Convexity properties of the normalized Steklov zeta function of a planar domain

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CONVEXITY PROPERTIES OF THE DIFFERENCE OVER THE REAL AXIS BETWEEN THE STEKLOV ZETA FUNCTIONS OF A SMOOTH PLANAR DOMAIN WITH 2π PERIMETER AND OF THE UNIT DISK

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Abstract. We consider the zeta function ζ_Ω for the Dirichlet-to-Neumann operator of a simply connected planar domain Ω bounded by a smooth closed curve of perimeter 2π. We prove that $\zeta''_\Omega(0) \geq \zeta''_{\mathbb{D}}(0)$ with equality if and only if Ω is a disk where $\mathbb{D}$ denotes the closed unit disk. We also provide an elementary proof that for a fixed real s satisfying $s \leq -1$ the estimate $\zeta''_\Omega(s) \geq \zeta''_{\mathbb{D}}(s)$ holds with equality if and only if Ω is a disk. We then bring examples of domains Ω close to the unit disk where this estimate fails to be extended to the interval $(0, 2)$. Other computations related to previous works are also detailed in the remaining part of the text.

1. Introduction

Let Ω be a simply connected planar domain bounded by a C^∞-smooth closed curve $\partial\Omega$. The *Dirichlet-to-Neumann operator* of the domain

$$\Lambda_\Omega : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$$

is defined by $\Lambda_\Omega f = \frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$, where ν is the outward unit normal to $\partial\Omega$ and u is the solution to the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = f.$$

The Dirichlet-to-Neumann operator is a first order pseudodifferential operator. Moreover, it is a non-negative self-adjoint operator with respect to the L^2-product

$$\langle u, v\rangle = \int_{\partial\Omega} u\bar{v}\, ds,$$

where ds is the Euclidean arc length of the curve $\partial\Omega$. In particular, the operator Λ_Ω has a non-negative discrete eigenvalue spectrum

$$\mathrm{Sp}(\Omega) = \{0 = \lambda_0(\Omega) < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots\},$$

where each eigenvalue is repeated according to its multiplicity. The spectrum is called the *Steklov spectrum* of the domain Ω. Steklov eigenvalues depend on the size of Ω in the obvious manner: $\lambda_k(c\Omega) = c^{-1}\lambda_k(\Omega)$ for $c > 0$. Therefore it suffices to consider domains satisfying the *normalization condition*

$$\mathrm{Length}(\partial\Omega) = 2\pi. \tag{1.1}$$

Let $\mathbb{S} = \partial\mathbb{D} = \{e^{i\theta}\} \subset \mathbb{C}$ be the unit circle. The Dirichlet-to-Neumann operator of the unit disk $\mathbb{D} = \{(x, y) \mid x^2 + y^2 \leq 1\}$ will be denoted by $\Lambda : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$, i.e., $\Lambda = \Lambda_{\mathbb{D}}$.

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The alternative definition of the operator is given by the formula $\Lambda e^{in\theta} = |n|e^{in\theta}$ for an integer n. Then the Steklov eigenvalues of the disk are given by

$$\lambda_k(\mathbb{D}) = \left\lfloor \frac{k+1}{2} \right\rfloor, k \in \mathbb{N},$$

where $\lfloor x \rfloor$ stands for the integer part of $x \in \mathbb{R}$.

Under condition (1.1), Steklov eigenvalues of the domain Ω have the following asymptotics [2, Theorem 1]:

$$\lambda_k(\Omega) = \lambda_k(\mathbb{D}) + O(k^{-\infty}) \quad \text{as} \quad k \to \infty, \tag{1.2}$$

Due to the asymptotics, the *zeta function of the domain* Ω

$$\zeta_\Omega(s) = \mathrm{Tr}[\Lambda_\Omega^{-s}] = \sum_{k=1}^{\infty} \big(\lambda_k(\Omega)\big)^{-s}$$

is well defined for $\Re s > 1$. Then ζ_Ω extends to a meromorphic function on $\mathbb{C}$ with the unique simple pole at $s = 1$. The zeta function $\zeta_\mathbb{D}$ of the unit disk is equal to $2\zeta_R$, where $\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}$ is the classical Riemann zeta function

Moreover, the difference $\zeta_\Omega(s) - \zeta_\mathbb{D}(s)$ is an entire function [2]. Observe also that $\zeta_\Omega(s)$ is real for a real s.

The main result of the present paper is the following

Theorem 1.1. *For a smooth simply connected bounded planar domain Ω of perimeter 2π, the inequality*

$$\sum_{k=1}^{\infty} \Big[\ln(\lambda_k)^2 - \ln\Big(\Big\lfloor \frac{k+1}{2} \Big\rfloor \Big)^2 \Big] = (\zeta_\Omega - \zeta_\mathbb{D})''(0) \geq 0 \tag{1.3}$$

holds. Moreover equality in (1.3) *holds if and only if Ω is a round disk.*

Inequality (1.3) is a straighforward consequence of the identity $\zeta_\Omega(0) = \zeta_\mathbb{D}(0)$ and of the estimate $(\zeta_\Omega - \zeta_\mathbb{D}) \geq 0$ on the real axis $\mathbb{R}$ [6, Theorem 1.1]. Equality in (1.3) trivially holds if Ω is a round disk (in that case $\zeta_\Omega = \zeta_\mathbb{D}$). Hence the only statement that remains to be proved is the "only if" part. The proof relies on the same deformation argument we used to prove [6, Theorem 1.1].

The above result proves the strict convexity of $\zeta_\Omega - \zeta_\mathbb{D}$ around 0 when the planar domain Ω is not a disk. It is in fact easier to prove convexity of $\zeta_\Omega - \zeta_\mathbb{D}$ on $(-\infty, -1]$. We have the following result.

Proposition 1.2. *Let Ω be a smooth simply connected bounded domain with 2π perimeter. Let $s \in (-\infty, -1]$. We have*

$$(\zeta_\Omega - \zeta_\mathbb{D})''(s) \geq 0,$$

and there is equality if and only if Ω is a round disk.

Convexity near $+\infty$ is also granted by Weinstock's inequality [8] and we have the following result.

Proposition 1.3. *Let Ω be a smooth simply connected bounded domain with 2π perimeter. Assume that Ω is not a round disk. Then there exists a positive real s_Ω so that for any $s \in [s_\Omega, +\infty)$ the inequality*

$$(\zeta_\Omega - \zeta_\mathbb{D})''(s) > 0 \tag{1.4}$$

holds.

It is then questionable whether one can extend the statement to the whole real axis. We exhibit counterexamples in the following Proposition.

Proposition 1.4. *There exist a smooth simply connected bounded planar domain Ω of perimeter 2π and a real number $s \in (0,2)$ so that*

$$(\zeta_\Omega - \zeta_\mathbb{D})''(s) < 0.$$

Now, we discuss an alternative approach to the same results which are of a more analytical character.

For a function $b \in C^\infty(\mathbb{S})$, we write $b(\theta)$ instead of $b(e^{i\theta})$ and use the same letter b for the operator $b : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ of multiplication by the function b.

Given a positive function $a \in C^\infty(\mathbb{S})$, the operator $\Lambda_a = a^{1/2}\Lambda a^{1/2}$ has the non-negative discrete eigenvalue spectrum

$$\mathrm{Sp}(\Lambda_a) = \{0 = \lambda_0(a) < \lambda_1(a) \le \lambda_2(a) \le \dots\}$$

which is called the *Steklov spectrum of the function a* (or of the operator Λ_a).

Two kinds of the Steklov spectrum are related as follows. Given a smooth simply connected planar domain Ω, choose a biholomorphism $\Phi : \mathbb{D} \to \Omega$ and define the function $0 < a \in C^\infty(\mathbb{S})$ by $a(\theta) = |\Phi'(e^{i\theta})|^{-1}$. Let $\phi : \mathbb{S} \to \partial\Omega$ be the restriction of Φ to $\mathbb{S}$. Then $\Lambda_a = a^{-1/2}\phi^*\Lambda_\Omega\,\phi^{*-1}a^{1/2}$ and $\mathrm{Sp}(\Lambda_a) = \mathrm{Sp}(\Omega)$. Two latter equalities make sense for an arbitrary positive function $a \in C^\infty(\mathbb{S})$ if we involve multi-sheet domains into our consideration. See [4, Section 3] for details. Theorem 1.1 is true for multi-sheet domains as well. The normalization condition (1.1) is written in terms of the function a as follows:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{a(\theta)} = 1. \tag{1.5}$$

The biholomorphism Φ of the previous paragraph is defined up to a conformal transformation of the disk $\mathbb{D}$, this provides examples of functions with the same Steklov spectrum. Two functions $a, b \in C^\infty(\mathbb{S})$ are said to be *conformally equivalent*, if there exists a conformal or anticonformal transformation Ψ of the disk $\mathbb{D}$ such that $b = |d\psi/d\theta|^{-1} a \circ \psi$, where the function $\psi(\theta)$ is defined by $e^{i\psi(\theta)} = \Psi(e^{i\theta})$ (Ψ is anticonformal if $\bar\Psi$ is conformal). If two positive functions $a, b \in C^\infty(\mathbb{S})$ are conformally equivalent, then $\mathrm{Sp}(a) = \mathrm{Sp}(b)$.

Under condition (1.5), Steklov eigenvalues $\lambda_k(a)$ have the same asymptotics (1.2). The *zeta function of a* is defined by

$$\zeta_a(s) = \mathrm{Tr}[\Lambda_a^{-s}] = \sum_{k=1}^{\infty} \big(\lambda_k(a)\big)^{-s} \tag{1.6}$$

for $\Re(s) > 1$. It again extends to a meromorphic function on $\mathbb{C}$ with the unique simple pole at $s = 1$ such that $\zeta_a(s) - 2\zeta_R(s)$ is an entire function. Here the Steklov zeta function $\zeta_{\mathbf{1}}$ of the constant function $\mathbf{1}$ (= the constant function identically equal to 1) is equal to $\zeta_\mathbb{D} (= 2\zeta_R)$.

The analytical versions of Theorem 1.1, Propositions 1.2, 1.3 and 1.4 sound as follows:

Theorem 1.5. *For a positive function $a \in C^\infty(\mathbb{S})$ satisfying the normalization condition* (1.5)*, the inequality*

$$(\zeta_a - 2\zeta_R)''(0) \ge 0 \tag{1.7}$$

holds. Moreover equality in (1.7) *holds if and only if a is conformally equivalent to the constant function* $\mathbf{1}$*.*

Proposition 1.6. *Let $s \in (-\infty, -1]$. For a positive function $a \in C^\infty(\mathbb{S})$ satisfying the normalization condition* (1.5)*, the inequality*

$$(\zeta_a - 2\zeta_R)''(s) \geq 0 \tag{1.8}$$

holds. Moreover equality in (1.8) *holds if and only if a is conformally equivalent to the constant function* 1*.*

Proposition 1.7. *Let a be a positive function $a \in C^\infty(\mathbb{S})$ satisfying the normalization condition* (1.5)*. Assume that a is not conformally equivalent to* **1***. Then there exists a positive real s_a so that for any $s \in [s_a, +\infty)$ the inequality*

$$(\zeta_a - 2\zeta_R)''(s) > 0 \tag{1.9}$$

holds.

Proposition 1.8. *Let $\mathcal{U}$ be an open neighborhood of* 1 *in $C^\infty(\mathbb{S})$. There exist a smooth positive function $a \in \mathcal{U}$ satisfying the normalization condition* (1.5) *and a real number $s \in (0, 2)$ so that*

$$(\zeta_a - 2\zeta_R)''(s) < 0.$$

Actually Proposition 1.8 is a strengthened version of Proposition 1.4.

