}_l$ and the graph of $U_3^T P_e U_1$ have the same edge set. Thus, the graph of the matrix $U_3^T P_e U_1$ is undirected and connected. It means that the graphs of $U_3^T P_e U_1$ has at least one spanning tree.

Based on the above analysis, we showed that the matrix $U_3^T P_e U_1$ is a row stochastic matrix with positive diagonal elements and its graph has at least one spanning tree. Then, according to *Lemma 2*, the matrix $U_3^T P_e U_1$ is SIA.

Therefore, from *Lemma 1*, there exits a column vector c^T such that

$$\lim_{k\to\infty} (U_3^T P_e U_1)^k = \mathbf{1}_m c^T. \tag{36}$$

Then, by substituing (36) and (26) into (25), one has

$$\psi_1 = \mathbf{1}_m c^T \otimes e^{A(t-t_0)}\psi_{10}. \tag{37}$$

Second, by introducing the variables

$$\mu_1 = (U \otimes I_n)\begin{bmatrix}\psi_1 \\ 0\end{bmatrix}, \mu_2 = (U \otimes I_n)\begin{bmatrix}0 \\ \psi_2\end{bmatrix}, \tag{38}$$

one has $z = \mu_1 + \mu_2$. Then, according to (38) the variable μ_1 is written such as

$$\mu_1 = [U_1 \otimes I_n \; U_2 \otimes I_n] \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} = (U_1 \otimes I_n)\psi_1. \tag{39}$$

It follows that $\mu_2 = z - \mu_1$, where $\mu_2 = (U_2 \otimes I_n)\psi_2$. And, by using (37) and (39), we obtain

$$\begin{aligned}\mu_1 &= U_1 \mathbf{1}_m c^T \otimes e^{A(t-t_0)}\psi_{10}, \\ &= \mathbf{1}_N c^T \otimes e^{A(t-t_0)}\psi_{10}. \end{aligned} \tag{40}$$

If it follows $\lim_{t\to\infty}\psi_2 \to 0$ and recalling that $(U \otimes I_n)$ is non-singular, then follows from (38) that $\lim_{t\to\infty}\mu_2 \to 0$. Finally, it follows from (40) that

$$\begin{aligned}\lim_{t\to\infty}\mu_2 &= \lim_{t\to\infty}(z-\mu_1) \\ &= \lim_{t\to\infty}(x - R - \mathbf{1}_N \underbrace{c^T \otimes e^{A(t-t_0)}\psi_{10}}_{h(t)}) \to 0, \\ &= \lim_{t\to\infty}(x_i - r_i - \underbrace{c^T \otimes e^{A(t-t_0)}\psi_{10}}_{h(t)}) \to 0, \end{aligned} \tag{41}$$

which implies that the system (11) can achieve state formation anticipated by R, meaning the hybrid robust formation control problem was solved. This completes the proof. □

Remark 2. *$h(t)$ in* (8) *generally can be used to guide a group of agents to achieve an anticipated formation specified by R as shown in Fig.2 and it is considered as the formation position function. Moreover, the formation function $h(t)$ in considered clustered network is also described as* (41)*, which depend on agents initial states and formation vector, agent's dynamics, communication networks' cluster and leaders.*

Remark 3. *According to Thorem 1, one sees that to ensure the state formation R, not only the communication topology is required to be connected and the Laplacian matrix is a symmetric matrix, but also the formation vector should satisfy the constraint* (19) *. Therefore, Theorem 1 establishes the relationship between the formability and the communication topology, the agents' dynamics and the formation vector.*

3.3. Robust Stabilization Controller Design

Based on the above analysis in Subsection B, the objective now is to design the matrix $K \in \mathbb{R}^{p\times n}$, such that the system (15) is robustly stable, i.e., $\lim_{t\to\infty}\psi_2 \to 0$. The design of such robust stabilization controller gain K is given in the following theorem.

Theorem 2. *Consider the system (15) satisfying assumptions **A1–A3** and condition (19). It is robustly stable if there exist positive-definite and diagonal matrices $P, \Pi, Z \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{p\times n}$ such that the following LMIs are feasible,*

$$\begin{bmatrix} \Xi_1 & \gamma_2 BX + \frac{(\Upsilon_1-\Upsilon_2)}{2}Z \\ * & -Z \end{bmatrix} \le 0, \tag{42}$$

$$\begin{bmatrix} \Xi_2 & -\gamma_N BX + \frac{(\Upsilon_1-\Upsilon_2)}{2}Z \\ * & -Z \end{bmatrix} \le 0, \tag{43}$$

$$\begin{bmatrix} Z & P \\ P & \Pi^{-1} \end{bmatrix} \ge 0, \tag{44}$$

where

$$\begin{aligned}\Xi_1 &= sym(AP + \gamma_2 BX\Upsilon_2) + \alpha P,\ \gamma_2 = min\{\Gamma\}, \\ \Xi_2 &= sym(AP + \gamma_N BX\Upsilon_2) + \alpha P,\ \gamma_N = max\{\Gamma\}.\end{aligned}$$

Furthermore, $K = XP^{-1}$.

