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## AN EFFICIENT TRANSMISSION OPERATOR FOR COMPUTING WAVE PROPAGATION BY DOMAIN DECOMPOSITION

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**Abstract.** *For large size problems, domain decompositions can be used to solve wave propagation problems such as the Helmholtz equation. Generally, this leads to an iterative process where data are exchanged at the boundary between subdomains. Depending on the quality of this exchange the number of iterations is more or less. Moreover, this number of iterations can depend on the frequency and on the number of domains. Here, we propose transmission operators approximating the Dirichlet to Neumann (DtN) operator which is known to be near optimal. We show this can be done using only the solution of problems involving sparse matrices and so keeping the computational time at an acceptable level. When this is combined with the double sweep preconditioner and that the computational domain is decomposed into a sequel of slices this results in an algorithm with a low number of iterations. Different examples are presented to support the precedent analysis.*

## 1 INTRODUCTION

Solving wave propagation problems described by the Helmholtz equation is important in many domains such as acoustics or elastic wave propagation. This can be difficult as the frequency increases because fine meshes have to be used in these cases. One possibility to increase the computing power is to use domain decompositions for which the domain is decomposed into subdomains on which the solutions can be computed more easily by solving small size problems. This generally involves an iterative scheme with a communication between the subdomains at each step, for instance by the transmission of data across the boundary of subdomains.

Many works have been done in the past on this subject where we find methods in which the subdomains overlap or not. In the field of non-overlapping methods we find the works on FETI methods as in [1] or for the application to the Helmholtz equation in [2] where interdomains fields continuity is obtained by Lagrange multipliers. These Lagrange multipliers are computed by solving a relatively small dual problem by a preconditioned conjugate gradient algorithm.

Another possibility is to use overlapping or non-overlapping Schwartz methods as in [3, 4, 5] where the solution is computed iteratively in each subdomain and transmission conditions are defined at the boundary of each domain from the solutions in the subdomains at the precedent step. The convergence rate mainly depends on this transmission operator at the boundary, the best one being the DtN map, see [6, 7], but it is too complicated to compute for being usable. Different methods have been proposed to approximate this complex non-local operator by a simpler one, often by local operators on the boundary, see for instance [8] and [9] for an update view on the subject. A PML layer can also be used as in [10].

Here we propose a new transmission operator defined from the solution of a wave problem in domains around the computed subdomain as in the PML but with a simpler formulation which does not need modifications to the problem formulation or computations of new stiffness matrices. For being efficient, a preconditionner must be used with these methods such as the double sweep preconditionner found in [11, 12, 13].

In section 2, the method is described for the case of a problem posed on two half planes where analytical results can be found following the approach of [5] but with a new transmission operator, then this is applied to more general domains which can be decomposed into a sequel of slices. Finally, in section 3, numerical results are presented before a conclusion.

## 2 COMPUTATION BY DOMAIN DECOMPOSITION

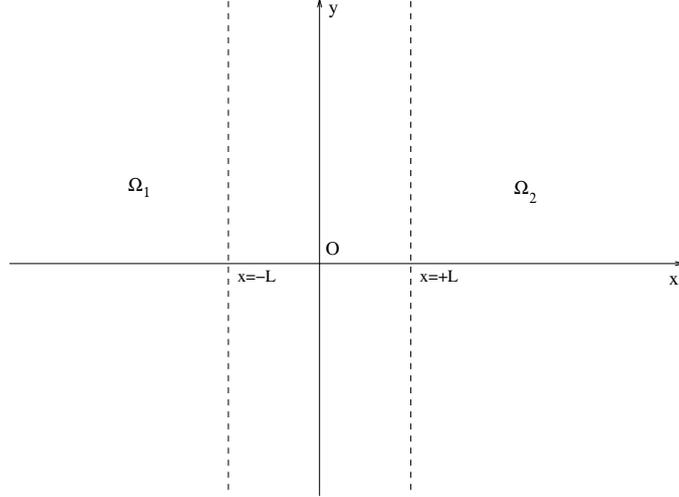
The objective is to solve a wave propagation problem by domain decomposition. We consider here the case of the Helmholtz equation given by

$$\Delta p + k^2 p = 0 \tag{1}$$

with  $p$  the pressure,  $k = \omega/c$  the wavenumber,  $c$  the sound velocity,  $\omega = 2\pi f$  and  $f$  the frequency. This is defined on a domain  $\Omega$  with a Neumann boundary condition as  $\frac{\partial p}{\partial n} = q$  on the boundary  $\Gamma$  of  $\Omega$ .

