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# Chapter 6

## Locating-Domination and Identification

Antoine Lobstein, Olivier Hudry, and Irène Charon

**Abstract** Locating-domination and identification are two particular, related, types of domination: a set  $C$  of vertices in a graph  $G = (V, E)$  is a locating-dominating code if it is dominating and any two vertices of  $V \setminus C$  are dominated by distinct sets of codewords;  $C$  is an identifying code if it is dominating and any two vertices of  $V$  are dominated by distinct sets of codewords. This chapter presents a survey of the major results on locating-domination and on identification.

### 6.1 Introduction

Locating-dominating codes were introduced by Slater in 1983 [168], but for more easily accessible sources, see Rall & Slater [155], or Colbourn, Slater & Stewart [60]. The term “identifying code” is used in the 1998 paper [134] by Karpovsky, Chakrabarty & Levitin, which certainly marks the starting point for the blossoming of works on this topic, but the concept is already contained in [158] (Rao, 1993). For both locating-dominating and identifying codes, see the ongoing bibliography at [142].

The graphs  $G = (V, E)$  that we shall consider will usually be finite, undirected and connected. Before we proceed, and since we consider domination at distance  $r$ , we extend the notion of neighborhood: for any integer  $r \geq 1$ , the *open  $r$ -neighborhood* of a vertex  $v \in V$  is the set  $N_r(v) = \{u : 0 < d(u, v) \leq r\}$ . The set  $N_r[v] = N_r(v) \cup \{v\}$  is called the *closed  $r$ -neighborhood* of  $v$ . A *code* is simply a set of vertices, whose elements are called *codewords*.

Formally, an  *$r$ -locating-dominating code*  $C \subseteq V$ , abbreviated  *$r$ -LD code*, is a distance- $r$  dominating code such that:

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$$\forall v_1 \in V \setminus C, \forall v_2 \in V \setminus C, v_1 \neq v_2 : N_r(v_1) \cap C \neq N_r(v_2) \cap C.$$

An  $r$ -locating-dominating code always exists (e.g.,  $C = V$ , or  $|C| = |V| - 1 > 0$  whenever  $|E| > 0$ ).

An  $r$ -identifying code  $C \subseteq V$ , abbreviated  $r$ -Id code, is a distance- $r$  dominating code such that:

$$\forall v_1 \in V, \forall v_2 \in V, v_1 \neq v_2 : N_r[v_1] \cap C \neq N_r[v_2] \cap C.$$

Identifying codes are sometimes called *differentiating*, mostly when  $r = 1$ . We can see that it is possible to retrieve a vertex  $v$  simply by knowing which codewords  $r$ -dominate  $v$ . Thus, if each codeword sends an alarm to a central controller whenever there is a malfunctioning vertex in its closed neighborhood, then, only knowing which codewords gave the alarm, the controller can unambiguously retrieve that vertex (if there is at most one). The same is true for a non-codeword in the case of  $r$ -LD codes. See also [149] for an illustration with smoke detectors.

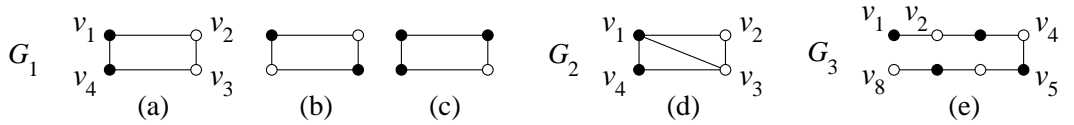
Two distinct vertices  $v_1 \in V, v_2 \in V$ , are said to be  $r$ -twins if  $N_r[v_1] = N_r[v_2]$ . It is easy to see that a graph  $G$  admits an  $r$ -identifying code if and only if it has no  $r$ -twins, i.e.,

$$\forall v_1 \in V, \forall v_2 \in V, v_1 \neq v_2 : N_r[v_1] \neq N_r[v_2]. \quad (6.1)$$

A graph satisfying (6.1) is usually called  $r$ -identifiable,  $r$ -twin-free or  $r$ -distinguishable, or also *point-distinguishing* ([175], for  $r = 1$ ). If three vertices  $x, y, z$  are such that  $z \in N_r[x]$  and  $z \notin N_r[y]$ , we say that  $z$   $r$ -separates  $x$  and  $y$  in  $G$  (note that  $z = x$  is possible). A set of vertices  $r$ -separates  $x$  and  $y$  if at least one of its elements does. So we can rephrase the definitions: an  $r$ -LD code (respectively, an  $r$ -Id code)  $r$ -dominates every vertex and  $r$ -separates every pair of non-codewords (respectively, of vertices).

Usually, one is interested in finding the smallest possible size of a code, LD or Id, in a given graph. Such codes are called *optimal*. Several notations exist; in this chapter, we denote by  $LD_r(G)$  (respectively,  $Id_r(G)$ ) the smallest possible cardinality of an  $r$ -locating-dominating code (respectively, an  $r$ -identifying code when  $G$  is  $r$ -twin-free), and we call these numbers the  $r$ -locating-domination number (respectively, the  $r$ -identification number) of  $G$ . They are abbreviated as  $r$ -LD and  $r$ -Id numbers, respectively. We may drop  $r$  when  $r = 1$  or when it is irrelevant.

### Example



**Fig. 6.1** Different graphs and codes. Black vertices represent codewords.

$r = 1$ : (a)  $\{v_1, v_4\}$  is locating-dominating, not identifying. (b)  $\{v_1, v_3\}$  is neither locating-dominating nor identifying. (c)  $\{v_1, v_2, v_4\}$  is both locating-dominating and identifying. We have:  $LD(G_1) = 2$ ,  $Id(G_1) = 3$ . (d)  $v_1$  and  $v_3$  are twins;  $\{v_1, v_4\}$  is locating-dominating, and  $LD(G_2) = 2$ .  $r = 4$ : (e)  $v_4$  and  $v_5$  are 4-twins;  $\{v_1, v_3, v_5, v_7\}$  is 4-locating-dominating, and  $LD_4(G_3) = 4$ .

The decision problems classically associated with the search of optimal  $r$ -LD and  $r$ -Id codes are NP-complete for all  $r \geq 1$ , see Sect. 6.8 on complexity. The following inequalities hold for any  $r \geq 1$ :

$$\gamma_r(G) \leq LD_r(G) \leq Id_r(G). \quad (6.2)$$

Almost all of this chapter is devoted to undirected graphs, but the above definitions can be extended to *digraphs*, by considering, e.g., *insets* instead of neighborhoods; see [39], [88], or [59] for an illustration.

The following three theorems are probably as old as the definitions of  $r$ -dominating,  $r$ -locating-dominating or  $r$ -identifying codes.

**Theorem 1** *Let  $r \geq 2$  be any integer and  $G$  be any graph. A code is 1-dominating (respectively, 1-locating-dominating, 1-identifying) in  $G^r$ , the  $r$ -th power of  $G$ , if and only if it is  $r$ -dominating (respectively,  $r$ -locating-dominating,  $r$ -identifying) in  $G$ .*

**Theorem 2** *Let  $r \geq 1$  be any integer. Any connected  $r$ -twin-free graph has order  $n = 1$  or  $n \geq 2r + 1$ . The only connected  $r$ -twin-free graph with order  $n = 2r + 1$  is the path  $P_{2r+1}$ .*

*Any cycle with order  $n \geq 2r + 2$  is  $r$ -twin-free.*

**Theorem 3** *Let  $r \geq 1$  be any integer. Every  $r$ -dominating code is  $(r + 1)$ -dominating. If  $C$  is  $r$ -dominating (respectively,  $r$ -locating-dominating,  $r$ -identifying), then so is any superset  $C^* \supseteq C$ .*

## 6.2 Possible Values for LD and Id Numbers

Almost everyone's first results on LD and Id numbers are the following two theorems, which give three easy bounds, whereas Th. 6 is more difficult.

**Theorem 4** (a) *For any integer  $r \geq 1$  and any graph  $G = (V, E)$  of order  $n$ , we have*

$$LD_r(G) \geq \lceil \log_2(n - LD_r(G) + 1) \rceil. \quad (6.3)$$

(b) *For any integer  $r \geq 1$  and any  $r$ -twin-free graph  $G = (V, E)$  of order  $n$ , we have*

$$Id_r(G) \geq \lceil \log_2(n + 1) \rceil.$$

**Proof.** (a) Let  $C$  be any  $r$ -LD code in  $G$ . All the  $n - |C|$  non-codewords  $v \in V \setminus C$  must be given nonempty and distinct sets  $N_r(v) \cap C$ , constructed with the  $|C|$  codewords, so  $2^{|C|} - 1 \geq n - |C|$ , from which (6.3) follows when  $C$  is optimal; (b) the argument is the same, but we have to consider all the  $n$  vertices  $v \in V$ , so  $2^{|C|} - 1 \geq n$ .  $\square$

**Theorem 5** *For any  $r \geq 1$  and connected graph  $G$  of order  $n \geq 2$ , we have  $LD_r(G) \leq n - 1$ .*

**Theorem 6** ([15]) *For any  $r \geq 1$  and connected  $r$ -twin-free graph  $G$  of order  $n \neq 1$ , we have  $Id_r(G) \leq n - 1$ .*

**Proof.** We give the short elegant proof by Gravier & Moncel [98] in 2007, six years after the first proof in [15]. Using Th. 1, we can see that it is sufficient to prove the case  $r = 1$ . We assume that  $G = (V, E)$  is a connected 1-twin-free graph, of order  $n \geq 3$ . Let  $C_1 = V \setminus \{a\}$  for  $a \in V$  with

maximum degree in  $G$ . If  $C_1$  is identifying, we are done, so we assume that it is not. Because  $G$  is connected,  $C_1$  is a dominating code. If  $u$  and  $v$  belong to  $N[a]$  and because  $N[u] \neq N[v]$ , we have  $N[u] \setminus \{a\} \neq N[v] \setminus \{a\}$ , so  $C_1$  separates  $u$  and  $v$ , and the same is true for  $u \notin N[a]$  and  $v \notin N[a]$ . Therefore, there must be  $u \in N[a]$  and  $v \notin N[a]$  which are not separated by  $C_1$ , i.e.,  $N[u] \cap C_1 = N[v] \cap C_1$ , or  $N[u] \setminus \{a\} = N[v]$ .

We claim that  $C_2 = V \setminus \{v\}$  is an identifying code. By the previous discussion with  $a$ , which did not use the maximum degree assumption yet, it is sufficient to check that every vertex in  $N[v]$  is separated from every vertex not in  $N[v]$ . And indeed: (i) since  $N[u] = N[v] \cup \{a\}$ , each vertex in  $V \setminus (N[v] \cup \{a\})$  is separated from each vertex in  $N[v]$  by  $u$ ; (ii) the vertex  $a$  is separated from  $v$  by itself; (iii) the vertex  $a$  is separated from each vertex  $a' \in N[v] \setminus \{v\}$ , for otherwise,  $N[a'] = N[a] \cup \{v\}$ , i.e.,  $a'$  has degree greater than  $a$ , a contradiction.  $\square$

The previous result actually holds for all graphs, finite or infinite, with bounded degree. There exist (infinite) graphs (with unbounded degree) such that the only 1-identifying code is the whole vertex set [15]: take two copies,  $G_1$  and  $G_2$ , of the infinite complete graph with vertex set  $\mathbb{Z}$ , and link  $i \in V(G_1)$  to  $j \in V(G_2)$  if and only if  $i \geq j$ .

### 6.2.1 Reaching the Bounds

Graphs exist that meet the lower and upper bounds given previously. Theorem 9(a) even characterizes the graphs  $G$  of order  $n$  such that  $LD(G) = n - 1$ , as will Th. 12 for 1-identifying codes.

**Theorem 7** ([43]) *Let  $r \geq 1$  and  $n$  be integers such that  $n \geq 2^{2r+1} + 2r + 1$ . There exists a connected graph of order  $n$  admitting an  $r$ -locating-dominating code achieving the lower bound (6.3).*

**Theorem 8** ([134, for  $r = 1$ ]), ([64]), ([43]) *Let  $r \geq 1$  and  $n$  be integers such that  $n \geq 2^{2r}$ . There exists a connected graph of order  $n$  admitting an  $r$ -identifying code of size  $\lceil \log_2(n + 1) \rceil$ .*

For  $r = 1$ , Th. 8 holds for all  $n \geq 3$ ; for  $r = 2$ , it holds if and only if  $n \geq 6$ .

**Theorem 9** (a) ([169]), ([170]) *A connected graph  $G$  of order  $n \geq 2$  is such that  $LD(G) = n - 1$  if and only if  $G$  is the star  $K_{1,n-1}$  or the complete graph  $K_n$ .*

(b) *For all  $r \geq 1$  and  $n \geq 2$ , we have  $LD_r(K_n) = n - 1$ .*

**Theorem 10** *For all  $n \geq 3$ , there exists a connected graph  $G$  of order  $n$  such that  $Id(G)$  achieves the upper bound  $n - 1$ .*

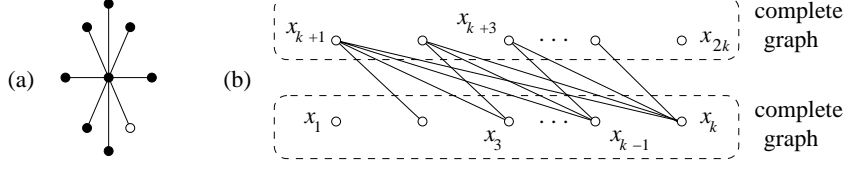
An easy example of such a graph is the star, see Fig. 6.2(a). We give two more examples, which will be used for the construction of graphs  $G$  such that  $Id_r(G) = n - 1$  for all  $r$ .

In the following, a vertex is  $r$ -universal if it  $r$ -dominates all the vertices in the graph. Let  $p \geq 2$ .

(a) Take  $G_{2p}^* = (V_{2p}^*, E_{2p}^*)$ , with  $V_{2p}^* = \{v_0, v_1, \dots, v_{2p-1}\}$ ,  $E_{2p}^* = \{v_i v_j : v_i \in V_{2p}^*, v_j \in V_{2p}^*, i \neq j, i \neq j + p \bmod 2p\}$ : the graph  $G_{2p}^*$  has even order and is the complete graph  $K_{2p}$  minus the edges of a perfect matching.

(b) Take  $G_{2p+1}^*$ , obtained from  $G_{2p}^*$  simply by adding one 1-universal vertex; its order is odd.

In view of Sect. 6.3, the following theorem gives the domination, locating-domination and identification numbers of these two classes of graphs, as well as for the star.



**Fig. 6.2** (a) The star  $K_{1,8}$ , with an optimal identifying code (of size 8), which is also an optimal locating-dominating code. (b) The graph  $A_k$ .

**Theorem 11** Let  $p \geq 2$  and  $n \geq 3$ . Then

- (a)  $\gamma(G_{2p}^*) = 2$ ,  $LD(G_{2p}^*) = p$  and  $Id(G_{2p}^*) = 2p - 1$ ;
- (b)  $\gamma(G_{2p+1}^*) = 1$ ,  $LD(G_{2p+1}^*) = p$  and  $Id(G_{2p+1}^*) = 2p$ ;
- (c)  $\gamma(K_{1,n-1}) = 1$ ,  $LD(K_{1,n-1}) = n - 1$  and  $Id(K_{1,n-1}) = n - 1$ .

It is an interesting fact, published in 2010 [82], that it is possible to characterize the graphs  $G$  with  $Id(G) = n - 1$ . For two graphs  $G = (V, E)$  and  $G' = (V', E')$ , let  $G \bowtie G' = (V \cup V', E \cup E' \cup \{vv' : v \in V, v' \in V'\})$  be their join graph. Let  $A_k = (V_k, E_k)$ , with  $V_k = \{x_1, \dots, x_{2k}\}$  and  $E_k = \{x_i x_j : |i - j| \leq k - 1\}$ , see Fig. 6.2(b). Note that  $A_1$  consists of two isolated vertices, and for  $k \geq 2$ ,  $A_k$  is the  $(k - 1)$ -th power of the path  $P_{2k}$ . Finally, let  $\mathcal{A}$  be the closure of all the graphs  $A_i$  for  $\bowtie$ .

**Theorem 12** (Foucaud *et al.* [82]), (Foucaud *et al.* [83]) A connected graph  $G$  of order  $n \geq 3$  is such that  $Id(G) = n - 1$  if and only if  $G$  is the star or belongs to the set of graphs  $\mathcal{A} \cup \{A \bowtie K_1 : A \in \mathcal{A}\}$ .