Proof. Firstly, we define

$$V = V(\psi) = \psi^T(\Theta \otimes P^{-1})\psi, \tag{45}$$

where

$$P = P^T > 0, \Theta = \begin{bmatrix} 0_m & 0 \\ 0 & I_{N-m} \end{bmatrix}.$$

Obviously, V is positive semi-definite. If we can prove that between impulses t_k and t_{k+1}, the function V is decreasing

$$\dot{V} < 0, \quad \forall t \in (t_k, t_{k+1}), \tag{46}$$

where

$$\begin{aligned}\dot{V} =& \psi^T[\Theta \otimes (A^T P^{-1} + P^{-1}A)]\psi - \\ &- \psi^T[(\Theta\Lambda U^{-1} + U\Lambda\Theta) \otimes P^{-1}BK]w. \end{aligned} \tag{47}$$

and at reset time t_k

$$V(t_k) \ge V(t_k^+). \tag{48}$$

Then according to the Lasalle's invariance principle, $\psi(t)$ globally exponentially converges to the largest invariance set contained in $\{\psi \in \mathbb{R}^{Nn} | \dot{V}(\psi) = 0\}$ for any initial conditions. It can be seen from (47) and definition of matrix Θ in (45) that $\dot{V}(\psi) = 0$ if and only if $\lim_{t\to\infty}\psi_2 \to 0$, where $\psi_2 = [\psi_{m+1}, \cdots, \psi_N]^T \in \mathbb{R}^{Nn-mn}$.

In the following, the condition (46) is equivalent to $\dot{V} + \alpha V \le 0$, where $\alpha > 0$. Thus, Eq. (47) becomes

$$\begin{aligned}\dot{V} + \alpha V =& \psi^T[\Theta \otimes (A^T P^{-1} + P^{-1}A + \alpha P^{-1})]\psi - \\ &- \psi^T[(\Theta\Lambda U^{-1} + U\Lambda\Theta) \otimes P^{-1}BK]w \le 0. \end{aligned} \tag{49}$$

Then, using the *S-procedure* and the sector-bounded conditions (4), there exits a diagonal matrix $\Pi \in \mathbb{R}^{n\times n}, \Pi \ge 0$, such that

$$\begin{aligned}&\dot{V} + \alpha V - \sum_{j=1}^{N}(w_j - \Upsilon_1 z_j)^T \Pi (w_j - \Upsilon_2 z_j) \le 0. \\ \Leftrightarrow &\dot{V} + \alpha V - w^T(I_N \otimes \Pi)w - z^T(I_N \otimes \Pi\Upsilon_1\Upsilon_2)z + \\ &+ w^T[I_N \otimes \Pi(\Upsilon_1 + \Upsilon_2)]z \le 0. \end{aligned} \tag{50}$$

Then, with $z = (U \otimes I_n)\psi, U^T = U^{-1}$ E.q (50) becomes

$$\begin{aligned}&\dot{V} + \alpha V - w^T(I_N \otimes \Pi)w - \psi^T(I_N \otimes \Pi\Upsilon_1\Upsilon_2)\psi + \\ &+ w^T[U \otimes \Pi(\Upsilon_1 + \Upsilon_2)]\psi \le 0. \\ &\Leftrightarrow \begin{bmatrix} \psi \\ w \end{bmatrix}^T \begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_3 & \Psi_4 \end{bmatrix} \begin{bmatrix} \psi \\ w \end{bmatrix} \le 0. \\ &\Leftrightarrow \begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_3 & \Psi_4 \end{bmatrix} \le 0. \end{aligned} \tag{51}$$

where

$$\Psi_1 = \Theta \otimes (A^T P^{-1} + P^{-1}A + \alpha P^{-1}) - I_N \otimes \Pi\Upsilon_1\Upsilon_2,$$
$$\Psi_2 = -\Theta\Lambda U^{-1} \otimes P^{-1}BK + \frac{1}{2}U^T \otimes \Pi(\Upsilon_1 + \Upsilon_2),$$
$$\Psi_3 = -U\Lambda\Theta \otimes K^T B^T P^{-1} + \frac{1}{2}U \otimes \Pi(\Upsilon_1 + \Upsilon_2),$$
$$\Psi_4 = -I_N \otimes \Pi.$$

Subsequently, taking the Schur complement to (51) results in $\Psi_1 - \Psi_2\Psi_4^{-1}\Psi_3 \leq 0$, in which

$$\begin{aligned} -\Psi_2\Psi_4^{-1}\Psi_3 &= \Theta\Lambda^2\Theta \otimes P^{-1}BK\Pi^{-1}K^T B^T P^{-1} + \\ &+ \frac{1}{4}I_N \otimes \Pi(\Upsilon_1 + \Upsilon_2)^2 + \frac{1}{2}\Lambda\Theta \otimes (\Upsilon_1 + \Upsilon_2)K^T B^T P^{-1} + \\ &+ \frac{1}{2}\Theta\Lambda \otimes P^{-1}BK(\Upsilon_1 + \Upsilon_2). \end{aligned}$$

