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ABSTRACT. The binormal flow is a model for the dynamics of a vortex filament in a 3-D inviscid incompressible fluid. The flow is also related with the classical continuous Heisenberg model in ferromagnetism, and the 1-D cubic Schrödinger equation. We consider a class of solutions at the critical level of regularity that generate singularities in finite time. One of our main results is to prove the existence of a natural energy associated to these solutions. This energy remains constant except at the time of the formation of the singularity when it has a jump discontinuity. When interpreting this conservation law in the framework of fluid mechanics, it involves the amplitude of the Fourier modes of the variation of the direction of the vorticity.

1. INTRODUCTION

In this paper we focus on qualitative and quantitative properties of singular solutions of the binormal flow. This geometric flow describes the evolution in time of a curve $\chi(t,x)$ in $\mathbb{R}^3$ that is parametrized by arclength x, via the equation

$$\chi_t = \chi_x \wedge \chi_{xx}. \tag{1}$$

If in a 3-D fluid the vorticity is concentrated initially along a curve, it is expected that at least in some situations the vorticity at later times is still concentrated along another curve, whose evolution is dictated by the binormal flow. This was formally derived by Da Rios in [11] after truncating the integral given by Biot-Savart's law (see also [27], [1],[5]). A more rigorous argument, but still under some strong assumptions, has been recently given by Jerrard and Seis in [20].

The binormal flow is linked to the 1-D cubic Schrödinger equation (NLS) in the following way. Taking the derivative in x of χ we obtain that the tangent vector $T(t,x) \in \mathbb{S}^2$ satisfies the classical continuous Heisenberg model used in ferromagnetism

$$T_t = T \wedge T_{xx}. \tag{2}$$

Next, by considering the curvature and torsion of $\chi(t,x)$, Hasimoto constructed, in the spirit of the Madelung transform, a complex valued function that satisfies the focusing 1-D cubic NLS ([19]). Conversely, given a real function of time $a(t)$, a solution u of

$$iu_t + u_{xx} + \frac{1}{2}(|u|^2 - a(t))u = 0, \tag{3}$$

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a point $P \in \mathbb{R}^3$, and an $\mathbb{R}^3$-orthonormal basis (v_1, v_2, v_3), one can construct a solution of (1) as follows. First define parallel frames $(T, e_1, e_2)(t, x)$ as the solutions of

$$\begin{cases} T_x = \Re(\overline{u}\, N), \quad N_x = -u\, T, \\ T_t = \Im(\overline{u_x}\, N), \quad N_t = -iu_x T + \dfrac{i}{2}(|u|^2 - a(t))N, \end{cases} \tag{4}$$

with $N = e_1 + ie_2$ and initial data $(T, e_1, e_2)(t_0, x_0) = (v_1, v_2, v_3)$. It follows that T constructed this way satisfies the Schrödinger map (2). Finally, setting

$$\chi(t, x) = P + \int_{t_0}^{t} (T \wedge T_x)(\tau, x_0) d\tau + \int_{x_0}^{x} T(t, s) ds,$$

we obtain that $\chi(t, x)$ satisfies the binormal flow (1). Note that the construction of $\chi(t)$ is not obvious if the solution u of (3) is not too regular. This is precisely the scenario considered in this paper.

Regarding (3) note that since $a(t)$ is real, the corresponding term can be easily removed from the equation by a change of function. From the gauge invariance in (4) this will lead to the construction of the same curve. In this way we obtain the cubic NLS

$$iu_t + u_{xx} + \frac{1}{2}|u|^2 u = 0, \tag{5}$$

that is invariant under the scaling

$$u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x). \tag{6}$$

At this respect we shall say that the solutions of (1) are critical if they are constructed from NLS solutions in a functional setting that is invariant by scaling.

Let us recall here that (5) is well-posed in H^s, for any $s \geq 0$ ([12],[7]), and that for $s < 0$ the Cauchy problem is ill-posed ([22],[10],[6],[24],[28],[23],[26]). We recall also that well-posedness holds for data with Fourier transform in L^p spaces, $p < +\infty$ ([29],[18],[8]).

It is well known that equation (5) is also invariant under Galilean transformations

$$u_\eta(t, x) = e^{-i\eta^2 t + i\eta x} u(t, x - 2\eta t). \tag{7}$$

One of the problems with the Sobolev class is that it is not invariant under translation in Fourier space, except of course L^2 that is not invariant under (6). As a consequence the Sobolev class is not well suited with respect to Galilean transformations. This is the reason why in our previous work [4] we consider initial data whose Fourier transforms are L^2 periodic, possibly smooth, functions. Another possibility is to measure the Fourier transform in the L^∞ norm because this topology is critical for cubic NLS with respect to both symmetries (6) and (7). One of the issues that we address in this paper is the possible growth in this latter topology.

The binormal flow is known to develop singularities in finite time. An important class of singular solutions is the family of self-similar solutions $\{\chi_\alpha\}_{\alpha>0}$, that are determined for $t > 0$ by the values of their curvature and torsion, $\frac{\alpha}{\sqrt{t}}$ and $\frac{x}{2t}$ respectively. The curve $\chi_\alpha(t)$

is smooth for $t > 0$ and, as proved in [16], it has a trace at $t = 0$ given by a polygonal line with just one corner of angle θ, such that

$$\sin\frac{\theta}{2} = e^{-\pi\frac{\alpha^2}{2}}. \tag{8}$$

The corresponding 1-D cubic NLS solution is $u_\alpha(t,x) = \alpha\frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}}$, taking $a(t) = \frac{\alpha^2}{t}$ in (3).

Recently, we constructed in [4] a class of smooth solutions of the binormal flow that generate several corners in finite time. More precisely, take a polygonal line with corners located at $x = j \in \mathbb{Z}$ and angles θ_j, and choose $\{\alpha_j\}$ using the relation (8). Then, under the assumption that some moments of the sequence $\{\alpha_j\}$ are squared integrable, we construct a strong smooth solution of the binormal flow for $t \neq 0$, that is a weak solution for all t. This solution has the given polygonal line as trace at $t = 0$. For this purpose we first construct for $t \neq 0$ and $a(t) = \frac{\sum_j |\alpha_j|^2}{t} := \frac{M}{t}$ a unique solution of (3) of the form

$$u(t,x) = \sum_j e^{-i(|\alpha_j|^2 - M)\log\sqrt{t}}\tilde{A}_j(t)\frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}} := \sum_j A_j(t)\frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}}, \tag{9}$$

such that $\lim_{t\to 0}\tilde{A}_j(t) = \alpha_j$, and $R_j(t) := \tilde{A}_j(t) - \alpha_j$ satisfies

$$\sup_{0<t<1} t^{-\gamma}\|\{R_j(t)\}\|_{l^{2,s}} + t\,\|\{\partial_t R_j(t)\}\|_{l^{2,s}} < C(\{\alpha_j\}), \tag{10}$$

for $0 < \gamma < 1$ (see also [25] for the subcubic case). Here $s \geq 3$, $\|(\beta_j)\|_{l^{2,s}} := (\sum_j (1 + |j|)^{2s}\,|\beta_j|^2)^{1/2}$, and the coefficients

$$e^{-i|\alpha_j|^2\log\sqrt{t}}\tilde{A}_j(t)$$

solve the non-autonomous Hamiltonian system:

$$i\partial_t A_k(t) = \frac{1}{4\pi t}\sum_{k-j_1+j_2-j_3=0} e^{-i\frac{k^2-j_1^2+j_2^2-j_3^3}{4t}} A_{j_1}(t)\overline{A_{j_2}(t)}A_{j_3}(t) - \frac{\sum_j |\alpha_j|^2}{2\pi t}A_k(t). \tag{11}$$

Moreover, the solution satisfies the mass conservation law:

$$M = \sum_j |\alpha_j|^2 = \sum_j |A_j(t)|^2. \tag{12}$$

Then, given this unique solution of (3) we construct the solution of the binormal flow as explained above. This solution has as initial data the given polygonal line. We refer the reader to Theorem 1.1 and Theorem 1.4 in [4] for the precise statements.

Our main result in this paper is to see if there are quantities as (12) associated to (1) and (2) that are also conserved. Recall that for smooth solutions of (2) the energy density is given by

$$c^2\,dx = |T_x|^2\,dx,$$

where c stands for the curvature. As a consequence, those solutions of (2) that are constructed from solutions of (3) which have finite L^2 norm will have energy that is also finite. But this is not the case for the solutions considered in this article.

It turns out that the right way of interpreting (12) is to look at the Fourier transform in space of T_x. Then, the energy appears as a scattering energy that is preserved as long as $t \neq 0$, while it has a jump at $t = 0$. More concretely, we have the following result.

Theorem 1.1. *Let χ be a binormal flow solution with initial data a polygonal line, as introduced above, and T its tangent vector. We define*

$$\Xi(T(t)) := \lim_{k\to\infty} \int_k^{k+1} |\widehat{T_x}(t,\xi)|^2 d\xi. \tag{13}$$

For $t > 0$ we have the following conservation law:

$$\Xi(T(t)) = 4\pi \sum_j |\alpha_j|^2. \tag{14}$$

At $t = 0$ when singularities are created for the binormal flow solution χ we have

$$\int_k^{k+1} |\widehat{T_x}(0,\xi)|^2 d\xi = 4 \sum_j (1 - e^{-\pi|\alpha_j|^2}) \quad \forall k \in \mathbb{Z}. \tag{15}$$

Therefore there is a jump discontinuity of $\Xi(T(t))$ at time $t = 0$, showing an instantaneous growth for positive times at large frequencies:

$$\Xi(T(0)) = 4 \sum_j (1 - e^{-\pi|\alpha_j|^2}) < 4\pi \sum_j |\alpha_j|^2 = \Xi(T(t)). \tag{16}$$

The proof of the theorem is based on a careful decomposition of $\widehat{T_x}(t,\xi)$ in principal terms that eventually give $\Xi(T(t))$ and terms for which we get either a constant type upper-bound or a logarithmic type upper-bound depending on $d(4\pi\xi, \frac{\mathbb{Z}}{t})$, and that become negligible in the computation of $\Xi(T(t))$.

Remark 1.2. *Observe that on the one hand that the quantity $\Xi(T(t))$ involves $\hat{T}_x(t,\xi)$ for large ξ, and therefore it measures the size of the amplitude of the large frequency waves of the variation of T. On the other hand T, when interpreted at the level of fluid mechanics, gives the direction of the vorticity. At this respect Constantin-Fefferman-Majda's criterion* [9] *states that the growth in the variation of the direction of the vorticity is necessary to produce singularities in Euler equations in three dimensions.*

Remark 1.3. *A similar statement holds for the normal vector, namely for $t > 0$*

$$\Xi(N(t)) := \lim_{k\to\infty} \int_k^{k+1} |\widehat{N_x}(t,\xi)|^2 d\xi = 4\pi \sum_j |\alpha_j|^2, \tag{17}$$

but

$$\Xi(\tilde{N}(0)) = 4 \sum_j (1 - e^{-\pi|\alpha_j|^2}), \tag{18}$$

where $\tilde{N}(0,x)$ *is the limit at* $t=0$ *of* [1]

$$\tilde{N}(t,x) = e^{i\sum_{r\in\mathbb{Z},r\neq x}|\alpha_r|^2\log\frac{|x-r|}{\sqrt{t}}}N(t,x).$$

Remark 1.4. *Theorem 1.1 applies in particular to the case of self-similar solutions of the binormal flow that are generated by polygonal lines with only one corner. Moreover, using a perturbation argument, we constructed in* [2] *solutions of the binormal flow that are smooth except at one time when they generate a corner. For these perturbed solutions we managed to show in* [3] *that*

$$\lim_{\xi\to\infty}|\widehat{T_x}(t,\xi)|^2 = 4\pi|\alpha_0|^2,$$

and that there exists $\epsilon>0$*, depending on the perturbation of the initial data with respect to the self-similar case, such that for any* $\xi\in\mathbb{R}$

$$|\widehat{T_x}(0,\xi)|^2 < 4(1-e^{-\pi|\alpha_0|^2})+\epsilon.$$

In particular for small perturbations we obtain for any $t>0$

$$\Xi(T(0)) < 4\pi|\alpha_0|^2 = \Xi(T(t)).$$

A similar statement holds for the normal vector $N(t)$*.*

Our final result is an observation that uses Theorem 1.1 to reinforce the conjecture done in [13] about the evolution of a regular planar polygon according to the binormal flow (see also [17], [21], [15]). In that paper, and after some theoretical arguments, it is conjectured that the evolution of a regular polygon is periodic in time, and that at rational multiples of the time period the curve is a skew polygon with the same angle between consecutive sides. In [13] the size of this angle is guessed from the data obtained in the numerical simulations, while in this paper we obtain it from the energy $\Xi(T(t))$.

