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ON REGULAR CURVES HOMEOMORPHISMS WITHOUT PERIODIC POINTS

AYMEN DAGHAR

ABSTRACT. The purpose of this paper is to study the dynamic of regular curves homeomorphisms without periodic points. We show mainly that they behave similarly like circle's homeomorphisms without periodic points. For instance, we prove that they are extensions of irrational rotation of the circle via a monotone factor map collapsing proximal pairs and we prove also the absence of Li-Yorke pairs. Furthermore, we give a characterisation of minimal sets, in particular we get that the circle is the only regular curve admitting a minimal (or a transitive) \mathbb{Z} -action. At the end of the paper, we give some counter-examples on rational curves.

1. Introduction

A *continuum* is a non-empty metric compact connected set. A continuum is a *regular curve* if any point has an ϵ -open neighborhood with finite boundary, for any $\epsilon > 0$. Due to the rim-finite property, this type of continua are known to be a 1-dimensional Peano-continua (see [9]). Recall that the class of regular curve is a larger class than completely regular continua, graph and local dendrite.

Recently many authors were interested on \mathbb{Z} -actions on regular curves. In [19] Seilder proved that those systems have zero topological entropy and in [15] Glasner and Megrelishvili proved that they are even null (i.e their sequence topological entropy is zero). In [10, 11] Naghmouchi showed that any ω -limit set is minimal and the absence of periodic point allows the presence for one and only one minimal set that coincides with the set of non-wandering points. In [3], we proved that infinite ω -limit sets have adding machine structure if we allow periodic points to exist, moreover we proved the absence of Li-Yorke pairs in this case. In the case of the circle, it is known that the absence of periodic points is enough to have an irrational rotation factor via a monotone factor map (and if the system is minimal then a conjugation occurs). It is also worth mentioning that the absence of proper proximal pair characterizes the conjugation with an irrational rotation (for more on this see [6, 4]). In the present paper, we will generalize this characterization to regular curves and we show that many dynamical

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properties displayed by regular curves homeomorphisms are false in general if we replace regular curve by a rational ones.

Plan of the paper: In section 2, we review some results already known that will be used later on. In section 3, we prove that the proximal pair relation is a closed monotone equivalence relation which will enable us to prove that the quotient map semi-conjugate the system to an irrational rotation and the absence of Li-York pair. Section 4 will be devoted to give a topological characterisation of the unique minimal set and in section 5 we prove the existence of an invariant simple closed curve. In section 6, we give some examples of \mathbb{Z} -action on rational curve where the results proven here and in [10, 11] are no more longer true. We also give an example of a \mathbb{Z} -action on rational continuum with a positive sequence topological entropy.

2. Preliminaries

Let X be a compact metric space. The closure (respectively, the boundary set) of a subset A of X is denoted by \bar{A} (respectively $\partial(A)$). We denote with 2^X (respectively $C(X)$) the set of all non-empty compact subsets (respectively compact connected subsets) of X and we endow them with the Hausdorff metric d_H defined as follows:

$$d_H(A, B) = \max(\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A))$$

where $A, B \in 2^X$ and $d(x, M) = \inf_{y \in M} d(x, y)$ for any $x \in X$ and $M \in 2^X$. Notice that both $(C(X), d_H)$ and $(2^X, d_H)$ are compact metric space (for more details see [16]).

Let $A \subset X$ we define $diam(A) = \sup_{x, y \in A} d(x, y)$.

A dynamical system is a pair (X, f) , where X is a compact metric space and $f : X \rightarrow X$ is a continuous map. Let (X, f) be a dynamical system. The ω -limit set of a given point $x \in X$ is defined as follow:

$$\omega_f(x) = \overline{\bigcap_{n \in \mathbb{N}} \{f^k(x) : k \geq n\}}$$

$$= \{y \in X : \exists n_1 < n_2 < \dots : \lim_{i \rightarrow +\infty} d(f^{n_i}(x), y) = 0\}$$

If f is a homeomorphism then the full orbit (resp. the backward orbit)(under f) of a given point $x \in X$ is $O_f(x) := \{f^n(x) : n \in \mathbb{Z}\}$ (resp. $O_f^-(x) := \{f^n(x) : n \in \mathbb{Z}_-\}$). The α -limit set of x is $\alpha_f(x) := \omega_{f^{-1}}(x)$.

A point x in X is called

- *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$,

- *wandering* for if there exists some neighborhood U , of x such that $f^{-n}(U) \cap U = \emptyset$, for every $n \in \mathbb{N}$. Otherwise, the point x is said to be *non-wandering*.

We denote by $P(f)$ and $\Omega(f)$ the sets of periodic points and the non-wandering points of f respectively. The forward orbit of a given point $x \in X$ is the subset $\{f^n(x) : n \in \mathbb{Z}_+\}$. A subset M is called *minimal* if it is non-empty, closed f -invariant (*i.e.* $f(A) \subset A$) and there is no proper subset of M having these properties. A subset M of X is called *totally minimal* if it is f^n -minimal for all $n \in \mathbb{N}$. The system (X, f) is said to be transitive if for any two non-empty open sets U, V of X there exists $n \in \mathbb{N}$ such that $f^{-n}(U) \cap V \neq \emptyset$.

A pair $(x, y) \in X \times X$ is called *proximal* if $\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0$ otherwise it is called *distal*. If $\limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0$, then (x, y) is called *asymptotic*. A pair (x, y) is called a *Li-Yorke pair* if it is proximal but not asymptotic. The dynamical system (X, f) is called *almost distal* if it has no Li-Yorke pair. A pair $(x, y) \in X \times X$ is called *regionally proximal* if for each $\epsilon > 0$ and any pair of open set O_x, O_y containing respectively x and y there exist $x' \in O_x, y' \in O_y$ and some $n \in \mathbb{N}$ such that $d(f^n(x'), f^n(y')) \leq \epsilon$.

Denote by $P(X, f)$ the subset of $X \times X$ of proximal pair of (X, f) , $A(X, f)$ the subset of $X \times X$ of asymptotic pair of (X, f) and by $RP(X, f)$ the set of regionally proximal pair. It is easy to see that if $(x, y) \in A(X, f)$ and $(z, y) \in P(X, f)$ then $(x, z) \in P(X, f)$, clearly $P(X, f)$ and $A(X, f)$ are $f \times f$ invariant.

Let (X, f) and (Y, g) be two dynamical systems. If there is a homeomorphism $\pi : X \rightarrow Y$ which intertwines the actions (*i.e.*, $\pi \circ f = g \circ \pi$), then we say that (X, f) and (Y, g) are (*topologically*) *conjugated*, if π is only onto continuous then we say that (X, f) and (Y, g) are (*topologically*) *semi-conjugated* or (X, f) is an extension of (Y, g) .