Next, by considering $\Psi_1 - \Psi_2\Psi_4^{-1}\Psi_3 \leq 0$, and since Λ, and Γ in (13) are diagonal, one obtains

$$\begin{aligned} &(A^T P^{-1} + P^{-1}A + \alpha P^{-1}) - \Pi\Upsilon_1\Upsilon_2 + \\ &+ \gamma_k^2 P^{-1}BK\Pi^{-1}K^T B^T P^{-1} + \frac{1}{4}\Pi(\Upsilon_1 + \Upsilon_2)^2 + \\ &+ \frac{1}{2}\gamma_k(\Upsilon_1 + \Upsilon_2)K^T B^T P^{-1} + \frac{1}{2}\gamma_k P^{-1}BK(\Upsilon_1 + \Upsilon_2) \leq 0, \end{aligned} \tag{52}$$

where $\gamma_k, k = m+1, \cdots, N$. After, multiplying both sides (52) with P, we get

$$\begin{aligned} &PA^T + AP + \alpha P + \frac{1}{4}P\Pi(\Upsilon_1 - \Upsilon_2)^2 P + \\ &+ \gamma_k^2 BK\Pi^{-1}K^T B^T + \frac{1}{2}\gamma_k P(\Upsilon_1 + \Upsilon_2)K^T B^T + \\ &+ \frac{1}{2}\gamma_k BK(\Upsilon_1 + \Upsilon_2)P \leq 0. \end{aligned} \tag{53}$$

Since $P, \Upsilon_1, \Upsilon_2$ are diagonal matrices, $P\Upsilon_1 = \Upsilon_1 P, P\Upsilon_2 = \Upsilon_2 P$, and (53) is equivalent to

$$\begin{aligned} &PA^T + AP + \alpha P + \gamma_k\Upsilon_2 PK^T B^T + \gamma_k BKP\Upsilon_2 + \\ &+ \frac{1}{4}P\Pi(\Upsilon_1 - \Upsilon_2)^2 P + \gamma_k^2 BK\Pi^{-1}K^T B^T + \\ &+ \frac{1}{2}\gamma_k P(\Upsilon_1 - \Upsilon_2)K^T B^T + \frac{1}{2}\gamma_k BK(\Upsilon_1 - \Upsilon_2)P \leq 0. \end{aligned} \tag{54}$$

It leads to

$$\begin{aligned} &sym(AP + \gamma_k BKP\Upsilon_2) + \alpha P + \\ &+ [\gamma_k BKP + \frac{1}{2}(\Upsilon_1 - \Upsilon_2)\Pi P^2] \times \\ &\times (\Pi P^2)^{-1}[\gamma_k BKP + \frac{1}{2}(\Upsilon_1 - \Upsilon_2)\Pi P^2]^T \leq 0. \end{aligned} \tag{55}$$

Taking $Z \geq \Pi P^2, Z > 0$ and $K = XP^{-1}$. Then, using the Schur complement again with (55) leads to

$$\begin{bmatrix} \Xi & \gamma_k BX + \frac{(\Upsilon_1 - \Upsilon_2)}{2}Z \\ * & -Z \end{bmatrix} \leq 0, \tag{56}$$

where $\Xi = sym(AP + \gamma_k BX\Upsilon_2) + \alpha P$ and $\gamma_k, k = m+1, \cdots, N$ are eigenvalues of Laplacian matrix $\mathcal{L}$.

Since $\gamma_2 = min\{\Gamma\}, \gamma_N = max\{\Gamma\}$ and $\gamma_2 \leq \gamma_{m+1} \leq \cdots \leq \lambda_N$, we can represent $\gamma_p, p = m+2, \cdots, N-1$ as convex combination of γ_2 and γ_N. Thus, we derive (42) and (43). The LMI (44) is obtained straigthforward from $Z \geq \Pi P^2, Z > 0$.

On the other hand, at the reset time $t = t_k$ one has

$$\begin{aligned} &V(\psi(t_k^+)) - V(\psi(t_k)) \\ &= \psi(t_k)^T[(P_\psi \otimes I_n)^T(\Theta \otimes P^{-1}(P_\psi \otimes I_n) - \\ &\quad - (\Theta \otimes P^{-1})]\psi(t_k). \end{aligned}$$

Then, to guarantee the second condition (48), one needs

$$(P_\psi \otimes I_n)^T(\Theta \otimes P^{-1})(P_\psi \otimes I_n) - (\Theta \otimes P^{-1}) \leq 0. \tag{57}$$

by multiplying both sides Eq. (57) by $(I_N \otimes P) > 0$, we obtain

$$(P_\psi \otimes I_n)^T(\Theta \otimes P)(P_\psi \otimes I_n) - (\Theta \otimes P) \leq 0. \tag{58}$$

by employing $P_\psi = U^{-1}P_e U, U^{-1} = U^T$ and using (12), and (9), it is easy to verify that $P_e^T\Theta P_e - \Theta \leq 0$. Thus, the condition (48) is always true. □

