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# Closed-Form Expressions of Ergodic Capacity and MMSE Achievable Sum Rate for MIMO Jacobi and Rayleigh Fading Channels

AMOR NAFKHA<sup>1</sup>, (Senior member, IEEE), AND NIZAR DEMNI<sup>2</sup>

<sup>1</sup>SCEE/IETR, CentraleSupélec, Avenue de Boulais, Cesson Sévigné, 35576, France (e-mail:amor.nafkha@centralesupelec.fr).

<sup>2</sup>IRMAR, Université de Rennes 1, Campus de Beaulieu, Rennes, 35042, France (e-mail:nizar.demni@univ-rennes1.fr).

Corresponding author: Amor Nafkha (e-mail: amor.nafkha@centralesupelec.fr).

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**ABSTRACT** Multimode/multicore fibers are expected to provide an attractive solution to overcome the capacity limit of the current optical communication system. In the presence of strong crosstalk between modes and/or cores, the squared singular values of the input/output transfer matrix follow the law of the Jacobi ensemble of random matrices. Assuming that the channel state information is only available at the receiver, we derive a new expression for the ergodic capacity of the MIMO Jacobi fading channel. The proposed expression involves double integrals which can be easily evaluated for a high-dimensional MIMO scenario. Moreover, the method used in deriving this expression does not appeal to the classical one-point correlation function of the random matrix model. Using a limiting transition between Jacobi and associated Laguerre polynomials, we derive a similar formula for the ergodic capacity of the MIMO Rayleigh fading channel. Moreover, we derive a new exact closed-form expressions for the achievable sum rate of MIMO Jacobi and Rayleigh fading channels employing linear minimum mean squared error (MMSE) receivers. The analytical results are compared to the results obtained by Monte Carlo simulations and the related results available in the literature, which shows perfect agreement.

**INDEX TERMS** Additive white noise, Channel capacity, Detection algorithms, MIMO, Optical fiber communication, Optical crosstalk, Probability density function, Rayleigh channels,

## I. INTRODUCTION

TO accommodate the exponential growth of data traffic over the last few years, the space division multiplexing (SDM) based on multicore optical fiber (MCF) or multimode optical fiber (MMF) is expected to overcome the barrier from capacity limit of single core fiber [1]–[3]. Recently, dense space division multiplexing (DSDM) with a large spatial multiplicity exceeding 30 was demonstrated with multicore technology [4], [5]. The main challenge in SDM occurs due to in-band crosstalk between multiple parallel transmission channels (cores and/or modes). This strong crosstalk can be dealt with using multiple-input multiple-output (MIMO) signal processing techniques [6]–[11]. Those techniques are widely used for wireless communication systems and they helped to drastically increase channel capacity. Assuming important crosstalk between cores and/or modes, negligible

back-scattering and near lossless propagation, we can model the transmission optical channel as a random complex unitary matrix [12]–[14].

In [12], authors appealed to the Jacobi unitary ensemble (JUE) to establish the propagation channel model for MIMO communications over multimode and/or multicore optical fibers. As suggested in [17, Section I.C], the Jacobi fading channel can be used to accurately model the interference-limited multiuser MIMO system. From mathematical point of view, the JUE is a matrix-variate analogue of the beta random variable and consists of complex Hermitian random matrices which can be realized at least in two different ways [18], [22]: (i) We mimic the construction of a Beta distribution random variable  $B$  as a quotient of two independent Gamma random variables  $B = X_1/(X_1 + X_2)$  where  $X_1$  and  $X_2$  are replaced by two independent complex central Wishart matrices [22].

We assume that the sum  $(X_1 + X_2)$  is reversible. (ii) We can draw a Haar distributed unitary matrix then take the square of the radial part of an upper-left sub-matrix [18]. By a known fact for unitarily invariant-random matrices [22], the average of any symmetric function with respect to the eigenvalues density can be expressed through the one-point correlation function, also known as the single-particle density. In particular, the ergodic capacity of a matrix drawn from the JUE can be represented by an integral where the integrand involves the Christoffel-Darboux kernel associated with Jacobi polynomials ([22], p.384). The drawback of this representation is the dependence of this kernel on the size of the matrix. Indeed, its diagonal is written either as a sum of squares of Jacobi polynomials and the number of terms in this sum equals the size of the matrix least one, or by means of the Christoffel-Darboux formula as a difference of the product of two Jacobi polynomials whose degrees depend on the size of the matrix. To the best of our knowledge, this is the first study that derives exact expression of the ergodic capacity as a double integral over a suitable region. Recently in [19], [20], the authors derived expressions for the exact moments of the mutual information in the high-SNR regime for MIMO Jacobi fading channel. The obtained exact moments lead to closed-form approximations to the outage probability.

In this paper, we provide a new expression for the ergodic capacity of the MIMO Jacobi fading channel relying this time on the formula derived in [24] for the moments of the eigenvalues density of the Jacobi random matrix. The obtained expression shows that the ergodic capacity is an average of some function over the signal to noise ratio (SNR), and it has the merit to have a simple dependence on the size of the matrix which allows for easier and more precise numerical simulations. By a limiting transition between Jacobi and associated Laguerre polynomials [25], we derive a similar expression for the ergodic capacity of the MIMO Rayleigh fading channel [21]. Using the derived expressions and the work of McKay *et al.* [41], we are able to derive closed-form formulas for the achievable sum-rate of MIMO Jacobi and Rayleigh fading channels employing linear minimum mean squared error (MMSE) receivers.

The paper is organized as follows. In Section II, we recall some notations, definitions of random matrices and special functions occurring in the remainder of the paper. Section III introduces the MIMO Jacobi fading channel and the discrete-time input-output relation. In Section IV, an exact closed-form expression is derived for the ergodic capacity of MIMO Jacobi fading channel. Using the results of the previous section, we derive a new exact closed-form expression of the ergodic capacity of the MIMO Rayleigh fading channel in Section V. In both MIMO Jacobi and Rayleigh fading channels, we provide new closed-form expressions for the achievable sum rate of MIMO MMSE receivers in Section VI. In Section VII, we demonstrate the accuracy of the analytical expressions through Monte Carlo simulations. Finally, Section VIII is devoted to concluding remarks, while mathematical proofs are deferred to the appendices.

## II. BASIC DEFINITIONS AND NOTATIONS

Throughout this paper, the following notations and definitions are used. We start with those concerned with special functions for which the reader is referred to the original book of Ismail [25]. The Pochhammer symbol  $(x)_k$  with  $x \in \mathbb{R}$  and  $k \in \mathbb{N}$  is defined by

$$(x)_k = x(x+1) \dots (x+k-1); (x)_0 = 1 \quad (1)$$

For  $x > 0$ , it is clear that

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} \quad (2)$$

where  $\Gamma(\cdot)$  is the Gamma function. Note that if  $x = -q$  is a non positive integer then

$$(-q)_k = \begin{cases} (-1)^k \frac{q!}{(q-k)!} & \text{if } k \geq q \\ 0 & \text{if } k < q \end{cases} \quad (3)$$

The Gauss hypergeometric function  ${}_2F_1(\cdot)$  is defined for complex  $|z| < 1$  by the following convergent power series

$${}_2F_1(\theta, \sigma, \gamma, z) = \sum_{k=0}^{\infty} \frac{(\theta)_k (\sigma)_k}{(\gamma)_k k!} z^k \quad (4)$$