By a *continuum* we mean a compact connected metric space. An *arc* is a continuum homeomorphic to the closed interval $[0, 1]$. A *simple closed curve* is a continuum homeomorphic to the circle S^1 . Let X be continuum and let α be a cardinal number such that $\alpha \leq \Omega$, where Ω is the cardinality of \mathbb{R} or the ordinal number ω . Let $x \in X$, we say that x is of order less or equal to α in X provided that for each open subset U containing x , there exists an open neighborhood V of x such that $V \subset U$ and $\text{card}(\partial(V)) \leq \alpha$. We say that x is of order α in X if x is of order $\leq \alpha$ and for any cardinal number $\gamma < \alpha$, x is not of order $\leq \gamma$. We denote by X^α (resp. X_α) the set of points of order $\leq \alpha$ (resp. of order α) in X .

Note that $\text{card}(\partial(V)) \leq \omega$, mean that $\partial(V)$ is finite. A point of order 1 in X is said to be an end point of X .

A *regular curve* is a continuum such that $X = X^\omega$. A *finitely suslinean continuum* is a continuum such that for any infinite pairwise disjoint family

of sub-continuum $(C_n)_{n \geq 0}$ we have $\lim_{n \rightarrow +\infty} \text{diam}(C_n) = 0$. A *hereditary locally connected continuum* is a continuum such that any sub-continuum is locally connected.

Recall that each regular curve is finitely suslinean (see [14] 1.6) and hence hereditary locally connected and a hereditary locally connected continuum is an uniformly arcwise connected and locally arcwise connected continuum (for more details see [16] and [9]). A map $\phi : X \rightarrow Y$ is said to be *pointwise monotone* if for any $y \in Y$ the set $\phi^{-1}(y)$ is a connected subset of X and *monotone* if for any C connected subset of Y we have $\phi^{-1}(C)$ is a connected subset of X .

Here in this Theorem we review some known results for the dynamic of regular curves homeomorphisms.

Theorem 2.1. (Naghmouchi [10, 11]) *Let (X, f) be a dynamical system such that $f : X \rightarrow X$ is a homeomorphism of a regular curve X then we have the following assertions:*

- (i) $\forall x \in X$, $\omega_f(x)$ as well as $\alpha_f(x)$ is minimal.
- (ii) If $P(f) = \emptyset$ then f has an unique minimal set $M = \Omega(f)$ and f is totally minimal on M .

Recall that a \mathbb{Z} -action on X generated by a self homeomorphism f of X is said to be *equicontinuous* if the family $\{f^n, n \in \mathbb{Z}\}$ is equicontinuous.

Theorem 2.2. (Qiu, Zhao, [13]) *Any distal null system is equicontinuous.*

Theorem 2.3. (Glasner, Megrelishvili, [15]) *Every action of a group G on a regular curve is null. In particular, every \mathbb{Z} -action on a regular curve is null.*

Remark 2.4. As a direct consequence from this two results, any distal system on a regular curve generated by a homeomorphism is equicontinuous.

3. On the proximal relation

In this section, we are going to prove that the proximal relation is a closed invariant equivalence relation and the naturel induced map is monotone for any regular curve homeomorphism without periodic points. In fact, by ([1] corollary 1), it suffices to prove that the set of proximal pair is a closed subset of $X \times X$.

Theorem 3.1. *Let (X, f) be a dynamical system where X is a regular curve and $f : X \rightarrow X$ is a homeomorphism without periodic point. Then $P(X, f)$ is a closed $f \times f$ -invariant equivalence relation and the quotient map π is monotone.*

We will need the following lemma:

Lemma 3.2. *Let M be the unique minimal set of (X, f) , if $M \subsetneq X$ then we have the following assertions:*

(i) $X \setminus M$ is not connected.

(ii) For each connected component C of $X \setminus M$, we have $\overline{C} \times \overline{C} \subset A(X, f)$.

Proof. (i) If $X \setminus M$ is connected, then $(X \setminus M) \times (X \setminus M) \subset A(X, f)$. Indeed, fix a point $x \in X \setminus M$ and denote by A the set of points in $X \setminus M$ asymptotic to x . Clearly, A is non empty and we will show that it is a clopen subset of $X \setminus M$. It is easy to see that A is open in X . In fact, any point $y \in A$ is wandering thus it admits a wandering neighborhood and as X is a regular curve we may choose this neighborhood to be connected and so this neighborhood is a part from the asymptotic class of y .

Let $(y_n)_{n \geq 0}$ be a sequence in A converging to a point y in $X \setminus M$. Similarly as above we can find an open wandering connected neighborhood O of y in X , so $\lim_{|n| \rightarrow +\infty} \text{diam}(f^n(O)) = 0$. For some $N \in \mathbb{N}$, $y_N \in O$ and then (y_N, y) is an asymptotic pair and so does (x, y) which implies that $y \in A$. Therefore A is a closed subset of $X \setminus M$. It turns out that $X \setminus M = A$ is f -invariant, in particular $f(x) \in A$ which is a contradiction since f has no periodic points.

(ii) Let C be a connected component of $X \setminus M$ and assume that $f^k(C)$ meets C for some integer $k > 0$, then $f^k(C) = C$ and so \overline{C} is a subcontinuum of X f^k -invariant. Since C is a connected component of $X \setminus M$ then $\partial(C) \subset M = \partial(X \setminus M)$. By Theorem 2.1, M is also the unique minimal set of f^k then $\overline{C} = C \cup M$ which implies that $f^k_{\overline{C}}$ is a regular curve homeomorphism of \overline{C} without periodic point such that $\overline{C} \setminus M = C$ is connected which gives a contradiction with assertion (i). It follows that the family $(f^n(C))_{n \in \mathbb{Z}}$ is pairwise disjoint, hence $\lim_{|n| \rightarrow +\infty} \text{diam}(f^n(C)) = 0$. \square

Proof of theorem. 3.1. First we prove that $P(X, f)$ is closed in $X \times X$. If $M = X$ is minimal then by Theorems 4.4 and 4.3 in [13], $P(X, f)$ is closed in $X \times X$. Now suppose that $M \subsetneq X$. Again by ([13] theorems 4.4 and 4.3), $P(X, f) \cap (M \times M) = P(M, f)$ is closed in $X \times X$. Let $(x_n, y_n)_{n \geq 0}$ be a sequence in $P(X, f)$ that converges to (x, y) with respect to the product topology, we are going to prove that $(x, y) \in P(X, f)$, we distinguish three cases:

Case 1. $(x, y) \in (X \setminus M) \times (X \setminus M)$: Let C_x and C_y be respectively the connected components of x and y in $X \setminus M$. There exists $N > 0$ such that $(x_N, y_N) \in C_x \times C_y$ and then by Lemma 3.2, $(x, x_N) \in A(X, f)$ and $(y, y_N) \in A(X, f)$ which implies that $(x, y) \in P(X, f)$.