The paper is organized as follows. In the next section we prove the asymptotic behavior in space of the tangent and modulated normal vectors, and see that this behavior is independent of time. This information allows us to prove Theorem 1.1 in §3. Finally, in the last section we make the observation about planar regular polygons mentioned above.

2. Asymptotic behavior in space of the orthonormal frame

Lemma 2.1. *There exist* $T^{\pm\infty}$ *with* $|T^{\pm\infty}|=1$ *such that for all* $t>0$

$$T^{\pm\infty} = \lim_{x\to\pm\infty}T(t,x). \tag{19}$$

Moreover,

$$|T(t,x)-T^{\pm\infty}| \leq \frac{C(t,\{\alpha_j\})}{\langle x\rangle}, \forall x\in\mathbb{R}^*, \pm x>0,$$

where $\langle x\rangle = 1+|x|$*.*

[1] the existence of $\tilde{N}(0,x)$ is proved in Lemmas 4.5 in [4].

Proof. We shall first prove that for fixed $t > 0$ there exists a unit vector $T^\infty(t)$ which is the limit of $T(t,x)$ as x goes to ∞ ; the asymptotic behavior at $-\infty$ can be treated in the same way.

As $T_x = \Re(\overline{u}N)$ we get for $0 < x_1 < x_2$:

$$T(t,x_2) - T(t,x_1) = \Re \int_{x_1}^{x_2} \sum_j \overline{A_j(t)} \frac{e^{-i\frac{(x-j)^2}{4t}}}{\sqrt{t}} N(t,x)dx.$$

We perform an integration by parts using the quadratic oscillatory phase to get $\frac{1}{x}$ decay in space:

$$T(t,x_2) - T(t,x_1) = \left[\Re \sum_j \overline{A_j(t)} e^{-i\frac{x^2}{4t}} \frac{4t}{-i2x} \frac{e^{i\frac{xj}{2t} - i\frac{j^2}{4t}}}{\sqrt{t}} N(t,x) \right]_{x_1}^{x_2}$$

$$-\Im 2\sqrt{t} \sum_j \overline{A_j(t)} e^{-i\frac{j^2}{4t}} \int_{x_1}^{x_2} e^{-i\frac{x^2}{4t}} \left(\frac{e^{i\frac{xj}{2t}}}{x} N(t,x) \right)_x dx.$$

Since $N_x = -uT$,

$$\left| T(t,x_2) - T(t,x_1) - \Im \frac{i}{\sqrt{t}} \sum_j j \overline{A_j(t)} e^{-i\frac{j^2}{4t}} \int_{x_1}^{x_2} e^{-i\frac{x^2}{4t}} \frac{e^{i\frac{xj}{2t}}}{x} N(t,x)dx \right.$$

$$\left. -\Im 2\sqrt{t} \sum_j \overline{A_j(t)} e^{-i\frac{j^2}{4t}} \int_{x_1}^{x_2} e^{-i\frac{x^2}{4t}} \frac{e^{i\frac{xj}{2t}}}{x} \sum_k A_k(t) \frac{e^{i\frac{(x-k)^2}{4t}}}{\sqrt{t}} T(t,x)dx \right| \leq C \frac{\sqrt{t}\|\{A_j(t)\}\|_{l^1}}{x_1}.$$

In the first integral we perform again an integration by parts using the quadratic phase to obtain integrability in space:

$$\left| T(t,x_2) - T(t,x_1) - 2\Im \sum_{j \neq k} \overline{A_j(t)} A_k(t) e^{i\frac{k^2-j^2}{4t}} \int_{x_1}^{x_2} e^{i\frac{x(j-k)}{2t}} \frac{T(t,x)}{x} dx \right|$$

$$\leq C \left(\frac{\sqrt{t}\|\{A_j(t)\}\|_{l^1}}{x_1} + \frac{\sqrt{t}\|\{jA_j(t)\}\|_{l^1}}{x_1^2} + \frac{\|\{j^2A_j(t)\}\|_{l^1}}{x_1\sqrt{t}} + \frac{\|\{jA_j(t)\}\|_{l^1}\|\{A_j(t)\}\|_{l^1}}{x_1} \right).$$

Above we have used that the term $j = k$ cancels. Now we perform an integration by parts using the linear phase, even though we don't improve the decay in x:

$$\left| T(t,x_2) - T(t,x_1) + 2\Im \sum_{j \neq k} \overline{A_j(t)} A_k(t) e^{i\frac{k^2-j^2}{4t}} \int_{x_1}^{x_2} e^{i\frac{x(j-k)}{2t}} \frac{2t}{i(j-k)} \left(\frac{T(t,x)}{x} \right)_x dx \right|$$

$$\leq C \left(\frac{\sqrt{t}\|\{A_j(t)\}\|_{l^1}}{x_1} + \frac{\sqrt{t}\|\{jA_j(t)\}\|_{l^1}}{x_1^2} + \frac{\|\{j^2A_j(t)\}\|_{l^1}}{x_1\sqrt{t}} + \frac{\|\{jA_j(t)\}\|_{l^1}\|\{A_j(t)\}\|_{l^1}}{x_1} \right).$$

In this way we can use that $T_x = \Re(\overline{u}N)$, so that a new oscillatory term with a quadratic phase appears:

$$|T(t,x_2) - T(t,x_1)$$

$$+2\sqrt{t}\Im\sum_{j\neq k;r}\overline{A_j(t)}A_k(t)\overline{A_r(t)}\frac{e^{i\frac{-r^2+k^2-j^2}{4t}}}{i(j-k)}\int_{x_1}^{x_2}e^{-i\frac{x^2}{4t}}\frac{e^{i\frac{x(j-k+r)}{2t}}}{x}N(t,x)dx$$

$$+2\sqrt{t}\Im\sum_{j\neq k;r}\overline{A_j(t)}A_k(t)A_r(t)\frac{e^{i\frac{r^2+k^2-j^2}{4t}}}{i(j-k)}\int_{x_1}^{x_2}e^{i\frac{x^2}{4t}}\frac{e^{i\frac{x(j-k-r)}{2t}}}{x}\overline{N(t,x)}dx\Bigg|$$

$$\leq C\left(\frac{\sqrt{t}\|\{A_j(t)\}\|_{l^1}}{x_1}+\frac{\sqrt{t}\|\{jA_j(t)\}\|_{l^1}}{x_1^2}+\frac{\|\{j^2A_j(t)\}\|_{l^1}}{x_1\sqrt{t}}+\frac{\|\{jA_j(t)\}\|_{l^1}\|\{A_j(t)\}\|_{l^1}}{x_1}\right).$$

Hence, we can perform again an integration by parts to get decay in space:

$$|T(t,x_2)-T(t,x_1)$$

$$-4t\sqrt{t}\Re\sum_{j\neq k;r}\overline{A_j(t)}A_k(t)\overline{A_r(t)}\frac{e^{i\frac{-r^2+k^2-j^2}{4t}}}{i(j-k)}\int_{x_1}^{x_2}e^{-i\frac{x^2}{4t}}\left(\frac{e^{i\frac{x(j-k+r)}{2t}}}{x^2}N(t,x)\right)_x dx$$

$$+4t\sqrt{t}\Re\sum_{j\neq k;r}\overline{A_j(t)}A_k(t)A_r(t)\frac{e^{i\frac{r^2+k^2-j^2}{4t}}}{i(j-k)}\int_{x_1}^{x_2}e^{i\frac{x^2}{4t}}\left(\frac{e^{i\frac{x(j-k-r)}{2t}}}{x^2}\overline{N(t,x)}\right)_x dx\Bigg|$$

$$\leq C\left(\frac{\sqrt{t}\|\{A_j(t)\}\|_{l^1}}{x_1}+\frac{\sqrt{t}\|\{jA_j(t)\}\|_{l^1}}{x_1^2}+\frac{\|\{j^2A_j(t)\}\|_{l^1}}{x_1\sqrt{t}}+\frac{\|\{jA_j(t)\}\|_{l^1}\|\{A_j(t)\}\|_{l^1}}{x_1}+\frac{\sqrt{t}\|\{A_j(t)\}\|_{l^1}^3}{x_1^2}\right).$$

As $N_x=-uT$ and as $|T(t,x_2)-T(t,x_1)|\leq 2$ we have obtained for $0<x_1<x_2$:

$$|T(t,x_2)-T(t,x_1)|\leq\frac{C(t,\{\alpha_j\})}{\langle x_1\rangle},$$

with

$$C(t,\{\alpha_j\})=C\left(1+\sqrt{t}\|\{jA_j(t)\}\|_{l^1}+\frac{\|\{j^2A_j(t)\}\|_{l^1}}{\sqrt{t}}+\|\{jA_j(t)\}\|_{l^1}\|\{A_j(t)\}\|_{l^1}\right.$$

$$\left.+\sqrt{t}\|\{A_j(t)\}\|_{l^1}^3+\frac{\|\{A_j(t)\}\|_{l^1}^3}{\sqrt{t}}+t\|\{A_j(t)\}\|_{l^1}^4\right). \tag{20}$$

By making $x_1,x_2\to\infty$ we thus obtain the existence of

$$T^\infty(t):=\lim_{x\to\infty}T(t,x), \tag{21}$$

with the desired rate of convergence of the statement.