Remark 4. *In case of homogeneous constraints, the upper and lower sectors and bounds for state constraints of all agents are the same, then Υ_1 and Υ_2 are multiple of identity matrices, i.e., $\Upsilon_1 = \upsilon_1 I_n$ and $\Upsilon_2 = \upsilon_2 I_n$. Then, the variable $P \in \mathbb{R}^{n\times n}$ in Theorem 2 is not required to be diagonal. Thus, the associated LMI problem is less conservative and its feasibility would be improved.*

Remark 5. *According to the LMIs (42)–(44), one sees that the dimension of variables $P \in \mathbb{R}^{n\times n}, X \in \mathbb{R}^{p\times n}$ are just equal to that of the matrix $A \in \mathbb{R}^{n\times n}$ of each agent. Thus, the complexity of those LMI problems is low. If γ_2, γ_N are computed by a given Laplacian matrix $\mathcal{L}$ with respect to graph $\mathcal{G}$, then we can solve LMIs (42)–(44) in fully distributed fashion i.e., each agent can compute the gain matrix K by itself and implement the consensus protocol (3) using only local information (its information and its neighbors' information).*

4. Application to Formation of UAVs

In this section, we consisder a group of N UAV's motion in $d-$dimensional Euclidean space, which is modeled as the second order dynamics in [32], where the state variable consists of the configuration states (position-p_i^x, p_i^y) and their derivatives (velocity-v_{x_i}, v_{y_i}), the control input $u_i \in \mathbb{R}^p$ denotes the acceleration commands. Finally, the system matrices are given such as

$$A = I_d \otimes \begin{bmatrix} 0 & 1 \\ a_{21}^1 & a_{22}^1 \end{bmatrix}, B = I_d \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{59}$$

In order to illustrate that the proposed approaches are implemented in the complex network, we consider the following

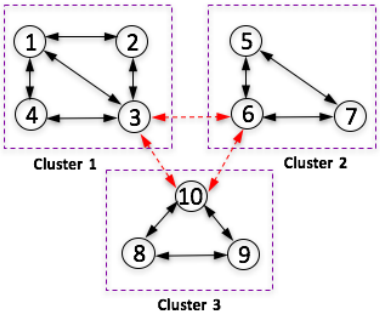

Figure 3: The communication of the network

network which has ten UAVs, and the network is divided into three clusters in Fig. 3.

The dynamics of the agents and the Laplacian matrix of leader network $\mathcal{G}_l$ are given by

$$A = I_2 \otimes \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, B = I_2 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\mathcal{L}_l = \begin{bmatrix} 0.9 & -0.4 & -0.5 \\ -0.4 & 0.7 & -0.3 \\ -0.5 & -0.3 & 0.8 \end{bmatrix} \Rightarrow P_l = \begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.4 & 0.3 & 0.3 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}.$$

In this example, we suppose that agents 3, 6 and 10 have more communication capability than other agents. Thus, we choose them as leaders of clusters 1, 2 and 3, respectively.

The formation considered this time is pentacle. Then, the formation specified by $R(m)$ are given by

$$r_1 = [0\ 4\sqrt{3}], r_2 = [2\ 2\sqrt{3}], r_3 = [4\sqrt{3}\ 2\sqrt{3}], r_4 = [2\sqrt{3}\ 0],$$
$$r_5 = [4\ -4\sqrt{3}], r_6 = [0\ -2\sqrt{3}], r_7 = [-4\ -4\sqrt{3}],$$
$$r_8 = [-2\sqrt{3}\ 0], r_9 = [-4\sqrt{3}\ 2\sqrt{3}], r_{10} = [-2\ \ 2\sqrt{3}].$$

and $R = [r_1\ r_2\ r_3\ r_4\ r_5\ r_6\ r_7\ r_8\ r_9\ r_{10}]^T \otimes [1\ 0]^T$. It is clear from R that

$$\left[\begin{bmatrix} r_1 \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} r_{10} \\ 0 \end{bmatrix}\right]^T = \left[\begin{bmatrix} r_1^x & r_1^y \\ 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} r_{10}^x & r_{10}^y \\ 0 & 0 \end{bmatrix}\right]^T.$$

It means that two scenarios related to UAV's positions and velocities are taken into account: ten UAVs will be controlled to reach a regular pentacle formation in the 2D plane corresponding to their positions, and all of ten UAV's velocites will be achieve a common value such as

$$\lim_{t\to\infty} \|(p_j - p_i) - (r_j - r_i)\| = 0,$$
$$\lim_{t\to\infty} \|(v_j - v_i)\| = 0.$$

Each cluster is color coded, where the first cluster is red, second is blue and third is black.