### 2.1 Case of two half-planes

Consider first the Helmholtz equation defined on a plane  $(x, y)$  which is divided into two half-plane subdomains by the line  $x = 0$ , see figure 1. Taking the Fourier transform in the  $y$  variable with the wavenumber  $k_y$ , one obtains the following sequel of one-dimensional problems in the


 Figure 1: Two half-planes with additional slices of thickness  $L$ .

two sub-domains, which are respectively

$$\begin{aligned} \frac{\partial^2 p_1^{(n+1)}}{\partial x^2} + (k^2 - k_y^2)p_1^{(n+1)} &= 0 \quad \text{for } x < 0 \\ \frac{\partial p_1^{(n+1)}}{\partial x} + \sigma_1(k_y)p_1^{(n+1)} &= \frac{\partial p_2^{(n)}}{\partial x} + \sigma_1(k_y)p_2^{(n)} \quad \text{at } x = 0 \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{\partial^2 p_2^{(n+1)}}{\partial x^2} + (k^2 - k_y^2)p_2^{(n+1)} &= 0 \quad \text{for } x > 0 \\ -\frac{\partial p_2^{(n+1)}}{\partial x} + \sigma_2(k_y)p_2^{(n+1)} &= -\frac{\partial p_1^{(n)}}{\partial x} + \sigma_2(k_y)p_1^{(n)} \quad \text{at } x = 0 \end{aligned} \quad (3)$$

with  $\sigma_1(k_y)$  and  $\sigma_2(k_y)$  two well chosen functions to assure the convergence of the iterative process. The solutions are respectively

$$\begin{aligned} p_1^{(n+1)}(x) &= e^{-\lambda(k_y)x} p_1^{(n+1)}(0) \quad \text{for } x < 0 \\ p_2^{(n+1)}(x) &= e^{\lambda(k_y)x} p_2^{(n+1)}(0) \quad \text{for } x > 0 \end{aligned} \quad (4)$$

with  $\lambda(k_y) = i\sqrt{k^2 - k_y^2}$ . From the transmission conditions (second lines of 2 and 3), one gets

$$\begin{aligned} p_1^{(n+1)}(0) &= \frac{\lambda(k_y) + \sigma_1(k_y)}{-\lambda(k_y) + \sigma_1(k_y)} p_2^{(n)}(0) \\ p_2^{(n+1)}(0) &= \frac{\lambda(k_y) + \sigma_2(k_y)}{-\lambda(k_y) + \sigma_2(k_y)} p_1^{(n)}(0) \end{aligned} \quad (5)$$

so that

$$p_1^{(2n)}(0) = \rho^n(k_y) p_1^{(0)}(0) \quad (6)$$

with

$$\rho(k_y) = \frac{\lambda(k_y) + \sigma_1(k_y)}{-\lambda(k_y) + \sigma_1(k_y)} \cdot \frac{\lambda(k_y) + \sigma_2(k_y)}{-\lambda(k_y) + \sigma_2(k_y)} \quad (7)$$

Different possibilities for choosing the values of  $\sigma_1$  and  $\sigma_2$  are discussed in [5]. Here, we propose another way of building these functions. The value of  $\sigma_1(k_y)$  is obtained from the solution of the problem in the slice  $0 < x < L$  with absorbing boundary conditions at  $x = L$ , more precisely from the solution of the problem

$$\begin{aligned} \frac{\partial^2 p_1}{\partial x^2} + (k^2 - k_y^2)p_1 &= 0 \quad \text{for } 0 < x < L \\ p_1 &= p \quad \text{for } x = 0 \\ \frac{\partial p_1}{\partial x} - \gamma_1(k_y)p_1 &= 0 \quad \text{for } x = L \end{aligned} \quad (8)$$

and  $\sigma_1(k_y)$  is defined such that  $\frac{\partial p_1}{\partial x}(0) + \sigma_1(k_y)p = 0$ . The solution of problem 8 is given by

$$p_1(x) = ae^{\lambda(k_y)x} + be^{-\lambda(k_y)x} \quad (9)$$

with the coefficients  $a$  and  $b$  such that

$$\begin{aligned} a + b &= p \\ \lambda(k_y)(ae^{\lambda(k_y)L} - be^{-\lambda(k_y)L}) - \gamma_1(k_y)(ae^{\lambda(k_y)L} + be^{-\lambda(k_y)L}) &= 0 \end{aligned} \quad (10)$$