Now we shall prove that this vector limit is independent of $t>0$. Let $0<t_1<t_2$ and $\epsilon>0$. In view of (21) we can choose x_0 such that for all $x\geq x_0$ we have

$$|T(t_1,x)-T^\infty(t_1)|+|T(t_2,x)-T^\infty(t_2)|\leq\epsilon.$$

Thus in order to get the conclusion (19) of the Lemma, it will be enough to find $x\geq x_0$ such that

$$|T(t_2,x)-T(t_1,x)|\leq\epsilon. \tag{22}$$

To this purpose we use that $T_t = \Im(\overline{u_x}N), N_t = -iu_xT + i\left(\frac{|u|^2}{2} - \frac{M}{2t}\right)N$. These expressions involve a loss of x. However, if a quadratic oscillatory phase $e^{-i\frac{x^2}{4t}}$ is present, integrating it in time yields $\frac{1}{x^2}$ decay, so eventually we gain $\frac{1}{x}$ decay with each such integration by parts:

$$T(t_2,x) - T(t_1,x) = \Im \int_{t_1}^{t_2} \sum_j e^{i(|\alpha_j|^2 - M)\log\sqrt{t}} \overline{\tilde{A}_j(t)} \frac{e^{-i\frac{(x-j)^2}{4t}}}{\sqrt{t}} (-i) \frac{x-j}{2t} N(t,x) dt$$

$$= O(\frac{1}{x}) - 2\Im \int_{t_1}^{t_2} \sum_j e^{-i\frac{x^2}{4t}} \frac{x-j}{x^2} \left(e^{i(|\alpha_j|^2 - M)\log\sqrt{t}} \overline{\tilde{A}_j(t)} e^{i\frac{xj}{2t} - i\frac{j^2}{4t}} \sqrt{t} N(t,x) \right)_t dt$$

$$= O(\frac{1}{x}) + \Re \int_{t_1}^{t_2} \sum_j e^{-i\frac{x^2}{4t}} \frac{x-j}{x} e^{i(|\alpha_j|^2 - M)\log\sqrt{t}} j \overline{\tilde{A}_j(t)} \frac{e^{i\frac{xj}{2t} - i\frac{j^2}{4t}}}{t\sqrt{t}} N(t,x) dt$$

$$-2\Im \int_{t_1}^{t_2} \sum_j e^{-i\frac{x^2}{4t}} \frac{x-j}{x^2} e^{i(|\alpha_j|^2 - M)\log\sqrt{t}} \overline{\tilde{A}_j(t)} e^{i\frac{xj}{2t} - i\frac{j^2}{4t}} \sqrt{t} N_t(t,x) dt.$$

In the first integral we perform again an integration by parts from the quadratic case to get the desired $\frac{1}{x}$ decay, while for the second integral we have to treat only the iu_xT part of N_t:

$$T(t_2,x) - T(t_1,x) = O(\frac{1}{x})$$

$$+2\Im \int_{t_1}^{t_2} \sum_{j\neq k} \frac{(x-j)(x-k)}{x^2} e^{i(|\alpha_j|^2 - |\alpha_k|^2)\log\sqrt{t}} \overline{\tilde{A}_j(t)} \tilde{A}_k(t) e^{i\frac{x(j-k)}{2t} - i\frac{j^2-k^2}{4t}} \frac{T(t,x)}{t} dt.$$

Now we perform an integration by parts using the linear phase in x to get:

$$T(t_2,x) - T(t_1,x) = O(\frac{1}{x})$$

$$+4\Re \int_{t_1}^{t_2} \sum_{j\neq k} \frac{(x-j)(x-k)}{x^3(j-k)} e^{i\frac{x(j-k)}{2t}} \left(e^{i(|\alpha_j|^2 - |\alpha_k|^2)\log\sqrt{t}} \overline{\tilde{A}_j(t)} \tilde{A}_k(t) e^{-i\frac{j^2-k^2}{4t}} tT(t,x) \right)_t dt$$

$$= O(\frac{1}{x})$$

$$+4\Re \int_{t_1}^{t_2} \sum_{j\neq k} \frac{(x-j)(x-k)}{x^3(j-k)} e^{i\frac{x(j-k)}{2t}} e^{i(|\alpha_j|^2 - |\alpha_k|^2)\log\sqrt{t}} \overline{\tilde{A}_j(t)} \tilde{A}_k(t) e^{-i\frac{j^2-k^2}{4t}} t \times$$

$$\times \Im \left(\sum_r e^{i(|\alpha_r|^2 - M)\log\sqrt{t}} \overline{\tilde{A}_r(t)} \frac{e^{-i\frac{(x-r)^2}{4t}}}{\sqrt{t}} (-i) \frac{x-r}{2t} N(t,x) \right) dt.$$

Although we still do not get enough decay in x we have got a quadratic phase in x. Hence, we perform another integration by parts using it to get an extra $\frac{1}{x}$ decay:

$$T(t_2,x) - T(t_1,x) = O(\frac{1}{x}).$$

Therefore we can find x depending on x_0, t_1, t_2 and $\{\alpha_j\}$ such that (22) holds, and the Lemma follows.

□

Lemma 2.2. *There exist $N^{\pm\infty} \in \mathbb{S}^2 + i\mathbb{S}^2$, $\mathbb{S}^2$ denoting the unit sphere in $\mathbb{R}^3$, such that for all $t > 0$*

$$N^{\pm\infty} = \lim_{x \to \pm\infty} N_M(t, x), \tag{23}$$

where for $x \neq 0$

$$N_M(t, x) = e^{iM \log \frac{|x|}{\sqrt{t}}} N(t, x).$$

As a consequence we also have

$$N^{\pm\infty} = \lim_{x \to \pm\infty} e^{iM \log \frac{\langle x \rangle}{\sqrt{t}}} N(t, x).$$

Moreover, we have the following rate of convergence

$$|e^{iM \log \frac{\langle x \rangle}{\sqrt{t}}} N(t, x) - N^{\pm\infty}| \leq \frac{C(t, \{\alpha_j\})}{\langle x \rangle}, \forall x \in \mathbb{R}^{*\pm}.$$

Proof. As done for the tangent vector, we shall first prove that for fixed $t > 0$ there exists a limit vector $N^{\infty}(t)$ for $N_M(t, x)$ as x goes to ∞ ; the asymptotic at $-\infty$ can be treated in the same way.

As for $x > 0$

$$(N_M)_x = (-uT + i\frac{M}{x}N)e^{iM \log \frac{x}{\sqrt{t}}}, \quad T_x = \Re(\overline{u}N),$$

we get for $0 < x_1 < x_2$ by integrating by parts:

$$\begin{aligned}
&N_M(t, x_2) - N_M(t, x_1) \\
&= \int_{x_1}^{x_2} \left(-\sum_j A_j(t) \frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}} T(t, x) + i\frac{M}{x} N(t, x) \right) e^{iM \log \frac{x}{\sqrt{t}}} dx \\
&= \left[-\sum_j A_j(t) e^{i\frac{x^2}{4t}} \frac{2t}{ix} \frac{e^{-i\frac{xj}{2t} + i\frac{j^2}{4t}}}{\sqrt{t}} T(t, x) e^{iM \log \frac{x}{\sqrt{t}}} \right]_{x_1}^{x_2} \\
&- \int_{x_1}^{x_2} 2i\sqrt{t} \sum_j e^{i\frac{j^2}{4t}} A_j(t) e^{i\frac{x^2}{4t}} \left(e^{-i\frac{xj}{2t}} T(t, x) \frac{e^{iM \log \frac{x}{\sqrt{t}}}}{x} \right)_x dx \\
&+ \int_{x_1}^{x_2} i\frac{M}{x} N(t, x) e^{iM \log \frac{x}{\sqrt{t}}} dx.
\end{aligned}$$

Thus

$$\left| N_M(t, x_2) - N_M(t, x_1) + \int_{x_1}^{x_2} \frac{1}{\sqrt{t}} \sum_j e^{i\frac{j^2}{4t}} j A_j(t) e^{i\frac{x^2}{4t}} e^{-i\frac{xj}{2t}} T(t, x) \frac{e^{iM \log \frac{x}{\sqrt{t}}}}{x} dx \right.$$

$$+\int_{x_1}^{x_2} 2i\sum_j A_j(t)e^{i\frac{(x-j)^2}{4t}}\Re\left(\sum_k \overline{A_k(t)}e^{-i\frac{(x-k)^2}{4t}}N(t,x)\right)\frac{e^{iM\log\frac{x}{\sqrt{t}}}}{x}dx$$
$$-\int_{x_1}^{x_2} i\frac{M}{x}N(t,x)e^{iM\log\frac{x}{\sqrt{t}}}dx\Bigg| \leq C\frac{\sqrt{t}\|\{A_j(t)\}\|_{l^1}}{x_1}.$$

In the first integral we perform again an integration by parts using the quadratic phase x^2, and get integrability with a $\frac{1}{x_1}$ decay. In the second integral we develop the real part. The diagonal $k=j$ terms of its non-conjugated part cancel with the third integral, as we have the conservation law $M=\sum_j|\alpha_j|^2=\sum_j|A_j(t)|^2$. We are left with:

$$\Bigg|N_M(t,x_2)-N_M(t,x_1)+i\sum_{j\neq k}A_j(t)\overline{A_k(t)}e^{i\frac{j^2-k^2}{4t}}\int_{x_1}^{x_2}e^{-i\frac{x(j-k)}{2t}}N(t,x)\frac{e^{iM\log\frac{x}{\sqrt{t}}}}{x}dx$$
$$+i\sum_{j,k}A_j(t)A_k(t)e^{i\frac{j^2+k^2}{4t}}\int_{x_1}^{x_2}e^{i\frac{x^2}{2t}}e^{-i\frac{x(j+k)}{2t}}\overline{N(t,x)}\frac{e^{iM\log\frac{x}{\sqrt{t}}}}{x}dx\Bigg|.$$
$$\leq C\left(\frac{\sqrt{t}\|\{A_j(t)\}\|_{l^1}}{x_1}+\frac{\sqrt{t}\|\{jA_j(t)\}\|_{l^1}}{x_1^2}+\frac{\|\{j^2A_j(t)\}\|_{l^1}}{x_1\sqrt{t}}+\frac{\|\{jA_j(t)\}\|_{l^1}\|\{A_j(t)\}\|_{l^1}}{x_1}\right)$$

In the second integral, a new integration by parts using the quadratic phase x^2 yields integrability with a $\frac{1}{x_1}$ decay. In the first integral we integrate by parts using the linear phase $x(j-k)$:

$$\left|N_M(t,x_2)-N_M(t,x_1)+2t\sum_{j\neq k}\frac{A_j(t)\overline{A_k(t)}}{j-k}e^{i\frac{j^2-k^2}{4t}}\int_{x_1}^{x_2}e^{-i\frac{x(j-k)}{2t}}N_x(t,x)\frac{e^{iM\log\frac{x}{\sqrt{t}}}}{x}dx\right|$$
$$\leq C\left(\frac{\sqrt{t}\|\{A_j(t)\}\|_{l^1}}{x_1}+\frac{\sqrt{t}\|\{jA_j(t)\}\|_{l^1}}{x_1^2}+\frac{\|\{j^2A_j(t)\}\|_{l^1}}{x_1\sqrt{t}}+\frac{\|\{jA_j(t)\}\|_{l^1}\|\{A_j(t)\}\|_{l^1}}{x_1}+\frac{\sqrt{t}\|\{A_j(t)\}\|_{l^1}^3}{x_1^2}\right).$$

As $N_x(t,x)=-uT(t,x)$ contains $e^{i\frac{x^2}{4t}}$, we perform a last integration by parts using this quadratic phase to get for all $0<x_1<x_2$:

$$|N_M(t,x_2)-N_M(t,x_1)|\leq\frac{C(t,\{\alpha_j\})}{\langle x_1\rangle},$$

with the same constant $C(t,\{\alpha_j\})$ as in (20):

$$C(t,\{\alpha_j\})=C\left(1+\sqrt{t}\|\{jA_j(t)\}\|_{l^1}+\frac{\|\{j^2A_j(t)\}\|_{l^1}}{\sqrt{t}}+\|\{jA_j(t)\}\|_{l^1}\|\{A_j(t)\}\|_{l^1}\right.$$
$$\left.+\sqrt{t}\|\{A_j(t)\}\|_{l^1}^3+\frac{\|\{A_j(t)\}\|_{l^1}^3}{\sqrt{t}}+t\|\{A_j(t)\}\|_{l^1}^4\right).$$

It follows that we have a limit

$$N^\infty(t):=\lim_{x\to\infty}N_M(t,x), \tag{24}$$

with a rate of convergence in space as in the statement of the lemma.

