



**HAL**  
open science

# Finite-deformation second-order micromorphic theory and its relations to strain and stress gradient models

Samuel Forest, Karam Sab

► **To cite this version:**

Samuel Forest, Karam Sab. Finite-deformation second-order micromorphic theory and its relations to strain and stress gradient models. *Mathematics and Mechanics of Solids*, 2020, 25 (7), pp.1429-1449. <10.1177/1081286517720844>. <hal-02919094>

**HAL Id: hal-02919094**

**<https://hal.science/hal-02919094v1>**

Submitted on 8 Jun 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Forest S, Sab K. *Finite-deformation second-order micromorphic theory and its relations to strain and stress gradient models*. Mathematics and Mechanics of Solids. 2020, vol 25, pp. 1429-1449. doi:10.1177/1081286517720844

## Finite deformation second order micromorphic theory and its relations to strain and stress gradient models

Samuel Forest<sup>1a</sup>, Karam Sab<sup>b</sup>

<sup>a</sup>MINES ParisTech CNRS  
Centre des matériaux, CNRS UMR 7633  
BP 87, 91003 Evry Cedex, France

<sup>b</sup>Laboratoire Navier (UMR 8205), Université Paris-Est, ENPC, IFSTTAR, CNRS  
77455 Marne-la-Vallée, France

---

### Abstract

Germain's general micromorphic theory of order  $n$  is extended to fully non-symmetric higher order tensor degrees of freedom. An interpretation of the microdeformation kinematic variables as relaxed higher order gradients of the displacement field is proposed. Dynamical balance laws and hyperelastic constitutive equations are derived within the finite deformation framework. Internal constraints are enforced to recover strain gradient theories of grade  $n$ . An extension to finite deformations of a recently developed stress gradient continuum theory is then presented, together with its relation to the second order micromorphic model. The linearization of the combination of stress and strain gradient models is then shown to deliver formulations related to Eringen's and Aifantis well-known gradient models involving the Laplacians of stress and strain tensors. Finally, the structure of the dynamical equations is given for strain and stress gradient media, showing fundamental differences in the dynamical behavior of these two classes of generalized continua.

*Keywords:* Micromorphic continuum, Strain gradient theory, Stress gradient theory, Finite deformation, Method of virtual power, Generalized continua, Dispersion relations

---

<sup>1</sup>Corresponding author. samuel.forest@ensmp.fr

## 1. Introduction

The micromorphic theory currently arouses strong interest in the mechanics of materials community due to its ability to account for size effects in the continuum modeling of many physical phenomena like strain and damage localization or dispersion of elastic waves, e.g. (Hütter et al., 2015) and (Madeo et al., 2016b). Eringen and Mindlin’s original model goes back to the early 1960s, see (Mindlin, 1964) and (Eringen and Suhubi, 1964) for the presentation at finite deformations. The generally incompatible microdeformation field variable  $\chi$  was introduced by these authors to represent the deformation of a triad of directors attached to the material’s microstructure, like lattice vectors in crystalline solids or fiber directions in composite materials. This represents a drastic enhancement of the continuum theory by 9 additional degrees of freedom complementing the displacement vector of the material point.

The most ambitious extension of the classical Cauchy continuum model is probably Germain’s general micromorphic theory which introduces higher order microdeformations,  $\chi_{ij}, \chi_{ijk}, \chi_{ijkl}, \dots$ , of increasing tensor order up to order  $n$  (Germain, 1973b). Germain’s vision of this hierarchy of additional degrees of freedom is related to a Taylor expansion of the description around the material point. Due to their definition in terms of a Taylor expansion, the general microdeformations are symmetric with respect to all the indices except the first one. Germain’s paper provides the hierarchy of balance of momentum equations. However, it does not provide constitutive equations for this class of media. It should be noted that Eringen himself also extended the original micromorphic model to higher order microdeformation tensors based on averaging procedures (Eringen, 1967, 1970). However, his theory was shown to be incomplete by Germain, leaving indeterminate higher order micromorphic variables.

The multipolar continuum mechanics proposed by (Green and Rivlin, 1964) also represents one of the most general continuum model involving both higher order field variables and higher order gradients. Green and Rivlin’s multipolar theory was compared to the general micromorphic one by (Germain, 1973b). Such a comparison will also be drawn for the theory proposed in the present work.

Suitable internal constraints on the microdeformation can be introduced so that Eringen’s micromorphic model reduces to the strain gradient theory proposed by (Mindlin, 1965), which is identical to the second gradient of displacement model. Such a reduction has been proposed by (Bleustein, 1967) and recently discussed by (Madeo et al., 2016c; Broese et al., 2016). This reduction can be applied to Germain’s general micromorphic media. For instance, the second order micromorphic theory ( $n = 2$ ) can be shown to degenerate into Mindlin’s second strain gradient model by constraining the first and second microdeformation to coincide with the deformation first and second gradients, respectively, see (Mindlin, 1965; Javili et al., 2013; Cordero et al., 2016). In gradient theories, the micromorphic degrees of freedom are therefore eliminated and a single balance equation of higher order remains to solve for the

displacement field. Finite element simulations of strain gradient materials very often rely on the use of such a constrained micromorphic theory, see (Dillard et al., 2006; Enakoutsu and Leblond, 2009; Ferretti et al., 2014; Madeo et al., 2015, 2016a) and the discussion in (Bergheau et al., 2014). Mindlin’s second strain gradient theory (Mindlin, 1965) was recently applied to the elasticity of nano–objects with surface effects (Cordero et al., 2016) and to the numerical analysis of singularities at edges and corners (Reiher et al., 2017). The associated computational analysis was based on a constrained second order micromorphic theory where the first and second order microdeformations are constrained to coincide with the first and second gradients of the displacement field (Forest et al., 2011; Cordero et al., 2016). These internal constraints were only considered for micromorphic media at small strains. Conversely, the micromorphic continua can be seen as relaxed higher grade materials.

A third rank tensor as additional independent kinematic degrees of freedom is also present in the stress gradient theory recently proposed by (Forest and Sab, 2012; Sab et al., 2016). The stress gradient model is a completely new continuum theory which was shown to be fundamentally different from the strain gradient approach. It was inspired by the bending gradient theory for thick plates according to (Lebée and Sab, 2011a,b). The third order degrees of freedom are conjugate to the gradient of the classical stress tensor in the work of internal forces. They have therefore a different physical unit as Germain’s second order microdeformation. An alternative stress gradient theory was designed by (Polizzotto, 2014, 2015) where third order kinematic test functions also arise but are not treated as independent degrees of freedom of the theory. The original stress gradient elasticity model was then shown to lead to a well–posed boundary value problem with the new boundary conditions proposed by (Forest and Sab, 2012; Sab et al., 2016). In particular, in a static stress gradient medium, the full stress tensor can be prescribed at the boundary, in contrast to Cauchy’s model for which only the traction vector is controlled. The stress gradient theories were originally formulated in the context of linear elasticity and a complete stress gradient theory at finite deformation is still missing.

The theory of elastoviscoplasticity for first order micromorphic media at finite deformation is well–established since the works by (Forest and Sievert, 2003, 2006; Regueiro, 2010; Sansour et al., 2010) etc. First strain gradient theory was also explored at finite elastoviscoplastic deformations in (Forest and Sievert, 2003; Bertram, 2015, 2016). Second strain gradient hyperelasticity at finite deformation was then considered by (Javili et al., 2013). However the reduction of the hyperelastic laws based on invariance requirements was not performed in the latter reference so that the suited Lagrangian strain measures remain to be defined. The treatment of internal constraints in gradient continua at finite strains was presented recently by (Bertram and Glüge, 2016).

The objective of the present work is to formulate a phenomenological theory of higher order micro–

morphic media generalizing Germain’s one, and to establish the links to strain and stress gradient models. The formulation is presented in the finite deformation setting. For that purpose, a constitutive framework is proposed for hyperelasticity, taking into account possible internal constraints. Its linearization will be shown to provide general equations for the statics and dynamics of second strain gradient and of stress gradient media. In particular, the dynamical stress gradient theory will be formulated at finite strain for the first time, providing the balance, boundary and Lagrangian constitutive equations.

The proposed extension of Germain’s theory consists in abandoning the symmetry requirements for the microdeformation tensors, thus departing from the Taylor expansion approach. The theory is presented for the order  $n = 2$  (first and second order microdeformation tensors of orders 2 and 3 respectively) and limited to the grade  $p = 1$ , considering only the first gradient of all degrees of freedom. Extensions to order  $n$  and grade  $p$  are possible but not considered in the present work for the sake of simplicity.

The higher order kinematics is presented in Section 2. Following (Germain, 1973b; Maugin, 1980), the method of virtual power is used to derive balance equations and the associated boundary conditions. Lagrangian strain measures are proposed to formulate hyperelastic constitutive equations in Section 3, based on the Helmholtz or Gibbs free energy potentials. This section ends with the consideration of general internal constraints and the consequences on the formulation of hyperelastic laws based on the exploitation of the second principle of thermodynamics. Specific internal constraints are then discussed in section 4 in order to relate the general theory to strain gradient and stress gradient models. The subsection 4.2 is devoted to the formulation of the stress gradient theory at finite deformation. This theory is then linearized and shown to coincide with the already existing linear elastic stress gradient model. The applications presented in Section 5 deal with the statics and dynamics of combined linear stress and strain gradient media.

### *Notations*

The material points of the body are labeled according to their position vectors  $\underline{\mathbf{X}}$  with respect to a reference configuration  $\Omega_0$ . They occupy the positions  $\underline{\mathbf{x}} = \Phi(\underline{\mathbf{X}}, t)$  in the current configuration  $\Omega$  of the body at time  $t$ . Their coordinates are expressed in two distinct Cartesian orthonormal bases:

$$\underline{\mathbf{X}} = X_I \underline{\mathbf{E}}_I, \quad \underline{\mathbf{x}} = x_i \underline{\mathbf{e}}_i \tag{1}$$

where upper-case (resp. lower-case) letters are used for indices referring to the reference (resp. current) configuration of the body. The components  $X_I$  (resp.  $x_i$ ) are called Lagrangian (resp. Eulerian) coordinates. Einstein’s convention on summation of repeated indices is enforced.

The Lagrangian (resp. Eulerian) volume and surface elements are  $dV$  (resp.  $dv$ ) and  $dS$  (resp.  $ds$ ),

respectively. The gradient operators are written

$$\nabla^0 = \frac{\partial}{\partial X_I} \underline{\mathbf{E}}_I, \quad \nabla = \frac{\partial}{\partial x_i} \underline{\mathbf{e}}_i \quad (2)$$

In the index notation, we write

$$u_{i,J} = \frac{\partial u_i}{\partial X_J} \quad (3)$$

An intrinsic notation is used whereby tensors of order 1, 2, 3, 4 and 6 are respectively denoted by  $\underline{\mathbf{a}}$ ,  $\underline{\underline{\mathbf{A}}}$ ,  $\underline{\underline{\underline{\mathbf{A}}}}$  (or  $\underline{\underline{\underline{\underline{\mathbf{A}}}}}$ ),  $\underline{\underline{\underline{\underline{\underline{\mathbf{A}}}}}}$ , and  $\underline{\underline{\underline{\underline{\underline{\underline{\mathbf{A}}}}}}}$ . To avoid any ambiguity, the corresponding index notation is often provided together with the intrinsic one.

## 2. Kinematics and balance laws for higher order micromorphic media

### 2.1. Higher order micromorphic degrees of freedom

Each material point is endowed with the following set of generalized degrees of freedom:

$$\begin{aligned} DOF &= \{ \underline{\mathbf{u}}, \quad \underline{\underline{\chi}}, \quad \underline{\underline{\underline{\chi}}}, \quad \underline{\underline{\underline{\underline{\chi}}}}, \quad \dots \} \\ DOF &= \{ u_i, \quad \chi_{iJ}, \quad \chi_{iJK}, \quad \chi_{iJKL}, \quad \dots \} \end{aligned} \quad (4)$$

where  $\underline{\mathbf{u}} = \Phi(\underline{\mathbf{X}}, t) - \underline{\mathbf{X}}$  is the displacement vector and the micromorphic degrees of freedom are independent tensors of increasing order. Following similar notations as introduced by (Mindlin, 1965), the same letter  $\chi$  is used for the independent micromorphic degrees of freedom of various order. These variables can be distinguished by the indication of respective tensor rank in the tensor notation or from the number of indices.

The first and higher order microdeformation tensor variables are generally non compatible fields. They do not possess any symmetry property *a priori*. In particular, the second order microvariable  $\chi_{iJK}$  does not exhibit any symmetry with respect to the last two indices. This is in contrast to Germain's general micromorphic medium where the higher order microvariables are conceived as the coefficients in a Taylor expansion of the relative motion of the material particle with respect to its center of mass. Such symmetry properties also hold in Eringen's higher grade micromorphic theory where higher order moments of the microfields are introduced (Eringen, 1970). The physical meaning and physical dimension of the higher order micromorphic variables are left unspecified since they will depend on the specific application. However, if

$$\underline{\underline{\mathbf{F}}} = \underline{\underline{\mathbf{1}}} + \underline{\mathbf{u}} \otimes \nabla^0 = F_{iJ} \underline{\mathbf{e}}_i \otimes \underline{\mathbf{E}}_J, \quad J = \det \underline{\underline{\mathbf{F}}} > 0 \quad (5)$$

is the deformation gradient of the continuum, the first interpretation of the micromorphic degrees of freedom proposed in this work is the relaxation of  $\underline{\underline{\mathbf{F}}}$  and its gradients of increasing order. The microdeformation tensor  $\underline{\underline{\chi}}$  is viewed as the relaxed counterpart of  $\underline{\underline{\mathbf{F}}}$  meaning that it is a generally incompatible

deformation field, in contrast to  $\underline{\mathbf{F}}$ . The reference state of microdeformation is  $\underline{\underline{\chi}} = \underline{\underline{\mathbf{1}}}$  and it is assumed that  $\det \underline{\underline{\chi}} > 0$ . Similarly, the second order microdeformation  $\chi_{iJK}$  represents the relaxed microdeformation gradient  $\chi_{iJ,K}$ , and consequently the relaxation of the second gradient  $F_{iJ,K} = u_{i,JK}$  which is symmetric w.r.t. the last two indices. The reference states of the second and higher order microdeformation tensors are zero. According to this definition, the physical dimension of each higher order microdeformation tensor is that of the corresponding higher gradient of  $\underline{\mathbf{F}}$ :  $\underline{\underline{\chi}}$  is dimensionless,  $\underline{\underline{\chi}}$  is in  $\text{m}^{-1}$ , etc.

