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## STRONG APPROXIMATIONS FOR THE GENERAL BOOTSTRAP OF EMPIRICAL PROCESSES WITH APPLICATIONS IN SELECTED TOPICS OF NONPARAMETRIC STATISTICS

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*Abstract* The purpose of this note is to provide an approximation for the generalized bootstrapped empirical process achieving the rate in [38]. The proof is based on the same arguments used in [36]. As a consequence, we establish an approximation of the bootstrapped kernel distribution estimation. Furthermore, our results are applied to two-sample testing procedures as well as to change-point problems. We end with establishing strong approximations of the bootstrapped empirical process when the parameters are estimated.

### 1. Introduction and Main Results

Let  $X_1, X_2, \dots$  be a sequence of independent, identically distributed [i.i.d.] random variables with common distribution function  $F(t) = P(X_1 \leq t)$ . The empirical distribution function of  $X_1, \dots, X_n$  is

$$(1.1) \quad F_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}\{X_i \leq t\}, \quad -\infty < t < \infty,$$

where  $\mathbb{I}\{A\}$  stands for the indicator function of the event  $A$ . Given the sample  $X_1, \dots, X_n$ , let  $X_1^*, \dots, X_m^*$  be conditionally independent random variables with common distribution function  $F_n(\cdot)$ . Let

$$(1.2) \quad F_{m,n}(t) = \frac{1}{m} \sum_{i=1}^m \mathbb{I}\{X_i^* \leq t\}, \quad -\infty < t < \infty,$$

denote the *classical* Efron (or multinomial) bootstrap (see, e.g. [29] and [30] for more details). Define the *bootstrapped empirical process*,  $\hat{\alpha}_{m,n}(\cdot)$ , by

$$(1.3) \quad \alpha_{m,n}(t) := \sqrt{n}(F_{m,n}(t) - F_n(t)), \quad -\infty < t < \infty.$$

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Among many other things, [12] established weak convergence of the process in (1.3), which enabled them to deduce the asymptotic validity of the bootstrap method in forming confidence bounds for  $F(\cdot)$ . [46, Section 23.1] provided a simple proof of weak convergence of the process in (1.3). The Bickel and Freedman result for  $\alpha_{m,n}(\cdot)$  has been subsequently generalized for empirical processes based on observations in  $\mathbb{R}^d$ ,  $d > 1$ , as well as in very general sample spaces and for various set and function-indexed random objects [see, for example [8], [9], [10], [31], [39]]. This line of research found its "final results" in the work of [33, 34] and [26].

By now, the bootstrap is a widely used tool and, therefore, the properties of  $\alpha_{m,n}(\cdot)$  are of great interest in applied as well as in theoretical statistics. In fact, several procedures can actually be described in terms of the empirical process  $\alpha_n(\cdot)$ , the limit distributions being functionals of  $B(F(\cdot))$ , where  $B(\cdot)$  is a Brownian bridge. The fact that the limits may depend on the unknown distribution  $F(\cdot)$  makes it important that good approximations of these limiting distributions be found and that is where the bootstrap proved to be a very effective tool. There is a huge literature on the application of the bootstrap methodology to nonparametric kernel density and regression estimation, among other statistical procedures, and it is not the purpose of this paper to survey this extensive literature. This being said, it is worthwhile mentioning that the bootstrap as per Efron's original formulation (see [29]) presents some drawbacks. Namely, some observations may be used more than once while others are not sampled at all. To overcome this difficulty, a more general formulation of the bootstrap has been devised: the *weighted* (or *smooth*) bootstrap, which has also been shown to be computationally more efficient in several applications. For a survey of further results on weighted bootstrap the reader is referred to [7]. Exactly as for Efron's bootstrap, the question of rates of convergence is an important one (both in probability and in statistics) and has occupied a great number of authors (see [25], [38], [36] and the references therein).

In this paper, we will consider a version of the Mason-Newton bootstrap (see [40], and the references therein). As will be clear, this approach to bootstrap is very general and allows for a great deal of flexibility in applications. Let  $(X_n)_{n \geq 1}$  be a sequence of i.i.d. random variables defined on a probability space  $(\Omega, \mathcal{A}, P)$ . We extend  $(\Omega, \mathcal{A}, P)$  to obtain a probability space  $(\Omega^{(\pi)}, \mathcal{A}^{(\pi)}, P)$ . The latter will carry the independent sequences  $(X_n)_{n \geq 1}$  and  $(Z_n)_{n \geq 1}$  (defined below) and will be considered rich enough as to allow the definition of another sequence  $(B_n^*)$  of Brownian bridges, independent of all the preceding sequences. The possibility of such an extension is discussed in detail in literature; the reader is referred, e.g., to [25], [38] and [11]. In the sequel, whenever an almost sure property is stated, it will be tacitly assumed that it holds with respect to the p.m.  $P$  defined on the extended space. We extend and complement, in a non trivial way, the applications and the results in [3] and [5] to the setting of weighed bootstrap of empirical process, we provide also some new results. Even the list of applications is by no means exhaustive, it is sufficient to point out how to apply our results in concrete situations that they stand as archetypes for a variety statistical tests that can be investigated in a similar way.

Define a sequence  $(Z_n)_{n \geq 1}$  of i.i.d. replicæ of a *strictly positive* random variable  $Z$  with distribution function  $G(\cdot)$ , independent of the  $X_n$ 's. In the sequel, the following assumptions on the  $Z_n$ 's will prevail:

(A1)  $E(Z) = 1$ ;  $E(Z^2) = 2$  (or, equivalently,  $\mathbf{Var}(Z) = 1$ ).

(A2) There exists an  $\varepsilon > 0$ , such that  $E(e^{tZ}) < \infty$  for all  $|t| \leq \varepsilon$ .

For all  $n \geq 1$ , let  $T_n = Z_1 + \dots + Z_n$  and define the random weights,

$$(1.4) \quad \mathscr{W}_{i;n} := \frac{Z_i}{T_n}, \quad i = 1, \dots, n.$$

The quantity

$$(1.5) \quad F_n^*(t) = \sum_{i=1}^n \mathscr{W}_{i;n} \mathbb{I}\{X_i \leq t\}, \quad \text{for } -\infty < t < \infty.$$

will be called *generalized (or weighted) bootstrapped empirical distribution function*. Analogously, recalling the empirical process based on  $X_1, \dots, X_n$ ,

$$(1.6) \quad \alpha_n(t) = n^{1/2}(F_n(t) - F(t)), \quad -\infty < t < \infty,$$

define the corresponding *generalized (or weighted) bootstrapped empirical process* by

$$(1.7) \quad \alpha_n^*(t) = n^{1/2}(F_n^*(t) - F_n(t)), \quad -\infty < t < \infty.$$

The system of weights defined in (1.4) appears in [40], p.1617, where it is shown that it satisfies assumptions  $(\mathscr{W}_I)$ ,  $(\mathscr{W}_{II})$  and  $(\mathscr{W}_{III})$  on p.1612 of the same reference, so that all the results therein hold for the objects to be treated in this note. In particular, weak convergence for the process  $\alpha_n^*(\cdot)$  to a Brownian bridge is proved. For more results concerning this version of the the weighted bootstrapped empirical process, we refer the reader to [27]. Note that, as a special case of the system of weights we are considering, one can obtain the one used for Bayesian bootstrap (see [45]).

The paper is organized as follows: in Section 2, we state and discuss our main strong approximation results of the Mason-Newton bootstrapped empirical process. The main tools used to obtain such results are contained in [36]. In Section 3, we state a strong approximation of a bootstrapped version of kernel-type distribution estimators. Sections 4 and 5 are devoted to the two-sample and to change-point problems, respectively. Finally, in Section 6, we deal with the strong approximation of the integrated empirical process when parameters are estimated. In section 7, we discuss some possible extensions. To prevent from interrupting the flow of the presentation, all proofs are gathered in Section 8.

## 2. Approximations for general bootstrap of empirical processes

In what follows, we obtain a KMT rate of convergence for the process  $\{\alpha_n^*(t) : t \in \mathbb{R}\}$  in sup norm. More precisely, we consider deviations between the generalized bootstrapped empirical process  $\{\alpha_n^*(t) : t \in \mathbb{R}\}$  and a sequence of approximating Brownian bridges  $\{B_n^*(F(t)) : t \in \mathbb{R}\}$  on  $\mathbb{R}$ . Our main result goes as follows.

**Theorem 1.** *Let assumptions (A1) and (A2) hold. Then, it is possible to define a sequence of Brownian bridges  $\{B_n^*(y) : 0 \leq y \leq 1\}$  such that, for  $x > 0$ , for  $n$  large enough, we have*

$$(2.1) \quad P \left( \sup_{-\infty < t < \infty} |\alpha_n^*(t) - B_n^*(F(t))| > n^{-1/2}(K_1 \log n + x) \right) \leq K_2 \exp(-K_3 x),$$

where  $K_1, K_2$  and  $K_3$  are positive universal constants.

The proof of Theorem 1 is given in Section 8.

**Remark 1** Theorem 1 implies the following approximation of the weighted bootstrap:

$$(2.2) \quad \sup_{-\infty < t < \infty} |\alpha_n^*(t) - B_n^*(F(t))| = O \left( \frac{\log n}{n^{1/2}} \right).$$

**Corollary 2.** *Let assumptions (A1) and (A2) hold. If  $\Phi(\cdot)$  is a Lipschitz functional defined on  $D[0, +\infty)$  such that the r.v.  $\Phi(B_n^*(F(\cdot)))$  admits a bounded density function, then, as  $n \rightarrow \infty$ ,*

$$(2.3) \quad \sup_{x \in \mathbb{R}} |P\{\Phi(\alpha_n^*(\cdot)) \leq x\} - P\{\Phi(B_n^*(F(\cdot))) \leq x\}| = O \left( \frac{\log n}{n^{1/2}} \right).$$

For more comments on this kind of results, we may refer the reader to [23], Corollary 1.1 and p. 2459.