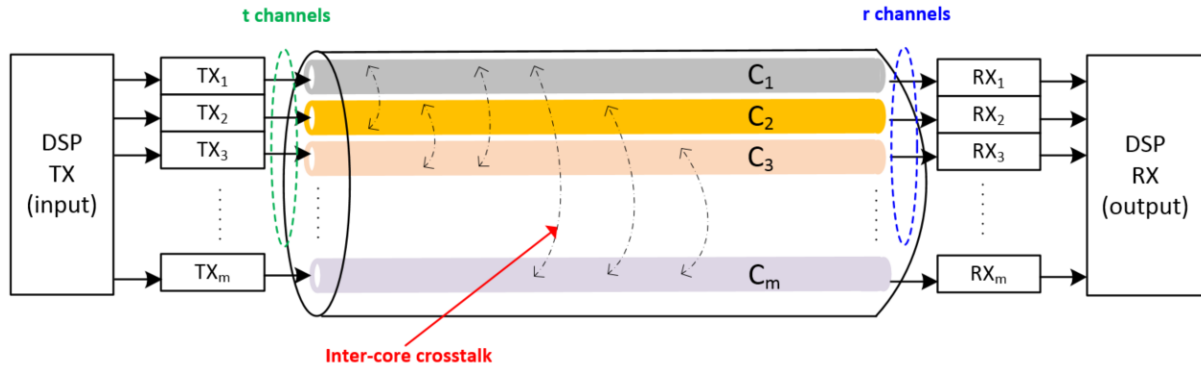
where  $(\cdot)_k$  denotes the Pochhammer symbol defined in (1) and  $\theta, \sigma, \gamma$  are real parameters with  $\gamma \neq \{0, -1, -2, \dots\}$ . The function  ${}_2F_1(\cdot)$  has an analytic continuation to the complex plane cut along the half-line  $[1, \infty[$ . In particular, the Jacobi polynomials  $P_q^{\alpha, \beta}(x)$  of degree  $q$  and parameters  $\alpha > -1, \beta > -1$  can also be expressed in terms of the Gauss hypergeometric function (4) as follows

$$P_q^{\alpha, \beta}(x) = \frac{(\eta)_q}{q!} {}_2F_1(-q, q + \eta + \beta, \eta; \frac{1-x}{2}) \quad (5)$$

where  $\eta = \alpha + 1$ . An important asymptotic property of the Jacobi polynomial is the fact that it can be reduced to the  $q$ -th associated Laguerre polynomial of parameter  $\alpha \geq 0$  through the following limit

$$L_q^\alpha(x) = \lim_{\beta \rightarrow \infty} P_q^{\alpha, \beta} \left( 1 - \frac{2x}{\beta} \right), x > 0 \quad (6)$$

Now, we come to the notations and the definitions related with random matrices, and refer the reader to [18], [22], [23]. Firstly, the Hermitian transpose and the determinant of a complex matrix  $\mathbf{A}$  are denoted by  $\mathbf{A}^\dagger$  and  $\det(\mathbf{A})$  respectively. Secondly, the Laguerre unitary ensemble (LUE) is formed out of non negative definite matrices  $\mathbf{A}^\dagger \mathbf{A}$  where  $\mathbf{A}$  is a rectangular  $m \times n$  matrix, with  $m \geq n$ , whose entries are complex independent Gaussian random variables. A matrix from the LUE is often referred to as a complex Wishart matrix and  $(m, n)$  are its degrees of freedom and its size respectively. Finally, let  $\mathbf{X} = \mathbf{A}^\dagger \mathbf{A}$  and  $\mathbf{Y} = \mathbf{B}^\dagger \mathbf{B}$  be two independent  $(m_1, n)$  and  $(m_2, n)$  complex Wishart matrices. Assume  $m_1 + m_2 \geq n$ , then  $\mathbf{X} + \mathbf{Y}$  is positive definite and the random matrix  $\mathbf{J}$ , defined as  $\mathbf{J} = (\mathbf{X} + \mathbf{Y})^{-1/2} \mathbf{X} (\mathbf{X} + \mathbf{Y})^{-1/2}$ , belongs to the JUE. The matrix  $\mathbf{J}$  is unitarily-invariant and satisfies  $\mathbf{0}_n \leq \mathbf{J} \leq \mathbf{I}_n$  where  $\mathbf{0}_n, \mathbf{I}_n$



**FIGURE 1.** Schematic of an optical MIMO system with an MCF,  $C_k$  indicates  $k^{th}$  fiber core, with  $k \in \{1, 2, \dots, m\}$

stand for the null and the identity matrices respectively<sup>1</sup>. If  $m_1, m_2 \geq n$  then the matrix  $\mathbf{J}$  and the matrix  $(\mathbf{I}_n - \mathbf{J})$  are positive definite and the joint distribution of the ordered eigenvalues of  $\mathbf{J}$  has a probability density function given by

$$\mathcal{F}_{a,b,n}(\lambda_1, \dots, \lambda_n) = Z_{a,b,n}^{-1} \prod_{1 \leq j \leq n} \lambda_j^{a-1} (1 - \lambda_j)^{b-1} \times [V(\lambda_1, \dots, \lambda_n)]^2 \mathbf{1}_{\{0 < \lambda_1 < \dots < \lambda_n < 1\}} \quad (7)$$

with respect to Lebesgue measure  $d\lambda = d\lambda_1 \dots d\lambda_n$ . Here,  $a = m_1 - n + 1$ ,  $b = m_2 - n + 1$ ,  $Z_{a,b,n}$  is a normalization constant read off from the Selberg integral [23], [24]:

$$Z_{a,b,n} = \prod_{j=1}^n \frac{\Gamma(a+j-1)\Gamma(b+j-1)\Gamma(j)}{\Gamma(a+b+n+j-2)},$$

$\mathbf{1}_{\{A\}}$  stands for the indicator function: given a set  $A$

$$\mathbf{1}_{\{x \in A\}} = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise,} \end{cases}$$

and  $V(\lambda_1, \dots, \lambda_n) = \prod_{1 \leq j < k \leq n} (\lambda_j - \lambda_k)$  is the Vandermonde polynomial. As suggested in [18], we can construct the matrix  $\mathbf{J}$  from the JUE ensemble as follows: let  $\mathbf{U}$  be an  $m \times m$  Haar-distributed unitary matrix. Let  $t$  and  $r$  be two positive integers such that  $t + r \leq m$  and  $t \leq r$ . Let also  $\mathbf{H}$  be the  $r \times t$  upper-left corner of  $\mathbf{U}$ , then the joint distribution of the ordered eigenvalues of matrix  $\mathbf{J} = \mathbf{H}^\dagger \mathbf{H}$  is given by (7) with parameters  $a = r - t + 1$ ,  $b = m - r - t + 1$ , and  $n = t$ .

In the sequel the following notation will be used.  $\mathbb{E}_\nu[\cdot]$  will denote the expectation with respect to the random variable  $\nu$ . We will denote the matrix determinant by  $\det(\cdot)$ , and the matrix inverse by  $[\cdot]^{-1}$ . The  $(i, j)$ -th element of a matrix  $\mathbf{A}$  is indicated by  $[\mathbf{A}]_{i,j}$ .