The transformation of the deformation gradient by a change of observer represented by the time-dependent rotation  $\underline{\underline{\mathbf{Q}}}(t)$  is as follows:

$$\underline{\mathbf{F}} \longrightarrow \underline{\underline{\mathbf{Q}}} \cdot \underline{\mathbf{F}} \quad (6)$$

The following transformation rules are assumed for the micromorphic degrees of freedom:

$$\underline{\underline{\chi}} \longrightarrow \underline{\underline{\mathbf{Q}}} \cdot \underline{\underline{\chi}} \quad \chi_{iJ} \longrightarrow Q_{ik} \chi_{kJ} \quad (7)$$

$$\underline{\underline{\underline{\chi}}} \longrightarrow \underline{\underline{\mathbf{Q}}} \cdot \underline{\underline{\underline{\chi}}} \quad \chi_{iJK} \longrightarrow Q_{il} \chi_{lJK} \quad (8)$$

...

The generalized degrees of freedom  $\chi_{iJK\dots}$  have the same structure and transformation rules as the multipolar displacements introduced by (Green and Rivlin, 1964), later called *multipolar deformation fields* in (Green and Rivlin, 1967). The fundamental difference with the present theory lies in the generalization of the balance of momentum equation which is absent in Green and Rivlin's theory, as recognized by (Germain, 1973b). Green and Rivlin's interpretation of the multipolar fields, presented in the appendix of (Green and Rivlin, 1964), involves a collection of particles and is based on a Taylor expansion whereby the multipolar tensors must be completely symmetric w.r.t. to all indices excepted the first. Their multipolar displacements represent therefore a generalization of this concept to multiparticle systems.

The interpretation of the first microdeformation tensor as the linear transformation of a triad of directors attached to the microstructure is particularly illustrative. Extensions to higher order microdeformation is possible based on higher order tensor products of directors as proposed by (Green et al., 1965).

## 2.2. Lagrangian generalized strain measures

The proposed theory is a continuum model of order  $n$  and of grade 1, meaning that only the first gradient of all degrees of freedom is considered. The set  $GRAD$  contains all the available constitutive variables of the model:

$$GRAD = \{\underline{\mathbf{u}}, \underline{\mathbf{u}} \otimes \nabla^0, \underline{\underline{\chi}}, \underline{\underline{\chi}} \otimes \nabla^0, \underline{\underline{\underline{\chi}}}, \underline{\underline{\underline{\chi}}} \otimes \nabla^0, \dots\} \quad (9)$$

$$GRAD = \{u_i, u_{i,J}, \chi_{iJ}, \chi_{iJ,K}, \chi_{iJK}, \chi_{iJK,L}, \dots\}$$

Appropriate strain measures are obtained by considering the invariance properties of the constitutive functions. In the case of hyperelasticity, the constitutive function is the Helmholtz free energy potential  $\psi$  per unit volume. It is *a priori* a function depending on all the variables contained in  $GRAD$ . According to the requirement of Galilean invariance, it must be invariant w.r.t. all Galilean transformations. The translation invariance excludes the presence of the displacement in the arguments of the constitutive function. The invariance w.r.t. to all constant rotations  $\underline{Q}_0$  writes

$$\begin{aligned}
& \psi(\underline{F}, \underline{\chi}, \underline{\chi} \otimes \nabla^0, \underline{\underline{\chi}}, \underline{\underline{\chi}} \otimes \nabla^0, \dots) = \\
& = \psi(\underline{Q}_0 \cdot \underline{F}, \underline{Q}_0 \cdot \underline{\chi}, \underline{Q}_0 \cdot \underline{\chi} \otimes \nabla^0, \underline{Q}_0 \cdot \underline{\underline{\chi}}, \underline{Q}_0 \cdot \underline{\underline{\chi}} \otimes \nabla^0, \dots) \\
& = \psi(\underline{U}, \underline{R}^T \cdot \underline{\chi}, \underline{R}^T \cdot \underline{\chi} \otimes \nabla^0, \underline{R}^T \cdot \underline{\underline{\chi}}, \underline{R}^T \cdot \underline{\underline{\chi}} \otimes \nabla^0, \dots) \quad (10) \\
& = \psi(U_{IJ}, R_{Ik}^T \chi_{kJ}, R_{Ij}^T \chi_{jK,L}, R_{Ij}^T \chi_{jKL}, R_{Ij}^T \chi_{jKL,M}, \dots)
\end{aligned}$$

where the specific choice  $\underline{Q}_0 = \underline{R}^T$  was made among all possible rotations,  $\underline{R}$  being the rotation part in the polar decomposition of the deformation gradient  $\underline{F} = \underline{R} \cdot \underline{U}$ , with  $\underline{U}$  the Lagrangian stretch tensor. The last expression shows that the arguments of the free energy potential are Lagrangian strain measures appropriate for constitutive modeling. In the present work, the following set of Lagrangian strain measures is adopted, without loss of generality:

$$\begin{aligned}
& \psi(\underline{C}, \underline{\Upsilon} := \underline{\chi}^{-1} \cdot \underline{F}, \underline{K} := \underline{\chi}^{-1} \cdot (\underline{\chi} \otimes \nabla^0), \underline{\underline{\Upsilon}} := \underline{\chi}^{-1} \cdot \underline{\underline{\chi}}, \underline{K} := \underline{\chi}^{-1} \cdot (\underline{\underline{\chi}} \otimes \nabla^0), \dots) \quad (11) \\
& \psi(C_{IJ}, \Upsilon_{IJ} := \chi_{Ik}^{-1} F_{kJ}, K_{IJK} := \chi_{Il}^{-1} \chi_{lJK}, \underline{\underline{\Upsilon}}_{IJK} := \chi_{Il}^{-1} \chi_{lJK}, K_{IJKL} := \chi_{Im}^{-1} \chi_{mJK,L}, \dots)
\end{aligned}$$

The pull-back of all quantities by  $\underline{\chi}^{-1}$  provides Lagrangian strain measures. The right Cauchy-Green strain is  $\underline{C} = \underline{F}^T \cdot \underline{F}$ . The generalized strain measures  $\underline{\Upsilon}$  and  $\underline{K}$  as defined in Eq. (11) are the ones used in Eringen's original micromorphic theory (Eringen, 1999; Forest and Sievert, 2003). All the arguments of the constitutive function (11) can be expressed in terms of the arguments found in the reduced constitutive law (10), as it should. The Lagrangian generalized strain measure  $\underline{K}$  possesses the remarkable property that its time derivative can be directly and simply related to the Eulerian gradient of the microdeformation rate:

$$(\dot{\underline{\chi}} \cdot \underline{\chi}^{-1}) \otimes \nabla = \underline{\chi} \cdot \underline{\dot{K}} : (\underline{\chi}^{-1} \boxtimes \underline{F}^{-1}), \quad (\dot{\chi}_{iL} \chi_{Lj}^{-1})_{,k} = \chi_{iP} \dot{K}_{PQR} \chi_{Qj}^{-1} F_{Rk}^{-1} \quad (12)$$

Other pull-backs are possible and lead to different strain measures, as usual in finite deformation theories. However they lead to more complicated expressions as discussed by (Eringen, 1999; Forest and Sievert, 2003) and (Bertram, 2015) for the special case of strain gradient theory. The new tensors  $\underline{\underline{\Upsilon}}$  and  $\underline{\underline{K}}$  represent direct extensions to second order micromorphic media.

Note that Green and Rivlin (1964) define similar Lagrangian generalized strain measures pulled-back by means of  $\underline{F}^T$  instead of  $\underline{\chi}^{-1}$ .

### 2.3. Generalized principle of virtual power

The method of virtual power is used to introduce the generalized stress tensors of the theory. The virtual power density of internal forces is computed w.r.t. any subdomain  $\mathcal{D}_0$  of the reference configuration  $\Omega_0$ :

$$\mathcal{P}^{(i)}(\delta\mathbf{u}, \delta\mathbf{\underline{\chi}}, \delta\mathbf{\underline{\underline{\chi}}}, \dots) = - \int_{\mathcal{D}_0} Jp^{(i)}(\delta\mathbf{u}, \delta\mathbf{\underline{\chi}}, \delta\mathbf{\underline{\underline{\chi}}}, \dots) dV \quad (13)$$

where  $p^{(i)}$  is the virtual power density of internal forces per unit volume of the current configuration, and  $J$  is the Jacobian. It is introduced as a function of the virtual fields represented by variations of displacements and microdeformations of any order. The virtual power density of internal forces is taken as a linear form w.r.t. the variations of the Lagrangian strain measures (11):

$$\begin{aligned} Jp^{(i)} &= \frac{1}{2}\mathbf{\underline{\underline{\Pi}}} : \delta\mathbf{\underline{\underline{C}}} + \mathbf{\underline{\underline{T}}} : \delta\mathbf{\underline{\underline{\Upsilon}}} + \mathbf{\underline{\underline{M}}} : \delta\mathbf{\underline{\underline{K}}} + \mathbf{\underline{\underline{T}}} : \delta\mathbf{\underline{\underline{\Upsilon}}} + \mathbf{\underline{\underline{M}}} : \delta\mathbf{\underline{\underline{K}}} \\ &= \frac{1}{2}\Pi_{IJKL} \delta C_{IJKL} + T_{IJK} \delta \Upsilon_{IJK} + M_{IJKL} \delta K_{IJKL} + T_{IJK} \delta \Upsilon_{IJK} + M_{IJKL} \delta K_{IJKL} \end{aligned} \quad (14)$$

where  $\mathbf{\underline{\underline{\Pi}}}$  is the usual Piola stress tensor,  $\mathbf{\underline{\underline{T}}}$  and  $\mathbf{\underline{\underline{T}}}$  are the stress tensors conjugate to the microdeformations whereas  $\mathbf{\underline{\underline{M}}}$  and  $\mathbf{\underline{\underline{M}}}$  are generalized couple stress tensors. The variations of the strains can be expressed in terms of the virtual displacements and microdeformations:

$$\delta\mathbf{\underline{\underline{C}}} = \mathbf{\underline{\underline{F}}}^T \cdot \delta\mathbf{\underline{\underline{F}}} + \delta\mathbf{\underline{\underline{F}}}^T \cdot \mathbf{\underline{\underline{F}}} \quad (15)$$

$$\delta\mathbf{\underline{\underline{\Upsilon}}} = \mathbf{\underline{\underline{\chi}}}^{-1} \cdot \delta\mathbf{\underline{\underline{F}}} - \mathbf{\underline{\underline{\chi}}}^{-1} \cdot \delta\mathbf{\underline{\underline{\chi}}} \cdot \mathbf{\underline{\underline{\Upsilon}}} \quad (16)$$

$$\delta\mathbf{\underline{\underline{K}}} = \mathbf{\underline{\underline{\chi}}}^{-1} \cdot (\delta\mathbf{\underline{\underline{\chi}}} \otimes \mathbf{\underline{\underline{\nabla}}^0}) - \mathbf{\underline{\underline{\chi}}}^{-1} \cdot \delta\mathbf{\underline{\underline{\chi}}} \cdot \mathbf{\underline{\underline{K}}} \quad (17)$$

$$\delta\mathbf{\underline{\underline{\Upsilon}}} = \mathbf{\underline{\underline{\chi}}}^{-1} \cdot \delta\mathbf{\underline{\underline{\chi}}} - \mathbf{\underline{\underline{\chi}}}^{-1} \cdot \delta\mathbf{\underline{\underline{\chi}}} \cdot \mathbf{\underline{\underline{\Upsilon}}} \quad (18)$$

$$\delta\mathbf{\underline{\underline{K}}} = \mathbf{\underline{\underline{\chi}}}^{-1} \cdot (\delta\mathbf{\underline{\underline{\chi}}} \otimes \mathbf{\underline{\underline{\nabla}}^0}) - \mathbf{\underline{\underline{\chi}}}^{-1} \cdot \delta\mathbf{\underline{\underline{\chi}}} \cdot \mathbf{\underline{\underline{K}}} \quad (19)$$

The virtual power of contact forces acting on the boundary  $\partial\mathcal{D}_0$  takes the form:

$$\begin{aligned} \mathcal{P}^{(c)}(\delta\mathbf{u}, \delta\mathbf{\underline{\chi}}, \delta\mathbf{\underline{\underline{\chi}}}, \dots) &= \int_{\partial\mathcal{D}_0} p^{(c)}(\delta\mathbf{u}, \delta\mathbf{\underline{\chi}}, \delta\mathbf{\underline{\underline{\chi}}}, \dots) dS \\ &= \int_{\partial\mathcal{D}_0} \mathbf{\underline{\underline{t}}} \cdot \delta\mathbf{u} + \mathbf{\underline{\underline{t}}} : \delta\mathbf{\underline{\chi}} + \mathbf{\underline{\underline{t}}} : \delta\mathbf{\underline{\underline{\chi}}} + \dots dS \end{aligned} \quad (20)$$

The virtual power of external forces, acting at a distance is taken of the form:

$$\begin{aligned} \mathcal{P}^{(e)}(\delta\mathbf{u}, \delta\mathbf{\underline{\chi}}, \delta\mathbf{\underline{\underline{\chi}}}, \dots) &= \int_{\mathcal{D}_0} Jp^{(e)}(\delta\mathbf{u}, \delta\mathbf{\underline{\chi}}, \delta\mathbf{\underline{\underline{\chi}}}, \dots) dV \\ &= \int_{\mathcal{D}_0} \mathbf{\underline{\underline{f}}} \cdot \delta\mathbf{u} + \mathbf{\underline{\underline{f}}} : \delta\mathbf{\underline{\chi}} + \mathbf{\underline{\underline{f}}} : \delta\mathbf{\underline{\underline{\chi}}} + \dots dV \end{aligned} \quad (21)$$

Higher order volume forces working with the gradient of the microdeformation tensors could be introduced in the latter expression in the spirit of (Germain, 1973a).