The second relation of 10 leads to  $b = R_1 a$  with  $R_1 = \frac{\lambda(k_y) - \gamma_1(k_y)}{\lambda(k_y) + \gamma_1(k_y)} e^{2\lambda(k_y)L}$ , so that we get

$$\sigma_1(k_y) = -\lambda(k_y) \frac{1 - R_1}{1 + R_1} \quad (11)$$

Similarly one gets

$$\sigma_2(k_y) = -\lambda(k_y) \frac{1 - R_2}{1 + R_2} \quad (12)$$

In the case  $k_y = 0$ , taking  $\gamma_1 = \gamma_2 = ik$ , one gets  $R_1 = R_2 = R$  and the classical values  $\sigma_1 = \sigma_2 = -ik$ . From (11) and (12) we obtain the value of  $\rho(k_y)$  by relation 7 giving  $\rho(k_y) = R^2$  in the case  $\gamma_1 = \gamma_2 = ik$ . The function  $\rho(k_y)$  is plotted versus  $k_y/k$  for several values of  $kL$  in figure 2. For  $kL = 0$ , the evanescent waves are not attenuated and the process cannot converge. On the contrary, for  $kL > 0$ , the value of  $\rho(k_y)$  is lower than one for the entire spectrum leading to the convergence of the iterative process and all the more so as the value of  $kL$  is high.

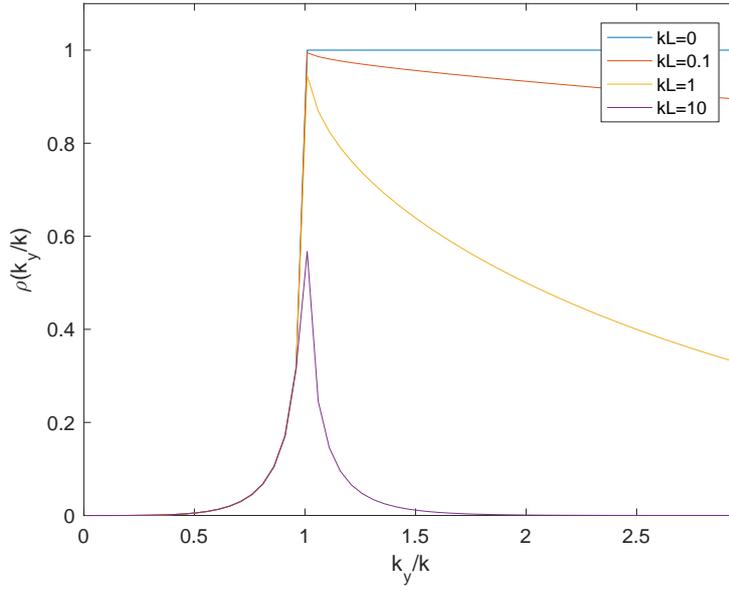
## 2.2 General method

We consider now a more general domain which is decomposed into vertical slices as shown in figure 3. In each vertical band  $i$ , one has to solve at step  $n + 1$

$$\begin{aligned} \Delta p_i^{(n+1)} + k^2 p_i^{(n+1)} &= 0 \quad \text{in } \Omega_i \\ \frac{\partial p_i^{(n+1)}}{\partial n_i} + S_{i,i-1} p_i^{(n+1)} &= g_{i,i-1}^{(n)} = -\frac{\partial p_{i-1}^{(n)}}{\partial n_{i-1}} + S_{i,i-1} p_{i-1}^{(n)} \quad \text{on } \Gamma_{i-1,i} \\ \frac{\partial p_i^{(n+1)}}{\partial n_i} + S_{i,i+1} p_i^{(n+1)} &= g_{i,i+1}^{(n)} = -\frac{\partial p_{i+1}^{(n)}}{\partial n_{i+1}} + S_{i,i+1} p_{i+1}^{(n)} \quad \text{on } \Gamma_{i,i+1} \end{aligned} \quad (13)$$

with the operators  $S$  as approximations of the DtN operators on the boundaries and the  $g_{i,i+1}$  are updated by

$$\begin{aligned} g_{i,i+1}^{(n+1)} &= -\frac{\partial p_{i+1}^{(n+1)}}{\partial n_{i+1}} + S_{i,i+1} p_{i+1}^{(n+1)} \\ &= -g_{i+1,i}^{(n)} + (S_{i,i+1} + S_{i+1,i}) p_{i+1}^{(n+1)} \end{aligned} \quad (14)$$


 Figure 2: Values of  $\rho$  for different  $kL$ 

with a similar formula for  $g_{i,i-1}^{(n+1)}$ .