Case 2. $(x, y) \in M \times M$: For any $n \geq 0$, let $c_n \in M$ (resp. $d_n \in M$) be such that $d(x_n, M) = d(x_n, c_n)$ (resp. $d(y_n, M) = d(y_n, d_n)$). By

([16], Theorem 4, page 257), there is a sequence of arcs $(I_n)_{n \geq 0}$ (resp. $(J_n)_{n \geq 0}$) joining x_n and c_n (resp. y_n and d_n) such that $\lim_{n \rightarrow +\infty} \text{diam}(I_n) = \lim_{n \rightarrow +\infty} \text{diam}(J_n) = 0$. For each n , let $a_n \in I_n \cap M$ (resp. $b_n \in J_n \cap M$) such that a_n (resp. b_n) is the first point of I_n (resp. J_n) meeting M if we start from x_n (resp. y_n). By Lemma 3.2, both of pairs (a_n, x_n) and (b_n, y_n) are in $A(X, f)$. It follows that $((a_n, b_n))_{n \geq 0}$ is a sequence in $P(M, f)$ that converges to (x, y) . Therefore $(x, y) \in P(M, f)$.

Case 3. $x \in M$ and $y \notin M$: Let C be the connected component of y in $X \setminus M$. As C is an open subset of X , it contains y_n eventually. Recall that $\overline{C} \cap M \neq \emptyset$ and $\overline{C} \times \overline{C} \subset A(X, f)$ (see Lemma 3.2). Pick any point $z \in M \cap \overline{C}$ then obviously, $(z, x_n) \in P(X, f)$ for each n eventually. By case 2, $(z, x) \in P(M, f)$ and as $(z, y) \in A(X, f)$, we get $(x, y) \in P(X, f)$.

Consequently, $P(X, f)$ is an equivalence relation in X with closed graph. It follows that π the naturel quotient map is a closed map from X to $\{\pi(x), x \in X\}$ endowed with the decomposition topology, thus $\tilde{X} = \pi(X)$ endowed with the decomposition topology is a continuum (see [16] Proposition 3.7 and Theorem 3.10). Now we are going to prove that π is monotone:

Let $x \in X$. If $\pi(x)$ is reduced to a point or the whole space then there is nothing to prove, so assume that $\pi(x)$ is a proper non degenerate subset of X . Recall that π is a closed map, then $\pi(x)$ is a compact subset of X . Suppose that $\pi(x)$ is not connected then there exists a point $y \in \pi(x) \setminus \{x\}$ such that the connected components C_x and C_y of $\pi(x)$ containing respectively x and y are disjoint. Then we can find an open set U of X with finite boundary such that $\pi(x) \subset U$ and for any arc I of X joining x, y we have $I \not\subset U$. Indeed, since $\pi(x)$ is a compact subset of the regular curve X we can find a decreasing sequence of open sets $(O_n)_{n \geq 0}$ with finite boundaries such that $\pi(x) = \bigcap_{n \geq 0} O_n$. If for infinitely many $n \geq 0$ there exists an arc I_n joining

x, y such that $I_n \subset O_n$, then there is a subcontinuum $I \subset \pi(x)$ containing x and y (it suffices to select a suitable converging subsequence of $(I_n)_{n \geq 0}$), a contradiction. Hence for some N , for any arc I of X joining x, y we have $I \not\subset O_n$ for any $n > N$. Consider then $U = O_N$. Now since the pair (x, y) is proximal we can find a sequence $(m_i)_{i \in \mathbb{N}}$ of positive integer and a point $c \in X$ such that $\lim_{i \rightarrow +\infty} f^{m_i}(\{x, y\}) = c$. By ([9], Theorem 4, page 257) we can find a sequence of arcs I_i joining $f^{m_i}(x)$ and $f^{m_i}(y)$ such that $\lim_{i \rightarrow +\infty} I_i = c$, with

respect to the Hausdorff metric. We have $f^{-m_i}(I_i)$ is an arc joining x and y then it should meet the finite set $\partial(U)$, hence there exists a point $z \in \partial(U)$ such that $z \in f^{-m_i}(I_i)$ for infinitely many $i \geq 0$ and then $f^{m_i}(z) \in I_i$ for infinitely many $i \geq 0$. Therefore a subsequence of $(f^{m_i}(z))_{i \geq 0}$ converges to c . Thus the pair (z, x) is proximal hence $z \in \pi(x) \cap \partial(U)$ which is a contradiction with the fact that $\pi(x) \subset U$.

In conclusion, the map π is an onto continuous pointwise monotone and so it is monotone by ([9] Theorem 9, page 131). \square

Corollary 3.3. *Let (X, f) be a dynamical system where X is a regular curve and $f : X \rightarrow X$ is a homeomorphism without periodic point then the following hold:*

- (i) $P(X, f) = P(X, f^{-1}) = A(X, f) = A(X, f^{-1})$.
- (ii) *There is no Li-yorke pair for both systems (X, f) and (X, f^{-1}) .*

Proof. (i) Let (x, y) be a proximal pair then $y \in \pi(x) \in C(X)$. Clearly the sequence $(f^n(\pi(x)))_{n \in \mathbb{Z}}$ is pairwise disjoint (otherwise there exist $z \in \pi(x)$ and an integer $k \in \mathbb{Z}^*$ such that the pair $(f^k(z), z)$ is proximal and so f has a periodic point which is a contradiction). Now as X is a regular curve then $\lim_{|n| \rightarrow +\infty} \text{diam}(f^n(\pi(x))) = 0$ and then any pair of points in $\pi(x)$ should be asymptotic for both f and f^{-1} .

- (ii) An immediate consequence of assertion (i). \square

Corollary 3.4. *There is no Li-yorke pair for any regular curve homeomorphism.*

Proof. The case where $P(f) = \emptyset$ is proven in Corollary 3.3 and for the case where f has periodic point, see corollary 4.7 in [3]. \square

As we see the proximal relation is an equivalence relation for regular curve homeomorphisms since it coincides with the asymptotic relation but it can be not closed if we allow periodic points to exist. Consider for example the closed interval $[0, 1]$ homeomorphism $t \rightarrow \sqrt{t}$, the asymptotic relation for this homeomorphism contains two classes that of 0 and of 1, where the class of 1 is reduced to the singleton $\{1\}$ and that of $\{0\}$ is $\{x \in [0, 1], 0 \leq x < 1\}$.