The virtual power of acceleration forces is taken of the form:

$$\begin{aligned}
\mathcal{P}^{(a)}(\delta \underline{\mathbf{u}}, \delta \underline{\underline{\boldsymbol{\chi}}}, \delta \underline{\underline{\boldsymbol{\chi}}}, \dots) &= - \int_{\mathcal{D}_0} J p^{(a)}(\delta \underline{\mathbf{u}}, \delta \underline{\underline{\boldsymbol{\chi}}}, \delta \underline{\underline{\boldsymbol{\chi}}}, \dots) dV \\
&= - \int_{\mathcal{D}_0} \rho_0 \left( \underline{\mathbf{a}} \cdot \delta \underline{\mathbf{u}} + (\underline{\underline{\boldsymbol{\chi}}} \cdot \underline{\mathbf{I}}) : \delta \underline{\underline{\boldsymbol{\chi}}} + (\underline{\underline{\boldsymbol{\chi}}} : \underline{\underline{\mathbf{I}}}) \dot{:} \delta \underline{\underline{\boldsymbol{\chi}}} + \dots \right) dV \\
&= - \int_{\mathcal{D}_0} \rho_0 (a_i \delta u_i + \ddot{\chi}_{iL} I_{LJ} \delta \chi_{iJ} + \ddot{\chi}_{iPQ} I_{PQJK} \delta \chi_{iJK} + \dots) dV
\end{aligned} \tag{22}$$

where  $\underline{\underline{\mathbf{I}}}$  and  $\underline{\underline{\mathbf{I}}}$  are Lagrangian generalized microinertia tensors. The mass densities per unit reference or current volume are called  $\rho_0$  and  $\rho$ , respectively. They are such that  $J = \det \underline{\underline{\mathbf{F}}} = \rho_0/\rho$  to comply with mass conservation. The acceleration vector is  $\underline{\mathbf{a}}$ . This Lagrangian formulation is in contrast to Germain's general micromorphic dynamics which is primarily introduced in the Eulerian framework, see (Germain, 1973b). Note that the acceleration of the multipolar deformations was not considered by Green and Rivlin (1964), even though these authors take the acceleration of directors into account in the subsequent works (Green et al., 1965; Green and Rivlin, 1967).

The principle of virtual power for general micromorphic media stipulates that

$$\mathcal{P}^{(i)} + \mathcal{P}^{(c)} + \mathcal{P}^{(e)} + \mathcal{P}^{(a)} = 0, \quad \forall \delta \underline{\mathbf{u}}, \delta \underline{\underline{\boldsymbol{\chi}}}, \delta \underline{\underline{\boldsymbol{\chi}}} \quad \text{and} \quad \forall \mathcal{D}_0 \subset \Omega_0 \tag{23}$$

#### 2.4. Balance laws of generalized moments of momentum

The exploitation of the principle of virtual power stated in the previous section leads to the derivation of the balance equations valid for all  $\underline{\mathbf{X}} \in \Omega_0$  in the form

$$\text{Div} (\underline{\underline{\mathbf{S}}} + \underline{\underline{\mathbf{T}}}^B) + \underline{\mathbf{f}} = \rho_0 \underline{\mathbf{a}}, \quad S_{iJ,J} + T_{iJ,J}^B + f_i = \rho_0 a_i \tag{24}$$

$$\text{Div} \underline{\underline{\mathbf{M}}}^B + \underline{\underline{\mathbf{T}}}^C + \underline{\mathbf{f}} = \rho_0 \underline{\underline{\boldsymbol{\chi}}} \cdot \underline{\underline{\mathbf{I}}}, \quad M_{iJK,K}^B + T_{iJ}^C + f_{iJ} = \rho_0 \ddot{\chi}_{iK} I_{KJ} \tag{25}$$

$$\text{Div} \underline{\underline{\mathbf{M}}}^B - \underline{\underline{\mathbf{T}}}^B + \underline{\mathbf{f}} = \rho_0 \underline{\underline{\boldsymbol{\chi}}} : \underline{\underline{\mathbf{I}}}, \quad M_{iJKL,L}^B - T_{iJK}^B + f_{iJK} = \rho_0 \ddot{\chi}_{iPQ} I_{PQJK} \tag{26}$$

and of the Neumann boundary conditions valid for all  $\underline{\mathbf{X}} \in \partial\Omega_0$ :

$$(\underline{\underline{\mathbf{S}}} + \underline{\underline{\mathbf{T}}}^B) \cdot \underline{\mathbf{N}} = \underline{\mathbf{t}}, \quad (S_{iJ} + T_{iJ}^B) N_J = t_i \tag{27}$$

$$\underline{\underline{\mathbf{M}}}^B \cdot \underline{\mathbf{N}} = \underline{\mathbf{t}}, \quad M_{iJK}^B N_K = t_{iJ} \tag{28}$$

$$\underline{\underline{\mathbf{M}}}^B \cdot \underline{\mathbf{N}} = \underline{\mathbf{t}}, \quad M_{iJKL}^B N_L = t_{iJK} \tag{29}$$

Tensor  $\underline{\underline{\mathbf{S}}} = \underline{\underline{\mathbf{F}}} \cdot \underline{\underline{\boldsymbol{\Pi}}}$  is the usual Boussinesq stress tensor, also called first Piola–Kirchhoff stress tensor.

The label  $^B$  refers to generalized Boussinesq tensors:

$$\underline{\underline{\mathbf{T}}}^B = \underline{\underline{\boldsymbol{\chi}}}^{-T} \cdot \underline{\underline{\mathbf{T}}}, \quad T_{iK}^B = \chi_{iJ}^{-T} T_{JK} \tag{30}$$

$$\underline{\underline{\mathbf{T}}}^B = \underline{\underline{\boldsymbol{\chi}}}^{-T} \cdot \underline{\underline{\mathbf{T}}}, \quad T_{iJK}^B = \chi_{iI}^{-T} T_{IJK} \tag{31}$$

$$\underline{\underline{\mathbf{M}}}^B = \underline{\underline{\boldsymbol{\chi}}}^{-T} \cdot \underline{\underline{\mathbf{M}}}, \quad M_{iJK}^B = \chi_{iI}^{-T} M_{IJK} \tag{32}$$

$$\underline{\underline{\mathbf{M}}}^B = \underline{\underline{\boldsymbol{\chi}}}^{-T} \cdot \underline{\underline{\mathbf{M}}}, \quad M_{iJKL}^B = \chi_{iI}^{-T} M_{IJKL} \tag{33}$$

The generalized stress tensor  $\underline{\underline{T}}^C$  arising in Eq. (25) couples all other micromorphic stress tensors in the following way:

$$\begin{aligned}\underline{\underline{T}}^C &= \underline{\underline{T}}^B \cdot \underline{\underline{\Upsilon}}^T + \underline{\underline{M}}^B : \underline{\underline{K}}^T + \underline{\underline{T}}^B : \underline{\underline{\Upsilon}}^T + \underline{\underline{M}}^B : \underline{\underline{K}}^T \\ T_{pQ}^C &= T_{pJ}^B \Upsilon_{JQ}^T + M_{pJK}^B K_{QJ,K} + T_{pJK}^B \Upsilon_{QJ,K} + M_{pJKL}^B K_{QJK,L}\end{aligned}\quad (34)$$

In contrast to the derived generalized balance of momentum equations, Green, Rivlin and Naghdi considered a single balance of momentum equation (Green and Rivlin, 1964; Green et al., 1965; Green and Rivlin, 1967). Relations are introduced involving the divergence of multipolar stresses defining auxiliary higher order stress tensors akin to the present tensors  $T_{iJK\dots}$ . These relations<sup>2</sup> are regarded as constitutive, not as balance equations. There are substituted into a single additional boundary condition involving the heat flux through the surface. Germain noted that this is due to their derivation of all equations from the energy conservation law which, in the classical case, leads to the same results as the method of virtual power, but to less equations in the case of micromorphic media, see (Germain, 1973b).

### 2.5. Linearized balance laws

The previous field equations are now linearized to obtain a more simple form and recover some existing balance laws for micromorphic continua. Deformations are small when

$$\|\underline{\underline{F}} - \underline{\underline{1}}\| \ll 1 \quad (35)$$

Considering that the microdeformations are relaxed strain gradient-like variables, the generalized microdeformations are small when:

$$\|\underline{\underline{\chi}} - \underline{\underline{1}}\| \ll 1, \quad L\|\underline{\underline{K}}\| \ll 1, \quad L\|\underline{\underline{\chi}}\| \ll 1, \quad L^2\|\underline{\underline{K}}\| \ll 1 \quad (36)$$

where  $L$  is a characteristic length related to the structure or to the wavelength of the applied loading conditions. In the following,  $\underline{\underline{\chi}} - \underline{\underline{1}}$  is replaced by the same notation  $\underline{\underline{\chi}}$ . The acceleration terms involve constitutive inertia tensors of order 2 and 4.

Within the context of small deformations and microdeformations, the generalized strain measures are linearized as follows:

$$\underline{\underline{C}} \simeq \underline{\underline{1}} + 2\underline{\underline{\varepsilon}}, \quad \text{with} \quad 2\underline{\underline{\varepsilon}} = \underline{\underline{u}} \otimes \underline{\underline{\nabla}} + \underline{\underline{\nabla}} \otimes \underline{\underline{u}} \quad (37)$$

$$\underline{\underline{\Upsilon}} \simeq \underline{\underline{u}} \otimes \underline{\underline{\nabla}} - \underline{\underline{\chi}}, \quad \underline{\underline{\Upsilon}} \simeq \underline{\underline{\chi}} \quad (38)$$

---

<sup>2</sup>see the equations (13.2) in (Green and Rivlin, 1964).

The linearized stress measures are then:

$$\underline{\underline{S}} = \underline{\underline{F}} \cdot \underline{\underline{\Pi}} = J \underline{\underline{\sigma}} \cdot \underline{\underline{F}}^{-T} \simeq \underline{\underline{\sigma}} \quad (39)$$

$$\underline{\underline{T}}^B \simeq \underline{\underline{T}}, \quad \underline{\underline{T}}^B \simeq \underline{\underline{T}}, \quad \underline{\underline{T}}^C \simeq \underline{\underline{T}} \quad (40)$$

$$\underline{\underline{M}}^B \simeq \underline{\underline{M}}, \quad \underline{\underline{M}}^B \simeq \underline{\underline{T}} \quad (41)$$

where  $\underline{\underline{\sigma}}$  is the usual Cauchy stress tensor.

As a result, the balance laws (24) to (26) reduce to

$$\operatorname{div}(\underline{\underline{\sigma}} + \underline{\underline{T}}) + \underline{\underline{f}} = \rho \underline{\underline{a}} \quad (42)$$

$$\operatorname{div} \underline{\underline{M}} - \underline{\underline{T}} + \underline{\underline{f}} = \rho \underline{\underline{\dot{\chi}}} \cdot \underline{\underline{I}} \quad (43)$$

$$\operatorname{div} \underline{\underline{M}} - \underline{\underline{T}} + \underline{\underline{f}} = \rho \underline{\underline{\dot{\chi}}} : \underline{\underline{I}} \quad (44)$$

These equations have the same form as the Eulerian balance laws according to Germain's general micromorphic theory, except that the present stress tensors  $\underline{\underline{T}}, \underline{\underline{T}}, \underline{\underline{M}}, \underline{\underline{M}}$  do not exhibit any symmetry property in contrast to Germain's ones.

### 3. Hyperelasticity of second order micromorphic media

The constitutive theory of hyperelastic second order micromorphic media is based on the objective Helmholtz free energy potential per unit volume:

$$\psi(\underline{\underline{C}}, \underline{\underline{\Upsilon}}, \underline{\underline{K}}, \underline{\underline{\Upsilon}}, \underline{\underline{K}}) \quad (45)$$

#### 3.1. Derivation of the hyperelastic state laws

The local form of the Clausius–Duhem inequality for second order micromorphic media is:

$$Jp^{(i)} - \dot{\psi} \geq 0 \quad (46)$$

Taking the expression (14) of the power of internal forces and the dependence (45) into account, it gives

$$\left( \frac{\underline{\underline{\Pi}}}{2} - \frac{\partial \psi}{\partial \underline{\underline{C}}} \right) : \underline{\underline{\dot{C}}} + \left( \underline{\underline{T}} - \frac{\partial \psi}{\partial \underline{\underline{\Upsilon}}} \right) : \underline{\underline{\dot{\Upsilon}}} + \left( \underline{\underline{M}} - \frac{\partial \psi}{\partial \underline{\underline{K}}} \right) : \underline{\underline{\dot{K}}} + \left( \underline{\underline{T}} - \frac{\partial \psi}{\partial \underline{\underline{\chi}}} \right) : \underline{\underline{\dot{\Upsilon}}} + \left( \underline{\underline{M}} - \frac{\partial \psi}{\partial \underline{\underline{K}}} \right) :: \underline{\underline{\dot{K}}} \geq 0 \quad (47)$$

The positiveness of this linear form w.r.t. to the strain rates requires that the conjugate quantities vanish.