The paper is organized as follows. We prove Propositions 1.6 and 1.7 in Section 2. We prove Theorem 1.5 in Section 3. We prove Proposition 1.8 in Section 4. The last Sections 5, 6 and 7 are apart from the convexity questions. We expand the quantities $\langle \ln(\Lambda_a + P_0)\phi_n, \phi_n\rangle$ in a C^∞ neighborhood of the constant function **1**. Here the Hilbert space $L^2(\mathbb{S})$ is considered with the scalar product

$$\langle u, v\rangle = \int_{\mathbb{S}} u(\theta)\overline{v(\theta)}d\theta,$$

and $(\phi_n)_{n\in\mathbb{Z}}$ is the orthonormal basis defined by

$$\phi_n(\theta) = \frac{1}{\sqrt{2\pi a(\theta)}} e^{in\int_0^\theta a^{-1}(s)ds},\ \theta \in [0, 2\pi),\ n \in \mathbb{Z}, \tag{1.10}$$

and P_0 is the orthogonal projection onto the kernel of Λ_a. In particular we prove that the identity $\langle \ln(\Lambda_a + P_0)\phi_n, \phi_n\rangle = \ln(|n|)$, $n \neq 0$, does not hold in general, which was the impetus for the deformation argument that leads to [6, Theorem 1.1], see the concluding remarks given in [6, Section 7].

2. Strict convexity on $(-\infty, -1]$ and near $+\infty$: Proof of Propositions 1.6 and 1.7

In this Section we first recall some notations and properties and we then prove Propositions 1.6 and 1.7.

2.1. Notations, powers and logarithm of operators.

We use the derivative

$$D = -i\frac{d}{d\theta} : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S}).$$

Operators D and Λ have the same one-dimensional null-space consisting of constant functions.

We define the first order differential operator $D_a : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ by

$$D_a = a^{1/2} D a^{1/2}.$$

The orthonormal basis $(\phi_n)_{n\in\mathbb{Z}}$ defined by (1.10) is an eigenbasis for D_a:

$$D_a\phi_n = n\phi_n,\ n \in \mathbb{Z}. \tag{2.1}$$

We will denote $|D_a| = (D_a^2)^{\frac{1}{2}}$. And we denote P_0 the orthogonal projection of $L^2(\mathbb{S})$ onto the one-dimensional space spanned by the function ϕ_0.

Let f be a function from $(0,+\infty)$ to $\mathbb{R}$ with at most a polynomial growth at $+\infty$: $|f(x)| = O(x^N)$ as $x \to +\infty$ for some integer N. Let A be a positive pseudodifferential operator of order one with a discrete eigenvalue spectrum: If $\{\psi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{S})$ consisting of eigenvectors of A with associated eigenvalues $\lambda_k > 0$, then

$$f(A)u = \sum_{k\in\mathbb{N}} f(\lambda_k)\langle u, \psi_k\rangle\psi_k \text{ for } u \in C^\infty(\mathbb{S}).$$

The operator $f(A) : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ defines a (possibly unbounded) selfadjoint operator in $L^2(\mathbb{S})$. In this paper we consider only the case when $A = \Lambda_a + P_0$ or $A = |D_a| + P_0$ and $f(x) = x^s \ln^m(x)$ for $s \in \mathbb{R}$ and $m = 0, 1, 2$.

For instance equality (2.1) implies

$$f(|D_a| + P_0)\phi_n = f\big(\max(|n|, 1)\big)\phi_n \quad n \in \mathbb{Z}. \tag{2.2}$$

When f is convex then we recall that

$$\langle f(A)u, v\rangle \geq f(\langle Au, v\rangle) \tag{2.3}$$

for $(u, v) \in C^\infty(\mathbb{S})^2$ so that $\langle u, v\rangle = 1$ and $\langle u, \psi_k\rangle\langle\psi_k, v\rangle \geq 0$ for every $k \in \mathbb{N}$ (see for instance the proof of [6, Lemma 5.2]).

Let $s \in \mathbb{R}$ and $m \in \mathbb{N}$. The difference

$$(\Lambda_a + P_0)^{-s}\ln^m(\Lambda_a + P_0) - (|D_a| + P_0)^{-s}\ln^m(|D_a| + P_0)$$

is a smoothing operator and

$$(-1)^m\frac{d^m(\zeta_a - 2\zeta_R)}{ds^m}(s) = \mathrm{Tr}\big[(\Lambda_a + P_0)^{-s}\ln^m(\Lambda_a + P_0) - (|D_a| + P_0)^{-s}\ln^m(|D_a| + P_0)\big], \tag{2.4}$$

see [6, Lemmas 3.4 and 3.5] where the operator "$H(\tau, z)$" is taken at $\tau = 0$ and $z = s$.

2.2. **Proof of Proposition 1.6.** First we use (2.4) when $m = 2$:

$$(\zeta_a'' - 2\zeta_R'')(-s) = \mathrm{Tr}((\Lambda_a + P_0)^s\ln(\Lambda_a + P_0)^2 - (|D_a| + P_0)^s\ln(|D_a| + P_0)^2)),$$

and we expand the trace with respect to the basis $(\phi_n)_{n\in\mathbb{Z}}$

$$(\zeta_a - 2\zeta_R)''(-s) = \sum_{n\in\mathbb{Z}\backslash\{0\}}\Big(\langle(\Lambda_a + P_0)^s\ln^2(\Lambda_a + P_0)\phi_n, \phi_n\rangle - |n|^s\ln(|n|)^2\Big). \tag{2.5}$$

Let $n \in \mathbb{Z}\backslash\{0\}$. By Cauchy-Bunyakovsky-Schwarz inequality we have

$$\begin{aligned}&\big(\langle(\Lambda_a + P_0)^s\ln(\Lambda_a + P_0)\phi_n, \phi_n\rangle\big)^2 = \big(\langle(\Lambda_a + P_0)^{\frac{s}{2}}\ln(\Lambda_a + P_0)\phi_n, (\Lambda_a + P_0)^{\frac{s}{2}}\phi_n\rangle\big)^2\\ \leq\ &\langle(\Lambda_a + P_0)^s\phi_n, \phi_n\rangle\langle(\Lambda_a + P_0)^s\ln^2(\Lambda_a + P_0)\phi_n, \phi_n\rangle\end{aligned} \tag{2.6}$$

for $s \in \mathbb{R}$. Now set $s \geq 1$. We recall the estimate [5, Lemmas 2.1 and 2.4]

$$\langle(\Lambda_a + P_0)^s\phi_n, \phi_n\rangle \geq |n|^s \geq 1. \tag{2.7}$$

We divide both sides of the inequality (2.6) by $\langle(\Lambda_a+P_0)^s\phi_n,\phi_n\rangle$ and we obtain

$$\begin{aligned}\langle(\Lambda_a+P_0)^s\ln^2(\Lambda_a+P_0)\phi_n,\phi_n\rangle &\geq \frac{\big(\langle(\Lambda_a+P_0)^s\ln(\Lambda_a+P_0)\phi_n,\phi_n\rangle\big)^2}{\langle(\Lambda_a+P_0)^s\phi_n,\phi_n\rangle}\\ &= \frac{1}{s^2}\frac{\big(\langle f(\Lambda_a+P_0)^s)\phi_n,\phi_n\rangle\big)^2}{\langle(\Lambda_a+P_0)^s\phi_n,\phi_n\rangle} \qquad (2.8)\end{aligned}$$

where f is the convex function $f(x)=x\ln(x)$, $x>0$. We used the identity $\ln(\Lambda_a+P_0)=s^{-1}\ln((\Lambda_a+P_0)^s)$. Then we use (2.3):

$$\langle f\big((\Lambda_a+P_0)^s\big)\phi_n,\phi_n\rangle \geq f(\langle(\Lambda_a+P_0)^s\phi_n,\phi_n\rangle)\geq 0. \qquad (2.9)$$

The nonnegativity in (2.9) follows from (2.7). Then we combine (2.8) and (2.9) and we obtain

$$\begin{aligned}\langle(\Lambda_a+P_0)^s\ln^2(\Lambda_a+P_0)\phi_n,\phi_n\rangle &\geq \frac{1}{s^2}\frac{f\big(\langle(\Lambda_a+P_0)^s\phi_n,\phi_n\rangle)\big)^2}{\langle(\Lambda_a+P_0)^s\phi_n,\phi_n\rangle}\\ &= \frac{1}{s^2}\langle(\Lambda_a+P_0)^s\phi_n,\phi_n\rangle\ln\Big(\langle(\Lambda_a+P_0)^s\phi_n,\phi_n\rangle\Big)^2\\ &\geq |n|^s\ln(|n|)^2. \qquad (2.10)\end{aligned}$$

We used (2.7) at the last line.

Inequality (1.8) follows from (2.5) and (2.10). Equality in (1.8) implies that each summand in (2.5) is zero: $\langle(\Lambda_a+P_0)^s\ln^2(\Lambda_a+P_0)\phi_n,\phi_n\rangle=|n|^s\ln(|n|)^2$ for $n\in\mathbb{Z}\backslash\{0\}$. In particular it implies equalities in (2.10). Therefore $\langle(\Lambda_a+P_0)^s\phi_n,\phi_n\rangle=|n|^s$ for $n\in\mathbb{Z}\backslash\{0\}$. The identity for $n=1$ is enough to conclude that a is conformally equivalent to $\mathbf{1}$ [5, Lemma 2.5]. □

2.3. **Proof of Proposition 1.7.** Assume that a is not conformally equivalent to the constant function $\mathbf{1}$. Weinstock's inequality [8] tells us that

$$\lambda_1(a)<1.$$

And by definition

$$\zeta_a(s)-2\zeta_R(s)=\sum_{k=1}^{\infty}\Big[\lambda_k(a)^{-s}-\big(\lfloor\frac{k+1}{2}\rfloor\big)^{-s}\Big],$$

$$\begin{aligned}\zeta_a''(s)-2\zeta_R''(s) &= \lambda_1(a)^{-s}\ln(\lambda_1(a))^2+\lambda_2(a)^{-s}\ln(\lambda_2(a))^2\\ &\quad+\sum_{k=3}^{\infty}\Big[\lambda_k(a)^{-s}\ln(\lambda_k(a))^2-\big(\lfloor\frac{k+1}{2}\rfloor\big)^{-s}\ln\big(\lfloor\frac{k+1}{2}\rfloor\big)^2\Big].\end{aligned}$$

Hence the leading order as $s\to+\infty$ in the above sum is $\lambda_1(a)^{-s}\ln(\lambda_1(a))^2$. The asymptotics makes obvious the existence of $s_a\in[0,+\infty)$ so that

$$\zeta_a''(s)-2\zeta_R''(s)>0, s\geq s_a. \qquad (2.11)$$

□

3. The second derivative ζ_Ω'' at 0: Proof of Theorem 1.5

The proof of Theorem 1.5 relies on the same deformation argument used to prove [6, Theorem 1.1]. We start this Section by recalling some definition of a variation of a function a. We give a proof of Theorem 1.5 at the end.