The quotient map π induces naturally a system (\tilde{X}, \tilde{f}) where $\tilde{X} = \pi(X)$ and $\tilde{f}(\pi(x)) = \pi(f(x))$, $\forall x \in X$. So we may conclude the following:

Corollary 3.5. *The factor system (\tilde{X}, \tilde{f}) of (X, f) is conjugate to an irrational rotation of the circle. Moreover, if the system (X, f) is distal then it is conjugated to an irrational rotation of the circle.*

Recall that a topological space Y is said to be *homogenous* if any pair of point of Y can be mapped to each other by a self homeomorphism of Y .

Proof. The quotient space \tilde{X} being continuous monotone image of a hereditary locally connected continuum is a hereditary locally connected continuum. As $P(X, f)$ is a closed subset of $X \times X$ and (X, f) is a null system then by ([13], Theorem 4.3) $RP(X, T) = P(X, T)$ thus (\tilde{X}, \tilde{f}) is the maximal equicontinuous factor of (X, f) (see [8], corollary 2.1). Furthermore

By Lemma 3.2, we have $\pi(\tilde{X}) = \pi(M)$. Then (\tilde{X}, \tilde{f}) is an equicontinuous minimal \mathbb{Z} -action, hence \tilde{X} should be an homogeneous space. Recall that \tilde{X} is non degenerate by the presence of distal pairs in (X, f) . According to [2], \tilde{X} is a simple closed curve. Now if (X, f) is distal then clearly π is a homeomorphism and so X is a simple closed curve and thus f is conjugated to an irrational rotation. \square

4. Topological structure of M .

The aim of this section is to show that M is either the simple closed curve or the cantor set. We begin with this key result which will enable us to conclude what we just stated. Let π be the monotone map from X onto the simple closed curve defined in the last section.

Theorem 4.1. *Let (X, f) be a dynamical system where X is a regular curve, f is a homeomorphism without periodic point and M is the unique minimal set of (X, f) then for any $a \in X$, we have $1 \leq \text{card}(\pi(a) \cap M) \leq 2$.*

We will need the following lemmas:

Lemma 4.2. *Let X be continuum, then X is hereditarily locally connected if and only if for any sequence $(A_n)_{n \geq 0}$ of continua in $C(X)$ that converges to $A \in C(X)$, we have $\lim_{n \rightarrow +\infty} \text{Mesh}(A_n \setminus A) = 0$, where $\text{Mesh}(B) = \sup\{\text{diam}(C) : C \text{ is a connected component of } B\}$.*

Proof. " \Rightarrow " Let $(A_n)_{n \geq 0}$ be a sequence in $C(X)$ that converges to A . Assume that $\limsup_{n \rightarrow +\infty} \text{Mesh}(A_n \setminus A) > 0$. Then $\lim_{i \rightarrow +\infty} \text{Mesh}(A_{n_i} \setminus A) = \gamma > 0$ for some increasing sequence $(n_i)_{i \geq 0}$ of positive integers. It follows that for some $k \in \mathbb{N}$ and for any $i \geq k$, there is a connected component C_i of $A_{n_i} \setminus A$ with $\text{diam}(C_i) > \frac{\gamma}{2}$. Recall that for any $i \in \mathbb{N}$, A_{n_i} is a locally connected continuum, and so C_i is an arcwise connected open subset of A_{n_i} (see Theorem 8.26 in [16]). For any $i \geq k$, let J_i be an arc in C_i with $\text{diam}(J_i) > \frac{\gamma}{4}$. By compactness of $C(X)$, we may assume that $(J_i)_{i \geq k}$ converges to some $J \in C(X)$. Clearly $J \subset A$ and $\text{diam}(J) \geq \frac{\gamma}{4}$. Moreover, for any $i \geq k$ we have $J_i \cap J \subset (A_{n_i} \setminus A) \cap A = \emptyset$. Hence J is a non degenerate convergence continuum which is a contradiction with (Theorem 2 page 269 in [9]).

" \Leftarrow " This is obvious from the definition of non degenerate convergence continua and the characterization of hereditary locally connected continua given in (Theorem 2, page 269 in [9]) \square

Lemma 4.3. *Let (X, f) be a dynamical system where X is a regular curve, f is a homeomorphism without periodic point. For any $x \in X$, $X \setminus \pi(x)$ is a connected open subset of X and there is a decreasing sequence of open sets*

$(U_n)_{n \geq 0}$ of X such that for any $n \geq 0$ the following hold:

- (i) $\text{card}(\partial(U_n)) = 2$.
- (ii) $\partial(U_n) \subset \{c \in X, \text{diam}(\pi(c)) = 0\}$.
- (iii) $\pi(x) = \bigcap_{n \in \mathbb{N}} \overline{U_n}$.
- (iv) U_n is a connected open of X .

Proof. The connectedness of $X \setminus \pi(x)$ can be easily deduced from the fact that π is monotone from X onto a simple closed curve. Now observe first that $A = \{s \in \pi(X), \text{diam}(\pi^{-1}(s)) > 0\}$ is at most countable, this is due to the fact that X is in particular Suslinean and π is monotone. It follows that in the simple closed curve \tilde{X} , we may choose a decreasing sequence of open arcs $(V_n)_{n \geq 0}$ with end points z_n and t_n lying outside $\{\pi(x)\} \cup A$ so that $\{\pi(x)\} = \bigcap_{n \in \mathbb{N}} V_n$. Let $U_n = \pi^{-1}(V_n)$, for all n . Thus we may check easily that $(U_n)_{n \geq 0}$ is a sequence of open sets of X with the desired properties. \square

Proof of Theorem 4.1. By Lemma 3.2, we have for any $x \in X, \pi(x) \cap M \neq \emptyset$. Let $x \in X$ and suppose that $L = \pi(x) \cap M$ with $\text{card}(L) \geq 3$. Let $J = \{j_0, j_1, j_2\} \subset L$ be such that $\epsilon = \min_{u \neq v \in J} d(u, v) > 0$.

Claim. $\partial(\pi(x)) = \pi(x) \cap M$. Let $a \in \pi(x) \setminus M$ then by Lemma 3.2, $a \in \text{int}_X(\pi(x))$, this implies that $\partial(\pi(x)) \subset (\pi(x) \cap M)$. On the other hand, $\text{int}_X(\pi(x))$ is disjoint from M (otherwise, we may find a proximal pair of two distinct points from the same orbit). It turns out that $\partial(\pi(x)) = \pi(x) \cap M$.