Note that  $G_4^* \simeq A_1 \bowtie A_1$ ,  $G_6^* \simeq G_4^* \bowtie A_1$ , ...,  $G_{2p}^* \simeq G_{2p-2}^* \bowtie A_1$ , and that  $G_{2p+1}^* \simeq G_{2p}^* \bowtie K_1$ . We now turn to the case  $r \geq 2$ : are there graphs  $G$  such that  $Id_r(G) = n - 1$ ? (as we have already seen, complete graphs give a trivial positive answer for  $r$ -LD codes). A crucial fact is that the two graphs  $G_{2p}^*$  and  $G_{2p+1}^*$  admit  $r$ -th roots for any  $r$ , if  $p$  is sufficiently large. More precisely:

**Theorem 13** ([43]) Let  $r \geq 2$  and  $p \geq 2$  be integers.

- (a) If  $2p \geq 3r^2$ , then there exists a graph  $G_{2p}$  of order  $2p$  such that  $(G_{2p})^r = G_{2p}^*$ .
- (b) If  $2p \geq 3r^2$ , then there exists a graph  $G_{2p+1}$  of order  $2p + 1$  such that  $(G_{2p+1})^r = G_{2p+1}^*$ .

**Corollary 1** (a) For  $n \geq 3r^2$ , there exists a graph  $G_n$  of even order  $n$  such that  $\gamma_r(G_n) = 2$ ,  $LD_r(G_n) = \frac{n}{2}$  and  $Id_r(G_n) = n - 1$ .

(b) For  $n \geq 3r^2 + 1$ , there exists a graph  $G_n$  of odd order  $n$  such that  $\gamma_r(G_n) = 1$ ,  $LD_r(G_n) = \frac{n-1}{2}$  and  $Id_r(G_n) = n - 1$ .

**Proof.** Use Ths. 11(a)–(b), 13 and 1. □

### 6.2.2 Reaching All Intermediate Values

As for the lower and upper bounds, constructions show how to achieve all intermediate values.

**Theorem 14** ([42]) Let  $r \geq 1$  and  $c \geq 5r^2 + 5r + 1$ . For  $n$  between  $c + 1$  and  $2^c + c - 1$ , there exists a connected graph  $G$  of order  $n$  such that  $LD_r(G) = c$ .

For Id codes, the case  $r = 1$  is easy and one can even address the special case of bipartite graphs.

**Theorem 15** ([41]) *For  $n \geq 3$  and for any integer  $c$  between  $\lceil \log_2(n+1) \rceil$  and  $n-1$ , there exists a connected bipartite graph  $G$  with  $n$  vertices such that  $Id(G) = c$ .*

**Theorem 16** ([41]) *Let  $r \geq 1$  and  $c \geq 5r^2 + 5r + 1$ . For  $n$  between  $c+1$  and  $2^c - 1$ , there exists a connected graph  $G$  of order  $n$  such that  $Id_r(G) = c$ .*

### 6.3 The Cost of Locating-Domination and Identification

The inequalities  $\gamma_r(G) \leq LD_r(G) \leq Id_r(G)$  in (6.2) express that locating-domination is more “expensive” than domination, and identification is more expensive than locating-domination. In this section, we compare the respective “costs” for these three notions. More precisely, denoting, for graphs of order  $n$ ,

$$\mathcal{G}_{r,n} = \{G : r\text{-twin-free, connected, } n \geq 2r+1\} \text{ and } \mathcal{G}_{r,n}^{tw} = \{G : \text{with } r\text{-twins, connected, } n \geq 2\},$$

we study the following maximum and minimum differences:

- $F_{Id,LD}(r,n) = \max_{G \in \mathcal{G}_{r,n}} \{Id_r(G) - LD_r(G)\}$ ,  $f_{Id,LD}(r,n) = \min_{G \in \mathcal{G}_{r,n}} \{Id_r(G) - LD_r(G)\}$ ,
- $F_{Id,\gamma}(r,n) = \max_{G \in \mathcal{G}_{r,n}} \{Id_r(G) - \gamma_r(G)\}$ ,  $f_{Id,\gamma}(r,n) = \min_{G \in \mathcal{G}_{r,n}} \{Id_r(G) - \gamma_r(G)\}$ .

In order to see the influence of the twin-free property on locating-domination and domination, we distinguish, for dominating and locating-dominating codes, between two cases, and study the following functions:

- $F_{LD,\gamma}(r,n) = \max_{G \in \mathcal{G}_{r,n}} \{LD_r(G) - \gamma_r(G)\}$ ,  $f_{LD,\gamma}(r,n) = \min_{G \in \mathcal{G}_{r,n}} \{LD_r(G) - \gamma_r(G)\}$ ;

these two functions are considered on the same set of graphs (the twin-free graphs) as the four functions involving identification, unlike the two functions below:

- $F_{LD,\gamma}^{tw}(r,n) = \max_{G \in \mathcal{G}_{r,n}^{tw}} \{LD_r(G) - \gamma_r(G)\}$ ,  $f_{LD,\gamma}^{tw}(r,n) = \min_{G \in \mathcal{G}_{r,n}^{tw}} \{LD_r(G) - \gamma_r(G)\}$ .

Finally, if we want to consider all the connected graphs of order  $n$ , twin-free or not, the result is obviously obtained by taking  $\max\{F_{LD,\gamma}(r,n), F_{LD,\gamma}^{tw}(r,n)\}$  and  $\min\{f_{LD,\gamma}(r,n), f_{LD,\gamma}^{tw}(r,n)\}$ .

Note that [20] characterizes the *trees*  $T$  such that  $LD_1(T) = \gamma_1(T)$  or  $Id_1(T) = \gamma_1(T)$ . Most results in this section on cost (namely, Ths. 19–26) are due to Hudry & Lobstein [124] in 2020.

#### 6.3.1 Preliminary Results

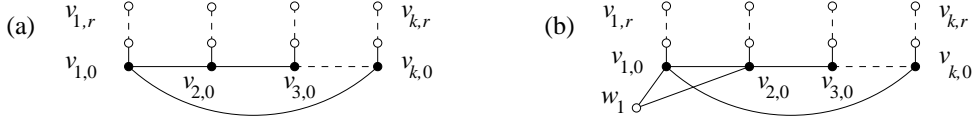
**Theorem 17** ([147]), ([102, p. 41]) *If  $G$  has order  $n$  and no isolated vertices, then  $\gamma(G) \leq \frac{n}{2}$ .*

**Theorem 18** ([96]) *If  $G$  is 1-twin-free, then  $Id(G) \leq 2LD(G)$ .*

Thanks to Th. 1 on the powers of graphs, the previous two results are true also for  $\gamma_r(G)$ ,  $Id_r(G)$  (for  $r$ -twin-free graphs) and  $LD_r(G)$ , for any  $r \geq 2$ .

### 6.3.2 Identification vs Domination

First, we construct an infinite family of graphs  $G_n^*$  of order  $n$  satisfying  $Id_r(G_n^*) = \gamma_r(G_n^*)$ . These graphs have order  $n = k(r + 1)$ ,  $k \geq 2r + 2$ , and are informally given by Fig. 6.3(a).



**Fig. 6.3** The  $k$  black vertices represent codewords. (a) The graph  $G_n^*$ . (b) The graph  $G_{n+1}$ .

**Theorem 19** (a) For all  $r \geq 1$ ,  $k \geq 2r + 2$  and  $n = k(r + 1)$ , we have  $\gamma_r(G_n^*) = Id_r(G_n^*) = k$ .

(b) For all  $r \geq 1$  and  $n \geq (2r + 2)(r + 1)$ , we have  $f_{Id,\gamma}(r, n) = 0$ , and obviously  $f_{Id,LD}(r, n) = f_{LD,\gamma}(r, n) = 0$ .

**Proof.** (a) The  $k$  vertices  $v_{i,r}$  must be  $r$ -dominated by at least one codeword, and no vertex can  $r$ -dominate two such vertices, so  $\gamma_r(G_n^*) \geq k$ . On the other hand, the code  $C = V_0$  represented by the black vertices on Fig. 6.3(a) has cardinality  $k$ , and it is straightforward to check that it is  $r$ -identifying. Note in particular that vertices in  $\{v_{1,0}, v_{2,0}, \dots, v_{k,0}\}$  are  $r$ -dominated by exactly  $2r + 1$  codewords (this is where the assumption  $k \geq 2r + 2$  is crucial), and more generally, vertices  $v_{i,j} \in \{v_{i,1}, v_{i,2}, \dots, v_{i,r}\}$  are  $r$ -dominated by an *odd* number, namely  $2r - 2j + 1$ , of codewords, for all  $i \in \{1, 2, \dots, k\}$  and  $j \in \{1, 2, \dots, r\}$ . So  $k \leq \gamma_r(G_n^*) \leq Id_r(G_n^*) \leq k$ .

(b) Then it is easy to see that we can reach all intermediate values between  $k(r + 1)$  and  $(k + 1)(r + 1) - 1$ . Figure 6.3(b) illustrates the case  $k(r + 1) + 1$ , with one additional vertex,  $w_1$ , which is  $r$ -dominated by an *even* number of codewords in  $G_{n+1}$ .  $\square$

For  $r = 1$ , Th. 19 starts at  $n = 8$ , for  $k = 4$ , but we can fill in the holes:  $f_{Id,\gamma}(1, n) = 1$  for  $n \in \{3, 4, 5\}$ , and for  $n \in \{6, 7\}$ ,  $f_{Id,\gamma}(1, n) = 0 = f_{Id,LD}(1, n) = f_{LD,\gamma}(1, n)$ .

Now how large can the difference  $Id_r(G) - \gamma_r(G)$  be? We know that it is at most  $n - 2$ , obtained by graphs  $G$  with  $Id_r(G) = n - 1$  and  $\gamma_r(G) = 1$ .

**Theorem 20** (a) For all  $n \geq 3$ , we have  $F_{Id,\gamma}(1, n) = n - 2$ .

(b) For all  $r \geq 2$  and even  $n \geq 3r^2$ , we have  $F_{Id,\gamma}(r, n) = n - 3$ .

(c) For all  $r \geq 2$  and odd  $n \geq 3r^2 + 1$ , we have  $F_{Id,\gamma}(r, n) = n - 2$ .

**Proof.** Use the star for (a), Cor. 1(b) for (c). For (b), observe that, among all the graphs  $G$  of even order  $n$  such that  $Id(G) = n - 1$  (see Th. 12), none of them, except the star, contains a 1-universal vertex, i.e., is such that  $\gamma(G) = 1$ . But the star cannot be the power of any graph. Therefore, for  $r \geq 2$ , there can exist no graph  $G$  with even order  $n$  such that  $Id_r(G) = n - 1$  and  $\gamma_r(G) = 1$ , since the  $r$ -th power of this graph would contradict our previous observation; consequently the difference  $Id_r(G) - \gamma_r(G)$  is at most  $n - 3$ . On the other hand, Cor. 1(a) gives an example achieving  $n - 3$ .  $\square$

### 6.3.3 Identification vs Locating-Domination

**Theorem 21** (a) (= Th. 19(b)) For all  $r \geq 1$  and  $n \geq (2r + 2)(r + 1)$ , we have  $f_{Id,LD}(r, n) = 0$ .



(b) For all  $n \geq 3$ , we have  $f_{Id,LD}(1, n) = 0$ .

**Theorem 22** (a) For all  $r \geq 1$  and  $n \geq 3r^2 + 1$ , we have  $F_{Id,LD}(r, n) = \lceil \frac{n}{2} \rceil - 1$ .

(b) We have  $F_{Id,LD}(1, 3) = 0$ .

**Proof.** We know (cf. Th. 18) that any (connected)  $r$ -twin-free graph  $G$  is such that  $Id_r(G) \leq 2LD_r(G)$ . Therefore,  $Id_r(G) - LD_r(G) \leq Id_r(G) - \frac{Id_r(G)}{2} \leq \frac{n-1}{2}$ , leading to  $Id_r(G) - LD_r(G) \leq \lceil \frac{n}{2} \rceil - 1$ . On the other hand, Cor. 1(a)–(b) provides graphs reaching  $\lceil \frac{n}{2} \rceil - 1$ .  $\square$

### 6.3.4 Domination vs Locating-Domination

The two cases, without and with twins, show quite a difference between  $F_{LD,\gamma}$  and  $F_{LD,\gamma}^{tw}$ .

#### 6.3.4.1 Domination vs Locating-Domination in Twin-Free Graphs

Using Th. 19(b), the sentence following its proof, and the paths  $P_4$  and  $P_5$ , we obtain the following.

**Theorem 23** (a) For all  $n \geq 4$ , we have  $f_{LD,\gamma}(1, n) = 0$ ;  $f_{LD,\gamma}(1, 3) = 1$ .

(b) For all  $r \geq 2$  and  $n \geq (2r + 2)(r + 1)$ , we have  $f_{LD,\gamma}(r, n) = 0$ .

**Theorem 24** (a) For all  $n \geq 3$ , we have  $F_{LD,\gamma}(1, n) = n - 2$ .

(b) For all  $r \geq 2$  and  $n \geq 3r^2 + 1$ , we have  $F_{LD,\gamma}(r, n) \geq \frac{n}{2} - 2$ .

(c) For all  $r \geq 2$  and  $n \geq 2r + 1$ , we have  $F_{LD,\gamma}(r, n) \leq n - 3$ .

**Proof.** (a) The star shows that  $F_{LD,\gamma}(1, n) = n - 2$ . (b) For any  $r$ , Cor. 1(a)–(b) immediately gives examples proving that  $F_{LD,\gamma}(r, n) \geq \frac{n}{2} - 2$ . (c) The characterization of the graphs  $G$  of order  $n$  such that  $Id(G) = n - 1$  (Th. 12) gives graphs which, apart from the star which is not the power of any graph, are such that  $LD(G) \leq n - 2$ . This allows to conclude that  $F_{LD,\gamma}(r, n) \leq n - 3$ , using the powers of graphs as in the proof of Th. 20.  $\square$

Improvements are possible on Th. 24(b) when  $r = 2$ .

**Theorem 25** (a) Let  $n = 8t \geq 24$ . Then  $F_{LD,\gamma}(2, n) \geq 5t - 3$ .

(b) Let  $n = 8t + i \geq 24$ , with  $1 \leq i \leq 7$ . Then  $F_{LD,\gamma}(2, n) \geq 5t + i - 6$ .

#### 6.3.4.2 Domination vs Locating-Domination in Graphs with Twins

**Theorem 26** (a) For  $r \geq 1$  and  $n \geq 2$ , we have  $F_{LD,\gamma}^{tw}(r, n) = n - 2$ .

(b) For  $n \in \{2, 5\}$  or  $n \geq 7$ , we have  $f_{LD,\gamma}^{tw}(1, n) = 0$ ; for  $n \in \{3, 4, 6\}$ , we have  $f_{LD,\gamma}^{tw}(1, n) = 1$ .

(c) For all  $r \geq 2$  and  $n \geq (2r + 2)(r + 1) + 1$ , we have  $f_{LD,\gamma}^{tw}(r, n) = 0$ .

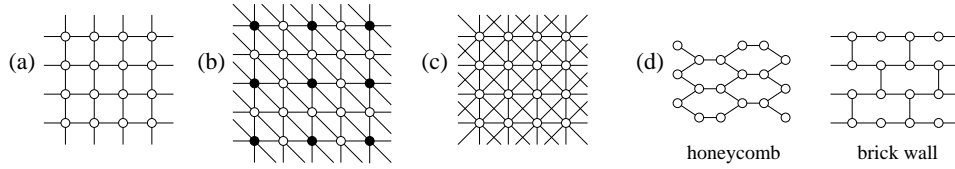
To conclude this section on compared costs, we can see that for  $r = 1$ , we have exact values for all  $n$  and functions  $f$  and  $F$ . For  $r \geq 2$ , most results are exact but valid for  $n$  large. One open problem is to establish results for all  $n$ , another to reduce the gap between lower and upper bounds for  $F_{LD,\gamma}(r, n)$ , cf. Ths. 24(b)–(c) and 25. Note that all open problems and conjectures mentioned throughout this chapter are gathered at its end.

## 6.4 Specific Families of Graphs

We survey some well-known and some not so well-known families of graphs.

