Liouville results and asymptotics of solutions of a quasilinear elliptic equation with supercritical source gradient term

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Abstract

We consider the elliptic quasilinear equation $-\Delta_m u = u^p |\nabla u|^q$ in $\mathbb{R}^N$ with $q \geq m$ and $p > 0$, $1 < m < N$. Our main result is a Liouville-type property, namely, all the positive C^1 solutions in $\mathbb{R}^N$ are constant. We also give their asymptotic behaviour : all the solutions in an exterior domain $\mathbb{R}^N \backslash B_{r_0}$ are bounded. The solutions in $B_{r_0} \backslash \{0\}$ can be extended as a continuous functions in B_{r_0}. The solutions in $\mathbb{R}^N \backslash \{0\}$ has a finite limit $l \geq 0$ as $|x| \to \infty$. Our main argument is a Bernstein estimate of the gradient of a power of the solution, combined with a precise Osserman's type estimate for the equation satisfied by the gradient.

Key Words: Liouville property, Bernstein method, Keller-Osserman estimates.

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Contents

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1 Introduction

In this paper we study local and global properties of positive solutions of the equation

$$-div\left(|\nabla u|^{m-2}\nabla u\right) := -\Delta_m u = u^p\,|\nabla u|^q\,, \tag{1.1}$$

in $\mathbb{R}^N$, ($N \geq 1$, $1 < m < N$ and $p > 0$) in the supercritical case

$$q \geq m. \tag{1.2}$$

We are concerned by the Liouville property in $\mathbb{R}^N$, which is wether all the positive C^1 solutions are constant. We also study the asymptotic behaviour of any solution of (1.1) near a singularity in the punctured ball $B_{r_0}\setminus\{0\}$, in $\mathbb{R}^N\setminus\{0\}$ or in an exterior domain $\mathbb{R}^N\setminus B_{r_0}$.

In the case $q = 0$, equation (1.1) reduces to the classical Lane-Emden-Fowler equation

$$-\Delta_m u = u^p, \tag{1.3}$$

which has already been the subject of countless publications. One of the questions solved is that the Liouville property holds if and only if

$$p < p_m^* := \frac{N(m-1)+m}{N-m}.$$

Note that p_m^* is the Sobolev exponent. Since it is impossible to quote all the articles on the subject, we only mention here the pioneering works and references therein. Gidas and Spruck [21] first showed the nonexistence of positive solutions in $\mathbb{R}^N$ for $m = 2$ and $p < p_2^*$. They combine the Bernstein technique applied in the equation satisfed by the gradient of a suitable power of u, with delicate integral estimates ensuring the Harnack inequality, see also [6]. Then the complete behaviour up to the case $p = p_2^*$ was obtained by moving plane methods by [14], see also [16]. In the general case $m > 1$, the nonexistence of nontrivial solutions for $p < p_2^*$ was proved in a beautiful article of Serrin and Zou [33], then the extension to the case $p = p_2^*$ was done by [30] for $m < 2$, then [36] for $1 < m < 2$, and finally [29] for any $m > 1$.

When $p = 0$, (1.1) reduces to the Hamilton-Jacobi equation

$$-\Delta_m u = |\nabla u|^q\,.$$

The Liouville property was proved in [24] for $m = 2$, and in [9] for any $m > 1$, using the Bernstein technique. In that case the nonexistence holds for any $q > m - 1$, without any sign condition on the solution. Estimates of the gradient for more general problems can be found in [23].

For the general case of equation (1.1), consider the range of exponents

$$p > 0, \qquad p + q + 1 - m > 0.$$

As in the case $q = 0$, there exists a "first subcritical case", where

$$p < \frac{N(m-1)}{N-m} - \frac{(N-1)q}{N-m},$$

for which any supersolution in $\mathbb{R}^N$ of equation (1.1) is constant, from [19]. Beyond this case, a second critical case appears when $0 \le q < m-1$: indeed there exist radial positive nonconstant solutions of (1.4) whenever $p \ge p^*_{m,q}$, where

$$p^*_{m,q} = \frac{N(m-1)+m}{N-m} - \frac{q((N-1)q - N(m-1)+m)}{(N-m)(m+1-q)},$$

see [15] and [8].

When $m = 2 < N$ and $p > 0$, equation

$$-\Delta u = u^p \left|\nabla u\right|^q \tag{1.4}$$

was studied in [8] for $0 < q \le 2$. The case $q = 2$ could be solved explicitely by a change of the unknown function, showing that the Liouville property holds for any $p > 0$. Using a direct Bernstein technique we obtained a first range of values of (p, q) for which the Liouville property holds, in particular it holds when $p + q - 1 < \frac{4}{N-1}$, covering the first subcritical case. Using an integral Bernstein technique in the spirit of [21] we obtained a wider range of (p, q) ensuring the Liouville property, recovering Gidas and Spruck result $p < \frac{N+2}{N-2}$ when $q = 0$. However some deep questions remained unsolved: Does the property hold for any $p < p^*_{2,q}$ when $q < 1$? Does it hold for any $p > 0$ when $1 \le q < 2$?

In a recent article, Filippucci, Pucci and Souplet [20, Theorem 1.1] considered the case $m = 2$, $q > 2$, of a superquadratic growth in the gradient, a case which was not covered by [8]. They proved the following:

Theorem [20, Theorem 1.1] *Any classical positive and* ***bounded*** *solution of equation (1.4) in* $\mathbb{R}^N$ *with* $q \ge 2$ *and* $p > 0$ *is constant.*

In this article, we prove that the Liouville property holds true not only for (1.4) but for the quasilinear equation (1.1) **without the assumption of boundedness** on the solution. Our main result is the following

Theorem 1.1 *Let* u *be any positive* $C^1(\mathbb{R}^N)$ *solution of equation (1.1), with* $1 < m < N$ *and*

$$q \ge m, \qquad p \ge 0. \tag{1.5}$$

Then u *is constant.*

We show that the case $q = m$ can still be solved explicitely, giving the complete behaviour of the solutions of the equation, see Theorem 2.1. Next we assume $q > m$. We prove that all the solutions in an exterior domain are **bounded**, and we give the asymptotic behaviour ($|x| \to 0$ and $|x| \to \infty$) of the solutions in $\mathbb{R}^N \setminus \{0\}$:

Theorem 1.2 *Assume* $1 < m < N$, $q > m$, $p \ge 0$. *Then any positive* C^1 *solution* u *of (1.1) in* $\mathbb{R}^N \setminus B_{r_0}$ *is bounded. If* u *is a non-constant solution, then* $|\nabla u|$ *does not vanish for* $|x| > r_0$. *Moreover any positive solution* u *in* $\mathbb{R}^N \setminus \{0\}$ *satisfies*

$$\lim_{|x|\to\infty} u(x) = l \ge 0. \tag{1.6}$$

If $l > 0$, *there exist constants* $C_1, C_2 > 0$ *such that for* $|x|$ *large enough,*

$$C_1 \left|x\right|^{\frac{N-m}{m-1}} \le |u(x) - l| \le C_2 \left|x\right|^{\frac{N-m}{m-1}}. \tag{1.7}$$

Concerning the solutions in $B_{r_0}\setminus\{0\}$ and in particular in $\mathbb{R}^N\setminus\{0\}$ we proved an estimate of the gradient, showing that the solution is continuous up to 0 but the gradient is singular at 0:

Theorem 1.3 *Assume $1 < m < N$, $q > m$, $p \geq 0$. Any positive solution u in $B_{r_0}\setminus\{0\}$ is bounded near 0, it can be extended as a continuous function in B_{r_0}, such that $u(0) > 0$, and for any $x \in B_{\frac{r_0}{2}}\setminus\{0\}$*

$$|\nabla u(x)| \leq C\,|x|^{-\frac{1}{q-m+1}}\,, \tag{1.8}$$

where $C = C(N,p,q,m,u)$. Finally

$$|u(x)-u(0)| \leq C\,|x|^{\frac{q-m}{q-m+1}}\,, \tag{1.9}$$

near 0, where $C = C(N,p,q,m,u(0))$. Moreover, if u is defined in $\mathbb{R}^N\setminus\{0\}$, then $u(x) \leq u(0)$ in $\mathbb{R}^N\setminus\{0\}$.

Note that the exponent involved in (1.8) is independent of p, actualy the solution behaves like a solution of the Hamilton-Jacobi equation

$$-\Delta_m u = c\,|\nabla u|^q\,, \tag{1.10}$$

with $c = u^p(0)$.

Finally we make an exhaustive study of the radial solutions for $q > m$, showing the sharpness of the nonradial results. We reduce the study to the one of an autonomous quadratic polynomial system of order 2, following the technique introduced in [10]. Compared to other classical techniques, it provides a complete description of all the positive solutions, in particular the global ones, without questions of regularity. We prove the following:

Theorem 1.4 *Assume $1 < m < N$, $q > m$, $p \geq 0$ and u is any positive non constant radial solution $r \mapsto u(r)$ of (1.1) in an interval $(a,b) \subseteq (0,\infty)$.*
(i) If $a = 0$, then u is bounded, decreasing and singular:

$$\lim_{r\to 0} u = u_0 > 0, \qquad \lim_{r\to 0} r\left|u'\right|^{q-m+1} = \frac{a_{m,q}}{u_0^p}, \qquad a_{m,q} = \frac{(N-1)q - N(m-1)}{q+1-m}. \tag{1.11}$$

And for given $u_0 > 0$, there exist infinitely many such solutions;
(ii) If $b = \infty$, then u admits a limit a limit $l \geq 0$ at infinity and

$$\lim_{r\to\infty} r^{\frac{N-m}{m-1}}\,|u(r)-l| = k > 0. \tag{1.12}$$

Furthermore, for given $l > 0$, $c \neq 0$ there exists a unique local solution near ∞, such that

$$\lim_{r\to\infty} r^{\frac{N-m}{m-1}}\,(u(r)-l) = c. \tag{1.13}$$

(iii) For any $u_0 > 0$, there exist infinitely many solutions in $(0,\infty)$, decreasing, such that $\lim_{r\to 0} u = u_0$, but a unique one, satisfying

$$\lim_{r\to 0} u = u_0 \quad \text{and} \quad \lim_{r\to\infty} u = 0. \tag{1.14}$$

There exist infinitely many solutions defined on an interval $(0,\rho)$, such that $\lim_{r\to\rho} u = 0$, and an infinity such that $\lim_{r\to\rho} u' = -\infty$. Finally, there exist an infinity of solutions in (ρ,∞) such that $\lim_{r\to\rho} u = 0$, and an infinity of solutions such that $\lim_{r\to\rho} u' = \infty$.

Note that Theorems 1.2 and 1.3 lead to the following natural question: are all the solutions in $\mathbb{R}^N \setminus \{0\}$ radially symmetric? This is still an open problem, even in the case $p = 0$ of the Hamilton-Jacobi equation.

To conclude this paper, we improve another result of [20], where it was noticed that [20, Theorem 1.1] was still valid for $p < 0$, $q \geq 2$. We prove here a much more general result covering the case $p = 0$.