Let $(U_n)_{n \geq 0}$ be the sequence of open sets defined in Lemma 4.3. We have then $\lim_{n \rightarrow +\infty} \overline{U_n} = \pi(x)$. By Lemma 4.2 and the fact that $\pi(x)$ is a proper subset of X , we can find an $N \geq 0$ such that $\text{Mesh}(\overline{U_N} \setminus \pi(x)) < \frac{\epsilon}{2}$ and $\overline{U_N} \subsetneq X$. By the Claim, $J \subset \partial(X \setminus \pi(x)) = \partial(\pi(x))$, hence by ([17], theorem 5) any point of J is accessible by arcs from $X \setminus \pi(x)$. It follows that for every $0 \leq i \leq 2$, $(X \setminus \pi(x)) \cup \{j_i\}$ is arc-wise connected. Let $r \notin \overline{U_N}$, thus as $(X \setminus \pi(x)) \cup \{j_i\}$ is arc-wise connected for any $0 \leq i \leq 2$, we can find an arc I_i in $(X \setminus \pi(x)) \cup \{j_i\}$ joining r to j_i such that $I_i \cap \pi(x) = \{j_i\}$. As $r \notin U_N$ and $J \subset U_N$, then we can find J_0, J_1 and J_2 sub-arc of I_0, I_1 and I_2 respectively such that for any $0 \leq i \leq 2$, J_i is an arc joining j_i to a point of $\partial(U_N)$ and $J_i \setminus \{j_i\} \subset \overline{U_N} \setminus \pi(x)$. As $\text{card}(\partial(U_N)) = 2$, there exist $0 \leq i < i' \leq 2$ such that $J_i \cap J_{i'} \cap \partial(U_N) \neq \emptyset$. As a consequence, there exists C a connected component of $\overline{U_N} \setminus \pi(x)$ such that $\overline{C} \cap J$ is not degenerate hence $\text{diam}(C) > \frac{\epsilon}{2}$, which is a contradiction \square

With the above results we can conclude the following:

Corollary 4.4. *Let (X, f) be a dynamical system where X is a regular curve, f is a homeomorphism without periodic point and M the unique minimal set of (X, f) then M is either the simple closed curve or the cantor set.*

Proof. We will consider the following two cases:

Case 1. Assume that M is connected. Then (M, f) is a minimal dynamical system generated by a regular curve's homeomorphism without periodic point. Let $x \in M$, by Theorem 4.1, we have $\pi(x)$ contains at most 2 points of M and then it must be reduced to a single point since the map $\pi|_M$ is monotone. It turns out that $\pi|_M$ is a homeomorphism and so M is a simple closed curve and $(M, f|_M)$ is conjugate to an irrational rotation of the circle.

Case 2. Assume now that M is not connected. Suppose that M admits a non degenerate connected component $C \subsetneq M$. As f is totally minimal on M , the family $(f^n(C))_{n \in \mathbb{Z}}$ is pairwise disjoint hence $\lim_{n \rightarrow +\infty} \text{diam}(f^n(C)) = 0$. Take any point $z \in C$ then $C \subset \pi(z)$, contradiction with Theorem 4.1. Thus the connected components of M are degenerate and so M is a Cantor set. \square

Remark 4.5. (i) Notice that $M \subset \overline{X_2}$. Indeed, since π is monotone there exists a point $x \in X$ such that $\pi(x) = \{x\}$. Recall that by Theorem 4.1, $x \in M$. By Lemma 4.3, $x \in X^2 \cap M$. Let L be an open arc in the simple closed curve \tilde{X} containing $\pi(x)$, we may easily check that $\pi^{-1}(L)$ intersects the boundary set of any open ϵ -neighborhood of x in at least two points for ϵ small enough. It follows that $M \subset \overline{X_2}$.

(ii) Consequently, on any regular curve R for which $X_2(R) = \emptyset$ (such as Sierpinsky Gasget and many other) there is always a periodic point for any \mathbb{Z} -action.

Corollary 4.6. *The following assertions are equivalent for a regular curve homeomorphism without periodic point :*

- (i) (X, f) is minimal.
- (ii) (X, f) is transitive.
- (iii) (X, f) is distal.
- (vi) (X, f) is equicontinuous.

All this dynamical property are equivalent for circle's homeomorphism, and if any of them is true for a regular curve homeomorphism without periodic points then it must be an irrational rotation of the circle (up to a conjugation).

5. CHARACTERIZATION OF THE DYNAMICAL SYSTEM $(M, f|_M)$

In this section, we will prove that there exists an invariant simple closed curve S of X and so $(M, f|_M)$ is a subsystem of $(S, f|_S)$. Thus dynamical

properties of the unique minimal M are the same as those minimal sets resulting from circle's homeomorphisms without periodic points.

Theorem 5.1. *Let (X, f) be a dynamical system such that X is a regular curve and f is a homeomorphism of X without periodic point then there is a simple closed curve $S \subset X$ such that $f(S) = S$.*

For the proof, we first need to establish other properties of the unique minimal set summarized in the following lemma and which are in themselves interesting:

Lemma 5.2. *Let (X, f) be a dynamical system such that X is a regular curve and f is a homeomorphism of X without periodic point. Suppose that the unique minimal set M of (X, f) is a Cantor set, then we have the following:*

- (i) $P(M, f|_M) \setminus \Delta_M$ is infinite and countable.
- (ii) For any infinite sequence $(x_n, y_n)_{n \geq 0} \in P(M, f) \setminus \Delta_M$ we have $\lim_{n \rightarrow +\infty} d(x_n, y_n) = 0$.

Proof. (i) Let $J \subset X$ be an arc in X joining two points x and y of M such that $J \setminus \text{End}(J) \subset X \setminus M$. By lemma 3.2, $\lim_{|n| \rightarrow +\infty} \text{diam}(f^n(J)) = 0$ and then $(x, y) \in P(M, f|_M) \setminus \Delta_M$. Hence $\{(f^n(x), f^n(y)) : n \in \mathbb{Z}\} \subset P(M, f|_M) \setminus \Delta_M$. Thus $P(M, f|_M) \setminus \Delta_M$ is infinite. We show now that $P(M, f|_M) \setminus \Delta_M$ is countable. Indeed, by Theorem 3.1, π is monotone and then $A = \{s \in \pi(X), \text{diam}(\pi^{-1}(s)) > 0\}$ is at most countable. This implies that $P(M, f|_M) \setminus \Delta_M$ is countable.

(ii) Notice that A is not empty since π is not a homeomorphism. As A is countable then $A = \{s_n : n \in \mathbb{N}\}$ such that $s_n \neq s_m$ if $n \neq m$. Thus $\lim_{n \rightarrow +\infty} \text{diam}(\pi^{-1}(s_n)) = 0$. It follows that for any infinite sequence $(x_n, y_n)_{n \geq 0} \in P(M, f) \setminus \Delta_M$ we have $\lim_{n \rightarrow +\infty} d(x_n, y_n) = 0$. \square

Proof of theorem 5.1. By Corollary 4.4, we need only to consider the case where M is a cantor set. Thus we will assume that M is a cantor set.