The operator  $S_{i,i+1}p_{i+1}^{(n+1)}$  is obtained by solving the Helmholtz equation in the domain  $\Omega_{i+1}$  with an absorbing boundary condition on  $\Gamma_{i+1,i+2}$ , that is

$$\begin{aligned} \Delta u + k^2 u &= 0 \text{ in } \Omega_{i+1} \\ \frac{\partial u}{\partial n_{i+1}} - iku + \frac{i}{4k} \frac{\partial^2 u}{\partial \tau^2} &= 0 \text{ on } \Gamma_{i+1,i+2} \\ u &= p_{i+1}^{(n+1)} \text{ on } \Gamma_{i,i+1} \end{aligned} \quad (15)$$

and  $S_{i,i+1}p_{i+1}^{(n+1)} = \frac{\partial u}{\partial n_{i+1}}$  on  $\Gamma_{i,i+1}$  giving an approximation of the DtN operator on that interface.

The operator  $S_{i+1,i}p_{i+1}^{(n+1)}$  is obtained by solving the Helmholtz equation in the domain  $\Omega_i$  with an absorbing boundary condition on  $\Gamma_{i-1,i}$ , that is

$$\begin{aligned} \Delta u + k^2 u &= 0 \text{ in } \Omega_i \\ \frac{\partial u}{\partial n_i} - iku + \frac{i}{4k} \frac{\partial^2 u}{\partial \tau^2} &= 0 \text{ on } \Gamma_{i-1,i} \\ u &= p_{i+1}^{(n+1)} \text{ on } \Gamma_{i,i+1} \end{aligned} \quad (16)$$

with  $\frac{\partial}{\partial \tau}$  the tangential derivative and  $S_{i+1,i}p_{i+1}^{(n+1)} = \frac{\partial u}{\partial n_i}$  on  $\Gamma_{i,i+1}$  giving an approximation of the DtN operator on that interface.

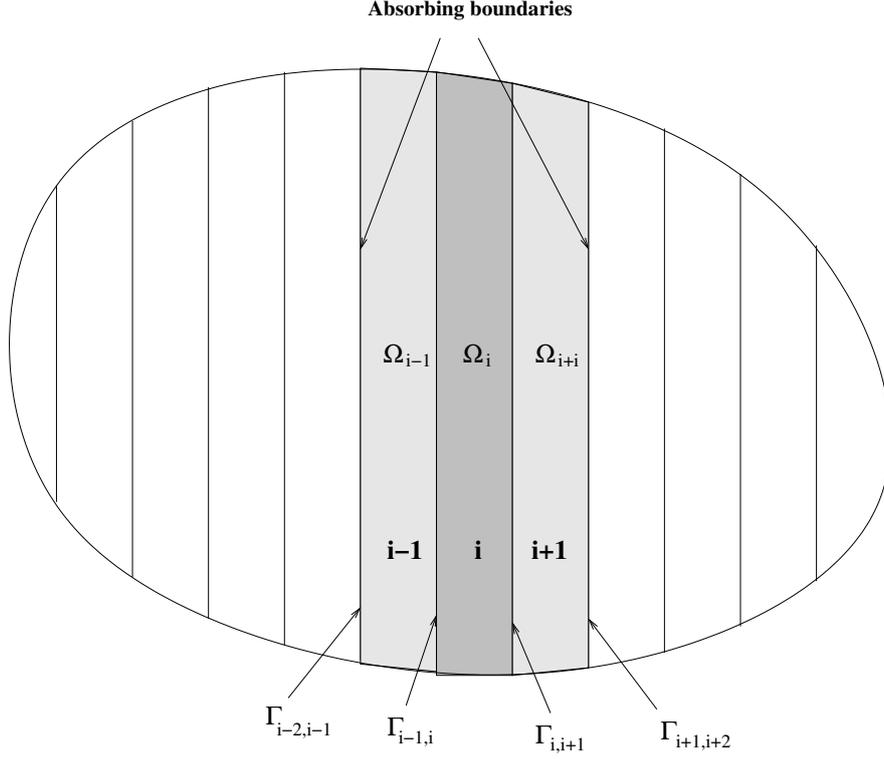


Figure 3: Domain and its decomposition into subdomains

Combining relations 13, 15 and 16, to update the solution in  $\Omega_i$ , one has to solve

