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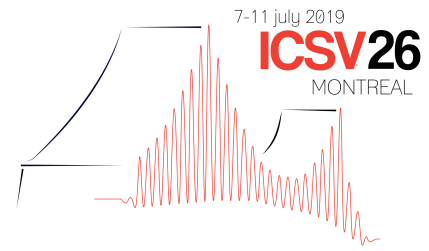
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INFLUENCES OF LATERAL WHEEL LOADS ON THE SLEEPER RESPONSES OF A RAILWAY TRACK

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Recently, the instrumented sleeper has been developed to measure the sleeper response in-situ. An analytical model of the dynamics of railway sleepers has been developed to calculate rapidly the sleeper response under vertical wheel loads. However, lateral wheel loads are not taken into account in this model but would influence the sleeper response. This paper presents a new analytical model of the dynamics of railway sleepers under lateral wheel loads. In this model, the sleeper is considered as a Euler-Bernoulli's beam resting on a visco-elastic foundation. Thereafter, by combining the relation between the reaction forces and displacements of the periodically supported beam, the sleeper response can be calculated with help of the Green's function. Finally, the numerical application shows the influence of the laterals wheel load on the sleeper response.

Keywords : railway track, analytical model, lateral loads, periodically supported beam, Green's function

1. Introduction

The railway track can be constructed by different technologies for each railway track component. The concrete monoblock sleepers remain popular for the ballasted railway. The study of the dynamic response of the sleeper is important because it affects the stability and confort of the railway track. A lot of researches have been carried out in this field, for example : the model of a railway track as an infinite beam placed on a continous foundation [1, 2] or a periodically supported beam [3–5].

The dynamic of the sleeper has been investigated with several different methods. The main aim is to analyze the sleeper behaviour and to model it in the case of different moving charges. Grassie [6] shows that a uniform beam can be used to model a non-uniform section sleeper. By using experimental and numerical methods, Laryea et al. [7] compared the performance of sleepers made out of different materials. Some works focus on the prestressed concrete sleeper by using linear FEM [8, 9] and non-linear FEM [10]. Recently, the dynamic response of a railway sleeper under vertical loads has been developed analytically by Tran et al. [11] in 2018. However, this model does not take into account the lateral loads. In a curved or in a crossing track, the wheel-rail contact generates an important lateral load

applied on the two rails. There are many researches on the lateral wheel loads because the ratio between the lateral (Y) and vertical (Q) loads which is called also Nadal's limit (Y/Q) is used extensively to monitor and predic derailments. The researchers show different methods to measurement the vertical load via rail web bending strains with the help of FEM [12–14]. Li et al. [15] showed the simulation of vertical dynamic vehicle-track interactions on a railway crossing in 2017. In 2019, Muñoz et al. [16] show a numerical simulation of a railway wagon with weakly coupled vertical and lateral dynamic loads.

In this paper, a sleeper response under lateral wheel loads is studied by considering a beam resting on a visco-elastic foundation. The dynamic equation of the beam is written in the frequency domain by using the Fourier transform. By modelling the rail as a periodically supported beam [17], the lateral train loads applied on the rail can be written with the help of the Dirac delta function. In the frequency domain, the dynamic equation for the sleeper together with the foundation is written and then solved analytically by using the Green's function.

The numerical application shows the influence of the lateral wheel-loads on the total response of a sleeper in the case of different charges on the two rails. This simple model shows an efficient and fast way to estimate the response of railway sleepers and can be developed for the analysis of more complex cases.