### III. SYSTEM MODEL

We consider an optical space division multiplexing where the multiple channels correspond to the number of excited modes and/or cores within the optical fiber. The coupling

<sup>1</sup>For two square matrices  $A$  and  $B$ , we write  $A \leq B$  when  $B - A$  is a non negative matrix.

between different modes and/or cores can be described by scattering matrix formalism as reported in [14], [34]–[36]. In this paper, we consider  $m$ -channel *near* lossless optical fiber with  $t \leq m$  transmitting excited channels and  $r \leq m$  receiving channels, as indicated in Fig. 1 for multicore optical fiber scenario. The scattering matrix formalism can describe very simply the propagation through the fiber using  $2m \times 2m$  scattering matrix  $\mathbf{S}$  given as

$$\mathbf{S} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{T}_2 \\ \mathbf{T}_1 & \mathbf{R}_2 \end{bmatrix}, \quad (8)$$

where the  $m \times m$  complex block matrices  $\mathbf{R}_1$  and  $\mathbf{R}_2$  describe the reflection coefficients in input and output ports of the fiber, respectively. Similarly, the  $m \times m$  complex block matrices  $\mathbf{T}_1$  and  $\mathbf{T}_2$  stand for the transmission coefficients through the fiber from input to output sides and vice versa, respectively. We assume a strong crosstalk between cores or modes, negligible backscattering, near-lossless propagation, and reciprocal characteristics of the fiber. Thus, we model the scattering matrix as a complex unitary symmetric matrix [16], (*i.e.*  $\mathbf{S}^\dagger \mathbf{S} = \mathbf{I}_{2m}$ ). Therefore, the four Hermitian matrices  $\mathbf{T}_1 \mathbf{T}_1^\dagger$ ,  $\mathbf{T}_2 \mathbf{T}_2^\dagger$ ,  $\mathbf{I}_m - \mathbf{R}_2 \mathbf{R}_2^\dagger$ , and  $\mathbf{I}_m - \mathbf{R}_1 \mathbf{R}_1^\dagger$  have the same set of eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$ . Each of these  $m$  transmission eigenvalues is a real number belong to the interval  $[0, 1]$ . Assuming a unitary coupling among all transmission modes the overall transfer matrix  $\mathbf{T}_1$  can be described by a  $m \times m$  unitary matrix, where each matrix entry  $[\mathbf{T}_1]_{ij}$  represents the complex path gain from transmitted mode  $i$  to received mode  $j$ . Moreover, the transmission matrix  $\mathbf{T}_1$  has a Haar distribution over the group of complex unitary matrices [12], [14]. Given the fact that only  $t \leq m$  and  $r \leq m$  modes are addressed by the transmitter and receiver, respectively, the effective transmission channel matrix  $\mathbf{H} \in \mathbb{C}^{r \times t}$  is a truncated<sup>2</sup> version of  $\mathbf{T}_1$ . As a result, the corresponding MIMO channel for this system reads

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z} \quad (9)$$

where  $\mathbf{y} \in \mathbb{C}^{r \times 1}$  is the received signal vector of dimension  $r \times 1$ ,  $\mathbf{x} \in \mathbb{C}^{t \times 1}$  is a  $t \times 1$  transmitted signal vector with

<sup>2</sup>Without loss of generality, the effective transmission channel matrix  $\mathbf{H}$  is the  $r \times t$  upper-left corner of the transmission matrix  $\mathbf{T}_1$  [18], [37]

TABLE 1. List of main variables

Variables	Descriptions
$m$	Number of overall available modes/cores
$t$	Number of transmitted modes/cores
$r$	Number of received modes/cores
$\mathbf{S}$	$\mathbb{C}^{2m \times 2m}$ scattering matrix
$\mathbf{R}_1$	$\mathbb{C}^{m \times m}$ matrix contains the reflection coefficients of the $m$ input modes/cores of the fiber
$\mathbf{T}_1$	$\mathbb{C}^{m \times m}$ matrix contains the transmission coefficients from the input to the output of the multi-mode/core fiber
$\mathbf{R}_2$	$\mathbb{C}^{m \times m}$ matrix contains the reflection coefficients of the $m$ output modes/cores of the fiber
$\mathbf{T}_2$	$\mathbb{C}^{m \times m}$ matrix contains the transmission coefficients from the output to the input of the multi-mode/core fiber
$\mathbf{H}$	$\mathbb{C}^{r \times t}$ matrix contains the transmission coefficients of the effective channel when $t \leq m$ transmitting modes/cores and $r \leq m$ receiving modes/cores are used.
$C_{t,r}^{m,\rho}$	Ergodic capacity of MIMO Jacobi fading channel with $t$ transmitting modes/cores, $r$ receiving modes/cores, $m$ overall available modes/cores (i.e. $m \geq t, m \geq r$ ), and a signal to noise ratio equal to $\rho$ .
$C_{t,r}^p$	Ergodic capacity of MIMO Rayleigh fading channel with $t$ transmitting antennas, $r$ receiving antennas, and a signal to noise ratio equal to $\rho$ .
$\mathcal{R}$	General expression of the achievable ergodic sum rate for the MMSE receiver under MIMO channel.
$\mathcal{R}_{t,r}^{m,\rho}$	Achievable ergodic sum rate of MMSE receiver under MIMO Jacobi fading channel with $t$ transmitting modes/cores, $r$ receiving modes/cores, $m$ available modes/cores, and a signal to noise ratio equal to $\rho$ .
$\mathcal{R}_{t,r}^p$	Achievable ergodic sum rate of MMSE receiver under MIMO Rayleigh fading channel with $t$ transmitting antennas, $r$ receiving antennas, and a signal to noise ratio equal to $\rho$ .

covariance matrix equal to  $\frac{\mathcal{P}}{t} \mathbf{I}_t$ , and  $\mathbf{z} \in \mathbb{C}^{r \times 1}$  is a  $r \times 1$  zero mean additive white circularly symmetric complex Gaussian noise vector with covariance matrix equal to  $\sigma^2 \mathbf{I}_r$  [46], [47]. The variable  $\mathcal{P}$  is the total transmit power across the  $t$  modes/cores, and  $\sigma^2$  is the Gaussian noise variance. Table 1 provides the list of main variables used in this manuscript.

#### IV. ERGODIC CAPACITY OF MIMO JACOBI CHANNEL

The expression of the ergodic capacity of the MIMO Jacobi fading channel was firstly expressed in [12] as an integral over  $[0, 1]$  of the sum of squares of  $\min(t, r)$  Jacobi polynomials with real coefficients, is the same theoretical approach adopted by Telatar [21]. Recently, ergodic capacity bounds (upper and lower) of the MIMO Jacobi fading channel were derived in [26] and [27]. In [26], authors derived lower bound and low SNR approximation of the ergodic capacity of MIMO Jacobi fading channel by rearranging the analytical expression given in [12, Eq. (11)]. Using recent results on the determinant of the Jacobi unitary ensemble and classical Jensen's and Minkowski's inequalities, the authors, in [27], derived tight closed-form bounds for the ergodic capacity [12, Eq. (11)]. In addition, they also provided accurate closed-form analytical approximations of ergodic capacity at high and low signal to noise ratio regimes.