The hyperelastic laws follow:

$$\underline{\underline{\Pi}} = 2 \frac{\partial \psi}{\partial \underline{\underline{C}}}, \quad \underline{\underline{T}} = \frac{\partial \psi}{\partial \underline{\underline{\Upsilon}}}, \quad \underline{\underline{M}} = \frac{\partial \psi}{\partial \underline{\underline{K}}}, \quad \underline{\underline{T}} = \frac{\partial \psi}{\partial \underline{\underline{\chi}}}, \quad \underline{\underline{M}} = \frac{\partial \psi}{\partial \underline{\underline{K}}} \quad (48)$$

### 3.2. Consideration of internal constraints

Internal constraints are envisaged in the form of relations that must be fulfilled by kinematic quantities. As an example, two constraints are introduced:

$$f(\underline{\Upsilon}) = 0, \quad g(\underline{\Upsilon}, \underline{\mathbf{K}}) = 0 \quad (49)$$

where  $f$  and  $g$  are differentiable scalar functions. This specific example is motivated by the applications in section 4, without loss of generality. Following the usual procedure, see for instance (Liu, 2002), extended for gradient materials by (Bertram and Glüge, 2016), the only evolutions of the strain variables compatible with the constraints are

$$\frac{\partial f}{\partial \underline{\Upsilon}} : \dot{\underline{\Upsilon}} = 0, \quad \frac{\partial g}{\partial \underline{\Upsilon}} : \dot{\underline{\Upsilon}} + \frac{\partial g}{\partial \underline{\mathbf{K}}} : \dot{\underline{\mathbf{K}}} = 0 \quad (50)$$

Some state laws (48) must be amended as follows to enforce the internal constraints:

$$\underline{\mathbf{T}} = \frac{\partial \psi}{\partial \underline{\Upsilon}} + \underline{\mathbf{T}}^R, \quad \underline{\mathbf{M}} = \frac{\partial \psi}{\partial \underline{\mathbf{K}}} + \underline{\mathbf{M}}^R, \quad \underline{\mathbf{T}} = \frac{\partial \psi}{\partial \underline{\chi}} + \underline{\mathbf{T}}^R, \quad (51)$$

where the quantities with the label  $R$  are reaction stresses of the form:

$$\underline{\mathbf{T}}^R = \lambda_f \frac{\partial f}{\partial \underline{\Upsilon}}, \quad \underline{\mathbf{M}}^R = \lambda_g \frac{\partial g}{\partial \underline{\mathbf{K}}}, \quad \underline{\mathbf{T}}^R = \lambda_g \frac{\partial g}{\partial \underline{\Upsilon}} \quad (52)$$

The reaction stresses involve the two Lagrange multipliers  $\lambda_f$  and  $\lambda_g$ . Their values must be determined from the resolution of the boundary value problem.

### 3.3. Hyperelasticity based on the Gibbs free energy potential

The Gibbs free energy potential is obtained by the Legendre transform of the Helmholtz potential:

$$\psi^*(\underline{\Pi}, \underline{\mathbf{T}}, \underline{\mathbf{M}}, \underline{\mathbf{T}}, \underline{\mathbf{M}}) = \underset{\{\underline{\mathbf{C}}, \underline{\Upsilon}, \underline{\mathbf{K}}, \underline{\Upsilon}, \underline{\mathbf{K}}\}}{\text{Max}} \left( \psi(\underline{\mathbf{C}}, \underline{\Upsilon}, \underline{\mathbf{K}}, \underline{\Upsilon}, \underline{\mathbf{K}}) - \underline{\Pi} : \frac{\underline{\mathbf{C}}}{2} - \underline{\mathbf{T}} : \underline{\Upsilon} - \underline{\mathbf{M}} : \underline{\mathbf{K}} - \underline{\mathbf{T}} : \underline{\Upsilon} - \underline{\mathbf{M}} : \underline{\mathbf{K}} \right) \quad (53)$$

The arguments of this constitutive function are the generalized stress tensors:  $(\underline{\Pi}, \underline{\mathbf{T}}, \underline{\mathbf{M}}, \underline{\mathbf{T}}, \underline{\mathbf{M}})$ .

The dissipation inequality (46) transforms into

$$\begin{aligned} Jp^{(i)} - \dot{\psi}^* - \underline{\Pi} : \frac{\dot{\underline{\mathbf{C}}}}{2} - \underline{\mathbf{T}} : \dot{\underline{\Upsilon}} - \underline{\mathbf{M}} : \dot{\underline{\mathbf{K}}} - \underline{\mathbf{T}} : \dot{\underline{\Upsilon}} - \underline{\mathbf{M}} : \dot{\underline{\mathbf{K}}} \\ - \underline{\dot{\Pi}} : \frac{\underline{\mathbf{C}}}{2} - \underline{\dot{\mathbf{T}}} : \underline{\Upsilon} - \underline{\dot{\mathbf{M}}} : \underline{\mathbf{K}} - \underline{\dot{\mathbf{T}}} : \underline{\Upsilon} - \underline{\dot{\mathbf{M}}} : \underline{\mathbf{K}} \geq 0 \end{aligned} \quad (54)$$

and consequently,

$$-\dot{\psi}^* - \underline{\dot{\Pi}} : \frac{\underline{\mathbf{C}}}{2} - \underline{\dot{\mathbf{T}}} : \underline{\Upsilon} - \underline{\dot{\mathbf{M}}} : \underline{\mathbf{K}} - \underline{\dot{\mathbf{T}}} : \underline{\Upsilon} - \underline{\dot{\mathbf{M}}} : \underline{\mathbf{K}} \geq 0$$

$$\left(\frac{\partial\psi^*}{\partial\tilde{\mathbf{\Pi}}} + \frac{\underline{\mathbf{C}}}{2}\right) : \dot{\tilde{\mathbf{\Pi}}} + \left(\frac{\partial\psi^*}{\partial\tilde{\mathbf{T}}} + \underline{\mathbf{\Upsilon}}\right) : \dot{\tilde{\mathbf{T}}} + \left(\frac{\partial\psi^*}{\partial\tilde{\mathbf{M}}} + \underline{\mathbf{K}}\right) : \dot{\tilde{\mathbf{M}}} + \left(\frac{\partial\psi^*}{\partial\tilde{\mathbf{T}}} + \underline{\mathbf{\Upsilon}}\right) : \dot{\tilde{\mathbf{T}}} + \left(\frac{\partial\psi^*}{\partial\tilde{\mathbf{M}}} + \underline{\mathbf{K}}\right) : \dot{\tilde{\mathbf{M}}} \leq 0 \quad (55)$$

from which the dual state laws are derived:

$$\underline{\mathbf{C}} = -2\frac{\partial\psi^*}{\partial\tilde{\mathbf{\Pi}}}, \quad \underline{\mathbf{\Upsilon}} = -\frac{\partial\psi^*}{\partial\tilde{\mathbf{T}}}, \quad \underline{\mathbf{K}} = -\frac{\partial\psi^*}{\partial\tilde{\mathbf{M}}}, \quad \underline{\mathbf{\Upsilon}} = -\frac{\partial\psi^*}{\partial\tilde{\mathbf{T}}}, \quad \underline{\mathbf{K}} = -\frac{\partial\psi^*}{\partial\tilde{\mathbf{M}}} \quad (56)$$

These constitutive laws must be amended in the presence of internal constraints. The consideration of internal constraints is usually limited to kinematic constraints, see (Bertram and Glüge, 2016). Static constraints can also be envisaged. As an example, let us consider a constraint linking two stress tensors:

$$f(\tilde{\mathbf{\Pi}}, \tilde{\mathbf{M}}) = 0 \quad (57)$$

The stress increments must then be such that:

$$\frac{\partial f}{\partial\tilde{\mathbf{\Pi}}} : \dot{\tilde{\mathbf{\Pi}}} + \frac{\partial f}{\partial\tilde{\mathbf{M}}} : \dot{\tilde{\mathbf{M}}} = 0 \quad (58)$$

The state laws (56)<sub>1</sub> and (56)<sub>5</sub> must then be amended as follows:

$$\underline{\mathbf{C}} = -2\frac{\partial\psi^*}{\partial\tilde{\mathbf{\Pi}}} + 2\lambda_f \frac{\partial f}{\partial\tilde{\mathbf{\Pi}}}, \quad \underline{\mathbf{K}} = -\frac{\partial\psi^*}{\partial\tilde{\mathbf{M}}} + \lambda_f \frac{\partial f}{\partial\tilde{\mathbf{M}}} \quad (59)$$

where the Lagrange multiplier  $\lambda_f$  must be determined from the resolution of the boundary value problem.

#### 4. Relation to strain and stress gradients theories

Specific internal constraints are now presented to retrieve the strain gradient and stress gradient models as limit cases.

##### 4.1. Strain gradient theory

The following constraints are considered:

$$f(\underline{\mathbf{\Upsilon}}) = \underline{\mathbf{\Upsilon}} - \underline{\mathbf{1}} = 0, \quad g(\underline{\mathbf{K}}, \underline{\mathbf{\Upsilon}}) = \underline{\mathbf{\Upsilon}} - \underline{\mathbf{K}} = 0 \quad (60)$$

The relative deformation tensor is assumed to remain the identity during any motion, which means that the microdeformation  $\underline{\boldsymbol{\chi}}$  coincides with the deformation gradient itself. As a result,

$$\underline{\boldsymbol{\chi}} \equiv \underline{\mathbf{F}}, \quad \underline{\mathbf{K}} \equiv \underline{\mathbf{F}} \otimes \nabla^0 \quad (61)$$

so that the theory includes the second gradient of displacement. The second order microdeformation is forced to coincide with the gradient of the microdeformation:

$$\underline{\underline{\chi}} \equiv \underline{\underline{\chi}} \otimes \nabla^0 \equiv \underline{\underline{F}} \otimes \nabla^0, \quad \underline{\underline{K}} \equiv \underline{\underline{F}} \otimes \nabla^0 \otimes \nabla^0 \quad (62)$$

The constrained second order micromorphic theory therefore reduces to the third displacement gradient model, or equivalently second strain gradient theory of (Mindlin, 1965).

In this case, the reaction stresses (52) become:

$$\underline{\underline{T}}^R = \lambda_f \underline{\underline{1}}, \quad \underline{\underline{M}}^R = \lambda_g \underline{\underline{1}}, \quad \underline{\underline{T}}^R = -\lambda_g \underline{\underline{1}} \quad (63)$$

The constitutive equations therefore leave the spherical part of the coupling stress tensor  $\underline{\underline{T}}$  indeterminate. It will be shown explicitly in the linearized case how the three balance equations (24) to (26) can be reduced to a single set of higher order partial differential equations for the unknown displacement components, see section 5.3.

As noted by (Seppecher, 2000), enforcing internal constraints makes it possible to derive the balance and constitutive equations in the bulk for gradient theories but it does not provide the corresponding boundary conditions. The latter are rather sophisticated and must be derived from the direct formulation of the grade  $n$  model, see (dell’Isola and Seppecher, 1995; Seppecher, 2000; dell’Isola et al., 2012; dell’Isola et al., 2016). A further discussion of the reduction of the first order micromorphic model to Mindlin’s strain gradient elasticity can be found in (Broese et al., 2016).

#### 4.2. Stress gradient theory at finite deformation

The static stress gradient theory was proposed for the first time in (Forest and Sab, 2012) and shown to be fundamentally distinct from the strain gradient approach. It was presented within the small strain framework and the objective of this subsection and of the next one is to extend it to the general finite deformation and dynamical cases. The theory is first introduced within the formalism of a constrained second order micromorphic model and then a direct construction is presented enforcing the constraint. However, it will become apparent that the mechanical meaning of the additional degrees of freedom arising in the stress gradient theory differs from that of the microdeformation tensors regarded as *relaxed deformation gradients* introduced in Section 2.1. As a result, alternative strain measures, different from (11), are proposed in the following.

In (Forest and Sab, 2012), the stress gradient model was interpreted as a constrained and truncated second order micromorphic medium where each material point is endowed with first and third rank tensors of degrees of freedom:  $u_i, \Phi_{iJK}$ , thus leaving aside first order additional degrees of freedom:

$$DOF = \{\underline{\underline{u}}, \quad \underline{\underline{\Phi}}\}, \quad DOF = \{u_i, \quad \Phi_{iJK}\}$$

$$GRAD = \{\underline{\mathbf{u}}, \underline{\mathbf{u}} \otimes \nabla^0, \underline{\Phi}, \underline{\Phi} \otimes \nabla^0\}, \quad GRAD = \{u_i, u_{i,L}, \Phi_{iJK}, \Phi_{iJK,L}\}$$

The first order microdeformation tensor being absent from this special theory, the Lagrangian strain measures (11) must be reconsidered. The three Lagrangian constitutive variables proposed for the stress gradient model are:

$$\begin{aligned} \underline{\mathcal{C}} &:= \underline{\mathbf{F}}^T \cdot \underline{\mathbf{F}}, \quad \underline{\Upsilon} := \underline{\mathbf{F}}^T \cdot \underline{\Phi}, \quad \underline{\mathcal{K}} := \underline{\mathbf{F}}^T \cdot \underline{\Phi} \otimes \nabla^0 \\ C_{IJ} &:= F_{Ik}^T F_{kJ}, \quad \Upsilon_{IJK} := F_{Ii}^T \Phi_{iJK}, \quad K_{IJKL} := F_{Ii}^T \Phi_{iJK,L} \end{aligned} \quad (64)$$

The motivation for the pull-back by means of the  $\underline{\mathbf{F}}^T$  variable, instead of  $\underline{\mathbf{F}}^{-1}$ , is the relation to the Boussinesq stress tensor, which will become apparent in the following. The conjugate stress tensors are introduced in the virtual power density of internal forces which is formally a truncation of (14):

$$Jp^{(i)} = \underline{\Pi} : \underline{\mathbf{F}}^T \cdot \delta \underline{\mathbf{F}} + \underline{\mathcal{T}} : \delta \underline{\Upsilon} + \underline{\mathcal{M}} :: \delta \underline{\mathcal{K}} \quad (65)$$

The elastic dual potential is then:  $\psi^*(\underline{\Pi}, \underline{\mathcal{T}}, \underline{\mathcal{M}})$ .