2. Governing equations

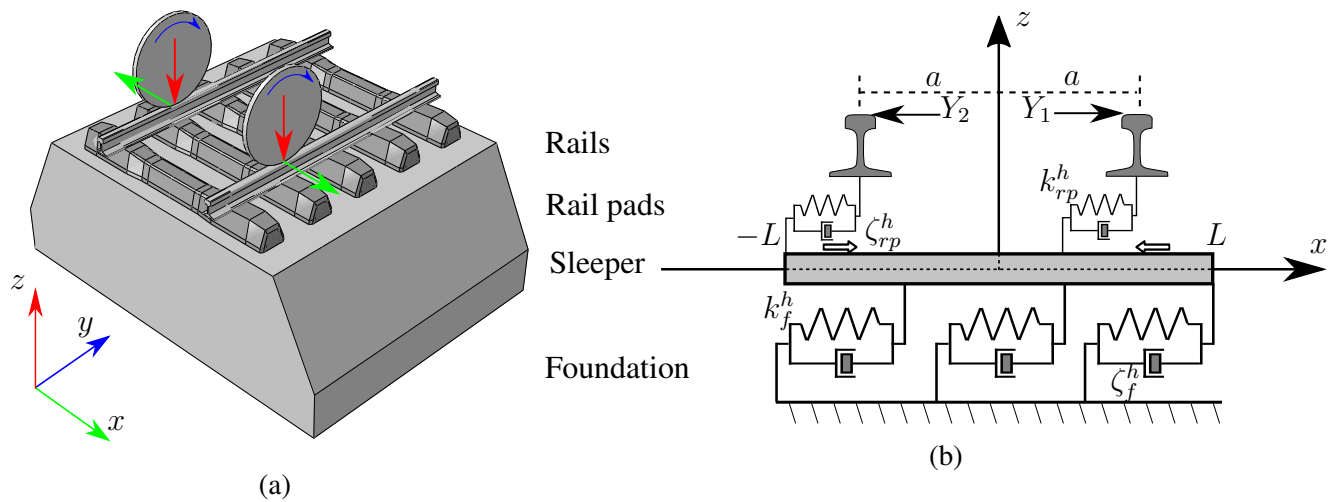


FIGURE 1 – Railway track (a) and the analytical model representation (b)

Let us consider the railway track as shown in Fig. 1. In this track, the sleeper and together a ballast foundation can be modeled with the help of an Euler-Bernoulli's beam resting on a Kelvin-Voigt's foundation. In this figure, we note that the sleeper length is $2L$ (from $-L$ to L) and the rail positions are $x = \pm a$. The loads Y_1 and Y_2 apply on the sleeper by the longitudinal direction. The dynamic equation of the sleeper with the boundary conditions of a free-free beam can be written by :

$$\begin{cases} \rho_s S_s \frac{\partial^2 w_s^h(x, t)}{\partial t^2} - E_s S_s \frac{\partial^2 w_s^h(x, t)}{\partial x^2} + k_b^h w_s^h(x, t) + \xi_b^h \frac{\partial w_s^h(x, t)}{\partial t} = F^h(x, t) \\ \frac{\partial w_s^h(L, t)}{\partial x} = 0 \\ \frac{\partial w_s^h(-L, t)}{\partial x} = 0 \end{cases} \quad (1)$$

where $w_s^h(x, t)$, ρ_s , E_s , S_s are the horizontal displacement, density, Young's modulus and section of the sleeper respectively. The notation k_b^h and ξ_b^h denote the stiffness and damping in the horizontal direction of the ballast layer respectively. Under the action of the lateral wheel loads, the sleeper is subjected to two horizontally reaction forces $R_1^h(t)$ and $R_2^h(t)$ from the two rails via a rail-pad which is modeled by the spring-damper system are shown in the Fig. 1 and the total horizontal force $F^h(x, t)$ can be written with the help of the Dirac's function as follows :

$$F^h(x, t) = R_1^h(t)\delta(x - a) + R_2^h(t)\delta(x + a) \quad (2)$$

Eqs. (1) and together Eq. (2) describe a dynamic model of a beam in a simple traction. By using the Fourier's transform, by combining the Eqs. (1), the dynamic equation of the sleeper can be rewritten in the frequency domain as follows :

$$\begin{cases} \frac{\partial^2 \hat{w}_s^h(x, \omega)}{\partial x^2} - \left(\frac{k_f^h - \omega^2 \rho_s S_s}{E_s S_s} \right) \hat{w}_s^h(x, \omega) = \frac{-\hat{R}_1^h(\omega)}{E_s S_s} \delta(x - a) + \frac{-\hat{R}_2^h(\omega)}{E_s S_s} \delta(x + a) \\ \frac{\partial \hat{w}_s^h(L, \omega)}{\partial x} = 0 \\ \frac{\partial \hat{w}_s^h(-L, \omega)}{\partial x} = 0 \end{cases} \quad (3)$$