**Theorem 3.** *Let assumptions (A1) and (A2) hold. There is a Kiefer process  $\{K(t; x) : 0 \leq t \leq 1; 0 \leq x \leq \infty\}$  such that*

$$(2.4) \quad \max_{1 \leq k \leq n} \sup_{-\infty < t < \infty} |k\alpha_k^*(t) - K(F(t), k)| \stackrel{a.s.}{=} O(n^{1/4}(\log n)^{1/2}).$$

The proof of Theorem 3 is also given in Section 8. Theorem 3, together with the law of the iterated logarithm for Gaussian sequences, gives, with probability 1, that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\sup_{t \in \mathbb{R}} |\alpha_n^*(t)|}{\sqrt{\log \log n}} &= \limsup_{n \rightarrow \infty} \frac{\sup_{t \in \mathbb{R}} |K(F(t), n)|}{\sqrt{n \log \log n}} \\ &= \sup_{t \in \mathbb{R}} \sqrt{2 \operatorname{Var}(\mathbb{K}(F(t), 1))} = \sup_{u \in [0, 1]} \sqrt{2 \operatorname{Var}(K(u, 1))}. \end{aligned}$$

Plainly,  $\operatorname{Var}(K(u, 1)) = u(1 - u)$  and  $\sup_{u \in [0, 1]} u(1 - u) = 1/4$ , so that (??) readily implies the following Corollary:

**Corollary 4.** *Let assumptions (A1) and (A2) hold. With probability 1, we have that*

$$(2.5) \quad \limsup_{n \rightarrow \infty} \frac{\sup_{t \in \mathbb{R}} |\alpha_n^*(t)|}{\sqrt{\log \log n}} = \frac{1}{2^{1/2}}.$$

Here is an immediate applications of our results to goodness of fit. Consider the problem of testing the null hypothesis

$$\mathcal{H}_0 : F = F_0,$$

Combining (2.2) and (2.5), it turns out that one could use the *bootstrapped Kolmogorov Smirnov statistic* and the *bootstrapped Cramér-von Mises statistic* defined, respectively, by

$$\mathbf{S}_n := \sup_{t \in \mathbb{R}} \left| n^{1/2} (F_n^*(t) - F_n(t)) \right|, \quad \text{and} \quad \mathbf{T}_n := n \int_{\mathbb{R}} ((F_n^*(t) - F_n(t)))^2 dF_n(t).$$

In fact, the following is in order.

**Corollary 5.** *Let assumptions (A1) and (A2) hold. Under  $\mathcal{H}_0$ , we have that, with probability 1, as  $n \rightarrow \infty$ ,*

$$(2.6) \quad \left| \mathbf{S}_n - \sup_{t \in \mathbb{R}} |B_n^*(F_0(t))| \right| = O\left(\frac{\log n}{n^{1/2}}\right),$$

$$(2.7) \quad \left| \mathbf{T}_n - \int_{\mathbb{R}} [B_n^*(F_0(t))]^2 dF_0(t) \right| = O\left(\sqrt{\frac{\log \log n}{n}} \log n\right).$$

Let consider the average absolute deviation from the sample mean

$$G_n(\bar{X}_n) = \frac{1}{n} \sum_{i=1}^n \left| X_i - \frac{1}{n} \sum_{j=1}^n X_j \right| = \frac{1}{n} \sum_{i=1}^n |X_i - \bar{X}_n|.$$

As in [43], the first approximation is to replace  $\bar{X}_n$  by the population mean,  $\mu$ , which suggests that  $G_n(\bar{X}_n)$  should be close to an average of independent random variables  $|X_i - \mu|$ . An interesting functional in this setting, see [43], is given by

$$G_n(t) = \frac{1}{n} \sum_{i=1}^n |X_i - t|.$$

At each fixed  $t$ , the law of large numbers implies that  $G_n(t)$  is eventually close to

$$G(t) = \int |x - t| dF(x).$$

The bootstrapped version is given by

$$G_n^*(t) = \sum_{i=1}^n \mathcal{W}_{i;n} |X_i - t|.$$

By using similar arguments as those used by Csörgő and Horváth in the discussion of the paper by [43], we have

$$\mathbb{G}_n^*(t) = n^{1/2}(G_n^*(t) - G_n(t)) = \int |x - t| d\alpha_n^*(x),$$

may be approximated by

$$\Gamma_n^*(t) = \int |x - t| dB_n^*(F(x)).$$

By imposing the condition that  $\mathbb{E}(X^2)$ , by following the proof of Csörgő and Horváth in the cited paper, we have

$$\sup_{t \in \mathbb{R}} |\mathbb{G}_n^*(t) - \Gamma_n^*(t)| = o_P(1).$$

We can write

$$\bar{X}_n = \int x dF_n(x), \quad \text{and} \quad \bar{X}_n^* = \int x dF_n^*(x).$$

By similar arguments to those used in the preceding result, we have

$$\left| n^{1/2}(\bar{X}_n - \bar{X}_n^*) - \int x dB_n^*(F(x)) \right| = o_P(1).$$

### 3. An application to kernel distribution estimation

Let  $X_1, \dots, X_n$  be independent random replicæ of a real-valued random variable with distribution function  $F(\cdot)$ . We assume that the distribution function  $F(\cdot)$  has a density  $f(\cdot)$  (with respect to the Lebesgue measure on  $\mathbb{R}$ ). First of all, we introduce a kernel density estimator of  $f(\cdot)$ . To this end, let  $K(\cdot)$  be a measurable function fulfilling the following condition.

$$(K1) \quad K(u) \geq 0 \quad \text{and} \quad \int K(u) du = 1.$$

Now, define the Akaike-Parzen-Rosenblatt kernel density estimator of  $f(\cdot)$  (see [2], [41] and [44]) as follows: for all  $x \in \mathbb{R}$ , estimate  $f(x)$  by

$$(3.1) \quad f_{n,h_n}(x) = \frac{1}{nh_n} \sum_{i=1}^n K\left(\frac{x - X_i}{h_n}\right),$$

where  $\{h_n : n \geq 1\}$  is a sequence of positive constants satisfying the conditions

$$h_n \downarrow 0 \quad \text{and} \quad nh_n \uparrow \infty, \quad \text{as} \quad n \rightarrow \infty.$$

Secondly, we define the bootstrapped version of  $f_{n,h_n}(\cdot)$  by setting for all  $h_n > 0$  and  $x \in \mathbb{R}$ ,

$$(3.2) \quad f_{n,h_n}^*(x) = \frac{1}{h_n} \sum_{i=1}^n \mathcal{W}_{i;n} K\left(\frac{x - X_i}{h_n}\right),$$

where  $\mathcal{W}_{i;n}$  is defined in (1.4). Let us define the smoothed empirical distribution and its bootstrap by

$$\widehat{F}_{n,h_n}(x) = \int_{-\infty}^x f_{n,h_n}(t)dt, \quad \text{and} \quad \widehat{F}_{n,h_n}^*(x) = \int_{-\infty}^x f_{n,h_n}^*(t)dt.$$

We will provide an approximation rate for the process  $\gamma_n^*$ , defined by

$$(3.3) \quad \gamma_n^*(x) = \sqrt{n} \left( \widehat{F}_{n,h_n}^*(x) - \widehat{F}_{n,h_n}(x) \right), \quad -\infty < x < \infty.$$

The following theorem, proved in Section 8, shows that a single bootstrap suffices to obtain the desired approximation for  $\{\gamma_n^*(x) : x \in \mathbb{R}\}$ .

**Theorem 6.** *Let conditions (A1), (A2), (K1) prevail. Then we can define Brownian bridges  $\{B_n^*(y) : 0 \leq y \leq 1\}$  such that almost surely along  $X_1, X_2, \dots$ , as  $n$  tends to infinity, we have*

$$(3.4) \quad \sup_{-\infty < x < \infty} \left| \gamma_n^*(x) - \int K \left( \frac{x-s}{h_n} \right) B_n^*(F(s)) ds \right| = O \left( \frac{\log n}{\sqrt{n}} \right).$$

If, moreover, we suppose boundedness of the unknown density,  $f$ , i.e., if we suppose the existence of  $M > 0$  such that  $\sup_{-\infty < x < \infty} f(x) \leq M$ , then, almost surely along  $X_1, X_2, \dots$ , as  $n$  tends to infinity,

$$(3.5) \quad \sup_{-\infty < x < \infty} |\gamma_n^*(x) - B_n^*(F(x))| = O_P \left( \frac{\log n}{\sqrt{n}} + h_n \sqrt{\log h_n^{-1}} \right).$$

#### 4. The two-sample problem

This section is devoted to a two-sample problem. Let  $m, n \in \mathbb{N}^*$ , and suppose  $X_1, \dots, X_m$  and  $Y_1, \dots, Y_n$  are independent random samples from continuous d.f.'s  $F$  and  $G$ , respectively. Denote by  $F_m^*$  and  $G_n^*$ , the bootstrapped empirical d.f. associated with  $F$  and  $G$ , respectively. As it turns out, tests for the null hypothesis

$$\mathcal{H}'_0 : F = G,$$

can be based on the *integrated two-sample empirical process* defined, for each  $m, n \in \mathbb{N}^*$ , by

$$\xi_{m,n}(t) := \sqrt{\frac{mn}{m+n}} (F_n^*(t) - G_m^*(t)) \quad \text{for } t \in \mathbb{R}.$$

We can use the following statistics for testing  $\mathcal{H}'_0$ : the *integrated two-sample Kolmogorov-Smirnov statistic* as well as the *integrated two-sample Cramér-von Mises statistic*, defined, respectively, by

$$\mathbf{S}_{m,n} := \sup_{t \in \mathbb{R}} |\xi_{m,n}(t)| \quad \text{and} \quad \mathbf{T}_{m,n} := \int_{\mathbb{R}} \xi_{m,n}(t)^2 dF_n(t).$$



Set, for any  $m, n \in \mathbb{N}^*$ ,

$$\varphi(m, n) := \max \left( \frac{\log m}{\sqrt{m}}, \frac{\log n}{\sqrt{n}} \right)$$

and

$$\phi(m, n) := \max \left( \sqrt{\frac{\log \log m}{m}} \log m, \sqrt{\frac{\log \log n}{n}} \log n \right).$$

The following results are consequences of Theorem 1.