Theorem 1.5 *Assume $1 < m < N$, $p \leq 0$ and $p + q + 1 - m > 0$. Then there exists a constant $C = C(N, p, q, m) > 0$ such that for any positive C^1 solution u of (1.1) in a bounded domain Ω,*

$$|\nabla u(x)| \leq C \ dist(x, \partial\Omega)^{-\frac{1}{q+1-m}}, \quad \forall x \in \Omega.$$

If $\Omega = \mathbb{R}^N$, then u is constant.

Let us give a brief comment on the analogous equation with an absorption term:

$$-\Delta_m u + u^p \, |\nabla u|^q = 0. \tag{1.15}$$

In the case $m = 2$, $0 < q < 2$, a complete classification of the solutions with isolated singularities was performed in [17]. A main contribution was recently given by the same authors in [18] where they obtained optimal estimates of the gradient for any $1 < m \leq N$, $p, q \geq 0$, $p + q - m + 1 > 0$, still by the Bernstein method.

Our paper is organized as follows. In Section 2 we first treat the case $q = m$. In Section 3 we give the main ideas of our proofs when $q > m = 2$, and we introduce some tools for the general case $q > m > 1$. Our main theorems are proved in Section 4, and Section 5 is devoted to the radial case. The extension to the case $p \leq 0$ is given in Section 6.

2 The case $q = m$

If $q = m$ we can express explicitely the solutions of (1.1). We prove the following:

Theorem 2.1 *Let $1 < m < N$, $p \geq 0, q = m$. Then*
(i) any C^1 positive solution in $\mathbb{R}^N$ is constant;
(ii) any nonconstant positive solution in $\mathbb{R}^N \setminus B_{r_0}$ has a limit l at ∞ and

$$\lim_{|x| \to \infty} |x|^{\frac{m-N}{m-1}} \, |u - l| = c > 0;$$

(iii) any positive solution in $B_{r_0} \setminus \{0\}$ extends as a continuous function in B_{r_0}, or satisfies

$$\lim_{x \to 0} \frac{u^{p+1}}{|\ln |x||} = \frac{(N - m)(p + 1)}{m - 1}; \tag{2.1}$$

(iv) any positive solution in $\mathbb{R}^N \setminus \{0\}$ is radial.

Proof. We use a change of variable already considered in [1]: the equation takes the form

$$-\Delta_m u = \beta(u)\,|\nabla u|^m\,,\ \text{with } \beta(u) = u^p. \tag{2.2}$$

We set $\gamma(\tau) = \int_0^\tau \beta(\theta)d\theta = \frac{\tau^{p+1}}{p+1}$, and

$$U(x) = \Psi(u(x)) = \int_0^{u(x)} e^{\frac{\gamma(\theta)}{m-1}}d\theta := \int_0^{u(x)} e^{\frac{\theta^{p+1}}{(p+1)(m-1)}}d\theta. \tag{2.3}$$

A function u is a solution of (1.1) if and only if the above function U satisfies

$$-\Delta_m U = 0,$$

and if u is nonnegative not identically 0, U is m-harmonic and positive. Conversely, u is derived from U by

$$u(x) = \Psi^{-1}(U(x)) = \int_0^{U(x)} \frac{ds}{1+g(s)} \ \text{ where } \ g(s) = \int_0^s \beta(\Psi(w))dw = \int_0^s \Psi^p(w)dw. \tag{2.4}$$

(i) If u is a solution in $\mathbb{R}^N$ of (2.2), it is constant. Indeed any nonnegative m-harmonic functions U defined in $\mathbb{R}^N$ is constant, from the Harnack inequality, see [28], [31] and [33, Theorem II].
(ii) If u is defined in $\mathbb{R}^N\backslash B_{r_0}$, then U is bounded, it admits a limit L at ∞ and there holds $|U(x)-L| \le C\,|x|^{\frac{p-N}{p-1}}$ near ∞, see [3] for more general results. Clearly the same properties hold for u (with another limit).
(iii) If u is defined in $B_{r_0}\backslash\{0\}$, it follows from [31] that, either U extends as a continuous m-harmonic function in B_{r_0}, or it behaves like $k\,|x|^{\frac{p-N}{p-1}}$ near 0, so (2.1) holds.
(iv) If u is a solution in $\mathbb{R}^N\backslash\{0\}$, it is proved in [27] that U is radial and endows the form

$$U(x) = k\,|x|^{\frac{m-N}{m-1}} + \lambda \qquad \text{with } k, \lambda \ge 0.$$

Then u is radial, and, using (2.4), it has the expression

$$u(x) = \int_0^\lambda \frac{ds}{1+g(s)} + \int_0^{k|x|^{\frac{m-N}{m-1}}} \frac{ds}{1+g(s-\lambda)}.$$

■

3 Main arguments of the proofs

3.1 Ideas in the case $m = 2$

Before detailling the proof of Theorem 1.1. for $q > m$, we give an overview of it in the simple case of equation (1.4), with $m = 2$, $p > 0, q > 2$. We set $u = v^b$, with $b \in (0,1)$, and obtain

$$-\Delta v = (b-1)\frac{|\nabla v|^m}{v} + b^{q-1}v^s\,|\nabla v|^q\,,$$

with $s = 1-q+b(p+q-1)$. Next we explicit the equation satisfied by $z = |\nabla v|^2$. Taking in account the Böchner formula and Cauchy-Schwarz inequality in $\mathbb{R}^N$,

$$-\frac{1}{2}\Delta z + \frac{1}{N}(\Delta v)^2 + < \nabla(\Delta v), \nabla v) \leq -\frac{1}{2}\Delta z + (Hess v)^2 + < \nabla(\Delta v), \nabla v >= 0,$$

we get an estimate of the form, with universal constants $C_i > 0$,

$$-\Delta z + C_1 v^{2s} z^q \leq C_2 \frac{z^2}{v^2} + C_3 \frac{1}{v} < \nabla z, \nabla v > + C_4 v^s z^{\frac{q-2}{2}} < \nabla z, \nabla v >,$$

then

$$-\Delta z + C_5 v^{2s} z^q \leq C_6 \frac{z^2}{v^2} + C_7 \frac{|\nabla z|^2}{z}. \tag{3.1}$$

Using the Hölder inequality we deduce,

$$-\Delta z + C_8 v^{2s} z^q \leq C_9 v^{-\frac{2(q+2s)}{q-2}} + C_7 \frac{|\nabla z|^2}{z}.$$

The crucial step is an estimate of Osserman's type in a ball B_ρ valid for functions satisfying the inequality

$$-\Delta z + \alpha(x) z^k \leq \beta(x) + d\frac{|\nabla z|^2}{z} \qquad \text{in } B_\rho,$$

where $k > 1$. This is proved in Lemma 3.1 below, and it asserts that

$$z(x) \leq C(N,k,d)\left(\frac{1}{\rho^2}\max_{B_\rho}\frac{1}{\alpha}\right)^{\frac{1}{k-1}} + \left(\max_{B_\rho}\frac{\beta}{\alpha}\right)^{\frac{1}{k}} \qquad \text{in } B_{\frac{\rho}{2}}.$$

Then we take $b = \frac{q-2}{p+q-1}$, in the same spirit as in [20], so that $\frac{B}{\alpha}$ is constant and $\alpha^{-1}(x) = v^2(x)$. We obtain an estimate

$$\max_{\bar{B}_{\frac{\rho}{2}}} |\nabla v| \leq C\left(\left(\frac{\max_{B_\rho} v}{\rho}\right)^{\frac{1}{q-1}} + 1\right).$$

But any solution in $\mathbb{R}^N$ satisfies for any $\rho \geq 1$

$$\max_{B_\rho} v \leq v(0) + C\rho \max_{B_\rho} |\nabla v| \leq C\rho(1 + \max_{B_\rho} |\nabla v|), \tag{3.2}$$

which yields

$$\max_{B_{\frac{\rho}{2}}} |\nabla v| \leq C((\max_{B_\rho} |\nabla v|)^{\frac{1}{q-1}} + 1).$$

Using the bootstrap method developped in [11] and [9] based upon the fact that $\frac{1}{q-1} < 1$, we deduce that $|\nabla v| \in L^\infty(\mathbb{R}^N)$. Note that the boundness of $|\nabla v|$ had been obtained in [20] but under the extra assumption $u \in L^\infty(\mathbb{R}^N)$, an assumption that we get rid of. Returning to $u = v^b$, it means that

$$-\Delta u = u^p |\nabla u|^q \leq C\frac{|\nabla u|^2}{u},$$

and the same happens for $u-l$, where $l=\inf_{\mathbb{R}^N} u$. It implies that $w_l=(u-l)^\sigma$ is subharmonic for σ large enough. Then from [9], see also Lemma 3.3 below, and since u is superharmonic,

$$\sup_{B_R} w_l \le C\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} w^{\frac{1}{\sigma}}\right)^\sigma = C\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}(u-l)\right)^\sigma \le C'(\inf_{B_R} u-l)^\sigma.$$

Since C' is independent of R, it follows that $\sup_{\mathbb{R}^N} w_l=0$, thus $u\equiv l$.

Next we consider a solution in an exterior domain and we replace (3.2) by a more precise comparison estimate between $v(x)$ and its infimum on a sphere of radius $|x|$, and use the fact that this infimum is bounded as $r\to\infty$. Then we can show that u is still bounded, and obtain the behaviour near ∞ by a careful study of u and w_l. Finally we study the behaviour in $B_{r_0}\setminus\{0\}$ by the Bernstein technique, not relative to v but directly to u, that means we take $b=1$, so that $s=p$. From (3.1) the function $\xi=|\nabla u|^2$ satisfies

$$-\Delta\xi + C_5 u^{2p}\xi^q \le C_6\frac{\xi^2}{u^2} + C_7\frac{|\nabla z|^2}{z},$$

and $k=\inf_{B_{\frac{r_0}{2}}\setminus\{0\}} u$ is positive by the strong maximum principle, thus

$$-\Delta\xi + C_8\xi^q \le C_9\xi^2 + C_7\frac{|\nabla z|^2}{z} \le \frac{C_8}{2}\xi^q + C_{11} + C_7\frac{|\nabla z|^2}{z},$$

from what we deduce the estimates of ξ.

3.2 Some tools

In the sequel we use the Bernstein method. In the case $p=0$, it appeared that the square of the gradient is a subsolution of an elliptic equation with absorption, for which one can find estimates from above of Osserman's type. In the case of equation (1.1), the problem is more difficult, but such upper estimates were also a main step in study of [8] of equation (1.4) for $q<2$. Here also they constitue a crucial step of our proofs below. The following Lemma gives an Osserman's type property of such equations, extending of [8, Lemma 2.2], see also used in [7, Proposition 2.1].