We are thus left to show the independence on time of $N^\infty(t)$. We fix $0 < t_1 < t_2$ and $\epsilon > 0$, choose x_0 such that

$$|N_M(t_1, x) - N^\infty(t_1)| + |N_M(t_2, x) - N^\infty(t_2)| \leq \epsilon.$$

To finish the proof of the lemma, it will be enough to find $x \geq x_0$ such that

$$|N_M(t_2, x) - N_M(t_1, x)| \leq \epsilon. \tag{25}$$

As the evolution in time laws are

$$T_t = \Im(\overline{u_x} N), \quad (N_M)_t = \left(-iu_x T + i\left(\frac{|u|^2}{2} - \frac{M}{2t}\right) N - i\frac{M}{2t} N\right) e^{iM \log \frac{x}{\sqrt{t}}},$$

we can write

$$N_M(t_2, x) - N_M(t_1, x) = \int_{t_1}^{t_2} \left(-i \sum_j e^{-i(|\alpha_j|^2 - M)\log\sqrt{t}} \tilde{A}_j(t) \frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}} i\frac{x-j}{2t} T(t,x) \right.$$

$$\left. + i \sum_{j \neq k} e^{-i(|\alpha_j|^2 - |\alpha_k|^2)\log\sqrt{t}} \tilde{A}_j(t) \overline{\tilde{A}_k(t)} \frac{e^{i\frac{j^2-k^2}{4t} - i\frac{x(j-k)}{2t}}}{2t} N - i\frac{M}{2t} N \right) e^{iM \log \frac{x}{\sqrt{t}}} dt.$$

In the first integral we perform an integration by parts using the quadratic phase, while in the second we use the linear one:

$$N_M(t_2, x) - N_M(t_1, x) = \left[\sum_j e^{-i(|\alpha_j|^2 - M)\log\sqrt{t}} \tilde{A}_j(t) \frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}} (-\frac{4t^2}{ix^2}) \frac{x-j}{2t} T(t,x) e^{iM \log \frac{x}{\sqrt{t}}} \right]_{t_1}^{t_2}$$

$$-2i \int_{t_1}^{t_2} \sum_j \frac{x-j}{x^2} e^{i\frac{x^2}{4t}} \left(e^{-i(|\alpha_j|^2 - M)\log\sqrt{t}} \tilde{A}_j(t) e^{-i\frac{xj}{2t} + i\frac{j^2}{4t}} \sqrt{t}\, T(t,x) e^{iM \log \frac{x}{\sqrt{t}}} \right)_t dt$$

$$+ \left[i \sum_{j \neq k} e^{-i(|\alpha_j|^2 - |\alpha_k|^2)\log\sqrt{t}} \tilde{A}_j(t) \overline{\tilde{A}_k(t)} \frac{e^{i\frac{j^2-k^2}{4t} - i\frac{x(j-k)}{2t}}}{2t} \frac{2t^2}{ix(j-k)} N e^{iM \log \frac{x}{\sqrt{t}}} \right]_{t_1}^{t_2}$$

$$- \int_{t_1}^{t_2} i \sum_{j \neq k} \frac{1}{x(j-k)} e^{-i\frac{x(j-k)}{2t}} \left(e^{-i(|\alpha_j|^2 - |\alpha_k|^2)\log\sqrt{t}} \tilde{A}_j(t) \overline{\tilde{A}_k(t)} e^{i\frac{j^2-k^2}{4t}} t\, N e^{iM \log \frac{x}{\sqrt{t}}} \right)_t dt$$

$$- \int_{t_1}^{t_2} i\frac{M}{2t} N e^{iM \log \frac{x}{\sqrt{t}}} dt$$

$$= O(\frac{1}{x}) + \int_{t_1}^{t_2} \sum_j \frac{x-j}{x} e^{i\frac{x^2}{4t}} e^{-i(|\alpha_j|^2 - M)\log\sqrt{t}} j \tilde{A}_j(t) e^{-i\frac{xj}{2t} + i\frac{j^2}{4t}} \frac{1}{t\sqrt{t}} T(t,x) e^{iM \log \frac{x}{\sqrt{t}}} dt$$

$$-2i \int_{t_1}^{t_2} \sum_j \frac{x-j}{x^2} e^{i\frac{x^2}{4t}} e^{-i(|\alpha_j|^2 - M)\log\sqrt{t}} \tilde{A}_j(t) e^{-i\frac{xj}{2t} + i\frac{j^2}{4t}} \sqrt{t}\, T_t(t,x) e^{iM \log \frac{x}{\sqrt{t}}} dt$$

$$-\int_{t_1}^{t_2} i\sum_{j\neq k}\frac{1}{x(j-k)}e^{-i\frac{x(j-k)}{2t}}e^{-i(|\alpha_j|^2-|\alpha_k|^2)\log\sqrt{t}}\tilde{A}_j(t)\overline{\tilde{A}_k(t)}e^{i\frac{j^2-k^2}{4t}}t\,N_t e^{iM\log\frac{x}{\sqrt{t}}}dt$$

$$-\int_{t_1}^{t_2} i\frac{M}{2t}Ne^{iM\log\frac{x}{\sqrt{t}}}dt := O(\frac{1}{x})+I_1+I_2+I_3+I_4.$$

In the first integral I_1 an integration by p arts using the quadratic phase gives us the $\frac{1}{x}$ decay. The second integral can be rewritten as

$$I_2 = O(\frac{1}{x})-\frac{2i}{x}\int_{t_1}^{t_2}\sum_j e^{i\frac{x^2}{4t}}e^{-i(|\alpha_j|^2-M)\log\sqrt{t}}\tilde{A}_j(t)e^{-i\frac{xj}{2t}+i\frac{j^2}{4t}}\sqrt{t}\,\Im(\overline{u_x}N(t,x))e^{iM\log\frac{x}{\sqrt{t}}}dt$$

$$=O(\frac{1}{x})+i\int_{t_1}^{t_2}\sum_{j,k}e^{i\frac{(x-j)^2-(x-k)^2}{4t}}e^{-i(|\alpha_j|^2-|\alpha_k|^2)\log\sqrt{t}}\tilde{A}_j(t)\overline{\tilde{A}_k(t)}\frac{1}{2t}N(t,x)e^{iM\log\frac{x}{\sqrt{t}}}dt$$

$$-i\int_{t_1}^{t_2}\sum_{j,k}e^{i\frac{(x-j)^2+(x-k)^2}{4t}}e^{-i(|\alpha_j|^2+|\alpha_k|^2-2M)\log\sqrt{t}}\tilde{A}_j(t)\tilde{A}_k(t)\frac{1}{2t}\overline{N(t,x)}e^{iM\log\frac{x}{\sqrt{t}}}dt$$

$$=O(\frac{1}{x})-I_4+i\int_{t_1}^{t_2}\sum_{j\neq k}e^{i\frac{x(j-k)}{2t}-i\frac{j^2-k^2}{4t}}e^{-i(|\alpha_j|^2-|\alpha_k|^2)\log\sqrt{t}}\tilde{A}_j(t)\overline{\tilde{A}_k(t)}\frac{1}{2t}N(t,x)e^{iM\log\frac{x}{\sqrt{t}}}dt$$

$$-i\int_{t_1}^{t_2}\sum_{j,k}e^{i\frac{x^2}{2t}}e^{i\frac{-x(j+k)}{2t}+i\frac{j^2+k^2}{4t}}e^{-i(|\alpha_j|^2+|\alpha_k|^2-2M)\log\sqrt{t}}\tilde{A}_j(t)\tilde{A}_k(t)\frac{1}{2t}\overline{N(t,x)}e^{iM\log\frac{x}{\sqrt{t}}}dt,$$

where we used the conservation law $M=\sum_j|\alpha_j|^2=\sum_j|\tilde{A}_j(t)|^2$. In the first integral we integrate by parts using the linear phase in x, that gives the decay $\frac{1}{x}$ except when the derivative in time falls on N. This term involves a power of x but also an oscillatory term with a quadratic phase in x. Another integration by parts gives eventually the decay $\frac{1}{x}$. In the last integral a new integration by parts using the quadratic phase gives immediately the decay $\frac{1}{x}$. Therefore

$$I_2+I_4=O(\frac{1}{x}).$$

Finally, in I_3 there is a factor $\frac{1}{x}$ and from N_t we loose a power of x just for the term $-u_xT$. However, this term introduces back the quadratic phase in x, and a new integration by parts yields the $\frac{1}{x}$ decay. Therefore

$$N_M(t_2,x)-N_M(t_1,x)=O(\frac{1}{x}),$$

so (25) follows. The proof of the lemma is over.

□

3. Proof of Theorem 1.1

3.1. **The result on the tangent vector.** We start with the proof of the results at time $t = 0$, namely (15). We will rely from section 4.6 in [4] that at $t = 0$ the curve is a polygonal line so that $T(0, x)$ is piecewise constant with jumps at the integers $j \in \mathbb{Z}$ and that

$$T_x(0) = \sum_j (T(0, j^+) - T(0, j^-))\delta_j = \sum_j \Theta_j (A^+_{|\alpha_j|} - A^-_{|\alpha_j|})\delta_j.$$

Here Θ_j denotes an appropriate rotation (see [4]) and $A^\pm_{|\alpha_j|}$ are the two unit vectors representing the limits at $\pm\infty$ of the tangent of the self-similar solution $\chi_{|\alpha_j|}$. Then, we have

$$\widehat{T_x}(0, \xi) = \sum_j \Theta_j (A^+_{|\alpha_j|} - A^-_{|\alpha_j|}) e^{i2\pi j\xi}.$$

In particular $\widehat{T_x}(0, \xi)$ is periodic in ξ and we get by Plancherel's theorem that for any k

$$\int_k^{k+1} |\widehat{T_x}(0, \xi)|^2 d\xi = \sum_j |\Theta_j (A^+_{|\alpha_j|} - A^-_{|\alpha_j|})|^2 = \sum_j |A^+_{|\alpha_j|} - A^-_{|\alpha_j|}|^2.$$

Therefore calling θ_j the angle between $A^+_{|\alpha_j|}$ and $A^-_{|\alpha_j|}$ and using (3) and (4) in [4] we have

$$|A^+_{|\alpha_j|} - A^-_{|\alpha_j|}|^2 = 2(1 - \cos\theta_j) = 4(1 - e^{-\pi|\alpha_j|^2}),$$

so that we obtain (15), and implicitly (16).

Now we fix $t > 0$ and our purpose it to compute $\Xi(t)$ and to obtain (14). Let $0 < \epsilon < 1$. In view of (10) we choose j_ϵ depending on ϵ, t and $\{\alpha_j\}$ such that

$$\sum_{|j| \geq j_\epsilon} |A_j(t)| \leq \epsilon. \tag{26}$$

In the following C will denote a generic constant dependent on t and $\{\alpha_j\}$, unless it is specified othewise.

Since $T_x(t, x) = \Re(\overline{u}N)(t, x)$ we have

$$\widehat{T_x}(t, \xi) = \int_{-\infty}^{\infty} e^{i2\pi x\xi}\, \Re(\overline{u}N)(t, x)dx$$

$$= \int_{-\infty}^{\infty} e^{i2\pi x\xi}\, \Re(\sum_j \overline{A_j(t)} \frac{e^{-i\frac{(x-j)^2}{4t}}}{\sqrt{t}} N(t, x))dx.$$

We denote η^+ a smooth function vanishing on $x < -\frac{1}{2}$ and valued 1 on $x > \frac{1}{2}$, and we denote $\eta^- = 1 - \eta^+$, so that

$$\widehat{T_x}(t, \xi) = \sum_\pm \int_{-\infty}^{\infty} e^{i2\pi x\xi}\, \Re(\sum_j \overline{A_j(t)} \frac{e^{-i\frac{(x-j)^2}{4t}}}{\sqrt{t}} N(t, x))\, \eta^\pm(x)dx.$$

With the notations from Lemma 2.2, on the integral involving $\eta^{\pm}$ we split

$$N(t,x) = N^{\pm\infty} e^{-iM\log\frac{\langle x\rangle}{\sqrt{t}}} + g_N^{\pm}(t,x),$$

where

$$g_N^{\pm}(t,x) := (N(t,x) - N^{\pm\infty} e^{-iM\log\frac{\langle x\rangle}{\sqrt{t}}}).$$

We define

$$\widehat{T_x}(t,\xi) = I(t,\xi) + J(t,\xi), \tag{27}$$

where $I(t,\xi)$ gathers the terms in $\widehat{T_x}(t,\xi)$ corresponding to $N^{\pm\infty}$ and $J(t,\xi)$ the ones corresponding to $g_N^{\pm}$. We shall start by estimating the second term $J(t,\xi)$.