**Corollary 7.** *Let assumptions (A1) and (A2) hold. On a suitable probability space, it is possible to define  $\{\xi_{m,n} : m, n \in \mathbb{N}^*\}$ , jointly with a sequence of Gaussian processes  $\{\mathbb{B}_{m,n}^* : m, n \in \mathbb{N}^*\}$ , such that, under  $\mathcal{H}'_0$ , with probability 1, as  $\min(m, n) \rightarrow \infty$ ,*

$$\sup_{t \in \mathbb{R}} |\xi_{m,n}(t) - \mathbb{B}_{m,n}^*(t)| = O(\varphi(m, n)),$$

where

$$\mathbb{B}_{m,n}^*(t) = F(t) \left( \sqrt{\frac{n}{n+m}} \mathbb{B}_m^{(1)}(F(t)) - \sqrt{\frac{m}{n+m}} \mathbb{B}_n^{(2)}(F(t)) \right),$$

the processes  $\{\mathbb{B}_m^{(1)} : m \in \mathbb{N}^*\}$  and  $\{\mathbb{B}_n^{(2)} : n \in \mathbb{N}^*\}$  consisting of two independent sequences of Brownian bridges constructed as in Theorem 1.

**Corollary 8.** *Let assumptions (A1) and (A2) hold. We have, under  $\mathcal{H}'_0$ , with probability 1, as  $\min(m, n) \rightarrow \infty$ ,*

$$\left| S_{m,n} - \sup_{t \in \mathbb{R}} |\mathbb{B}_{m,n}^*(t)| \right| = O(\varphi(m, n)), \quad \text{and} \quad \left| \mathbf{T}_{m,n} - \int_{\mathbb{R}} \mathbb{B}_{m,n}^*(t)^2 dF(t) \right| = O(\phi(m, n)).$$

As in [1], [16], consider the following *modified two-sample empirical process*, for a fixed positive integer  $q$ ,

$$\xi_{m,n}^{(q)}(t) := \sqrt{\frac{mn}{m+n}} (F_m^*(t)^q - G_n^*(t)^q) \quad \text{for } t \in \mathbb{R}.$$

Reasonable statistics for testing  $\mathcal{H}'_0$  would be the *modified Kolmogorov-Smirnov statistic* and the *modified Cramér-von Mises statistic*

$$S_{m,n}^{(q)} := \sup_{t \in \mathbb{R}} |\xi_{m,n}^{(q)}(t)| \quad \text{and} \quad \mathbf{T}_{m,n}^{(q)} := \int_{\mathbb{R}} \xi_{m,n}^{(q)}(t)^2 dF_n(t).$$

We extend Corollary 7 and 8 as follows.

**Corollary 9.** *Let assumptions (A1) and (A2) hold. On a suitable probability space, it is possible to define  $\{\xi_{m,n}^{(q)} : m, n \in \mathbb{N}^*\}$ , jointly with a sequence of Gaussian processes  $\{\mathbb{B}_{m,n}^{*(q)} : m, n \in \mathbb{N}^*\}$ , such that, under  $\mathcal{H}'_0$ , with probability 1, as  $\min(m, n) \rightarrow \infty$ ,*

$$\sup_{t \in \mathbb{R}} |\xi_{m,n}^{(q)}(t) - \mathbb{B}_{m,n}^{*(q)}(t)| = O(\varphi(m, n)),$$

where

$$\mathbb{B}_{m,n}^{*(q)}(t) := \frac{q}{2^{q-1}} F(t)^{2q-1} \left( \sqrt{\frac{n}{m+n}} \mathbb{B}_m^{(1)}(F(t)) - \sqrt{\frac{m}{m+n}} \mathbb{B}_n^{(2)}(F(t)) \right).$$

**Corollary 10.** *Let assumptions (A1) and (A2) hold. Under  $\mathcal{H}'_0$ , with probability 1, as  $\min(m, n) \rightarrow \infty$ , we have*

$$\left| \mathbf{S}_{m,n}^{(q)} - \sup_{t \in \mathbb{R}} |\mathbb{B}_{m,n}^{*(q)}(t)| \right| = O(\varphi(m, n)) \text{ and } \left| \mathbf{T}_{m,n}^{(q)} - \int_{\mathbb{R}} \mathbb{B}_{m,n}^{*(q)}(t)^2 dF(t) \right| = O(\phi(m, n)).$$

**Remark 11.** *The family of statistics indexed by  $q$  may be used to maximize the power of the statistical test for a specific alternative hypothesis as argued in [1].*

The proofs of the results of this section are similar to those in [16], [5] and [15].

## 5. The change-point problem

Here and elsewhere,  $[t]$  denotes the largest integer not exceeding  $t$ . In many practical applications, we assume the structural stability of statistical models and this fundamental assumption needs to be tested before it can be applied. This is called the analysis of structural breaks, or change-points, which has led to the development of a variety of theoretical and practical results. For good sources of references to research literature in this area along with statistical applications, the reader may wish to consult [17], [22] and [20] and the beautiful [37]. For recent references on the subject we may refer, among many others, to [13], [4] and [14].

In this section, we deal with testing changes in d.f.'s for a sequence of independent real-valued r.v.'s  $X_1, \dots, X_n$ . The corresponding null hypothesis that we want to test is

$$\mathcal{H}_0'' : X_1, \dots, X_n \text{ have d.f. } F.$$

As frequently done, the behavior of the derived tests will be investigated under the alternative hypothesis of a single change-point

$$\mathcal{H}_1'' : \exists k^* \in \{1, \dots, n-1\} \text{ such that } X_1, \dots, X_{k^*} \text{ have d.f. } F$$

$$\text{and } X_{k^*+1}, \dots, X_n \text{ have d.f. } G.$$

The d.f.'s  $F$  and  $G$  are assumed to be continuous. Since the critical integer  $k^*$  where the structural break occurs can always be written as  $[ns^*]$  for a certain  $s^* \in [0, 1)$ , one is convinced that it is only natural to test the null hypothesis  $\mathcal{H}_0''$  basing on functionals of the following (sequence of) process(es): set, for each  $n \in \mathbb{N}^*$ ,

$$(5.1) \quad \tilde{\alpha}_n(t, s) := \frac{[ns](n - [ns])}{n^{3/2}} \left( \mathbb{F}_{[ns]}^-(t) - \mathbb{F}_{n-[ns]}^+(t) \right) \quad \text{for } s \in [0, 1], t \in \mathbb{R},$$

where  $\mathbb{F}_k^-$  is the bootstrapped empirical d.f. based upon the  $k$  first observations and  $\mathbb{F}_{n-k}^+$  is that based upon the  $(n - k)$  last ones. In (5.1), we adopt the convention that, for  $k = 0$ ,  $\mathbb{F}_0^- = \mathbb{F}_0^+ = 0$ , so that  $\tilde{\alpha}_n(t, s) = 0$  if  $s \in (0, 1/n)$ . Now, it is possible to define the random variables

$$X_1, \mathscr{W}_{1;[ns]}, \dots, X_{[ns]}, \mathscr{W}_{[ns];[ns]} \quad \text{and} \quad X_{[ns]+1}, \mathscr{W}_{1;n-[ns]}, \dots, X_n, \mathscr{W}_{n-[ns];n-[ns]}$$

on the same probability space where two Kiefer processes  $\{\mathbb{K}_1(u, s) : s \in \mathbb{R}, u \in [0, 1]\}$  and  $\{\mathbb{K}_2(u, s) : s \in \mathbb{R}, u \in [0, 1]\}$ , are defined in such a manner that the “restricted” processes  $\{\mathbb{K}_1(u, s) : s \in [1, n/2], u \in [0, 1]\}$  and  $\{\mathbb{K}_2(u, s) : s \in [n/2, n], u \in [0, 1]\}$  are independent. It turns out that a natural approximation of  $\{\tilde{\alpha}_n : n \in \mathbb{N}^*\}$  is given by the sequence of Gaussian processes  $\{\bar{\mathbb{K}}_n(s, F(t)) : s \in [0, 1], t \in \mathbb{R}, n \in \mathbb{N}^*\}$  defined by

$$\bar{\mathbb{K}}_n(u, s) := \begin{cases} \frac{1}{\sqrt{n}} [\mathbb{K}_2(u, [ns]) - s(\mathbb{K}_1(u, [n/2]) + \mathbb{K}_2(u, [n/2]))] \\ \text{for } s \in [0, \frac{1}{2}], u \in [0, 1] \\ \frac{1}{\sqrt{n}} [-\mathbb{K}_1(u, [n(1-s)]) + (1-s)(\mathbb{K}_1(u, [n/2]) + \mathbb{K}_2(u, [n/2]))] \\ \text{for } s \in [\frac{1}{2}, 1], u \in [0, 1]. \end{cases}$$

More precisely, we have the following result.