4.1. Heterogeneous constraints

Due to a limited range of sensor, and the limited velocity of each UAV, the states of connected agents are bounded i.e., the

Table 1: The sate constraints of ten agents

1st state	$\underline{s}_j(m)$	$\overline{s}_j$	2nd state	$\underline{s}_j$	$\overline{s}_j(m/s)$
p_1^x, p_1^y	-5	15	v_{x1}, v_{y1}	-2	0
p_2^x, p_2^y	-10	0	v_{x2}, v_{y2}	-1	3
p_3^x, p_3^y	-19	15	v_{x3}, v_{y3}	-1.3	3.1
p_4^x, p_4^y	-12	5	v_{x4}, v_{y4}	-1.8	2
p_5^x, p_5^y	-17	12	v_{x5}, v_{y5}	0	3
p_6^x, p_6^y	-20	10	v_{x6}, v_{y6}	-4	5
p_7^x, p_7^y	-18	7	v_{x7}, v_{y7}	-3.7	1.2
p_8^x, p_8^y	-19	11	v_{x8}, v_{y8}	-4	3
p_9^x, p_9^y	-22	15	v_{x9}, v_{y9}	-1.5	1.3
p_{10}^x, p_{10}^y	-10	15	v_{x10}, v_{y10}	-3	3

state constraints are the saturation function (4) with heterogeneous constraints such as Table 1, and the initial conditions of three clusters are randomized.

In $t \in (t_k,\ t_{k+1})$, the control protocol now is

$$u_i = K \sum_{j=1}^{N} a_{(ij)} \begin{bmatrix} \phi_j(z_j^{p_x}) - \phi_i(z_i^{p_x}) & \phi_j(z_j^{p_y}) - \phi_i(z_i^{p_y}) \\ \phi_j(v_j^x) - \phi_i(v_i^x) & \phi_j(v_j^y) - \phi_i(v_i^y) \end{bmatrix} + Q \begin{bmatrix} r_i^x & r_i^y \\ 0 & 0 \end{bmatrix}, \quad (60)$$

where $z_i^{p_x} = p_i^x - r_i^x; z_i^{p_y} = p_i^y - r_i^y$. Then, employing (60), the dynamics of agent i can be reformulated as

$$\begin{bmatrix} \dot{p}_i^x & \dot{p}_i^y \\ \dot{v}_i^x & \dot{v}_i^y \end{bmatrix} = \begin{bmatrix} v_i^x & v_i^y \\ -v_i^x & -v_i^y \end{bmatrix} + Q \begin{bmatrix} r_i^x & r_i^y \\ 0 & 0 \end{bmatrix} + BK \sum_{j=1}^{N} a_{(ij)} \begin{bmatrix} \phi_j(z_j^{p_x}) - \phi_i(z_i^{p_x}) & \phi_j(z_j^{p_y}) - \phi_i(z_i^{p_y}) \\ \phi_j(v_j^x) - \phi_i(v_i^x) & \phi_j(v_j^y) - \phi_i(v_i^y) \end{bmatrix}.$$

At $t = t_k$, the interaction of leaders can expressed as

$$x_l(t_k^+) = (P_l \otimes I_2)x_l(t_k) + r_l - (P_l \otimes I_2)r_l,$$

where $l = 3, 6, 10$, and

$$x_{l_i} = \begin{bmatrix} p_{l_i}^x & p_{l_i}^y \\ v_{l_i}^x & v_{l_i}^y \end{bmatrix}, x_l = \begin{bmatrix} x_{l_1} \\ x_{l_2} \\ x_{l_3} \end{bmatrix}, \begin{array}{l} x_{l_1} = x_3, \\ x_{l_2} = x_6, \\ x_{l_3} = x_{10}. \end{array}$$

Choosing $Q = [0\ 1]$ to satisfy *Theorem 1*, and $c = 0.1, \Upsilon_1 = diag\{0, 0.1\}, \Upsilon_2 = diag\{0.1, 0.2\}$. Then, solving LMIs in *Theorem 2*, one has the feedback matrix $K = [-5.6625\ \ -6.0109]$.

The interaction among leaders occurs at some instant times. These are defined based on some events or particularly demand of systems, which is decided by an operator. In our simulation, we assume that the reset time of the leader's communication is periodic and it happens at each second as depicted in Fig. 4 (upper). The evolution of leaders' states is also depicted in Fig. 4. It is clear that leaders' values are updated at the reset time t_k through the communication graph $\mathcal{G}_l$.

Convergence of both the variable ψ_2 in (18) and the Lyapunov function (45) is depicted in Fig. 5. It again verifies that the clustered network achieves formation under robust formation control protocol (3) if $\lim_{t\to\infty} \psi_2 \to 0$, shown in *Theorem*