An internal constraint linking the fourth rank stress tensor  $\underline{\mathcal{M}}$  to the Piola stress tensor is now introduced:

$$f(\underline{\Pi}, \underline{\mathcal{M}}) = \underline{\mathcal{M}} - \underline{\Pi} \otimes \underline{\mathbf{1}} = 0 \quad (66)$$

A consequence of this constraint is that  $\underline{\mathcal{M}}$  becomes symmetric w.r.t. the first two indices and last two indices. It is equivalent to the constraint that the fourth rank Boussinesq stress tensor is directly related to the Boussinesq second rank stress tensor:

$$\underline{\mathcal{M}}^B = \underline{\mathbf{F}} \cdot \underline{\mathcal{M}} = \underline{\mathcal{S}} \otimes \underline{\mathbf{1}}, \quad M_{iJKL}^B = S_{iJ} \delta_{KL} \quad (67)$$

It must be noted that the physical dimension implied for the stress tensor components  $M_{iJKL}^B$  by the constraint (67) is different from the one proposed in the previous section in the case of micromorphic theory regarded as a relaxed strain gradient model:  $Pa$  instead of  $Pa.m^2 = N$ . An internal length could be introduced to restore the dimensional consistency but it is preferred to keep the proposed form (67) which will be related to a stress gradient theory. Accordingly, the physical dimension of the additional degrees of freedom,  $\Phi_{iJK}$ , is that of *microdisplacements*, in  $m$ , instead of *microdeformations* in the original theory of section 2.1.

The presence of the internal constraint leads to the modification (59) of the constitutive equations in the form:

$$\underline{\mathcal{C}} = -2 \frac{\partial \psi^*}{\partial \underline{\Pi}} - 2\lambda_f \underline{\mathbf{1}}, \quad \underline{\mathcal{T}} = -\frac{\partial \psi^*}{\partial \underline{\Upsilon}}, \quad \underline{\mathcal{K}} = -\frac{\partial \psi^*}{\partial \underline{\mathcal{M}}} + \lambda_f \underline{\mathbf{1}} \quad (68)$$

where  $\lambda_f$  is a Lagrange multiplier to be determined from the boundary conditions.

Due to this specific definition of the higher order stress tensors, and, as a consequence, to an alternative definition of the Lagrangian strain measures different from (11), the generalized balance of momentum equations must be derived anew. For that purpose, the constraint (67) is directly implemented in the virtual power density of internal forces (65):

$$\begin{aligned}
Jp^{(i)} &= \underline{\underline{\Pi}} : \underline{\underline{F}}^T \cdot \delta \underline{\underline{F}} + \underline{\underline{T}} : (\delta \underline{\underline{F}}^T \cdot \underline{\underline{\Phi}} + \underline{\underline{F}}^T \cdot \delta \underline{\underline{\Phi}}) + \underline{\underline{M}} :: (\delta \underline{\underline{F}}^T \cdot \underline{\underline{\Phi}} \otimes \nabla^0 + \underline{\underline{F}}^T \cdot \delta \underline{\underline{\Phi}} \otimes \nabla^0) \\
&= \Pi_{IJ} F_{Ik}^T \delta F_{kJ} + T_{IJK} (\delta F_{Ik}^T \Phi_{kJK} + F_{Ik}^T \delta \Phi_{kJK}) + \Pi_{IJ} \delta_{KL} (\delta F_{Ik}^T \Phi_{kJK,L} + F_{Ik}^T \delta \Phi_{kJK,L}) \\
&= \underline{\underline{\Pi}} : \left( \frac{\delta \underline{\underline{C}}}{2} + \delta(\underline{\underline{F}}^T \cdot \underline{\underline{\Phi}} \cdot \nabla^0) \right) + \underline{\underline{T}} : \delta(\underline{\underline{F}}^T \cdot \underline{\underline{\Phi}}) \\
&= (\Pi_{IJ} F_{Jk}^T + \Pi_{IJ} \Phi_{kJL,L} + T_{IJK} \Phi_{kJK}) \delta u_{k,I} + T_{IJK} F_{Ik}^T \delta \Phi_{kJK} + \Pi_{IJ} F_{Ik}^T \delta \Phi_{kJL,L} \\
&= (S_{kI} \delta u_k + \Pi_{IJ} \Phi_{kJL,L} \delta u_k + T_{IJK} \Phi_{kJK} \delta u_k)_I - (S_{kI} + \Pi_{IJ} \Phi_{kJL,L} + T_{IJK} \Phi_{kJK})_I \delta u_k \\
&+ T_{kJK}^B \delta \Phi_{kJK} + (S_{kJ} \delta \Phi_{kJL})_{,L} - S_{kJ,L} \delta \Phi_{kJK}
\end{aligned} \tag{69}$$

where the generalized Boussinesq tensor,  $\underline{\underline{T}}^B := \underline{\underline{F}} \cdot \underline{\underline{T}}$ , has been introduced. As a result, the virtual power of internal forces of a subdomain  $\mathcal{D}_0 \subset \Omega_0$  can be written as the sum of a volume and a surface contribution:

$$\begin{aligned}
\mathcal{P}^{(i)} &= \int_{\mathcal{D}_0} \delta \underline{\underline{u}} \cdot \left( \underline{\underline{S}} + (\underline{\underline{\Phi}} \cdot \nabla^0) \cdot \underline{\underline{\Pi}} + \underline{\underline{\Phi}} : \underline{\underline{T}}^T \right) \cdot \nabla^0 dV \\
&- \int_{\partial \mathcal{D}_0} \delta \underline{\underline{u}} \cdot \left( \underline{\underline{S}} + (\underline{\underline{\Phi}} \cdot \nabla^0) \cdot \underline{\underline{\Pi}} + \underline{\underline{\Phi}} : \underline{\underline{T}}^T \right) \cdot \underline{\underline{N}} dS \\
&- \int_{\mathcal{D}_0} \left( \underline{\underline{T}}^B - \underline{\underline{S}} \otimes \nabla^0 \right) : \delta \underline{\underline{\Phi}} dV - \int_{\partial \mathcal{D}_0} (\underline{\underline{S}} \otimes \underline{\underline{N}}) : \delta \underline{\underline{\Phi}} dS
\end{aligned} \tag{71}$$

where the transposition for third order tensors is taken as  $T_{JKI}^T = T_{IJK}$ . The latter expression of the power of internal forces dictates the form of the power of contact forces as:

$$\mathcal{P}^{(c)}(\delta \underline{\underline{u}}, \delta \underline{\underline{\Phi}}) = \int_{\mathcal{D}_0} \left( \underline{\underline{t}} \cdot \delta \underline{\underline{u}} + \underline{\underline{t}} : \delta \underline{\underline{\Phi}} \right) dS \tag{72}$$

where  $\underline{\underline{t}}$  and  $\underline{\underline{t}}$  are generalized surface traction tensors.

The virtual power of external and acceleration forces are taken as:

$$\mathcal{P}^{(e)}(\delta \underline{\underline{u}}, \delta \underline{\underline{\Phi}}) = \int_{\mathcal{D}_0} \underline{\underline{f}} \cdot \delta \underline{\underline{u}} + \underline{\underline{f}} : \delta \underline{\underline{\Phi}} dV \tag{73}$$

$$\mathcal{P}^{(a)}(\delta \underline{\underline{u}}, \delta \underline{\underline{\Phi}}) = - \int_{\mathcal{D}_0} \rho_0 \left( \underline{\underline{a}} \cdot \delta \underline{\underline{u}} + (\underline{\underline{\ddot{\Phi}}} : \underline{\underline{I}}) : \delta \underline{\underline{\Phi}} \right) dV \tag{74}$$

The resulting field equations are:

$$\text{Div} \left( \underline{\underline{S}} + (\text{Div} \underline{\underline{\Phi}}) \cdot \underline{\underline{\Pi}} + \underline{\underline{\Phi}} : \underline{\underline{T}}^T \right) + \underline{\underline{f}} = \rho_0 \underline{\underline{\ddot{u}}} \tag{75}$$

$$\underline{\underline{T}}^B - \underline{\underline{S}} \otimes \nabla^0 + \underline{\underline{f}} = \rho_0 \underline{\underline{\ddot{\Phi}}} : \underline{\underline{I}} \tag{76}$$

The gradient of the classical Boussinesq stress tensor  $\underline{\underline{S}}$  arises in the latter equation. In the static case in particular and in absence of third rank body forces, the generalized stress tensor  $\underline{\underline{T}}^B$  is nothing but the stress gradient, hence the name of *stress gradient theory*.

The exploitation of the principle of virtual power at the boundary provides the expression of the generalized traction tensors:

$$\underline{\underline{t}} = \left( \left( \underline{\underline{F}} + \underline{\underline{\Phi}} \cdot \underline{\underline{\nabla}}^0 \right) \cdot \underline{\underline{\Pi}} + \underline{\underline{\Phi}} : \underline{\underline{T}}^T \right) \cdot \underline{\underline{N}}, \quad t_i = ((F_{iJ} + \Phi_{iJL,L})\Pi_{JK} + \Phi_{iJL}T_{JLK})N_K \quad (77)$$

$$\underline{\underline{t}} = \underline{\underline{S}} \otimes \underline{\underline{N}}, \quad t_{iJK} = S_{iJ}N_K \quad (78)$$

The consideration of Eq. (75) and (77) leads to the definition of a new Boussinesq stress tensor for the stress gradient theory:

$$\underline{\underline{S}}^B := \underline{\underline{S}} + (\text{Div } \underline{\underline{\Phi}}) \cdot \underline{\underline{\Pi}} + \underline{\underline{\Phi}} : \underline{\underline{T}}^T = \left( \underline{\underline{F}} + \text{Div } \underline{\underline{\Phi}} \right) \cdot \underline{\underline{\Pi}} + \underline{\underline{\Phi}} : \underline{\underline{T}}^T \quad (79)$$

whose divergence balances volume and inertia forces and whose action on the normal vector at the boundary delivers the generalized traction.

#### 4.3. Hyperelastic stress gradient model and its linearization

The reduced hyperelastic laws of the stress gradient medium can be obtained by eliminating the Lagrange multiplier in Eq. (68). For that purpose, note that:

$$\underline{\underline{K}} : \underline{\underline{1}} = \underline{\underline{F}}^T \cdot \text{Div } \underline{\underline{\Phi}} \quad (80)$$

so that

$$\lambda_f \underline{\underline{1}} = \underline{\underline{F}}^T \cdot \text{Div } \underline{\underline{\Phi}} + \frac{\partial \psi^*}{\partial \underline{\underline{M}}} : \underline{\underline{1}} \quad (81)$$

Finally, it is found that:

$$\frac{\underline{\underline{C}}}{2} + \underline{\underline{F}}^T \cdot \text{Div } \underline{\underline{\Phi}} = - \left( \frac{\partial \psi^*}{\partial \underline{\underline{\Pi}}} + \frac{\partial \psi^*}{\partial \underline{\underline{M}}} : \underline{\underline{1}} \right) = - \frac{\partial \Psi^*}{\partial \underline{\underline{\Pi}}} \quad (82)$$

after recalling that  $\underline{\underline{M}} = \underline{\underline{\Pi}} \otimes \underline{\underline{1}}$  and introducing a reduced stress potential  $\Psi^*(\underline{\underline{\Pi}}, \underline{\underline{T}})$  such that

$$\underline{\underline{E}} = \frac{\underline{\underline{C}}}{2} + \underline{\underline{F}}^T \cdot \underline{\underline{\Phi}} \cdot \underline{\underline{\nabla}}^0 = - \frac{\partial \Psi^*}{\partial \underline{\underline{\Pi}}}, \quad \underline{\underline{\Upsilon}} = \underline{\underline{F}}^T \cdot \underline{\underline{\Phi}} = - \frac{\partial \Psi^*}{\partial \underline{\underline{T}}} \quad (83)$$

Accordingly, the dual potential  $\Psi(\underline{\underline{E}}, \underline{\underline{\Upsilon}})$  can be derived based on the reduced strain tensor  $\underline{\underline{E}}$  and the generalized strain measure  $\underline{\underline{\Upsilon}}$ :

$$\delta \Psi = \frac{\partial \Psi}{\partial \underline{\underline{E}}} : \delta \underline{\underline{E}} + \frac{\partial \Psi}{\partial \underline{\underline{\Upsilon}}} : \delta \underline{\underline{\Upsilon}} = \underline{\underline{\Pi}} : \delta \underline{\underline{E}} + \underline{\underline{T}} : \delta \underline{\underline{\Upsilon}} \quad (84)$$

after identification of the conjugate stress tensors as they appear in the increment of virtual work of internal forces, Eq. (69).