**Theorem 12.** *On a suitable probability space, it is possible to define  $\{\tilde{\alpha}_n : n \in \mathbb{N}^*\}$ , together with a sequence of Gaussian processes  $\{\bar{\mathbb{K}}_n : n \in \mathbb{N}^*\}$  as above, such that, under  $\mathcal{H}_0''$ , with probability 1, as  $n \rightarrow \infty$ ,*

$$\sup_{s \in [0,1]} \sup_{t \in \mathbb{R}} |\tilde{\alpha}_n(t, s) - \bar{\mathbb{K}}_n(F(t), s)| = O\left(\frac{\log^2 n}{\sqrt{n}}\right).$$

According to [24], a way to test change-point is to use the following statistics:

$$(5.2) \quad \tau_n := \sup_{s \in [0,1]} \sup_{t \in \mathbb{R}} |\tilde{\alpha}_n(t, s)|.$$

The corollary below, which is a consequence of Theorem 12, deals with the weak convergence of  $\tau_n$ , under  $\mathcal{H}_0''$ .

**Corollary 13.** *If  $\mathcal{H}_0''$  holds true, then we have the convergence in distribution, as  $n \rightarrow \infty$ ,  $\tau_n \xrightarrow{\mathcal{L}} \sup_{s,u \in [0,1]} |\bar{\mathbb{K}}(u, s)|$ , where  $\bar{\mathbb{K}} = \{\bar{\mathbb{K}}(u, s) : s, u \in [0, 1]\}$  is a Gaussian process with mean zero and covariance function  $\mathbb{E}(\bar{\mathbb{K}}(u, s)\bar{\mathbb{K}}(s', u')) = (u \wedge u' - uu')(s \wedge s' - ss')$ .*

In fact, straightforward algebra yields, for any  $s, t, u, v \in [0, 1]$ ,

$$\mathbb{E}(\bar{\mathbb{K}}_n(u, s)\bar{\mathbb{K}}_n(t, v)) = \frac{1}{n} (u \wedge v - uv) \psi_n(t, s)$$

with

$$\psi_n(t, s) = \begin{cases} [n(s \wedge t)] - s[nt] - t[ns] + 2[n/2]st & \text{for } s, t \in [0, 1/2], \\ [n(1 - s \vee t)] - (1 - s)[n(1 - t)] \\ - (1 - t)[n(1 - s)] + 2[n/2](1 - s)(1 - t) & \text{for } s, t \in [1/2, 1], \\ s[n(1 - t)] + (1 - t)[ns] - 2[n/2]s(1 - t) & \text{for } s \in [0, 1/2], t \in [1/2, 1] \\ (1 - s)[nt] + t[n(1 - s)] - 2[n/2](1 - s)t & \text{for } s \in [1/2, 1], t \in [0, 1/2] \end{cases}$$

We immediately see that  $\lim_{n \rightarrow \infty} \frac{1}{n} \psi_n(t, s) = s \wedge t - st$ , so that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left( \overset{\circ}{\mathbb{K}}_n(t, v) \overset{\circ}{\mathbb{K}}_n(u, s) \right) = (s \wedge t - st) (u \wedge v - uv) = \mathbb{E} \left( \overset{\circ}{\mathbb{K}}(t, v) \overset{\circ}{\mathbb{K}}(u, s) \right)$$

where  $\overset{\circ}{\mathbb{K}}$  is the tied-down Kiefer process on  $[0, 1] \times [0, 1]$ . This proves the convergence of Gaussian processes in distribution, as  $n \rightarrow \infty$ ,  $\overset{\circ}{\mathbb{K}}_n \xrightarrow{\mathcal{L}} \overset{\circ}{\mathbb{K}}$ , which in turn, together with Theorem 12, entails Corollary 13. Actually, according to [22], the most appropriate way to test change-point is to use the following weighted statistic:

$$(5.3) \quad \tau_{n,w} := \sup_{s \in [0,1]} \sup_{t \in \mathbb{R}} \frac{|\tilde{\alpha}_n(t, s)|}{w([ns]/n)},$$

where  $w(\cdot)$  is a positive function defined on  $(0, 1)$ , increasing in a neighborhood of zero and decreasing in a neighborhood of one, satisfying the condition

$$I(w, \varepsilon) := \int_0^1 \exp \left( -\frac{\varepsilon w^2(s)}{s(1-s)} \right) \frac{ds}{s(1-s)} < \infty$$

for some constant  $\varepsilon > 0$ . For a history and further applications of  $I(w, \varepsilon)$ , we refer to [21], Chapter 4. An example of such a function is given e.g. in [47]:

$$w(t) := \left( t(1-t) \log \log \frac{1}{t(1-t)} \right)^{1/2} \quad \text{for } t \in (0, 1).$$

Using similar techniques to those which are developed in [22], one is able to show that, as  $n \rightarrow \infty$ ,

$$\tau_{n,w} \xrightarrow{\mathcal{L}} \sup_{s,t \in [0,1]} \frac{|\overset{\circ}{\mathbb{K}}(t, s)|}{w(s)}.$$

For more details, we refer to [4]. One can use the maximally selected Cramér-von Mises statistic

$$\tau_{n;2} = \sup_{s \in [0,1]} \int_{\mathbb{R}} \tilde{\alpha}_n^2(t, s) dF_n(t),$$

and the analogue of the Kuiper statistic

$$\tau_{n;3} = \sup_{s \in [0,1]} \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{n^{3/2}} \times \left\{ \left( \sup_{t \in \mathbb{R}} \mathbb{F}_{\lfloor ns \rfloor}^-(t) - \mathbb{F}_{n - \lfloor ns \rfloor}^+(t) \right) - \inf_{t \in \mathbb{R}} \left( \mathbb{F}_{\lfloor ns \rfloor}^-(t) - \mathbb{F}_{n - \lfloor ns \rfloor}^+(t) \right) \right\}.$$

One can show that

$$\tau_{n;2} \xrightarrow{\mathcal{L}} \sup_{s \in [0,1]} \int_{\mathbb{R}} \overline{\mathbb{K}}_n^2(F(t), s) dF(t),$$

and

$$\tau_{n;3} \xrightarrow{\mathcal{L}} \sup_{s \in [0,1]} \left\{ \sup_{t \in \mathbb{R}} \overline{\mathbb{K}}_n(F(t), s) - \inf_{t \in \mathbb{R}} \overline{\mathbb{K}}_n(F(t), s) \right\}.$$

In a similar way as in Section 2.3 of [22], under  $\mathcal{H}_0$ , we have the following immediate consequences of Theorem 12, for fixed  $t_0 \in \mathbb{R}$  and  $0 < s_0 < 1$ , as  $n \rightarrow \infty$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} \int_0^1 \{ \tilde{\alpha}_n(t, s) \}^2 ds dt &\xrightarrow{\mathcal{L}} \int_{-\infty}^{\infty} \int_0^1 \{ \mathbb{K}(F(t), s) \}^2 ds dt, \\ (s_0(1 - s_0))^{-1/2} \left\{ \sup_{t \in \mathbb{R}} |\tilde{\alpha}_n(t, s_0)| \right\} &\xrightarrow{\mathcal{L}} \sup_{t \in \mathbb{R}} |\mathbb{B}(F(t))|, \\ (F(t_0)(1 - F(t_0)))^{-1/2} \left\{ \sup_{s \in [0,1]} |\tilde{\alpha}_n(t_0, s)| \right\} &\xrightarrow{\mathcal{L}} \sup_{s \in [0,1]} |\mathbb{B}(s)|, \\ 12 \int_{-\infty}^{\infty} \int_0^1 \{ \tilde{\alpha}_n(t, s) \} ds dx &\xrightarrow{\mathcal{L}} N(0, 1), \\ \left( \frac{12}{F(t_0)(1 - F(t_0))} \right) \int_0^1 \{ \tilde{\alpha}_n(t_0, s) \} ds &\xrightarrow{\mathcal{L}} N(0, 1), \\ \left( \frac{12}{s_0(1 - s_0)} \right) \int_{-\infty}^{\infty} \{ \tilde{\alpha}_n(t, s_0) \} dx &\xrightarrow{\mathcal{L}} N(0, 1). \end{aligned}$$

**Remark 14.** As already noted in [48], the statistic given by (5.2) should be more powerful for detecting changes that occur in the middle, i.e., near  $n/2$ , where  $k/n(1 - k/n)$  reaches its maximum, than for the ones occurring near the end points. The advantage of using the weighted statistic defined in (5.3) is the detection of changes that occur near the end points, while retaining the sensitivity to possible changes in the middle as well.

### 6. Strong approximation of the bootstrapped empirical process when parameters are estimated

In this section, we are interested in the strong approximation of the integrated empirical process when parameters are estimated. Our approach is in the same

spirit of [19]. Let us introduce, for each  $n \in \mathbb{N}^*$ , the *estimated bootstrapped empirical process*  $\hat{\alpha}_n$ :

$$(6.1) \quad \hat{\alpha}_n^*(t) := \sqrt{n} \left( F_n^*(t) - F(t, \hat{\theta}_n) \right) \quad \text{for } t \in \mathbb{R},$$

where  $\{\hat{\theta}_n : n \in \mathbb{N}^*\}$  is a sequence of estimators of a parameter  $\theta$  from a family of d.f.'s  $\{F(t, \theta) : t \in \mathbb{R}, \theta \in \Theta\}$  ( $\Theta$  being a parametric family, i.e. a subset of  $\mathbb{R}^d$  and  $d$  a fixed positive integer) related to a sequence of i.i.d. r.v.'s  $\{X_i : i \in \mathbb{N}^*\}$ . Let us mention that a general study of the weak convergence of the estimated empirical process was carried out by [28]. For a more recent reference, we may refer to [32] where the authors investigated the empirical processes with estimated parameters under auxiliary information and provided some results concerning the bootstrap in order to evaluate the limiting laws.

Let us introduce some notation.

(6.1) The transpose of a vector  $V$  of  $\mathbb{R}^d$  will be denoted by  $V^\top$ .

(6.2) The norm  $\|\cdot\|$  on  $\mathbb{R}^d$  is defined by  $\|(y_1, \dots, y_d)\| := \max_{1 \leq i \leq d} |y_i|$ .