In this section, we provide a novel and simple closed-form expression of the ergodic capacity in the setting of

MIMO Jacobi fading channel. We assume that the channel state information (CSI) is only known at the receiver, not at the transmitter. The investigation of the ergodic capacity of the MIMO Jacobi fading channel under unknown CSI at the receiver side is out of scope of the present work. Without loss of generality, in the sequel of the present paper, we shall assume that  $t \leq r$  and  $m \geq t + r$ . The channel ergodic capacity, under a total average transmit power constraint, is then achieved by taking  $\mathbf{x}$  as a vector of zero-mean circularly symmetric complex Gaussian components with covariance matrix  $\mathcal{P} \mathbf{I}_t / t$ , and it is given by [12, Eq. (10)]

$$C_{t,r}^{m,\rho} = \mathbb{E}_{\mathbf{H}} \left[ \ln \det \left( \mathbf{I}_t + \frac{\rho \mathbf{H}^\dagger \mathbf{H}}{t} \right) \right], \quad t \leq r, \quad (10)$$

where  $\mathbb{E}_{\mathbf{H}}[\cdot]$  denotes the expectation over all channel realizations,  $\ln$  is the natural logarithm function and  $\rho = \frac{\mathcal{P}}{\sigma^2}$  is the average signal-to-noise ratio (SNR). Given the fact that both matrices  $\mathbf{H}^\dagger \mathbf{H}$  and  $\mathbf{H} \mathbf{H}^\dagger$  share the same non zero eigenvalues even if  $m < t + r$ , the authors in [12, Theorem 2] shows that the ergodic capacity is given by

$$C_{t,r}^{m,\rho} = (t + r - m) C_{1,1}^{1,\rho} + C_{m-r, m-t}^{m,\rho}, \quad t \leq r. \quad (11)$$

In the sequel of this paper, we assume further that  $m > t + r \Leftrightarrow b \geq 2$  and the case  $m = r + t \Leftrightarrow b = 1$  can be dealt with by a limiting procedure. Actually, our formula for the ergodic capacity derived below is valid for real  $a > 0, b > 1$ , and we can consider its limit as  $b \rightarrow 1$ . However, for ease of reading, we postpone the details of the computations relative to this limiting procedure to a future forthcoming paper.

Now, recall that the random matrix  $\mathbf{H}^\dagger \mathbf{H}$  has the Jacobi distribution, then its ordered eigenvalues have the joint density given by (7) with parameters  $a = r - t + 1$  and  $b = m - t - r + 1$ . Using (7), we can explicitly express the ergodic capacity (10) as

$$C_{t,r}^{m,\rho} = \int \sum_{k=1}^t \ln(1 + \rho \lambda_k) \mathcal{F}_{a,b,t}(\lambda_1, \dots, \lambda_t) d\lambda_1 \dots d\lambda_t \quad (12)$$

A major step towards our main result is the following proposition.

*Proposition 1:* For any  $\rho \in (0, 1)$ ,

$$\Psi C_{t,r}^{m,\rho} = A_{t,r} \rho^{t-1} P_{t-1}^{r-t, m-t-r-1} \left( \frac{\rho + 2}{\rho} \right) {}_2F_1(t + 1, r + 1, m + 1; -\rho) \quad (13)$$

where the operator  $\Psi = [D_\rho(\rho D_\rho)]$  with  $D_\rho$  is the derivative operator with respect to  $\rho$ , and  $A_{t,r} = \frac{r! t!}{(m-t+1)_t}$ .

*Proof:* The full proof for Proposition 1 can be found at the Appendix A. ■

Using proposition 1, we are able to derive the following new expression of the ergodic capacity of MIMO Jacobi fading channel.



**Theorem 1:** Assume that  $r \geq t$ ,  $m > t + r$ , and  $\rho \geq 0$ , then the ergodic capacity of an uncorrelated MIMO Jacobi fading channel is given by

$$C_{t,r}^{m,\rho} = B_{t,r}^m \int_0^1 u^{a-1} (1-u)^{b-2} P_{t-1}^{a-1,b} (1-2u) \times P_t^{a-1,b-2} (1-2u) Li_2(-\rho u) du \quad (14)$$

where  $a = r - t + 1$ ,  $b = m - r - t + 1$ , and  $B_{t,r}^m = \frac{t! (m-t)!}{\Gamma(r) \Gamma(m-r)}$ . The function  $Li_2(\cdot)$  is the dilogarithm function [50] defined as

$$Li_2(z) = - \int_0^z \frac{\ln(1-v)}{v} dv, z \in \mathbb{C}$$

*Proof:* The appendix B contains proof of Theorem 1. ■

## VI. ERGODIC CAPACITY OF MIMO RAYLEIGH CHANNEL

The ergodic capacity of the MIMO Rayleigh fading channel was extensively examined in order to provide a compact mathematical expression in several papers [21], [28]–[32]. In [28], [29], the ergodic capacity is provided using the Christoffel-Darboux kernel, and the authors replaced the Laguerre polynomials by their expressions which is a known fact in invariant random matrix models. In [30]–[32], authors derived a closed form expression of moment generating function (MGF) so that the ergodic capacity may be derived by taking the first derivative. However, this expression of MGF relies on the Cauchy-Binet Theorem and only gives a hypergeometric function of matrix arguments [33], from which by derivatives, we can get again an alternating sum coming from the determinant. Consequently, we can not derive the proposed expression of the ergodic capacity (15) from this sum.

Using the limiting transition (6) between Jacobi and associated Laguerre polynomials, we are able to give another expression for the ergodic capacity expression of the wireless MIMO Rayleigh fading channel. Indeed, it was shown in [14], [15], that the parameter  $b$  in (14) can be interpreted as the power loss through the optical fiber. Therefore, as  $b$  becomes large, the channel matrix  $\mathbf{H}$  in (9) starts to look like a complex Gaussian matrix with independent and identically distributed entries. As a matter of fact, the MIMO Jacobi fading channel approaches the MIMO Rayleigh fading channel in the large  $b$ -limit corresponding to a huge waste of input power through the optical fiber. In particular, the ergodic capacity (14) converges as  $b \rightarrow \infty$  to the ergodic capacity of the uncorrelated MIMO Rayleigh fading channel already considered by Telatar in [21, Theorem 2], and we are able to derive the following new result. Note that the pioneer work of Telatar was recently revisited by Wei in [45].

**Theorem 2:** The ergodic capacity of the uncorrelated MIMO Rayleigh fading channel with  $t$  transmitters and  $r$  receivers, with  $r \geq t$ , can be expressed

$$C_{t,r}^\rho = \frac{t!}{(r-1)!} \int_0^{+\infty} u^{r-t} e^{-u} L_{t-1}^{r-t}(u) L_t^{r-t}(u) \times Li_2(-\rho u) du. \quad (15)$$

*Proof:* The reader can refer to Appendix C for the proof of Theorem 2. ■

## VI. ACHIEVABLE SUM RATE OF MIMO MMSE RECEIVER

In this section, we are interested in the performance of linear MMSE receivers. Assuming to employ a MMSE filter, and that each filter output is independently decoded. Let  $\rho_k$  denotes the instantaneous signal to interference-plus-noise ratio (SINR) to the  $k^{th}$  MIMO subchannel<sup>3</sup>. Minimizing the mean squared error between the output of a linear MMSE receiver and the actually transmitted symbol  $\mathbf{x}_k$  for  $1 \leq k \leq t$  leads to the filter vector

$$\mathbf{g}_k = \left( \mathbf{H}\mathbf{H}^\dagger + \frac{t}{\rho} \mathbf{I}_r \right)^{-1} \mathbf{h}_k \quad (16)$$

where  $\mathbf{h}_k$  is the  $k^{th}$  column of channel matrix  $\mathbf{H}$ . Applying this filter vector into (9) yields

$$\mathbf{x}_k^{mmse} = \mathbf{g}_k^\dagger \mathbf{y} \quad (17)$$

The achievable ergodic sum rate for the MMSE receiver can be expressed as

$$\mathcal{R} = \sum_{k=1}^t \mathbb{E}_{\rho_k} [\ln(1 + \rho_k)] \quad (18)$$

As shown in [38], [41], [42], [44], the instantaneous received SINR for the  $k^{th}$  MMSE filter output is given by

$$\rho_k = \frac{1}{\left[ (\mathbf{I}_t + (\rho/t) \mathbf{H}^\dagger \mathbf{H})^{-1} \right]_{k,k}} - 1 \quad (19)$$

In general, the analytical expression of the probability density function of  $\rho_k$  is difficult to determine. This situation makes the direct evaluation of the achievable ergodic MMSE sum rate (18) very difficult.