The linearization of the proposed theory is consistent with the linear stress gradient model presented in the static case by (Forest and Sab, 2012), as it will be shown. The small deformation framework for the stress gradient theory is characterized by:

$$\|\underline{\underline{F}} - \underline{\underline{1}}\| \ll 1, \quad L^{-1}\|\underline{\underline{\Phi}}\| \ll 1, \quad \|\underline{\underline{\Phi}} \otimes \underline{\underline{\nabla}}\| \ll 1 \quad (85)$$

The generalized strain measures are linearized as:

$$\underline{\underline{E}} - \underline{\underline{1}}/2 = (\underline{\underline{C}} - \underline{\underline{1}})/2 + \underline{\underline{F}}^T \cdot \underline{\underline{\Phi}} \simeq \underline{\underline{\varepsilon}} + \underline{\underline{\Phi}} \cdot \underline{\underline{\nabla}} \equiv \underline{\underline{e}}, \quad \underline{\underline{\Upsilon}} = \underline{\underline{F}}^T \cdot \underline{\underline{\Phi}} \simeq \underline{\underline{\Phi}} \quad (86)$$

where  $\underline{\underline{\varepsilon}}$  is the infinitesimal strain tensor and  $\underline{\underline{e}}$  is the generalized strain measure found in (Forest and Sab, 2012). Regarding stresses, we obtain

$$\underline{\underline{\Pi}} \simeq \underline{\underline{\sigma}}, \quad \underline{\underline{T}} \simeq \underline{\underline{\sigma}} \otimes \underline{\underline{\nabla}} \equiv \underline{\underline{R}} \quad (87)$$

where  $\underline{\underline{\sigma}}$  is the Cauchy stress tensor and  $\underline{\underline{R}}$  was the name given to the infinitesimal stress gradient in (Forest and Sab, 2012), in the static case. The linearization of the balance laws (75) and (76) proceeds as follows:

$$\underline{\underline{S}} + (\text{Div } \underline{\underline{\Phi}}) \cdot \underline{\underline{\Pi}} + \underline{\underline{\Phi}} : \underline{\underline{T}}^T \simeq \underline{\underline{\sigma}}, \quad \text{since } \|\underline{\underline{S}}\| \gg \|(\text{Div } \underline{\underline{\Phi}}) \cdot \underline{\underline{\Pi}}\|, \quad \|\underline{\underline{S}}\| \gg \|\underline{\underline{\Phi}} : \underline{\underline{T}}^T\| \quad (88)$$

As a consequence, the balance laws of the linear and static stress gradient theory by (Forest and Sab, 2012) are recovered:

$$\text{div } \underline{\underline{\sigma}} + \underline{\underline{f}} = 0, \quad \underline{\underline{R}} - \underline{\underline{\sigma}} \otimes \underline{\underline{\nabla}} + \underline{\underline{f}} = 0 \quad (89)$$

After linearization of the stress and strain measures and their substitution into the hyperelastic laws (84), the elastic laws proposed in (Forest and Sab, 2012) are also retrieved:

$$\underline{\underline{e}} = \frac{\partial w^*(\underline{\underline{\sigma}}, \underline{\underline{R}})}{\partial \underline{\underline{\sigma}}}, \quad \underline{\underline{\Phi}} = \frac{\partial w^*(\underline{\underline{\sigma}}, \underline{\underline{R}})}{\partial \underline{\underline{R}}}, \quad \underline{\underline{\sigma}} = \frac{\partial w(\underline{\underline{e}}, \underline{\underline{\Phi}})}{\partial \underline{\underline{e}}}, \quad \underline{\underline{R}} = \frac{\partial w(\underline{\underline{e}}, \underline{\underline{\Phi}})}{\partial \underline{\underline{\Phi}}} \quad (90)$$

It is worth considering now the linearization of the virtual power of contact forces taken from Eq. (71):

$$\delta \underline{\underline{u}} \cdot \left( \underline{\underline{S}} + (\underline{\underline{\Phi}} \cdot \underline{\underline{\nabla}}^0) \cdot \underline{\underline{\Pi}} + \underline{\underline{\Phi}} : \underline{\underline{T}}^T \right) \cdot \underline{\underline{N}} + (\underline{\underline{S}} \otimes \underline{\underline{N}}) : \delta \underline{\underline{\Phi}} \simeq \underline{\underline{\sigma}} : (\delta \underline{\underline{u}} \otimes \underline{\underline{N}} + \delta \underline{\underline{\Phi}} \cdot \underline{\underline{N}}) \quad (91)$$

which is the expression found in (Forest and Sab, 2012). This structure of the virtual work of contact forces allows to prescribe the full stress tensor along a boundary, which is a peculiar feature of the linear stress gradient theory. Existence and uniqueness theorems were established by (Sab et al., 2016) for such stress based boundary conditions. Corresponding existence theorems remain to be derived for the finite deformation stress gradient model proposed in the present work.

## 5. Applications

The proposed applications deal first with a combination of strain and stress gradient theories in the static case and then with the one-dimensional dynamics of second order micromorphic and stress gradient media. They are limited to the small strain case.

### 5.1. Combination of stress and strain gradient models

A second order micromorphic medium is considered that involves the following degrees of freedom

$$DOF = \{\underline{\boldsymbol{u}}, \quad \underline{\boldsymbol{\chi}}, \quad \underline{\boldsymbol{\Phi}}\} \quad (92)$$

The following expression of the work of internal forces, combining the features of first order micromorphic and stress gradient theories, is adopted:

$$p^{(i)} = \underline{\boldsymbol{\sigma}} : (\underline{\boldsymbol{\varepsilon}} + \underline{\boldsymbol{\Phi}} \cdot \nabla) + \underline{\boldsymbol{R}} \dot{ : } \underline{\boldsymbol{\Phi}} + \underline{\boldsymbol{s}} : (\underline{\boldsymbol{\chi}} - \underline{\boldsymbol{\varepsilon}}) + \underline{\boldsymbol{M}} \dot{ : } \underline{\boldsymbol{\chi}} \otimes \nabla \quad (93)$$

within the infinitesimal deformation framework. The field balance equations associated with such a medium are threefold, as deduced from the earlier analyses:

$$\operatorname{div}(\underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{s}}) = 0, \quad \underline{\boldsymbol{R}} = \underline{\boldsymbol{\sigma}} \otimes \nabla, \quad \operatorname{div} \underline{\boldsymbol{M}} + \underline{\boldsymbol{s}} = 0 \quad (94)$$

in the static case and in the absence of body forces. Implementing the internal constraint  $\underline{\boldsymbol{\chi}} \equiv \underline{\boldsymbol{\varepsilon}}$  corresponding to the strain gradient part of the model, the three previous equations reduce to 2:

$$\operatorname{div}(\underline{\boldsymbol{\sigma}} - \operatorname{div} \underline{\boldsymbol{M}}) = 0, \quad \underline{\boldsymbol{R}} = \underline{\boldsymbol{\sigma}} \otimes \nabla \quad (95)$$

The linear constitutive laws for the stress gradient part of the model are as in (Forest and Sab, 2012):

$$\underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{C}} \dot{ : } \underline{\boldsymbol{\varepsilon}} = \underline{\boldsymbol{C}} \dot{ : } (\underline{\boldsymbol{\varepsilon}} + \underline{\boldsymbol{\Phi}} \cdot \nabla), \quad \underline{\boldsymbol{R}} = \underline{\boldsymbol{D}} \dot{ : } \underline{\boldsymbol{\Phi}} \quad (96)$$

where  $\underline{\boldsymbol{C}}$  and  $\underline{\boldsymbol{D}}$  are fourth and sixth rank tensors of generalized elastic moduli. In fact, for the purpose of comparison with Eringen's model, it is sufficient to consider the simplified law:

$$\underline{\boldsymbol{\Phi}} = \ell_\sigma^2 \underline{\boldsymbol{C}}^{-1} \dot{ : } \underline{\boldsymbol{R}} \quad (97)$$

so that

$$\underline{\boldsymbol{\sigma}} - \ell_\sigma^2 \nabla^2 \underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{C}} \dot{ : } \underline{\boldsymbol{\varepsilon}} \quad (98)$$

which is identical to Eringen's model involving the Laplacian of the stress tensor, see (Eringen, 1983). The fact that Eringen's model can be retrieved from a stress gradient theory was recognized by (Polizzotto, 2015).

On the other hand, following (Forest and Aifantis, 2010), a simplified higher order elasticity law linking the hyperstress tensor to the strain gradient is:

$$\underline{\underline{M}} = \ell_\varepsilon^2 \underline{\underline{C}} : (\underline{\underline{\varepsilon}} \otimes \nabla) \quad (99)$$

which is similar to Eq. (97). This leads to a generalized elasticity law for the effective stress tensor,  $\underline{\underline{\tau}}$ , whose divergence vanishes under static conditions, according to (Germain, 1973a):

$$\underline{\underline{\tau}} = \underline{\underline{\sigma}} - \text{div } \underline{\underline{M}} = \underline{\underline{\sigma}} - \ell_\varepsilon^2 \underline{\underline{C}} : \nabla^2 \underline{\underline{\varepsilon}} \quad (100)$$

which coincides with Aifantis celebrated gradient elasticity model (Ru and Aifantis, 1993). Combining the constitutive laws (98) and (100), the following partial differential equation for stress and strain tensors is obtained:

$$\underline{\underline{\tau}} - \ell_\sigma^2 \nabla^2 \underline{\underline{\sigma}} = \underline{\underline{C}} : (\underline{\underline{\varepsilon}} - \ell_\varepsilon^2 \nabla^2 \underline{\underline{\varepsilon}}) \quad (101)$$

This equation involves the Laplacians of both stress and strain tensors. Note that both stress tensors  $\underline{\underline{\sigma}}$  and  $\underline{\underline{\tau}}$  are present.

In the context of a combined stress and strain gradient theory, the question arises of the choice of the proper stress tensor whose gradient should be incorporated in the model. Here, we have considered  $\underline{\underline{\sigma}} \otimes \nabla$  but the theory could involve instead  $\underline{\underline{\tau}} \otimes \nabla$ . For that purpose, an alternative combination of stress and strain gradient theories can be proposed from the following modification of the power of internal forces (93):

$$p^{(i)} = \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} + (\underline{\underline{\sigma}} + \underline{\underline{s}}) : \underline{\underline{\Phi}} \cdot \nabla + \underline{\underline{R}} : \underline{\underline{\Phi}} + \underline{\underline{s}} : (\underline{\underline{\chi}} - \underline{\underline{\varepsilon}}) + \underline{\underline{M}} : \underline{\underline{\chi}} \otimes \nabla \quad (102)$$

which leads to the following static balance equations:

$$\text{div } (\underline{\underline{\sigma}} + \underline{\underline{s}}) = 0, \quad \underline{\underline{R}} = (\underline{\underline{\sigma}} + \underline{\underline{s}}) \otimes \nabla, \quad \text{div } \underline{\underline{M}} + \underline{\underline{s}} = 0 \quad (103)$$

Eliminating the relative stress tensor  $\underline{\underline{s}}$  provides the balance equations with respect to the effective stress  $\underline{\underline{\tau}} = \underline{\underline{\sigma}} - \text{div } \underline{\underline{M}}$ :

$$\text{div } \underline{\underline{\tau}} = 0, \quad \underline{\underline{R}} = \underline{\underline{\tau}} \otimes \nabla \quad (104)$$

which shows that it is a theory for the effective stress gradient. Implementing again the internal constraint  $\underline{\underline{\chi}} \equiv \underline{\underline{\varepsilon}}$  corresponding to the strain gradient part of the model, the following constitutive equations are retained:

$$\underline{\underline{\tau}} = \underline{\underline{C}} : \underline{\underline{\varepsilon}} = \underline{\underline{C}} : (\underline{\underline{\varepsilon}} + \text{div } \underline{\underline{\Phi}}), \quad \underline{\underline{\Phi}} = \ell_\sigma^2 \underline{\underline{C}}^{-1} : \underline{\underline{R}} \quad (105)$$

The following relation between stress and strain gradients is deduced

$$\underline{\underline{\sigma}} - \ell_\sigma^2 \nabla^2 \underline{\underline{\tau}} = \underline{\underline{C}} : (\underline{\underline{\varepsilon}} + \ell_\varepsilon^2 \nabla^2 \underline{\underline{\varepsilon}}) \quad (106)$$

This equation differs from (101) by the interchange of  $\underline{\sigma}$  and  $\underline{\tau}$  on the left hand side and by the plus sign on the right hand side.

Within the small strain framework, (Gutkin and Aifantis, 1999; Aifantis, 2003) proposed a gradient elasticity constitutive law involving both strain and stress gradient variables in the form

$$(1 - l_\sigma^2 \nabla^2) \underline{\sigma} = (1 - l_\varepsilon^2 \nabla^2) (\lambda (\text{trace } \underline{\varepsilon}) \mathbf{1} + 2\mu \underline{\varepsilon}) \quad (107)$$

The equations (101) and (101) also involve Laplacians of both stress and strain tensors. They are reminiscent of Aifantis generalized gradient elasticity model (107). In this equation,  $\underline{\sigma}$  is presented as the usual stress tensor. Forest and Aifantis (2010) suggested however that it should be interpreted as the effective stress  $\underline{\tau}$  from the strain gradient model. The equation (107) is presented as a constitutive law. In fact, it is rather a combination of balance and constitutive equations as in the derivation of Eq. (101) and (106). The ambiguity on the nature of the stress tensor remains as long as the suitable balance, boundary and constitutive equations are not provided.

The merits of a combined stress and strain gradient model are suggested by the recent work of (Tran, 2016) where the stress gradient model was shown to predict softening effects at small scales. In contrast, stiffening effects are generally obtained with a strain gradient model. Depending on boundary or interface conditions related to either stress gradient or strain gradient types, such stiffening and softening effects may be competing in actual materials.

## 5.2. Higher order dynamics of linear micromorphic and strain gradient media

The dynamics of micromorphic media is derived in the case of linear elasticity. The obtained field equations are then specialized in the presence of internal constraints linking the micromorphic deformations to strain gradients.