(6.3) For a function  $(t, \theta) \mapsto g(t, \theta)$  where  $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{R}^d$ ,  $\nabla_\theta g(t, \theta_0)$  denotes the vector in  $\mathbb{R}^d$  of partial derivatives  $((\partial g / \partial \theta_1)(t, \theta), \dots, (\partial g / \partial \theta_d)(t, \theta))$  evaluated at  $\theta = \theta_0$ , and  $\nabla_\theta^2 g(t, \theta)$  denotes the  $d \times d$  matrix of second order partial derivatives  $((\partial^2 g / \partial \theta_i \partial \theta_j)(t, \theta))_{1 \leq i, j \leq d}$ .

(6.4) For a vector  $V = (v_1, \dots, v_d)$  of real-valued functions,  $\int V$  denotes the vector  $(\int v_1, \dots, \int v_d)$ .

Next, we write out the set of all conditions (those of [19]) which will be used in the sequel.

(i) The estimator  $\hat{\theta}_n$  admits the following form: for each  $n \in \mathbb{N}^*$ ,

$$\sqrt{n} (\hat{\theta}_n - \theta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n l(X_i, \theta_0) + \varepsilon_n,$$

where  $\theta_0$  is the theoretical true value of  $\theta$ ,  $l(\cdot, \theta_0)$  is a measurable  $d$ -dimensional vector-valued function, and  $\varepsilon_n$  converges to zero as  $n \rightarrow \infty$  in a manner to be specified later on. Notice that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n l(X_i, \theta_0) = \sqrt{n} \int_{-\infty}^t l(s, \theta_0) dF_n(s).$$

(ii) The mean value of  $l(X_i, \theta_0)$  vanishes:  $\mathbb{E}(l(X_i, \theta_0)) = 0$ .

(iii) The matrix  $M(\theta_0) := \mathbb{E}(l(X_i, \theta_0)^\top l(X_i, \theta_0))$  is a finite nonnegative definite  $d \times d$  matrix.

(iv) The vector-valued function  $(t, \theta) \mapsto \nabla_\theta F(t, \theta)$  is uniformly continuous in  $t \in \mathbb{R}$  and  $\theta \in \mathbf{V}$ , where  $\mathbf{V}$  is the closure of a given neighborhood of  $\theta_0$ .

- (v) Each component of the vector-valued function  $t \mapsto l(t, \theta_0)$  is of bounded variation in  $t$  on each finite interval of  $\mathbb{R}$ .
- (vi) The vector-valued function  $t \mapsto \nabla_{\theta} F(t, \theta_0)$  is uniformly bounded in  $t \in \mathbb{R}$ , and the vector-valued function  $(t, \theta) \mapsto \nabla_{\theta}^2 F(t, \theta)$  is uniformly bounded in  $t \in \mathbb{R}$  and  $\theta \in \mathbf{V}$ .
- (vii) Set  $\ell(s, \theta_0) := l(F^{-1}(s, \theta_0), \theta_0)$  for  $s \in (0, 1)$ , where  $F^{-1}(s, \theta_0) = \inf\{t \in \mathbb{R} : F(t, \theta_0) \geq s\}$ . The limiting relations below hold:

$$\lim_{s \searrow 0} \sqrt{s \log \log(1/s)} \|\ell(s, \theta_0)\| = 0,$$

and

$$\lim_{s \nearrow 1} \sqrt{(1-s) \log \log[1/(1-s)]} \|\ell(s, \theta_0)\| = 0,$$

- (viii) Set  $\ell'_s(s, \theta_0) := \frac{\partial \ell}{\partial s}(s, \theta_0)$  for  $s \in (0, 1)$ . The partial derivative  $\ell'_s(s, \theta_0)$  exist for every  $s \in (0, 1)$  and the bounds below hold: there is a positive constant  $C$  such that  $s \|\ell'_s(s, \theta_0)\| \leq C$  for all  $s \in (0, \frac{1}{2})$  and  $(1-s) \|\ell'_s(s, \theta_0)\| \leq C$  for all  $s \in (\frac{1}{2}, 1)$ .

The next result is an analogous of Theorem 3.1 in [19]. For each  $n \in \mathbb{N}^*$ , let  $G_n = \{G_n(t) : t \in \mathbb{R}\}$  be the process defined by

$$\begin{aligned} G_n(t) &:= \frac{1}{\sqrt{n}} \left( K(F(t, \theta_0), n) - \left( \int_{\mathbb{R}} l(s, \theta_0) d_s K(F(s, \theta_0), n) \right) \nabla_{\theta} F(t, \theta_0)^{\top} \right) \\ &= \frac{1}{\sqrt{n}} \left( K(F(t, \theta_0), n) - \mathbf{W}(n) \nabla_{\theta} F(t, \theta_0)^{\top} \right) \quad \text{for } t \in \mathbb{R}, \end{aligned}$$

where we set

$$\mathbf{W}(\tau) := \int_{\mathbb{R}} l(s, \theta_0) d_s K(F(s, \theta_0), \tau) \quad \text{for } \tau \geq 0.$$

The process  $\{\mathbf{W}(\tau) : \tau \geq 0\}$  is a  $d$ -dimensional Brownian motion with a covariance matrix of rank equal to that of  $M(\theta_0)$ . The estimated empirical process given by  $\hat{\alpha}_n^*$  defined by (6.1) will be approximated in the sequence of processes  $G_n = \{G_n(t) : t \in \mathbb{R}\}$ . Set  $\bar{\varepsilon}_n := \sup_{t \in \mathbb{R}} |\hat{\alpha}_n^*(t) - G_n(t)|$ .

**Theorem 15.** *Suppose that the sequence of estimators  $\{\hat{\theta}_n : n \in \mathbb{N}^*\}$  satisfies Conditions (i), (ii) and (iii). Then, as  $n \rightarrow \infty$ ,*

- (a)  $\bar{\varepsilon}_n \xrightarrow{P} 0$  if Conditions (iv), (v) hold and  $\varepsilon_n \xrightarrow{P} 0$ ;
- (b)  $\bar{\varepsilon}_n \xrightarrow{a.s.} 0$  if Conditions (vi)–(viii) hold and  $\varepsilon_n \xrightarrow{a.s.} 0$ ;
- (c)  $\bar{\varepsilon}_n = O(\max(h(n), n^{-\epsilon}))$  for some  $\epsilon > 0$  if Conditions (vi)–(viii) hold and  $\varepsilon_n = O(h(n))$  for some function  $h$  satisfying  $h(n) > 0$  and  $h(n) \rightarrow 0$ .

The proof of this theorem is very similar to that of [15], [5] and [6] hence will be omitted. The main idea is to rehearse the proof of Theorem 3.1 of [19] by replacing the strong approximations of the empirical process by their analogous obtained in Theorem 1.

**Remark 16.** It is well known that Theorem 1 can be used easily through routine bootstrap sampling, which we describe briefly as follows. Let  $N$  be a large integer. Let  $W_n^{(k)} = (\mathcal{W}_{1;n}^{(k)}, \dots, \mathcal{W}_{n;n}^{(k)})^\top$ , for  $k = 1, \dots, N$ , be vectors weights satisfying the preceding conditions, and being independent of  $X_1, \dots, X_n$ . Moreover, for any  $k = 1, \dots, N$ , let us define the weighted bootstrapped empirical process by  $\alpha_n^{*(k)}(t) = n^{1/2}(F_n^{*(k)}(t) - F_n(t))$ ,  $-\infty < t < \infty$ , where  $F_n^{*(k)}(t) = \sum_{i=1}^n \mathcal{W}_{i;n}^{(k)} \mathbb{I}\{X_i \leq t\}$ , for  $-\infty < t < \infty$ , is the generalized weighted bootstrapped empirical distribution function. Now, according to Theorem 1, we readily obtain the convergence of  $(\alpha_n(\cdot), \alpha_n^{*(1)}(\cdot), \dots, \alpha_n^{*(N)}(\cdot))$  to  $(B_n(\cdot), B_n^{*(1)}(\cdot), \dots, B_n^{*(N)}(\cdot))$  where  $B_n^{*(1)}(\cdot), \dots, B_n^{*(N)}(\cdot)$  are independent copies of  $B_n(\cdot)$ . In order to approximate the limiting distribution of  $\alpha_n(\cdot)$ , one can use the empirical distribution of  $\alpha_n^{*(1)}(\cdot), \dots, \alpha_n^{*(N)}(\cdot)$ , for  $N$  large enough. If we are interested to perform a statistical test based on a smooth functional  $S_n := \varphi(\alpha_n)$ , with the convention that large values of  $S_n$  lead to the rejection of the null hypothesis,  $\mathcal{H}_0$  say, under some regularity conditions, a valid approximation to the  $P$ -value for the test based on  $S_n$ , for  $N$  large enough, is given by  $\frac{1}{N} \sum_{k=1}^N \mathbb{I}\{S_n^{*(k)} \geq S_n\}$ , where  $S_n^{*(k)} := \varphi(\alpha_n^{*(k)})$ .

**7. Possible extensions**

**7.1. The multivariate case**

Let  $\mathbf{X}_k = (X_{1;k}, \dots, X_{d;k})$ ,  $k = 1, \dots, n$ , be i.i.d. random vectors with a  $d$ -dimensional continuous df  $\mathbb{F}(\cdot)$ . The joint empirical df.s is given, by  $\mathbb{F}_n(\mathbf{u}) = \frac{1}{n} \sum_{k=1}^n \mathbb{I}\{X_{1;k} \leq u_1, \dots, X_{d;k} \leq u_d\}$ . The empirical process  $\beta_n(\cdot)$  associated with  $\mathbb{F}(\cdot)$ , is defined, for  $\mathbf{u} \in [0, 1]^d$ , respectively, by

$$(7.1) \quad \beta_n(\mathbf{u}) = \sqrt{n}(\mathbb{F}_n(\mathbf{u}) - \mathbb{F}(\mathbf{u})).$$

Let us introduce the bootstrapped empirical distribution,  $\mathbb{F}_n^*(\mathbf{u}) = \sum_{i=1}^n \mathcal{W}_{i;n} \mathbb{I}\{X_{1;i} \leq u_1, \dots, X_{d;i} \leq u_d\}$ . We define the bootstrapped empirical process by, for  $\mathbf{u} \in [0, 1]^d$ ,  $\beta_n^*(\mathbf{u}) = \sqrt{n}(\mathbb{F}_n^*(\mathbf{u}) - \mathbb{F}_n(\mathbf{u}))$ . Assume that the following conditions, as in [18], are satisfied.