Let  $\mathbf{H}_k$  denotes the sub-matrix obtained by striking  $\mathbf{h}_k$  out of  $\mathbf{H}$ . As shown in [49, Theorem 1.33], the  $k^{th}$  diagonal term of the matrix,  $\left( \mathbf{I}_t + \frac{\rho \mathbf{H}^\dagger \mathbf{H}}{t} \right)^{-1}$ , can be expressed as

$$\left[ \left( \mathbf{I}_t + \frac{\rho \mathbf{H}^\dagger \mathbf{H}}{t} \right)^{-1} \right]_{k,k} = \frac{\det \left( \mathbf{I}_{t-1} + \frac{\rho \mathbf{H}_k^\dagger \mathbf{H}_k}{t} \right)}{\det \left( \mathbf{I}_t + \frac{\rho \mathbf{H}^\dagger \mathbf{H}}{t} \right)}, \quad (20)$$

where the matrix  $\mathbf{H}_k^\dagger \mathbf{H}_k$  is the  $k \times k$  principal minor of matrix  $\mathbf{H}^\dagger \mathbf{H}$  defined by striking out the  $k^{th}$  column of  $\mathbf{H}$ .

<sup>3</sup>In our case ( $t \leq r$ ), the MIMO channel can be decomposed into  $t$  parallel subchannels.

Similarly to what has been developed in [41], by substituting (19) and (20) in (18), we can obtain the following expression of the achievable ergodic sum rate for the MIMO MMSE receiver.

$$\mathcal{R} = t \mathbb{E}_{\mathbf{H}} \left[ \ln \det \left( \mathbf{I}_t + \frac{\rho \mathbf{H}^\dagger \mathbf{H}}{t} \right) \right] - \sum_{k=1}^t \mathbb{E}_{\mathbf{H}_k} \left[ \ln \det \left( \mathbf{I}_{t-1} + \frac{\rho \mathbf{H}_k^\dagger \mathbf{H}_k}{t} \right) \right] \quad (21)$$

By employing the Haar invariant property, exchanging any two different rows or/and exchanging two different columns do not change the joint distribution of the entries, the joint probability density function of the ordered eigenvalues of  $\mathbf{H}_k^\dagger \mathbf{H}_k$  is the same as  $\mathbf{H}_j^\dagger \mathbf{H}_j$  for all  $j \neq k$  and  $j \in \{1, \dots, t\}$ . Thus, the achievable ergodic sum rate for the MMSE receiver can be expressed as

$$\mathcal{R} = t \mathbb{E}_{\mathbf{H}} \left[ \ln \det \left( \mathbf{I}_t + \frac{\rho \mathbf{H}^\dagger \mathbf{H}}{t} \right) \right] - t \mathbb{E}_{\mathbf{H}_1} \left[ \ln \det \left( \mathbf{I}_{t-1} + \frac{\rho \mathbf{H}_1^\dagger \mathbf{H}_1}{t} \right) \right] \quad (22)$$

In case of MIMO Jacobi fading channel, the matrix  $\mathbf{H}_t$  is the  $r \times (t-1)$  left corner of the channel matrix  $\mathbf{H}$ . Then, the joint distribution density function of the ordered eigenvalues of  $\mathbf{H}_t^\dagger \mathbf{H}_t$  is given by (7) with parameters  $a = r - t + 2$ ,  $b = m - r - t + 2$ , and  $n = t - 1$ . The following result characterizes the achievable ergodic sum rate of the MIMO Jacobi fading channel when the linear MMSE filter is used at the receiver side.

**Theorem 3:** For any  $\rho \geq 0$ , The achievable ergodic sum rate of MMSE receiver under MIMO Jacobi fading channel is given by

$$\mathcal{R}_{t,r}^{m,\rho} = t \left[ C_{t,r}^{m,\rho} - C_{t-1,r}^{m, \frac{(t-1)\rho}{t}} \right] \quad (23)$$

*Proof:* By substituting (14) into (22). ■

Very recently, Lim *et al.* [42] proposed closed form expression of the achievable sum rate for MMSE MIMO systems in uncorrelated Rayleigh environments. However, the derived expression, [42, eq.(67)], is not closed form and does not allow a better understanding of the MMSE achievable sum rate due to the use of the sum of Meijer G-functions (or equivalent representation in terms of generalized hypergeometric functions). In following corollary, we presented a novel and exact closed-form formula for ergodic achievable sum rate for MMSE receiver under MIMO Rayleigh fading channels.

**Corollary 1:** For any  $\rho \geq 0$ , The achievable ergodic sum rate of MMSE receiver under MIMO Rayleigh fading channels with  $t \leq r$  can be expressed as

$$\begin{aligned} \mathcal{R}_{t,r}^\rho &= r [\Psi(t, r, \rho) - \Psi(t, r+1, \rho)] + \frac{t!}{(r-1)!} \\ &\times \int_0^{+\infty} u^{r-t+1} e^{-u} L_{t-2}^{r-t+1}(u) L_{t-1}^{r-t+1}(u) \\ &\times \left[ Li_2 \left( \frac{-\rho u}{t} \right) - Li_2 \left( \frac{-\rho(t-1)u}{t^2} \right) \right] du. \end{aligned} \quad (24)$$

where

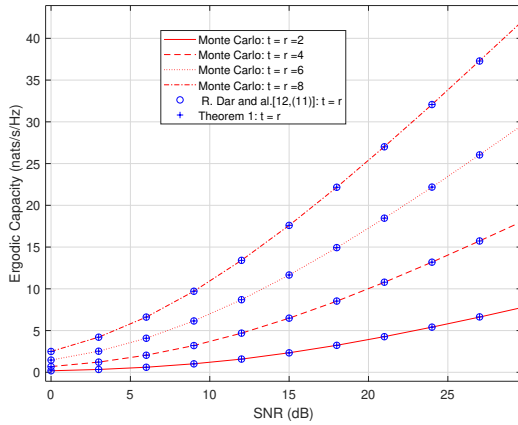
$$\begin{aligned} \Psi(t, r, \rho) &= \frac{t!}{(r-1)!} \int_0^{+\infty} u^{r-t} e^{-u} [L_{t-1}^{r-t}(u)]^2 \\ &\times Li_2 \left( \frac{-\rho u}{t} \right) du \end{aligned}$$