First, we complete the squares of the phases:

$$J(t,\xi) = \frac{1}{2}\sum_{\pm}\int_{-\infty}^{\infty} e^{i2\pi x\xi} \sum_j \overline{A_j(t)}\frac{e^{-i\frac{(x-j)^2}{4t}}}{\sqrt{t}} g_N^{\pm}(t,x)\,\eta^{\pm}(x)dx$$

$$+\frac{1}{2}\sum_{\pm}\int_{-\infty}^{\infty} e^{i2\pi x\xi} \sum_j A_j(t)\frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}} \overline{g_N^{\pm}(t,x)}\,\eta^{\pm}(x)dx$$

$$= \frac{e^{i4\pi^2 t\xi^2}}{2\sqrt{t}}\sum_{\pm,j} e^{i2\pi j\xi}\,\overline{A_j(t)}\int_{-\infty}^{\infty} e^{-i\frac{(x-j-4\pi t\xi)^2}{4t}} g_N^{\pm}(t,x)\,\eta^{\pm}(x)dx$$

$$+\frac{e^{-i4\pi^2 t\xi^2}}{2\sqrt{t}}\sum_{\pm,j} e^{i2\pi j\xi}\,A_j(t)\int_{-\infty}^{\infty} e^{i\frac{(x-j+4\pi t\xi)^2}{4t}} \overline{g_N^{\pm}(t,x)}\,\eta^{\pm}(x)dx.$$

We split now the summation into $|j| < j_\epsilon$ and $|j| \geq j_\epsilon$, and call the corresponding terms $J^l(t,\xi)$ and $J^h(t,\xi)$.

Lemma 3.1. *There exists $\xi(\epsilon,t,\{\alpha_j\}) \in \mathbb{R}$ such that for $\xi \geq \xi(\epsilon,t,\{\alpha_j\})$ and $4\pi t\xi \notin \mathbb{Z}$ we have the bounds*

$$|J^l(t,\xi)| \leq \begin{cases} C\epsilon, \ \text{if } d(2\pi\xi, \frac{\mathbb{Z}}{2t}) \geq 1, \\ C\epsilon\,|\log(d(2\pi\xi, \frac{\mathbb{Z}}{2t}))|, \ \text{if } d(2\pi\xi, \frac{\mathbb{Z}}{2t}) < 1. \end{cases}$$

Proof. In virtue of Lemma 2.2, $g_N^{\pm}$ are bounded functions with

$$|g_N^{\pm}(t,x)| \leq \frac{C}{\langle x\rangle}, \forall x \in \mathbb{R}^{*\pm},$$

so $g_N^{\pm}(t,x)\eta^{\pm}(x)$ converge to zero at both $-\infty$ and $+\infty$. Therefore we can remove from

$$J^l(t,\xi) = \frac{e^{i4\pi^2 t\xi^2}}{2\sqrt{t}}\sum_{\pm,|j|<j_\epsilon} e^{i2\pi j\xi}\,\overline{A_j(t)}\int_{-\infty}^{\infty} e^{-i\frac{(x-j-4\pi t\xi)^2}{4t}} g_N^{\pm}(t,x)\eta^{\pm}(x)dx$$

$$+\frac{e^{-i4\pi^2 t\xi^2}}{2\sqrt{t}}\sum_{\pm,|j|<j_\epsilon} e^{i2\pi j\xi}\,A_j(t)\int_{-\infty}^{\infty} e^{i\frac{(x-j+4\pi t\xi)^2}{4t}}\overline{g_N^{\pm}(t,x)}\eta^{\pm}(x)dx,$$

bounded pieces of the integrals in x located around $j\pm 4\pi t\xi$. Indeed on these parts, since $|j|\le j_\epsilon$, we have convergence to zero as ξ goes to infinity. Therefore there exists $\xi(\epsilon,t,\{\alpha_j\})$ such that for $\xi\ge\xi(\epsilon,t,\{\alpha_j\})$ we have

$$|J^l(t,\xi)-J_1^l(t,\xi)|\le\epsilon,$$

where

$$J_1^l(t,\xi)=\frac{e^{i4\pi^2 t\xi^2}}{2\sqrt{t}}\sum_{\pm,|j|<j_\epsilon} e^{i2\pi j\xi}\,\overline{A_j(t)}\int_{-\infty}^{\infty} e^{-i\frac{(x-j-4\pi t\xi)^2}{4t}}g_N^{\pm}(t,x)\eta^{\pm}(x)\chi(x-j-4\pi t\xi)dx$$

$$+\frac{e^{-i4\pi^2 t\xi^2}}{2\sqrt{t}}\sum_{\pm,|j|<j_\epsilon} e^{i2\pi j\xi}\,A_j(t)\int_{-\infty}^{\infty} e^{i\frac{(x-j+4\pi t\xi)^2}{4t}}\overline{g_N^{\pm}(t,x)}\eta^{\pm}(x)\chi(x-j+4\pi t\xi)dx,$$

and $\chi(s)$ is a smooth function vanishing on $\{x,|x|<\frac12\}$, and valued 1 on $\{x,|x|>1\}$. In particular the support of χ' is bounded. Now we integrate by parts using the quadratic phases. Again since $g_N^{\pm}(t,x)\eta^{\pm}(x)$ converge to zero at both $-\infty$ and $+\infty$ there are no boundary terms and we get:

$$J_1^l(t,\xi)=-i\sqrt{t}e^{i4\pi^2 t\xi^2}\sum_{\pm,|j|<j_\epsilon} e^{i2\pi j\xi}\,\overline{A_j(t)}\int_{-\infty}^{\infty} e^{-i\frac{(x-j-4\pi t\xi)^2}{4t}}\left(\frac{g_N^{\pm}(t,x)\eta^{\pm}(x)\chi(x-j-4\pi t\xi)}{x-j-4\pi t\xi}\right)_x dx$$

$$+i\sqrt{t}e^{-i4\pi^2 t\xi^2}\sum_{\pm,|j|<j_\epsilon} e^{i2\pi j\xi}\,A_j(t)\int_{-\infty}^{\infty} e^{i\frac{(x-j+4\pi t\xi)^2}{4t}}\left(\frac{\overline{g_N^{\pm}(t,x)}\eta^{\pm}(x)\chi(x-j+4\pi t\xi)}{x-j+4\pi t\xi}\right)_x dx.$$

When the derivative falls on χ or on the denominator, we get again smallness by using the dominated convergence theorem. We are left with the terms involving $(g_N^{\pm})_x=-uT+i\frac{M}{\langle x\rangle}N^{\pm\infty}e^{-iM\log\frac{\langle x\rangle}{\sqrt{t}}}$. Now we note that we can discard also the last term of $(g_N^{\pm})_x(t,x)$ as for instance [2]

$$\left|\int_{-\infty}^{\infty} e^{\mp i\frac{(x-j\mp 4\pi t\xi)^2}{4t}}\frac{e^{\mp iM\log\frac{\langle x\rangle}{\sqrt{t}}}}{(x-j\mp 4\pi t\xi)\langle x\rangle}\eta^{+}(x)\chi(x-j\mp 4\pi t\xi)dx\right|\le\frac{C}{|j\pm 4\pi t\xi|}\le\epsilon,$$

for $\xi>0$ far away from the finite set $\{j,|j|<j_\epsilon\}$ and choosing $\xi(\epsilon,t,\{\alpha_j\})$ larger if needed. Therefore, we are left with estimating the terms of $J_1^l(t,\xi)$ involving the $-uT$ part of $(g_N^{\pm})_x$: there exists $\xi(\epsilon,t,\{\alpha_j\})$ such that for $\xi\ge\xi(\epsilon,t,\{\alpha_j\})$ we have

$$|J^l(t,\xi)-J_2^l(t,\xi)|\le C\epsilon,$$

[2]Indeed, we can use for large a the fact that $\int e^{is^2}\frac{e^{iM\log\langle a+s\rangle}}{s\langle a+s\rangle}\eta^{\pm}(a+s)\chi(s)ds=O(\frac1a)+\int e^{is^2}\left(\frac{e^{iM\log\langle a+s\rangle}}{s^2\langle a+s\rangle}\eta^{\pm}(a+s)\chi(s)\right)_s ds$, and split the integral into regions $\frac12\le|s|\le\frac{a}{2},\frac{a}{2}\le|s|\le 2a,2a\le|s|$ to get a $\frac1a$-bound.

with

$$J_2^l(t,\xi)=i\sum_{\pm,|j|<j_\epsilon}\overline{A_j(t)}\int_{-\infty}^{\infty}\frac{\sum_r A_r(t)e^{i\frac{x(j-r+4\pi t\xi)}{2t}}e^{-i\frac{j^2-r^2}{4t}}}{x-j-4\pi t\xi}T(t,x)\eta^{\pm}(x)\chi(x-j-4\pi t\xi)dx$$

$$-i\sum_{\pm,|j|<j_\epsilon}A_j(t)\int_{-\infty}^{\infty}\frac{\sum_r \overline{A_r(t)}e^{i\frac{x(r-j+4\pi t\xi)}{2t}}e^{i\frac{j^2-r^2}{4t}}}{x-j+4\pi t\xi}T(t,x)\eta^{\pm}(x)\chi(x-j+4\pi t\xi)dx.$$

Note that the summation $\sum_{\pm}$ and $\eta^{\pm}$ can be now removed as $\eta^+ + \eta^- = 1$.

We treat first the terms involving $|r| < j_\epsilon$. If needed we choose $\xi(\epsilon, t, \{\alpha_j\})$ larger such that for $|r| < j_\epsilon$ and $\xi \geq \xi(\epsilon, t, \{\alpha_j\})$ we have:

$$\frac{1}{\pm(j-r)+4\pi t\xi}\leq\epsilon.$$

We perform in the corresponding integrals an integration by parts using the linear phase in x. Then, we get the ϵ-smallness from the above constraint, and the integral that yields is uniformly bounded. Indeed, when the derivative falls either on $\chi, \eta^{\pm}$ or on the denominator $\frac{1}{x-j\mp 4\pi t\xi}$ we get immediately an uniform bound on the integral. When the derivative falls on $T(t,x)$ it generates a quadratic phase. Hence we can first remove a bounded piece of the integral centered where the phase vanishes, and then we can integrate by parts to get again a uniform bound for the integral.