### 6.4.1 Infinite Grids and Strips

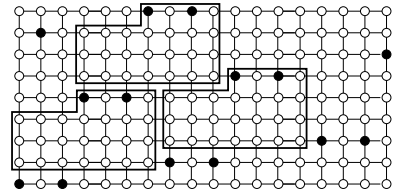
The four infinite 2-dimensional grids  $G_S$  (square),  $G_T$  (triangular),  $G_K$  (king), and  $G_H$  (hexagonal), partially represented in Fig. 6.4, have been much studied in the literature, specially with respect to identifying codes, and the densities  $\partial_r^{LD}(G_\pi)$  and  $\partial_r^{Id}(G_\pi)$  of optimal  $r$ -LD and  $r$ -Id codes investigated for  $\pi \in \{S, T, K, H\}$ . One can also consider the infinite strips  $G_\pi^{[k]}$  of height  $k \geq 1$  (the case  $k = 1$  gives the infinite path, see also below, Sect. 6.4.2). We remind that the *density* of a code  $C$  in  $G_\pi^{[k]}$  can be defined for every  $k \geq 1$  by  $\partial(C, G_\pi^{[k]}) = \limsup_{n \rightarrow \infty} \frac{|C \cap N_n[v]|}{|N_n[v]|}$ , where  $v$  is an arbitrary vertex.



**Fig. 6.4** Partial representations of the four grids: (a) the square grid; (b) the triangular grid: black vertices are codewords (cf. Th. 32); (c) the king grid; (d) the hexagonal grid (with two possible representations).

Some constructions of codes are obtained by heuristics searching for small subcodes inside tiles that will be repeated periodically [38], see Fig. 6.5 for a first example.

**Fig. 6.5** A periodic 5-identifying code in the square grid  $G_S$ , of density  $2/25$ . Codewords are in black.



We give, as far as we know, the best lower and upper bounds, for grids and strips. Note how few results there are for  $r$ -LD codes when  $r \neq 1$ .

### 6.4.1.1 The Square Grid

The square grid,  $G_S$ , has vertex set  $V_S = \mathbb{Z} \times \mathbb{Z}$  and edge set  $E_S = \{uv : u - v \in \{(1, 0), (0, 1)\}\}$ . Figure 6.6 will give constructions proving the upper bounds for Ths. 27(a) and 28, Fig. 6.7 for Ths. 27(c) and 30(b).

#### • Locating-dominating codes

**Theorem 27** ([171]), ([172]) (a) We have  $\partial_1^{LD}(G_S) = \frac{3}{10}$ .

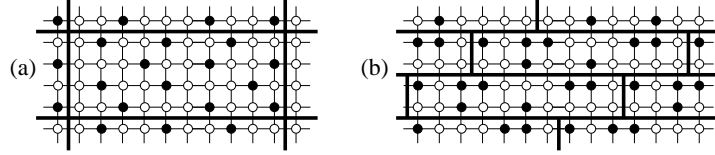
(b) We have  $\partial_1^{LD}(G_S^{[1]}) = \frac{2}{5}$ .

(c) We have  $\partial_1^{LD}(G_S^{[2]}) = \frac{3}{8} = 0.375$ ,  $\partial_1^{LD}(G_S^{[3]}) = \frac{1}{3}$ .

#### • Identifying codes

The following upper bound  $\partial_1^{Id}(G_S) \leq \frac{7}{20}$  is from [54] in 1999, the lower bound  $\partial_1^{Id}(G_S) \geq \frac{7}{20}$  by Ben-Haim & Litsyn [13] in 2005.

**Theorem 28** We have  $\partial_1^{Id}(G_S) = \frac{7}{20}$ .



**Fig. 6.6** Black vertices are the elements of a periodic code in  $G_S$ , which is (a) 1-LD with density  $\frac{3}{10}$ , (b) 1-Id with density  $\frac{7}{20}$ .

The following general lower bounds come from [33], the lower bound in (b) from [128], and all the upper bounds from [117].

**Theorem 29** (a) For every  $r \geq 1$ , we have  $\frac{3}{8r+4} \leq \partial_r^{Id}(G_S) \leq \begin{cases} \frac{2}{5r} & : r \text{ even} \\ \frac{2r}{5r^2-2r+1} & : r \text{ odd} \end{cases}$ . When  $r$  increases, these bounds are close to  $\frac{3}{8r} = \frac{0.375}{r}$  and  $\frac{2}{5r}$ .

(b) We have  $\frac{6}{35} \approx 0.17143 \leq \partial_2^{Id}(G_S) \leq \frac{r_5}{29} \approx 0.17241$ .

The previous upper bounds have been improved in [38], using heuristics, for  $r \in \{3, 4, 5, 6\}$ . The strips of all heights have also been studied, for  $r = 1$ ; in (d) below, the upper bound is from [22], the lower bound from [126].

**Theorem 30** (a) ([94]) We have  $\partial_1^{Id}(G_S^{[1]}) = \frac{1}{2}$ .

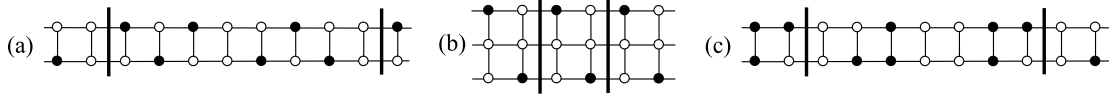
(b) ([64]), ([65]) We have  $\partial_1^{Id}(G_S^{[2]}) = \frac{3}{7} \approx 0.42857$ .

(c) ([22]) We have  $\partial_1^{Id}(G_S^{[3]}) = \frac{7}{18} \approx 0.38889$ .

(d) We have  $\partial_1^{Id}(G_S^{[4]}) = \frac{11}{28} \approx 0.39286$ .

(e) ([126]) We have  $\partial_1^{Id}(G_S^{[5]}) = \frac{19}{50} = 0.38$ .

(f) ([23]) For  $k \geq 6$ , we have  $\frac{7}{20} + \frac{1}{20k} \leq \partial_1^{Id}(G_S^{[k]}) \leq \frac{7}{20} + \frac{3}{10k}$ .



**Fig. 6.7** Black vertices are the elements of a periodic code which is (a) 1-LD with density  $\frac{3}{8}$  in  $G_S^{[2]}$ , (b) 1-LD with density  $\frac{1}{3}$  in  $G_S^{[3]}$ , (c) 1-ID with density  $\frac{3}{7}$  in  $G_S^{[2]}$ .

#### 6.4.1.2 The Triangular Grid

The triangular grid, or square grid with one diagonal,  $G_T$ , has vertex set  $V_T = \mathbb{Z} \times \mathbb{Z}$  and edge set  $E_T = \{uv : u - v \in \{(1, 0), (0, 1), (1, -1)\}\}$ . Figure 6.8 will give constructions proving the upper bounds in  $G_T^{[3]}$  for Ths. 31(b) and 34(a).

##### • Locating-dominating codes

**Theorem 31** (a) ([107]) *We have  $\partial_1^{LD}(G_T) = \frac{13}{57} \approx 0.22807$ .*

(b) ([22]) *We have  $\partial_1^{LD}(G_T^{[2]}) = \frac{1}{3}$ ,  $\partial_1^{LD}(G_T^{[3]}) = \frac{3}{10}$ .*

##### • Identifying codes

**Theorem 32** (Karpovsky, Chakrabarty & Levitin [134]) *We have  $\partial_1^{Id}(G_T) = \frac{1}{4}$ .*

**Proof.** The upper bound is proved by the construction given in Fig. 6.4(b), which has the property (P) that every codeword is dominated by exactly one codeword (itself) and every non-codeword is dominated by exactly two codewords (either horizontally or vertically or diagonally). This proves that the code is indeed 1-identifying, and moreover that it is best possible.  $\square$

Property (P) is at the root of Th. 70 when  $r = 1$ . Another example of a graph and a code having this property is the cycle of length  $2p \geq 6$ , with  $p$  pairwise nonadjacent vertices forming the code, see Th. 44. Property (P) cannot be true for  $r > 1$  [57].

**Theorem 33** (a) ([33]) *For every  $r \geq 1$ , we have  $\frac{2}{6r+3} \leq \partial_r^{Id}(G_T) \leq \begin{cases} \frac{1}{2r+2} : r \in \{1, 2, 3\} \bmod 4 \\ \frac{1}{2r+4} : r = 0 \bmod 4 \end{cases}$ .*

*When  $r$  increases, these bounds are close to  $\frac{1}{3r}$  and  $\frac{1}{2r}$ .*

(b) ([38]) *We have  $\partial_3^{Id}(G_T) \leq \frac{2}{17} \approx 0.11765$  and  $\partial_5^{Id}(G_T) \leq \frac{1}{13} \approx 0.07692$ .*

Results on the different triangular strips are available when  $r = 1$ .

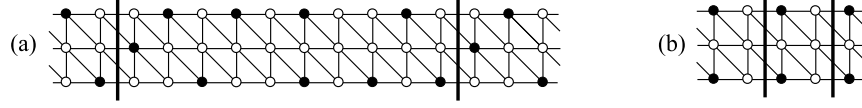
**Theorem 34** ([66]) (a) *We have  $\partial_1^{Id}(G_T^{[2]}) = \frac{1}{2}$ ,  $\partial_1^{Id}(G_T^{[3]}) = \partial_1^{Id}(G_T^{[4]}) = \frac{1}{3}$ ,  $\partial_1^{Id}(G_T^{[5]}) = \frac{3}{10}$ , and  $\partial_1^{Id}(G_T^{[6]}) = \frac{1}{3}$ .*

(b) *For odd  $k \geq 7$ , we have  $\partial_1^{Id}(G_T^{[k]}) = \frac{1}{4} + \frac{1}{4k}$ .*

(c) *For even  $k \geq 8$ , we have  $\frac{1}{4} + \frac{1}{4k} \leq \partial_1^{Id}(G_T^{[k]}) \leq \frac{1}{4} + \frac{1}{2k}$ .*

#### 6.4.1.3 The King Grid

The king grid, or square grid with two diagonals,  $G_K$ , has vertex set  $V_K = \mathbb{Z} \times \mathbb{Z}$  and edge set  $E_K = \{uv : u - v \in \{(1, 0), (0, 1), (1, -1), (1, 1)\}\}$ . Its name comes from the fact that, on an infinite



**Fig. 6.8** Black vertices are the elements of a periodic code in  $G_T^{[3]}$  which is (a) 1-LD with density  $\frac{3}{10}$ , (b) 1-Id with density  $\frac{1}{3}$ .

chessboard, the  $r$ -neighborhood of a vertex  $v$  is the set of vertices that a king, starting from  $v$ , can reach in at most  $r$  moves. Figure 6.9 will give constructions proving the upper bounds for Ths. 35, 37 and 38(b).

• **Locating-dominating codes**

**Theorem 35** ([113]) *We have  $\partial_1^{LD}(G_K) = \frac{1}{5}$ .*

For  $r > 1$ , there are good bounds on  $\partial_r^{LD}(G_K)$ , and the first two strips are solved for  $r = 1$ .

**Theorem 36** ([152]) *For every  $r \geq 1$ , we have  $\frac{1}{4r+2} \leq \partial_r^{LD}(G_K) \leq \begin{cases} \frac{1}{4r} & : r \text{ even} \\ \frac{r+1}{4r(r+1)+2} & : r \text{ odd} \end{cases}$ . When  $r$  increases, these bounds are all equivalent to  $\frac{1}{4r}$ .*

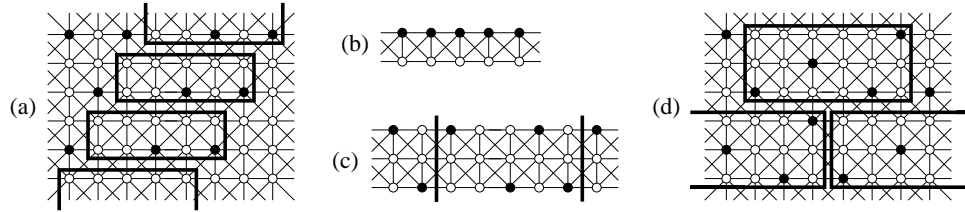
**Theorem 37** ([22]) *We have  $\partial_1^{LD}(G_K^{[2]}) = \frac{1}{2}$ ,  $\partial_1^{LD}(G_K^{[3]}) = \frac{4}{15} \approx 0.26667$ .*

• **Identifying codes**

It is remarkable that the best density is known for all  $r \geq 1$  for identification [34] (2004). In the following theorem, the lower bound on  $\partial_1^{Id}(G_K)$  comes from [58], the upper bound from [38].

**Theorem 38** (a) (Charon *et al.* [34]) *For every  $r > 1$ , we have  $\partial_r^{Id}(G_K) = \frac{1}{4r}$ .*

(b) *We have  $\partial_1^{Id}(G_K) = \frac{2}{9} \approx 0.22222$ .*



**Fig. 6.9** Black vertices are the elements of a periodic code which is (a) 1-LD with density  $\frac{1}{5}$  in  $G_K$ , (b) 1-LD with density  $\frac{1}{2}$  in  $G_K^{[2]}$ , (c) 1-LD with density  $\frac{4}{15}$  in  $G_K^{[3]}$ , (d) 1-Id with density  $\frac{2}{9}$  in  $G_K$ .

For  $r = 1$ , the strips have been investigated. Note that  $G_K^{[2]}$  is not 1-twin-free.

**Theorem 39** ([67]) (a) *We have  $\partial_1^{Id}(G_K^{[3]}) = \frac{1}{3}$ ;  $\partial_1^{Id}(G_K^{[4]}) = \frac{5}{16} = 0.3125$ ;  $\partial_1^{Id}(G_K^{[5]}) = \frac{4}{15} \approx 0.26667$ ;  $\partial_1^{Id}(G_K^{[6]}) = \frac{5}{18} \approx 0.27778$ .*

(b) *For  $k \geq 7$ , we have  $\frac{2}{9} + \frac{8}{81k} \leq \partial_1^{Id}(G_K^{[k]}) \leq \begin{cases} \frac{2}{9} + \frac{6}{18k} & : k \equiv 0 \pmod{3} \\ \frac{2}{9} + \frac{8}{18k} & : k \equiv 1 \pmod{3} \\ \frac{2}{9} + \frac{7}{18k} & : k \equiv 2 \pmod{3} \end{cases}$ .*

#### 6.4.1.4 The Hexagonal Grid

The hexagonal grid,  $G_H$ , has vertex set  $V_H = \mathbb{Z} \times \mathbb{Z}$  and edge set  $E_H = \{uv : u = (i, j) \text{ and } u - v \in \{(0, (-1)^{i+j+1}), (1, 0)\}\}$ . It is the grid for which one has the sparsest and weakest results. Figure 6.10 will give constructions proving the upper bounds for Ths. 40 and 41.

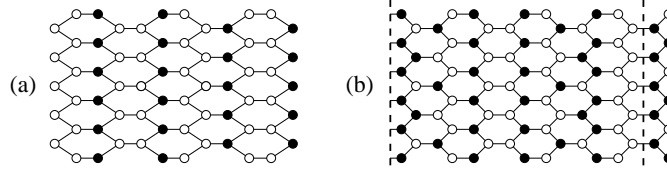
• **Locating-dominating codes**

**Theorem 40** ([113]) *We have  $\partial_1^{LD}(G_H) = \frac{1}{3}$ .*

• **Identifying codes**

The following upper bound is from [56], the lower bound from [61].

**Theorem 41** *We have  $\frac{5}{12} \approx 0.41667 \leq \partial_1^{Id}(G_H) \leq \frac{3}{7} \approx 0.42857$ .*



**Fig. 6.10** Black vertices are the elements of a periodic code in  $G_H$ , which is (a) 1-LD with density  $\frac{1}{3}$ , (b) 1-Id with density  $\frac{3}{7}$ .

In the following theorem, the lower bounds in (a) and (b) come from [131] and [144], respectively, and both upper bounds from [38]; the general lower bounds in (c) come from [33], and the upper bounds from [174].

**Theorem 42** (a) *We have  $\partial_2^{Id}(G_H) = \frac{4}{19} \approx 0.21053$ .*

(b) *We have  $\frac{3}{25} = 0.12 \leq \partial_3^{Id}(G_H) \leq \frac{1}{6} \approx 0.16667$ .*

(c) *For  $r \geq 8$ , we have  $\left. \begin{array}{l} \frac{2}{5r+3} : r \text{ even} \\ \frac{2}{5r+2} : r \text{ odd} \end{array} \right\} \leq \partial_r^{Id}(G_H) \leq \left\{ \begin{array}{ll} \frac{5r+3}{6r(r+1)} & : r \text{ even} \\ \frac{5r^2+10r-3}{(6r-2)(r+1)^2} & : r \text{ odd} \end{array} \right.$ . When  $r$  increases, these bounds are close to  $\frac{2}{5r}$  and  $\frac{5}{6r} \approx \frac{0.83333}{r}$ .*

There are many better upper bounds, obtained by the use of heuristics, in [38] for  $r \leq 30$ .