By assertion (i) of Lemma 5.2, we may write $P(M, f|_M) \setminus \Delta_M = \{(x_n, y_n) : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, let I_n be an arc joining x_n and y_n in X such that $I_n \cap M = \{x_n, y_n\}$ and $\lim_{|k| \rightarrow +\infty} \text{diam}(f^k(I_n)) = 0$. Let $S = M \cup (\bigcup_{n \in \mathbb{N}} O_f(I_n))$, obviously $f(S) = S$ and it is closed in X by assertion (ii) of Lemma 5.2. We have only to show that S is a simple closed curve.

First, it is clear that for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, $f^k(I_n) \setminus \{f^k(x_n), f^k(y_n)\}$ is open in S means that points in $f^k(I_n) \setminus \{f^k(x_n), f^k(y_n)\}$ have order 2 in S . Now lets check the order in S of points in M . It is easy to see that

$\pi_S : S \rightarrow \tilde{X}$ is onto and monotone, this gives that S is connected and so it is a regular curve. Now we may consider $g = f|_S$ instead of f . Thus by Remark 4.5, any point $x \in S$ with $\pi(x) = \{x\}$ has order 2 in S . Let $x \in S$ such that $\pi(x) \cap M = \{x, y\}$ with $x \neq y$. As g is a homeomorphism of S we may assume that $\pi(x) = I_m$ for some m where $End(I_m) = \{x, y\}$. By applying Lemma 4.3 to the dynamical system (S, g) , we get that $S \setminus I_m$ is an open connected subset of S with boundary set $\{x, y\}$. Thus we can find an arc L in $\overline{S \setminus I_m}$ joining x and y . Let now $(U_n)_{n \geq 0}$ be the sequence of open neighborhoods in S of $\pi(x)$ constructed in Lemma 4.3 with respect to (S, g) . If we suppose that x has order greater than 2 in S then by the n-Beisatz Theorem (see [9], 8, page 277) we may find a 3-star $T = T_1 \cup T_2 \cup T_3$ centered at x included in S where T_i is an arc with end points x and z_i and $T_i \cap T_j = \{x\}$ for $i \neq j$. As $I_m \setminus \{x, y\}$ is an open arc of S then we can assume that $(T_1 \cup T_2) \cap I_m = \{x\}$. So let J_i , $i = 1, 2, 3$ be respectively subarcs of T_1, T_2, T_3 such that:

- (i) J_1, J_2 are disjoint from J_3 .
- (ii) $J_1 \cap J_2 = \{x\}$.
- (iii) y is one of the end point J_3 .

It follows that for n large enough, $card(\partial(U_n)) \geq 3$, a contradiction. In result, x should have order ≤ 2 . Notice that S has no cut points. Indeed if $a \in M$ then $a \in \partial(S \setminus \pi(a))$ and so $S \setminus \{a\}$ is connected since $S \setminus \pi(a)$ is already connected (see Lemma 4.3). Now if $a \in int_S(I_m)$ for some m then obviously $S \setminus \{a\}$ is connected. As a consequence, S does not have end points (points of order 1). In conclusion, every point in S has order 2. By ([16], Corollary 9.6), S is a simple closed curve. \square

Note that in the case where the minimal set is a simple closed curve, the invariant simple closed curve is unique. But in the case where the minimal is a cantor set, the invariant simple closed curve is not always unique, in fact it can be infinitely many, for this we can consider the following example: Let $f : S^1 \rightarrow S^1$ be a circle homeomorphism without periodic points and having a cantor minimal set M . Fix a connected component I_0 of $S \setminus M$ and pick a cantor set $K(I_0) \subset I_0$. For each $n \in \mathbb{Z}$, let $I_n = f^n(I_0)$ and $K(I_n) = f^n(K(I_0))$ and attach a homeomorphic copy of the Gehman dendrite denoted by $G(I_n)$ to the circle S^1 such that $G(I_n) \cap S^1 = K(I_n) = End(G(I_n))$ and $\lim_{|n| \rightarrow \infty} diam(G(I_n)) = 0$.

Obviously the resulting space $X = S^1 \cup (\bigcup_{n \in \mathbb{Z}} G(I_n))$ is a regular curve. We will extend f to a homeomorphism g on X in the following way: $g|_{S^1} = f$ and for each $n \in \mathbb{Z}$, g sends $G(I_n)$ to $G(I_{n+1})$. The closure of I_0 is an arc with end points $a_0, b_0 \in M$ and clearly there is an uncountable way to join a_0 and b_0 by an arc included in their proximal class $I_0 \cup G(I_0)$.

If we take an arc J_0 included in the proximal class of a_0 joining a_0 and b_0 , then it is easy to see that $(S_1 \setminus (\bigcup_{n \in \mathbb{Z}} I_n)) \cup (\bigcup_{n \in \mathbb{Z}} g^n(J_0))$ is an invariant simple closed curve.

6. COUNTER-EXAMPLES IN RATIONAL CURVE

In this section, we will see that if we consider rational curve (even with finite rim-type) instead of regular ones, we may find counter-examples of results already shown for regular curves which illustrate the importance of the rim-finiteness property. We recall first the definition of a rational curve and the rim-type of a rational curve.

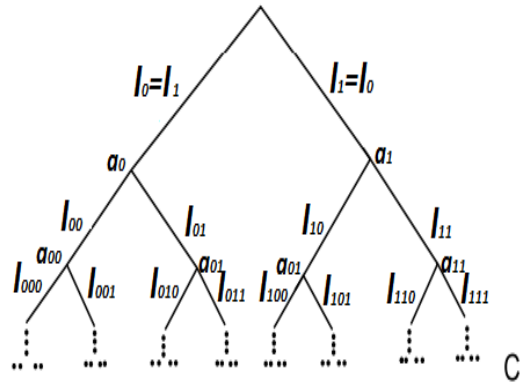
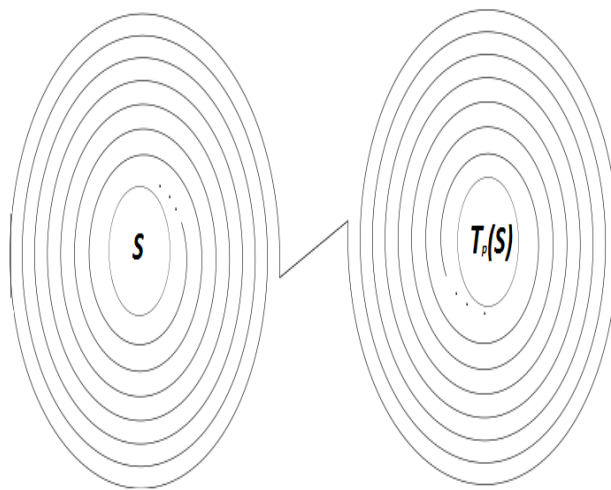
A continua X is said to be rational if any point have an ϵ -open neighborhood with countable boundary for any $\epsilon > 0$.