Lemma 3.1 *Let Ω be a domain of $\mathbb{R}^N$, and $z\in C(\Omega)\cap C^2(G)$, where $G=\{x\in\Omega : z(x)\neq 0\}$. Let $w\mapsto \mathcal{A}w=-\sum_{i,j=1}^N a_{ij}\frac{\partial^2 w}{\partial x_i\partial x_j}$ be a uniformly elliptic operator in the open set G :*

$$\theta\,|\xi|^2 \le \sum_{i,j=1}^N a_{ij}\xi_i\xi_j \le \Theta\,|\xi|^2, \qquad \theta>0. \tag{3.3}$$

Suppose that for any $x\subset G$,

$$\mathcal{A}(z)+\alpha(x)z^k \le \beta(x) + d\frac{|\nabla z|^2}{z},$$

with $k>1$, and $d=d(N,p,q)$, and α,β are continuous in Ω and α is positive. Then there exists $c=c(N,p,q,k)>0$ such that for any ball $\overline{B}(x_0,\rho)\subset\Omega$ there holds

$$z(x_0)\le c\left(\frac{1}{\rho^2}\max_{B_\rho(x_0)}\frac{1}{\alpha}\right)^{\frac{1}{k-1}} + \left(\max_{B_\rho(x_0)}\frac{\beta}{\alpha}\right)^{\frac{1}{k}}.$$

Proof. Let $\overline{B}(x_0,\rho)\subset\Omega$. We can assume that $z(x_0)\neq 0$. Let $r=|x-x_0|$. Let w be the function defined in $B_\rho(x_0)$ by

$$w(x)=\lambda(\rho^2-r^2)^{-\frac{2}{k-1}}+\mu,$$

where $\lambda,\mu>0$. Let G_1 be a connected component of $\{x\in B_\rho(x_0);z(x)>w(x)\}$. Then $G_1\subset G$ and $\overline{G_1}\subset\overline{B}(x_0,\rho)\subset G$. We define $\mathcal{L}w$ in $B_\rho(x_0)$ by

$$\mathcal{L}(w)=\mathcal{A}(w)+\alpha(x)w^k-\beta(x)-d\frac{|\nabla w|^2}{w}.$$

Then

$$w_{x_i}=\frac{4\lambda}{k-1}(\rho^2-r^2)^{-\frac{2}{k-1}-1}x_i,$$

$$w_{x_ix_j}=\frac{4\lambda}{k-1}(\rho^2-r^2)^{-\frac{2}{k-1}-1}\delta_{ij}+\frac{4\lambda(k+1)}{(k-1)^2}(\rho^2-r^2)^{-\frac{2}{k-1}-2}x_ix_j,$$

$$\begin{aligned}
\mathcal{A}(w) &= -\sum_{i,j=1}^N a_{ij}w_{x_ix_j}=\frac{4\lambda}{k-1}(\rho^2-r^2)^{-\frac{2}{k-1}-1}(-\sum_{i,j=1}^N a_{ij}\delta_{ij})\\
&\quad+\frac{4\lambda(k+1)}{(k-1)^2}(\rho^2-r^2)^{-\frac{2}{k-1}-2}(-\sum_{i,j=1}^N a_{ij}x_ix_j)\\
&\geq -\Theta(\frac{4\lambda N}{k-1}(\rho^2-r^2)^{-\frac{2}{k-1}-1}+\frac{4\lambda(k+1)}{(k-1)^2}(\rho^2-r^2)^{-\frac{2}{k-1}-2}r^2\\
&= -\Theta(\frac{4\lambda N}{k-1}(\rho^2-r^2)^{-\frac{2}{k-1}-2}(N(\rho^2-r^2)+\frac{k+1}{k-1}r^2)\\
&= -\Theta(\frac{4\lambda}{k-1}(\rho^2-r^2)^{-\frac{2}{k-1}-2}(N\rho^2+(\frac{k+1}{k-1}-N)r^2),
\end{aligned}$$

$$|\nabla w|^2=\frac{16\lambda^2}{(k-1)^2}(\rho^2-r^2)^{-\frac{4}{k-1}-2}r^2\Longrightarrow\frac{|\nabla w|^2}{w}\leq\frac{16\lambda}{(k-1)^2}(\rho^2-r^2)^{-\frac{2}{k-1}-2}r^2,$$

and

$$w^k=(\lambda(\rho^2-r^2)^{-\frac{2}{k-1}}+\mu)^k\geq\mu^k+\lambda^k(\rho^2-r^2)^{-\frac{2k}{k-1}}=\mu^k+\lambda^k(\rho^2-r^2)^{-\frac{2}{k-1}-2}.$$

We deduce from this series of inequalities,

$$\begin{aligned}
\mathcal{L}(w) &\geq \alpha(x)\mu^k-\beta(x)+\lambda(\rho^2-r^2)^{-\frac{2k}{k-1}}(\lambda^{k-1}C(x)\\
&\quad-\Theta(\frac{4}{k-1}(N\rho^2+(\frac{k+1}{k-1}-N)r^2)-\frac{16dr^2}{(k-1)^2}\\
&\geq \alpha(x)\mu^k-\beta(x)+\lambda(\rho^2-r^2)^{-\frac{2k}{k-1}}(\lambda^{k-1}C(x)-c'\rho^2),
\end{aligned}$$

where $c'=\Theta(\frac{4}{k-1}(2N+\frac{k+1}{k-1})+\frac{16d}{(k-1)^2}=c'(N,p,q,k)$. We deduce that $\mathcal{L}(w)\geq 0$ if we impose

$$\mu^k\geq\max_{B_\rho(x_0)}\frac{\beta}{\alpha}\ \text{ and }\ \lambda^{k-1}\geq c'\rho^2\max_{B_\rho(x_0)}\frac{1}{\alpha}.$$

If $x_1 \in G_1$ is such that $z(x_1) - w(x_1) = \max_{G_1}(z-w) > 0$, then $\nabla z(x_1) = \nabla w(x_1)$, and $\mathcal{A}(z-w)(x_1) \geq 0$. Therefore

$$0 \geq \mathcal{L}(z-w)(x_1)) = \mathcal{A}(z-w)(x_1) + \alpha(x)(z^k - w^k)(x_1) + d\left(\frac{|\nabla w|^2}{w} - \frac{|\nabla z|^2}{z}\right).$$

Since the last term is positive, it is a contradiction. Then $z \leq w$ in $B_\rho(x_0)$. In particular $z(x_0) \leq w(x_0)$. ∎

We also use a bootstrap argument, initialy used in [11, Lemma 2.2], and then in [9] in more general form.

Lemma 3.2 *Let $d, h \in \mathbb{R}$ with $d \in (0,1)$ and y be a positive nondecreasing function on some interval (r_1, ∞). Assume that there exist $K > 0$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ and $r > r_1$,*

$$y(r) \leq K\varepsilon^{-h} y^d(r(1+\varepsilon)).$$

Then there exists $C = C(K, d, h, \varepsilon_0)$ such that $\sup_{(r_1,\infty)} y \leq C$.

Proof. Consider the sequence $\{\varepsilon_n\} = \{\varepsilon_0 2^{-n}\}_{n\geq 1}$. Since the series $\sum \varepsilon_n$ is convergent, the sequence $\{P_m\} := \{\prod_{i=1}^m (1+\varepsilon_i)\}_{m\geq 1}$ is convergent too, with limit $P > 0$. Then there holds for any $r > r_1$

$$y(r) \leq K\varepsilon_1^{-h} y^d(r(1+\varepsilon_1)) = K\varepsilon_1^{-h} y^d(rP_1).$$

We deduce by induction,

$$y(r) \leq K^{1+d+..+d^m}(\varepsilon_1^{-h}\varepsilon_2^{-hd}..\varepsilon_m^{-hd^{m-1}}) y^{d^m}(P_m r) = (K\varepsilon_0^{-h})^{1+d+..+d^m}(2^{h(1+2d+..+md^{m-1})}) y^{d^m}(P_m r),$$

and $rP_m \to rP$, $d^m \to 0$, thus $(y(P_m r))^{d^m} \to 1$. Therefore we deduce that for any $r > r_1$,

$$y(r) \leq (K\varepsilon_0^{-h})^{\sum_{m=1}^\infty d^m} 2^{\sum_{m=1}^\infty m d^{m-1}} = (K\varepsilon_0^{-h})^{\frac{1}{1-d}} 2^{\frac{d}{(1-d)^2}}.$$

∎

We also mention below a property of m-subharmonic functions given in [9, Lemma 2.1]. It's proof is also based upon a boostrap method and is valid for more general quasilinear operators:

Lemma 3.3 *Let $u \in W^{1,m}_{loc}(\Omega)$ be nonnegative, m-subharmonic function in a domain Ω of $\mathbb{R}^N$. Then for any $\tau > 0$, there exists a constant $C = C(N, m, \tau) > 0$ such that for any ball $B_{2\rho}(x_0) \subset \Omega$ and any $\varepsilon \in \left(0, \frac{1}{2}\right]$,*

$$\sup_{B_\rho(x_0)} u \leq C\varepsilon^{-\frac{Nm^2}{\tau^2}} \left(\frac{1}{\left|B_{(1+\varepsilon)\rho}(x_0)\right|} \int_{B_{(1+\varepsilon)\rho}(x_0)} u^\tau\right)^{\frac{1}{\tau}}.$$

Finally we use some simple properties of mean value on spheres of m-superharmonic functions, in the same spirit as the ones given in [2, Lemmas 3.7, 3.8, 3.9] for mean values on annulus, and in [13] for $m = 2$. For the sake of completeness we recall their proofs.

Lemma 3.4 *Let $u \in C^1(\Omega)$ be nonnegative, m-superharmonic in Ω.*
(i) If $\Omega = \mathbb{R}^N \backslash B_{r_0}$, then

$$r \mapsto \mu(r) := \inf_{|x|=r} u,$$

is bounded in (r_0, ∞), and strictly monotone or constant for large r.
(ii) If $\Omega = B_{r_0} \backslash \{0\}$, then $r \mapsto \mu(r)$ is nonincreasing in $(0, r_0)$.

Proof. (i) Let $r > r_0$ be fixed. The function $f(x) = \mu(r)(1 - (\frac{|x|}{r_0})^{\frac{m-N}{m-1}})$ is m-harmonic, and $u \geq f$ on $\partial B_r \cup \partial B_{r_0}$, therefore $u \geq f$ in $\overline{B_r} \backslash B_{r_0}$. Let $k > 0$ large enough such that $1 - k^{\frac{p-N}{p-1}} \geq \frac{1}{2}$. If we take $r > kr_0$ and any x such that $|x| = kr_0$ we obtain

$$u(x) \geq \mu(kr_0) \geq f(x) = \mu(r)(1 - k^{\frac{p-N}{p-1}}) \geq \frac{1}{2}\mu(r),$$

so $\mu(r)$ is bounded for $r > kr_0$. For any $r_2 > r_1 > r_0$, $\varphi(r_1, r_2) := \inf_{\overline{B_{r_2}} \backslash B_1} u = \min(\mu(r_1), \mu(r_2))$ from the maximum principle. Then φ is nonincreasing in r_2 and nondecreasing in r_1. If μ has a strict local minimum at some point r, then for $0 < \delta < \delta_0$ small enough, $\mu(r) < \varphi(r - \delta_0, r + \delta_0) \leq \varphi(r - \delta, r + \delta)$, which yields a contradiction as $\delta \to 0$. Then μ is monotone. If it is constant on two intervals (a, b) and (a', b') with $b < a'$ and non-constant on (b, a') it follows by Vazquez's maximum principle [35] that u is constant on $\overline{B}_b \setminus B_a$ and on $\overline{B}_{b'} \setminus B_{a'}$ but non constant on $B_{a'} \setminus \overline{B}_b$. It means, always by Vazquez's maximum principle,
- either $\min\{\mu(r) : a < r < b'\} = \mu(a)$ (if μ is nondecreasing) and the minimum of u in $\overline{B}_{b'} \setminus B_a$ is achieved in any point in $B_b \setminus \overline{B}_a$, hence u is constant in $\overline{B}_{b'} \setminus B_a$,
- or $\min\{\mu(r) : a < r < b'\} = \mu(a)$ (if μ is nonincreasing) and the minimum of u in $\overline{B}_{b'} \setminus B_a$ is achieved in any point in $B_{b'} \setminus \overline{B}_{a'}$, hence u is constant in $\overline{B}_{b'} \setminus B_a$.
In both case we obtain a contradiction. Hence μ is either strictly monotone for r large enough, or it is constant, and so is u.