- (B.1) The sequences  $\{\mathbf{X}_k : 1 \leq k < \infty\}$  and  $\{Z_k : 1 \leq k < \infty\}$  are independent.
- (B.2) The  $\{\mathbf{X}_k : 1 \leq k < \infty\}$  are i.i.d. random vectors with values in  $[0, 1]^d$  and with distribution function  $\mathbb{F}(\cdot)$ .
- (B.3) The  $\{Z_k : 1 \leq k < \infty\}$  are i.i.d. random variables with  $\mathbb{E}(Z_k) = 0$ ,  $\mathbb{E}Z_k^2 = 1$ .
- (B.4) The  $Z_k$  have a finite moment generating function in an open interval containing the origin.

**Theorem 17.** Under conditions (B.1)-(B.2)-(B.3)-(B.4), one can construct processes  $\{\beta_n^*(\mathbf{u}) : \mathbf{u} \in [0, 1]^d\}$  and  $\{\mathbb{B}_{n,\mathbb{F}}^*(\mathbf{u}) : \mathbf{u} \in [0, 1]^d\}$  on some probability space, such that, almost surely as  $n \rightarrow \infty$ ,

$$(7.2) \quad \sup_{\mathbf{u} \in [0,1]^d} |\beta_n^*(\mathbf{u}) - \mathbb{B}_{n,\mathbb{F}}^*(\mathbf{u})| = O\left(n^{-1/2(2d-1)} \log n\right),$$



where  $\{\mathbb{B}_{n,\mathbb{F}}^*(\mathbf{u}) : \mathbf{u} \in [0, 1]^d\}$  is sequence of Brownian bridges fulfilling (7.3)

$$\mathbb{E}(\mathbb{B}_{n,\mathbb{F}}^*(\mathbf{u})) = 0, \quad \mathbb{E}(\mathbb{B}_{n,\mathbb{F}}^*(\mathbf{u})\mathbb{B}_{n,\mathbb{F}}^*(\mathbf{v})) = \mathbb{F}(\mathbf{u} \wedge \mathbf{v}) - \mathbb{F}(\mathbf{u})\mathbb{F}(\mathbf{v}), \quad \text{for } \mathbf{u}, \mathbf{v} \in [0, 1]^d.$$

**Theorem 18.** Under conditions (B.1)-(B.2)-(B.3)-(B.4), one can construct processes  $\{\beta_n^*(\mathbf{u}) : \mathbf{u} \in [0, 1]^d\}$  and  $\{\Gamma_{\mathbb{F}}^*(\mathbf{u}, n) : \mathbf{u} \in [0, 1]^d\}$  on some probability, space such that, almost surely as  $n \rightarrow \infty$ ,

$$(7.4) \quad \sup_{\mathbf{u} \in [0, 1]^d} |\sqrt{n}\beta_n^*(\mathbf{u}) - \Gamma_{\mathbb{F}}^*(\mathbf{u}, n)| = O(n^{1/2-1/(4d)}(\log n)^{3/2}),$$

where  $\{\Gamma_{\mathbb{F}}^*(\mathbf{u}, n) : \mathbf{u} \in [0, 1]^d\}$  is the Kiefer process that fulfills

$$\mathbb{E}(\Gamma_{\mathbb{F}}^*(\mathbf{u}, z)) = 0 \quad \text{and} \quad \mathbb{E}(\Gamma_{\mathbb{F}}^*(\mathbf{u}, z)\Gamma_{\mathbb{F}}^*(\mathbf{v}, t)) = (z \wedge t) \{\mathbb{F}(\mathbf{u} \wedge \mathbf{v}) - \mathbb{F}(\mathbf{u})\mathbb{F}(\mathbf{v})\},$$

for  $\mathbf{u}, \mathbf{v} \in [0, 1]^d$  and  $s, t \geq 0$ .

The results of this section may be proved by the techniques developed in the present paper in connection with the results of [18].

### 8. Proofs

This section is devoted to the detailed proofs of our results. The previously displayed notation continue to be used in the sequel.

#### Proof of Theorem 1.

In the sequel, we will write  $\|\cdot\|$  to indicate  $\sup_{-\infty < t < +\infty} |\cdot|$ . We have that

$$\|\alpha_n^*(t) - B_n^*(F(t))\| = \|\sqrt{n}(F_n^*(t) - F_n(t)) - B_n^*(F(t))\|.$$

Now, it is easily seen that

$$(8.1) \quad \sqrt{n}(F_n^*(t) - F_n(t)) = \left(\frac{n}{T_n}\right) \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( Z_i - \frac{1}{n} \sum_{j=1}^n Z_j \right) \mathbb{I}\{X_i \leq t\} \right],$$

Notice that

$$\begin{aligned} P\left(\|\alpha_n^*(t) - B_n^*(F(t))\| > n^{-1/2}(K_1 \log n + x)\right) \\ = P\left(\left\|S_n^{(1)}(t) + S_n^{(2)}(t)\right\| > n^{-1/2}(K_1 \log n + x)\right), \end{aligned}$$

where

$$\begin{aligned} \|\alpha_n^*(t) - B_n^*(F(t))\| &= \left\| \left(\frac{n}{T_n}\right) \left[ \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \left( Z_i - \frac{1}{n} \sum_{j=1}^n Z_j \right) \mathbb{I}\{X_i \leq t\} \right) - B_n^*(F(t)) \right] \right\| \\ &+ \left\| \left(\frac{n}{T_n} - 1\right) B_n^*(F(t)) \right\| := \left\| S_n^{(1)}(t) + S_n^{(2)}(t) \right\|. \end{aligned}$$

We will show that, for  $x > 0$ ,

$$(8.2) \quad P\left(\left\|S_n^{(1)}(t)\right\| > n^{-1/2}(K_4 \log n + x)\right) \leq K_5 \exp(-K_6 x),$$

and

$$(8.3) \quad P\left(\left\|S_n^{(2)}(t)\right\| > n^{-1/2}(K_7 \log n + x)\right) \leq K_8 \exp(-K_9 x),$$

where  $K_i$ ,  $i = 1, \dots, 9$  are positive universal constants. First, we show that (8.3) holds, for all  $0 < x < n$ . We note that there is a constant  $K_{10}$  such that

$$(8.4) \quad P\left(\|B_n^*\| \geq x^{1/2}\right) \leq K_{10} \exp(-x/2).$$

Recall that under conditions (A1) and (A2), there are constants  $K_{11}, \dots, K_{15}$  such that such that (see [42] Theorem 2.6., p. 55)

$$(8.5) \quad P\left(\left|\sum_{i=1}^n Z_i - n\right| > (xn)^{1/2}\right) \leq K_{11} \exp(-K_{12}x),$$

for all  $0 \leq x \leq K_{13}n$ , and

$$(8.6) \quad P\left(\left|\sum_{i=1}^n Z_i - n\right| > x\right) \leq K_{14} \exp(-K_{15}x),$$

[38] have proved that there exists a standard Wiener process  $\{W(s) : 0 \leq s < \infty\}$  so that for all  $x > 0$  and integers  $n \geq 1$ , we have

$$(8.7) \quad P(|T_n - n - W(n)| > K_{15} \log n + x) \leq K_{16} \exp(-K_{17}x)$$

where  $K_i$ ,  $i = 15, \dots, 17$  are positive universal constants. By using (8.7), we have

$$(8.8) \quad \begin{aligned} P(T_n - n \leq -n/2) &= P(T_n - n - W(n) + W(n) \leq -n/2) \\ &\leq P(|T_n - n - W(n)| \geq n/4) + P(|W(1)| \geq n^{1/2}/4) \\ &\leq K_{16} \exp(-K_{17}n) \leq K_{16} \exp(-K_{17}x). \end{aligned}$$

Making use of (8.5) in connection with (8.8), we have

$$(8.9) \quad \begin{aligned} P\left(\left|\frac{n}{T_n} - 1\right| \geq (x/n)^{1/2}\right) &= P\left(\left|\frac{n}{T_n} - 1\right| \geq (x/n)^{1/2}, \{T_n \geq n/2 \cup T_n < n/2\}\right) \\ &\leq P\left(|T_n - n| \geq \frac{(nx)^{1/2}}{2}\right) + P(T_n - n \leq -n/2) \\ &\leq K_{11} \exp(-x) + P(T_n - n \leq -n/2). \end{aligned}$$

By combining (8.4) and (8.9) we obtain, for all  $0 < x < K_{13}n$ ,

$$(8.10) \quad P\left(\left\|S_n^{(2)}(t)\right\| > n^{-1/2}(K_{16} \log n + x)\right) \leq K_{17} \exp(-K_{18}x).$$

where  $K_i$ ,  $i = 16, \dots, 18$  are positive universal constants. For  $K_{13}n < x < \infty$ , by applying (8.6) and using similar arguments as in (8.9), we infer that

$$(8.11) \quad P\left(\left|\frac{n}{T_n} - 1\right| \geq (x/n)\right) \leq K_{19} \exp(-K_{20}x),$$