We are thus left with estimating the terms involving $|r| \geq j_\epsilon$, for which the linear phase might approach zero: there exists $\xi(\epsilon, t, \{\alpha_j\})$ such that for $\xi \geq \xi(\epsilon, t, \{\alpha_j\})$ we have

$$|J^l(t,\xi)-J_3^l(t,\xi)|\leq C\epsilon,$$

with

$$J_3^l(t,\xi)=i\sum_{|j|<j_\epsilon,|r|\geq j_\epsilon}\overline{A_j(t)}A_r(t)\,I^+(t,\xi,j,r)-i\sum_{|j|<j_\epsilon,|r|\geq j_\epsilon}A_j(t)\overline{A_r(t)}\,I^-(t,\xi,j,r),\tag{28}$$

where

$$I^{\pm}(t,\xi,j,r):=e^{\mp i\frac{j^2-r^2}{4t}}\int_{-\infty}^{\infty}\frac{e^{ix(\pm\frac{j-r}{2t}+2\pi\xi)}}{x-j\mp 4\pi t\xi}\,T(t,x)\,\chi(x-j\mp 4\pi t\xi)\,dx.\tag{29}$$

We first note that in view of (26) we have $\epsilon-$smallness of $\sum_{|r|\geq j_\epsilon}|A_r(t)|$. For $4\pi t\xi\notin\mathbb{Z}$ we can integrate by parts in $I^{\pm}(t,\xi,j,r)$ using the linear phase to get the bound $\frac{C}{d(4\pi\xi,\frac{\mathbb{Z}}{t})}$. Therefore, we cannot control this way the $L^2(k,k+1)$ norm in ξ. To overcome this difficulty we shall prove that for $4\pi t\xi\notin\mathbb{Z}$:

$$|I^{\pm}(t,\xi,j,r)|\leq\begin{cases}C,\ \text{if}\ |\pm\frac{j-r}{t}+4\pi\xi|\geq 1,\\ C|\log(|\pm\frac{j-r}{t}+4\pi\xi|)|,\ \text{if}\ |\pm\frac{j-r}{t}+4\pi\xi|<1.\end{cases}\tag{30}$$

These bounds imply

$$|I^+(t,\xi,j,r)|+|I^-(t,\xi,j,r)|\leq\begin{cases} C,\ \text{if } |\frac{j-r}{t}+4\pi\xi|\geq 1, |\frac{j-r}{t}+4\pi\xi|\geq 1,\\ C|\log(|\frac{j-r}{t}+4\pi\xi|)|,\ \text{if } |\frac{j-r}{t}+4\pi\xi|<1,\\ C|\log(|-\frac{j-r}{t}+4\pi\xi|)|,\ \text{if } |-\frac{j-r}{t}+4\pi\xi|<1.\end{cases}$$

Note that for $0<t<1$ the last two regions intersect if and only if $|2(j-r)|<2t<2$, that is when $r=j$ and in that case the bound is the same, $C\log(4\pi|\xi|)$. Then, by summing in j and r, and by using (26) we get for $4\pi t\xi\notin\mathbb{Z}$ the bounds

$$|J_3^l(t,\xi)|\leq\begin{cases} C\epsilon,\ \text{if } d(4\pi\xi,\frac{\mathbb{Z}}{t})\geq 1,\\ C\epsilon\,|\log(d(4\pi\xi,\frac{\mathbb{Z}}{t}))|,\ \text{if } d(4\pi\xi,\frac{\mathbb{Z}}{t})<1,\end{cases}$$

thus the lemma follows from (28).

We are thus left with proving (30). We split the integral in $I^+(t,\xi,j,r)$ into the regions $x<0$ and $x>0$:

$$I^{\pm}(t,\xi,j,r)=e^{\mp i\frac{j^2-r^2}{4t}}\int_0^\infty\frac{e^{ix(\pm\frac{j-r}{2t}+2\pi\xi)}}{x-j\mp 4\pi t\xi}\,T(t,x)\,\chi(x-j\mp 4\pi t\xi)\,dx$$

$$+e^{\mp i\frac{j^2-r^2}{4t}}\int_{-\infty}^0\frac{e^{ix(\pm\frac{j-r}{2t}+2\pi\xi)}}{x-j\mp 4\pi t\xi}\,T(t,x)\,\chi(x-j\mp 4\pi t\xi)\,dx.$$

By using the convergence rate in Lemma 2.1:

$$|(T(t,x)-T^\infty)\mathbb{I}_{(0,\infty)}(x)|+|(T(t,x)-T^{-\infty})\mathbb{I}_{(-\infty,0)}(x)|\leq\frac{C}{\langle x\rangle},\forall x\in\mathbb{R},$$

and in view of the definition of χ we get

$$|I^{\pm}(t,\xi,j,r)-\tilde{I}^{\pm}(t,\xi,j,r)|\leq C,$$

where

$$\tilde{I}^{\pm}(t,\xi,j,r):=T^\infty e^{\mp i\frac{j^2-r^2}{4t}}\int_{x>0,|x-j\mp 4\pi t\xi|>1}\frac{e^{ix(\pm\frac{j-r}{2t}+2\pi\xi)}}{x-j\mp 4\pi t\xi}\,dx \tag{31}$$

$$+T^{-\infty}e^{\mp i\frac{j^2-r^2}{4t}}\int_{x<0,|x-j\mp 4\pi t\xi|>1}\frac{e^{ix(\pm\frac{j-r}{2t}+2\pi\xi)}}{x-j\mp 4\pi t\xi}\,dx.$$

If $|\pm\frac{j-r}{t}+4\pi\xi|\geq 1$ we perform an integration by parts using the linear phase and get the bound uniform in ξ,j, and r in (30).

If $|\pm\frac{j-r}{t}+4\pi\xi|<1$ we denote for simplicity $a=-j\mp 4\pi t\xi$ and $b=\pm\frac{j-r}{2t}+2\pi\xi$. We change variables $x+a=y$, $yb=s$ to rewrite $\tilde{I}^{\pm}(t,\xi,j,r)$ as:

$$e^{\mp i\frac{j^2-r^2}{4t}}e^{-iab}\,(T^\infty\int_{\frac{s}{b}>a,\,|s|>|b|}\frac{e^{is}}{s}\,ds+T^{-\infty}\int_{\frac{s}{b}<a,\,|s|>|b|}\frac{e^{is}}{s}\,ds), \tag{32}$$

On the region where $|s|>1$, due to the oscillatory phase we get a bound uniform in ξ, j, and r. Finally, on the region where $|b|<|s|<1$, if such regions exist, the integration of $\frac{e^{is}}{s}$ yields a $\log(|b|)$ bound. Therefore we have obtained (30) and the Lemma follows.

For further purposes we note that we have obtained for $|\pm\frac{j-r}{t}+4\pi\xi|<1$ the estimate

$$|\tilde{I}^{\pm}(t,\xi,j,r)-e^{\mp i\frac{j^2-r^2}{4t}}e^{-i(-j\mp 4\pi t\xi)(\pm\frac{j-r}{2t}+2\pi\xi)}\,(T^{\infty}-T^{-\infty}) \times\int_{s>(-j\mp 4\pi t\xi)(\pm\frac{j-r}{2t}+2\pi\xi),\,1>|s|>|\pm\frac{j-r}{2t}+2\pi\xi|}\frac{e^{is}}{s}\,ds|\leq C, \tag{33}$$

with C an universal constant. □

Lemma 3.2. *There exists $\xi(\epsilon,t,\{\alpha_j\})$ such that for $\xi\geq\xi(\epsilon,t,\{\alpha_j\})$ and $4\pi t\xi\notin\mathbb{Z}$ we have the bounds*

$$|J^h(t,\xi)|\leq\begin{cases} C\epsilon, \ \text{if } d(4\pi\xi,\frac{\mathbb{Z}}{t})\geq 1,\\ C\epsilon\,|\log(d(4\pi\xi,\frac{\mathbb{Z}}{t}))|, \ \text{if } d(4\pi\xi,\frac{\mathbb{Z}}{t})<1,\end{cases}$$

Proof. Recall that

$$J^h(t,\xi)=\frac{e^{i4\pi^2t\xi^2}}{2\sqrt{t}}\sum_{\pm,|j|\geq j_\epsilon}e^{i2\pi j\xi}\,\overline{A_j(t)}\int_{-\infty}^{\infty}e^{-i\frac{(x-j-4\pi t\xi)^2}{4t}}g_N^{\pm}(t,x)\eta^{\pm}(x)dx$$
$$+\frac{e^{-i4\pi^2t\xi^2}}{2\sqrt{t}}\sum_{\pm,|j|\geq j_\epsilon}e^{i2\pi j\xi}\,A_j(t)\int_{-\infty}^{\infty}e^{i\frac{(x-j+4\pi t\xi)^2}{4t}}\overline{g_N^{\pm}(t,x)}\eta^{\pm}(x)dx.$$

In this case we will get the ϵ-decay from $A_j(t)$ thanks to (26). We can remove the same pieces of the integrals as in the proof of the Lemma 3.1 to end with

$$i\sum_{|j|\geq j_\epsilon;r}\overline{A_j(t)}A_r(t)\,I^{+}(t,\xi,j,r)-i\sum_{|j|\geq j_\epsilon;r}A_j(t)\overline{A_r(t)}\,I^{-}(t,\xi,j,r), \tag{34}$$

where $I^{\pm}(t,\xi,j,r)$ were defined in (29). This can be handled the same way as was done for $J_3^l(t,\xi)$ in the proof of Lemma 3.1. □

Lemma 3.3. *For any $\xi\in\mathbb{R}$ we have:*

$$|I(t,\xi)|\leq C\|\{A_j(t)\}\|_{l^1},$$

with C an universal constant. Moreover, there exists $\xi(\epsilon,t,\{\alpha_j\})$ such that for $k\geq\xi(\epsilon,t,\{\alpha_j\})$

$$\left|\int_k^{k+1}|I(t,\xi)|^2d\xi-4\pi\sum_j|\alpha_j|^2\right|\leq C\,\epsilon.$$

Proof. We start by performing some change of variables in the expression of $I(t,\xi)$:

$$I(t,\xi)=\frac{1}{2}\sum_{\pm}\int_{-\infty}^{\infty}e^{i2\pi x\xi}\,\sum_j\overline{A_j(t)}\frac{e^{-i\frac{(x-j)^2}{4t}}}{\sqrt{t}}N^{\pm\infty}e^{-iM\log\frac{\langle x\rangle}{\sqrt{t}}}\eta^{\pm}(x)dx$$

$$+\frac{1}{2}\sum_{\pm}\int_{-\infty}^{\infty} e^{i2\pi x\xi}\sum_j A_j(t)\frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}}\overline{N^{\pm\infty}}e^{iM\log\frac{\langle x\rangle}{\sqrt{t}}}\eta^{\pm}(x)dx$$

$$= N^{\pm\infty}\frac{e^{i4\pi^2t\xi^2}}{2\sqrt{t}}\sum_{\pm,j} e^{i2\pi j\xi}\,\overline{A_j(t)}\int_{-\infty}^{\infty} e^{-i\frac{(x-j-4\pi t\xi)^2}{4t}}e^{-iM\log\frac{\langle x\rangle}{\sqrt{t}}}\eta^{\pm}(x)dx$$

$$+\overline{N^{\pm\infty}}\frac{e^{-i4\pi^2t\xi^2}}{2\sqrt{t}}\sum_{\pm,j} e^{i2\pi j\xi}\,A_j(t)\int_{-\infty}^{\infty} e^{i\frac{(x-j+4\pi t\xi)^2}{4t}}e^{iM\log\frac{\langle x\rangle}{\sqrt{t}}}\eta^{\pm}(x)dx.$$

$$= N^{\pm\infty}e^{i4\pi^2t\xi^2}\sum_{\pm,j} e^{i2\pi j\xi}\,\overline{A_j(t)}\int_{-\infty}^{\infty} e^{-iy^2}e^{-iM\log\frac{\langle 2\sqrt{t}y+j+4\pi t\xi\rangle}{\sqrt{t}}}\eta^{\pm}(2\sqrt{t}y+j+4\pi t\xi)dy$$

$$+\overline{N^{\pm\infty}}e^{-i4\pi^2t\xi^2}\sum_{\pm,j} e^{i2\pi j\xi}\,A_j(t)\int_{-\infty}^{\infty} e^{iy^2}e^{iM\log\frac{\langle 2\sqrt{t}y+j-4\pi t\xi\rangle}{\sqrt{t}}}\eta^{\pm}(2\sqrt{t}y+j-4\pi t\xi)dy.$$

We first note that the integrals are uniformly bounded in j and ξ: the contribution of the bounded region $|y|<1$ is bounded as the integrant is of modulus less than one, while the contribution of the region $|y|>1$ is bounded by doing integrations by parts using the quadratic phase. Therefore, we get the first bound of the Lemma.