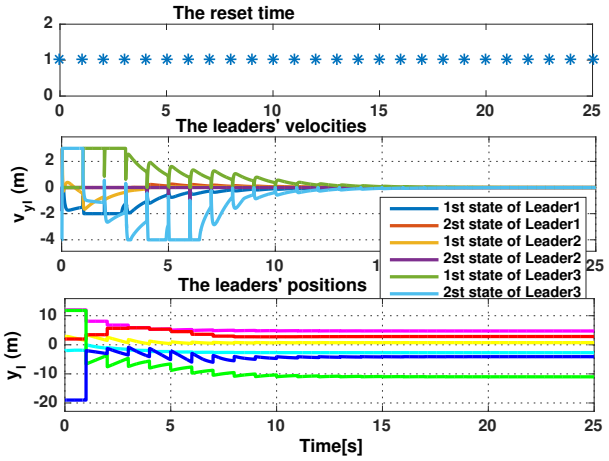

Figure 4: A reset signal and leaders' states of UAVs

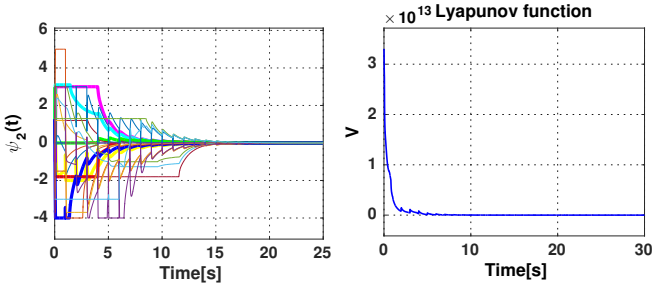

Figure 5: Convergence of the variable ψ_2 and the Lyapunov function.

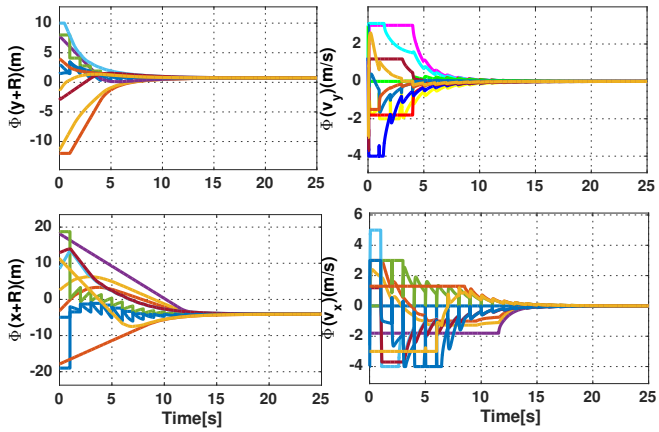

Figure 6: Ten-UAVs' positions (x_i, y_j) constraints (left) and ten-UAVs'velocities (v_{x_i}, v_{y_i}) constraints (right).

1, and the Lyapunov function satisfies the condtions (46) and (48). Moreover, according to the simulation results depicted respectively in Fig. 6, and Fig. 7, we see that all positions and velocities of UAVs in clustered network reach the pentacle formation and consensus under the state constraints.

4.2. Homogeneous constraints

Finally, in oder to investigate the influence of state constraints on the formation performance, we have carried out two

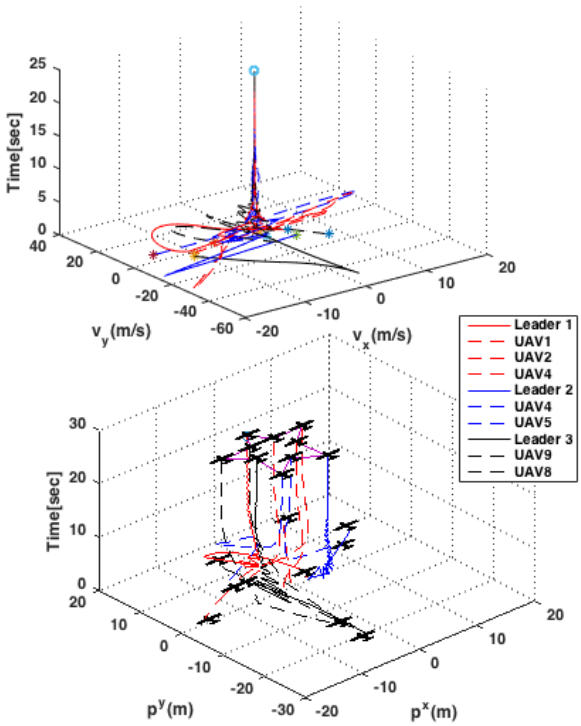

Figure 7: Pentacle formation of ten UAVs'positions (x_i, y_i) (lower) and consensus of ten UAVs'velocities (v_{x_i}, v_{y_i}) under state constraints (upper).

simulations. In the first simulation, we consider constraints on positions that belong to $[-15\ 15](m)$ and we suppose that the speed shoud be in the interval $[-15\ 8](m/s)$. The second simuation suppose that the constraints on positions belong to $[-3\ 3](m)$ and that speed should be in $[-3\ 3](m/s)$. The obtained simulation resluts are depicted in Fig.8 and Fig. 9. In both of cases, one can see that agents achieve and keep the desired formation and that the values of state variables are in the defined region. Moreover, one can remark that when the constraints on the speed are stricter (the second case) the achivement of formation takes more time.