Note that we have with probability 1  $\|B_n^*\| = O((\log n)^{1/2})$ . This when combined with (8.11) gives, for  $K_{13}n < x < \infty$ , the following

$$(8.12) \quad P\left(\left\|S_n^{(2)}(t)\right\| > n^{-1/2}(K_{21} \log n + x)\right) \leq K_{22} \exp(-K_{23}x),$$

where  $K_i$ ,  $i = 21, \dots, 23$  are positive universal constants. This completes the proof of (8.3). The proof of (8.2), we have

$$(8.13) \quad \begin{aligned} & P\left(\left\|S_n^{(1)}(t)\right\| > n^{-1/2}(K_4 \log n + x)\right) \\ & \leq P\left(\left\|\left(\frac{n}{T_n} - 1\right) \left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \left(Z_i - \frac{1}{n} \sum_{j=1}^n Z_j\right) \mathbb{I}\{X_i \leq t\}\right) - B_n^*(F(t))\right]\right\| \right. \\ & \left. > n^{-1/2}(K_{24} \log n + x)\right) + K_{25} \exp(-K_{26}x). \end{aligned}$$

where  $K_i$ ,  $i = 24, \dots, 26$  are positive universal constants. For  $0 \leq x \leq K_{13}n$ , we have

$$(8.14) \quad \begin{aligned} & P\left(\left\|\left(\frac{n}{T_n} - 1\right) \left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \left(Z_i - \frac{1}{n} \sum_{j=1}^n Z_j\right) \mathbb{I}\{X_i \leq t\}\right) - B_n^*(F(t))\right]\right\| \right. \\ & \left. > n^{-1/2}(K_4 \log n + x)\right) \\ & \leq P\left(\left\|(n - T_n) \left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{Z_i}{T_n} - \frac{1}{n}\right) \mathbb{I}\{X_i \leq t\}\right)\right]\right\| + \left\|\left(\frac{n}{T_n} - 1\right) B_n^*(F(t))\right\| \right. \\ & \left. > n^{-1/2}(K_4 \log n + x)\right) \\ & \leq P\left(\left\|(n - T_n) \left[\frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{Z_i}{T_n} - \frac{1}{n}\right)\right]\right\| + \left\|\left(\frac{n}{T_n} - 1\right) B_n^*(F(t))\right\| \right. \\ & \left. > n^{-1/2}(K_4 \log n + x)\right) \\ & (8.14) P\left(\left\|\left(\frac{n}{T_n} - 1\right) B_n^*(F(t))\right\| > n^{-1/2}(K_4 \log n + x)\right) \leq K_{17} \exp(-K_{18}x), \end{aligned}$$

For  $K_{13}n < x < \infty$ , by combining the fact that

$$\left\| \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( Z_i - \frac{1}{n} \sum_{j=1}^n Z_j \right) \mathbb{I}\{X_i \leq t\} \right) - B_n^*(F(t)) \right\| = O\left(\frac{\log n}{n^{1/2}}\right),$$

and equation (8.11), we readily infer that

$$P \left( \left\| \left( \frac{n}{T_n} - 1 \right) \left[ \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( Z_i - \frac{1}{n} \sum_{j=1}^n Z_j \right) \mathbb{I}\{X_i \leq t\} \right) - B_n^*(F(t)) \right] \right\| > n^{-1/2}(K_{19} \log n + x) \right) \leq K_{27} \exp(-K_{28}x),$$

where  $K_i, i = 27, 28$  are positive universal constants. Making use of (8.13), (8.14) and (8.15), we infer that

$$P \left( \left\| S_n^{(1)}(t) \right\| > n^{-1/2}(K_4 \log n + x) \right) \leq K_5 \exp(-K_6x).$$

Hence the proof is complete. □

**Proof of Theorem 3.**

The proof is largely inspired from [36]. By Theorem 2.2 in [35] there is a two-time parameter Wiener process  $\{W(t, x), 0 \leq t, x < \infty\}$  such that

$$(8.15) \quad \sup_{-\infty < t < \infty} \left| \sum_{i=1}^n Z_i \mathbb{I}\{X_i \leq t\} - W(F(t), n) \right| \stackrel{a.s.}{=} O(n^{1/4}(\log n)^{1/2}).$$

By the preceding proof we have

$$(8.16) \quad \sup_{-\infty < t < \infty} \left| n^{1/2} \alpha_n^*(t) - \left( \sum_{i=1}^n Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=1}^n Z_i \right) \left\{ \frac{n}{T_n} \right\} \right|$$

$$(8.17) \quad \stackrel{a.s.}{=} O(\log n).$$

and therefore (8.15) implies

$$\sup_{-\infty < t < \infty} \left| n^{1/2} \alpha_n^*(t) - (W(F(t), n) - F(t)W(1, n)) \right| \stackrel{a.s.}{=} O(n^{1/4}(\log n)^{1/2}).$$

Observing that  $K(t, x) = W(t, x) - tW(1, x)$  is a Kiefer process, the proof of Theorem 3 is achieved. □

### Proof of Corollary 2.

The proof follows the same lines of the of those in [5] and [6] that is included here in order to make our presentation more self-contained. The functional  $\Phi$  being Lipschitz, there exists a positive constant  $L$  such that, for any functions  $u, v$ ,

$$|\Phi(u) - \Phi(v)| \leq L \sup_{t \in \mathbb{R}} |u(t) - v(t)|,$$

inequality that we will use in the form

$$(8.18) \quad \Phi(v) - L \sup_{t \in \mathbb{R}} |u(t) - v(t)| \leq \Phi(u) \leq \Phi(v) + L \sup_{t \in \mathbb{R}} |u(t) - v(t)|.$$

Let us choose for  $u, v$  the processes  $U_n := \alpha_n^*(\cdot)$  and  $V_n := B_n^*(F(\cdot))$ . Applying the elementary inequality  $|P(A) - P(B)| \leq P(A \setminus B) + P(B \setminus A)$  to the events  $A = \{\Phi(U_n) \leq x\}$  and  $B = \{\Phi(V_n) \leq x\}$  provides, for any  $x \in \mathbb{R}$  and any  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} & |P\{\Phi(U_n) \leq x\} - P\{\Phi(V_n) \leq x\}| \\ & \leq P\{\Phi(U_n) \leq x \leq \Phi(V_n)\} + P\{\Phi(V_n) \leq x \leq \Phi(U_n)\}. \end{aligned}$$

By (8.18), we see that

$$\begin{aligned} P\{\Phi(U_n) \leq x \leq \Phi(V_n)\} & \leq P\left\{\Phi(V_n) - L \sup_{t \in \mathbb{R}} |U_n(t) - V_n(t)| \leq x \leq \Phi(V_n)\right\}, \\ P\{\Phi(V_n) \leq x \leq \Phi(U_n)\} & \leq P\left\{\Phi(V_n) \leq x \leq \Phi(V_n) + L \sup_{t \in \mathbb{R}} |U_n(t) - V_n(t)|\right\}, \end{aligned}$$

from which we deduce, by addition, that

(8.19)

$$|P\{\Phi(U_n) \leq x\} - P\{\Phi(V_n) \leq x\}| \leq P\left\{|\Phi(V_n) - x| \leq L \sup_{t \in \mathbb{R}} |U_n(t) - V_n(t)|\right\}.$$

On the other hand, by choosing  $x = c \log n$  for a suitable constant  $c$  in (2.1) that will be specified below and putting  $\epsilon_n := (A + c) \log n / \sqrt{n}$ , we obtain the estimate below valid for large enough  $n$ :

$$P\left\{\sup_{t \in \mathbb{R}} |U_n(t) - V_n(t)| \geq \epsilon_n\right\} \leq \frac{B}{n^c c}.$$

By choosing  $c > 1/(2C)$ , we have

$$(8.20) \quad P\left\{\sup_{t \in \mathbb{R}} |U_n(t) - V_n(t)| \geq \epsilon_n\right\} = o\left(\frac{1}{\sqrt{n}}\right).$$

Now, by (8.19), we write

$$\begin{aligned}
 & |P\{\Phi(U_n) \leq x\} - P\{\Phi(V_n) \leq x\}| \\
 &= P\left\{\sup_{t \in \mathbb{R}} |U_n(t) - V_n(t)| < \epsilon_n, |\Phi(V_n) - x| \leq L \sup_{t \in \mathbb{R}} |U_n(t) - V_n(t)|\right\} \\
 &\quad + P\left\{\sup_{t \in \mathbb{R}} |U_n(t) - V_n(t)| \geq \epsilon_n, |\Phi(V_n) - x| \leq L \sup_{t \in \mathbb{R}} |U_n(t) - V_n(t)|\right\} \\
 (8.21) \leq & P\{|\Phi(V_n) - x| \leq L\epsilon_n\} + P\left\{\sup_{t \in \mathbb{R}} |U_n(t) - V_n(t)| \geq \epsilon_n\right\}.
 \end{aligned}$$

Noticing that the distribution of  $B_n^*$  does not depend on  $n$ , which entails the equality  $P\{|\Phi(V_n) - x| \leq L\epsilon_n\} = P\{|\Phi(V) - x| \leq L\epsilon_n\}$ , where  $V := B(F(\cdot))$ , and recalling the assumption that the r.v.  $\Phi(V)$  admits a density function bounded by  $M$  say, we get that, for any  $x \in \mathbb{R}$  and any  $n \in \mathbb{N}^*$ ,

$$(8.22) \quad P\{|\Phi(V_n) - x| \leq L\epsilon_n\} \leq 2LM\epsilon_n.$$

Finally, putting (8.20) and (8.22) into (8.21) leads to (2.3), which completes the proof of Corollary 2.  $\square$

### Proof of Theorem 6.

We start by proving (3.4). We have for  $x \in \mathbb{R}$

$$\sqrt{nh_n^2} \left( \widehat{F}_{n,h_n}^*(x) - \widehat{F}_{n,h_n}(x) \right) = \int K((x-s)/h_n) \alpha_n^*(s) ds.$$

Now, Theorem 1 together with condition (K1) give

$$\begin{aligned}
 & \sup_{-\infty < x < \infty} \left| \int \alpha_n^*(x - th_n) K(t) dt - \int B_n^*(F(x - th_n)) K(t) dt \right| \\
 (8.23) \quad & \leq \sup_{-\infty < u < \infty} |\alpha_n^*(u) - B_n^*(F(u))| \int K(t) dt = O\left(\frac{\log n}{\sqrt{n}}\right),
 \end{aligned}$$

thus proving (3.4).