where : ω is the angular velocity; $k_f^h = k_b^h + i\omega\xi_b^h$ is the dynamic stiffness of the foundation; $\hat{w}_s^h(x, \omega)$, \hat{R}_1^h and \hat{R}_2^h are the horizontal sleeper displacements and two reaction forces applied on the sleeper at two rails in the frequency domain respectively. Eq. (3) describes the dynamics of the sleeper on the ballast in the horizontal direction and the solution of this equation is calculated with the help of the Green's function which is given by :

$$\frac{\partial^2 G_a^h(x, \omega)}{\partial x^2} - \gamma_s^2 G_a^h(x, \omega) = \delta(x - a) \quad (4)$$

where $\gamma_s^2 = \frac{k_f^h - \omega^2 \rho_s S_s}{E_s S_s}$. This is a 2nd order linear differential equation and the solution $G_a^h(x, \omega)$ is shown in the Appendix A. Thus, with the help of the Green's function, the horizontal displacement of the sleeper can be calculated as follows :

$$\hat{w}_s^h(x, \omega) = \frac{-\hat{R}_1^h(\omega)}{E_s S_s} G_a^h(x, \omega) + \frac{-\hat{R}_2^h(\omega)}{E_s S_s} G_{-a}^h(x, \omega) \quad (5)$$

The horizontal displacement of the sleeper at the two rail positions ($x = \pm a$) can be explained in this expression :

$$\begin{cases} \hat{w}_s^h(a, \omega) = \frac{-\hat{R}_1^h}{E_s S_s} G_a^h(a, \omega) + \frac{-\hat{R}_2^h}{E_s S_s} G_{-a}^h(a, \omega) \\ \hat{w}_s^h(-a, \omega) = \frac{-\hat{R}_1^h}{E_s S_s} G_a^h(-a, \omega) + \frac{-\hat{R}_2^h}{E_s S_s} G_{-a}^h(-a, \omega) \end{cases} \quad (6)$$

In an other way, when the rails are modeled by periodically supported beams, the reaction forces R_1^h and R_2^h at the two rail positions of the sleeper in the frequency domain can be calculated as follows (see Appendix B) :

$$\begin{cases} \hat{R}_1^h(\omega) = \mathcal{K}^h(\omega) \hat{w}_1^h(\omega) + \mathcal{P}_1^h(\omega) \\ \hat{R}_2^h(\omega) = \mathcal{K}^h(\omega) \hat{w}_2^h(\omega) + \mathcal{P}_2^h(\omega) \end{cases} \quad (7)$$

where $\mathcal{K}^h(\omega)$ and $\mathcal{P}_i^h(\omega)$ are the equivalent stiffness and equivalent charges for two rails calculated by Eq. (18); $\hat{w}_1^h(\omega)$ and $\hat{w}_2^h(\omega)$ are the displacements of rail 1 and rail 2 at the sleeper position respectively. Moreover, the two forces R_1^h and R_2^h can be expressed by the constitutive law of the rail pads in the frequency domain as follows :

$$\begin{cases} \hat{R}_1^h(\omega) &= -k_p^h (\hat{w}_1^h(\omega) - \hat{w}_s^h(a, \omega)) \\ \hat{R}_2^h(\omega) &= -k_p^h (\hat{w}_2^h(\omega) - \hat{w}_s^h(-a, \omega)) \end{cases} \quad (8)$$

where $k_p^h = k_{rp}^h + i\omega\xi_{rp}^h$ is the horizontal dynamic stiffness of the rail pad and k_{rp}^h, ξ_{rp}^h are the horizontal stiffness and horizontal damping coefficients of the rail pad. By substituting Eq. (7) into Eq. (8), the reaction forces \hat{R}_1^h and \hat{R}_2^h can be expressed by :

$$\begin{cases} \hat{R}_1^h = \frac{k_p^h \mathcal{K}^h}{k_p^h + \mathcal{K}^h} \hat{w}_s^h(a, \omega) + \frac{k_p^h}{k_p^h + \mathcal{K}^h} \mathcal{P}_1^h \\ \hat{R}_2^h = \frac{k_p^h \mathcal{K}^h}{k_p^h + \mathcal{K}^h} \hat{w}_s^h(-a, \omega) + \frac{k_p^h}{k_p^h + \mathcal{K}^h} \mathcal{P}_2^h \end{cases} \quad (9)$$