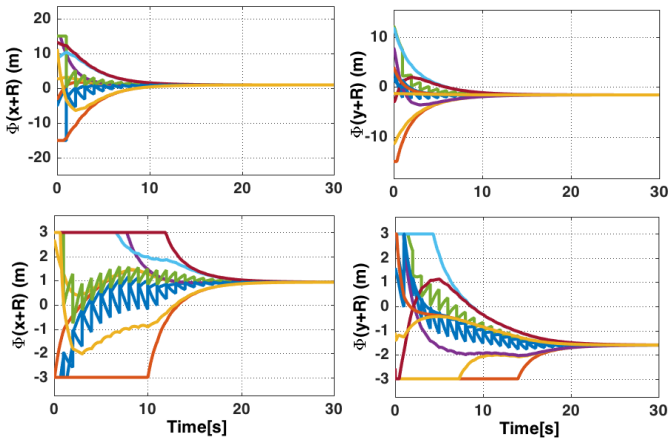

Figure 8: Ten-UAVs' positions (x_i, y_j) under state constraints belonging to $[-15\ 15](m)$ (upper); Ten-UAVs' positions (x_i, y_j) under state constraints belonging to $[-3\ 3](m)$ (lower).

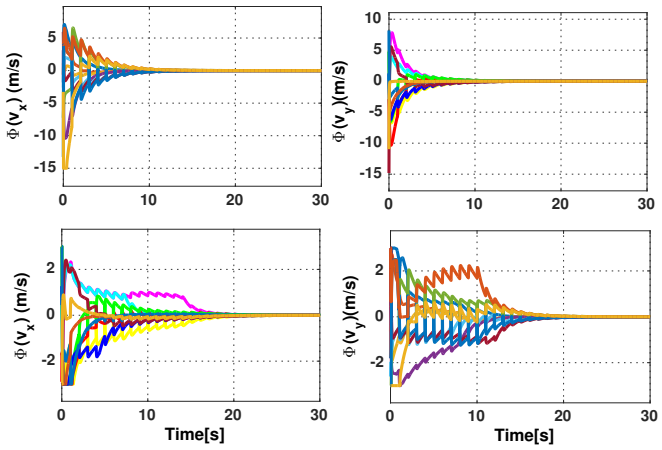

Figure 9: Ten-UAVs' velocities (v_{x_i}, v_{y_i}) under state constraints belonging to $[-15\ 8](m/s)$ (upper); Ten-UAVs' velocities (v_{x_i}, v_{y_i}) under state constraints belonging to $[-3\ 3](m/s)$ (lower).

5. Conclusion

In this paper, a novel approach has been proposed to design distributed robust formation controllers for general linear MASs under state constraints with the following features. First, the considered networks are partitioned into clusters, where the communication between agents inside each cluster is continuous, but the cluster leaders interact at some reset times. Second, it is shown that the robust formation design with state constraints can be indirectly solved by considering the stability of an equivalent system. Third, sufficient conditions for the robust stability of this equivalent system were derived from solutions of local convex LMI problems, which can be solved in a distributed manner. A possible application of our proposed approaches to the UAVs formation flying was illustrated.

Our future work will address the case of output constraints in clustered network with switching topology.

Acknowledgment

The authors would like to thank the Editor and all the reviewers for their constructive suggestions.

Reference

[1] Yu Zhao, Zhisheng Duan, Guanghui Wen, and Yanjiao Zhang. Distributed finite-time tracking control for multi-agent systems: An observer-based approach. *Systems & Control Letters*, 62(1):22–28, 2013.

[2] Yongfang Liu and Zhiyong Geng. Finite-time formation control for linear multi-agent systems: A motion planning approach. *Systems & Control Letters*, 85:54–60, 2015.

[3] Jing Guo, Gangfeng Yan, and Zhiyun Lin. Local control strategy for moving-target-enclosing under dynamically changing network topology. *Systems & Control Letters*, 59(10):654–661, 2010.

[4] Xiwang Dong, Fanlin Meng, Zongying Shi, Geng Lu, and Yisheng Zhong. Output containment control for swarm systems with general linear dynamics: A dynamic output feedback approach. *Systems & Control Letters*, 71:31–37, sep 2014.

[5] Wei Ren. Multi-vehicle consensus with a time-varying reference state. *Systems & Control Letters*, pages 56(7): 474–483, 2007.

[6] Wei Ren and Ella Atkins. Distributed multi-vehicle coordinated control via local information exchange. *International Journal of Robust and Nonlinear Control*, 17(November 2006):557–569, 2007.

[7] Kwang-Kyo Oh and Hyo-Sung Ahn. Formation Control and Network Localization via Orientation Alignment. *IEEE Transactions on Automatic Control*, 59(2):540–545, 2014.

[8] Xiwang Dong, Bocheng Yu, Zongying Shi, and Yisheng Zhong. Time-varying formation control for unmanned aerial vehicles: Theories and applications. *IEEE Transactions on Control Systems Technology*, 23(1):340–348, 2015.

[9] Cheng Lin Liu and Yu Ping Tian. Formation control of multi-agent systems with heterogeneous communication delays. *International Journal of Systems Science*, 40(6):627–636, 2009.