Once (3.4) is at hand, to prove (3.5), it suffices to bound

$$\left| \int B_n^*(F(x - th_n)) dK(t) - B_n^*(F(x)) \right| \leq \int |B_n^*(F(x - th_n)) - B_n^*(F(x))| K(t) dt,$$

in probability. By condition (K1), and provided the unknown density  $f$  is bounded (by a strictly positive constant, say  $M$ ), for  $n$  large enough,

$$(8.24) \quad |B_n^*(F(x - th_n)) - B_n^*(F(x))| \leq \sup_{|u-v| \leq \delta_n} |B_n^*(u) - B_n^*(v)|$$

where  $\delta_n = Mh_n$ . Now, it is always possible to define a Brownian Bridge,  $\{B^*(y) : 0 \leq y \leq 1\}$ , on the same probability space carrying the sequence of Brownian Bridges  $\{B_n^*(y) : 0 \leq y \leq 1\}_{n \geq 1}$ , such that for all  $n$ , and all  $\varepsilon > 0$

$$\begin{aligned}
 & P \left( \{2\delta_n \log \delta_n^{-1}\}^{-1/2} \sup_{|u-v|<h} \sup_{h \in [0, \delta_n]} |B_n^*(u) - B_n^*(v)| > 1 + \varepsilon \right) \\
 &= P \left( \{2\delta_n \log \delta_n^{-1}\}^{-1/2} \sup_{|u-v|<h} \sup_{h \in [0, \delta_n]} |B^*(u) - B^*(v)| > 1 + \varepsilon \right).
 \end{aligned}$$

Since  $\delta_n \rightarrow 0$ , by Theorem 1.4.1 in [25], we have with probability one

$$(8.25) \quad \lim_{n \rightarrow \infty} \{2\delta_n \log \delta_n^{-1}\}^{-1/2} \sup_{|u-v|<h} \sup_{h \in [0, \delta_n]} |B^*(u) - B^*(v)| = 1.$$

Thus, as  $n \rightarrow \infty$ ,

$$P \left( \{2\delta_n \log \delta_n^{-1}\}^{-1/2} \sup_{|u-v|<h} \sup_{h \in [0, \delta_n]} |B_n^*(u) - B_n^*(v)| > 1 + \varepsilon \right) \rightarrow 0,$$

giving

$$(8.26) \quad \sup_{|u-v| \leq h} \sup_{h \in [0, \delta_n]} |B_n^*(u) - B_n^*(v)| = O_P \left( \sqrt{2\delta_n \log \delta_n^{-1}} \right).$$

Put (8.24), (8.24) and (8.26) together to obtain

$$\sup_{-\infty < x < \infty} |\gamma_n^*(x) - B_n^*(F(x))| = O_P \left( \frac{\log n}{\sqrt{n}} + h_n \sqrt{\log h_n^{-1}} \right),$$

thus completing the proof of Theorem. □

**Proof of Theorem 12.**

In the computations below, the superscript “-” in the quantities  $F_n$ ,  $F_n^*$  and  $\alpha$  refers to the first  $k$  observations while the superscript “+” refers to the last  $n - k$  observations. We have the following representation for  $\tilde{\alpha}_n(t, s)$ : with probability 1, for  $n \in \mathbb{N}^*$ ,  $t \in \mathbb{R}$  and  $s \in (0, 1)$ ,

$$\begin{aligned}
 (8.27) \quad \tilde{\alpha}_n(t, s) &= \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{n^{3/2}} \left( (\mathbb{F}_{\lfloor ns \rfloor}^-(t) - F_n(t)) - (\mathbb{F}_{n - \lfloor ns \rfloor}^+(t) - F_n(t)) \right) \\
 &= \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{n^{3/2}} \left( (\mathbb{F}_{\lfloor ns \rfloor}^-(t) - F_n(t)) \right) \\
 &\quad + \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{n^{3/2}} \left( \mathbb{F}_{n - \lfloor ns \rfloor}^+(t) - F_n(t) \right) \\
 &= \frac{\sqrt{\lfloor ns \rfloor} (n - \lfloor ns \rfloor)}{n^{3/2}} \alpha_{\lfloor ns \rfloor}^-(t) - \frac{\lfloor ns \rfloor \sqrt{n - \lfloor ns \rfloor}}{n^{3/2}} \alpha_{n - \lfloor ns \rfloor}^+(t).
 \end{aligned}$$

Notice that

$$\sup_{-\infty < t < \infty} \left| n^{1/2} \alpha_n^*(t) - \left( \sum_{i=1}^n Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=1}^n Z_i \right) \left\{ \frac{n}{T_n} \right\} \right| \stackrel{a.s.}{=} O(\log n).$$

Making use of (8.16), we have

$$\begin{aligned} & \frac{\sqrt{[ns]}(n - [ns])}{n^{3/2}} \alpha_{[ns]}^-(t) \\ &= \frac{n - [ns]}{n^{3/2}} \left( \sum_{i=1}^{[ns]} Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=1}^{[ns]} Z_i \right) + O(\log n/n^{-1/2}), \\ & \frac{[ns] \sqrt{n - [ns]}}{n^{3/2}} \alpha_{n-[ns]}^+(t) \\ &= \frac{[ns]}{n^{3/2}} \left( \sum_{i=[ns]+1}^n Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=[ns]+1}^n Z_i \right) + O(\log n/n^{-1/2}). \end{aligned}$$

With probability 1, as  $n \rightarrow \infty$ , uniformly in  $s$  and  $t$ ,

$$(8.28) \quad \tilde{\alpha}_n(t, s) = \delta_n(F(t), s) + O\left(\frac{\log n}{\sqrt{n}}\right).$$

Now, observe that

$$\begin{aligned} (8.29) \quad \delta_n(F(t), s) &= \frac{1}{\sqrt{n}} \left( \left( \sum_{i=1}^{[ns]} Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=1}^{[ns]} Z_i \right) \right. \\ &\quad \left. - \frac{[ns]}{n} \left( \sum_{i=1}^n Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=1}^n Z_i \right) \right) \\ &= \frac{1}{\sqrt{n}} \left( \frac{n - [ns]}{n} \left( \sum_{i=n}^n Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=1}^n Z_i \right) \right. \\ &\quad \left. - \left( \sum_{i=[ns]+1}^n Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=[ns]+1}^n Z_i \right) \right). \end{aligned}$$

By Theorem 2.2 in [35], we have almost surely, as  $n \rightarrow \infty$ ,

$$(8.30) \quad \sup_{s \in [0, 1/2]} \sup_{t \in \mathbb{R}} \left| \left( \sum_{i=1}^{[ns]} Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=1}^{[ns]} Z_i \right) - \mathbb{K}_2(F(t), [ns]) \right| = O((\log n)^2),$$

$$(8.31) \quad \sup_{s \in [1/2, 1]} \sup_{t \in \mathbb{R}} \left| \left( \sum_{i=[ns]+1}^n Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=[ns]+1}^n Z_i \right) - \mathbb{K}_1(F(t), [ns]) \right| = O((\log n)^2).$$



Notice that we have the following decomposition

$$\begin{aligned} \sum_{i=n}^n Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=1}^n Z_i &= \sum_{i=1}^{\lfloor n/2 \rfloor} Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=1}^{\lfloor n/2 \rfloor} Z_i \\ &+ \sum_{i=\lfloor n/2 \rfloor + 1}^n Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=\lfloor n/2 \rfloor + 1}^n Z_i. \end{aligned}$$

Hence, by adding (8.30) and (8.31), we readily infer that, almost surely, as  $n \rightarrow \infty$ ,

$$\sup_{t \in \mathbb{R}} \left| \sum_{i=n}^n Z_i \mathbb{I}\{X_i \leq t\} - F(t) \sum_{i=1}^n Z_i - (\mathbb{K}_1(F(t), \lfloor n/2 \rfloor) + \mathbb{K}_2(F(t), \lfloor n/2 \rfloor)) \right| = O((\log n)^2).$$

As a byproduct, from (8.29)–(8.32) and recalling the definition of  $\overline{\mathbb{K}}_n$  given just before Theorem 12, we deduce that, almost surely, as  $n \rightarrow \infty$ ,

$$(8.32) \quad \sup_{s \in [0,1]} \sup_{t \in \mathbb{R}} |\delta_n(F(t), s) - \overline{\mathbb{K}}_n(F(t), s)| = O\left(\frac{(\log n)^2}{\sqrt{n}}\right).$$

We finally conclude from (8.28) and (8.32) by using the triangle inequality: almost surely, as  $n \rightarrow \infty$ ,

$$\begin{aligned} \sup_{s \in [0,1]} \sup_{t \in \mathbb{R}} |\tilde{\alpha}_n(t, s) - \overline{\mathbb{K}}_n(F(t), s)| &\leq \sup_{s \in [0,1]} \sup_{t \in \mathbb{R}} |\tilde{\alpha}_n(t, s) - \delta_n(F(t), s)| \\ &+ \sup_{s \in [0,1]} \sup_{u \in [0,1]} |\delta_n(F(t), s) - \overline{\mathbb{K}}_n(F(t), s)| \\ &= O\left(\frac{(\log n)^2}{\sqrt{n}}\right). \end{aligned}$$

This completes the proof of Theorem 12. □

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