3.1. **Deformation of a function a and the Hilbert transform $\mathcal{H}$.** Let $l = 0$ or $l = \infty$ and let $\varepsilon > 0$. A real function $\alpha \in C^l\big([0,\varepsilon), C^\infty(\mathbb{S})\big)$ is called a C^l-*deformation* (or C^l-*variation*) of a positive function $a \in C^\infty(\mathbb{S})$ when it satisfies the 3 conditions: $\alpha(0,\theta) = a(\theta)$; For any $\tau \in [0,\varepsilon)$ the function $\alpha_\tau = \alpha(\tau,\cdot) \in C^\infty(\mathbb{S})$ is positive and it satisfies the normalization condition

$$\int_{\mathbb{S}} \alpha_\tau^{-1}(\theta)d\theta = 2\pi. \tag{3.1}$$

The entire function ζ_{α_τ} has the following smoothness along the deformation α [6, Lemma 3.5]

$$\zeta_{\alpha_\tau} - 2\zeta_R \in C^l([0,\varepsilon)_\tau, \mathcal{F}(\mathbb{C})). \tag{3.2}$$

Here $\mathcal{F}(\mathbb{C})$ denotes the space of entire functions on the complex plane.

The *Hilbert transform* $\mathcal{H}$ is the linear operator on $L^2(\mathbb{S})$ defined by

$$\mathcal{H}(\mathbf{1}) = 0, \quad \mathcal{H}e^{in\theta} = \operatorname{sgn}(n)e^{in\theta} \text{ for an integer } n \neq 0.$$

We will use the identities

$$D = \mathcal{H}\Lambda = \Lambda\mathcal{H},\ D_a = \Lambda_a a^{-1/2}\mathcal{H}a^{1/2}. \tag{3.3}$$

3.2. **Preliminary Lemma.**

Lemma 3.1. *Let $a \in C^\infty(\mathbb{S})$ be positive and satisfy the normalization condition* (1.5). *Then*

$$\operatorname{Tr}(\ln(\Lambda_a + P_0)(\Lambda_a + P_0)^{-1}(\Lambda_a^2 - D_a^2)) \geq 0 \tag{3.4}$$

with equality if and only if a is conformally equivalent to the constant function $\mathbf{1}$.

The operator inside the trace in (3.4) is trace class. Indeed it is the product of the bounded operator $\ln(\Lambda_a + P_0)(\Lambda_a + P_0)^{-1}$ and of the smoothing operator $\Lambda_a^2 - D_a^2$ (see Section 2.1).

Proof of Lemma 3.1. We expand the trace with respect to the orthonormal basis (ϕ_n):

$$\begin{aligned} &\operatorname{Tr}(\ln(\Lambda_a + P_0)(\Lambda_a + P_0)^{-1}(\Lambda_a^2 - D_a^2)) \\ = &\sum_{n\in\mathbb{Z}\setminus\{0\}} \big(\langle \Lambda_a \ln(\Lambda_a + P_0)\phi_n, \phi_n\rangle - n^2\langle \ln(\Lambda_a + P_0)(\Lambda_a + P_0)^{-1}\phi_n, \phi_n\rangle\big). \end{aligned} \tag{3.5}$$

We used the identity $(\Lambda_a + P_0)^{-1}\Lambda_a^2 = \Lambda_a$ and we used (2.1). We prove that the summands are nonnegative.

Let $n \in \mathbb{N}\setminus\{0\}$. First we use (2.9) for $s = 1$:

$$\langle \Lambda_a \ln(\Lambda_a + P_0)\phi_n, \phi_n\rangle \geq \langle \Lambda_a\phi_n, \phi_n\rangle \ln(\langle \Lambda_a\phi_n, \phi_n\rangle). \tag{3.6}$$

In addition we use (2.1) and (3.3) and the identity $(\Lambda_a + P_0)^{-1}\Lambda_a = I - P_0$ where I is the identity operator and we obtain

$$\begin{aligned} n\langle \ln(\Lambda_a + P_0)(\Lambda_a + P_0)^{-1}\phi_n, \phi_n\rangle &= \langle \ln(\Lambda_a + P_0)(\Lambda_a + P_0)^{-1}D_a\phi_n, \phi_n\rangle \\ &= \langle \ln(\Lambda_a + P_0)a^{-1/2}\mathcal{H}a^{1/2}\phi_n, \phi_n\rangle. \end{aligned} \tag{3.7}$$

Let

$$\delta_n = n^{-1}\langle \Lambda_a\phi_n, \phi_n\rangle,\ u = \delta_n^{-1}a^{-1/2}\mathcal{H}a^{1/2}\phi_n,\ v = \phi_n.$$

We apply (2.3) when $f(x) = -\ln(x)$ and $A = \Lambda_a + P_0$ (see details for the sign of $\langle u, \psi_k\rangle\langle \psi_k, v\rangle$ in [6, Section 5, part 5]) and we obtain

$$\langle \ln(\Lambda_a + P_0)\delta_n^{-1}a^{-1/2}\mathcal{H}a^{1/2}\phi_n, \phi_n\rangle \leq \ln\big(\langle \Lambda_a\delta_n^{-1}a^{-1/2}\mathcal{H}a^{1/2}\phi_n, \phi_n\rangle\big) = \ln\big(\delta_n^{-1}n\big). \tag{3.8}$$

We combine (3.7) and (3.8) and we obtain

$$n^2\langle \ln(\Lambda_a+P_0)(\Lambda_a+P_0)^{-1}\phi_n,\phi_n\rangle \leq n\delta_n \ln\left(\delta_n^{-1}n\right) = \langle \Lambda_a\phi_n,\phi_n\rangle \ln\left(\frac{n^2}{\langle \Lambda_a\phi_n,\phi_n\rangle}\right)$$
$$\leq \langle \Lambda_a\phi_n,\phi_n\rangle \ln(n) \leq \langle \Lambda_a\phi_n,\phi_n\rangle \ln(\langle \Lambda_a\phi_n,\phi_n\rangle). \tag{3.9}$$

We used the growth of the logarithm and we used the estimates

$$\langle \Lambda_a\phi_n,\phi_n\rangle \geq n,\ \delta_n \geq 1,$$

see (2.7) for $s=1$. We combine (3.6) and (3.9) and we obtain

$$\langle \Lambda_a \ln(\Lambda_a+P_0)\phi_n,\phi_n\rangle \geq n^2\langle \ln(\Lambda_a+P_0)(\Lambda_a+P_0)^{-1}\phi_n,\phi_n\rangle,\ n\in\mathbb{N}\backslash\{0\}.$$

We obtain the same estimates for negative integers n by complex conjugation invariance. Then we use again (3.5) and we obtain (3.4).

Now equality in (3.4) means equalities in the last line of (3.9). Hence

$$\langle \Lambda_a\phi_n,\phi_n\rangle = n,\ n\in\mathbb{N}\backslash\{0\}.$$

The identity for $n=1$ is enough to conclude that a is conformally equivalent to $\mathbf{1}$ [5, Lemma 2.5]. □

3.3. **Proof of Theorem 1.5.** The inequality

$$(\zeta_a - 2\zeta_R)(s) \geq 0,\ s\in\mathbb{R},$$

holds by [6, Theorem 1.1]. Since $(\zeta_a-2\zeta_R)(0)=0$ we obtain the inequality (1.7). In addition $\zeta_a = 2\zeta_R$ when a is conformally equivalent to the constant function $\mathbf{1}$.

Hence we only have to prove that $(\zeta_a-2\zeta_R)''(0)=0$ implies that a is conformally equivalent to a constant function.

Consider the deformation $\alpha\in C^\infty([0,\infty)_\tau, C^\infty(\mathbb{S}))$ introduced in [6, Theorem 1.3]. It satisfies the evolution equation

$$\frac{\partial \alpha_\tau}{\partial\tau} = -\alpha_\tau\Lambda\alpha_\tau + \mathcal{H}\alpha_\tau D\alpha_\tau, \tau\geq 0, \tag{3.10}$$

with initial condition $\alpha_0 = a$, and by [6, Theorem 4.1]

$$\frac{\partial \zeta_{\alpha_\tau}}{\partial\tau}(s) = s\mathrm{Tr}((\Lambda_{\alpha_\tau}+P_{0,\tau})^{-s-1}(\Lambda_{\alpha_\tau}^2 - D_{\alpha_\tau}^2)), \tag{3.11}$$

for $s\in\mathbb{R}$ and $\tau\in[0,\infty)$. Here $P_{0,\tau}$ is the orthogonal projection of $L^2(\mathbb{S})$ onto the one-dimensional space spanned by the function $(2\pi\alpha_\tau)^{-1/2}$.

In addition $\alpha_\tau\to\mathbf{1}$ as $\tau\to\infty$ in C^∞-topology.

Let $s\in\mathbb{R}$. Set $N=|s|+1$. The operator $(\Lambda_{\alpha_\tau}+P_{0,\tau})^N(\Lambda_{\alpha_\tau}^2-D_{\alpha_\tau}^2)$ is a smoothing operator, see Section 2.1, while $(\Lambda_{\alpha_\tau}+P_{0,\tau})^{-\sigma-1-N}$ is a family of bounded operators in $L^2(\mathbb{S})$ that is smooth with respect to σ in a neighborhood of s. Hence we can intertwin the trace on the right hand side of (3.11) and any derivative with respect to the s-variable. We derive (3.11) with respect to s and we denote $'$ or $\frac{d}{ds}$ the derivative with respect to the real variable s and we have

$$\begin{aligned}\frac{\partial \zeta'_{\alpha_\tau}}{\partial\tau}(s) &= \mathrm{Tr}((\Lambda_{\alpha_\tau}+P_{0,\tau})^{-s-1}(\Lambda_{\alpha_\tau}^2-D_{\alpha_\tau}^2))\\ &\quad + s\mathrm{Tr}(\frac{d}{ds}(\Lambda_{\alpha_\tau}+P_{0,\tau})^{-s}(\Lambda_{\alpha_\tau}+P_{0,\tau})^{-1}(\Lambda_{\alpha_\tau}^2-D_{\alpha_\tau}^2))\\ &= \mathrm{Tr}((\Lambda_{\alpha_\tau}+P_{0,\tau})^{-s-1}(\Lambda_{\alpha_\tau}^2-D_{\alpha_\tau}^2))\\ &\quad - s\mathrm{Tr}(\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})(\Lambda_{\alpha_\tau}+P_{0,\tau})^{-s-1}(\Lambda_{\alpha_\tau}^2-D_{\alpha_\tau}^2))\end{aligned}$$

We derive once more in s and we obtain

$$\begin{aligned}\frac{\partial \zeta''_{\alpha_\tau}}{\partial \tau}(s) &= -2\mathrm{Tr}(\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})(\Lambda_{\alpha_\tau}+P_{0,\tau})^{-s-1}(\Lambda^2_{\alpha_\tau}-D^2_{\alpha_\tau}))\\ &\quad +s\mathrm{Tr}(\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})^2(\Lambda_{\alpha_\tau}+P_{0,\tau})^{-s-1}(\Lambda^2_{\alpha_\tau}-D^2_{\alpha_\tau}).\end{aligned}$$

Therefore

$$\frac{\partial \zeta''_{\alpha_\tau}}{\partial \tau}(0) = -2\mathrm{Tr}(\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})(\Lambda_{\alpha_\tau}+P_{0,\tau})^{-1}(\Lambda^2_{\alpha_\tau}-D^2_{\alpha_\tau})). \tag{3.12}$$

We apply Lemma 3.1 to obtain that

$$\zeta''_{\alpha_\tau}(0) \text{ is nonincreasing in } \tau. \tag{3.13}$$

Moreover since $\alpha_\tau \to 1$ as $\tau \to \infty$ in C^∞-topology, we obtain that

$$\zeta''_{\alpha_\tau}(0) \to 2\zeta''_R(0), \text{ as } \tau \to \infty. \tag{3.14}$$

Indeed we consider the continuous path $\beta \in C([0,\infty)_\varepsilon, C^\infty(\mathbb{S}))$ defined by

$$\beta_0 = \mathbf{1},\ \beta_\varepsilon = \alpha_{\frac{1}{\varepsilon}} \text{ for } \varepsilon > 0.$$

Then (3.2) yields

$$\zeta_{\beta_\varepsilon} \in C([0,\infty)_\varepsilon, C^\infty(\mathbb{C}\backslash\{1\})) \text{ and } \frac{d^j\zeta_{\beta_\varepsilon}}{ds^j}(0) \to 2\frac{d^j\zeta_R}{ds^j}(0) \text{ as } \varepsilon \to 0^+$$

for any $j \in \mathbb{N}$. Hence we proved statement (3.14).