#### 6.4.2 Paths and Cycles

For  $r = 1$ , the values of  $LD(P_n)$ ,  $LD(C_n)$ ,  $Id(P_n)$  and  $Id(C_n)$  have been completely determined.

**Theorem 43** ([169]), ([170]) *For every path or cycle  $G_n$  of order  $n \geq 1$ , we have  $LD(G_n) = \lceil \frac{2n}{5} \rceil$ .*

**Proof.** For the lower bound, we give an alternative by Bertrand *et al.* [16] in 2004 to the 1988 proof by Slater. This proof uses the following counting argument, which can be applied to graphs other than paths or cycles, and can be adapted to Id codes, cf. proof of Th. 32. Let  $C$  be a 1-LD code



The case  $r = 2$  is completely settled in [160]. And in 2011–12, the complete results were given independently by Chen, Lu & Miao [52] and by Junnila & Laihonon [130].

**Theorem 49** ([52]), ([130]) *Let  $r \geq 2$  and  $n = (2r + 1)p + q$ ,  $p \geq 1$ ,  $0 \leq q < 2r + 1$ .*

- (a) *If  $q = 0$ , then  $Id_r(P_n) = \frac{(2r+1)p}{2} + 1$  if  $p$  is even;  $Id_r(P_n) = \frac{(2r+1)(p-1)}{2} + 2r$  if  $p$  is odd.*
- (b) *If  $1 \leq q \leq r + 1$ , then  $Id_r(P_n) = \frac{(2r+1)p}{2} + q$  if  $p$  is even;  $Id_r(P_n) = \frac{(2r+1)(p-1)}{2} + 2r + 1$  if  $p$  is odd.*
- (c) *If  $r + 2 \leq q \leq 2r$ , then  $Id_r(P_n) = \frac{(2r+1)p}{2} + q - 1$  if  $p$  is even;  $Id_r(P_n) = \frac{(2r+1)(p-1)}{2} + 2r + 1$  if  $p$  is odd.*

This means that  $Id_r(P_n)$  lies between  $\frac{n+1}{2}$  and  $\frac{n}{2} + r$ , according to the values of  $p$  and  $q$ .

#### 6.4.2.2 Cycles with $r > 1$

##### • Locating-dominating codes

**Theorem 50** ([16]) *Let  $r \geq 2$ . For all  $n \geq 1$ , we have  $\frac{n}{3} \leq LD_r(C_n)$ , and this bound can be achieved for infinitely many values of  $n$ .*

In [52], the exact value of  $LD_2(C_n)$  is given for  $n \geq 1$ , and in [72], the same is done for  $LD_3(C_n)$  and  $LD_4(C_n)$ ; furthermore, the exact values of  $LD_r(C_n)$  are given for  $r \geq 5$ ,  $n \in [3, 2r+3] \cup \{3r, 3r+3\}$ , and the following is proved.

**Theorem 51** ([72]) *Let  $r \geq 5$  and  $n \geq 12r + 5 + 6r((r-3)(2r+1) + r-1)$ . Then we have  $LD_r(C_n) = \lceil \frac{n}{3} \rceil$  if  $n \not\equiv 3 \pmod{6}$ , and  $\frac{n}{3} \leq LD_r(C_n) \leq \frac{n}{3} + 1$  otherwise.*

The previous result is improved in [70]: the bound on  $n$  becomes a polynomial in  $r^2$  instead of  $r^3$ .

##### • Identifying codes

The crucial following lower bound is from [64] and solves the case  $n$  even.

**Theorem 52** ([64]), ([65]) *For all  $r \geq 2$  and  $n \geq 2r + 2$ , we have  $Id_r(C_n) \geq \gcd(2r+1, n) \times \lceil \frac{n}{2\gcd(2r+1, n)} \rceil$ . If  $n \geq 2r + 4$  is even, then  $Id_r(C_n) = \frac{n}{2}$ , and  $Id_r(C_{2r+2}) = 2r + 1$ .*

Then [99] provides cases when the exact value of  $Id_r(C_n)$  is known. The case  $r = 2$  is completely settled in [160]. More cases giving the exact value for  $Id_r(C_n)$  are given in [178], further results of this type are given in [52], and finally in 2012, Junnila & Laihonon [130] closed the case: for all  $r \geq 2$  and  $n \geq 2r + 2$ ,  $Id_r(C_n)$  is known; but the many cases do not allow to give the results in a compact way. Let us simply mention here that  $Id_r(C_n)$  lies between  $\frac{n+1}{2}$  and  $\frac{n}{2} + r$ .

#### 6.4.3 Trees

The following upper bound comes from [20], the lower bound from [166].

**Theorem 53** *Let  $T$  be a tree of order  $n$  with  $\ell$  leaves and  $s$  support vertices.*

- (a) *We have  $\frac{n+2(\ell-s)+1}{3} \leq LD_1(T) \leq \frac{n+\ell-s}{2}$ .*
- (b) ([153]) *We have  $LD_1(T) \leq \frac{2n+3\ell-2}{5}$ .*



Moreover, the trees achieving the above bounds can be characterized. When  $\ell = s$ , the lower bound reads  $LD_1(T) > n/3$ . The star  $K_{1,n-1}$  has  $s = 1$ ,  $\ell = n - 1$  and  $LD_1(K_{1,n-1}) = n - 1$ . All the intermediate values between  $\lceil \frac{n+1}{3} \rceil$  and  $n - 1$  can be reached [166].

**Theorem 54** ([26]) *For any tree  $T$  of order  $n$ , we have  $LD_2(T) > \frac{n}{4}$ , and  $LD_3(T) > \frac{n}{5}$ .*

From this it is conjectured that for any tree  $T$  of order  $n$ ,  $LD_r(T) > \frac{n}{r+2}$ ; if true, the conjecture is sharp [26]. The following upper bound comes from [94], the lower bound from [20].

**Theorem 55** *For any tree  $T$  of order  $n$  with  $\ell$  leaves and  $s$  support vertices, we have*

$$(a) \frac{n+2\ell-2}{2} \geq Id_1(T) \geq \frac{3(n+\ell-s+1)}{7}. \text{ Both bounds are sharp.}$$

$$(b) ([154]) \frac{3n+2\ell-1}{5} \geq Id_1(T) \geq \frac{2n-s+3}{4}.$$

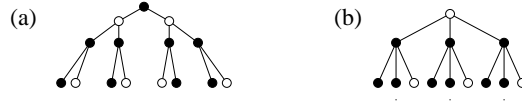
The trees achieving the above bounds in (b) can be characterized. All the intermediate values between  $\lceil \frac{3(n+1)}{7} \rceil$  and  $n - 1$  can be reached [41].

The following result *corrects* and completes the study of complete (or balanced)  $q$ -ary trees for 1-Id codes in [134], and also settles the case of 1-LD codes. It is illustrated by Fig. 6.12.

**Theorem 56** ([17]) *Let  $h \geq 1$ ,  $q \geq 2$ ,  $CT_h^q$  be the complete  $q$ -ary tree of depth  $h$ , and  $\xi_h^q = \frac{q^h-1}{q-1}$  be its order.*

$$(a) \text{ We have } LD(CT_h^q) = \left\lceil \frac{q^2 \xi_h^q}{q^2 + q + 1} \right\rceil.$$

$$(b) \text{ We have } Id(CT_h^2) = \left\lceil \frac{20\xi_h^2}{31} \right\rceil; \text{ if } q > 2, \text{ then } Id(CT_h^q) = \left\lceil \frac{q^2 \xi_h^q}{q^2 + q + 1} \right\rceil.$$



**Fig. 6.12** Codewords are in black. (a) The complete binary tree of depth 4, with an optimal 1-LD code (of size 9) which is not 1-Id. (b) The complete ternary tree of depth 3, with an optimal 1-Id code (of size 9). This code is also an optimal 1-LD code.

Comparing (a) and (b) above, one can see that the complete nonbinary trees give yet another example of graphs in which 1-identification costs the same as 1-locating-domination—see Th. 21(b).

Although the computing problems of finding optimal  $r$ -LD and  $r$ -Id codes are generally NP-hard, even for some restricted classes of graphs (see Sect. 6.8 below), trees (and forests) form a class for which polynomial and even linear algorithms exist that output an optimal 1-LD or 1-Id code: see [169] for LD codes, [4] for Id codes, and [32] for Id codes in oriented trees.

#### 6.4.4 The $q$ -ary $n$ -Cube

Here, the standard notation  $n$  is not for the order of the cube, but for its dimension: the  $q$ -ary  $n$ -cube is denoted  $\mathbb{F}_q^n$ , where  $\mathbb{F}_q = \{0, 1, \dots, q-1\}$ ; it has  $q^n$   $q$ -ary vectors  $v_1 v_2 \dots v_n$ , and the distance is usually the *Hamming distance*  $d_H$  defined by  $d_H(u, v) = |\{i : u_i \neq v_i, 1 \leq i \leq n\}|$ . In this setting,  $r$ -dominating codes are rather called  *$r$ -covering codes*, and constitute an important topic

inside coding theory. Their strong algebraic structure makes them very particular: for instance, a code may be a vector subspace of dimension  $k$  in  $\mathbb{F}_q^n$ , in which case it has size  $q^k$  and is said to be *linear*. Since a generator matrix with  $k$  rows suffices to describe the code, this affects the size of an instance when considering complexity issues, see, e.g., [143], [91], [118], [119]. See also [55] for an overview of covering codes, with tables giving bounds on the sizes of optimal codes, linear or not, for the first values of  $r$  and  $n$ . We shall restrict ourselves to the case  $q = 2$ , although there are some works on the nonbinary cube, and denote  $\mathbb{F}_2^n$  simply  $\mathbb{F}^n$ . Almost all the results given below concern identification. There are strong links between Id codes and  $\mu$ -fold coverings (where each vertex is dominated by at least  $\mu$  codewords), but we do not have enough space to discuss them here; see, e.g., [76]. The following theorem connects the 1-LD number in  $\mathbb{F}^n$  and the 2-domination number in  $\mathbb{F}^{n-1}$ .

**Theorem 57** ([116]) *For all  $n \geq 5$ , we have  $\frac{n^2 2^{n+1}}{n^3 + 2n^2 + 3n - 2} \leq LD_1(\mathbb{F}^n) \leq (2n - 3)\gamma_2(\mathbb{F}^{n-1})$ .*

The exact values of  $LD_1(\mathbb{F}^n)$  are known for  $1 \leq n \leq 6$ ; for instance,  $\{u \in \mathbb{F}^5 : d_H(u, 00000) = 1 \text{ or } 4\}$  is an optimal 1-LD code (of size 10) [116]. See also [73], [133]. The following result gives the complete answer for linear codes.

**Theorem 58** ([116]) *Let  $n = 3 \times 2^k - 5 + s$ , for  $k \geq 1$  and  $0 \leq s < 3 \times 2^k$ . Then the size of an optimal 1-LD linear code in  $\mathbb{F}^n$  is  $2^{n-k}$ .*

We now turn to Id codes. The following theorem links the  $r$ -Id and  $(2r)$ -domination numbers.

**Theorem 59** ([134]) *For  $n \geq 3$ , we have  $\frac{n 2^{n+1}}{n(n+1)+2} \leq Id(\mathbb{F}^n)$  and  $2^{n+1}/(1 + \sum_{j=0}^r \binom{n}{j}) \leq Id_r(\mathbb{F}^n)$ . If  $r < n/2$ , then  $Id_r(\mathbb{F}^n) \leq \binom{n}{r} \gamma_{2r}(\mathbb{F}^n)$ .*

The above lower bound on  $Id_r(\mathbb{F}^n)$  is improved in [77] (but both bounds coincide for  $r$  fixed and  $n$  large enough), then in [75]. The next result uses the direct sum construction (DSC)  $C \oplus \mathbb{F}^s = \{c|u : c \in C \subseteq \mathbb{F}^n, u \in \mathbb{F}^s\}$ , where  $|$  stands for concatenation. This is a classical tool, often used in coding theory, allowing to go from  $\mathbb{F}^n$  to  $\mathbb{F}^{n+s}$ ,  $s \geq 1$ .

**Theorem 60** ([18]) *For  $n \geq 2$ , we have  $Id(\mathbb{F}^{n+2}) \leq 4Id(\mathbb{F}^n)$ . If  $C$  is an optimal 1-Id code in  $\mathbb{F}^n$  such that  $d_H(c, C \setminus \{c\}) = 1$  for every  $c \in C$ , then  $Id(\mathbb{F}^{n+1}) \leq 2Id(\mathbb{F}^n)$ .*

Since  $\mathbb{F}^s$  is a 0-Id code of size  $2^s$ , a generalization of the previous result would be the following conjecture:  $Id_{r_1+r_2}(\mathbb{F}^{n_1+n_2}) \leq Id_{r_1}(\mathbb{F}^{n_1}) \times Id_{r_2}(\mathbb{F}^{n_2})$ . The case  $r_1 = r_2 = 1$  is proved in [76], which by iteration leads to

$$Id_r(\mathbb{F}^{\sum_{i=1}^r n_i}) \leq \prod_{i=1}^r Id_1(\mathbb{F}^{n_i}).$$

Also, by refining the condition on  $C$  in Th. 60, one can widen the possibilities of the DSC, see, e.g., [76], [74], [31]. But  $Id(\mathbb{F}^{n+1}) \leq 2Id(\mathbb{F}^n)$  remains a conjecture; to our knowledge, the closest result obtained so far is the following.

**Theorem 61** ([74]) *For  $n \geq 2$ , we have  $Id(\mathbb{F}^{n+1}) \leq \left(2 + \frac{1}{n+1}\right) Id(\mathbb{F}^n)$ .*

There are also asymptotic results (with  $n$  going to infinity) in [134], [118], [74], or [125].

In 2010, Charon *et al.* [31] gave tables with bounds on  $Id_r(\mathbb{F}^n)$ ,  $1 \leq r \leq 5$ ,  $n \leq 21$ ; the same was done for  $r \in \{2, 3\}$  and  $n \leq 30$  by Exoo *et al.* [75] in the same year; all sources are given. The lower bounds stem from the bounds discussed above, and from more ad hoc methods using the topology

of the cube and studying local situations, possibly with the help of a computer. The upper bounds use widely the DSC, often enhanced by computer, or heuristics such as simulated annealing or the noising method.

In the linear case, the complete answer is known for  $r = 1$  (see [157] for  $r > 1$ ), and is very similar to Th. 58 for 1-LD codes. In particular, it uses the second part of Th. 60, which always works with linear codes and allows to go from  $\mathbb{F}^n$  to  $\mathbb{F}^{n+1}$  by doubling the size of the code.

**Theorem 62** ([156]) *Let  $n = 3 \times 2^k - 3 + s$ , for  $k \geq 1$  and  $0 \leq s < 3 \times 2^k$ . Then the size of an optimal 1-Id linear code in  $\mathbb{F}^n$  is  $2^{n-k}$ .*

In conclusion, we offer an open problem about the monotonicity of  $Id_r(\mathbb{F}^n)$ . It is true that for all  $n \geq 2$ , we have  $Id_1(\mathbb{F}^n) \leq Id_1(\mathbb{F}^{n+1})$  (Moncel [145] in 2006), but this is not the case for all  $r$ . For instance,  $Id_5(\mathbb{F}^6) = 63$ ,  $Id_5(\mathbb{F}^7) \in \{31, 32\}$ ,  $Id_5(\mathbb{F}^8) \in [19, 21]$ ,  $Id_5(\mathbb{F}^9) \in [12, 17]$ . We conjecture however that for a fixed  $r > 1$ , there exists  $n(r)$  such that for all  $n \geq n(r)$ , we have  $Id_r(\mathbb{F}^n) \leq Id_r(\mathbb{F}^{n+1})$ .

#### 6.4.5 Bipartite Graphs and Discriminating Codes

There are scattered results regarding bipartite graphs (see Ths. 15 and 100(c), Sect. 6.6.2.3, and the second sentence after Th. 107), but these graphs are also of interest to us because they lend themselves to the natural definition of *discriminating codes*: consider a bipartite graph  $G = (I \cup A, E)$  where  $I$  represents individuals and  $A$  their attributes (hair color, age, glasses, ...). A code  $C \subseteq A$  is discriminating if every individual has at least one attribute in  $C$ , and no two individuals have the same set of attributes in  $C$  [30].

**Theorem 63** ([28]) *If  $C$  is an optimal discriminating code in  $G$ , then  $\lceil \log_2(|I| + 1) \rceil \leq |C| \leq |I|$ . Both bounds are sharp.*

Discriminating codes may be seen as a generalization of 1-Id codes, since a 1-Id code in a graph  $G = (V, E)$  is clearly a discriminating code in the bipartite graph consisting of  $V$  on the one hand, and vertices representing the closed neighborhoods in  $G$  on the other hand, with edges linking each closed neighborhood to its members.