To estimate $\int_k^{k+1}|I(t,\xi)|^2d\xi$ we shall split $I(t,\xi)$ into a function of size of order ϵ and a function of $L^2(k,k+1)$-norm equal to $4\pi\sum_j|\alpha_j|^2$. In view of the definition (26) of j_ϵ, the terms in $I(t,\xi)$ involving $|j|>j_\epsilon$ can be upper-bounded by $C\epsilon$. We are left with the terms involving $|j|\le j_\epsilon$. Observe that[3]

$$\lim_{a\to\pm\infty}\left(\int_{-\infty}^{\infty}e^{iy^2}e^{iM\log\langle 2y+a\rangle}\eta^{\pm}(2\sqrt{t}y+\sqrt{t}a)dy - e^{iM\log\langle a\rangle}\int_{-\infty}^{\infty}e^{iy^2}dy\right)=0,$$

and

$$\lim_{a\to\pm\infty}\int_{-\infty}^{\infty}e^{iy^2}e^{iM\log\langle 2y+a\rangle}\eta^{\mp}(2\sqrt{t}y+\sqrt{t}a)dy=0.$$

Hence, choosing $\xi(\epsilon,t,\{\alpha_j\})$ larger if needed, for $|j|\le j_\epsilon$ and $\xi\ge\xi(\epsilon,t,\{\alpha_j\})$ we get:

$$\left|\int_{-\infty}^{\infty}e^{\mp iy^2}e^{\mp iM\log\frac{\langle 2\sqrt{t}y+j\pm 4\pi t\xi\rangle}{\sqrt{t}}}\eta^{\pm}(2\sqrt{t}y+j\pm 4\pi t\xi)ds - e^{\mp iM\log\frac{\langle j\pm 4\pi t\xi\rangle}{\sqrt{t}}}\int_{-\infty}^{\infty}e^{\mp is^2}ds\right|\le\epsilon,$$

and

$$\left|\int_{-\infty}^{\infty}e^{\mp iy^2}e^{\mp iM\log\frac{\langle 2\sqrt{t}y+j\pm 4\pi t\xi\rangle}{\sqrt{t}}}\eta^{\mp}(2\sqrt{t}y+j\pm 4\pi t\xi)ds\right|\le\epsilon.$$

Therefore we have for $\xi\ge k_\epsilon$

$$|I(t,\xi)-I^1(t,\xi)-I^2(t,\xi)|\le C\,\epsilon, \tag{35}$$

[3]Indeed, the integral $\int_{-\infty}^{\infty}e^{iy^2}(e^{iM\log\langle 2y+a\rangle}-e^{iM\log\langle a\rangle})\eta^{\pm}(2\sqrt{t}y+\sqrt{t}a)dy$ can be upper-bounded by $\frac{C}{|a|}$ on $|s|<1$, while on $|s|>1$ by performing integration by parts from the quadratic phase and by using the dominated convergence theorem we get decay to zero as $|a|\to\infty$. Then by the same type of arguments we have $\lim_{a\to\pm\infty}\left(\int_{-\infty}^{\infty}e^{iy^2}\eta^{\pm}(2\sqrt{t}y+\sqrt{t}a)dy-\int_{-\infty}^{\infty}e^{iy^2}dy\right)=0$ and $\lim_{a\to\pm\infty}\int_{-\infty}^{\infty}e^{iy^2}\eta^{\mp}(2\sqrt{t}y+\sqrt{t}a)dy=0$.

where

$$I^1(t,\xi)=N^{\infty}e^{i4\pi^2t\xi^2}\sqrt{\pi}e^{i\frac{\pi}{4}}\sum_{|j|\leq j_\epsilon}e^{i2\pi j\xi}\,e^{-iM\log\frac{\langle j+4\pi t\xi\rangle}{\sqrt{t}}}\,\overline{A_j(t)},$$

and

$$I^2(t,\xi)=\overline{N^{-\infty}}e^{-i4\pi^2t\xi^2}\sqrt{\pi}e^{-i\frac{\pi}{4}}\sum_{|j|\leq j_\epsilon}e^{i2\pi j\xi}\,e^{iM\log\frac{\langle j-4\pi t\xi\rangle}{\sqrt{t}}}\,A_j(t).$$

Since $I^1_1(t,\xi)$ and $I^2_1(t,\xi)$ are uniformly bounded by $2\sqrt{\pi}\sum_j|A_j(t)|$, we obtain from (35)

$$\left|\int_k^{k+1}|I(t,\xi)|^2d\xi-\int_k^{k+1}|I^1(t,\xi)|^2d\xi-\int_k^{k+1}|I^2(t,\xi)|^2d\xi-\int_k^{k+1}I^1(t,\xi)\overline{I^2(t,\xi)}d\xi\right|\leq C\,\epsilon.$$

Then, as $|N^{\pm\infty}|=2$, Plancherel's formula gives us for $k\geq k_\epsilon$

$$\left|\int_k^{k+1}|I(t,\xi)|^2d\xi-4\pi\sum_{|j|\leq j_\epsilon}|A_j(t)|^2-\int_k^{k+1}I^1(t,\xi)\overline{I^2(t,\xi)}d\xi\right|\leq C\,\epsilon.$$

Now we see that the crossed terms are

$$\int_k^{k+1}I^1(t,\xi)\overline{I^2(t,\xi)}d\xi=N^{\infty}.N^{-\infty}\,\pi e^{i\frac{\pi}{2}}\sum_{|j_1|,|j_2|\leq j_\epsilon}\overline{A_{j_1}(t)A_{j_2}(t)}\times$$

$$\times\int_k^{k+1}e^{i8\pi^2t\xi^2}e^{i2\pi(j_1-j_2)\xi}e^{-iM\log\frac{\langle j+4\pi t\xi\rangle}{\sqrt{t}}}e^{-iM\log\frac{\langle j-4\pi t\xi\rangle}{\sqrt{t}}}d\xi.$$

One single integration by parts using the quadratic phase in ξ gives us decay in k, so choosing $\xi(\epsilon,t,\{\alpha_j\})$ larger if needed we obtain for $k\geq\xi(\epsilon,t,\{\alpha_j\})$

$$\left|\int_k^{k+1}|I(t,\xi)|^2d\xi-4\pi\sum_{|j|\leq j_\epsilon}|A_j(t)|^2\right|\leq C\,\epsilon.$$

Recalling the choice (26) of j_ϵ and the conservation law (12) we get for $k\geq\xi(\epsilon,t,\{\alpha_j\})$

$$\left|\int_k^{k+1}|I(t,\xi)|^2d\xi-4\pi\sum_j|\alpha_j|^2\right|\leq C\,\epsilon.$$

□

Summarizing we have decomposed

$$\widehat{T_x}(t,\xi)=:I(t,\xi)+J(t,\xi),$$

and proved in Lemmas 3.1-3.2-3.3 that there exists $\xi(\epsilon,t,\{\alpha_j\})\in\mathbb{R}$ such that for $\xi\geq\xi(\epsilon,t,\{\alpha_j\})$ and $4\pi t\xi\notin\mathbb{Z}$ we have the bounds:

$$|J(t,\xi)|\leq\begin{cases}C\epsilon,\ \text{if } d(4\pi\xi,\frac{\mathbb{Z}}{t})\geq 1,\\ C\epsilon\,|\log(d(4\pi\xi,\frac{\mathbb{Z}}{t}))|,\ \text{if } d(4\pi\xi,\frac{\mathbb{Z}}{t})<1;\end{cases}$$

$$|I(t,\xi)|\leq C,$$

and for all $k \geq \xi(\epsilon, t, \{\alpha_j\})$:

$$\left| \int_k^{k+1} |I(t,\xi)|^2 d\xi - 4\pi \sum_j |\alpha_j|^2 \right| \leq C\,\epsilon.$$

We note that for ξ in an interval of size one, there are only a finite number of possible locations where $d(4\pi\xi, \frac{\mathbb{Z}}{t}) < 1$, depending only on t, and on these regions $J(t,\xi)$ is square integrable. Therefore

$$\left| \int_k^{k+1} |\widehat{T_x}(t,\xi)|^2 d\xi - 4\pi \sum_j |\alpha_j|^2 \right| \leq C\,\epsilon, \forall k \geq \xi(\epsilon, t, \{\alpha_j\}).$$

The value of $0 < \epsilon < 1$ was arbitrary, the constant C is independent of ϵ, so we obtain the conservation law (14), and the proof of Theorem 1.1 is complete.

3.2. **The result on the normal vectors.** In this subsection we obtain the results (17) and (18) from Remark 1.3. We recall from Lemmas 4.5-4.7 in [4] that we have a limit $\tilde{N}(0)$ at $t = 0$ of

$$\tilde{N}(t,x) = e^{i\sum_{r\neq x} |\alpha_r|^2 \log \frac{|x-r|}{\sqrt{t}}} N(t,x),$$

that is piecewise constant

$$\tilde{N}(0,x) = \tilde{N}(0,x'), \forall x, x' \in (j, j+1), \forall j \in \mathbb{Z},$$

and

$$\tilde{N}(0,j^{\pm}) = e^{i\sum_{r\neq j} |\alpha_r|^2 \log|r-j|} e^{iArg(\alpha_j)} \Theta_j(B^{\pm}_{|\alpha_j|}).$$

Here $B^{\pm}_{|\alpha_j|} \in \mathbb{S}^2 + i\mathbb{S}^2$ are defined in [16] in terms of the asymptotics at $\pm\infty$ of the normal vectors of the self-similar solution $\chi_{|\alpha_j|}$. It follows that at $t = 0$ we have

$$\tilde{N}_x(0) = \sum_j (\tilde{N}(0,j^+) - \tilde{N}(0,j^-))\delta_j = \sum_j e^{i\sum_{r\neq j} |\alpha_r|^2 \log|r-j|} e^{iArg(\alpha_j)} \Theta_j (B^+_{|\alpha_j|} - B^-_{|\alpha_j|})\delta_j,$$

so

$$\widehat{\tilde{N}_x}(0,\xi) = \sum_j e^{i\sum_{r\neq j} |\alpha_r|^2 \log|r-j|} e^{iArg(\alpha_j)} \Theta_j (B^+_{|\alpha_j|} - B^-_{|\alpha_j|}) e^{i2\pi j}.$$

As $\widehat{\tilde{N}_x}(0,\xi)$ is periodic in ξ, we get by Plancherel's theorem that for any k

$$\int_k^{k+1} |\widehat{\tilde{N}_x}(0,\xi)|^2 d\xi = \sum_j |\Theta_j (B^+_{|\alpha_j|} - B^-_{|\alpha_j|})|^2 = \sum_j |B^+_{|\alpha_j|} - B^-_{|\alpha_j|}|^2.$$

Therefore, as we know from [16] that

$$|B^+_{|\alpha_j|} - B^-_{|\alpha_j|}|^2 = 4|B^+_{|\alpha_j|,1}|^2 = 4(1 - (A^+_{|\alpha_j|,1})^2) = 4(1 - e^{-\pi|\alpha_j|^2}),$$

we obtain (18).