Now assume that $\zeta''_a(0) = 2\zeta''_R(0)$. Then we obtain by (3.13) and (3.14) $(\zeta_{\alpha_\tau} - 2\zeta_R)''(0) = 0$ for any τ and

$$\mathrm{Tr}(\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})(\Lambda_{\alpha_\tau}+P_{0,\tau})^{-1}(\Lambda^2_{\alpha_\tau}-D^2_{\alpha_\tau})) = -\frac{1}{2}\frac{\partial \zeta''_{\alpha_\tau}}{\partial \tau}(0) = 0.$$

Therefore we apply again Lemma 3.1 and we obtain that α_τ is conformally equivalent to a constant valued function for any τ. In particular, a is conformally equivalent to $\mathbf{1}$. □

4. The difference $\zeta_a - 2\zeta_R$ may not be convex everywhere on the real axis: Proof of Proposition 1.8

We recall the following result [6, Proposition 3.8].

Proposition 4.1 (see [6]). *Let α_τ be a C^∞-variation of the function $a = \mathbf{1}$. Then, for every $z \in \mathbb{C}$,*

$$\left.\frac{\partial\big(\zeta_{\alpha_\tau}(z)\big)}{\partial\tau}\right|_{\tau=0} = 0, \tag{4.1}$$

$$\left.\frac{\partial^2\big(\zeta_{\alpha_\tau}(z)\big)}{\partial\tau^2}\right|_{\tau=0} = 4z \sum_{\substack{(n,p)\in\mathbb{N}^2\\ p>0,\ n>0,\ p\neq n}} \frac{n^{-z}-p^{-z}}{p^2-n^2}\, pn\, |\hat\beta_{p+n}|^2 + 2z^2 \sum_{n>0} n^{-z}\, |\hat\beta_{2n}|^2, \tag{4.2}$$

where $\beta(\theta) = \left.\frac{\partial\alpha_\tau(\theta)}{\partial\tau}\right|_{\tau=0}$ (and $\alpha_0 = \mathbf{1}$).

The proof of Proposition 1.8 relies on the analysis of the right hand side of (4.2). From now on we consider only C^∞-variation α_τ of the function $a = \mathbf{1}$ so that

$$\beta(e^{i\theta}) = 2\cos((2r+1)\theta),\ \theta \in \mathbb{R}, \tag{4.3}$$

for some large integer r. Take for instance the smooth variation

$$\alpha_\tau(e^{i\theta}) = \big(1 - 2\tau\cos((2r+1)\theta)\big)^{-1},\ \tau \in (-1/2, 1/2),\ \theta \in \mathbb{R}. \tag{4.4}$$

The right hand side of (4.2) becomes

$$\left.\frac{\partial^2\big(\zeta_{\alpha_\tau}(s)\big)}{\partial\tau^2}\right|_{\tau=0} = -4s \sum_{\substack{(n,p)\in\mathbb{N}^2\\ p>0,\ n>0,\ p+n=2r+1}} \frac{p^{-s}-n^{-s}}{p^2-n^2}\,pn,$$

for a real s (we used $\hat\beta_{2r+1}=1$). Hence

$$\left.\frac{\partial^2\big(\zeta_{\alpha_\tau}(s)\big)}{\partial\tau^2}\right|_{\tau=0} = -8s \sum_{n>0,\ p>n,\ p+n=2r+1} \frac{p^{-s}-n^{-s}}{p^2-n^2}pn.$$

We derive with respect to s:

$$\begin{aligned}\left.\frac{\partial^2\big(\zeta'_{\alpha_\tau}(s)\big)}{\partial\tau^2}\right|_{\tau=0} &= \frac{8}{2r+1}\sum_{p=r+1}^{2r}\frac{p(2r+1-p)}{2p-2r-1} \\ &\quad\times(p^{-s}(-1+s\ln(p))-(2r+1-p)^{-s}(-1+s\ln(2r+1-p))).\end{aligned} \tag{4.5}$$

(We substitued n by $2r+1-p$.)

Let us make an asymptotic analysis as $r\to\infty$, $0<s<2$.

$$\begin{aligned}&\sum_{p=r+1}^{2r}\frac{p(2r+1-p)}{2p-2r-1}(p^{-s}(-1+s\ln(p))-(2r+1-p)^{-s}(-1+s\ln(2r+1-p))) \\ =\ & (2r+1)^{1-s}\sum_{p=r+1}^{2r}\frac{\frac{p}{2r+1}\big(1-\frac{p}{2r+1}\big)}{2\frac{p}{2r+1}-1}\Big(\big(\frac{p}{2r+1}\big)^{-s}\big(-1+s\ln\big(\frac{p}{2r+1}\big)\big) \\ &-\big(1-\frac{p}{2r+1}\big)^{-s}\big(-1+s\ln\big(1-\frac{p}{2r+1}\big)\big)\Big) \\ &+s(2r+1)^{1-s}\ln(2r+1)\sum_{p=r+1}^{2r}\frac{\frac{p}{2r+1}\big(1-\frac{p}{2r+1}\big)}{2\frac{p}{2r+1}-1}\Big(\big(\frac{p}{2r+1}\big)^{-s}-\big(1-\frac{p}{2r+1}\big)^{-s}\Big).\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{p=r+1}^{2r}\frac{p(2r+1-p)}{2p-2r-1}(p^{-s}(-1+s\ln(p))-(2r+1-p)^{-s}(-1+s\ln(2r+1-p))) \\ =\ & (2r+1)^{2-s}\Big[\int_{1/2}^1\frac{x(1-x)}{2x-1}(x^{-s}(-1+s\ln(x)) \\ &-(1-x)^{-s}(-1+s\ln(1-x)))dx+o(1)\Big] \\ &+s(2r+1)^{2-s}\ln(2r+1)\Big[\int_{1/2}^1\frac{x(1-x)}{2x-1}(x^{-s}-(1-x)^{-s})+o(1)\Big]\end{aligned} \tag{4.6}$$

as $r\to+\infty$. We used the following elementary statement for the singularity near $x=1$, see for instance [1, Section 2.12.7]: For a continuous function $\eta\in C((0,1),\mathbb{R})$ so that $\eta(x)=O(x^{-\rho})$, $\eta(1-x)=O(x^{-\rho})$ as $x\to0^+$, $0<\rho<1$, then

$$\int_0^1\eta=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N-1}\eta(\frac{i}{N}).$$

Then the leading order is given by $(2r+1)^{2-s}\ln(2r+1)$ as $r \to +\infty$ and the coefficient in front of it is

$$s\int_{1/2}^{1} \frac{x(1-x)}{2x-1}(x^{-s} - (1-x)^{-s})dx < 0 \tag{4.7}$$

since $x > 1-x$ and $x^{-s} < (1-x)^{-s}$ for $x \in (1/2, 1)$ and $s \in (0,2)$.

Now we combine (4.5), (4.6) and (4.7) and we obtain that at fixed $s \in (0,2)$ there exists a large integer r_s so that

$$\left.\frac{\partial^2\big(\zeta'_{\alpha_\tau}(s)\big)}{\partial\tau^2}\right|_{\tau=0} < 0 \tag{4.8}$$

for any integer $r \geq r_s$ (let us remind that the path α_τ is defined by the integer r).

From (4.1) it also follows that

$$\left.\frac{\partial\big(\zeta'_{\alpha_\tau}(s)\big)}{\partial\tau}\right|_{\tau=0} = 0. \tag{4.9}$$

At $\tau = 0$, $\zeta'_{\alpha_\tau}(s) = 2\zeta'_R(s)$.

Therefore we make a Taylor expansion of $\zeta'_{\alpha_\tau}(s)$ with respect to τ in a neighborhood of 0 and we obtain that there exists τ_s

$$\zeta'_{\alpha_\tau}(s) < 2\zeta'_R(s)$$

for any integer $r \geq r_s$ and any $\tau \in (0, \tau_s)$.

Now we are ready to conclude the proof of Proposition 1.8. Take $\mathcal{U}$ any neighborhood of $\mathbf{1}$ in $C^\infty(\mathbb{S})$. Let $s \in (0,2)$. Then define the integer r_s and the positive real number τ_s as above and choose $\tau \in (0, \tau_s)$ small enough so that

$$\alpha_\tau \in \mathcal{U}.$$

Such an α_τ plays the role of a in the statement of Proposition 1.8. Indeed $\zeta'_{\alpha_\tau}(s) < 2\zeta'_R(s)$. Since $\zeta'_{\alpha_\tau}(0) = 2\zeta'_R(0)$ (see [3, 6]) the function $\zeta_{\alpha_\tau} - 2\zeta_R$ is not convex in $(0,2)$. □

We can go beyond the interval $(0,2)$. Now let $s > 2$

$$\begin{aligned}\left.\frac{\partial^2\big(\zeta'_{\alpha_\tau}(s)\big)}{\partial\tau^2}\right|_{\tau=0} &= \frac{8}{2r+1}\sum_{p=r+1}^{2r}\frac{p(2r+1-p)}{2p-2r-1} \\ &\quad\times(p^{-s}(-1+s\ln(p)) - (2r+1-p)^{-s}(-1+s\ln(2r+1-p))) \\ &= \frac{8}{2r+1}\sum_{p=1}^{2r}\frac{(2r+1-p)}{2p-2r-1}p^{1-s}(-1+s\ln(p))\end{aligned}$$

(We make a change of variable "p"$= 2r+1-p$ at the last line.) Then

$$\begin{aligned}&\sum_{p=1}^{2r}\frac{(2r+1-p)}{2p-2r-1}p^{1-s}(-1+s\ln(p)) \\ =\ &\sum_{p=1}^{2r}p^{1-s}(1-s\ln(p)) - \sum_{p=1}^{2r}\frac{p^{2-s}(1-s\ln(p))}{2p-2r-1} \\ =\ &\sum_{p=1}^{2r}p^{1-s}(1-s\ln(p)) - \sum_{p=\lfloor\sqrt{r}\rfloor+1}^{2r}\frac{p^{2-s}(1-s\ln(p))}{2p-2r-1} - \sum_{p=1}^{\lfloor\sqrt{r}\rfloor}\frac{p^{2-s}(1-s\ln(p))}{2p-2r-1}.\end{aligned}$$

We conclude using the elementary facts: Note that

$$\sum_{p=1}^{2r} p^{1-s}(1-s\ln(p)) \to \zeta_R(s-1)+s\zeta_R'(s-1) \text{ as } r\to\infty,$$