The definition can be extended to any odd  $r \geq 1$ , by asking that for  $i \in I$ ,  $i_1 \in I$ ,  $i_2 \in I$ ,  $i_1 \neq i_2$ , we have  $N_r(i) \cap C \neq \emptyset$  and  $N_r(i_1) \cap C \neq N_r(i_2) \cap C$ . The associated decision problem is NP-complete for any fixed  $r \geq 1$  [28], but for  $r = 1$  there is a linear algorithm in the case of trees [30].

Special cases of bipartite graphs have been investigated: the infinite square and hexagonal grids [28], bipartite planar graphs [30], and the binary  $n$ -cube, for which it can be proved that for any odd  $r$ , there is a bijection between the set of  $r$ -Id codes in  $\mathbb{F}^n$  and the set of  $r$ -discriminating codes in  $\mathbb{F}^{n+1}$  [29].

#### 6.4.6 Line Graphs and Edge Identification

We can *identify an edge with edges* in a graph  $G = (V, E)$ : for  $e \in E$ , we denote  $N(e)$  the set of edges which are adjacent to  $e$ , and  $N[e] = N(e) \cup \{e\}$ . A code  $C_E \subseteq E$  is edge-identifying

if for every  $e \in E$ , we have  $N[e] \cap C_E \neq \emptyset$ , and for every  $e_1 \in E$ ,  $e_2 \in E$ ,  $e_1 \neq e_2$ , we have  $N[e_1] \cap C_E \neq N[e_2] \cap C_E$ . Clearly,  $C_E$  is edge-identifying in  $G$  if and only if, in the line graph  $L(G)$ , the vertices corresponding to the edges in  $C_E$  form a 1-Id code. This motivates the study of Id codes in the class of line graphs. For instance, one has the following results.

**Theorem 64** ([81]) (a) *If  $G$  is a 1-twin-free line graph of order  $n$ , then  $Id(G) > \frac{3\sqrt{2}}{4}\sqrt{n} \approx 1.06066\sqrt{n}$ .*

(b) *For  $n \geq 4$ , the minimum size of an edge-identifying code in the binary cube  $\mathbb{F}^n$  is  $2^{n-1}$ .*

The decision problem associated to edge identification is NP-complete [148].

### 6.4.7 Miscellaneous

#### • Planar and outerplanar graphs

**Theorem 65** ([155]) *For any planar graph  $G$  of order  $n \geq 18$ , we have  $LD(G) \geq \frac{n+10}{7}$ ; for any outerplanar graph  $G$  of order  $n$ , we have  $LD(G) \geq \frac{2n+3}{7}$ . Both bounds are sharp.*

#### • Split graphs

Split graphs have been studied in several works; in particular, complete suns have been intensively investigated, both for 1-LD and 1-Id codes: see, e.g., [2], [122].

• In the context of LD and Id codes, attention has been given in the existing literature to the following families: graph products (mostly Cartesian, but also corona, lexicographic, Kronecker or direct), cubic graphs, rotagraphs and fasciagraphs, finite rectangles extracted from the infinite grids ( $k_1$  rows,  $k_2$  columns),  $n$ -dimensional infinite grids, Kneser graphs, circulant networks, complementary prisms, the  $n$ -cube endowed with the Lee metric, triangle-free graphs, cographs, Sierpiński graphs, interval graphs, permutation graphs, triangular graphs, vertex-transitive graphs, fractal graphs, block graphs, chordal graphs, series parallel graphs, and others. Random graphs have also been studied.

See also Tables 1.3–1.5 in [79] for a survey of the different lower and upper bounds, and their tightness, for different classes of graphs.

## 6.5 Relationships with Other Parameters

We study the different relationships linking the LD and Id numbers of a graph to different parameters such as maximum and minimum degree, girth, domination number, independent domination numbers, or diameter, mostly for  $r = 1$ .

### 6.5.1 Locating-Dominating Codes

**Theorem 66** ([170]) *For any graph  $G$  of order  $n \geq 2$  and maximum degree  $\Delta$ , we have  $n \leq LD_1(G) + \sum_{i=1}^{\Delta} \binom{LD_1(G)}{i}$ .*

For  $\Delta = 2$  (paths and cycles), this can be improved to  $n \leq \frac{5}{2}LD_1(G)$ , see Th. 43.

Theorem 67 below gives upper bounds on  $LD_1(G)$ , with conditions on minimum degree and girth, then the best known constructions, for  $n$  small or arbitrarily large.

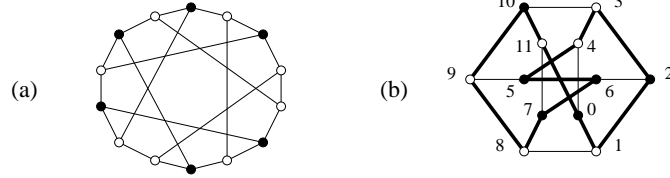
**Theorem 67** *Let  $G$  be a graph of order  $n$ , minimum degree  $\delta$ , and girth  $g \geq 5$ .*

(a<sub>1</sub>) ([50]) *If  $\delta \geq 2$ , then  $LD_1(G) \leq \frac{n}{2}$ .*

(a<sub>2</sub>) ([10]) *If  $\delta = 3$ , then  $LD_1(G) \leq \frac{72n}{45} \approx 0.4889n$ .*

(b) ([10]) *The cycle  $C_6$  has  $n = 6$ ,  $\delta = 2$ ,  $g = 6$ , and  $LD_1(C_6) = 3 = \frac{n}{2}$ ; there are infinitely many connected graphs  $G$  of order  $n$  with  $\delta = 2$ ,  $g = 5$ , and  $LD_1(G) = \frac{n-1}{2}$ .*

*The Heawood graph  $H_{14}$  (see Fig. 6.13(a)) has  $n = 14$ ,  $\delta = 3$ ,  $g = 6$ , and  $LD_1(H_{14}) = 6 = \frac{3n}{7} \approx 0.4286n$ ; there are infinitely many connected graphs  $G$  of order  $n$  with  $\delta = 3$ ,  $g = 5$ , and  $LD_1(G) = \frac{4(n-1)}{11} \approx 0.3636n$ .*



**Fig. 6.13** Black vertices form optimal codes. (a) The Heawood graph  $H_{14}$ ; (b) The graph  $G_{12}$ .

We recall that  $i(G)$  (respectively,  $\beta(G)$ ) denotes the minimum (respectively, maximum) cardinality of an independent dominating set in a graph  $G$ .

**Theorem 68** ([50]) (a) *If  $T$  is a nontrivial tree, then  $i(T) \leq LD_1(T) \leq \beta(T)$ .*

(b) *For every nontrivial tree  $T$ , we have  $LD_1(T) + \gamma_1(T) \leq n$ , with equality if and only if  $T$  is a tree consisting only of leaves and support vertices.*

**Theorem 69** ([25]) *Let  $G$  be a graph of order  $n$  and diameter  $D \geq 3$ . Then  $LD_1(G) \leq n - \lceil \frac{3D-1}{5} \rceil$ , and the bound is tight. Consequently, if  $LD_1(G) \in \{n-2, n-1\}$ , then  $G$  has diameter at most 3.*

### 6.5.2 Identifying Codes

We first give a lower bound involving the maximum degree, then discuss different upper bounds, first in terms of maximum degree, then of minimum degree and girth.

**Theorem 70** ([134]) *Let  $r \geq 1$ . If  $G$  is an  $r$ -twin-free graph of order  $n$  and maximum degree  $\Delta$ , then  $Id_r(G) \geq \frac{2n}{\Delta+2}$ .*

For  $r = 1$ , [79] gives a full characterization of the graphs reaching the previous lower bound.

**Theorem 71** ([83]) *Let  $r \geq 1$  and  $G$  be a connected  $r$ -twin-free graph of order  $n$  and maximum degree  $\Delta$ . Then  $Id_r(G) \leq n - \frac{n(\Delta-2)}{\Delta(\Delta-1)^{5r}-2} = n - \frac{n}{\Theta(\Delta^{5r})}$ .*

It is conjectured in [84] that there exists a constant  $c$  such that, for every nontrivial connected 1-twin-free graph  $G$ , we have  $Id_1(G) \leq n - \frac{n}{\Delta} + c$ . Graphs such that  $Id_1(G) = n - \frac{n}{\Delta}$  exist (e.g.,  $K_{\Delta, \Delta}$  with  $\Delta \geq 3$ ). The complete  $q$ -ary trees  $T$  have  $\Delta = q + 1$  and, by Th. 56(b), we have  $Id_1(T) = \lceil n - \frac{n}{\Delta-1+1/\Delta} \rceil$ . The conjecture holds for graphs with very high  $\Delta$ , and for  $\Delta = 2$ , and the following results are intended to further substantiate it.

**Theorem 72** ([84]) *Let  $G$  be a connected, 1-twin-free, triangle-free graph with order  $n$  and maximum degree  $\Delta \geq 3$ .*

(a) *We have  $Id_1(G) \leq n - \frac{n}{\Delta + \frac{3\Delta}{\ln \Delta - 1}} = n - \frac{n}{\Delta + o(\Delta)}$ .*

(b) *If moreover  $G$  is a nontrivial planar or bipartite graph, then  $Id_1(G) \leq n - \frac{n}{\Delta + 9}$ .*

Other graphs can be exhibited, with  $Id_1(G) \leq n - \frac{n}{o(\Delta)}$ . In the following theorem, a vertex  $z$  is said to be *forced* if there are two vertices  $u$  and  $v$  with symmetric difference  $N[u] \Delta N[v]$  equal to  $\{z\}$ , which implies that  $z$  belongs to any 1-Id code in  $G$ .

**Theorem 73** ([89]) (a) *Let  $G$  be a connected, 1-twin-free graph, with order  $n$  and maximum degree  $\Delta \geq 3$ . Then  $Id_1(G) \leq n - \frac{nf^2}{103\Delta}$ , where  $f$  is the ratio over  $n$  of the number of non-forced vertices. In particular, if  $G$  is  $\Delta$ -regular, then  $Id_1(G) \leq n - \frac{n}{103\Delta} = n - \frac{n}{\Theta(\Delta)}$ .*

(b) *There exists an integer  $\Delta_0$  such that for every connected, 1-twin-free graph with order  $n$  and maximum degree  $\Delta \geq \Delta_0$ , we have  $Id_1(G) \leq n - \frac{n}{103\Delta(\Delta+1)^2} = n - \frac{n}{\Theta(\Delta^3)}$ .*

The following result can be compared to Th. 67.

**Theorem 74** ([10]) *Let  $G$  be a twin-free graph of order  $n$ , minimum degree  $\delta$ , and girth  $g \geq 5$ .*

(a<sub>1</sub>) *If  $\delta \geq 2$ , then  $Id_1(G) \leq \frac{5n}{7} \approx 0.7143n$ .*

(a<sub>2</sub>) *If  $\delta = 3$ , then  $Id_1(G) \leq \frac{31n}{45} \approx 0.6889n$ .*

(b) *The cycle  $C_7$  has  $n = 7$ ,  $\delta = 2$ ,  $g = 7$ , and  $Id_1(C_7) = 5 = \frac{5n}{7}$ ; there are infinitely many connected graphs  $G$  of order  $n$  with  $\delta = 2$ ,  $g = 5$ , and  $Id_1(G) = \frac{3(n-1)}{5}$ .*

*The graph  $G_{12}$  with vertex set  $\{x_i : 0 \leq i \leq 11\}$  and the edges of the Hamiltonian cycle  $x_0x_1 \dots x_{11}x_0$  plus the six edges  $x_0x_4$ ,  $x_1x_8$ ,  $x_2x_6$ ,  $x_3x_{10}$ ,  $x_5x_9$  and  $x_7x_{11}$  (see Fig. 6.13(b)) has  $n = 12$ ,  $\delta = 3$ ,  $g = 5$ , and  $Id_1(G_{12}) = 6 = \frac{n}{2}$ ; there are infinitely many connected graphs  $G$  of order  $n$  with  $\delta = 3$ ,  $g = 5$ , and  $Id_1(G) = \frac{5(n-1)}{11} \approx 0.4545n$ .*

### 6.5.3 Twin-Free Graphs and Parameters

From now on and until the end of this section, we investigate the extremal values that the following parameters: size, order, maximum and minimum degree, size of a maximum clique, radius, diameter, size of a maximum independent set, and identification number, can achieve in a connected  $r$ -twin-free graph. If  $\pi$  stands for such a parameter, we fix  $r$  and search for the smallest value  $f_r(\pi)$  that  $\pi$  can reach, or we fix  $r$  and  $n$  and search for the smallest and largest values  $f_{r,n}(\pi)$  and  $F_{r,n}(\pi)$ . Therefore,  $f_r(\pi) = \min\{\pi(G) : G \text{ connected, } r\text{-twin-free with at least } 2r + 1 \text{ vertices}\}$ ,  $f_{r,n}(\pi) = \min\{\pi(G) : G \text{ connected, } r\text{-twin-free with } n \geq 2r + 1\}$ , and  $F_{r,n}(\pi)$  is defined similarly. The first case, when  $\pi = \varepsilon$  is the size of the graph, will help understand easily these notions.

#### • Number of edges $\varepsilon$

For any connected graph of order  $n$ , the size is between  $n - 1$  (trees) and  $\frac{n(n-1)}{2}$  (the complete

graph). The size of a connected  $r$ -twin-free graph of order at least  $2r + 1$  is at least  $2r$ , and if the order is  $n \geq 2r + 1$ , then this number is at least  $n - 1$ ; moreover, the paths  $P_{2r+1}$  and  $P_n$  meet these bounds. So  $f_r(\varepsilon) = 2r$  and  $f_{r,n}(\varepsilon) = n - 1$ . The maximum number of edges possible for  $r$ -twin-free graphs of order  $n$  is known for  $r = 1$ , and can be achieved only by the complete graphs minus a maximum matching [64]. For  $r > 1$ , we are close to the exact value.

**Theorem 75** ([64]) *For all  $n \geq 3$ , we have  $F_{1,n}(\varepsilon) = \frac{n(n-1)}{2} - \lfloor \frac{n}{2} \rfloor$ .*

**Theorem 76** ([5]) (a) *For  $n$  large enough, we have  $\frac{n^2}{2} - 2n \log_2 n \lesssim F_{2,n}(\varepsilon) \lesssim \frac{n^2}{2} - \frac{1}{2}n \log_2 n$ .*

(b) *For  $r > 2$  and  $n$  large enough with respect to  $r$ , we have  $\frac{n^2}{2} - rn \log_2 n \lesssim F_{r,n}(\varepsilon) \lesssim \frac{n^2}{2} - 0.63(r - 0.915)n \log_2 n$ .*

For  $r > 2$ , the gap between the above lower and upper bounds is about  $(0.37r + 0.58)n \log_2 n$ . For  $r = 10$  and  $n$  around 60 000, it represents 0.2%.

• **Number of vertices  $n$**

Obviously,  $f_{r,n}(n) = F_{r,n}(n) = n$  for all  $r \geq 1$ ,  $n \geq 2r + 1$ , and the following comes from Th. 2.

**Theorem 77** *For all  $r \geq 1$ , we have  $f_r(n) = 2r + 1$ .*

• **Maximum degree  $\Delta$**

In any connected graph of order  $n$ , the maximum degree is between 1 (if  $n = 2$ ) or 2 (the path, the cycle,  $n \geq 3$ ), and  $n - 1$  (the complete graph, the star, every graph  $G$  with  $\gamma_1(G) = 1$ ).

**Theorem 78** ([45]) (a) *For all  $r \geq 1$ , we have  $f_r(\Delta) = 2$ ; for all  $r \geq 1$  and  $n \geq 2r + 1$ , we have  $f_{r,n}(\Delta) = 2$ .*

(b) *For all  $n \geq 3$ , we have  $F_{1,n}(\Delta) = n - 1$ .*

(c) *For all  $p \geq 2$  and  $n \geq 5$ , if  $2^{p-1} + p - 1 < n \leq 2^p + p$ , then  $F_{2,n}(\Delta) = n - p - 1$ .*

(d) *For all  $r \geq 3$  and  $n \geq 2r + 1$ , we have  $F_{r,n}(\Delta) \leq k$ , where  $k$  is the largest integer such that  $k + (r - 2)\lceil \log_3(k + 1) \rceil + \lceil \log_2(k + 1) \rceil \leq n - 1$ .*

When  $r > 2$ , there are too many cases for the lower bounds on  $F_{r,n}(\Delta)$  to be given here. Let us only mention that: (i) we have the exact value of  $F_{r,n}(\Delta)$  for infinitely many values of  $n$  and  $r$ —for instance, if  $r = 100$  and  $60\,050 \leq n \leq 60\,143$ , then  $F_{100,n}(\Delta) = 59\,048$ ; (ii) if  $n$  is large with respect to  $r$ , the lower bound on  $F_{r,n}(\Delta)$  approximately behaves like  $n - r \log_3 n$ , whereas the upper bound in Case (d) above behaves like  $n - (r - 2 + \log_2 3) \log_3 n$ , which gives a gap between the bounds roughly equal to  $0.415 \times \log_3 n$  (independent of  $r$ ).