(ii) For given $r_1 < r_0$, and $\delta > 0$, there exists $\varepsilon_\delta \leq r_1$ such that for $0 < \varepsilon < \varepsilon_\delta$, such that $\delta \varepsilon^{\frac{m-N}{m-1}} \geq \mu(r_1)$. Let $h(x) = \mu(r_1) - \delta |x|^{\frac{m-N}{m-1}}$. Then $u \geq h$ on $\partial B_{r_1} \cup \partial B_\varepsilon$, then $u \geq h$ in $\overline{B_{r_1}} \backslash B_\varepsilon$. Making $\varepsilon \to 0$ and then $\delta \to 0$, one gets $u \geq \mu(r_1)$ in $B_{r_1} \backslash \{0\}$, thus $\mu(r) \geq \mu(r_1)$ for $r < r_1$. ∎

4 Proof of the main results

4.1 Proof of the Liouville property for $q > m$

We first give a general Bernstein estimate for solutions of equation (1.1):

Lemma 4.1 *Let u be any C^1 positive solution of ((1.1) in a domain Ω, with $m > 1$ and p, q arbitrary real numbers. Let $G = \{x \in \Omega : |\nabla u(x)| \neq 0\}$. Let $u = v^b$ with $b \in \mathbb{R} \backslash \{0\}$ and $z = |\nabla v|^2$. Then the operator*

$$w \mapsto \mathcal{A}(w) = -\Delta w - (m-2)\frac{D^2 w(\nabla v, \nabla v)}{|\nabla v|^2} = -\sum_{i,j=1}^{N} a_{ij} v_{x_i x_j}, \tag{4.1}$$

with coefficients $a_{i,j}$ depending on ∇v, is uniformly elliptic in G, and for any $\varepsilon > 0$, there exists $C_\varepsilon = C_\varepsilon(N, m, p, q, b, \varepsilon)$ such that

$$-\frac{1}{2}\mathcal{A}(z) + \left(\frac{1-\varepsilon}{N}(b-1)^2(m-1)^2 - (1-b)(m-1)\right)\frac{z^2}{v^2} + \frac{1-2\varepsilon}{N}|b|^{2(q-m+1)}\, v^{2s} z^{q+2-m} + \left(\frac{1}{N}2(b-1)(m-1) - s\right)|b|^{q-m}\, bv^{s-1} z^{\frac{q+4-m}{2}} \leq C_\varepsilon \frac{|\nabla z|^2}{z}. \tag{4.2}$$

Proof. The following identities hold if $u = v^b$: $\nabla u = bv^{b-1}\nabla v$,

$$|\nabla u|^{m-2}\nabla u = |b|^{m-2}\, bv^{(b-1)(m-1)}\,|\nabla v|^{m-2}\nabla v,$$

$$\Delta_m u = |b|^{m-2}\, b(v^{(b-1)(m-1)}\Delta_m v + (b-1)(m-1)v^{(b(m-1)-m}\,|\nabla v|^m),$$

$$-v^{(b-1)(m-1)}\Delta_m v = (b-1)(m-1)v^{(b(m-1)-m}\,|\nabla v|^m + |b|^q\, v^{bp+(b-1)q}\,|\nabla v|^q,$$

and finally

$$-\Delta_m v = (b-1)(m-1)\frac{|\nabla v|^m}{v} + |b|^{q-m}\, bv^s\,|\nabla v|^q, \tag{4.3}$$

with

$$s = m - 1 - q + b(p + q - m + 1). \tag{4.4}$$

We set $z = |\nabla v|^2$. Then in G,

$$-\Delta_m v = f \Longleftrightarrow -\Delta v - (m-2)\frac{D^2 v(\nabla v, \nabla v)}{|\nabla v|^2} = f\,|\nabla v|^{2-m},$$

from which identity we infer

$$-\Delta v = (m-2)\frac{D^2 v(\nabla v, \nabla v)}{|\nabla v|^2} + (b-1)(m-1)\frac{|\nabla v|^2}{v} + |b|^{q-m}\, bv^s\,|\nabla v|^{q+2-m}$$

where

$$< Hess\, v(\nabla v), \nabla v >= D^2 v(\nabla v, \nabla v) = \frac{1}{2} < \nabla z, \nabla v > .$$

We recall the Böchner formula combined with Cauchy-Schwarz inequality,

$$-\frac{1}{2}\Delta z + \frac{1}{N}(\Delta v)^2 + < \nabla(\Delta v), \nabla v) \leq -\frac{1}{2}\Delta z + (Hess\, v)^2 + < \nabla(\Delta v), \nabla v >= 0.$$

Since

$$-\Delta v = \frac{m-2}{2}\frac{< \nabla z, \nabla v >}{z} + (b-1)(m-1)\frac{z}{v} + |b|^{q-m}\, bv^s z^{\frac{q+2-m}{2}},$$

we deduce

$$< \nabla(\Delta v), \nabla v >= -\frac{m-2}{2} < \nabla\frac{< \nabla z, \nabla v >}{z}, \nabla v > + (1-b)(m-1) < \nabla\frac{z}{v}, \nabla v > - |b|^{q-m}\, b(sv^{s-1} z^{\frac{q+4-m}{2}} + \frac{q+2-m}{2} v^s z^{\frac{q-m}{2}} < \nabla z, \nabla v >).$$

we observe that

$$< \nabla \frac{z}{v}, \nabla v >= \frac{< \nabla z, \nabla v >}{v} - \frac{z^2}{v^2} \quad \text{and} \quad \frac{< \nabla z, \nabla v >^2}{z^2} \leq \frac{|\nabla z|^2}{z},$$

thus

$$\begin{aligned} -\frac{m-2}{2} < \nabla \frac{< \nabla z, \nabla v >}{z}, \nabla v >= -\frac{m-2}{2} \left(\frac{D^2 z(\nabla v, \nabla v)}{z} + \frac{1}{2} \frac{|\nabla z|^2}{z} - \frac{< \nabla z, \nabla v >^2}{z^2} \right) \\ \geq -\frac{m-2}{2} \frac{D^2 z(\nabla v, \nabla v)}{z} - |m-2| \frac{|\nabla z|^2}{z}. \end{aligned}$$

We define the operator $\mathcal{A}$ by (4.1); it satisfies (3.3) with $\theta = \min(1, m-1)$ and $\Theta = \max(1, m-1)$, so it is uniformly elliptic in G. Therefore

$$\begin{aligned} -\frac{1}{2}\mathcal{A}(z) + \frac{1}{N}(\Delta v)^2 - (1-b)(m-1)\frac{z^2}{v^2} - |b|^{q-m} bsv^{s-1} z^{\frac{q+4-m}{2}} \\ \leq (b-1)(m-1)\frac{< \nabla z, \nabla v >}{v} + \frac{(q+2-m)|b|^{q-m} b}{2} v^s z^{\frac{q-m}{2}} < \nabla z, \nabla v > \\ + |m-2| \frac{|\nabla z|^2}{z}. \end{aligned} \tag{4.5}$$

For $\varepsilon > 0$ there holds by Hölder's inequality,

$$\frac{q+2-m}{2} v^s z^{\frac{q-m}{2}} < \nabla z, \nabla v > \leq \frac{\varepsilon}{N} |b|^{2(q-m+1)} v^{2s} z^{q+2-m} + C_\varepsilon \frac{|\nabla z|^2}{z},$$

$$\begin{aligned} (\Delta v)^2 &= \left(\frac{m-2}{2} \frac{< \nabla z, \nabla v >}{z} + (b-1)(m-1)\frac{z}{v} + |b|^{q-m} bv^s z^{\frac{q+2-m}{2}} \right)^2 \\ &\geq (b-1)^2 (m-1)^2 \frac{z^2}{v^2} + |b|^{2(q-m+1)} v^{2s} z^{q+2-m} + 2(b-1)(m-1)|b|^{q-m} bv^{s-1} z^{\frac{q+4-m}{2}} \\ &\quad -(m-2)\frac{|\nabla z|}{\sqrt{z}} (|b-1|(m-1)\frac{z}{v} + |b|^{q-m+1} v^s z^{\frac{q+2-m}{2}}), \end{aligned}$$

and for any $\varepsilon > 0$,

$$(m-2)\frac{|\nabla z|}{\sqrt{z}} (|b-1|(m-1)\frac{z}{v} \leq \frac{\varepsilon}{N}(b-1)^2 (m-1)^2 \frac{z^2}{v^2} + C_\varepsilon \frac{|\nabla z|^2}{z},$$

$$(m-2)\frac{|\nabla z|}{\sqrt{z}} |b|^{q-m+1} v^s z^{\frac{q+2-m}{2}} \leq \frac{\varepsilon}{N} |b|^{2(q-m+1)} v^{2s} z^{q+2-m} + C_\varepsilon \frac{|\nabla z|^2}{z},$$

thus (4.2) follows. ∎

Proof of Theorem 1.1. We use Lemma 4.1 with $b \in (0,1)$, combined with the estimate

$$\left(\frac{1}{N} 2(b-1)(m-1) - s \right) |b|^{q-m} bv^{s-1} z^{\frac{q+4-m}{2}} \leq \frac{\varepsilon}{N} |b|^{2(q-m+1)} v^{2s} z^{q+2-m} + C_\varepsilon \frac{z^2}{v^2}.$$

Then there exist constants $C_i > 0$ depending only on m, b, N, p, q, such that

$$\frac{1}{2}\mathcal{A}(z) + C_1 v^{2s} z^{q+2-m} \leq C_2 \frac{z^2}{v^2} + C_3 \frac{|\nabla z|^2}{z}.$$

Next we choose $s = -1$ in (4.4), thus

$$b(p + q - m + 1) = q - m,$$

which is positive because $q > m$. We deduce using Hölder inequality,

$$\mathcal{A}(z) + \frac{C_4 z^{q+2-m} - C_5}{v^2} \leq \mathcal{A}(z) + \frac{C_1 z^{q+2-m} - C_2 z^2}{v^2} \leq C_3 \frac{|\nabla z|^2}{z}.$$