The generalized Hooke laws are given in the case of a centro-symmetrical second order micromorphic material for the sake of brevity. The retained form is that proposed in (Cordero et al., 2016):

$$\underline{\sigma} = \underline{\underline{c}} : \underline{\varepsilon} \quad (108)$$

$$\underline{\underline{T}} = \underline{\underline{a}} : (\underline{\underline{u}} \otimes \nabla - \underline{\underline{\chi}}) \quad (109)$$

$$\underline{\underline{M}} = \underline{\underline{\underline{b}}} : \underline{\underline{K}} \quad (110)$$

$$\underline{\underline{T}} = \underline{\underline{\underline{a}}} : (\underline{\underline{K}} - \underline{\underline{\chi}}) \quad (111)$$

$$\underline{\underline{M}} = \underline{\underline{\underline{b}}} :: \underline{\underline{K}} \quad (112)$$

The relations (109) and (111) penalize the difference between the first (resp. second) order microdeformation and the first (resp. second) gradient of the displacement fields. Very large values of the higher

order moduli  $\underline{\underline{\mathbf{a}}}$  and  $\underline{\underline{\mathbf{a}}}$  amount to prescribing the constraints (60) which reduce the second micromorphic medium to a second strain gradient material. Odd rank tensors of elasticity are excluded in the previous laws by the hypothesis of centrosymmetry (Auffray et al., 2013). The full classification of anisotropy classes remains to be done for second order micromorphic media and is not discussed here (Olive and Auffray, 2013).

It is shown now how the three linearized balance laws (42) to (44) can be combined, after substitution of the constitutive equations (108) to (112) to derive a single partial differential equation involving all degrees of freedom and in which internal constraints can be enforced. The derivation is essentially formal due to the complexity of tensor combinations. Eliminating the coupling stress  $\underline{\underline{\mathbf{T}}}$  in Eq. (42) by means of Eq. (43) yields

$$\operatorname{div}(\underline{\underline{\boldsymbol{\sigma}}} + \operatorname{div} \underline{\underline{\mathbf{M}}} - \rho I \underline{\underline{\boldsymbol{\chi}}}) = \rho \underline{\underline{\mathbf{u}}}$$

The third rank stress tensor can be eliminated from the previous expression using the constitutive law (110) and the balance law (44):

$$\underline{\underline{\mathbf{M}}} = \underline{\underline{\mathbf{b}}} : \underline{\underline{\mathbf{K}}} = \underline{\underline{\mathbf{b}}} : \left( \underline{\underline{\boldsymbol{\chi}}} + \underline{\underline{\mathbf{a}}}^{-1} : \underline{\underline{\mathbf{T}}} \right) = \underline{\underline{\mathbf{b}}} : \left( \underline{\underline{\boldsymbol{\chi}}} + \underline{\underline{\mathbf{a}}}^{-1} : \left( \operatorname{div} \underline{\underline{\mathbf{M}}} - \rho \underline{\underline{\boldsymbol{\chi}}} : \underline{\underline{\mathbf{I}}} \right) \right)$$

and finally

$$\operatorname{div} \left( \underline{\underline{\boldsymbol{\sigma}}} + \operatorname{div} \left( \underline{\underline{\mathbf{b}}} : \left( \underline{\underline{\boldsymbol{\chi}}} + \underline{\underline{\mathbf{a}}}^{-1} : \left( \operatorname{div} \underline{\underline{\mathbf{M}}} - \rho \underline{\underline{\boldsymbol{\chi}}} : \underline{\underline{\mathbf{I}}} \right) \right) \right) \right) - \rho I \underline{\underline{\boldsymbol{\chi}}} = \rho \underline{\underline{\mathbf{u}}} \quad (113)$$

At this stage, the linearized internal constraints (60):

$$\underline{\underline{\boldsymbol{\chi}}} \equiv \underline{\underline{\mathbf{u}}} \otimes \nabla, \quad \underline{\underline{\boldsymbol{\chi}}} \equiv \underline{\underline{\mathbf{u}}} \otimes \nabla \otimes \nabla \quad (114)$$

are substituted into (113) to obtain the dynamical equations of second strain gradient media. This equation is written in symbolic form in order to draw the attention on the order of the space and time derivatives of the displacement variable in each term:

$$Au^{(2)} + Bu^{(4)} + Cu^{(6)} - \rho I \ddot{u}^{(4)} - D\ddot{u}^{(2)} - \rho \ddot{u} = 0 \quad (115)$$

where the integers in parentheses denote the order of the spatial derivative and the coefficients symbolically represent appropriate combinations of elastic moduli. It is remarkable that the dynamical equation contains spatial derivatives of even order up to 6 and mixed space and time derivatives of order 2 to 6. The fourth order terms are characteristic of Mindlin's first strain gradient dynamics. The direct derivation of the mixed term  $\ddot{u}^{(2)}$  in the strain gradient model requires the introduction of an hypermomentum tensor that is very often forgotten, especially in the dynamics of Euler–Bernoulli beams, see (Metrikine, 2006; Rosi and Auffray, 2016). It arises here as a natural consequence of the micromorphic dynamics. The introduction of the second order microdeformation leads to the terms of sixth order. The previous

equation represents one of the most general one for the analysis of the dispersion of waves in gradient media (Mindlin, 1972; Askes and Aifantis, 2011). It contains terms that have been introduced heuristically in several dynamical equations of the literature, see the discussions in (Engelbrecht et al., 2005; Askes et al., 2008b,a; Papargyri-Beskou et al., 2009).

### 5.3. Dynamics of linear stress gradient media

Some specific aspects of the dynamical behavior of linear stress gradient media are now unraveled by combining the linear balance and constitutive equations including inertia contributions. The linearized dynamical equations of stress gradient media are deduced from Eq. (75) and (76) as follows:

$$\operatorname{div} \underline{\boldsymbol{\sigma}} = \rho \ddot{\underline{\mathbf{u}}}, \quad \underline{\mathbf{R}} - \underline{\boldsymbol{\sigma}} \otimes \nabla = \rho I \ddot{\underline{\boldsymbol{\Phi}}} \quad (116)$$

in the absence of volume forces. Note that the inertia coefficient  $I$  in Eq. (116)<sub>2</sub> has no physical dimension since  $\underline{\mathbf{u}}$  and  $\underline{\boldsymbol{\Phi}}$  share the same physical dimension of a length. The linearized elasticity laws are given by Eq. (96). They relate the stress and stress gradients to the strain and microdisplacements. For the sake of brevity, the derivation of the dynamic equation for displacement is presented in the one-dimensional case. The Eq. (116) then reduce to

$$\sigma' = \rho \ddot{u}, \quad R - \sigma' = \rho I \ddot{\Phi} \quad (117)$$

where  $(\bullet)'$  denotes the one-dimensional derivative. The constitutive laws (96) become

$$\sigma = E(u' + \Phi'), \quad R = D\Phi \quad (118)$$

where  $E$  and  $D$  are constitutive elasticity moduli, in  $Pa$ . Substitution of the constitutive laws into the balance leads to the generalized Navier equations:

$$E(u^{(2)} + \Phi^{(2)}) = \rho \ddot{u}, \quad D\Phi - E(u^{(2)} + \Phi^{(2)}) = \rho I \ddot{\Phi} \quad (119)$$

From the two previous equations, the following consequences are respectively deduced:

$$\Phi^{(2)} = \frac{\rho}{E} \ddot{u} - u^{(2)}, \quad D\Phi^{(2)} - \rho \ddot{u}^{(2)} = \rho I \ddot{\Phi}^{(2)} \quad (120)$$

Substitution of the former into the latter equation provides the linear differential equation for the displacement variable:

$$Du^{(2)} + \rho(1 - I)\ddot{u}^{(2)} + \frac{\rho^2 I}{E} \ddot{\ddot{u}} - \frac{D\rho}{E} \ddot{u} = 0 \quad (121)$$

The dispersion of one-dimensional elastic waves according to Eq. (115) and (121) should be analysed in a way similar to the work done by Metrikine (2006) and Berezovski (2015) for other microstructured continua. Metrikine (2006) examined the question of the causality of such equations. He postulates that  $a$

*partial differential equation that governs the dynamical behaviour of a one-dimensional model must be of the same order with respect to the spatial coordinate and with respect to time.* This is not the case of Eq. (115) due to the higher order spatial derivation. This feature is well-known in strain gradient media and is encountered in the Euler-Bernoulli beam model. It corresponds to the possibility of waves travelling with infinite speed. This paradoxon is solved by the use of a Timoshenko beam model. Equivalently, the unconstrained micromorphic media can be used to regularize the wave behavior of gradient continua. The postulated causality condition is not satisfied either for stress gradient continua, as it can be seen from Eq. (121) but for a different reason. The highest time derivative is 4 whereas the spatial order is 2. This is a remarkable feature distinguishing the stress gradient from the strain gradient models.

## 6. Conclusions

The main contributions of the present work are the following:

- *A second order micromorphic model at finite deformation* introducing generalized Boussinesq stress tensors for balance and boundary conditions, and generalized Lagrangian strain measures for constitutive laws. It is more general than the previous Green and Naghdi, Eringen and Germain theories. The higher order microdeformations are introduced as relaxed deformation gradients of suitable order.
- *A reduction of the general micromorphic model to the grade  $n$  continuum model* by suitable constraints on the microdeformations tensors. In strain gradient theories, the higher order stress tensor is NOT equal to the gradient of the usual stress tensor. That is why strain and stress gradient theories are distinct models of the continuum.
- *A stress gradient theory at finite deformation including inertia terms.* It involves a third order tensor of additional degrees of freedom like the second order micromorphic model. However, its mechanical meaning is different: it represents *microdisplacements* and NOT a relaxed strain gradient.
- *A one-dimensional linearized dispersion equation for the stress gradient model* that essentially differs from the strain gradient prediction. The highest order in time (resp. space) derivation is two orders larger than the spatial (resp. time) derivation order in linear stress (resp. strain) gradient media.

These new theories are excellent candidates for the study of wave dispersion in generalized continua, as started in (Madeo et al., 2014, 2016b) for first order micromorphic continua. Micromorphic models are necessary to overcome the paradoxon of the infinite wave speed of some elastic waves in gradient media and to obtain both acoustic and optical branches in the dispersion diagrams (Papargyri-Beskou et al., 2009).

A central remaining question is the determination of the higher order elastic moduli arising in such theories. Enhanced homogenization methods have been proposed recently to derive them from the microstructure of periodic heterogeneous material, see (Forest and Trinh, 2011; Trinh et al., 2012; Jänicke and Steeb, 2012) for the construction of gradient and micromorphic models from the underlying heterogeneous Cauchy materials. Most approaches remain heuristic and the question is still largely open. Nassar et al. (2016) and Reda et al. (2017) recently proposed alternative asymptotic methods for the determination of dynamical properties of gradient and micromorphic media with application to composite materials. The found properties are inevitably anisotropic which has implications for the dispersion of waves, see (Rosi and Auffray, 2016).

The strain gradient and stress gradient models emerge as distinct and rather complementary approaches to material behavior. Stiffening effects are expected in strain gradient media at small scales, whereas softening effects were derived in the recent homogenization results by Tran (2016). It seems that both models should be combined in order to represent the competition of stiffening and softening effects present in heterogeneous materials, as sketched in Section 5.1.

The proposed theories were presented within the finite deformation framework for application in the size-dependent plasticity and fracture of metals that generally occur at large deformations. They could be applied to the ductile fracture of porous materials as initiated in (Hütter, 2017a,b) or to the deformation of metal polycrystals (Poh, 2013).

## Acknowledgements

The first author is indebted to Gérard A. Maugin for many fascinating discussions, in the last twenty years, on the past and future of continuum physics.

## References

- Aifantis, E., 2003. Update on a class of gradient theories. *Mechanics of Materials* 35, 259–280. doi:[http://dx.doi.org/10.1016/S0167-6636\(02\)00278-8](http://dx.doi.org/10.1016/S0167-6636(02)00278-8).
- Askes, H., Aifantis, E.C., 2011. Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results. *International Journal of Solids and Structures* 48, 1962–1990. doi:<http://dx.doi.org/10.1016/j.ijsolstr.2011.03.006>.
- Askes, H., Metrikine, A.V., Pichugin, A.V., Bennett, T., 2008a. Four simplified gradient elasticity models for the simulation of dispersive wave propagation. *Philosophical Magazine* 88, 3415–3443.

- Askes, H., Wang, B., Bennett, T., 2008b. Element size and time step selection procedures for the numerical analysis of elasticity with higher-order inertia. *Journal of Sound and Vibration* 314, 650–656. doi:<http://dx.doi.org/10.1016/j.jsv.2007.12.034>.
- Auffray, N., Le Quang, H., He, Q., 2013. Matrix representations for 3D strain-gradient elasticity. *Journal of the Mechanics and Physics of Solids* 61, 1202–1223.
- Berezovski, A., 2015. Nonlinear dispersive wave equations for microstructured solids. *Proceedings of the Estonian Academy of Sciences* 64, 203–211.
- Bergheau, J., Leblond, J., Perrin, G., 2014. A new numerical implementation of a second-gradient model for plastic porous solids, with an application to the simulation of ductile rupture tests. *Computer Methods in Applied Mechanics and Engineering* 268, 105 – 125. URL: <http://www.sciencedirect.com/science/article/pii/S0045782513002338>, doi:<http://dx.doi.org/10.1016/j.cma.2013.09.006>.
- Bertram, A., 2015. Finite gradient elasticity and plasticity: a constitutive mechanical framework. *Continuum Mechanics and Thermodynamics* 27, 1039–1058.
- Bertram, A., 2016. Finite gradient elasticity and plasticity: a constitutive thermodynamical framework. *Continuum Mechanics and Thermodynamics* 28, 819–883.
- Bertram, A., Glüge, R., 2016. Gradient materials with internal constraints. *Mathematics and Mechanics of Complex Systems* 4, 1–15.
- Bleustein, J., 1967. A note on the boundary conditions of Toupin’s strain–gradient theory. *International Journal of Solids Structures* 3, 1053–1057.
- Broese, C., Tsakmakis, C., Beskos, D., 2016. Mindlin’s micro-structural and gradient elasticity theories and their thermodynamics. *Journal of Elasticity* 125, 87–132. doi:[10.1007/s10659-016-9572-7](https://doi.org/10.1007/s10659-016-9572-7).
- Cordero, N., Forest, S., Busso, E., 2016. Second strain gradient elasticity of nano-objects. *Journal of the Mechanics and Physics of Solids* 97, 92–124. doi:[10.1016/j.jmps.2015.07.012](https://doi.org/10.1016/j.jmps.2015.07.012).
- dell’Isola, F., Madeo, A., Seppecher, P., 2016. Cauchy tetrahedron argument applied to higher contact interactions. *Archive for Rational Mechanics and Analysis* 219, 1305–1341. doi:[10.1007/s00205-015-0922-6](https://doi.org/10.1007/s00205-015-0922-6).
- dell’Isola, F., Seppecher, P., 1995. The relationship between edge contact forces, double forces and interstitial working allowed by the principle of virtual power. *C.R. Acad. Sci. Paris Iib* 321, 303–308.