• **Minimum degree  $\delta$**

In any connected graph of order  $n$ , the minimum degree is between 1 and  $n - 1$  (the complete graph). The first part of Case (b) below relies on the fact that  $K_n$  minus a maximum matching is 1-twin-free, cf. Th. 75.

**Theorem 79** ([5]) (a) *For all  $r \geq 1$ , we have  $f_r(\delta) = 1$ ; for all  $r \geq 1$  and  $n \geq 2r + 1$ , we have  $f_{r,n}(\delta) = 1$ .*

(b) *For all  $n \geq 3$ , we have  $F_{1,n}(\delta) = n - 2$ ; for all  $n \geq 5$ , we have  $F_{2,n}(\delta) = \lfloor \frac{n-2}{2} \rfloor$ .*

(c) *For all  $r \geq 3$ , we have  $F_{r,2r+1}(\delta) = 1$ .*

(d) *Let  $r \geq 3$ ,  $n \geq 2r + 2$  and  $k = \lfloor \frac{n-2}{r} \rfloor$ .*

(d<sub>1</sub>) *We have  $k - 1 \leq F_{r,n}(\delta)$  if  $k$  is odd, and  $k \leq F_{r,n}(\delta)$  if  $k$  is even.*

(d<sub>2</sub>) *For  $r \in \{3, 4, 5, \dots, 24\}$ , we have  $F_{r,n}(\delta) \leq \frac{n}{\lfloor \frac{r}{2} \rfloor + 1} - 1$ .*

(d<sub>3</sub>) *For  $r \geq 25$ , we have  $F_{r,n}(\delta) \leq \min \left\{ \frac{n}{\lfloor \frac{r}{2} \rfloor + 1} - 1, \frac{3n-r+2}{2(r-5)} \right\}$ .*

We conjecture that  $F_{r,n}(\delta)$  is close, possibly equal, to  $\lfloor \frac{n-2}{r} \rfloor$ .

• **Size  $\omega$  of a maximum clique**

In any connected graph of order  $n$ , a maximum clique has size between 2 and  $n$ . The clique  $K_n$  is far from being  $r$ -twin-free; actually any graph whose  $r$ -th power is the clique has  $r$ -twins. However, a twin-free graph may contain quite a large clique, and it is remarkable that we can exactly determine its size. Case (b) below is connected to Th. 95. Case (c) depends on some conditions on  $k$  and  $n$  that we do not give here.

**Theorem 80** (Charon, Hudry & Lobstein 2011 [44]) (a) For all  $r \geq 1$ , we have  $f_r(\omega) = 2$ ; for all  $r \geq 1$  and  $n \geq 2r + 1$ , we have  $f_{r,n}(\omega) = 2$ .

(b) For all  $n \geq 3$ , we have  $F_{1,n}(\omega) = k$ , where  $k$  is the largest integer such that  $k + \lceil \log_2 k \rceil \leq n$ .

(c) For all  $r \geq 2$  and  $n \geq 2r + 1$ , let  $k$  be the largest integer such that  $k + r \lceil \log_2 k \rceil \leq n - 1$ .

Then, according to conditions on  $n$  and  $k$ , we have  $F_{r,n}(\omega) = k$  or  $k + 1$ .

If  $n$  is large with respect to  $r$ , then  $F_{r,n}(\omega)$  behaves like  $n - r \log_2 n$ . If  $r$  is a fraction of  $n$ , then  $F_{r,n}(\omega)$  is bounded by above by a constant; the extremal case is when  $r = \frac{n-1}{2}$  and  $F_{r,n}(\omega) = 2$  (path  $P_{2r+1}$ ).

• **Radius  $\rho$**

In any connected graph of order  $n$ , the radius is between 1 (the complete graph, the star) and  $\lfloor \frac{n}{2} \rfloor$  (the path, the cycle). The study of the radius in twin-free graphs is easy, the results are complete.

**Theorem 81** ([5]) (a) For all  $r \geq 1$ , we have  $f_r(\rho) = r$ ; for all  $r \geq 1$  and  $n \geq 2r + 1$ , we have  $f_{r,n}(\rho) = r$ .

(b) For all  $r \geq 1$  and  $n \geq 2r + 1$ , we have  $F_{r,n}(\rho) = \lfloor \frac{n}{2} \rfloor$ .

• **Diameter  $D$**

In any connected graph of order  $n$ , the diameter is between 1 (the complete graph) and  $n - 1$  (the path). The results on the diameter in twin-free graphs are complete. Figure 6.14 illustrates Case (c).

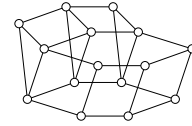
**Theorem 82** ([5]) (a) For all  $r \geq 1$ , we have  $f_r(D) = r + 1$ .

(b) For all  $r \geq 1$ , we have  $f_{r,2r+1}(D) = 2r$ .

(c) For all  $r \geq 1$  and  $n \geq 2r + 2$ , we have  $f_{r,n}(D) = r + 1$ .

(d) For all  $r \geq 1$  and  $n \geq 2r + 1$ , we have  $F_{r,n}(D) = n - 1$ .

**Fig. 6.14** The case  $n = 15$ ,  $r = 3$ : the graph is 3-twin-free and has diameter  $r + 1 = 4$ .



• **Size  $\alpha$  of a maximum independent set**

In any connected graph of order  $n$ ,  $\alpha$  lies between 1 (the complete graph) and  $n - 1$  (the star). Theorem 89 contributes to Case (a) in the following theorem.

**Theorem 83** ([5]) (a) For all  $r \geq 1$ , we have  $f_r(\alpha) = r + 1$ .

(b) For all  $n \geq 3$ , we have  $f_{1,n}(\alpha) = 2$ .

(c) For all  $r \geq 2$ , we have  $f_{r,2r+1}(\alpha) = r + 1$ .

(d) Let  $r \geq 2$ ,  $n \geq 2r + 2$  and  $k = \lfloor \frac{n-2}{r} \rfloor$ . We have  $r + 1 \leq f_{r,n}(\alpha) \leq \begin{cases} \frac{2n}{k+2} : k \text{ even} \\ \frac{2n}{k+1} : k \text{ odd} \end{cases}$ .



The star gives Case (a) in the following theorem.

**Theorem 84** ([5]) (a) For all  $n \geq 3$ , we have  $F_{1,n}(\alpha) = n - 1$ .

(b) For all  $r \geq 2$  and  $n \geq 2r + 1$ , we have  $\max \left\{ \lceil \frac{n}{2} \rceil, k + \lceil \log_2 k \rceil \lfloor \frac{r}{2} \rfloor \right\} \leq F_{r,n}(\alpha) \leq n - r$ , where  $k$  is the largest integer such that  $k + r \lceil \log_2 k \rceil \leq n - 1$ .

If  $n$  is large with respect to  $r$ , then  $k + \lceil \log_2 k \rceil \lfloor \frac{r}{2} \rfloor$  behaves approximately like  $n - \frac{r}{2} \log_2 n$ .

• **Identification number  $Id_r$**

We start with  $F_{r,n}(Id_r)$  and  $f_{r,n}(Id_r)$ , and conclude with  $f_r(Id_r)$ , the most interesting of the three functions. From Sect. 6.2.1, in particular Ths. 8 and 13, we derive the following results.

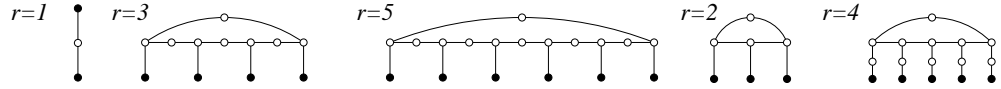
**Theorem 85** (a) For all  $r \geq 1$ , if  $n = 2r + 1$ , or if  $n \geq 3r^2$  is even, or if  $n \geq 3r^2 + 1$  is odd, we have  $F_{r,n}(Id_r) = n - 1$ . In particular, for all  $n \geq 3$ , we have  $F_{1,n}(Id_1) = n - 1$ .

(b) For all  $r \geq 1$  and  $n \geq 2^{2r}$ , we have  $f_{r,n}(Id_r) = \lceil \log_2(n + 1) \rceil$ .

**Theorem 86** ([44]) (a) We have  $f_1(Id_1) \geq 2$  and for all  $r \geq 2$ ,  $f_r(Id_r) \geq \lceil \log_2(2r + 4) \rceil$ .

(b) For all  $r \geq 1$ , we have  $f_r(Id_r) \leq r + 1$ .

The constructive upper bound is illustrated by Fig. 6.15.



**Fig. 6.15** Different graphs with optimal  $r$ -Id codes of size  $r + 1$ . Codewords are in black. For  $r \geq 3$  odd, there is a cycle  $C_{2r+2}$  and  $n = (2r + 2) + r + 1$ . For  $r$  even, there is a cycle  $C_{r+2}$  and  $n = (r + 2) + \frac{r}{2}(r + 1)$ .

The lower and upper bounds in Th. 86 coincide for  $r \in \{1, 2, 3\}$ . For  $r = 4$ , ad hoc arguments show that no graph admits a 4-Id code of size 4. Therefore,

**Theorem 87** ([44]) For  $r \in \{1, 2, 3, 4\}$ , we have  $f_r(Id_r) = r + 1$ .

Open problem: for  $r \geq 5$ , find the exact value of  $f_r(Id_r)$ , knowing that it lies between  $\lceil \log_2(2r + 4) \rceil$  and  $r + 1$ .

## 6.6 Structural Issues

This section is mostly devoted to identifying codes, either because identifying codes lead to the problem of their very existence, i.e., to the important issue of when a graph is twin-free, or simply because in the literature, more attention has been devoted to these topics for identifying codes, compared to locating-dominating codes.

We shall study problems such as the existence of a chordless path in twin-free graphs, whether a twin-free graph deprived of a vertex remains twin-free, the consequence of adding or deleting an edge or a vertex on the identification number, or graphs that are critical, in some sense, with respect to locating-domination or identification.

Locating-dominating and identifying codes do not lend themselves well to Nordhaus-Gaddum-type bounds, and there are very few results. We give two examples.

**Theorem 88** ([106]) *For any graph  $G$  of order  $n \geq 1$ , we have  $|LD_1(G) - LD_1(\overline{G})| \leq 1$ .*

Also, because  $P_4 = \overline{P}_4$  and  $Id_1(P_4) = 3$ , the trivial bounds  $2\lceil \log_2(n+1) \rceil \leq Id_1(G) + Id_1(\overline{G}) \leq 2(n-1)$  and  $\lceil \log_2(n+1) \rceil^2 \leq Id_1(G) \times Id_1(\overline{G}) \leq (n-1)^2$  are tight.

### 6.6.1 Structural Properties of Twin-Free Graphs

The first part of Th. 2 is an obvious consequence of the following result, due to Auger in 2008 [3], which is not obvious.

**Theorem 89** *Let  $r \geq 1$  and  $G$  be a connected  $r$ -twin-free graph. Then  $G$  admits  $P_{2r+1}$ , the path on  $2r+1$  vertices, as an induced subgraph, i.e., there is a chordless path of length  $2r$  in  $G$ .*

#### 6.6.1.1 Deletion of a Vertex in a Twin-Free Graph

Theorems 90–94, due to Charon *et al.* [35] in 2007, give results which vary widely with  $r$ . If  $G = (V, E)$  is  $r$ -twin-free, we say that  $G$  is  $r$ -terminal if for all  $v \in V$ ,  $G - v$  is not  $r$ -twin-free (so  $G$  is not  $r$ -terminal if there exists  $v \in V$  such that  $G - v$  is also  $r$ -twin-free). An alternative would be  $r$ -twin-free critical, see also the discussion before Th. 96.

If  $n = 2r + 1$ , the only connected  $r$ -twin-free graph is the path  $P_{2r+1}$  for  $r \geq 1$ , and the only  $r$ -terminal graph is  $P_{2r+1}$  for  $r > 1$  (the case of  $P_3$  is particular, because removing the middle vertex yields two isolated vertices, constituting a 1-twin-free graph). We address the following questions: (a) are there 1-terminal graphs? (b) For  $r > 1$ , are there  $r$ -terminal graphs other than  $P_{2r+1}$ ?

The answer to (a) is negative, and the answer to (b) is multifold: it is negative if we restrict ourselves to trees; it is positive if  $r \geq 3$ . The case  $r = 2$  remains open.

#### • The case $r = 1$

**Theorem 90** ([35]) *Let  $n \geq 3$  and  $G = (V, E)$  be any connected 1-twin-free graph of order  $n$ . Then  $G$  is not 1-terminal.*

**Proof.** If  $n = 3$ , then  $G = P_3$  is not 1-terminal, as already seen; so we can assume that  $n \geq 4$ . By Th. 6, there is a 1-Id code  $C$  of size  $n - 1$  in  $G$ . Consider  $G - v$ , where  $\{v\} = V \setminus C$  ( $G - v$  may be connected or not); then  $C$  is still 1-identifying in  $G - v$ , because removing  $v$  did not cut connexions of length  $r (= 1)$  between pairs of vertices not containing  $v$  itself—this explains why the cases  $r = 1$  and  $r > 1$  are different. Therefore,  $G - v$  is 1-twin-free.  $\square$

The following theorem sharpens the previous one with respect to connectivity.

**Theorem 91** ([35]) *Let  $n \geq 4$  and  $G = (V, E)$  be any connected 1-twin-free graph of order  $n$ . Then there exists  $v \in V$  such that  $G - v$  is 1-twin-free and connected.*

#### • The case of trees

**Theorem 92** ([35]) *Let  $r \geq 1$ ,  $n \geq 2r + 2$  and  $T = (V, E)$  be any (connected)  $r$ -twin-free tree of order  $n$ . Then there exists a leaf  $v \in V$  such that  $T - v$  is  $r$ -twin-free (and connected). Consequently, the only  $r$ -terminal trees are the paths  $P_{2r+1}$ , for  $r > 1$ .*

• **The case  $r \geq 3$**

We now consider general graphs, for  $r \geq 3$ .

**Theorem 93** ([35]) *For each integer  $r \geq 3$ , there is a graph  $G$ ,  $G \neq P_{2r+1}$ , which is  $r$ -terminal.*

**Sketch of proof.** We search for a connected  $r$ -twin-free graph  $G = (V, E)$ , with  $|V| \geq 2r + 2$ ,  $r \geq 3$ , such that for all  $v \in V$ ,  $G - v$  is not  $r$ -twin-free. One possible construction is the following: take a cycle of length  $2r$  with vertices  $c_i$  ( $i \in \{0, 1, \dots, 2r - 1\}$ ), and add one vertex  $s_i$  together with the edge  $c_i s_i$  for every value of  $i$  but one.  $\square$

• **Other values of  $r$  and open problems**

The construction of Th. 93 does not work for  $r = 2$ . Other constructions have been tried and failed, and the problem remains open: Apart from  $P_5$ , do 2-terminal graphs exist?

**Theorem 94** ([35]) *For each integer  $r \geq 6$ , there are infinitely many  $r$ -terminal graphs.*

Now, another open problem is the situation for  $r \in \{3, 4, 5\}$ : there exist  $r$ -terminal graphs, but are they in finite or infinite number? Conversely, there is a bound on the number of vertices that need to be added in order to obtain a 1-twin-free graph.

**Theorem 95** ([94]) *If  $G$  has order  $n$ , then  $G$  embeds as an induced subgraph in a 1-twin-free graph with order at most  $n + \lceil \log_2 n \rceil$ . The bound is tight.*

### 6.6.1.2 Deletion of an Edge in a Twin-Free Graph

The graphs that we have called terminal in the previous section might as well have been called *vertex-terminal*, and the  $r$ -twin-free graphs  $G = (V, E)$  such that for all  $e \in E$ ,  $G - e$  is not  $r$ -twin-free could then have been called  *$r$ -edge-terminal*. Back to 1974 however, the expression “line-critical point distinguishing graph” was coined in [68] (for  $r = 1$  only) and the following result was proved.