By combining the two Eqs. (6) and (9), the reaction forces of the sleeper on two rails can be deduced as :

$$\begin{cases} \hat{R}_1^h = \frac{E_s S_s [\chi^h + G_{-a}^h(-a, \omega)] \mathcal{P}_1^h - G_{-a}^h(a, \omega) \mathcal{P}_2^h}{\mathcal{K}^h \tilde{D}^h} \\ \hat{R}_2^h = \frac{E_s S_s [\chi^h + G_a^h(a, \omega)] \mathcal{P}_2^h - G_a^h(-a, \omega) \mathcal{P}_1^h}{\mathcal{K}^h \tilde{D}^h} \end{cases} \quad (10)$$

where $\tilde{D}^h = [\chi^h + G_a^h(a, \omega)] [\chi^h + G_{-a}^h(-a, \omega)] - G_a^h(-a, \omega) G_{-a}^h(a, \omega)$ and $\chi^h = E_s S_s \left(\frac{k_p^h + \mathcal{K}^h}{k_p^h \mathcal{K}^h} \right)$.

Eq. (10) defines the reaction force of the sleeper on two rails. By substituting this equation into the Eq. (5), we obtain the horizontal displacement of the sleeper due to the lateral wheel loads. Thus, the sleeper strain due to the lateral wheel loads in the frequency domain can be calculated analytically as follows :

$$\hat{\varepsilon}_{xx}^h(x, \omega) = \frac{\partial \hat{w}_s^h(x, \omega)}{\partial x} = \frac{-\hat{R}_1^h(\omega) \partial G_a^h(x, \omega)}{E_s S_s \partial x} + \frac{-\hat{R}_2^h(\omega) \partial G_{-a}^h(x, \omega)}{E_s S_s \partial x} \quad (11)$$

3. Numerical application

3.1 The sleeper response under a lateral static load

Consider a railway track with parameters given in Table. 1. The sleeper response due to the lateral force is calculated in the case of a curved track where the wheel loads applied on each rail are modeled by $Y_1 = 8$ kN, $Y_2 = 5$ kN. In this case, Fig. 2a shows that the sleeper is in traction where the sleeper strain is positive. The sleeper strain is maximum at the rail 1 ($1.73 \mu\text{m/m}$) where the applied force is bigger and the sleeper is in traction. At the rail 2, the sleeper is in compression where the sleeper strain is $-0.95 \mu\text{m/m}$. Fig. 2b shows two reaction forces at the rail seat and $R_1^h = 6.52$ kN and $R_2^h = 4.59$ kN on the rail 1 and rail 2 respectively.

TABLE 1 – Parameters of the railway track

Content	Unit	Notation	Value
Young's modulus of rail	GPa	E_r	210
Cross-sectional moment inertia of rail (transversal direction)	m^4	I_r^h	$5E^{-6}$
Rail density	kgm^{-3}	ρ_r	7850
Rail section area	m^2	S_r	$7.69E^{-3}$
Young's modulus of sleeper	GPa	E_s	48
Density of sleeper	kgm^{-3}	ρ_s	2475
Sleeping section area	m^2	S_s	$54.9E^{-3}$
Length of sleeper	m	$2L$	2.41
Track gauge	m	$2a$	1.435
Stiffness of ballast (vertical/horizontal)	MNm^{-1}	$k_f^{v/h}$	240 / 160
Damping coefficient of ballast (vertical/horizontal)	$kNsm^{-1}$	$\zeta_f^{v/h}$	58.8 / 90
Stiffness of rail pad (vertical/horizontal)	MNm^{-1}	$k_{rp}^{v/h}$	175 / 62
Damping coefficient of rail pad (vertical/horizontal)	$kNsm^{-1}$	$\zeta_{rp}^{v/h}$	10 / 30
Train speed	ms^{-1}	v	42.5
Sleeper spacing	m	l	0.6

3.2 Influence of lateral wheel loads on a sleeper response

Thereafter, by varying the Nadal's limit (Y/Q), the influence of the loads is studied for two equal vertical loads on each rail ($Q_1 = Q_2 = Q$). The sleeper strains under a vertical load are calculated with the help of an analytic dynamic model of the railways sleepers [11]. The track parameters are the same as in Table. 1. The influence of the sleeper strain is calculated at the middle and the two rail seats of the sleeper and shown in the Table. 2.