A rational curve is said to be of rim-type α , where α is an ordinal number if α is the smallest ordinal number such that for any $\epsilon > 0$, any point have an ϵ -open neighborhood with countable boundary D , such that D^α is empty, where D^α is the α -derived set of D . (For the existence of such an ordinal see [9], Theorem 13 page 290). Note that if $\alpha = 1$ then X is a regular curve. Glasner and Megrelishvili showed in [15] that every \mathbb{Z} -action on a regular curve is null. The aim of our first example is to give a \mathbb{Z} -action on some rational curve with positive sequence entropy

Example 1. A \mathbb{Z} -action on a rational curve with positive topological sequence entropy: Let X be a compact countable space with derived degree 2, then by [20], there exists a homeomorphism $h : X \rightarrow X$ with positive sequence topological entropy. In the same manner as Theorem 3.1 in [19] we can find a rational curve F_X (the fan of X) and an homeomorphism g of F_X with the same sequence topological entropy as h (see [5]). \square

According to Theorem 3.1 and the results in [3], the proximal relation is always an equivalence relation for regular curve homeomorphism and there is no Li-Yorke pair regardless of the existence of periodic point. However there is a rational curve homeomorphism (see Example 2) which is Li-Yorke chaotic and the proximal relation is not transitive.

Example 2. In [12], the author gave a Li-Yorke chaotic homeomorphism F on a rational curve Y such that there exists a point $y \in Y$ satisfying $\omega_f(y) \subset \text{Fix}(f)$ and $\omega_f(y)$ is infinite. In one hand y is proximal to any point $a \in \omega_f(y)$, on the other hand any proper pair of point in $\omega_f(y)$ is distal hence the proximal relation is not transitive. \square

FIGURE 1. The Gehman dendrite D FIGURE 2. The continuum A

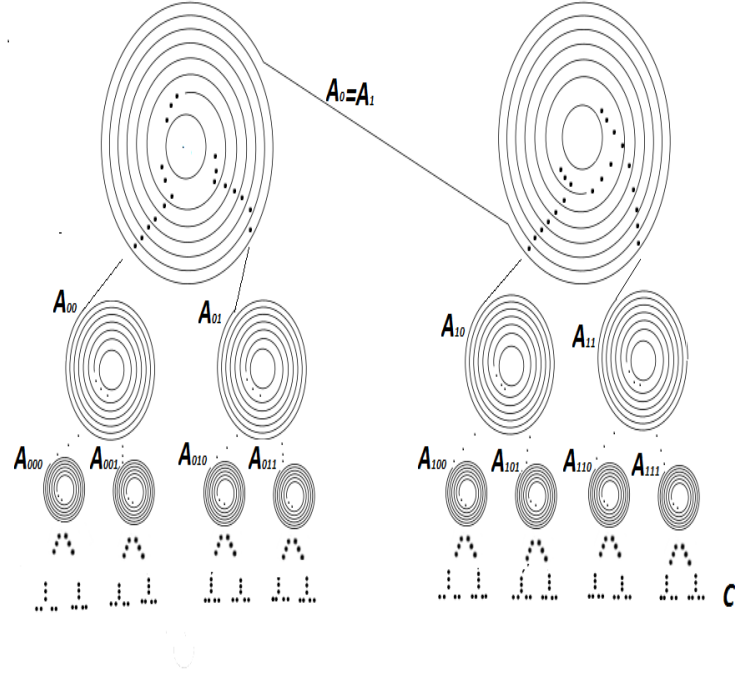


FIGURE 3. The continuum X

It was shown in [3] that for any \mathbb{Z} -action on a regular, infinite minimal sets possess adding machine structure provided that the set of periodic points is not empty. However, when there is no periodic points then there is a unique minimal set and the adding machine structure is not allowed see [11]. The aim of our third example is to give a \mathbb{Z} -action on some rational curve without periodic points but having several minimal sets, one of them is a simple closed curve (thus the restriction to this minimal set is conjugates to an irrational rotation of the circle) and another is an adding machine Cantor set, this examples combines two types of minimal sets which could never appear simultaneously on regular curves.

Example 3. A \mathbb{Z} -action on some rational curve without periodic points but having a simple closed curve minimal set and an adding machine Cantor set: First we will outline the idea of the construction as follow: The idea consists to deform the Gehman dendrite homeomorphism constructed in [7] by replacing each arc I_α for $\alpha \in \{0, 1\}^n$ by a homeomorphic copy A_α of a rational curve A having two disjoint simple closed curves S_α^1 and S_α^2 such that $H^{2^n}(A_\alpha) = A_\alpha$ and $\omega_{H^{2^n}}(x) = S_\alpha^1, \forall x \in A_\alpha \setminus S_\alpha^2$. We start by introducing the rational curve A illustrated in the following figure as a subset of \mathbb{R}^3 :

$$A = R_1 \cup R_2 \cup \{(0, \cos(\theta), \sin(\theta)), \theta \in \mathbb{R}\} \cup T_P(\{(0, \cos(\theta), \sin(\theta)), \theta \in \mathbb{R}\})$$

where $R_1 = \{(\theta, \cos(\frac{1}{\theta}), \sin(\frac{1}{\theta})) : 0 < \theta \leq \frac{2}{\pi}\}$, $R_2 = T_P(R_1)$ and T_P is the symmetry with respect to the plan $x = \frac{2}{\pi}$. Let $\phi : [0, \frac{2}{\pi}] \rightarrow [0, \frac{2}{\pi}]$ be defined as $\phi(t) = \frac{t}{1+2\sqrt{2}\pi t}$ and Φ_1 be the self-mapping of $R_1 \cup \{(0, \sin(\theta), \cos(\theta)), \theta \in \mathbb{R}\}$ defined as follow:

$$\Phi_1(a) = \begin{cases} (\phi(\theta), \cos(\frac{1}{\phi(\theta)}), \sin(\frac{1}{\phi(\theta)})) & \text{if } a \in R_1 \\ (0, \cos(\theta + 2\pi\sqrt{2}), \sin(\theta + 2\pi\sqrt{2})) & \text{if } a \in S \end{cases}$$

Where $S = \{(0, \cos(\theta), \sin(\theta)), \theta \in \mathbb{R}\}$.

Let now $a \in R_2 \cup T_P(S)$, we define $\Phi_2(a) = T_P(\Phi_1(T_P^{-1}(a)))$.