**Theorem 96** ([68]) *A nontrivial connected graph  $G$  is line-critical point distinguishing [or 1-edge-terminal] if and only if  $G = P_3$ .*

*A nonempty graph is line-critical point distinguishing if and only if it is the disjoint union of paths of length two and isolated vertices.*

To our knowledge, everything remains to be done for  $r > 1$ .

### 6.6.2 Adding or Deleting Edges or Vertices Can Save Codewords

The question is to determine how the identification or locating-domination numbers change when we add or delete edges or vertices. We first deal with identifying codes.

#### 6.6.2.1 Identifying Codes: Adding or Deleting an Edge

The problem is the following: given an  $r$ -twin-free graph  $G = (V, E)$  and an edge  $e \in E$ , and assuming that the graph  $G - e$  is  $r$ -twin-free, what can be said about the relationship between

$Id_r(G)$  and  $Id_r(G - e)$ ? The answer widely depends on  $r$ , as show the following two theorems by Charon *et al.* [37] in 2014.

• **The case  $r = 1$**

**Theorem 97** *If  $G$  and  $G - e$  are 1-twin-free, then  $Id(G) - Id(G - e) \in \{-2, -1, 0, 1, 2\}$ . Moreover, pairs of connected graphs  $G$  and  $G - e$  such that  $Id(G) - Id(G - e) = 0$ ,  $Id(G) - Id(G - e) = \pm 1$  or  $Id(G) - Id(G - e) = \pm 2$  exist.*

• **The case  $r \geq 2$**

Now the differences  $Id_r(G) - Id_r(G - e)$  and  $Id_r(G - e) - Id_r(G)$  can be large, and we obtain results which slightly vary with  $r$ .

**Theorem 98** *Let  $k \geq 2$  be an arbitrary integer.*

(a1) *Let  $r \geq 2$ . There exist two  $r$ -twin-free graphs,  $G$  and  $G - e$ , with  $(r+1)k + r\lceil \log_2(k+2) \rceil + 2r$  vertices, such that  $Id_r(G) \geq k$  and  $Id_r(G - e) \leq r\lceil \log_2(k+2) \rceil + r + 3$ .*

(a2) *Let  $r \geq 5$ . There exist two  $r$ -twin-free graphs,  $G$  and  $G - e$ , with  $(2r-2)k + r\lceil \log_2(k+2) \rceil + r + 3$  vertices, such that  $Id_r(G) \geq k$  and  $Id_r(G - e) \leq r\lceil \log_2(k+2) \rceil + r + 1$ .*

(b1) *There exist two 2-twin-free graphs,  $G$  and  $G - e$ , with  $3k + 2\lceil \log_2(k+2) \rceil + 5$  vertices, such that  $Id_2(G - e) \geq k$  and  $Id_2(G) \leq 2\lceil \log_2(k+2) \rceil + 5$ .*

(b2) *Let  $r \geq 3$ . There exist two  $r$ -twin-free graphs,  $G$  and  $G - e$ , with  $(2r-2)k + r\lceil \log_2(k+2) \rceil + r + 2$  vertices, such that  $Id_r(G - e) \geq k$  and  $Id_r(G) \leq r\lceil \log_2(k+2) \rceil + r + 1$ .*

Very close results can be obtained with pairs of connected graphs. Whether these inequalities can be substantially improved is an open problem.

Note that  $k$  can be taken arbitrarily and is linked to the order  $n$  of  $G$  and  $G - e$  by the relation  $n = (c_1r + c_2)k + r\lceil \log_2(k+2) \rceil + (c_3r + c_4)$  where the integer quadruple  $(c_1, c_2, c_3, c_4)$  takes different values in (a1), (a2), (b1) and (b2) above. This means, roughly speaking, that  $k$  is a fraction, depending on  $r$ , of  $n$ ; therefore, Th. 98 implies that, given  $r \geq 2$ , there is an infinite sequence of graphs  $G$  of order  $n$  and two positive constants  $\alpha$  and  $\beta$  such that  $Id_r(G) \geq \alpha n$  and, after deletion of a suitable edge  $e$ ,  $Id_r(G - e) \leq \beta \log_2 n$  (or the other way round:  $Id_r(G) \leq \beta \log_2 n$  and  $Id_r(G - e) \geq \alpha n$ ). We can see that adding or deleting one edge can lead to quite a drastic difference for the identification numbers. On this topic, see also [90] for a probabilistic approach.

### 6.6.2.2 Identifying Codes: Adding or Deleting one Vertex or More

• **The case  $r = 1$**

If  $S \subset V$ , we denote by  $G - S$  the graph obtained from  $G$  by deleting the vertices of  $S$ .

**Theorem 99** ([83]) *If  $G$  and  $G - S$  are 1-twin-free, then  $Id(G) - Id(G - S) \leq |S|$ . In particular, if  $v \in V$ , we have  $Id(G) - Id(G - v) \leq 1$ ; moreover, the two inequalities are tight.*

**Theorem 100** ([151]) *Let  $n$  be the order of  $G$ ,  $S \subset V$  and  $v \in V$ .*

(a) *If  $n \geq 2^{|S|-1}$ , then  $Id(G - S) - Id(G) \leq n - 2|S| - \lfloor \frac{n-|S|}{2^{|S|}} \rfloor$ ; the inequality is tight for  $n$  sufficiently large with respect to  $|S|$ .*

(b) *We have  $Id(G - v) - Id(G) \leq \frac{n}{2} - \varepsilon$ , with  $\varepsilon = 2$  if  $n \in \{2, 4, 5, 6, 8\}$ ,  $\varepsilon = 1$  otherwise; the inequality is tight.*

(c) *If  $G$  is bipartite, then  $Id(G - v) - Id(G) \leq \frac{n - \log_2(n - \log_2 n)}{2} - 1$ ; the inequality is tight.*

So, if one deletes a vertex, the identification number cannot drop by more than one, and can increase by a quantity close to  $n/2$ .

• **The case  $r \geq 2$**

Now both  $Id_r(G-v) - Id_r(G)$  and  $Id_r(G) - Id_r(G-v)$  can be large: the difference  $Id_r(G-v) - Id_r(G)$  can increase to approximately  $\frac{n}{4}$  (for even  $r$ ) and  $\frac{n(3r-1)}{12r}$  (for odd  $r$ ), or  $\frac{n(2r-2)}{2r+1}$ , according to whether we want the graphs to be connected or not, and  $Id_r(G) - Id_r(G-v)$  to approximately  $\frac{n(r-1)}{r}$ .

**Theorem 101** ([36]) *Let  $n$  be the order of  $G$  and  $v \in V$ .*

(a) *Let  $r \geq 2$  be even. There exist two connected graphs,  $G$  and  $G-v$ , such that  $Id_r(G-v) - Id_r(G) \geq \frac{n}{4} - (r+1)$ .*

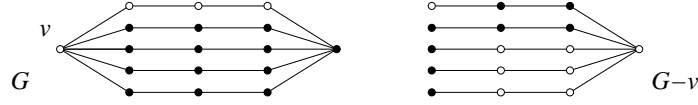
(b) *Let  $r \geq 3$  be odd. There exist two connected graphs,  $G$  and  $G-v$ , such that  $Id_r(G-v) - Id_r(G) \geq \frac{n(3r-1)}{12r} - r$ .*

(c) *Let  $r \geq 2$ . There exist two graphs,  $G$  and  $G-v$ , such that  $Id_r(G-v) - Id_r(G) \geq \frac{(n-1)(2r-2)}{2r+1} - 2r$ .*

**Theorem 102** ([36]) *Let  $n$  be the order of  $G$  and  $v \in V$ . Then there exist two connected graphs,  $G$  and  $G-v$ , such that  $Id_r(G) - Id_r(G-v) \geq \frac{(n-3r-1)(r-1)+1}{r}$ .*

Whether all these inequalities can be substantially improved is an open problem.

Figure 6.16 illustrates Th. 102 for  $n = 17$ ,  $r = 3$ .



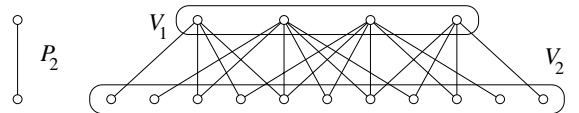
**Fig. 6.16** In each graph, the black vertices form an optimal 3-Id code, and  $Id_3(G) - Id_3(G-v) = 13 - 8 = 5$ .

### 6.6.2.3 Criticality Concepts

In this section,  $r = 1$ . Let  $\pi$  stand for  $LD = LD_1$  or  $Id = Id_1$ . A graph  $G = (V, E)$  is said to be  $\pi^+$ -edge removal critical, or  $\pi^+$ -ER-critical, if  $\pi(G-e) > \pi(G)$  for all  $e \in E$ , and  $\pi^-$ -ER-critical if  $\pi(G-e) < \pi(G)$  for all  $e \in E$ . Similarly, one can define  $\pi^+$ -vertex removal critical graphs and  $\pi^-$ -vertex removal critical graphs.

For instance, it is possible to characterize all graphs that are  $LD^+$ -ER-critical. Let  $H = (V_1 \cup V_2, E)$  be a connected bipartite graph such that: for every  $w \in V_2$  and for every nonempty subset  $V_1' \subseteq N(w)$  there exists a unique  $w' \in V_2$  such that  $N(w') = V_1'$ . Let  $\mathcal{H}$  be the set of all such graphs, see Fig. 6.17 for an example.

**Fig. 6.17** Two graphs belonging to  $\mathcal{H}$  [21].



**Theorem 103** ([21]) *A nontrivial connected graph  $G$  is  $LD^+$ -ER-critical if and only if  $G \in \mathcal{H}$ .*

*A nonempty graph is  $LD^+$ -ER-critical if and only if it is the disjoint union of independent sets and of graphs in  $\mathcal{H}$ .*

For more results, see, e.g., [92] for graphs which are critical for identification, or [62] and [63] for locating-domination.

In Sect. 6.6.1, we mostly showed what a twin-free graph can become when we delete vertices; so far, in Sect. 6.6.2 we studied what the identification and locating-domination numbers can become when adding or deleting vertices or edges; but we can also be interested in what an *existing* (optimal) code becomes when edges or vertices are deleted or added—this leads naturally to the notion of *robustness*: an  $r$ -Id code  $C$  is  $t$ -edge-robust [111] in  $G$  if  $C$  remains  $r$ -Id in all the graphs obtained from  $G$  by adding or deleting edges, with a total amount of additions and deletions at most  $t$ ; see also, among others, [139], [114]. Different definitions exist for  $t$ -vertex-robust codes, see [159], [111].

## 6.7 Number of Optimal Codes

In this section, we are interested in graphs with many optimal codes—or only one.

In order to obtain a large number of optimal 1-Id codes, Honkala, Hudry & Lobstein [110] considered in 2015 the binary 3-cube and used it to build a graph with  $8k + 1$  vertices,  $k \geq 1$ , admitting at least  $56^k + 32k \times 56^{k-1}$  optimal 1-Id codes, of size  $4k$ , then they built graphs of higher orders and numbers of optimal codes, until they obtained the following result.

**Theorem 104** ([110]) *There exist infinitely many connected graphs of order  $n$  admitting approximately  $2^{0.770 \times n}$  different optimal 1-identifying codes.*

For  $r > 1$ , they used trees admitting many optimal codes, and combined them.

**Theorem 105** ([110]) *Let  $r \geq 1$  be an integer and  $\varepsilon > 0$  be a real. There exist infinitely many connected graphs of order  $n$  admitting  $2^{(\frac{1+\log_2 5}{2}-\varepsilon)n}$  different optimal  $r$ -identifying codes.*

Note that  $\frac{1+\log_2 5}{2} \approx 0.664$ . An obvious upper bound is  $\binom{n}{\lceil \frac{n}{2} \rceil}$ , which, using Stirling's formula, can be approximated above by  $2^{n-\frac{1}{2}\log_2 n}$ ; to reduce the gap between these lower and upper bounds remains an open problem.

In [95] there is the description of graphs  $G$  where every  $k$ -set of vertices is a 1-Id code, but the codes are not necessarily optimal, that is,  $k$  may be different from  $Id_1(G)$ . See a small example in Fig. 6.18(a).

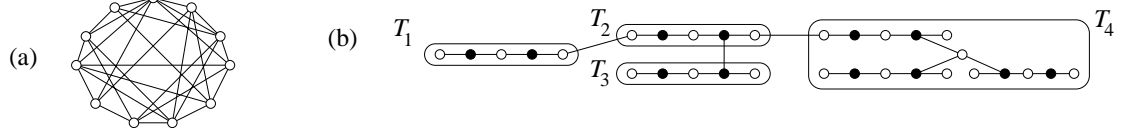
On the other hand, a characterization of trees admitting a unique optimal 1-LD code is given in [19], see examples of such trees in Fig. 6.18(b).

For complexity issues related to the uniqueness of optimal codes, see Sect. 6.8.2.

## 6.8 Complexity

### 6.8.1 How Hard is it to Find Optimal Codes?

For any integer  $r \geq 1$ , we consider the following two decision problems.



**Fig. 6.18** (a) A graph where all sets with 7 vertices are 1-identifying codes (not optimal). (b) Codewords are in black. Each code inside  $T_1, T_2, T_3, T_4, T_1 \cup T_2, T_2 \cup T_3, T_2 \cup T_4, T_1 \cup T_2 \cup T_3, T_1 \cup T_2 \cup T_4, T_2 \cup T_3 \cup T_4, T_1 \cup T_2 \cup T_3 \cup T_4$  is a unique optimal 1-locating-dominating code.

#### **$r$ -LD CODE**

**Instance:** A graph  $G = (V, E)$  and an integer  $k \leq |V|$ .

**Question:** Does  $G$  admit an  $r$ -locating-dominating code of size at most  $k$ ?

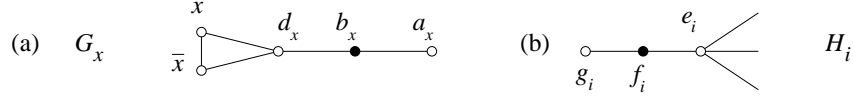
#### **$r$ -Id CODE**

**Instance:** A graph  $G = (V, E)$  and an integer  $k \leq |V|$ .

**Question:** Does  $G$  admit an  $r$ -identifying code of size at most  $k$ ?

**Theorem 106** ([60]) *The problem 1-LD CODE is NP-complete.*

**Proof.** We simplify the 1987 proof by Colbourn, Slater & Stewart in [60]. We take an arbitrary instance of 3-SAT, that is, a set  $X = \{x_1, \dots, x_n\}$  of variables and a set  $\mathcal{C} = \{c_1, \dots, c_m\}$  of clauses of size 3. For each variable  $x$ , we construct the 5-vertex graph  $G_x = (V_x, E_x)$  given in Fig. 6.19(a), in which additional edges may be incident only with  $x$  or  $\bar{x}$ .



**Fig. 6.19** Black vertices will be part of a LD code. (a) The graph  $G_x$ . (b) The graph  $H_i$ .

For each clause  $c_i = \{\ell_{i,1}, \ell_{i,2}, \ell_{i,3}\}$ , we construct the 3-vertex graph  $H_i = (W_i, F_i)$  given by Fig. 6.19(b), where the 3 edges incident with  $e_i$  and not with  $f_i$ , are incident with  $\ell_{i,1}, \ell_{i,2}$ , and  $\ell_{i,3}$ . We claim that the resulting graph  $G$ , whose order  $5n + 3m$  is polynomial with respect to the size of the instance of 3-SAT, admits a LD code of size  $k = 2n + m$  if and only if the set of clauses  $\mathcal{C}$  is satisfiable.

(a) If  $\mathcal{C}$  is satisfiable, take as codewords the  $n$  vertices  $b_x$ , the  $m$  vertices  $f_i$ ,  $1 \leq i \leq m$ , then for each variable  $x \in X$ , whichever of  $x$  and  $\bar{x}$  is true. This code, of size  $k$ , is locating-dominating, in particular because  $e_i$  and  $g_i$  are separated by at least one vertex corresponding to a true literal, i.e., are separated by a codeword.

(b) Let  $C$  be a LD code in  $G$ , of size  $k$ . Obviously,  $|C \cap V_x| \geq 2$  and  $|C \cap W_i| \geq 1$ ; therefore, thanks to the choice of  $k$ , we have  $|C \cap V_x| = 2$  and  $|C \cap W_i| = 1$ . Also, because  $g_i$  must be dominated by  $C$ , we have  $|C \cap \{g_i, f_i\}| = 1$  and none of the vertices  $e_i$  is in  $C$ . This implies that  $|C \cap \{x, \bar{x}\}| = 1$ , because if  $C \cap \{x, \bar{x}\} = \emptyset$ , then the non-codewords  $x$  and  $\bar{x}$  are not separated by  $C$  (the vertex  $d_x$  cannot do it), and if  $|C \cap \{x, \bar{x}\}| = 2$ , then  $b_x$  and  $a_x$  are not dominated by  $C$ . So we can define a valid truth assignment of the variables by setting  $x$  true if and only if  $x \in C$ .