(here the series is absolutely convergent) and

$$\begin{aligned} &|\sum_{p=\lfloor\sqrt{r}\rfloor+1}^{2r} \frac{p^{-s+2}}{2p-2r-1}(1-s\ln(p))| \leq r^{1-s/2}(1+|s|\ln(2r)) \sum_{p=\lfloor\sqrt{r}\rfloor+1}^{2r} \frac{1}{|2p-2r-1|} \\ \leq\ & 2r^{1-s/2}(1+|s|\ln(2r))(1+\int_1^{2r}\frac{dt}{t}) = 2r^{1-s/2}(1+|s|\ln(2r))(1+\ln(2r)) \to 0, \end{aligned}$$

$$\begin{aligned} |\sum_{p=1}^{\lfloor\sqrt{r}\rfloor} \frac{p^{2-s}(1-s\ln(p))}{2p-2r-1}| &\leq (2r+1-2\sqrt{r})^{-1}\sum_{p=1}^{\lfloor\sqrt{r}\rfloor} p^{2-s}(1+|s|\ln(p)) \qquad (4.10)\\ &\leq \frac{\sqrt{r}}{2r+1-2\sqrt{r}}\sum_{p=1}^{\lfloor\sqrt{r}\rfloor} p^{1-s}(1+|s|\ln(p)) \to 0, \qquad (4.11) \end{aligned}$$

as $r\to+\infty$. Hence we finally obtain that

$$\frac{2r+1}{8}\left.\frac{\partial^2\left(\zeta'_{\alpha_\tau}(s)\right)}{\partial\tau^2}\right|_{\tau=0} \to \zeta_R(s-1)+s\zeta_R'(s-1) \text{ as } r\to\infty. \qquad (4.12)$$

We recall the formula [7, Chapter 2, Section 2.1]:

$$\zeta_R(z) = \frac{z}{z-1} - z\int_1^\infty \frac{u-\lfloor u\rfloor}{u^{1+z}}du.$$

There is a single pole at $z=1$ and there exists $s_0\in(2,\infty)$ so that

$$\zeta_R(s-1)+s\zeta_R'(s-1)<0,\ s\in(2,s_0).$$

Note that $s_0<+\infty$ since $\zeta_R(z)\to 1$ and $(z+1)\zeta_R'(z)\to 0$ as $\Re z\to+\infty$.

Let $s\in(2,s_0)$ and for r large enough we again obtain (4.8). Repeating the proof of Proposition 1.8 we obtained the following result.

Proposition 4.2. *Let $\mathcal{U}$ be an open neighborhood of 1 in $C^\infty(\mathbb{S})$ and let $s\in(2,s_0)$. There exists a smooth positive function $a\in\mathcal{U}$ so that*

$$(\zeta_a'-2\zeta_R')(s)<0.$$

5. Addenda: Comments

We proved in [6] that

$$(\zeta_a-2\zeta_R)(s)\geq 0$$

for any $s\in\mathbb{R}$ and any smooth positive function $a\in C^\infty(\mathbb{S})$ satisfying the normalizing condition (1.5). We used a path of deformaton α_τ (see Section 2). The estimate $(\zeta_a-2\zeta_R)(s)\geq 0$ was actually proved in an easier way when $|s|\geq 1$ in [5]: The proof relied on the estimates

$$\langle(\Lambda_a+P_0)^s\phi_n,\phi_n\rangle \geq |n|^s \text{ for } (n,s)\in\mathbb{Z}\times(-\infty,-1]\cup[1,+\infty).$$

In this Section we focus on the loss of these estimates for s in a neighborhood of 0.

5.1. **Main result.** Let $\alpha_\tau \in C^\infty(\mathbb{S})$, $\tau \in [0,\varepsilon)$, be a C^∞-variation of **1**. We denote $P_{0,\tau}$ the orthogonal projection of $L^2(\mathbb{S})$ onto the one-dimensional space spanned by the function $(2\pi\alpha_\tau)^{-1/2}$. We recall that the operator $\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})-\ln(|D_{\alpha_\tau}|+P_{0,\tau})$ is a smoothing operator at each τ, see Section 2.1. Actually it defines a family of smoothing operators which is smooth with respect to τ:

$$\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})-\ln(|D_{\alpha_\tau}|+P_{0,\tau})\in C^\infty([0,\varepsilon)_\tau,\mathcal{L}(H^l(\mathbb{S}),H^{l'}(\mathbb{S}))), \tag{5.1}$$

for any nonnegative real l,l' where $\mathcal{L}(H^l(\mathbb{S}),H^{l'}(\mathbb{S}))$ denotes the Banach space of bounded operators from the Sobolev space $H^l(\mathbb{S})$ of order l to the one of order l', see [6, Lemma 3.4] ("$\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})-\ln(|D_{\alpha_\tau}|+P_{0,\tau})=-\frac{\partial H}{\partial z}(\tau,0)$" there).

For each τ we consider the orthornomal basis $(\phi_{m,\tau})_{m\in\mathbb{Z}}$ of eigenvectors of $|D_{\alpha_\tau}|$:

$$\phi_{m,\tau}(\theta)=\frac{1}{\sqrt{2\pi\alpha_\tau(\theta)}}e^{im\int_0^\theta\alpha_\tau^{-1}(s)ds},\ m\in\mathbb{Z},\ \theta\in\mathbb{R}.$$

We remind that $\ln(|D_{\alpha_\tau}|+P_{0,\tau})\phi_m=\ln|m|\phi_m$. By (5.1)

$$\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{m,\tau},\phi_{m,\tau}\rangle\in C^\infty([0,\varepsilon)_\tau,\mathbb{R}).$$

for $m\in\mathbb{Z}\backslash\{0\}$.

We expand $\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{m,\tau},\phi_{m,\tau}\rangle$ at order 2 in a neighborhood of $\tau=0$.

Theorem 5.1. *Let m be a positive integer. We have*

$$\frac{d}{d\tau}\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{m,\tau},\phi_{m,\tau}\rangle_{|\tau=0}=0, \tag{5.2}$$

$$\frac{1}{4}\frac{d^2}{d\tau^2}\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{m,\tau},\phi_{m,\tau}\rangle_{|\tau=0}=\sum_{p>0,\ p\neq m}\gamma(p,m)|\hat\beta_{m+p}|^2, \tag{5.3}$$

where

$$\gamma(p,m)=\frac{pm\Big(2\ln(p/m)mp+(p+m)(m-p)\Big)}{(m-p)^2(m+p)^2}$$

for $(p,m)\in(\mathbb{N}\backslash\{0\})^2$, $p\neq m$. Therefore the following Taylor expansion at 0 holds

$$\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{m,\tau},\phi_{m,\tau}\rangle=\ln m+2\tau^2\sum_{p>0,\ p\neq m}\gamma(p,m)|\hat\beta_{m+p}|^2+o(\tau^2),\ \tau\to0^+. \tag{5.4}$$

5.2. **Examples of smooth variations α_τ.** Let m be a positive integer. The sign of m is not relevant since $\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{m,\tau},\phi_{m,\tau}\rangle$ has the same value when m is replaced by its opposite.

When

$$\alpha_\tau=\big(1-2\tau\cos(r\theta)\big)^{-1}$$

for a positive integer $r>m$ then (5.3) gives

$$\begin{aligned}\frac{1}{4}\frac{d^2}{d\tau^2}\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{m,\tau},\phi_{m,\tau}\rangle_{|\tau=0}&=\gamma(r-m,m)\\&=\frac{(r-m)m\Big(2\ln(\frac{r-m}{m})m(r-m)+r(2m-r)\Big)}{(2m-r)^2r^2}\end{aligned}$$

and

$$\frac{1}{4}\frac{d^2}{d\tau^2}\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{r-m,\tau},\phi_{r-m,\tau}\rangle_{|\tau=0}=\gamma(m,r-m)=-\gamma(r-m,m).$$

We consider the asymptotic regime when $r \to \infty$. In that case we have

$$\gamma(r-m,m) = -r^{-1}m + o(r^{-1}) \text{ as } r \to +\infty.$$

Therefore for r large enough with respect to m

$$\frac{d^2}{d\tau^2}\langle \ln(\Lambda_{\alpha_\tau} + P_{0,\tau})\phi_{m,\tau}, \phi_{m,\tau}\rangle_{|\tau=0} < 0,$$

and

$$\frac{d^2}{d\tau^2}\langle \ln(\Lambda_{\alpha_\tau} + P_{0,\tau})\phi_{r-m,\tau}, \phi_{r-m,\tau}\rangle_{|\tau=0} > 0,$$

Then the expansion (5.4) implies that for small enough positive τ

$$\langle \ln(\Lambda_{\alpha_\tau} + P_{0,\tau})\phi_{m,\tau}, \phi_{m,\tau}\rangle < \ln m,\ \langle \ln(\Lambda_{\alpha_\tau} + P_{0,\tau})\phi_{r-m,\tau}, \phi_{r-m,\tau}\rangle > \ln(r-m).$$

Hence for small enough positive real s and small enough positive τ

$$\langle (\Lambda_{\alpha_\tau} + P_{0,\tau})^s\phi_{m,\tau}, \phi_{m,\tau}\rangle < m^s,\ \langle (\Lambda_{\alpha_\tau} + P_{0,\tau})^{-s}\phi_{r-m,\tau}, \phi_{r-m,\tau}\rangle < (r-m)^{-s}.$$

5.3. **Consequence.** We collect the example of deformations of the previous subsection and we obtain the following result. In the statement given below $\tilde{m}$ is the integer $r-m$ of the previous subsection.

Corollary 5.2. *Let $\mathcal{U}$ be an open neighborhood of 1 in $C^\infty(\mathbb{S})$ and let m be a positive integer. There exists a smooth positive function $a \in \mathcal{U}$ that satisfies the normalizing condition (1.5) and there exist $s_m \in (0,1)$ and a positive integer $\tilde{m}$ so that*

$$\langle (\Lambda_a + P_0)^s\phi_m, \phi_m\rangle < m^s,\ \langle (\Lambda_a + P_0)^{-s}\phi_{\tilde{m}}, \phi_{\tilde{m}}\rangle < \tilde{m}^{-s}$$

for any $s \in (0, s_m)$.

Section 6 is devoted to preliminary Lemmas for the proof of Theorem 5.1. We conclude the proof of Theorem 5.1 in Section 7. Both Sections 6 and 7 consist mainly in elementary and technical computations.

6. Preliminary Lemmas for the proof of Theorem 5.1

Let α_τ, $\tau \in [0,\varepsilon)$, be a C^∞-variation of $\mathbf{1}$ for some $\varepsilon > 0$. We recall the definition of the smooth functions $\phi_{m,\tau} \in C^\infty(\mathbb{S})$ and the definition of the operators $P_{0,\tau},\ \Lambda_{\alpha_\tau} : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ that depend smoothly on τ:

$$\phi_{m,\tau}(e^{i\theta}) = \frac{1}{\sqrt{2\pi}}\alpha_\tau^{-1/2}e^{im\int_0^\theta \alpha_\tau^{-1}(e^{is})ds},\ m \in \mathbb{Z},$$

and

$$P_{0,\tau} = \alpha_\tau^{-1/2}P_{e_0}\alpha_\tau^{-1/2},\ \Lambda_{\alpha_\tau} = \alpha_\tau^{1/2}\Lambda\alpha_\tau^{1/2}.$$

Here P_{e_0} is the orthogonal projection onto the line spanned by $e_0 = (2\pi)^{-1/2}$.

Straightforward computations yield the following Lemma.