If we apply Lemma 3.1 with

$$\alpha(x) = \frac{C_4}{v^2(x)}, \qquad \beta(x) = \frac{C_5}{v^2(x)}, \qquad k = q + 2 - m,$$

we deduce that any solution in $\overline{B}_\rho(x_0)$ $\rho > 0$, satisfies

$$z(x_0) \leq C_6 \left(\frac{1}{\alpha \rho^2}\right)^{\frac{1}{k-1}} + \left(\frac{C_5}{C_4}\right)^{\frac{1}{k}} \leq C_7 \left(\frac{\max_{B_\rho(x_0)} v}{\rho}\right)^{\frac{2}{q+1-m}} + 1),$$

which yields

$$|\nabla v(x_0)| \leq C_8 \left(\left(\frac{\max_{\overline{B}_\rho(x_0)} v}{\rho}\right)^{\frac{1}{q+1-m}} + 1\right), \tag{4.6}$$

where we observe that $\frac{1}{q+1-m} < 1$, since $q > m$. Let $\varepsilon \in \left(0, \frac{1}{2}\right]$. As a consequence, for any solution in B_{2R}, (or even $\overline{B}_{\frac{3R}{2}}$) considering any $x_0 \in \overline{B}_R$ and taking $\rho = R\varepsilon$, we get

$$\max_{\overline{B}_R} |\nabla v| \leq c \left(\left(\frac{\max_{\bar{B}_{R(1+\varepsilon)}} v}{\varepsilon R}\right)^{\frac{1}{q+1-m}} + 1\right) \leq c\varepsilon^{-\frac{1}{q+1-m}} \left(\left(\frac{\max_{\bar{B}_{R(1+\varepsilon)}} v}{R}\right)^{\frac{1}{q+1-m}} + 1\right), \tag{4.7}$$

$$\max_{\bar{B}_{R(1+\varepsilon)}} v \leq v(0) + R(1+\varepsilon) \max_{\bar{B}_{R(1+\varepsilon)}} |\nabla v|,$$

$$\frac{\max_{\bar{B}_{R(1+\varepsilon)}} v}{R} \leq \frac{1 + v(0)}{R} + (1+\varepsilon) \max_{\bar{B}_{R(1+\varepsilon)}} |\nabla v| \leq c_0 \left(\frac{1}{R} + \max_{\bar{B}_{R(1+\varepsilon)}} |\nabla v|\right),$$

where $c_0 = 2 + v(0)$ depends on $v(0)$. If $R \geq 1$,

$$\left(\frac{\max_{\bar{B}_{R(1+\varepsilon)}} v}{R}\right)^{\frac{1}{q+1-m}} + 1 \leq c_0^{\frac{1}{q+1-m}} \left(1 + \max_{\bar{B}_{R(1+\varepsilon)}} |\nabla v|\right)^{\frac{1}{q+1-m}}\right) + 1 \leq c_1 \left(1 + \max_{\bar{B}_{R(1+\varepsilon)}} |\nabla v|\right)^{\frac{1}{q+1-m}}.$$

Then from (4.7),

$$y(R) := 1 + \max_{\overline{B}_R} |\nabla v| \leq 1 + c_2 \varepsilon^{-\frac{1}{q+1-m}} (1 + \max_{\bar{B}_{R(1+\varepsilon)}} |\nabla v|)^{\frac{1}{q+1-m}} \leq c_3 \varepsilon^{-\frac{1}{q+1-m}} (1 + \max_{\bar{B}_{R(1+\varepsilon)}} |\nabla v|)^{\frac{1}{q+1-m}}.$$

Using the definition of y, this is

$$y(R) \le c\varepsilon^{-\frac{1}{q+1-m}} \left(y((1+\varepsilon)R)\right)^{\frac{1}{q+1-m}},$$

where c depends on $v(0)$. Therefore $y(R)$ is bounded as a consequence of Lemma 3.2. Thus $|\nabla v|$ is bounded and using the definition of v with the value of b,

$$|\nabla v|^{q-m} = u^{p+1} |\nabla u|^{q-m} \in L^\infty(\mathbb{R}^N).$$

Next we consider any $l \ge 0$ such that $u - l > 0$. The function $u_l = u - l$ satisfies

$$0 \le -\Delta_m u_l \le C_\infty \frac{|\nabla u|^m}{u} \le C_\infty \frac{|\nabla u_l|^m}{u_l},$$

with $C_\infty = \left\| u^{p+1} |\nabla u|^{q-m} \right\|_{L^\infty(\mathbb{R}^N)}$. Then the function $w_l = u_l^\sigma$ with $\sigma > 1$ to be specified below, satisfies

$$\begin{aligned} -\Delta_m w_l &= \sigma^{m-1} u_l{}^{(\sigma-1)(m-1)} (-\Delta_m u_l + (\sigma-1)(m-1)\frac{|\nabla u_l|^m}{u_l}) \\ &\le \sigma^{m-1}((\sigma-1)(m-1) - C_\infty) u_l^{\sigma(m-1)-m} |\nabla u_l|^m. \end{aligned}$$

Therefore w_l is m-subharmonic for σ large enough.

We first take $l = 0$, so $w = u^\sigma$. By Lemma 3.3, for any $\tau > 0$, there exists a constant $C_\tau = C_\tau(N, m, \tau)$ such that

$$\sup_{B_R} w \le C_\tau \left(\frac{1}{|B_{2R}|} \int_{B_{2R}} w^\tau \right)^{\frac{1}{\tau}} = C_\tau \left(\frac{1}{|B_{2R}|} \int_{B_{2R}} u^{\tau\sigma} \right)^{\frac{1}{\tau}}, \tag{4.8}$$

and since u is m-superharmonic, there holds for any $\theta \in (0, \frac{N(m-1)}{N-m})$, [34]

$$\inf_{B_R} u \ge c_\theta \left(\frac{1}{|B_{2R}|} \int_{B_{2R}} u^\theta \right)^{\frac{1}{\theta}}. \tag{4.9}$$

Taking $\tau = \frac{\theta}{\sigma}$, we deduce

$$\sup_{B_R} u = (\sup_{B_R} w)^{\frac{1}{\sigma}} \le C_\tau^{\frac{1}{\sigma}} \left(\frac{1}{|B_{2R}|} \int_{B_{2R}} u^{\tau\sigma} \right)^{\frac{1}{s\sigma}} \le \frac{C_\tau^{\frac{1}{\sigma}}}{c_\theta} \inf_{B_R} u. \tag{4.10}$$

This means that u, and also w, satisfies the Harnack inequality in $\mathbb{R}^N$:

$$\sup_{B_R} w \le \frac{C_\tau}{c_\theta^\sigma} \inf_{B_R} w.$$

But $r \mapsto \mu(r) = \inf_{|x|=r} u = \inf_{B_r} u$ from the maximum principle, is nonincreasing, so it has a limit $L \ge 0$ as $r \to \infty$. This implies that u is bounded and $l = \inf_{\mathbb{R}^N} u \ge 0$. If we replace u by u_l and w by w_l, then (4.8) holds with w and u replaced respectively by w_l and u_l since w_l is m-subharmonic, but also (4.9) holds with u replaced by u_l since u_l is m-superharmonic. Thus

$$\sup_{B_R} w_l \le C(\inf_{B_R} u_l)^\sigma.$$

Therefore $\sup_{B_R} w_l$ tends to 0 as $R \to \infty$. Then $w_l \equiv 0$, thus $u \equiv l$. ∎

4.2 Asymptotic behaviour near ∞

In this section we consider the behaviour of solutions defined in an exterior domain.

Proof of Theorem 1.2. Consider a nonnegative solution $u = v^b$ $(0 < b < 1)$ of (1.1) in $\mathbb{R}^N \backslash B_{r_0}$. From (4.6) the function v satisfies in $\overline{B}_\rho(x_0)$ $(\rho > 0)$,

$$|\nabla v(x_0)| \le C \left(\left(\frac{\max_{\overline{B}_\rho(x_0)} v}{\rho} \right)^{\frac{1}{q+1-m}} + 1 \right), \tag{4.11}$$

with $C = C(N, p, q, m)$. Here we denote by c_i some positive constants depending on r_0, N, p, q, m. Let $R > 4r_0$ and $0 < \varepsilon \le \frac{1}{4}$. Applying (4.11) with $\rho = \varepsilon R$, we get

$$|\nabla v(x_0)| \le c_1 \left(\left(\frac{\max_{\overline{B}_{\varepsilon R}(x_0)} v}{\varepsilon R} \right)^{\frac{1}{q+1-m}} + 1 \right) \le c_1 \varepsilon^{-\frac{1}{q+1-m}} \left(\left(\frac{\max_{\overline{B}_{\varepsilon R}(x_0)} v}{R} \right)^{\frac{1}{q+1-m}} + 1 \right),$$

then

$$\max_{|x|=R} |\nabla v| \le c_2 \left(\left(\frac{\max_{R(1-\frac{\varepsilon}{2}) \le |x| \le R(1+\frac{\varepsilon}{2})} v}{\varepsilon R} \right)^{\frac{1}{q+1-m}} + 1 \right) \le c_3 \left(\left(\frac{\max_{\frac{R}{1+\varepsilon} \le |x| \le R(1+\varepsilon)} v}{\varepsilon R} \right)^{\frac{1}{q+1-m}} + 1 \right),$$

$$\max_{\frac{R}{2} \le |x| \le 2R} |\nabla v| \le c_4 \left(\left(\frac{\max_{\frac{R}{2(1+\varepsilon)} \le |x| \le 2R(1+\varepsilon)} v}{\varepsilon R} \right)^{\frac{1}{q+1-m}} + 1 \right),$$

and finally,

$$1 + \max_{\frac{R}{2} \le |x| \le 2R} |\nabla v| \le c_5 \varepsilon^{-\frac{1}{q+1-m}} \left(\left(\frac{\max_{\frac{R}{2(1+\varepsilon)} \le |x| \le 2R(1+\varepsilon)} v}{R} \right)^{\frac{1}{q+1-m}} + 1 \right).$$

From Lemma 3.4-(i), $\mu(r) = \inf_{|x|=r} u = (\inf_{|x|=r} v)^b$ is bounded : let $M = \max_{r \ge r_0} \mu(r)$. Note that M depends on u. Now for any x such that $|x| = \rho$, there exists at least one point x_ρ where $v(x_\rho) = \inf_{|x|=\rho} v$. We can join any point $x \in S_\rho$ to x_ρ by a connected chain of balls of radius $\varepsilon\rho$ with at points $x_i \in S_\rho$ and this chain can be constructed so that it has at most $\frac{\pi}{\varepsilon}$ elements. Considering one ball containing x and joining it to a ball containing x_ρ, we get that

$$v(x) \le v(x_\rho) + C_N \varepsilon^{-1} \rho \max_{\frac{\rho}{1+\varepsilon} \le |x| \le \rho(1+\varepsilon)} |\nabla v| \le M^{\frac{1}{b}} + C_N \varepsilon^{-1} \rho \max_{\frac{\rho}{1+\varepsilon} \le |x| \le \rho(1+\varepsilon)} |\nabla v| .$$