[10] Liang Han, Xiwang Dong, Qingdong Li, and Zhang Ren. Formation tracking control for time-delayed multi-agent systems with second-order dynamics. *Chinese Journal of Aeronautics*, 30(1):348–357, 2017.

[11] Pu Yang, Ben Ma, Yan Dong, and Jianwei Liu. Fault-tolerant Consensus of Leader-following Multi-agent Systems Based on Distributed Fault Estimation Observer. *International Journal of Control, Automation and Systems*, 16(61203090):1–9, 2018.

[12] K. Li, S. E. Li, F. Gao, Z. Lin, J. Li and Q. Sun. Robust Distributed Consensus Control of Uncertain Multi-Agents Interacted by Eigenvalue-Bounded Topologies. *IEEE Internet of Things Journal*, 2020 (Early Access).

[13] F. Gao, F. Lin and B. Liu. Distributed H_∞ Control of Platoon Interacted by Switching and Undirected Topology. *International Journal of Automotive Technology*, 21, 259268 (2020).

[14] Angelia Nedi, Asuman Ozdaglar, and Pablo A Parrilo. Constrained Consensus and Optimization in Multi-Agent Networks. *IEEE Transactions on Automatic Control*, 55(4):922–938, 2010.

[15] W. Wei, J. Xiang, and Y. Li. Consensus problems for linear time-invariant multi-agent systems with saturation constraints. *IET Control Theory & Applications*, 5(6):823–829, 2011.

[16] Lipo Mo and Peng Lin. Distributed consensus of second-order multiagent systems with nonconvex input constraints. *International Journal of Robust and Nonlinear Control*, 28(11):3657–3664, 2018.

[17] Zheqing Zhou and Xiaofeng Wang. Constrained consensus in continuous-time multiagent systems under weighted graph. *IEEE Transactions on Automatic Control*, 63(6):1686–1693, 2018.

[18] Dinh Hoa Nguyen, Tatsuo Narikiyo, and Michihiro Kawanishi. Multi-agent system consensus under input and state constraints. *2016 European Control Conference, ECC 2016*, pages 537–542, 2017.

[19] Dinh Hoa Nguyen, Tatsuo Narikiyo, and Michihiro Kawanishi. Robust consensus analysis and design under relative state constraints or uncertainties. *IEEE Transactions on Automatic Control*, 63(6):1694–1700, 2018.

[20] Zhi Hong Guan, Yonghong Wu, and Gang Feng. Consensus analysis based on impulsive systems in multiagent networks. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 59(1):170–178, 2012.

[21] Zhi-Hong Guan, Zhi-Wei Liu, Gang Feng, and Ming Jian. Impulsive consensus algorithms for second-order multi-agent networks with sampled information. *Automatica*, 48(7):1397–1404, jul 2012.

[22] Hong-xiang Hu, Andong Liu, Qi Xuan, Li Yu, and Guangming Xie. Second-order consensus of multi-agent systems in the cooperationcompetition network with switching topologies: A time-delayed impulsive control approach. *Systems & Control Letters*, 62(12):1125–1135, dec 2013.

[23] Xinzhi Liu, Kexue Zhang, and Wei Chau Xie. Consensus seeking in multi-agent systems via hybrid protocols with impulse delays. *Nonlinear Analysis: Hybrid Systems*, 25:90–98, 2017.

[24] A. Jadbabaie and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.

[25] Wei Ren and Randal W Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, 2005.

[26] Marcos Cesar Bragagnolo, Irinel Constantin Morarescu, Jamal Daafouz, and Pierre Riedinger. Reset strategy for consensus in networks of clusters. *Automatica*, 65:53–63, 2016.

[27] Irinel-constantin Morarescu, Samuel Martin, Antoine Girard, and Aurélie

Muller-gueudin. Coordination in Networks of Linear Impulsive Agents. *IEEE Transactions on Automatic Control*, 61(9):2402–2415, 2016.

[28] Van Thiem Pham, Nadhir Messai, and Noureddine Manamanni. Consensus of Multi-Agent Systems in Clustered Networks. *2019 18th European Control Conference (ECC)*, pages 1085–1090, 2019.

[29] Van Thiem Pham, Nadhir Messai, and Noureddine Manamanni. Impulsive Observer-Based Control in Clustered Networks of Linear Multi-Agent Systems. *IEEE Transactions on Network Science and Engineering*, pages 1–1, 2019 (Early Access).

[30] Van Thiem Pham, Nadhir Messai, D Hoa Nguyen, and Noureddine Manamanni. Adaptive Output Consensus Design in Clustered Networks of Heterogeneous Linear Multi-Agent Systems. *2019 58th IEEE Conference on Decision and Control (CDC)*, Nice, France, pages 5426-5431, 2019.

[31] Reza Olfati-Saber and Richard M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.

[32] G. Lafferriere, A. Williams, J. Caughman, and J.J.P. Veerman. Decentralized control of vehicle formations. *Systems & Control Letters*, 54(9):899–910, 2005.