Lemma 6.1. *Let $m \in \mathbb{Z}$. We have*

$$\Big(\frac{d}{d\tau}\phi_{m,\tau}\Big)_{|\tau=0} = (2\pi)^{-1/2}\Big(\frac{1}{2}f + im\int_0^\theta f\Big)e^{im\theta}, \tag{6.1}$$

$$\Big(\frac{d^2}{d\tau^2}\phi_{m,\tau}\Big)_{|\tau=0} = (2\pi)^{-1/2}\Big[\Big(\frac{1}{2}f + im\int_0^\theta f\Big)^2 + \frac{1}{2}F - \frac{1}{2}f^2 + im\int_0^\theta F\Big]e^{im\theta}, \tag{6.2}$$

where $f(\theta) = \frac{\partial\alpha_\tau^{-1}}{\partial\tau}_{|\tau=0}(\theta) = -\beta(\theta)$ and $F(\theta) = \frac{\partial^2\alpha_\tau^{-1}}{\partial\tau^2}_{|\tau=0}(\theta)$. In addition

$$(\Lambda_{\alpha_\tau})_{|\tau=0} = \Lambda,\ (P_{0,\tau})_{|\tau=0} = P_{e_0}, \tag{6.3}$$

$$\big(\frac{d}{d\tau}\Lambda_{\alpha_\tau}\big)_{|\tau=0} = -\frac{1}{2}(f\Lambda + \Lambda f),\ \big(\frac{d}{d\tau}P_{0,\tau}\big)_{|\tau=0} = \frac{1}{2}(fP_{e_0} + P_{e_0}f), \tag{6.4}$$

$$\big(\frac{d^2}{d\tau^2}\Lambda_{\alpha_\tau}\big)_{\varepsilon=0} = -\frac{1}{2}\Big(F\Lambda + \Lambda F\Big) + \frac{1}{4}(3f^2\Lambda + 2f\Lambda f + 3\Lambda f^2), \tag{6.5}$$

$$\big(\frac{d^2}{d\tau^2}P_{0,\tau}\big)_{|\tau=0} = \frac{1}{2}\Big(FP_{e_0} + P_{e_0}F\Big) + \frac{1}{4}(-f^2P_{e_0} + 2fP_{e_0}f - P_{e_0}f^2). \tag{6.6}$$

Next we consider the operator $\ln(\Lambda_{\alpha_\tau} + P_{0,\tau}) : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ that depends smoothly on τ, see Section 5.1. Smoothness in τ is understood here as $\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi \in C^\infty([0,\varepsilon)_\tau \times \mathbb{S})$ for any $\phi \in C^\infty(\mathbb{S})$. We compute the first and second derivatives of $\ln(\Lambda_{\alpha_\tau} + P_{0,\tau})$ at $\tau = 0$. We introduce the function $\rho : \mathbb{Z}^2 \times (0,+\infty) \to [0,+\infty)$ defined by

$$\rho(n,p,s) = \int_0^s \max(|n|,1)^t \max(|p|,1)^{s-t}dt,\ (n,p,s) \in \mathbb{Z}^2 \times (0,+\infty).$$

In other words

$$\rho(n,p,s) = s\max(|p|,1)^s \text{ when } \max(|n|,1) = \max(|p|,1)$$

and

$$\rho(n,p,s) = \frac{\max(|n|,1)^s - \max(|p|,1)^s}{\ln(\max(|n|,1)) - \ln(\max(|p|,1))} \text{ otherwise.} \tag{6.7}$$

We also introduce the function $h : \mathbb{N}\backslash\{0\} \times \mathbb{Z} \to [0,+\infty)$ defined by

$$h(m,p) = \frac{1}{2m^2} \text{ when } \max(|p|,1) = m,$$

and

$$h(m,p) = \frac{\ln(m) - \ln(\max(|p|,1))}{(m - \max(|p|,1))^2} - \frac{1}{m(m - \max(|p|,1))} \text{ otherwise.} \tag{6.8}$$

Lemma 6.2. *Let $(n,p) \in \mathbb{Z}^2$ and let m be a positive integer. We have*

$$\frac{1}{2\pi}\langle\frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau} + P_{0,\tau})_{|\tau=0}e^{ip\theta}, e^{in\theta}\rangle = \frac{1}{2}\rho(n,p,1)^{-1}(-|n| - |p| + \delta_n + \delta_p)\hat{f}_{n-p}, \tag{6.9}$$

$$\begin{aligned}\frac{1}{2\pi}\langle\big[\frac{\partial^2}{\partial\tau^2}\ln(\Lambda_{\alpha_\tau} + P_{0,\tau})\big]_{|\tau=0}e^{im\theta}, e^{im\theta}\rangle &= m^{-1}\Big[\frac{3m}{4\pi}\int_{\mathbb{S}} f^2 + \frac{1}{2}\sum_{p\in\mathbb{Z}}|p||\hat{f}_{m-p}|^2 + \frac{1}{2}|\hat{f}_m|^2\Big] \\ &\quad -\frac{1}{2}\sum_{p\in\mathbb{Z}} h(m,p)(|m| + |p| - \delta_p)^2|\hat{f}_{m-p}|^2.\end{aligned} \tag{6.10}$$

Here δ_n is the Kronecker symbol: $\delta_n = 1$ when $n = 0$ and $\delta_n = 0$ otherwise.

Proof of Lemma 6.2. We recall that the operator $(\Lambda_{\alpha_\tau}+P_{0,\tau})^t-(|D_{\alpha_\tau}|+P_{0,\tau})^t$ is a smoothing operator at each τ and $t \in \mathbb{R}$ [6, Lemma 3.4]. Now we take into account that

$$(|D_{\alpha_\tau}| + P_{0,\tau})^t\phi = \sum_{l\in\mathbb{Z}}(\max(|l|,1))^t\langle\phi,\phi_{l,\tau}\rangle\phi_{l,\tau} \in C^\infty([0,\varepsilon)_\tau \times \mathbb{R}_t \times \mathbb{S}, \mathbb{C}),$$

and it follows that $(\Lambda_{\alpha_\tau} + P_{0,\tau})^t\phi \in C^\infty([0,\varepsilon)_\tau \times \mathbb{R}_t \times \mathbb{S}, \mathbb{C})$ for any $\phi \in C^\infty(\mathbb{S})$. In this Section any operator is considered as a linear operator from $C^\infty(\mathbb{S})$ to $C^\infty(\mathbb{S})$ and smoothness with respect to either the t-variable or τ-variable refers to the pointwise smoothness, i.e. when the operator is applied to any $\phi \in C^\infty(\mathbb{S})$.

Moreover

$$\frac{\partial}{\partial t}(\Lambda_{\alpha_\tau} + P_{0,\tau})^t = (\Lambda_{\alpha_\tau} + P_{0,\tau})^t\ln(\Lambda_{\alpha_\tau} + P_{0,\tau}),$$

and

$$\frac{\partial}{\partial t}\big(\frac{\partial}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau})^t\big)=\big(\frac{\partial}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau})^t\big)\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})+(\Lambda_{\alpha_\tau}+P_{0,\tau})^t\frac{\partial\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})}{\partial\tau}.$$

Hence

$$\frac{\partial}{\partial t}\Big[\big(\frac{\partial}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau})^t\big)(\Lambda_{\alpha_\tau}+P_{0,\tau})^{1-t}\Big]=(\Lambda_{\alpha_\tau}+P_{0,\tau})^t\frac{\partial\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau})^{1-t}.$$

We integrate over $t\in[0,1]$

$$\int_0^1(\Lambda_{\alpha_\tau}+P_{0,\tau})^t\frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})(\Lambda_{\alpha_\tau}+P_{0,\tau})^{1-t}dt=\frac{\partial}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau}),\tag{6.11}$$

for $\tau\in[0,\varepsilon)$.

Note that

$$(\Lambda_{\alpha_\tau}+P_{0,\tau})^s_{|\tau=0}e^{il\theta}=\max(1,|l|)^se^{il\theta}$$

for any $l\in\mathbb{Z}$ and any $s\in\mathbb{R}$ by (6.3). Therefore we set $\tau=0$ on the left hand side of (6.11) and we apply it to the vector $(2\pi)^{-1/2}e^{ip\theta}$ and we take the scalar product with $(2\pi)^{-1/2}e^{in\theta}$ and we obtain by linearity of the integral over t and selfadjointness of $(\Lambda_{\alpha_\tau}+P_{0,\tau})^t$

$$\begin{aligned}
&(2\pi)^{-1}\langle\int_0^1\big[(\Lambda_{\alpha_\tau}+P_{0,\tau})^t\big]_{|\tau=0}\big[\frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\big]_{|\tau=0}\big[(\Lambda_{\alpha_\tau}+P_{0,\tau})^{1-t}\big]_{|\tau=0}dte^{ip\theta},e^{in\theta}\rangle\\
=\;&(2\pi)^{-1}\int_0^1\max(1,|p|)^{1-t}\langle\big[(\Lambda_{\alpha_\tau}+P_{0,\tau})^t\big]_{|\tau=0}\big[\frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\big]_{|\tau=0}e^{ip\theta},e^{in\theta}\rangle\\
=\;&(2\pi)^{-1}\int_0^1\max(1,|p|)^{1-t}\langle\big[\frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\big]_{|\tau=0}e^{ip\theta},(\Lambda_{\alpha_\tau}+P_{0,\tau})^te^{in\theta}\rangle dt\\
=\;&(2\pi)^{-1}\int_0^1(\max(1,|n|)^t\max(1,|p|))^{1-t}dt\langle\frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})_{|\tau=0}e^{ip\theta},e^{in\theta}\rangle.
\end{aligned}\tag{6.12}$$

Now set $\tau=0$ on the right hand side of (6.11) and apply it to the vector $(2\pi)^{-1/2}e^{ip\theta}$ and take the scalar product with $(2\pi)^{-1/2}e^{in\theta}$ and use (6.4) to obtain

$$(2\pi)^{-1}\langle\frac{\partial}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau})_{|\tau=0}e^{ip\theta},e^{in\theta}\rangle=-\frac{1}{2}(|n|+|p|)\hat f_{n-p}+\frac{1}{2}\delta_p\hat f_n+\frac{1}{2}\delta_n\hat f_{-p}.\tag{6.13}$$

We equate (6.12) and (6.13) and this gives (6.9).