Φ_1, Φ_2 are one to one continuous maps and both of them have an arc I_1 (resp I_2) that meets only at their end point $(\frac{2}{\pi}, 0, 1)$ with empty pre-image, let Φ_3 be an affine homeomorphism from I_2 to I_1 .

Finally we define Φ on A as follow

$$\Phi(a) = \begin{cases} \Phi_1(a) & \text{if } a \in R_1 \\ \Phi_2^{-1}(a) & \text{if } a \in R_2 \setminus I_2 \\ \Phi_3(a) & \text{if } a \in I_2 \\ (0, \cos(\theta + 2\sqrt{2}\pi), \sin(\theta + 2\sqrt{2}\pi)) & \text{if } a \in S \\ T_P(\Phi_1(T_P^{-1}(a))) & \text{if } a \in T_P(S) \end{cases}$$

Clearly Φ is an homeomorphism of A having two minimal sets which are the simple closed curves, $S, T_P(S)$.

We use example (5.4) In [7] of a Gehman homeomorphism $f : D \rightarrow D$ which have an infinite minimal set endowed with an adding machine, we recall the definition of the adding machine:

Let $C = \{0, 1\}^{\mathbb{N}}$, we endowed C with the metric d_C on C given by

$$d_C((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sum_{i=1}^{\infty} \frac{\delta(x_i, y_i)}{2^i}$$

where $\delta(x_i, y_i) = 1$ if $x_i \neq y_i$ and $\delta(x_i, y_i) = 0$ if $x_i = y_i$. The addition in C is defined as follows:

$$(x_1, x_2, \dots) + (y_1, y_2, \dots) := (z_1, z_2, \dots)$$

where $z_1 = (x_1 + y_1) \bmod j_1$ and $z_2 = (x_2 + y_2 + t_1) \bmod j_2$, with $t_1 = 0$ if $x_1 + y_1 < j_1$ and $t_1 = 1$ if $x_1 + y_1 \geq j_1$, where $j_1 = 0$ and $j_2 = 1$. So, we carry a one in the second case. Continue adding and carrying in this way for the whole sequence.

We define $\sigma : C \rightarrow C$ by

$$\sigma((x_1, x_2, \dots)) = (x_1, x_2, \dots) + \bar{1}$$

where $\bar{1} = (1, 0, 0, \dots)$, we will refer to the system (C, σ) as the *adding machine*.

Let C be the triadic cantor included in $[0, 1] \times \{0\}$ and $(\alpha_n)_{n \geq 1}$ be the sequence of accessible points of C . We can see C as the set $\{0, 1\}^{\mathbb{N}}$ and $(\alpha_n)_{n \geq 1}$ as the sequence of finite words in C , organised in a way such that for each $n \geq 1$, $\Gamma_n = \{\alpha_{2^n}, \dots, \alpha_{2^{n+1}-1}\} = \{0, 1\}^n$ is the set of finite words of C of length n .

For each $n \geq 1$, let f_n be the self homeomorphism of Γ_n defined in the following way :

$$f_n(\alpha) = \begin{cases} \alpha + \bar{1} & \text{if } \alpha \neq (1, 1, \dots, 1) \\ (0, 0, \dots, 0) & \text{if } \alpha = (1, 1, \dots, 1) \end{cases}$$

Let now $\epsilon_n = \min_{p \neq q \in \Gamma_n} d(p, q)$, for each $n \geq 1$ and $\alpha_n \in \Gamma_n$, we define S_{α_n} as a simple closed curve included in the set $\{(x, y) \in [0, 1] \times [0, 1], -|\alpha_n - \epsilon_n| < x < |\alpha_n - \epsilon_n|, \frac{1}{n+1} < y < \frac{1}{n}\}$, in a way such that the family $\{S_{\alpha_n}, \alpha_n \in \Gamma(n), n \geq 1\}$ is pairwise disjoint.

For each $n \geq 1$ and $\alpha_n \in \Gamma_n$, let $(\alpha_n, 0), (\alpha_n, 1) \in \Gamma_{n+1}$ and let $R_{(\alpha_n, 0)}$ and $R_{(\alpha_n, 1)}$ be two disjoint rays (i.e continuous one to one image of \mathbb{R}) such that the following hold :

(i) $A_{(\alpha_n, 0)} = R_{(\alpha_n, 0)} \cup S_{\alpha_n} \cup S_{(\alpha_n, 0)}$ and $A_{(\alpha_n, 1)} = R_{(\alpha_n, 1)} \cup S_{\alpha_n} \cup S_{(\alpha_n, 1)}$ are homeomorphic copy of A .

(ii) $diam(A_{(\alpha_n, l)}) \leq 2\epsilon_{\alpha_n}$, for $l \in \{0, 1\}$ where $\epsilon_{\alpha_n} = \sup\{d(a, b), a \in S_{\alpha_n}, b \in S_{(\alpha_n, l)}\}$.

(iii) For any distinct finite words x and y of C , A_x and A_y are disjoint or their intersection is one of their simple closed curve if $x \in \{(y, 1), (y, 0)\}$ or $y \in \{(x, 1), (x, 0)\}$.

Finally let X be the subset of the plane defined by

$$X = \left(\bigcup_{\alpha \in \Gamma(n), n \geq 1} A_{\alpha} \right) \cup C$$

It is clear that X is a rational curve and by collapsing each S_{α_n} for each $n \geq 1$ and $\alpha_n \in \Gamma_n$ to a point the quotient space will be exactly the Gehman dendrite D . Denote by Ψ the quotient map, We have $C = End(X)$ is a cantor set satisfying $\Psi|_C : C \rightarrow End(D)$ is a homeomorphism. Let L_n be the affine homeomorphism of $\bigcup_{\alpha \in \Gamma(n)} A_{\alpha}$ that maps each $A_{(a_1, \dots, a_n)}$ to $A_{f_n((a_1, \dots, a_n))}$ and

denote by Φ_{α} the self homeomorphism of A_{α} corresponding to Φ defined on A . Finally we define H the self-mapping of X as follow :

$$H(x) = \begin{cases} \Psi|_C^{-1}(f(\Psi|_C(x))) & \text{if } x \in C, \\ L_n(\Phi_{n, \alpha}(x)) & \text{if } x \in A_{n, \alpha}. \end{cases}$$

It is easy to see that H is a homeomorphism on X and that Ψ semi-conjugated H to a Gehman dendrite homeomorphism such that the restriction of on $End(D)$ is the adding machine. Although H does not have any periodic point, it has infinitely many minimal sets. Each one of them is either a finite disjoint sum of simple closed curves or a cantor set endowed with an adding machine.

Note that all the rational curves given above have finite rim-type. The continua in example 1 is a 3 rim-type and the others are 2 rim-type, which show the importance of the rim-finite property.

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