TABLE 2 – Influence of lateral loads on a sleeper response

Train loads			Nadal's ratio		Influence on a sleeper strain (%)		
Q (kN)	Y_1 (kN)	Y_2 (kN)	Y_1/Q	Y_2/Q	Rail seat 1	Middle	Rail seat 2
75	7	5	0.093	0.067	1.89	2.15	0.46
100	7	5	0.07	0.05	1.41	1.62	0.35
125	7	5	0.056	0.04	1.12	1.30	0.28
100	7	-5	0.07	0.05	3.39	8.99	3.09
100	5	-5	0.05	0.05	2.68	7.61	2.68
100	3	-5	0.03	0.05	1.98	6.18	2.29

When the Nadal's ratio increases, the lateral wheel loads are more important. When the rails are subjected to two lateral forces by different directions, the sleeper strain at the middle of the sleeper has less influence.

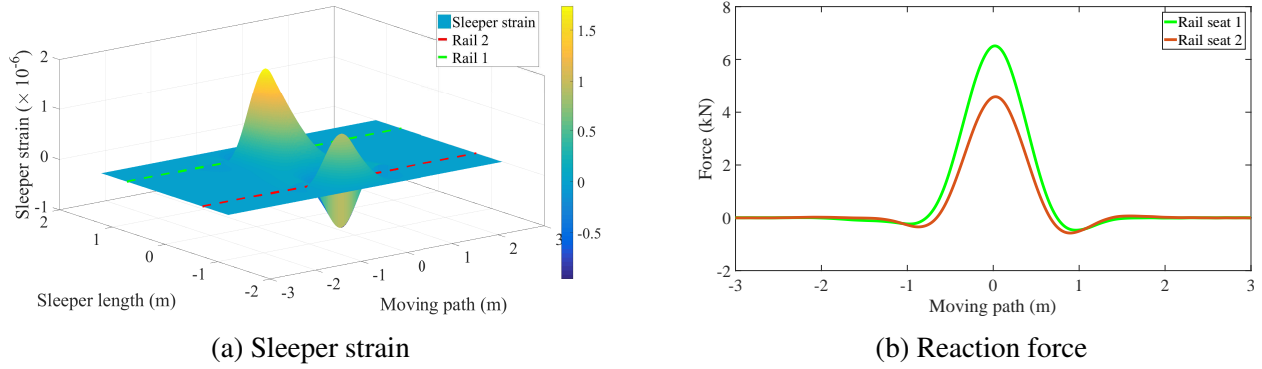


FIGURE 2 – Sleeper response under the lateral wheel loads

4. Conclusions

In this study, the analytical model for the dynamics of railway sleepers under the lateral wheel loads has been developed by considering a model of a beam resting on a Kelvin-Voigt foundation. By using the relation between the reaction force and displacement of the rail from the periodically supported beam, the sleeper response is calculated rapidly by using the Green's function. By considering the beam as in simple traction, the sleeper strains due to the lateral forces are uniform distributed when two balanced forces act on a sleeper. The study of the influence of the lateral wheel loads on the sleeper response has been realized to better understand the total sleeper strain when the train passes. In the future works, an inverse problem will be built to obtain the lateral wheel loads from the sleeper responses.

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A. Calculation of the Green's function

The characteristic function of the Eq. (4) is given by :

$$P(\lambda^h) = (\lambda^h)^2 - \gamma_s^2 \quad (12)$$

This equation admits two complex roots λ_1^h and λ_2^h which are defined as $(\lambda_{1,2}^h)^2 = \gamma_s^2$. The general form of the Green's function $G_a^h(x, \omega)$ is given by :

$$G_a^h(x, \omega) = \begin{cases} A_1(\omega)e^{\lambda_1^h x} + A_2(\omega)e^{\lambda_2^h x} & (\forall x \in [-L, a]) \\ B_1(\omega)e^{\lambda_1^h x} + B_2(\omega)e^{\lambda_2^h x} & (\forall x \in [a, L]) \end{cases} \quad (13)$$

In addition, the Green's function $G_a^h(x, \omega)$ has to satisfy the boundary condition of the free-free beam and the continuity at $x = a$. Thus, the four coefficients $A_i(\omega)$ and $B_i(\omega)$ are evaluated such that the Green's function $G_a^h(x, \omega)$ satisfies the following conditions :