Then

$$\begin{aligned} \max_{\frac{R}{2(1+\varepsilon)} \le |x| \le 2R(1+\varepsilon)} v &\le c_M^1 \left(1 + \varepsilon^{-1} R \max_{\frac{R}{2(1+3\varepsilon)} \le |x| \le 2R(1+3\varepsilon)} |\nabla v| \right) \\ &\le c_M^2 \varepsilon^{-1} R \left(1 + \max_{\frac{R}{2(1+3\varepsilon)} \le |x| \le 2R(1+3\varepsilon)} |\nabla v| \right), \end{aligned}$$

and

$$\frac{1}{\varepsilon R} \max_{\frac{R}{2(1+\varepsilon)} \le |x| \le 2R(1+\varepsilon)} v \le c_M^3 \varepsilon^{-2} \left(1 + \max_{\frac{R}{2(1+3\varepsilon)} \le |x| \le 2R(1+3\varepsilon)} |\nabla v| \right).$$

Using estimate (4.7) we obtain

$$1+\max_{\frac{R}{2}\le|x|\le 2R}|\nabla v|\le c_M^4\varepsilon^{-\frac{2}{q+1-m}}\left(1+\max_{\frac{R}{2(1+3\varepsilon)}\le|x|\le 2R(1+3\varepsilon)}|\nabla v|\right)^{\frac{1}{q+1-m}}. \tag{4.12}$$

Let $\{\varepsilon_n\}_{n\ge1}$ be a positive decreasing sequence such that $P_n:=\prod_{j=1}^n(1+\varepsilon_j)\to 2$ and $\Theta_n:=\prod_{j=1}^n\varepsilon_{j+1}^{d^j}\to\Theta>0$ when $n\to\infty$. It is easy to find such sequences such that $\varepsilon_j\sim 2^{-j}$. For $\frac{R}{2}\le a<2R\le b$ we set

$$y(a,b)=1+\max_{a\le|x|\le b}|\nabla v| \quad .$$

Then (4.12) with $(a,b)=(\frac{R}{2},2R)$ and $\varepsilon_1=3\varepsilon$ asserts that

$$y(\tfrac{R}{2},2R)\le c_5\varepsilon_1^{-h}\left(y(\tfrac{R}{2(1+\varepsilon_1)},2R(1+\varepsilon_1))\right)^d \quad\text{with}\quad h=\frac{2}{q+1-m}\quad\text{and}\quad d=\frac{1}{q+1-m}\in(0,1).$$

Applying (4.12) with $(a,b)=(\frac{R}{2P_n},2RP_n)$ we obtain

$$y(\tfrac{R}{2P_n},2RP_n)\le c_5\varepsilon_{n+1}^{-h}\left(y(\tfrac{R}{2P_{n+1}},2RP_{n+1})\right)^d.$$

By induction, we deduce

$$y(\tfrac{R}{2},2R)\le c_5^{1+d+d^2+...+d^n}\Theta_n^{-h}\left(y(\tfrac{R}{2P_{n+1}},2RP_{n+1})\right)^{d^{n+1}}. \tag{4.13}$$

Since $y(\frac{R}{2P_{n+1}},2RP_{n+1})\to y(\frac{R}{4},4R)$, we obtain that

$$1+\max_{\frac{R}{2}\le|x|\le 2R}|\nabla v|\le c_5^{\frac{d}{1-d}}\Theta^{-h}:=C(M,N,p,q,m). \tag{4.14}$$

Then we conclude again that $|\nabla v|$ is bounded for $|x|\ge 4r_0$, then in $\mathbb{R}^N\backslash B_{r_0}$ since we have assumed that $u\in C^1\mathbb{R}^N\backslash B_{r_0}$. We consider again the function $w=u^\sigma$, for σ depending of r_0, large enough so that $(\sigma-1)(m-1)\ge\left\|u^{p+1}\left|\nabla u\right|^{q-m}\right\|_{L^\infty(\mathbb{R}^N\backslash B_{r_0})}$. As in the proof of Theorem 1.1 we conclude that w is m-subharmonic in $\mathbb{R}^N\backslash\bar B_{r_0}$. Hence u satisfies the Harnack inequality, using the estimate (4.10). Therefore, for any $R>2r_0$,

$$\sup_{\frac{R}{2}\le|x|\le 3\frac{R}{2}}u\le C\inf_{\frac{R}{2}\le|x|\le 3\frac{R}{2}}u.$$

Since u is is m-superharmonic, it follows by the strong maximum principle, that it cannot have any local minimum in $\mathbb{R}^N\backslash\bar B_{r_0}$. Since u^σ is m-subharmonic it cannot have any local maximum too, and u shares this property. As a consequence $|\nabla u|$ does not vanish in $\mathbb{R}^N\backslash\bar B_{r_0}$. The function $r\mapsto\mu$ is bounded by Lemma 3.4, hence u is also bounded by the above Harnack inequality. Finally $\mu(r)$ is monotone for large r, so it admits a limit $l\ge0$ when $r\to\infty$.

If $\mu(r)$ is nonincreasing for $r\ge r_1>r_0$, then $u-l\ge0$, so we can consider the function w_l instead of w. Then

$$\max_{R\le|x|\le 2R}w_l\le C(\inf_{R\le|x|\le 2R}(u-l))^\sigma,$$

Then w_l tends to 0, thus u tends to l as $|x| \to \infty$. Since $u - l$ is m-superharmonic in $\mathbb{R}^N \backslash B_{r_0}$, then there holds

$$u(x) - l \geq C\,|x|^{\frac{m-N}{m-1}}\,,$$

with $C = C(r_0, N, m, u)$, see for example [5, Proposition 2.6], [33, Lemma 2.3]. It is the case in particular when u is a solution in $\mathbb{R}^N \backslash \{0\}$. Note that the radial solutions such that μ is nonincreasing are precisely defined in $(0, \infty)$.

Now, it follows from the upper estimate of $y(R)$, that the function u satisfies

$$-\Delta_m u = u^p\,|\nabla u|^q \leq C\frac{|\nabla u|^m}{u},$$

in $\mathbb{R}^N \backslash B_{r_0}$. Next suppose that $l > 0$. Then

$$-\Delta_m u \leq C'\,|\nabla u|^m\,.$$

The function U (still used in case $q = m$), defined by

$$U = (m-1)(e^{\frac{u-l}{m-1}} - 1),$$

satisfies $-\Delta_m U \leq 0$ and U tend to 0 at ∞. Then there exists $R_\varepsilon > 0$ such that $U(x) \leq \varepsilon$ for $|x| \geq R_\varepsilon$. For $R > R_\varepsilon$, the function $x \mapsto \omega(x) := \varepsilon + (\sup_{|z|=r_0} U(z))(\frac{|x|}{r_0})^{\frac{m-N}{m-1}}$ is a m-harmonic in $B_R \backslash B_{r_0}$, hence it is larger than U. Letting $\varepsilon \to 0$ we get $U \leq C\,|x|^{\frac{m-N}{m-1}}$ near ∞; and U has the same behaviour as $u - l$, so we deduce the estimate from above,

$$u(x) - l \leq C\,|x|^{\frac{m-N}{m-1}}\,.$$

Then we get the estimate (1.7). ∎

Remark 4.2 *(i) In case u is defined in $\mathbb{R}^N \backslash \{0\}$ and $l = 0$, we obtain the estimates*

$$C_1\,|x|^{\frac{m-N}{m-1}} \leq u(x) \leq C_2\,|x|^{\frac{1}{\sigma}\frac{m-N}{m-1}}\,.$$

It would be interesting to improve the estimate from above.

(ii) If u is defined in $\mathbb{R}^N \backslash B_{r_0}$ and if μ is nonincreasing, we have proved that u has a limit $l \geq 0$ as $|x| \to \infty$. If μ is nondecreasing, we only obtain that $\mu(r) = \inf_{|x|=r} u(x)$ has a limit l, and $\sup_{|x|=r} u(x)$ has a limit $\lambda \geq l$. Indeed the function w is m-subharmonic positive and bounded, so the function $r \mapsto \sup_{|x|=r} w = (\sup_{|x|=r} u)^\sigma$ is also monotone for large r and has a limit λ^σ. We have $w = u^\sigma \leq \lambda^\sigma$, so $\sup_{|x|=r} u$ is also nondecreasing. But we cannot prove that $\lambda = l$.

4.3 Behaviour near an isolated singularity

In this section we study the behaviour of solutions with an isolated singularity at the origin.

Proof of Theorem 1.3 Let u be a nonnegative solution u of (1.1) in $B_{r_0} \backslash \{0\}$. We apply directly the Bernstein method to u : we obtain by Lemma 4.1 with $b = 1$, and then $s = p$. Setting $\xi = |\nabla u|^2$, we get

$$\frac{1}{2}\mathcal{A}(\xi) + C_1 u^{2p} z^{q+2-m} \leq C_2\frac{\xi^2}{u^2} + C_3\frac{|\nabla \xi|^2}{\xi}.$$

By the strong maximum principle, there exists a constant $a_0 > 0$ depending on r_0 and N, p, q, such that $u \geq a_0$ in $B_{\frac{r_0}{2}} \setminus \{0\}$. Therefore, there holds

$$\frac{1}{2}\mathcal{A}(\xi) + C_1^{2p} a_0^{2p} z^{q+2-m} \leq C_2 \frac{\xi^2}{a_0^2} + C_3 \frac{|\nabla \xi|^2}{\xi},$$

in $B_{\frac{r_0}{2}} \setminus \{0\}$. Then from Lemma 3.1, we deduce the inequality

$$z(x_0) \leq c \left(\left(\frac{1}{a_0^{2p} \rho^2} \right)^{\frac{1}{q+1-m}} + \left(\frac{1}{a_0^{2(p+1)}} \right)^{\frac{1}{q+2-m}} \right) \leq c_0^2 \left(\frac{1}{\rho^{\frac{2}{q+1-m}}} + 1 \right),$$

for any ball $\overline{B}_\rho(x_0) \subset B_{\frac{r_0}{2}} \setminus \{0\}$, with $c = c(N, p, q, m)$ and $c_0^2 = c \left(a_0^{-\frac{p}{q+1-m}} + a_0^{-\frac{p+1}{q+2-m}} \right)$. Hence for any $x \in \overline{B}_R \setminus \{0\}$, with $R \leq \min(1, \frac{r_0}{8})$,

$$|\nabla u(x)| \leq \frac{2c_0}{|x|^{\frac{1}{q+1-m}}} \leq \frac{2c_0}{|x|^{\frac{1}{q+1-m}}}.$$

As a consequence, considering any x_R such that $x, x' \in \overline{B}_{\frac{R}{2}} \setminus \{0\}$, there holds

$$\left| u(x') - u(x) \right| \leq 2c_0 R^{\frac{q-m}{q+1-m}}.$$

Since $q > m$, u is bounded near 0. Then, with constants $C > 0$ depending on a_0,

$$-\Delta_m u = f \leq C \left| \nabla u \right|^q \leq C \left| x \right|^{-\frac{q}{q+1-m}}.$$

Then $f \in L^{\frac{N}{m}+\varepsilon}_{loc}(B_{\frac{R}{4}})$, since $N - \frac{N}{m}\frac{q}{q+1-m} = \frac{N(m-1)(q-m)}{m(q+1-m)} > 0$. Thus from [31] u can be extended as a continuous function, solution of the equation in the sense of distributions. Then we deduce that for any $x \in \overline{B}_{\frac{R}{2}} \setminus \{0\}$,

$$|u(0) - u(x)| \leq c_0 \left| x \right|^{\frac{q-m}{q+1-m}}.$$

Moreover, replacing r_0 by $\rho > 0$ small enough such that $u(x) \geq \frac{u(0)}{2}$ in B_ρ, then $a_0 \geq \frac{u(0)}{2}$, hence $c_0 \leq C(N, p, q, m, u(0))$, and for $|x| \leq \min(1, \frac{\rho}{8})$, we infer

$$|u(0) - u(x)| \leq C \left| x \right|^{\frac{q-m}{q+1-m}}.$$

Next assume that u is defined in $\mathbb{R}^N \setminus \{0\}$ and is not constant. Then u is bounded, since it is bounded near 0 and ∞. Then $r \mapsto \mu(r) = \inf_{|x|=r} u$ is nonincreasing, thus $\mu(r) \leq u(0)$: indeed $\forall \varepsilon > 0$, we have $\mu(|x|) \leq u(x) \leq u(0) + \varepsilon$ for any $|x| \leq r_\varepsilon$, then from the monotone decreasingness, $\mu(r) \leq u(0) + \varepsilon$ for any $r > 0$. From Theorem 1.2 $\lim_{|x|\to\infty} u = l = \lim_{r\to\infty} \mu$. and then necessarily $l \leq u(0)$. Suppose that there exists $x \neq 0$ such that $u(x) > u(0)$; then u has a maximum in $\mathbb{R}^N \setminus \{0\}$, but $|\nabla u|$ cannot vanish in $\mathbb{R}^N \setminus \{0\}$ by Theorem 1.2, so we get a contradiction.