*Proof:* By substituting (15) into (22). ■

## VII. NUMERICAL RESULTS AND DISCUSSION

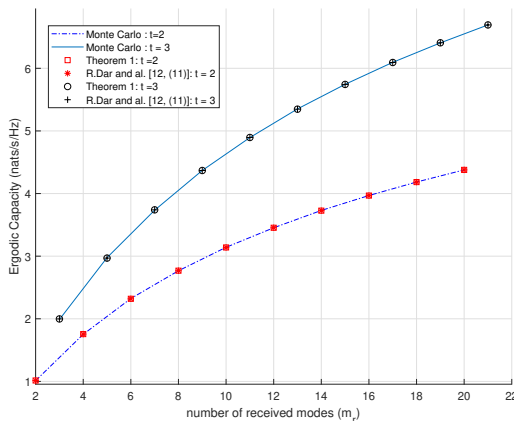
In this section, we present numerical results supporting the analytical expressions derived in Section IV and Section V. All of the Monte Carlo simulation results were obtained by averaging over  $10^5$  independent channel realization. For MIMO Rayleigh fading channels, the entries in  $\mathbf{H} \in \mathbb{C}^{r \times t}$  are independent and identically distributed complex, zero mean Gaussian random variables with normalized unit magnitude variance, and they can be obtained using a built-in MATLAB function (*i.e.* "randn"). For the MIMO Jacobi fading channels, the simulation process is initialized firstly by creating a random complex Gaussian matrix  $\mathbf{G} \in \mathbb{C}^{m \times m}$  with independent and identically distributed entries that are complex circularly symmetric Gaussian with zero mean and  $1/2$  variance per dimension. Then, using QR decomposition then matrix  $\mathbf{G}$  can be decomposed as  $\mathbf{G} = \mathbf{Q}\mathbf{R}$  where  $\mathbf{Q} \in \mathbb{C}^{m \times m}$  is a unitary matrix and  $\mathbf{R} \in \mathbb{C}^{m \times m}$  is upper triangular matrix. Finally, the MIMO Jacobi fading channel  $\mathbf{H}$  was constructed by taking the  $r \times t$  sub-matrix in the upper-left corner of matrix  $\mathbf{Q}$ . In both MIMO channel cases, the ergodic capacity and achievable sum rate with MMSE receivers can be obtained by averaging (10) and (22), respectively, over all realization of the channel matrix  $\mathbf{H}$ . Herein, we consider the case where the channel state information is available at the receiver side. Figure 2 examines the ergodic capacity of the MIMO Jacobi fading channel as a function of the SNR, when the number of parallel transmission paths is fixed to  $m = 20$  and the number of transmit modes equal to the number of receive modes  $r = t$ . It is evident that when we increase the number of transmitted and received modes, we improve the ergodic capacity of the system. As expected, the ergodic capacity increases with SNR. Figure 2 is also shown that the two theoretical expressions curves of the ergodic capacity (14) and [12, (11)] perfectly matched the simulation results.

Figure 3 shows the theoretical and simulated ergodic capacity of MIMO Jacobi channel as a function of the number



**FIGURE 2.** The variation of the ergodic capacity of MIMO Jacobi channel as a function of  $\rho$  for  $m = 20$

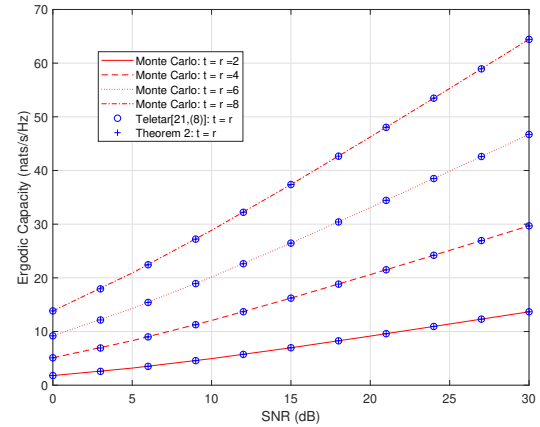
of received modes. Here, we fixed the number of parallel transmission paths to  $m = 25$ , the SNR to  $\rho = 10$  dB, and the number of transmit modes  $t$  to have following values  $\{2, 3\}$ . It is shown that every simulated curve is in excellent agreement with the theoretical curves calculated from (14) and [12, (11)]. The relationship between the channel capacity and the number of received modes is logarithmic. This implies that trying to improve the channel capacity by just increasing the number of received modes or cores is not efficient in the sense that the capacity increases logarithmically with  $r$ . The same relationship has been noted and discussed in the case of the uncorrelated MIMO Rayleigh fading channel (see Fig. 5, [21], and [40]).



**FIGURE 3.** Ergodic capacity of MIMO Jacobi channel as function of receive cores and/or modes with  $\rho = 10$  dB and  $m = 25$ .

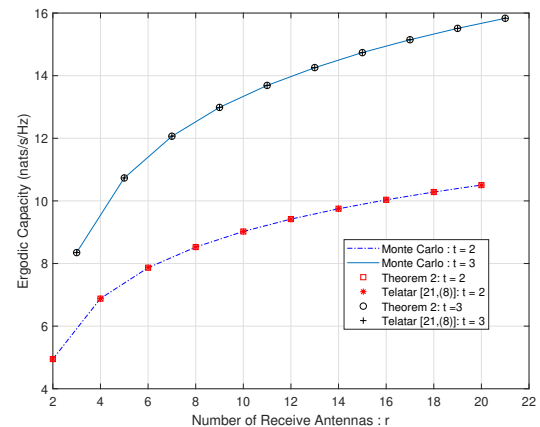
For the uncorrelated MIMO Rayleigh fading channel, the proposed expression of the ergodic capacity was verified through Monte Carlo experiments and it is shown in Fig. 4. In Fig. 4, the comparisons are shown between theoretical expressions and simulation values of the ergodic capacity as a function of the SNR. As we can observe in Fig. 4, for a given

SNR, the capacity increases as the numbers of transmit and receive antennas grow. In all cases, the results demonstrate an excellent agreement between analytical expressions and Monte-Carlo simulations. Moreover, We can observe that the expression in (15) matches perfectly with the expression introduced by Telatar [21, Eq. (8)]. For the cases where  $t = r = 2$  and  $t = r = 4$ , the obtained results are consistent with simulation results reported in [39], [40].



**FIGURE 4.** Ergodic capacity of the uncorrelated MIMO Rayleigh fading channel versus SNR for different numbers of transmit and receive antennas.

Figure 5 shows the ergodic capacity of uncorrelated MIMO Rayleigh fading channel of as the number of receive antennas  $r$  increases. As expected, we observe that the ergodic capacity increases in logarithmic scale with respect to  $r$ , this tallies with the result reported in [48, Eq. (6)]. As for optical MIMO channel, the three different ways to compute the uncorrelated MIMO Rayleigh fading channel capacity give the same results. These simulations were carried out to verify the mathematical derivation and no inconsistencies were noted.



**FIGURE 5.** Ergodic capacity of MIMO Rayleigh channel when the number of received antennas increases and  $\rho = 10$  dB.

We now focus on the ergodic sum rate for the MMSE



receiver. We first consider the MIMO Jacobi fading channel. Fig. 6 shows the evolution of the ergodic sum rate for the MMSE receiver versus the SNR over the optical MIMO channel. For these results, we suppose that either  $m = 20$  or  $m = 8$ . As expected, the ergodic sum rate increases with increasing SNR. Moreover, our simulation results show that the formula derived in Theorem 3 and Monte Carlo simulations provide the same results.

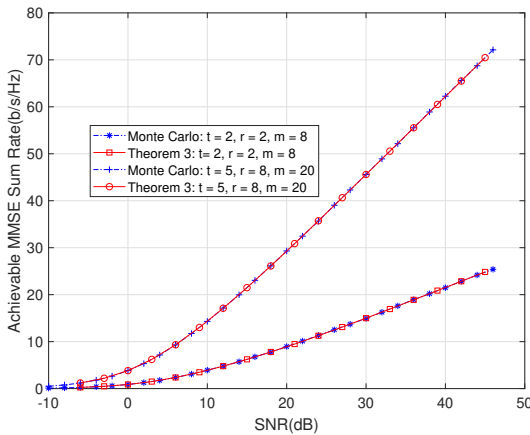


FIGURE 6. Evolution of the ergodic sum rate for the MMSE receiver over MIMO Jacobi fading channel.