We prove (6.10). First we replace the operator $\Lambda_{\alpha_\tau}+P_{0,\tau}$ by the operator $(\Lambda_{\alpha_\tau}+P_{0,\tau})^t$ in (6.11) and we obtain that

$$\frac{\partial}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau})^t=\int_0^t(\Lambda_{\alpha_\tau}+P_{0,\tau})^{r-t}\frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})(\Lambda_{\alpha_\tau}+P_{0,\tau})^rdr.$$

We repeat the same reasoning as in (6.12) and we obtain

$$\begin{aligned}
\langle\frac{\partial}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau})^t_{|\tau=0}e^{ip\theta},e^{in\theta}\rangle&=\rho(n,p,t)\langle\frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})e^{ip\theta},e^{in\theta}\rangle\\
&=\frac{1}{2}\frac{\rho(n,p,t)}{\rho(n,p,1)}(-|n|-|p|+\delta_n+\delta_p)\hat f_{n-p}.
\end{aligned}\tag{6.14}$$

Now we derive (6.11) with respect to τ and we obtain

$$\begin{aligned}
&\int_0^1 (\Lambda_{\alpha_\tau}+P_{0,\tau})^t \frac{\partial^2}{\partial\tau^2}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})(\Lambda_{\alpha_\tau}+P_{0,\tau})^{1-t}dt\\
=\ & \frac{\partial^2}{\partial\tau^2}(\Lambda_{\alpha_\tau}+P_{0,\tau}) - \int_0^1 \frac{\partial}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau})^t \frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})(\Lambda_{\alpha_\tau}+P_{0,\tau})^{1-t}dt\\
&-\int_0^1 (\Lambda_{\alpha_\tau}+P_{0,\tau})^t \frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\frac{\partial}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau})^{1-t}dt. \qquad (6.15)
\end{aligned}$$

Set $\tau=0$ on the left hand side of (6.15) and apply it to the vector $(2\pi)^{-1/2}e^{im\theta}$ and take the scalar product with $e^{im\theta}$:

$$\begin{aligned}
&(2\pi)^{-1}\langle\int_0^1 \big[(\Lambda_{\alpha_\tau}+P_{0,\tau})^t\big]_{|\tau=0}\big[\frac{\partial^2}{\partial\tau^2}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\big]_{|\tau=0}\big[(\Lambda_{\alpha_\tau}+P_{0,\tau})^{1-t}\big]_{|\tau=0}dte^{im\theta},e^{im\theta}\rangle\\
=\ & m\langle\big[\frac{\partial^2}{\partial\tau^2}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\big]_{|\tau=0}e^{im\theta},e^{im\theta}\rangle. \qquad (6.16)
\end{aligned}$$

Then we consider the first term on the right hand side of (6.15). We use (6.5) and (6.6) and we use the identity $\langle Fe^{im\theta},e^{im\theta}\rangle=\int_{\mathbb{S}}F=\big[\frac{\partial^2}{\partial\tau^2}\int_{\mathbb{S}}\alpha_\tau^{-1}\big]_{|\tau=0}=0$ by the normalizing condition (1.5), and we obtain

$$(2\pi)^{-1}\langle\frac{\partial^2}{\partial\tau^2}(\Lambda_{\alpha_\tau}+P_{0,\tau})_{|\tau=0}e^{im\theta},e^{im\theta}\rangle=\frac{1}{4}(\frac{3m}{\pi}\int_{\mathbb{S}}f^2+2\sum_p|p||\hat{f}_{m-p}|^2)+\frac{1}{2}|\hat{f}_m|^2. \quad (6.17)$$

(We also used that $P_{e_0}e^{im\theta}=0$ for the positive integer m.)

Now we consider the second and third terms on the right hand side of (6.15). Set $\tau=0$ in the second term and use (6.9) and (6.14)

$$\begin{aligned}
&(2\pi)^{-1}\langle\int_0^1 \big[\frac{\partial}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau})^t\big]_{|\tau=0}\big[\frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\big]_{|\tau=0}(\Lambda+P_0)^{1-t}dte^{im\theta},e^{im\theta}\rangle\\
&=(2\pi)^{-2}\sum_{p\in\mathbb{Z}}\int_0^1 m^{1-t}\langle\big[\frac{\partial}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau})^t\big]_{|\tau=0}e^{ip\theta},e^{im\theta}\rangle\langle\big[\frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\big]_{|\tau=0}e^{im\theta},e^{ip\theta}\rangle dt\\
=\ & \frac{m}{4}\sum_{p\in\mathbb{Z}}\rho^{-2}(m,p,1)(m+|p|-\delta_p)^2|\hat{f}_{m-p}|^2\int_0^1\rho(m,p,t)m^{-t}dt\\
=\ & \frac{m}{4}\sum_{p\in\mathbb{Z}}\frac{\tilde{\rho}(m,p)}{\rho^2(m,p,1)}(m+|p|-\delta_p)^2|\hat{f}_{m-p}|^2. \qquad (6.18)
\end{aligned}$$

Here we introduced the function $\tilde{\rho}:\mathbb{N}\backslash\{0\}\times\mathbb{Z}\to[0,+\infty)$ defined by

$$\tilde{\rho}(m,p)=\int_0^1\rho(m,p,s)m^{-s}ds,\ m\in\mathbb{N}\backslash\{0\},\ p\in\mathbb{Z}.$$

From (6.7) it follows that

$$\tilde{\rho}(m,p)=\frac{1}{2}\text{ when }\max(|p|,1)=m,$$

and

$$\tilde{\rho}(m,p)=\frac{1}{\ln(m)-\ln(\max(|p|,1))}+\frac{\max(|p|,1)-m}{m(\ln(m)-\ln(\max(|p|,1)))^2}\text{ otherwise.}$$

From (6.8) it follows that

$$\frac{\tilde\rho(m,p)}{\rho^2(m,p,1)} = h(m,p),\ p\in\mathbb{Z}.$$

We similarly deal with the third term and we obtain

$$\begin{aligned}(2\pi)^{-1}\langle\int_0^1 \big[(\Lambda_{\alpha_\tau}+P_{0,\tau})^t\big]_{|\tau=0}\big[\frac{\partial}{\partial\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\big]_{|\tau=0}\big[\frac{\partial}{\partial\tau}(\Lambda_{\alpha_\tau}+P_{0,\tau})^{1-t}\big]_{|\tau=0}dt e^{im\theta}, e^{im\theta}\rangle\\ = \frac{m}{4}\sum_{p\in\mathbb{Z}} h(m,p)(|m|+|p|-\delta_p)^2|\hat f_{m-p}|^2. \qquad (6.19)\end{aligned}$$

We collect (6.15)–(6.19) and we use (6.7) and (6.8) and we obtain (6.10).

□

7. Proof of Theorem 5.1

First we write

$$\begin{aligned}\frac{d}{d\tau}\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{m,\tau},\phi_{m,\tau}\rangle &= \langle\big(\frac{d}{d\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\big)\phi_{m,\tau},\phi_{m,\tau}\rangle\\ &\quad +2\Re\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\frac{d}{d\tau}\phi_{m,\tau},\phi_{m,\tau}\rangle.\end{aligned}$$

And we set $\tau=0$, and we use (6.1) and (6.9) and we use the identities $\phi_{m,0}=(2\pi)^{-1/2}e^{im\theta}$, $2\pi\hat f_0=\int_0^{2\pi} f=0$ and we obtain (5.2). (The imaginary term $im\int_0^\theta f$ on the right hand side of (6.9) is disregarded when we take the real part.)

Now we prove (5.3). We derive once more in τ the above formula and we obtain

$$\begin{aligned}&\frac{d^2}{d\tau^2}\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{m,\tau},\phi_{m,\tau}\rangle\\ =\ &\langle\big(\frac{d^2}{d\tau^2}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\big)\phi_{m,\tau},\phi_{m,\tau}\rangle + 4\Re\langle\frac{d}{d\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\frac{d}{d\tau}\phi_{m,\tau},\phi_{m,\tau}\rangle \qquad (7.1)\\ &+2\Re\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\frac{d^2}{d\tau^2}\phi_{m,\tau},\phi_{m,\tau}\rangle + 2\Re\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\frac{d}{d\tau}\phi_{m,\tau},\frac{d}{d\tau}\phi_{m,\tau}\rangle\end{aligned}$$

Let us compute the three last terms of the right hand side of (7.1) at $\tau=0$. We use (6.1) for the fourth term and we obtain

$$\begin{aligned}\big[\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\frac{d}{d\tau}\phi_{m,\tau},\frac{d}{d\tau}\phi_{m,\tau}\rangle\big]_{|\tau=0} &= (2\pi)^{-1}\sum_{p\in\mathbb{Z}}\ln(\max(|p|,1))|\langle e^{ip\theta},\frac{d}{d\tau}\phi_{m,\tau}\rangle|^2_{\tau=0}\\ &= \sum_{p\in\mathbb{Z}\setminus\{0\}}\ln(|p|)\Big|\frac{1}{2}\hat f_{p-m}+im\widehat{\big(\int_0^\theta f\big)}_{p-m}\Big|^2. \qquad (7.2)\end{aligned}$$

We use (6.2) for the third term and we obtain

$$\begin{aligned}&\big[\Re\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\frac{d^2}{d\tau^2}\phi_{m,\tau},\phi_{m,\tau}\rangle\big]_{|\tau=0} = (2\pi)^{-1/2}\ln(|m|)\Re\langle\big(\frac{d^2}{d\tau^2}\phi_{m,\tau}\big)_{|\tau=0},e^{im\theta}\rangle\\ =\ &(2\pi)^{-1}\ln(|m|)\Re\int_0^{2\pi}\Big[\big(\frac{1}{2}f+im\int_0^\theta f\big)^2+\frac{1}{2}F-\frac{1}{2}f^2+im\int_0^\theta F\Big]d\theta\\ =\ &(2\pi)^{-1}\ln(|m|)\int_0^{2\pi}\Big[-\frac{1}{4}f^2-m^2\big(\int_0^\theta f\big)^2\Big]d\theta. \qquad (7.3)\end{aligned}$$

(We used that $\int_{\mathbb{S}} f = \int_{\mathbb{S}} F = 0$). We use (6.1) and (6.9) for the second term and we obtain

$$\begin{aligned}
&\Re\langle \frac{d}{d\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\frac{d}{d\tau}\phi_{m,\tau},\phi_{m,\tau}\rangle_{|\tau=0}\\
=\;&(2\pi)^{-1}\Re\langle\big(\frac{d}{d\tau}\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\big)_{\tau=0}(\frac{1}{2}f+im\int_0^\theta f)e^{im\theta},e^{im\theta}\rangle\\
=\;&(4\pi)^{-1}\Re\sum_{p\in\mathbb{Z}}\frac{(-m-|p|+\delta_p)}{\rho(m,p,1)}\hat{f}_{m-p}\langle(\frac{1}{2}f+im\int_0^\theta f)e^{im\theta},e^{ip\theta}\rangle\\
=\;&\frac{1}{2}\Re\sum_{p\in\mathbb{Z}}\frac{(-m-|p|+\delta_p)}{\rho(m,p,1)}\big(\frac{1}{2}|\hat{f}_{m-p}|^2+im\hat{f}_{m-p}\widehat{\big(\int_0^\theta f\big)}_{p-m}\big).
\end{aligned}\tag{7.4}$$

The first term on the right hand side of (7.1) is given by (6.10) at $\tau=0$.
We collect (7.1), (7.2), (7.3), (7.4) and (6.10) and we obtain

$$\begin{aligned}
&\Big[\frac{d^2}{d\tau^2}\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{m,\tau},\phi_{m,\tau}\rangle\Big]_{|\tau=0}\\
&=m^{-1}\Big[\frac{3m}{4\pi}\int_{\mathbb{S}}f^2+\frac{1}{2}\sum_p|p||\hat{f}_{m-p}|^2+\frac{1}{2}|\hat{f}_m|^2\Big]-\frac{1}{2}\sum_{p\in\mathbb{Z}}h(m,p)(m+|p|-\delta_p)^2|\hat{f}_{m-p}|^2\\
&+2\sum_{p\in\mathbb{Z}\setminus\{0\}}\ln(|p|)\Big|\frac{1}{2}\hat{f}_{p-m}+im\widehat{\big(\int_0^\theta f\big)}_{p-m}\Big|^2+\pi^{-1}\ln(m)\int_0^{2\pi}\Big[-\frac{1}{4}f^2-m^2\big(\int_0^\theta f\big)^2\Big]d\theta\\
&+2\Re\sum_{p\in\mathbb{Z}}\frac{(-m-|p|+\delta_p)}{\rho(m,p,1)}\big(\frac{1}{2}|\hat{f}_{m-p}|^2+im\hat{f}_{m-p}\widehat{\big(\int_0^\theta f\big)}_{p-m}\big).
\end{aligned}\tag{7.5}$$