5 Radial case

If u is a positive radial solution of (1.1), and if we denote for simplicity $u(r) = u(x)$ with $r = |x|$, then u satisfies the following o.d.e.

$$(\left|u'\right|^{m-2}u')' + \frac{N-1}{r}\left|u'\right|^{m-2}u' + u^p\left|u'\right|^q = r^{1-N}(r^{N-1}\left|u'\right|^{m-2}u')' + u^p\left|u'\right|^q = 0. \tag{5.1}$$

We begin with a simple observation about the set of zeros of u'. We have shown above that any solution of the exterior problem is either constant, or its gradient does not vanish. In the radial case, the proof is elementary:

Proposition 5.1 *Assume $q > m-1$, $p \geq 0$. Then any nonnegative radial solution of (5.1) on a segment $[r_1, r_2] \subset (0,\infty)$ is constant, or strictly monotone.*

Proof. By the strong maximum principle [35], we can assume that $u > 0$ on (r_1, r_2). The function

$$r \mapsto W(r) := r^{N-1}\left|u'(r)\right|^{m-2}u'(r)$$

is nonincreasing. Suppose that u' has two zeros ρ_1 and ρ_2 in (r_1, r_2), then by integrating W' and using the equation, we deduce that $u' \equiv 0$ on $[\rho_1, \rho_2]$, hence u is constant therein, therefore we can assume that $[\rho_1, \rho_2]$ is the maximal subinterval of $[r_1, r_2]$ where u' vanishes. If $[r_1, \rho_1] \neq [r_1, r_2]$, for example $r_1 < \rho_1$, then $u' > 0$ on (r_1, ρ_1) where $u'(r) = (r^{1-N}W(r))^{\frac{1}{m-1}}$. By (5.1),

$$\frac{m-1}{m-1-q}\left(W^{\frac{m-1-q}{m-1}}\right)' = |W|^{-\frac{q}{m-1}}W' = -r^{N-1-(N-1)\frac{q}{m-1}}u^p, \tag{5.2}$$

on (r_1, ρ_1) and $\lim_{r\to\rho_1} u'r) = 0$. Since $m-1-q <$ this implies $\lim_{r\to\rho_1} W^{\frac{m-1-p}{m-1}}r) = \infty$, a contradiction since u is bounded on $[r_1, \rho_1]$. We proceed similarly if $r_1 = \rho_1$ but $\rho_2 < r_2$ or if $\rho_1 = \rho_2$. Hence either u is constant or it is strictly monotone. ∎

Next we make a complete description of the radial solutions for $p \geq 0$, $q > m$.

5.1 The case $p = 0$

This case $p = 0$ of the Hamilton-Jacobi equation is well known, since equation (5.2) can be directly integrated, so the solutions are explicit, and are a Ariadne's thread for studying the case $p > 0$. We find different types of nonconstant solutions according to the sign of u' :

$$\begin{cases} u' = r^{\frac{1-N}{m-1}}(C_1 - a_{m,q}^{-1}r^{-a_{m,q}})^{-\frac{1}{q-m+1}} \\ u' = -r^{\frac{1-N}{m-1}}(C_2 + a_{m,q}^{-1}r^{-a_{m,q}})^{-\frac{1}{q-m+1}} \end{cases}$$

where $a_{m,q} = \frac{(N-1)q-N(m-1)}{m-1} > 0$ since $q > \frac{N(m-1)}{N-1} > m-1$; and the value of u follows by integration, with the requirement that $u > 0$. The solutions such that $C_1 > 0$ satisfy $\lim_{r\to 0} r^{\frac{1}{q-m+1}}u'(r) = -a_{m,q}^{\frac{1}{q-m+1}}$, then $\lim_{r\to 0} u(r) = u_0 > 0$, since $q > m$. The conclusions of theorem 1.4 follow in that case.

5.2 The case $p > 0$

Equation (5.1) can be reduced to an autonomous system, since it is invariant by the transformation $u \mapsto T_\lambda u$ ($\lambda > 0$) given by

$$T_\lambda u(x) = \lambda^{-\frac{q-m}{p+q+1-m}} u(\lambda x). \tag{5.3}$$

Here we perform a change of unknown, introduced in [10], which consists in a differentiation of the equation, as in the Bernstein technique. We set

$$X(t) = -r\frac{u'(r)}{u(r)}, \qquad Z(t) = -ru^p \left|u'\right|^{q-m} u', \qquad t = \ln r, \tag{5.4}$$

and obtain the following quadratic system of Kolmogorov type, valid any point t where $u'(t) \neq 0$, and any reals m, p, q,

$$\left\{ \begin{array}{rcl} X_t &=& X(X - \frac{N-m}{m-1} + \frac{Z}{m-1}) \\ Z_t &=& Z(N - \frac{N-1}{m-1}q - pX + \frac{q+1-m}{m-1}Z), \end{array} \right. \tag{5.5}$$

in the region $Q = \left\{(X, Z) \in \mathbb{R}^2 : XZ > 0\right\}$. Note that the trajectories $X = 0$ and $Z = 0$ are not admissible in our study. Since $p + q \neq m - 1$, we can recover u and u' by

$$u = (r^{q-m} \left|Z\right| \left|X\right|^{m-1-q})^{\frac{1}{p+q-m+1}}, \quad u' = -\frac{Xu}{r} = (r^{-(p+1)} \left|Z\right| \left|X\right|^p)^{\frac{1}{p+q-m+1}} \mathrm{sign}(-X). \tag{5.6}$$

The fixed points of the system in $\overline{Q}$ are

$$N_0 = (0, a_{m,q}) = (0, \frac{(N-1)q - N(m-1)}{q+1-m}), \qquad O = (0,0), \qquad A_0 = (\frac{N-m}{m-1}, 0).$$

We begin by a local study of the different points and the correponding results for the solutions of (5.1):

Lemma 5.2 *(i) The point N_0 is a source, with eigenvalues $0 < \lambda_1 = \frac{q-m}{q+1-m} < \lambda_2 = \frac{(N-1)q-N(m-1)}{m-1}$ and eigenvectors $v_1 = (1, c_{m,q})$ with $c_{m,q} > 0$ and $v_2 = (0,1)$. Then there exist infinitely many singular decreasing solutions u of (5.1) defined near 0, satisfying (1.11).*

(ii) The point O is a sink, with eigenvalues $0 > \xi_1 = -\frac{N-m}{m-1} > \xi_2 = N - \frac{N-1}{m-1}q$, and eigenvectors $u_1 = (1,0)$ and $u_2 = (0,1)$. Then there exist infinitely many solutions u of (1.1) defined for large r and either increasing or decreasing near ∞, satisfying (1.12) and (1.13)

(iii) The point A_0 is a saddle point, with eigenvalues $\mu_1 = -\frac{(N-m)p+(N-1)q-(m-1)N}{m-1} < 0 < \mu_2 = \frac{N-m}{m-1}$ and eigenvectors $w_1 = (1, -d_{m,q})$ with $d_{m,q} > 0$ and $w_2 = (1,0)$. Then for any $c > 0$ there exists a unique solution u of (5.1), defined at least for large r, such that $\lim_{r\to\infty} r^{\frac{N-m}{m-1}} u = c > 0$.

Proof. (i) We perform the linearization at N_0 : setting $Z = a_{m,q} + \overline{Z}$, we get, with $X > 0$ and $Z > 0$,

$$\left\{ \begin{array}{rcl} X_t &=& \frac{q-m}{q+1-m} X \\ \overline{Z}_t &=& a_{m,q}(-pX + \frac{q+1-m}{m-1}\overline{Z}), \end{array} \right.$$

which gives the eigenvalues λ_1, λ_2 and their respective eigenvectors, with the value of $c_{m,q}$

$$c_{m,q} = \frac{p(N-1)q - N)(m-1)}{(N-1)q^2 - 2N(m-1)q + (m-1)(N(m-1)+m)}.$$

So N_0 is a source; the particular trajectory $X = 0$ associated to λ_2 is not admissible. There exists an infinity of trajectories starting from N_0 as $t \to -\infty$, associated to the eigenvalue λ_1; the solutions (X, Z) satisfy $X > 0$, $\lim_{t\to-\infty} e^{-\frac{q-m}{q+1-m}t} X = C_0$, where $C_0 > 0$ is arbitrary and $\lim_{t\to-\infty} Z = a_{m,q}$; then from (5.6) and the definition of Z , there exist infinitely many decreasing singular solutions u of (5.1) defined near 0, satisfying (1.11).

(ii) The linearisation at O gives the system

$$\left\{ \begin{array}{rcl} X_t &=& -\frac{N-m}{m-1} X \\ Z_t &=& (N - \frac{N-1}{m-1} q) Z, \end{array} \right.$$

with admits the eigenvalues ξ_1, ξ_2. So O is a sink, two particular trajectories are the axis $X = 0$ and $Z = 0$ which not admissible. There is an infinity of trajectories converging to O as $t \to \infty$, tangent to the axis $Z = 0$, associated to the eigenvalue ξ_1, with either $X, Z > 0$, or $X, Z < 0$. They satisfy

$$X \sim_{t\to\infty} C_1 e^{-\frac{N-m}{m-1}t}, \quad Z =\sim_{t\to\infty} C_2 e^{(N-\frac{N-1}{m-1}q)t}, \text{ with } C_1, C_2 > 0. \tag{5.7}$$

The corresponding solutions u of (5.1) are defined for large r, and either decreasing or increasing; from (5.6), we obtain $\lim_{r\to\infty} u = (C_1^{m-1-q} C_2)^{\frac{1}{p+q-m+1}} = l > 0$ and $\lim_{r\to\infty} r^{\frac{N-1}{m-1}} u' = -lC_1$, thus $\lim_{r\to\infty} r^{\frac{N-m}{m-1}} (u - l) = -lC_1$. Thus (1.12) and (1.13) follow. The uniqueness property follows from the uniqueness of a trajectory satisfying (5.7) for given C_1, C_2, see also Remark 5.3 below.