Finally, Fig. 7 shows the evolution of the ergodic sum rate for the MMSE receiver versus the SNR over an uncorrelated MIMO Rayleigh fading channel. We compare the sum rate obtained by means of Monte Carlo simulations and the one obtained with the formula derived in Corollary 1. Fig. 7 shows a perfect match between Monte Carlo and analytical result given in (24). It worth noting that, for  $t = r = 2$  and  $t = r = 4$ , the obtained simulation results are the same as reported in [41], [42] and [43].

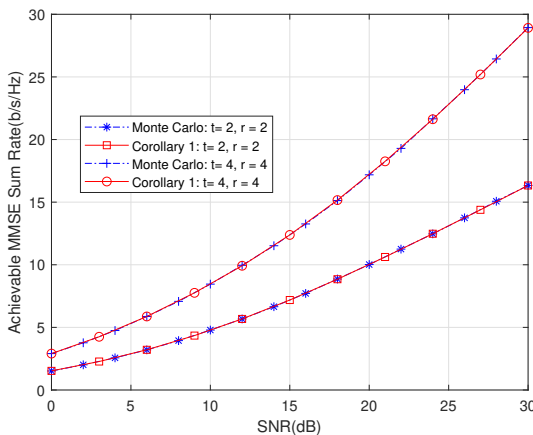


FIGURE 7. Evolution of the ergodic sum rate for the MMSE receiver over an uncorrelated MIMO Rayleigh fading channel.

## VIII. CONCLUSIONS

This paper has investigated the ergodic capacity of MIMO Jacobi fading channel which can be used to model accurately multimode and/or multicore optical fibers with the following characteristics: high crosstalk between modes and/or cores, negligible backscattering and near-lossless propagation. We assumed that a perfect channel state information (CSI) is only available at the receiver side, by using the joint distribution of eigenvalues of the Jacobi unitary ensemble, an exact expression of the ergodic capacity has been derived. By appealing to the limit relation between Jacobi and associated Laguerre polynomials, an exact expression of the ergodic capacity of MIMO Rayleigh fading channels has further been obtained. Furthermore, the above results led to exact expressions of the achievable sum rate for MIMO MMSE receiver in both fading channels. Monte Carlo simulations have been conducted to check the validity of the analytical results. Theoretical results show perfect matching with those obtained by simulations, and allow to derive tight bounds on the ergodic capacity for both MIMO fading channels. Considering the fact that wireless or fiber channels are subject to eavesdropping, we will address the MIMO secrecy capacity problem in future research papers.

## APPENDIX A PROOF OF PROPOSITION 1

For ease of reading, we simply denote below the ergodic capacity by  $C(\rho)$ . Moreover, the reader can easily check that our computations are valid for real  $a > 0$ ,  $b > 1$ . We start by recalling from [24, Corollary 2.3] that for any  $k \geq 1$ ,

$$\int \left( \sum_{i=1}^t \lambda_i^k \right) \mathcal{F}_{a,b,t}(\lambda) d\lambda = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \prod_{j=-i}^{k-i-1} \frac{(t+j)(a+t+j-1)}{(a+b+2t+j-2)}.$$

Now, let  $\rho \in [0, 1]$  and use the Taylor expansion

$$\ln(1 + \rho \lambda_i) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(\rho \lambda_i)^k}{k}$$

to get

$$\sum_{i=1}^t \ln(1 + \rho \lambda_i) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\rho^k}{k} \left( \sum_{i=1}^t \lambda_i^k \right).$$

Consequently,

$$C(\rho) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \rho^k}{k} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \prod_{j=-i}^{k-i-1} \frac{(t+j)(a+t+j-1)}{(a+b+2t+j-2)}. \quad (25)$$

Changing the summation order and performing the index change  $k \mapsto k + i + 1$  in (25), we get

$$C(\rho) = \sum_{i=0}^{\infty} (-1)^i \sum_{k=0}^{\infty} \frac{(-1)^{k+i}}{(k+i+1)} \frac{\rho^{k+i+1}}{(k+i+1)!} \binom{k+i}{i} \prod_{j=-i}^k \frac{(t+j)(a+t+j-1)}{(a+b+2t+j-2)}.$$

Now, one can observe that the product displayed in the right hand side of the last equality vanishes whenever  $i \geq t$  due to the presence of the factor  $j+t, -i \leq j \leq k$ . Thus, the first series terminates at  $i = t-1$  and together with the index change  $j \mapsto t+j$  in the product lead to

$$C(\rho) = \sum_{i=0}^{t-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+i+1)} \frac{\rho^{k+i+1}}{(k+i+1)!} \binom{k+i}{i} \prod_{j=t-i}^{k+t} \frac{(j)(a+j-1)}{(a+b+t+j-2)}.$$

Next, we compute for each  $t-i \leq j \leq t+k$

$$\prod_{j=t-i}^{t+k} (j) = \frac{(t+k)!}{(t-i-1)!} = \frac{(t+1)_k t!}{(t-i-1)!},$$

and similarly

$$\prod_{j=t-i}^{t+k} (a+j-1) = \frac{(a+t)_k (a)_t}{(a)_{t-i-1}}$$

$$\prod_{j=t-i}^{t+k} (a+b+t+j-2) = \frac{(a+b+t-1)_{t+k}}{(a+b+t-1)_{t-i-1}}$$

Altogether, the ergodic capacity reads

$$\frac{(a)_t}{(a+b+t-1)_t} \sum_{i=0}^{t-1} \frac{t!}{(t-1-i)! i!} \frac{(a+b+t-1)_{t-i-1}}{(a)_{t-i-1}} \sum_{k \geq 0} \frac{(-1)^k \rho^{k+i+1}}{(k+i+1)^2} \frac{(t+1)_k (a+t)_k}{(a+b+2t-1)_k k!}.$$

But the series

$$\sum_{k \geq 0} \frac{(-1)^k \rho^{k+i+1}}{(k+i+1)^2} \frac{(t+1)_k (a+t)_k}{(a+b+2t-1)_k k!}$$

as well as its derivatives with respect to  $\rho$  converge uniformly in any closed sub-interval in  $]0, 1[$ . It follows that

$$D_\rho(\rho D_\rho) \sum_{k \geq 0} \frac{(-1)^k \rho^{k+i+1}}{(k+i+1)^2} \frac{(t+1)_k (a+t)_k}{(a+b+2t-1)_k k!} = \rho^i {}_2F_1(t+1, a+t, a+b+2t-1; -\rho)$$

where  $D_\rho$  is the derivative operator acting on the variable  $\rho$ . Finally, the index change  $i \mapsto t-i-1$  together with

$$(1-t)_i = (-1)^i \frac{(t-1)!}{(t-1-i)!}$$

yield

$$\sum_{i=0}^{t-1} \frac{t!}{(t-1-i)! i!} \frac{(a+b+t-1)_{t-i-1}}{(a)_{t-i-1}} \rho^i = \frac{t! \rho^{t-1}}{(a)_{t-1}} P_{t-1}^{a-1, b} \left( \frac{\rho+2}{\rho} \right).$$

Since

$$\frac{t! \rho^{t-1}}{(a)_{t-1}} \frac{(a)_t}{(a+b+t-1)_t} = \frac{t!(a+t-1) \rho^{t-1}}{(a+b+t-1)_t},$$

The statement of the proposition 1 corresponds to the special parameters  $a = r-t+1$  and  $b = m-t-r+1$ .