For $t>0$ we fix $\epsilon\in(0,1)$. We split:

$$\widehat{N_x}(t,\xi)=-\sum_{\pm}\int_{-\infty}^{\infty}e^{i2\pi x\xi}\,u(t,x)(T^{\pm\infty}+(T(t,x)-T^{\pm\infty}))\eta^{\pm}(x)dx$$

$$=-\int_{-\infty}^{\infty}e^{i2\pi x\xi}\sum_{\pm,j}A_j(t)\frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}}(T^{\pm\infty}+(T(t,x)-T^{\pm\infty}))\eta^{\pm}(x)dx=:\tilde{I}(t,\xi)+\tilde{J}(t,\xi).$$

Proceeding as above for $J(t,\xi)$ we get the existence of $\xi(\epsilon,t,\{\alpha_j\})$ such that

$$\tilde{J}(t,\xi)=-\int_{-\infty}^{\infty}e^{i2\pi x\xi}\sum_{\pm,j}A_j(t)\frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}}g_T^{\pm}(t,x)\eta^{\pm}(x)dx$$

$$=-e^{-i4\pi^2t\xi^2}\sum_{\pm,j}e^{i2\pi j\xi}\,A_j(t)\int_{-\infty}^{\infty}e^{i\frac{(x-j+4\pi t\xi)^2}{4t}}g_T^{\pm}(t,x)\eta^{\pm}(x)dx$$

satisfies, for $\xi\geq\xi(\epsilon,t,\{\alpha_j\})$ and $4\pi t\xi\notin\mathbb{Z}$,

$$|\tilde{J}(t,\xi)|\leq\begin{cases}C\epsilon,\ \text{if } d(4\pi\xi,\frac{\mathbb{Z}}{t})\geq 1,\\ C\epsilon\,|\log(d(4\pi\xi,\frac{\mathbb{Z}}{t}))|,\ \text{if } d(4\pi\xi,\frac{\mathbb{Z}}{t})<1.\end{cases}$$

For $\tilde{I}(t,\xi)$ we make the changes of variable $x=j+2\sqrt{t}y$ and $s=y-2\pi\sqrt{t}\xi$:

$$\tilde{I}(t,\xi)=-2\sum_{\pm,j}T^{\pm\infty}e^{i2\pi j\xi}\,e^{-i|\alpha_j|^2\log\sqrt{t}}A_j(t)e^{-i\frac{j^2}{4t}}\int_{-\infty}^{\infty}e^{iy^2-i4\pi\sqrt{t}\xi y}\eta^{\pm}(j+2\sqrt{t}y)dy$$

$$=-2\sum_{\pm,j}T^{\pm\infty}e^{i2\pi j\xi}\,A_j(t)e^{-i\frac{j^2}{4t}}e^{-i4\pi^2t\xi^2}\int_{-\infty}^{\infty}e^{is^2}\eta^{\pm}(j+4\pi t\xi+2\sqrt{t}s)ds.$$

Since for $|j|>j_\epsilon$ we get $\epsilon-$smallness from the A_j's, and in view of the definition of $\eta^{\pm}$, we have

$$|\tilde{I}(t,\xi)+2T^{\infty}e^{-i4\pi^2t\xi^2}\sum_j e^{i2\pi j\xi}\,A_j(t)e^{-i\frac{j^2}{4t}}\sqrt{\pi}e^{i\frac{\pi}{4}}|\leq C\epsilon.$$

In particular, we note that all the terms are uniformly bounded, so that by Plancherel's theorem we have

$$\left|\int_k^{k+1}|\tilde{I}(t,\xi)|^2d\xi-4\pi\sum_j|A_j(t)|^2\right|\leq C\epsilon.$$

Therefore, as in the case of the tangent vector T we get that for $k\geq\xi(\epsilon,t,\{\alpha_j\})$

$$|\int_k^{k+1}|\widehat{N_x}(t,\xi)|^2d\xi-4\pi\sum_j|A_j(t)|^2|\leq C\epsilon.$$

As $\epsilon\in(0,1)$ was arbitrary we get (17) by the conservation of mass (12).

Remark 3.4. *In view of the estimates we have obtained on the $J(t,\xi)$, it is natural to look for a logarithmic growth of $\hat{T}_x(t,\xi)$ in terms of the distance $d(4\pi\xi, \frac{\mathbb{Z}}{t})$. Moreover, the numerical computations given in* [14] *suggest the unboundedness of $\|\widehat{T}_x\|_\infty$ in the case of a regular planar polygon as initial data.*

By doing similar computations to the ones in this section, and by using in particular (33)*, we obtain for values of ξ such that there exists $n \in \mathbb{N}$, $d \in (0,1)$ satisfying*

$$4\pi\xi = \frac{n}{t} + d,$$

the estimate:

$$\left| \hat{T}_x(t,\xi) - i \sum_j \overline{A_j(t)} A_{j+n}(t)\, e^{-i\frac{j^2-(j+n)^2}{4t}} \, (T^\infty - T^{-\infty}) \right. \tag{36}$$

$$\left. \times e^{ij\frac{d}{2}} \left(e^{in\frac{d}{2}} \int_{s>(-j-n)\frac{d}{2}, 1>|s|>\frac{d}{2}} \frac{e^{is}}{s}\, ds - \int_{s>-j\frac{d}{2}, 1>|s|>\frac{d}{2}} \frac{e^{is}}{s}\, ds \right) \right| \leq K(t, \{\alpha_j\}).$$

For instance, in the case of initial data $\alpha_0^n = \alpha_n^n = \delta$ and $\alpha_j^n = 0$ for $j \notin \{0, n\}$, that corresponds to a polygonal line with two corners separated by a distance of size n, in (36) *the sum reduces to the case $j = 0$, and we get:*

$$\left| \hat{T}_x(t,\xi) - i\overline{A_0(t)} A_n(t)\, e^{i\frac{n^2}{4t}} (T^\infty - T^{-\infty}) \left(e^{in\frac{d}{2}} - 1 \right) e^{i\frac{d}{2}} \log \frac{d}{2} \right| \leq K(t, \{\alpha_j^n\}). \tag{37}$$

For $d \ll \frac{1}{n}$ the factor $e^{in\frac{d}{2}} - 1$ ruins the $\log d$ growth. Instead, for $d \approx \frac{1}{n}$ we could look for a $\log n$ growth. Unfortunately, the results we have at hand about the IVP of (3) *are not good enough, and we get a constant $K(t, \{\alpha_j^n\})$ in n that grows faster than $\log n$. On the other hand it seems rather natural to be able to solve* (3) *and the corresponding equation* (1) *just under the condition that $\sum_j |\alpha_j|^2$ is finite. This question will be studied elsewhere.*

4. An observation about the dynamics of a regular polygon

In this section we give some evidence that supports the conjecture made in [13] about the evolution of a regular planar polygon according to the binormal flow.

As recalled in the Introduction, the case when the initial curve in (1) is a broken line with just one corner of angle θ located at $x = 0$ was considered in [16]. In that paper the Hasimoto transformation is still used, and a solution is found considering as initial condition for (3) $\alpha\delta_0$, where

$$\sin\frac{\theta}{2} = e^{-\pi\frac{\alpha^2}{2}},$$

$u_\alpha(t,x) = \alpha \frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}}$, and $a(t) = \frac{\alpha^2}{t}$. As a consequence, and except in the trivial situation of one straight line where $\theta = \pi$, the filament function of the initial curve $\chi_\alpha(0)$, i.e. $\theta\delta_0$, is not the limit of the filament functions of $\chi_\alpha(t)$. Nevertheless, it was proved in [2] that this solution is unique and the corresponding initial value problem is well posed in an appropriate sense.

Similarly, if $\chi(0)$ is a broken line with several corners of angles θ_j located at the integers $x = j$ it was proved in [4] that one has to consider the sequence $\{\alpha_j\}$ with modulus defined by

$$\sin\frac{\theta_j}{2} = e^{-\pi\frac{|\alpha_j|^2}{2}}.$$

The phases are determined in a more complicated way involving the curvature and torsion angles of $\chi(0)$. Nevertheless, if $\chi(0)$ is a planar polygon $\{\alpha_j\}$ can be taken real. Then we construct a solution of (3) with $a(t) = \frac{\sum_j |\alpha_j|^2}{t}$, and datum at time zero given by $\sum_j \alpha_j \delta_j$.

It is then natural to expect that in the case of a planar regular polygon with N sides as initial data of (1) one has to consider as initial data for (3)

$$\sum_j \alpha\, \delta_{\frac{j}{N}} \tag{38}$$

with $\alpha > 0$ defined by

$$\sin(\frac{\pi}{N}) = e^{-\pi\frac{\alpha^2}{2}}.$$

By using the Galilean invariance and assuming uniqueness, it was shown in [13] that the corresponding solution of (3) has to be written as

$$\psi(t,x) = \hat{\psi}(t,0)\sum_j e^{it(2\pi Nj)^2 + i(2\pi Nj)x}.$$

In view of (38) and the Poisson summation formula $\sum_j e^{i(2\pi Nj)x} = \frac{1}{N}\sum_j \delta_{\frac{j}{N}}$, we have

$$\hat{\psi}(t,0) = \alpha N,$$

which therefore does not depend on time.

So, on one hand we have a behavior of the linear evolution

$$\psi(t,x) = \sum_j \hat{\psi}(t,0) e^{it\Delta}\delta_{\frac{j}{N}},$$

and we can think that the conservation law (12) also holds in the periodic setting[4].

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[4]We recall that the sequence $\{A_j(t)\}_{j\in\mathbb{N}}$ was found by doing a fixed point argument on $\tilde{A}_k(t)$ for the equation (24) in [4]:

$$i\partial_t \tilde{A}_k(t) = f_k(t) - \frac{1}{8\pi t}(|\tilde{A}_k(t)|^2 - |\alpha_k|^2)\tilde{A}_k(t),$$

where

$$f_k(t) = \frac{1}{8\pi t}\sum_{(j_1,j_2,j_3)\in NR_k} e^{-i\frac{k^2 - j_1^2 + j_2^2 - j_3^2}{4t}} e^{-i\frac{|\alpha_k|^2 - |\alpha_{j_1}|^2 + |\alpha_{j_2}|^2 - |\alpha_{j_3}|^2}{4\pi}\log\sqrt{t}} \tilde{A}_{j_1}(t)\overline{\tilde{A}_{j_2}(t)}\tilde{A}_{j_3}(t),$$

with initial data $\tilde{A}_k(0) = \alpha_k$. In particular we remark that for $N \in \mathbb{N}$, $\{B_j(t)\}_{j\in\mathbb{R}}$ with $B_j(t) := \tilde{A}_{N+j}(t)$ solves also the equation. Therefore if the initial data satisfies $\alpha_{k+N} = \alpha_k$ for all k, and there is uniqueness of the solution, then we conclude that $\tilde{A}_{k+N}(t) = \tilde{A}_k(t)$ for all k and t, so the periodic setting is preserved.

As a consequence we would get

$$N(\alpha N)^2 = N|\hat{\psi}(t,0)|^2.$$

On the other hand, it was proved in [13] making use again of the Poisson summation formula, that for rational times $t_{p,q}$ the Talbot effect holds: if q is odd

$$\psi(t_{p,q},x) = \frac{\hat{\psi}(t_{p,q},0)}{Nq}\sum_l \sum_{m=0}^{q-1} G(p,q,m)\delta_{l+\frac{m}{Nq}}(x) =: \sum_l \sum_{m=0}^{q-1} \alpha_{l,m}\delta_{l+\frac{m}{Nq}}(x),$$

with

$$|\alpha_{l,m}| = \frac{|\hat{\psi}(t_{p,q},0)|}{N\sqrt{q}}.$$

Then

$$|\alpha_{l,m}|^2 = \frac{|\hat{\psi}(t_{p,q},0)|^2}{N^2 q},$$

so

$$e^{-\pi\frac{|\alpha_{l,m}|^2}{2}} = e^{-\pi\frac{|\hat{\psi}(t_{p,q},0)|^2}{2N^2 q}} = (e^{-\pi\frac{\alpha^2}{2}})^{\frac{1}{q}},$$

therefore the angles $\theta_{p,q}$ of the skew polygon at time $t_{p,q}$ satisfy

$$\sin(\frac{\theta_{p,q}}{2}) = \sin(\frac{\pi}{N})^{\frac{1}{q}},$$

that is precisely the value given in [13] and obtained from the numerical data. Similarly one can repeat the argument if q is even.

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