(iii) Linearisation at A_0 : setting $X = \frac{N-m}{m-1} + \overline{X}$, we get

$$\left\{ \begin{array}{rcl} \overline{X}_t &=& \frac{N-m}{m-1} (\overline{X} + \frac{Z}{m-1}) \\ Z_t &=& -\frac{(N-m)p+(N-1)q-(m-1)N}{m-1} Z, \end{array} \right.$$

which admits the eigenvalues $\mu_1 < 0 < \mu_2$ and the eigenvectors, with

$$d_{m,q} = \frac{m-1}{N-m} (N - m + |\mu_1| (m - 1)$$

It is a saddle point. The trajectory $X = 0$ associated to μ_2 is not admissible. Then a unique trajectory $\mathcal{T}_{A_0}$ converging to A_0 as $t \to \infty$. By the scaling (5.3), we deduce the uniqueness property for u. ∎

Next we a complete description of the local and global solutions in the phase-plane leading to the conclusions of Theorem 1.4:

Proof of Theorem 1.4 when $p > 0$. We consider the sets

$$\mathcal{L}_X = \{(X, Z) \in Q : X_t = 0\} = \left\{ (X, Z) \in Q : X - \frac{N-m}{m-1} + \frac{Z}{m-1} = 0 \right\},$$

$$\mathcal{L}_Z = \{(X, Z) \in Q : Z_t = 0\} = \left\{ (X, X) \in Q : N - \frac{N-1}{m-1} q - pX + \frac{q+1-m}{m-1} Z \right\}.$$

The straight line $\mathcal{L}_X$ has an extremity at A_0, with slope $-(m-1)$, and the slope of $\mathcal{T}_{A_0}$ is $-d_{m,q} < -(m-1)$, so $\mathcal{T}_{A_0}$ is above $\mathcal{L}_X$ near $t = \infty$. The line $\mathcal{L}_X$ has an extremity at N_0 , with slope $\frac{p(m-1)}{q+1-m}$,

is located above $\mathcal{L}_X$ for $X > 0$. The trajectories issued from N_0 have the slope $c_{m,q}$, and we check that it is greater than $\frac{p(m-1)}{q+1-m}$ because $q > 1-m$; so they start above $\mathcal{L}_Z$.

(i) The trajectory $\mathcal{T}_{A_0}$ stays in the region $\mathcal{R} = \left\{0 < X < \frac{N-m}{m-1}; X - \frac{N-m}{m-1} + \frac{Z}{m-1} > 0\right\}$ which is negatively invariant. Then X_t stays positive, thus X is increasing, hence bounded. Either $\mathcal{T}_{A_0}$ stays under $\mathcal{L}_Z$, then $Z_t < 0$ and Z is bounded, thus $\mathcal{T}_{A_0}$ converges to N_0, or it crosses the line $\mathcal{L}_Z$ at time t_0, and for $t < t_0$ there holds $Z_t > 0$ so that Z stays bounded, and $\mathcal{T}_{A_0}$ still converges to N_0; in fact the second eventuality holds, because of the slope of the eigenvector at N_0. So the trajectory, $\mathcal{T}_{A_0}$ joins N_0 to A_0. By scaling, for any $u_0 > 0$ there exists a unique solution u defined in $(0, \infty)$ satisfying (1.14).

(ii) All the trajectories with one point in the bounded invariant region $\mathcal{R}'$ delimitated by the axis $X = 0, Z = 0$ and $\mathcal{T}_{A_0}$, join N_0 to O, and the corresponding solutions u are positive on $(0, \infty)$, decreasing, and satisfy (1.11). The trajectories with one point in the region $\mathcal{R}'' \subset Q$ above $\mathcal{T}_{A_0}$ converge to N_0 as $t \to -\infty$, and satisfy $X_t > 0$, since $\mathcal{T}_{A_0}$ is above $\mathcal{L}_X$, and cannot be bounded, since there is no fixed point in this region. They can be of two types:

• Either they cross $\mathcal{L}_Z$, then after crossing Z is decreasing, necessarily to 0; then from (5.6), u is defined in a maximal interval $(0, \rho)$ with $u(\rho) = 0$. Such solutions exists because by any point on $\mathcal{L}_Z$ passes a trajectory.

• Or they stay above $\mathcal{L}_Z$, thus Z increases to ∞; in this case from (5.6) u is defined in a maximal interval $(0, \rho)$ with $\lim_{r\to\rho} u' = -\infty$; Let us show the existence of such solutions: For given $c > 0$, we define

$$\mathcal{L}_c = \{(X, Z) \in Q : X > 0, Z = cX + a_{m,q}\}.$$

We compute the field on this line, and show that it is entering the region above $\mathcal{L}_c$ for c large enough: indeed we obtain,

$$\begin{aligned}\frac{Z_t}{X_t} - c &= \frac{Z(\frac{q+1-m}{m-1}(Z - a_{m,q}) - pX)}{X(X - \frac{N-m}{m-1} + \frac{cX+a_{m,q}}{m-1})} - c \\ &= \frac{Z(c\frac{q+1-m}{m-1} - p)}{X - \frac{N-m}{m-1} + \frac{cX+a_{m,q}}{m-1}} - c > \frac{(c(q+1-m) - p(m-1))(cX + a_{m,q})}{(m-1+c)X + a_{m,q} - (N-m)} - c,\end{aligned}$$

and

$$\begin{aligned}&(c(q+1-m) - p(m-1))(cX + a_{m,q}) - c((m-1+c)X + a_{m,q} - (N-m)) \\ &= cX(c(q-m) - (p+1)(m-1) - (p + a_{m,q}(m-1)) + c(q-m)a_{m,q} + N - m) - p(m-1)a_{m,q}\end{aligned}$$

is positive for large c, since $q - m > 0$. All the solutions with one point above $\mathcal{L}_c$ stay in this region, so above $\mathcal{L}_Z$, which proves the existence.

(iii) All the trajectories with one point in $\{(X, Z) \in Q : X < 0\}$ satisfy $X_t > 0$ from (5.5). Then X increases necessarily up to 0, and then $Z_t > 0$ for large t, thus (X, Z) converges to O, and u is defined for r large enough, increasing and $\lim_{r\to\infty} u = l > 0$.

• Either they cross $\mathcal{L}_Z$, then before crossing Z is decreasing, necessarily to 0; then from (5.6), u is defined in a maximal interval $(0, \rho)$ with $u(\rho) = 0$. Such solutions exist as above.

• Or they stay under $\mathcal{L}_Z$, thus X and Z decrease to $-\infty$; in this case from (5.6) u is defined in a maximal interval $(0, \rho)$ with $\lim_{r\to\rho} u' = -\infty$. Let us show their existence: for given $k > 0$ we

compute the field on the line $\mathcal{L}^k = \{(X,Z) \in Q : X < 0, Z = kX\}$. On this line $X_t > 0$ and

$$\begin{aligned} Z_t - kX_t &= Z(N - \tfrac{N-1}{m-1}q - pX + \tfrac{q+1-m}{m-1}Z - X + \tfrac{N-m}{m-1} - \tfrac{Z}{m-1}) \\ &= Z((\tfrac{q-m}{m-1} - \tfrac{p+1}{k})Z - \tfrac{N-1}{m-1}(q-m)) = Z^2(\tfrac{q-m}{m-1} - \tfrac{p+1}{k}) + \tfrac{N-1}{m-1}(q-m)\,|Z|\,, \end{aligned}$$

is positive for large k. The region below $\mathcal{L}^k$ is therefore negatively invariant, then the existence is folllows. This conclude the proof. ∎

Remark 5.3 *The change of variable $u(r) = \tilde{u}(s), s = r^{\frac{m-N}{m-1}}$, introduced in [22], and also used in [13] in case $m = 2 < N$, leads to the equation*

$$(|\tilde{u}_s|_s^{m-2}\,\tilde{u})_s + \left(\frac{m-N}{m-1}\right)^{q-m} s^{\frac{N-1}{N-m}(q-m)}\tilde{u}^p\,|\tilde{u}_s|^q = 0. \tag{5.8}$$

Hence if u is not constant $\tilde{u}_s$ does not vanish, from Remark 5.1, and (5.8) is equivalent to

$$(m-1)\tilde{u}_{ss} + \left(\frac{m-N}{m-1}\right)^{q-m} s^{\frac{N-1}{N-m}(q-m)}\tilde{u}^p\,|\tilde{u}_s|^{q-m+2} = 0. \tag{5.9}$$

In particular we find again the existence and uniqueness of local solutions near ∞, satisfying (1.13) for given $l \geq 0$ and $c \neq 0$ ($c > 0$ if $l = 0$); indeed the problem reduces to the equation (5.9) with the initial conditions $\tilde{u}(0) = l$ and $\tilde{u}_s(0) = c$.

6 The case $p < 0$

Proof of Theorem 1.5. We still consider $u = v^b$, with $b > 0$: we recall that from (4.3) (4.4)

$$-\Delta_m v = (b-1)(m-1)\frac{|\nabla v|^m}{v} + b^{q-m+1}v^s\,|\nabla v|^q\,,$$

with $s = 1 - q + m + b(p + q - m + 1)$. Next we take

$$b = \frac{q+1-m}{p+q-m+1},$$

thus here $b \geq 1$ and $s = 0$, so,

$$-\Delta_m v = (b-1)(m-1)\frac{|\nabla v|^m}{v} + b^{q-m+1}\,|\nabla v|^q\,, \tag{6.1}$$

where the two terms have the same sign. Then $z = |\nabla v|^2$ satisfies

$$\mathcal{A}(v) = -\Delta v - \frac{m-2}{2}\frac{<\nabla z, \nabla v>}{z} = (b-1)(m-1)\frac{z}{v} + b^{q-m+1}z^{\frac{q+2-m}{2}}.$$

Setting , we get from (4.5)

$$\begin{aligned} & -\frac{1}{2}\mathcal{A}(z) + \frac{1}{N}(\Delta v)^2 + (b-1)(m-1)\frac{z^2}{v^2} \\ \leq\ & (b-1)(m-1)(\frac{<\nabla z, \nabla v>}{v} + \frac{q+2-m}{2}b^{q-m+1}z^{\frac{q-m}{2}} < \nabla z, \nabla v >, \end{aligned}$$

where now the term in $\frac{z^2}{v^2}$ has a positive coefficient. Since $b \geq 1$, we get an estimate of the form

$$\mathcal{A}(z) + C_1 z^{q+2-m} \leq C_3 \frac{|\nabla z|^2}{z}.$$

Since $q + 2 - m > 1$, we deduce the estimate in any ball $B_\rho(x_0)$,

$$|\nabla v(x_0)| \leq C \left(\frac{1}{\rho}\right)^{\frac{1}{q+1-m}},$$

from Lemma 3.1, where C is a universal constant, which leads to the conclusions. ∎

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