## APPENDIX B PROOF OF THEOREM 1

Let's  $n = t$  and  $\rho \in [0, 1]$ . From [25, Eq. (4.4.6)], we readily deduce that the hypergeometric function

$${}_2F_1(n+1, a+n, a+b+2n-1; -\rho)$$

coincides up to a multiplicative factor with the Jacobi function of the second kind  $Q_n^{a-1, b-2}$  in the variable  $x$  related to  $\rho$  by

$$-\rho = \frac{2}{1-x} \Leftrightarrow x = \frac{\rho+2}{\rho}.$$

Consequently,

$$[D_\rho(\rho D_\rho)] C(\rho) = 2B_{a,b,n} \frac{(1+\rho)^{b-2}}{\rho^{a+b-1}} P_{n-1}^{a-1, b} \left( \frac{\rho+2}{\rho} \right) Q_n^{a-1, b-2} \left( \frac{\rho+2}{\rho} \right)$$

where

$$B_{a,b,n} = \frac{n! \Gamma(a+b+n-1)}{\Gamma(a+n-1) \Gamma(N+n-1)}.$$

Moreover, recall from [25, Eq. (4.4.2)], that (note that  $(\rho+2)/\rho > 1$ )

$$Q_n^{a-1, b-2} \left( \frac{\rho+2}{\rho} \right) = \frac{\rho^{a+b-3}}{2^{a+b-4} (\rho+1)^{b-2}} \int_{-1}^1 (1-u)^{a-1} \times (1+u)^{b-2} \frac{P_n^{a-1, b-2}(u)}{((\rho+2)/\rho) - u} du.$$

After some mathematical manipulation and since

$$u \mapsto \frac{1}{((\rho+2)/\rho) - u} \left( P_{n-1}^{a-1, b} \left( \frac{\rho+2}{\rho} \right) - P_{n-1}^{a-1, b}(u) \right)$$

is a polynomial of degree  $n-2$ , then the orthogonality of the Jacobi polynomials entails

$$[D_\rho(\rho D_\rho)] C(\rho) = \frac{B_{a,b,n}}{2^{a+b-3}} \int_{-1}^1 (1-u)^{a-1} (1+u)^{b-2} \times P_{n-1}^{a-1, b}(u) \frac{P_n^{a-1, b-2}(u)}{\rho(\rho+2-\rho u)} du.$$

Writing

$$\frac{1}{\rho(\rho+2-\rho u)} = \frac{1}{2} \left[ \frac{1}{\rho} - \frac{(1-u)}{\rho+2-\rho u} \right], \quad u \in [-1, 1],$$

and using again the orthogonality of Jacobi polynomials, we get

$$[D_\rho(\rho D_\rho)]C(\rho) = -\frac{B_{a,b,n}}{2^{a+b-2}} \int_{-1}^1 (1-u)^a(1+u)^{b-2} \times \frac{P_{n-1}^{a-1,b}(u)P_n^{a-1,b-2}(u)}{(\rho(1-u)+2)} du$$

which is still defined at  $\rho = 0$ . A first integration with respect to  $\rho$  gives

$$[\rho D_\rho]C(\rho) = -\frac{B_{a,b,n}}{2^{a+b-2}} \int_{-1}^1 (1-u)^{a-1}(1+u)^{b-2} P_{n-1}^{a-1,b}(u) \times P_n^{a-1,b-2}(u) [\ln(\rho(1-u)+2) - \ln 2] du$$

and a second integration leads to

$$C(\rho) = -\frac{B_{a,b,n}}{2^{a+b-2}} \int_{-1}^1 (1-u)^{a-1}(1+u)^{b-2} P_{n-1}^{a-1,b}(u) \times P_n^{a-1,b-2}(u) \left\{ \int_0^\rho \frac{\ln(v(1-u)/2+1)}{v} dv \right\} du.$$

Performing the variable changes  $u \mapsto 1-2u$  in the last expression, we end up with

$$C(\rho) = -B_{a,b,n} \int_0^1 u^{a-1}(1-u)^{b-2} P_{n-1}^{a-1,b}(1-2u) \times P_n^{a-1,b-2}(1-2u) \left\{ \int_0^\rho \frac{\ln(vu+1)}{v} dv \right\} du$$

for any  $\rho \in [0, 1[$ . By analytic continuation, this formula extends to the cut plane  $\mathbb{C} \setminus (-\infty, 0)$  and is in particular is valid for  $\rho \geq 0$ . Specializing it to  $a = r - t + 1$ , and  $b = m - t - r + 1$  completes the proof of the Theorem 1.

### APPENDIX C PROOF OF THEOREM 2

Perform the variable change  $\rho \mapsto b\rho$  in the definition of  $C_{t,r}^{m,\rho}$ :

$$\begin{aligned} C(b\rho) &= Z_{a,b,n}^{-1} \int \ln \left( \prod_{i=1}^n (1+b\rho\lambda_i) \right) \prod_{i=1}^n \lambda_i^{a-1} (1-\lambda_i)^{b-1} \\ &\quad \times V(\lambda)^2 \mathbf{1}_{\{0 < \lambda_1 < \dots < \lambda_n < 1\}} d\lambda \\ &= \frac{Z_{a,b,n}^{-1}}{b^{(a+n)(n-1)}} \int \ln \left( \prod_{i=1}^n (1+\rho\lambda_i) \right) \prod_{i=1}^n \lambda_i^{a-1} \\ &\quad \times \left( 1 - \frac{\lambda_i}{b} \right)^{b-1} V(\lambda)^2 \mathbf{1}_{\{0 < \lambda_1 < \dots < \lambda_n < b\}} d\lambda. \end{aligned}$$

On the other hand, our obtained expression for the ergodic capacity together with the variable change  $v \mapsto bv$  entail:

$$\begin{aligned} C(b\rho) &= -\frac{B_{a,b,n}}{b^a} \int_0^1 u^{a-1} \left( 1 - \frac{u}{b} \right)^{b-2} P_{n-1}^{a-1,b} \left( 1 - \frac{2u}{b} \right) \\ &\quad \times P_n^{a-1,b-2} \left( 1 - \frac{2u}{b} \right) \left\{ \int_0^\rho \frac{\ln(vu+1)}{v} dv \right\} du \end{aligned}$$

Now

$$\lim_{b \rightarrow \infty} \frac{B_{a,b,n}}{b^a} = \frac{n!}{\Gamma(a+n-1)}$$

and similarly

$$\lim_{b \rightarrow \infty} \frac{Z_{a,b,n}^{-1}}{b^{n(a+n-1)}} = \prod_{i=1}^n \frac{1}{\Gamma(i)\Gamma(a+i-1)}$$

Moreover, the limiting transition (6) yields

$$\begin{aligned} \lim_{b \rightarrow \infty} P_{n-1}^{a-1,b} \left( 1 - \frac{2u}{b} \right) &= L_{n-1}^{a-1}(u) \\ \lim_{b \rightarrow \infty} P_n^{a-1,b-2} \left( 1 - \frac{2u}{b} \right) &= L_n^{a-1}(u). \end{aligned}$$

As a result,

$$\begin{aligned} \lim_{b \rightarrow \infty} C(b\rho) &= -\frac{n!}{\Gamma(a+n-1)} \int_0^{+\infty} u^{a-1} e^{-u} L_{n-1}^{a-1}(u) \\ &\quad \times L_n^{a-1}(u) \left\{ \int_0^\rho \frac{\ln(vu+1)}{v} dv \right\} du. \end{aligned}$$

where  $\prod_{i=1}^n \frac{1}{\Gamma(i)\Gamma(a+i-1)}$  is the normalization constant of the density of the joint distribution of the ordered eigenvalues of a complex Wishart matrix [23]. The theorem is proved.

### ACKNOWLEDGMENT

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