(i) If  $C \cap \{g_i, f_i\} = \{g_i\}$ , then  $e_i$  must be dominated by a codeword of type  $x$  or  $\bar{x}$ ; (ii) the same is true if  $C \cap \{g_i, f_i\} = \{f_i\}$ , because  $g_i$  and  $e_i$  must be separated by the code. This shows that every clause  $c_i$  is satisfied by this assignment.  $\square$

The same result for  $r \geq 2$  was stated in [26] in 1995. We indicate the 2003 reference [40], which gives a proof.

**Theorem 107** (a) ([26]), ([40]) *For any  $r > 1$ , the problem  $r$ -LD CODE is NP-complete.*

(b) ([57] for  $r = 1$ ), ([40]) *For any  $r \geq 1$ , the problem  $r$ -Id CODE is NP-complete.*

This means that, given  $G$  and  $r$ , determining  $LD_r(G)$  or  $Id_r(G)$ , or finding an optimal  $r$ -LD or  $r$ -Id code, is NP-hard.

Note that Th. 107 holds even if the instances are restricted to bipartite graphs; it also exists for bipartite oriented graphs [39]. For more on complexity, see [4], [6] for identification in planar graphs with arbitrarily high girth or low maximal degree, or [87] for interval and permutation graphs; Table 1 in [80] extends Table 1.8 in [79] and summarizes what was known around 2015 about the complexity of certain classes of graphs, for LD and Id codes, as well as dominating codes, for comparison.

See [118], [119] for identification in the binary  $n$ -cube, and [121] for problems closely related, such as the search for an  $r$ -Id code containing a prescribed vertex subset. For concepts close to locating-domination and identification that will be surveyed in Sect. 6.9: complexity results on  $(r, \leq \ell)$ -identifying codes are given in [6], on watching systems in [7], on open neighborhood locating-dominating codes in [164], on metric bases in [93, p. 204]; usually, the decision problems are NP-complete, even for some restricted classes of graphs, but linear for trees (when  $r = 1$ ). The problem 1-LD CODE is also linear for series parallel graphs [60], and so is 1-Id CODE for block graphs [1].

For approximability issues, see, e.g., [80] (where Table 2 gives a state of play) or [87].

Complexity can bring what may look like surprises [79]: there are classes of graphs in which 1-LD CODE is polynomial and 1-Id CODE is NP-complete, and, if we define the decision problem 1-DOM CODE similarly to 1-LD CODE or 1-Id CODE, there are even classes of graphs in which 1-LD CODE and 1-Id CODE are polynomial, whereas 1-DOM CODE is NP-complete.

### 6.8.2 Uniqueness of Optimal Codes

We consider the following decision problems, stated for fixed  $r \geq 1$ .

<b>UNIQUE OPTIMAL <math>r</math>-LD CODE</b> <b>Instance:</b> A graph $G = (V, E)$ . <b>Question:</b> Does $G$ admit a <i>unique optimal</i> $r$ -locating-dominating code?	<b>UNIQUE OPTIMAL <math>r</math>-Id CODE</b> <b>Instance:</b> A graph $G = (V, E)$ . <b>Question:</b> Does $G$ admit a <i>unique optimal</i> $r$ -identifying code?
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In [123], it is proved, among other results, that for every  $r \geq 1$ , these two problems are NP-hard and belong to the class  $L^{NP}$  (also denoted by  $P^{NP[O(\log n)]}$  or  $\Theta_2$ ), which contains the decision problems which can be solved by applying, with a number of calls which is logarithmic with respect to the size of the instance, a subprogram able to solve an appropriate problem in NP.



## 6.9 Related Concepts, Generalizations

- One of the first possible generalizations is to consider that *more than one vertex* may need to be retrieved by the code, and usually one puts a limit  $\ell$  on the number of these vertices. Therefore, an  $(r, \leq \ell)$ -*locating-dominating code*  $C$  in a graph  $G = (V, E)$  is an  $r$ -dominating code such that for all  $X \subseteq V \setminus C$ ,  $Y \subseteq V \setminus C$ , such that  $X \neq Y$ ,  $|X| \leq \ell$ ,  $|Y| \leq \ell$ , we have

$$\bigcup_{u \in X} N_r(u) \cap C \neq \bigcup_{v \in Y} N_r(v) \cap C.$$

One can similarly define  $(r, \leq \ell)$ -*identifying codes*. Both generalizations, and other similar ones, have been studied mostly in the  $n$ -cube and in the grids; we refer to, e.g., [134], [137], [112], [116], [97], [73], [70], [150], [86].

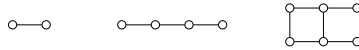
- Next, considering that using  $r$ -LD or  $r$ -Id codes amounts to using  $r$ -neighborhoods centered at the codewords, one can think of *changing the pattern* surrounding a codeword.

One choice of pattern is, given  $G = (V, E)$  and  $r \geq 1$ , to consider that every codeword  $c \in C$  can check a connected *subset* of  $N_r[c]$ , instead of the whole set  $N_r[c]$  for an  $r$ -identifying code, and that several codewords can be on the same vertex. This leads to the notion of *watching systems* [7], which are more complex but can be very efficient: consider for instance the star  $K_{1,n-1}$ , which requires as many as  $n-1$  codewords for 1-identification and even for 1-locating-domination, and as few as  $\lceil \log_2(n+1) \rceil$  if one puts suitable watchers at the centre of the star. See also [9], [8]. When a watching system has at most one codeword on each vertex, we talk of *choice identification* [27].

Other patterns for different graphs can be thought of: cycles, paths (including paths of even order in the infinite path, as opposed to odd paths which are closed neighborhoods around their centres), squares of even side in  $\mathbb{Z} \times \mathbb{Z}$ , as opposed to odd squares, which are closed neighborhoods in the king grid, etc. See, e.g., [120], [162], [85].

- A slight modification in the definition of a 1-identifying code leads to *open neighborhood locating-dominating codes* (OLD codes): an OLD code  $C$  is such that for every two vertices  $u, v$  of  $V(G)$ , the sets  $N(u) \cap C$  and  $N(v) \cap C$  are nonempty and different. This was introduced in [115] (for  $r \geq 1$  and for identification of sets of vertices) in the binary  $n$ -cube, then extended to all graphs in [164] (for  $r = 1$ ). This describes a situation where an intruder at a vertex  $v$  can prevent the detection device at  $v$  from signalling the intrusion, that is,  $v$  can only check  $N(v)$ . In most of our models, we assumed that a codeword correctly sent a 1 if it detected something in its neighborhood, a 0 otherwise. The definition of OLD codes is one of several definitions for so-called *fault-tolerant codes*, where different scenarii are considered for the alarms given by the codewords.

If  $OLD(G)$  denotes the minimum size of an OLD code in  $G$ , the only graphs with  $OLD(G) = |V(G)|$  are the three graphs given by Fig. 6.20 and the only graphs with  $OLD(G) = 2$  are the complete graphs  $K_2$  and  $K_3$  [51]. Another representative result is that the minimum density of an OLD code in the triangular grid is  $4/13 \approx 0.3077$  [136], to be compared with 0.25 for a 1-identifying code, see Th. 32. For more on this topic, see also, e.g., [165], [176], [163].



**Fig. 6.20** The three graphs  $G = (V, E)$  with  $OLD(G) = |V|$ .

One can also mention here the *liar's problem*, when the vertices of  $C$  can tell precisely where the malfunctioning vertex  $v$  is if it belongs to their closed neighborhoods, so a dominating code would be enough, except that any one codeword in the closed neighborhood of  $v$  can lie, that is, either misidentify any vertex in its closed neighborhood as  $v$ , or fail to report any vertex. For instance, three codewords are necessary and sufficient in the complete graph  $K_n$  with  $n \geq 3$ . See [173] or [161].

- Another related definition is the following: a code  $C$  is *strongly  $r$ -identifying* [115] if for all  $v_1 \in V$ ,  $v_2 \in V$ ,  $v_1 \neq v_2$ , the sets  $\{N_r[v_1] \cap C, N_r(v_1) \cap C\}$  and  $\{N_r[v_2] \cap C, N_r(v_2) \cap C\}$  are disjoint; this can be extended to codes identifying more than one vertex. See [109], where the best density for a strongly 1-Id code in the triangular grid is proved to be  $6/19 \approx 0.3158$ , or [138], [141].

- Using identifying codes in search of one defective vertex can be seen as asking to the codewords, all at one time: “Is there a defective vertex in your closed neighborhood?” *Adaptive identification* consists in asking the queries one by one, taking into account the previous answers. So, after the first yes answer is obtained in one neighborhood, we dichotomize this neighborhood with other neighborhoods (= new queries) so as to minimize the maximum number of queries [12], [127], [135].

- The notions of 1-LD and 1-Id codes can be modified by adding the condition that the code, instead of being dominating, must be total dominating: a total dominating code  $C$  of a graph  $G = (V, E)$  is a *locating-total dominating code* if for every pair of distinct vertices  $u$  and  $v$  in  $V \setminus C$ , one has  $N(u) \cap C \neq N(v) \cap C$ , and  $C$  is a *differentiating-total dominating code*, or an *identifying-total dominating code*, if for every pair of distinct vertices  $u$  and  $v$  in  $V$ ,  $N[u] \cap C \neq N[v] \cap C$ . See, e.g., [103], [49], [53], [22], [105], [146], [129].

- An older way of locating vertices is the following: given a graph  $G = (V, E)$  and a code  $C = \{c_1, c_2, \dots, c_k\}$ , the  $C$ -location of a vertex  $v \in V$  is the distance vector  $(d(v, c_1), d(v, c_2), \dots, d(v, c_k))$ . If no two vertices have the same  $C$ -location, then  $C$  is said to be *resolving*, as introduced in [101], or *locating*, as introduced independently in [167], around 1975. An optimal resolving code is called a *metric basis* and its size is the *metric dimension* of the graph. For example, the only graphs with metric dimension 1 are the paths, for which each endvertex constitutes a resolving code. See also [24], [87], or [100]; see [104] for *metric-locating-domination* (where the code must also be 1-dominating), and [48] for the related concepts of *distance-location* and *external distance-location*.

Moreover, if we have a proper vertex coloring  $c$  with  $k$  colors, and if  $\Pi = (C_1, C_2, \dots, C_k)$  is the resulting partition of  $V$ , we can define, for every  $v \in V$ , the *color code* of  $v$ , that is, the distance vector  $(d(v, C_1), d(v, C_2), \dots, d(v, C_k))$ . If distinct vertices have distinct color codes, then  $c$  is called a *locating-coloring*. The *locating-chromatic number* of  $G$  is the minimum number of colors in a locating-coloring. See, e.g., [46], [47], [11]. Other more or less distant concepts using colorings exist, such as *locally identifying colorings* [69], and many more, see, e.g., [181], [148], [180].

- For the model of *information retrieval* in associative memories, see, e.g., [179], [132], [140].
- The  *$d$ -identifying codes* [177] generalize Id codes by endowing every vertex with a positive integer cost, which will give a global cost to the code that may be different from its size, and with a positive integer weight which, together with  $d$ , allows and measures a degree of uncertainty in the identification.

- Another parameter, the *propagation time*, can be considered; in this model, a fault at one vertex  $v$  in a weighted digraph spreads along the arcs and reaches any out-neighbor  $w$  in a time equal to the weight of the arc  $vw$ , then spreads from  $w$ , and so on. See, e.g., [158], [14].

- Other definitions: *paired-LD* or Id codes (the code must be paired-dominating), *co-isolated LD* or Id codes (the code  $C$  is such that there exists at least one isolated vertex in  $\langle V - C \rangle$ ), *independent LD* or Id codes, *connected codes*, *weak  $r$ -codes* and *light  $r$ -codes*, *self-LD* or Id codes, *solid-LD* codes.

It is also possible to generalize these concepts to *hypergraphs*, and of course, some of these definitions can be combined, e.g., connected vertex-robust identifying codes [78].

## 6.10 Concluding Remarks, Open Problems, Conjectures

We have tried to give an overview of the major results on locating-domination and on identification. We apologize in advance if your favorite variation or result is omitted. We simply refer people interested to know more about these concepts to [142] for more literature on the topic; they can search for key-words there, or on the web for even more references.

We conclude with two seemingly paradoxical remarks partly collected from [160]:

(a) For some graphs, *increasing*  $r$ , i.e., increasing the power of the codewords, can require *more* codewords; this was first noticed in [26], where a tree  $T$  is given, such that  $LD_2(T) = LD_3(T) = 6$  and  $LD_4(T) = 5$ , while  $LD_7(T) > 6$ . Paths provide similar cases for identification; for instance, if  $k \geq 1$ , then by Th. 44 we have  $Id_1(P_{10k+5}) = 5k + 3$ , whereas by Th. 49(a) we have  $Id_2(P_{10k+5}) = 5k + 4$ . In the case of the binary  $n$ -cube, we have  $Id_1(\mathbb{F}^6) = 19$ ,  $Id_2(\mathbb{F}^6) = 8$ ,  $Id_3(\mathbb{F}^6) = 7$ ,  $Id_4(\mathbb{F}^6) = 18$ ,  $Id_5(\mathbb{F}^6) = 63$ .

This paradox can be understood when  $r$  becomes relatively large with respect to the graph: if the  $r$ -neighborhoods grow too much, they cannot separate anything anymore; the extremal case is when  $r$  reaches the diameter of the graph, where  $n - 1$  codewords are needed for an  $r$ -LD code, and no  $r$ -Id code exists. But this is not the case for the example of the path  $P_{10k+5}$  above.

(b) If we take a path or a cycle and make it *longer*, it may need *fewer* codewords; for instance, for  $k \geq 1$ , one has  $Id_2(C_{10k+6}) = Id_2(C_{10k+5}) - 2$ .

The discussion following Th. 62 gives another example, which however can be explained by the fact that increasing the dimension  $n$  of the cube makes  $r$  not too large with respect to  $n$  anymore.

Finally, we recapitulate the open problems and conjectures given throughout the text.

**Open Problem 1** [2020] (see End of Section 6.3). For  $r \geq 2$ , most results on the costs of locating-domination vs domination are exact but valid for  $n$  large. (a) Establish results for all  $n$ . (b) Reduce the gap between lower and upper bounds for  $F_{LD,\gamma}(r, n)$ .

**Open Problem 2** [2011] (see Th. 87). For  $r \geq 5$ , find the exact value of  $f_r(Id_r)$ , knowing that it lies between  $\lceil \log_2(2r + 4) \rceil$  and  $r + 1$ .

**Open Problem 3** [2007] (see End of Section 6.6.1.1). (a) Apart from  $P_5$ , do 2-terminal graphs exist? (b) There exist  $r$ -terminal graphs for  $r \in \{3, 4, 5\}$ , but are they in finite or infinite number?

**Open Problem 4** [2013]. Improve the bounds in Ths. 98, 101 and 102.

**Open Problem 5** [2015] (see Ths. 104 and 105). Reduce the gap between the lower and upper bounds on the number of different optimal identifying codes.

**Conjecture 1** [1995] (see Th. 54). For any tree  $T$  of order  $n$ , we have  $LD_r(T) > \frac{n}{r+2}$ .

**Conjecture 2** [1999] (see the discussion following Th. 60). (a) We have  $Id_{r_1+r_2}(\mathbb{F}^{n_1+n_2}) \leq Id_{r_1}(\mathbb{F}^{n_1}) \times Id_{r_2}(\mathbb{F}^{n_2})$ . (b) We have  $Id(\mathbb{F}^{n+1}) \leq 2Id(\mathbb{F}^n)$ .

**Conjecture 3** [2010] (see the discussion after Th. 62). For a fixed  $r > 1$ , there exists  $n(r)$  such that for all  $n \geq n(r)$ , we have  $Id_r(\mathbb{F}^n) \leq Id_r(\mathbb{F}^{n+1})$ .

**Conjecture 4** [2012] (see the discussion following Th. 71). There exists a constant  $c$  such that, for every nontrivial connected 1-twin-free graph  $G$ , with maximum degree  $\Delta$ , we have  $Id_1(G) \leq n - \frac{n}{\Delta} + c$ .

**Conjecture 5** [2009] (see Th. 79). The value  $F_{r,n}(\delta)$  is close, possibly equal, to  $\lfloor \frac{n-2}{r} \rfloor$ .

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