From now on every computations intend to simplify the above formula. We introduce the function G

$$G(\theta)=\int_0^\theta f,\;\hat{f}_k=ik\hat{G}_k,\;k\in\mathbb{Z}.$$

Then (7.5) becomes

$$\begin{aligned}
&\Big[\frac{d^2}{d\tau^2}\langle\ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{m,\tau},\phi_{m,\tau}\rangle\Big]_{|\tau=0}\\
=\;&m^{-1}\Big[\frac{3m}{4\pi}\int_{\mathbb{S}}f^2+\frac{1}{2}\sum_p|p|(m-p)^2|\hat{G}_{m-p}|^2)+\frac{1}{2}m^2|\hat{G}_m|^2\Big]\\
&-\frac{1}{2}\sum_{p\in\mathbb{Z}}h(m,p)(m+|p|-\delta_p)^2(m-p)^2|\hat{G}_{m-p}|^2\\
&+\frac{1}{2}\sum_{p\in\mathbb{Z}\setminus\{0\}}(p+m)^2\ln(|p|)|\hat{G}_{p-m}|^2+\pi^{-1}\ln(m)\int_0^{2\pi}\Big[-\frac{1}{4}f^2-m^2G^2\Big]d\theta\\
&+\sum_{p\in\mathbb{Z}}\frac{m+|p|-\delta_p}{\rho(m,p,1)}(m-p)(m+p)|\hat{G}_{m-p}|^2.
\end{aligned}$$

Next we change p in $-p$ and we use the identities $\int_S f^2 = 2\pi \sum_{p\in\mathbb{Z}} (m+p)^2 |\hat{G}_{m+p}|^2$ and $\int_S G^2 = 2\pi \sum_{p\in\mathbb{Z}} |\hat{G}_{m+p}|^2$ and we obtain

$$\begin{aligned}
&\Big[\frac{d^2}{d\tau^2}\langle \ln(\Lambda_{\alpha_\tau} + P_{0,\tau})\phi_{m,\tau}, \phi_{m,\tau}\rangle\Big]_{|\tau=0} \\
= \;& \sum_{p\in\mathbb{Z}} \big(\frac{3}{2} + \frac{1}{2m}|p|\big)(m+p)^2 |\hat{G}_{m+p}|^2 + \frac{1}{2} m |\hat{G}_m|^2 \\
& - \frac{1}{2}\sum_{p\in\mathbb{Z}} h(m,p)(m+|p|-\delta_p)^2 (m+p)^2 |\hat{G}_{m+p}|^2 \\
& + \frac{1}{2}\sum_{p\in\mathbb{Z}\setminus\{0\}} (m-p)^2 \ln(|p|) |\hat{G}_{m+p}|^2 - 2\ln(m)\sum_{p\in\mathbb{Z}} \big(\frac{1}{4}(m+p)^2 + m^2\big)|\hat{G}_{m+p}|^2 \\
& + \sum_{p\in\mathbb{Z}} \frac{m+|p|-\delta_p}{\rho(m,p,1)}(m-p)(m+p)|\hat{G}_{m+p}|^2. \qquad (7.6)
\end{aligned}$$

The contribution over negative integers p is

$$\begin{aligned}
&\sum_{p<0} |\hat{G}_{m+p}|^2 \Big[\big(\frac{3}{2} - \frac{1}{2m}p\big)(m+p)^2 - \frac{1}{2}h(m,p)(m-p)^2(m+p)^2 \\
&+ \frac{1}{2}(m-p)^2 \ln(|p|) - \frac{1}{2}\ln(m)\big(5m^2 + 2mp + p^2\big) + \frac{(m-p)^2(m+p)}{\rho(m,p,1)}\Big].
\end{aligned}$$

We sustitute the values for ρ and h (6.7), (6.8). The contribution over the negative integers p becomes

$$\begin{aligned}
&\sum_{p<0} |\hat{G}_{m+p}|^2 \Big[\big(\frac{3}{2} - \frac{1}{2m}p\big)(m+p)^2 - \frac{1}{2}(m-p)^2(\ln m - \ln|p|) \\
&+ \frac{1}{2m}(m-p)^2(m+p) + \frac{1}{2}(m-p)^2\ln(|p|) - \frac{1}{2}\ln(m)\big(5m^2+2mp+p^2\big) \\
&+ (m-p)^2(\ln m - \ln|p|)\Big] \\
= \;& 2m(-\ln(m)+1)\sum_{p<0}(m+p)|\hat{G}_{m+p}|^2. \qquad (7.7)
\end{aligned}$$

Then we note that

$$0 = \frac{1}{2}\int_0^{2\pi} (G^2)' = \int_0^{2\pi} fG$$

which is written in Fourier series as

$$m|\hat{G}_m|^2 + \sum_{p<0}(m+p)|\hat{G}_{m+p}|^2 + \sum_{p>0}(m+p)|\hat{G}_{m+p}|^2 = 0. \qquad (7.8)$$

Hence we combine (7.7) and (7.8) and the contribution in (7.6) over the negative integers is given by

$$2m^2(\ln(m)-1)|\hat{G}_m|^2 + 2m(\ln(m)-1)\sum_{p>0}(m+p)|\hat{G}_{m+p}|^2. \qquad (7.9)$$

Now let us look at the contribution when $p=0$ in (7.6). We split this case in two: When $m=1$ and when $m>1$. First when $(m,p)=(1,0)$ the contribution in (7.6) is

$$|\hat{G}_1|^2(\frac{3}{2}+\frac{1}{2}) = 2|\hat{G}_1|^2. \qquad (7.10)$$

Next when $p = 0$ and $m > 1$ the contribution of p in (7.6) is given by

$$|\hat{G}_m|^2\big[\frac{3}{2}m^2 + \frac{1}{2}m - \frac{1}{2}h(m,0)(m-1)^2m^2 - \frac{5}{2}\ln(m)m^2 + m^2\frac{m-1}{\rho(m,0,1)}\big].$$

We sustitute the value for $h(m,0) = \frac{\ln(m)}{(m-1)^2} - \frac{1}{m(m-1)}$ and $\rho(m,0,1) = \frac{m-1}{\ln(m)}$, see (6.7)–(6.8), and the contribution becomes

$$\begin{aligned} & |\hat{G}_m|^2\big[\frac{3}{2}m^2 + \frac{1}{2}m - \frac{1}{2}\ln(m)m^2 + \frac{1}{2}m(m-1) - \frac{5}{2}\ln(m)m^2 + m^2\ln(m)\big] \\ = \;& 2m^2(1-\ln(m))|\hat{G}_m|^2. \end{aligned} \tag{7.11}$$

Then setting $m = 1$ in (7.11) yields the same contribution as (7.10).

Next we substitute (7.9), (7.11) into (7.6) and we obtain

$$\begin{aligned} & \Big[\frac{d^2}{d\tau^2}\langle\ln(\Lambda_{\alpha_\tau} + P_{0,\tau})\phi_{m,\tau}, \phi_{m,\tau}\rangle\Big]_{|\tau=0} \\ = \;& 2m(\ln(m)-1)\sum_{p=1}^{\infty}(m+p)|\hat{G}_{m+p}|^2 \\ & + \sum_{p=1}^{\infty}|\hat{G}_{m+p}|^2\big[(\frac{3}{2} + \frac{1}{2m}p\big)(m+p)^2 - \frac{1}{2}h(m,p)(m+p)^4 \\ & + \frac{1}{2}(m-p)^2\ln(p) - 2\ln(m)\big(\frac{1}{4}(m+p)^2 + m^2\big) + \frac{(m+p)^2}{\rho(m,p,1)}(m-p)\big]. \end{aligned} \tag{7.12}$$

Let us look at the contribution of the m-th summand (when $p = m$): This is given by

$$\begin{aligned} & |\hat{G}_{2m}|^2\big(4m^2(\ln(m)-1) + (\frac{3}{2}+\frac{1}{2})4m^2 - 8h(m,m)m^4 - 4\ln(m)m^2\big) \\ = \;& |\hat{G}_{2m}|^2\big(4m^2(\ln(m)-1) + (\frac{3}{2}+\frac{1}{2})4m^2 - 4m^2 - 4\ln(m)m^2\big) = 0. \end{aligned}$$

Therefore we rewrite (7.12) as

$$\begin{aligned} & \Big[\frac{d^2}{d\tau^2}\langle\ln(\Lambda_{\alpha_\tau} + P_{0,\tau})\phi_{m,\tau}, \phi_{m,\tau}\rangle\Big]_{|\tau=0} \\ = \;& \sum_{p\in\mathbb{N},\ p>0, p\neq m}|\hat{G}_{m+p}|^2\big[2m(\ln(m)-1)(m+p) + (\frac{3}{2}+\frac{1}{2m}p)(m+p)^2 \\ & -\frac{1}{2}h(m,p)(m+p)^4 + \frac{1}{2}(m-p)^2\ln(p) \\ & -2\ln(m)\big(\frac{1}{4}(m+p)^2 + m^2\big) + \frac{(m+p)^2}{\rho(m,p,1)}(m-p)\big]. \end{aligned}$$

Then we replace ρ and h by their definitions (6.7) and (6.8) and we obtain

$$\begin{aligned} & \Big[\frac{d^2}{d\tau^2}\langle\ln(\Lambda_{\alpha_\tau} + P_{0,\tau})\phi_{m,\tau}, \phi_{m,\tau}\rangle\Big]_{|\tau=0} \\ = \;& \sum_{p\in\mathbb{N},\ p>0, p\neq m}|\hat{G}_{m+p}|^2\big[2m(\ln(m)-1)(m+p) \\ & +(\frac{3}{2}+\frac{1}{2m}p)(m+p)^2 - \frac{1}{2}(\frac{\ln m - \ln p}{(m-p)^2} - \frac{1}{m(m-p)})(m+p)^4 \\ & +\frac{1}{2}(m-p)^2\ln(p) - 2\ln(m)\big(\frac{1}{4}(m+p)^2 + m^2\big) + (\ln m - \ln p)(m+p)^2\big]. \end{aligned}$$

We regroup the 2 lonely terms in $\ln(m)$ and we obtain

$$\begin{aligned}
&\Big[\frac{d^2}{d\tau^2}\langle \ln(\Lambda_{\alpha_\tau}+P_{0,\tau})\phi_{m,\tau},\phi_{m,\tau}\rangle\Big]_{|\tau=0}\\
=&\sum_{p\in\mathbb{N},\ p>0,p\neq m}|\hat{G}_{m+p}|^2\big[-2m(m+p)+(\frac{3}{2}+\frac{1}{2m}p)(m+p)^2\\
&-\frac{1}{2}(\frac{\ln m-\ln p}{(m-p)^2}-\frac{1}{m(m-p)})(m+p)^4\\
&+\frac{1}{2}(m-p)^2(\ln(p)-\ln(m))+(\ln m-\ln p)(m+p)^2\big].
\end{aligned}$$

Further elementary computations give (5.3). In particular we regroup the terms in $\ln(p)-\ln(m)$ and we easily obtain that the coefficient in front of this difference is given by

$$\frac{8p^2m^2}{(m-p)^2}|\hat{G}_{m+p}|^2=\frac{8p^2m^2}{(m-p)^2(m+p)^2}|\hat{f}_{m+p}|^2=\frac{8p^2m^2}{(m-p)^2(m+p)^2}|\hat{\beta}_{m+p}|^2.$$

Regrouping the others terms yields

$$4pm\frac{m+p}{m-p}|\hat{G}_{m+p}|^2=4pm\frac{(m+p)(m-p)}{(m-p)^2(m+p)^2}|\hat{\beta}_{m+p}|^2.$$

□

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