- dell’Isola, F., Sepecher, P., Madeo, A., 2012. How contact interactions may depend on the shape of Cauchy cuts in N-th gradient continua: approach “à la D’Alembert”. *Zeitschrift für Angewandte Mathematik und Physik* 63, 1119–1141.
- Dillard, T., Forest, S., Ienny, P., 2006. Micromorphic continuum modelling of the deformation and fracture behaviour of nickel foams. *European Journal of Mechanics A/Solids* 25, 526–549.
- Enakoutsu, K., Leblond, J., 2009. Numerical implementation and assessment of the GLPD micromorphic model of ductile rupture. *European Journal of Mechanics A/solids* 28, 445–460.
- Engelbrecht, J., Berezovski, A., Pastrone, F., Braun, M., 2005. Waves in microstructured materials and dispersion. *Philosophical Magazine* 85, 4127–4141.
- Eringen, A., 1967. Mechanics of micromorphic continua, in: Kröner, E. (Ed.), *Mechanics of Generalized Continua*, IUTAM Symposium, Freudenstadt, Springer–Verlag Berlin Heidelberg. pp. 18–35.
- Eringen, A., 1970. Laws of micromorphic mechanics. *Int. J. Engng Sci.* 8, 819–828.
- Eringen, A., 1999. *Microcontinuum field theories*. Springer, New York.
- Eringen, A., Suhubi, E., 1964. Nonlinear theory of simple microelastic solids. *Int. J. Engng Sci.* 2, 189–203, 389–404.
- Eringen, A.C., 1983. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. *Journal of Applied Physics* 54, 4703–4710. doi:<http://dx.doi.org/10.1063/1.332803>.
- Ferretti, M., Madeo, A., dell’Isola, F., Boisse, P., 2014. Modeling the onset of shear boundary layers in fibrous composite reinforcements by second-gradient theory. *Zeitschrift für angewandte Mathematik und Physik* 65, 587–612. doi:[10.1007/s00033-013-0347-8](https://doi.org/10.1007/s00033-013-0347-8).
- Forest, S., Aifantis, E.C., 2010. Some links between recent gradient thermo-elasto-plasticity theories and the thermomechanics of generalized continua. *International Journal of Solids and Structures* 47, 3367–3376.
- Forest, S., Cordero, N., Busso, E., 2011. First vs. second gradient of strain theory for capillarity effects in an elastic fluid at small length scales. *Computational Materials Science* 50, 1299–1304.
- Forest, S., Sab, K., 2012. Continuum stress gradient theory. *Mechanics Research Communications* 40, 16–25.
- Forest, S., Sievert, R., 2003. Elastoviscoplastic constitutive frameworks for generalized continua. *Acta Mechanica* 160, 71–111.

- Forest, S., Sievert, R., 2006. Nonlinear microstrain theories. *International Journal of Solids and Structures* 43, 7224–7245.
- Forest, S., Trinh, D.K., 2011. Generalized continua and non-homogeneous boundary conditions in homogenization methods. *ZAMM Zeitschrift fuer Angewandte Mathematik und Mechanik* 91, 90–109.
- Germain, P., 1973a. La méthode des puissances virtuelles en mécanique des milieux continus, première partie : théorie du second gradient. *J. de Mécanique* 12, 235–274.
- Germain, P., 1973b. The method of virtual power in continuum mechanics. part 2 : Microstructure. *SIAM J. Appl. Math.* 25, 556–575.
- Green, A., Naghdi, P., Rivlin, R., 1965. Directors and multipolar displacements in continuum mechanics. *International Journal of Engineering Science* 2, 611–620. doi:[http://dx.doi.org/10.1016/0020-7225\(65\)90039-X](http://dx.doi.org/10.1016/0020-7225(65)90039-X).
- Green, A., Rivlin, R., 1964. Multipolar continuum mechanics. *Arch. Rational Mech. and Anal.* 17, 113–147.
- Green, A.E., Rivlin, R.S., 1967. The relation between director and multipolar theories in continuum mechanics. *Zeitschrift für angewandte Mathematik und Physik ZAMP* 18, 208–218. doi:[10.1007/BF01596913](https://doi.org/10.1007/BF01596913).
- Gutkin, M.Y., Aifantis, E.C., 1999. Dislocations in the theory of gradient elasticity. *Scripta Materialia* 40, 559 – 566. URL: <http://www.sciencedirect.com/science/article/pii/S1359646298004242>, doi:[http://dx.doi.org/10.1016/S1359-6462\(98\)00424-2](http://dx.doi.org/10.1016/S1359-6462(98)00424-2).
- Hütter, G., 2017a. Homogenization of a cauchy continuum towards a micromorphic continuum. *Journal of the Mechanics and Physics of Solids* 99, 394 – 408. doi:<http://dx.doi.org/10.1016/j.jmps.2016.09.010>.
- Hütter, G., 2017b. A micromechanical gradient extension of Gurson’s model of ductile damage within the theory of microdilational media. *International Journal of Solids and Structures* 110-111, 15–23. doi:<http://dx.doi.org/10.1016/j.ijsolstr.2017.02.007>.
- Hütter, G., Mühlich, U., Kuna, M., 2015. Micromorphic homogenization of a porous medium: elastic behavior and quasi-brittle damage. *Continuum Mech. Thermodyn.* 27, 1059–1072.
- Jänicke, R., Steeb, H., 2012. Minimal loading conditions for higher order numerical homogenisation schemes. *Archives of Applied Mechanics* 82, i1075–1088.

- Javili, A., dell’Isola, F., Steinmann, P., 2013. Geometrically nonlinear higher-gradient elasticity with energetic boundaries. *Journal of the Mechanics and Physics of Solids* 61, 2381–2401.
- Lebée, A., Sab, K., 2011a. A bending-gradient model for thick plates, Part I: Theory. *International Journal of Solids and Structures* 48, 2878–2888.
- Lebée, A., Sab, K., 2011b. A bending-gradient model for thick plates, Part II: Closed-form solutions for cylindrical bending of laminates. *International Journal of Solids and Structures* 48, 2889–2901.
- Liu, I.S., 2002. *Continuum Mechanics*. Springer.
- Madeo, A., Barbagallo, G., D’Agostino, M.V., Boisse, P., 2016a. Continuum and discrete models for unbalanced woven fabrics. *International Journal of Solids and Structures* 94–95, 263 – 284. doi:<http://dx.doi.org/10.1016/j.ijsolstr.2016.02.005>.
- Madeo, A., Barbagallo, G., d’Agostino, M.V., Placidi, L., Neff, P., 2016b. First evidence of non-locality in real band-gap metamaterials: determining parameters in the relaxed micromorphic model. *Proceedings of the Royal Society of London A* 472, 20160169. doi:[10.1098/rspa.2016.0169](https://doi.org/10.1098/rspa.2016.0169).
- Madeo, A., Ferretti, M., dell’Isola, F., Boisse, P., 2015. Thick fibrous composite reinforcements behave as special second-gradient materials: three-point bending of 3d interlocks. *Zeitschrift für angewandte Mathematik und Physik* 66, 2041–2060. doi:[10.1007/s00033-015-0496-z](https://doi.org/10.1007/s00033-015-0496-z).
- Madeo, A., Ghiba, I., Neff, P., Münch, I., 2016c. A new view on boundary conditions in the grioli-koiter-mindlin-toupin indeterminate couple stress model. *European Journal of Mechanics - A/Solids* 59, 294 – 322. doi:<http://dx.doi.org/10.1016/j.euromechsol.2016.02.009>.
- Madeo, A., Neff, P., Ghiba, I.D., Placidi, L., Rosi, G., 2014. Wave propagation in relaxed micromorphic continua: modeling metamaterials with frequency band-gaps. *Continuum Mechanics and Thermodynamics* 27, 551–570. doi:[10.1007/s00161-013-0329-2](https://doi.org/10.1007/s00161-013-0329-2).
- Maugin, G.A., 1980. The method of virtual power in continuum mechanics: Application to coupled fields. *Acta Mechanica* 35, 1–70.
- Metrikine, A., 2006. On causality of the gradient elasticity models. *Journal of Sound and Vibration* 297, 727–742. doi:<http://dx.doi.org/10.1016/j.jsv.2006.04.017>.
- Mindlin, R., 1964. Micro-structure in linear elasticity. *Arch. Rat. Mech. Anal.* 16, 51–78.
- Mindlin, R., 1965. Second gradient of strain and surface-tension in linear elasticity. *Int. J. Solids Structures* 1, 417–438.

- Mindlin, R., 1972. Elasticity, piezoelectricity and crystal lattice dynamics. *Journal of Elasticity* 2, 217–282.
- Nassar, H., He, Q.C., Auffray, N., 2016. A generalized theory of elastodynamic homogenization for periodic media. *International Journal of Solids and Structures* 84, 139 – 146. doi:<http://dx.doi.org/10.1016/j.ijsolstr.2016.01.022>.
- Olive, M., Auffray, N., 2013. Symmetry classes for even-order tensors. *Mathematics and Mechanics of Complex Systems* 1, 177–210.
- Papargyri-Beskou, S., Polyzos, D., Beskos, D., 2009. Wave dispersion in gradient elastic solids and structures: A unified treatment. *International Journal of Solids and Structures* 46, 3751 – 3759. doi:<http://dx.doi.org/10.1016/j.ijsolstr.2009.05.002>.
- Poh, L., 2013. Scale transition of a higher order plasticity model – A consistent homogenization theory from meso to macro. *Journal of the Mechanics and Physics of Solids* 61, 2692–2710. doi:<http://dx.doi.org/10.1016/j.jmps.2013.09.004>.
- Polizzotto, C., 2014. Stress gradient versus strain gradient constitutive models within elasticity. *International Journal of Solids and Structures* 51, 1809–1818. doi:[10.1016/j.ijsolstr.2014.01.021](http://dx.doi.org/10.1016/j.ijsolstr.2014.01.021).
- Polizzotto, C., 2015. A unifying variational framework for stress gradient and strain gradient elasticity theories. *European Journal of Mechanics A-Solids* 49, 430–440. doi:[10.1016/j.euromechsol.2014.08.013](http://dx.doi.org/10.1016/j.euromechsol.2014.08.013).
- Reda, H., Goda, I., Ganghoffer, J., L’Hostis, G., Lakiss, H., 2017. Dynamical analysis of homogenized second gradient anisotropic media for textile composite structures and analysis of size effects. *Composite Structures* 161, 540 – 551. doi:<http://dx.doi.org/10.1016/j.compstruct.2016.10.068>.
- Regueiro, R., 2010. On finite strain micromorphic elastoplasticity. *International Journal of Solids and Structures* 47, 786–800.
- Reiher, J.C., Giorgio, I., Bertram, A., 2017. Finite-element analysis of polyhedra under point and line forces in second-strain gradient elasticity. *International Journal of Engineering Mechanics* 143, 04016112–1–13. doi:[https://doi.org/10.1061/\(ASCE\)EM.1943-7889.0001184](https://doi.org/10.1061/(ASCE)EM.1943-7889.0001184).
- Rosi, G., Auffray, N., 2016. Anisotropic and dispersive wave propagation within strain-gradient framework. *Wave Motion* 63, 120–134. doi:<http://dx.doi.org/10.1016/j.wavemoti.2016.01.009>.

- Ru, C., Aifantis, E., 1993. A simple approach to solve boundary value problems in gradient elasticity. *Acta Mechanica* 101, 59–68.
- Sab, K., Legoll, F., Forest, S., 2016. Stress gradient elasticity theory: Existence and uniqueness of solution. *Journal of Elasticity* 123, 179–201. doi:[10.1007/s10659-015-9554-1](https://doi.org/10.1007/s10659-015-9554-1).
- Sansour, C., Skatulla, S., Zbib, H., 2010. A formulation for the micromorphic continuum at finite inelastic strains. *Int. J. Solids Structures* 47, 1546–1554.
- Seppecher, P., 2000. Second gradient theory: Application to Cahn–Hilliard fluids, in: Maugin, G., Drouot, R., Sidoroff, F. (Eds.), *Continuum Thermomechanics, The Art and Science of Modelling Material Behaviour, Paul Germain’s Anniversary Volume*, Kluwer Academic Publishers. pp. 379–388.
- Tran, V.P., 2016. Stochastic modeling of random heterogeneous materials. Phd thesis. Université Paris Est.
- Trinh, D.K., Jänicke, R., Auffray, N., Diebels, S., Forest, S., 2012. Evaluation of generalized continuum substitution models for heterogeneous materials. *International Journal of Multiscale Computational Engineering* 10, 527–549. doi:[10.1615/IntJMultCompEng.2012003105](https://doi.org/10.1615/IntJMultCompEng.2012003105).