1. Two boundary conditions at each end of the beam depending on the type of end support, for free-

free beam :

$$\begin{cases} \frac{\partial G_a^h(L, \omega)}{\partial x} = 0 \\ \frac{\partial G_a^h(-L, \omega)}{\partial x} = 0 \end{cases} \quad (14)$$

2. Continuity condition of displacement at $x = a$:

$$G_a^h(a^+, \omega) - G_a^h(a^-, \omega) = 0 \quad (15)$$

3. Discontinuity of strain at $x = a$:

$$\frac{\partial G_a^h(a^+, \omega)}{\partial x} - \frac{\partial G_a^h(a^-, \omega)}{\partial x} = 1 \quad (16)$$

The aforementioned Eqs. (14), (15) and (16) are linear with four unknowns and we can solve them by an analytic or numerical method.

B. Periodically supported beam model under wheel loads

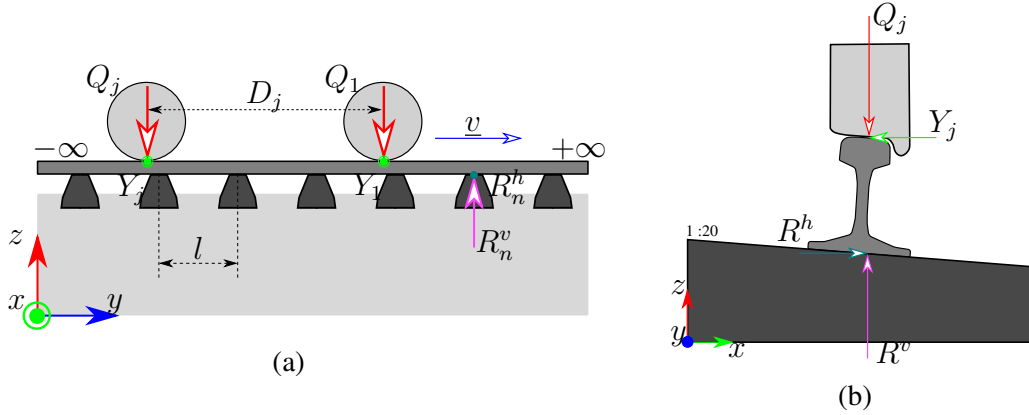


FIGURE 3 – Periodically supported beam subjected to moving loads (a) and (b)

When each rail is modeled by an infinite beam and the sleeper reaction by concentrated forces, we have a system as shown in Fig. 3 where each rail is subjected to two moving loads by two directions (vertical Q_j and horizontal force Y_j) by each wheel. The horizontal force is also called a lateral wheel load. Each wheel is characterized by its initial position D_j and train speed v . A relation between the horizontal beam displacement $\hat{w}_r^h(\omega)$ at the sleeper position and the vertical sleeper reaction force $\hat{R}^h(\omega)$ due to the vertical force in the frequency domain have been proven by Hoang et al. [17] :

$$\hat{R}^h(\omega) = \mathcal{K}^h(\omega)\hat{w}_r^h(\omega) + \mathcal{P}^h(\omega) \quad (17)$$

where $\mathcal{K}^h(\omega)$ and $\mathcal{P}^h(\omega)$ are the so-called equivalent stiffness and equivalent charge of the periodically supported beam due to the lateral wheel loads. By taking the case of an Euler-Bernoulli's beam for the rail, the two parameters can be expressed by :

$$\begin{cases} \mathcal{K}^h(\omega) = 4\lambda_r^3 E_r I_r^h \left[\frac{\sin l\lambda_r}{\cos l\lambda_r - \cos \frac{\omega l}{v}} - \frac{\sinh l\lambda_r}{\cosh l\lambda_r - \cos \frac{\omega l}{v}} \right]^{-1} \\ \mathcal{P}^h(\omega) = \frac{\mathcal{K}^h(\omega)}{v E_r I_r^h \left[\left(\frac{\omega}{v} \right)^4 - \lambda_r^4 \right]} \sum_{j=1}^K Y_j e^{-i\omega \frac{D_j}{v}} \end{cases} \quad (18)$$

where $\lambda_r = \sqrt[4]{\frac{\rho_r S_r \omega^2}{E_r I_r^h}}$ and I_r^h , E_r , S_r , ρ_r are the cross-sectional inertia in the horizontal direction, Young's modulus, section, density of the rail respectively and l is the sleeper spacing.

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