$$\begin{aligned}
 \Delta p_i^{(n+1)} + k^2 p_i^{(n+1)} &= 0 \text{ in } \Omega_i \\
 \Delta u_{i-1}^{(n+1)} + k^2 u_{i-1}^{(n+1)} &= 0 \text{ in } \Omega_{i-1} \\
 \Delta u_{i+1}^{(n+1)} + k^2 u_{i+1}^{(n+1)} &= 0 \text{ in } \Omega_{i+1} \\
 \frac{\partial p_i^{(n+1)}}{\partial n_i} + \frac{\partial u_{i-1}^{(n+1)}}{\partial n_{i-1}} &= g_{i,i-1}^{(n)} \text{ on } \Gamma_{i-1,i} \\
 \frac{\partial p_i^{(n+1)}}{\partial n_i} + \frac{\partial u_{i+1}^{(n+1)}}{\partial n_{i+1}} &= g_{i,i+1}^{(n)} \text{ on } \Gamma_{i,i+1} \\
 \frac{\partial u_{i-1}^{(n+1)}}{\partial n_{i-1}} - i k u_{i-1}^{(n+1)} + \frac{i}{4k} \frac{\partial^2 u_{i-1}^{(n+1)}}{\partial \tau^2} &= 0 \text{ on } \Gamma_{i-2,i-1} \\
 \frac{\partial u_{i+1}^{(n+1)}}{\partial n_{i+1}} - i k u_{i+1}^{(n+1)} + \frac{i}{4k} \frac{\partial^2 u_{i+1}^{(n+1)}}{\partial \tau^2} &= 0 \text{ on } \Gamma_{i+1,i+2}
 \end{aligned} \tag{17}$$

The last equations of (15) and (16) provide the relations of continuity between the solutions in the adjacent domains. At the discrete level this problem is solved by assembling the dynamic stiffness matrices of the domains  $\Omega_{i-1}$ ,  $\Omega_i$  and  $\Omega_{i+1}$  and putting forces on the interfaces  $\Gamma_{i-1,i}$  and  $\Gamma_{i,i+1}$  defined respectively by the functions  $g_{i,i-1}^{(n)}$  and  $g_{i,i+1}^{(n)}$ . For external boundaries, the boundary condition of the problem should be used, for instance  $\frac{\partial p}{\partial n} = q_0$  for a Neumann boundary defined by a function  $q_0$ .

The problem is then solved by the GMRES algorithm associated to the double sweep preconditionner described in [11, 12, 13]. For the updating of  $g_{i,i+1}$ , one has to solve the problems

(15) and (16) and use their solutions in (14).

### 3 NUMERICAL EXAMPLES

#### 3.1 Rectangle

We first consider the case of a rectangular domain of size  $1m \times 1m$  with a boundary condition given by a plane wave  $e^{ikx}$  on the left and right boundaries as  $\frac{\partial p}{\partial n} = ikn_x e^{ikx}$ . The mesh is created with triangular elements of degree 2 (52633 nodes for 5 subdomains up to 80401 nodes for 100 subdomains). The mesh for 100 subdomains is shown on figure 4 where each domain is a fine vertical band. Computations are done for different frequencies and number of subdomains. Table 1 shows that in this case the number of iterations is very low and do not depend on the frequency or the number of subdomains. The solution for 100 subdomains and the frequency 2000Hz is shown on the right of figure 5.

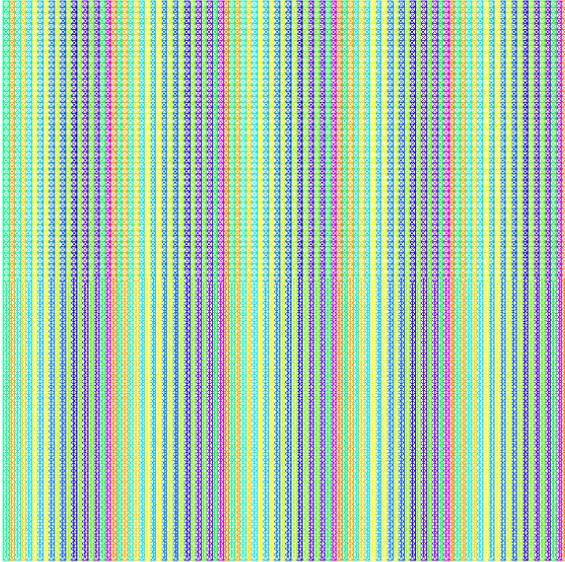


Figure 4: Rectangular domain with 100 subdomains

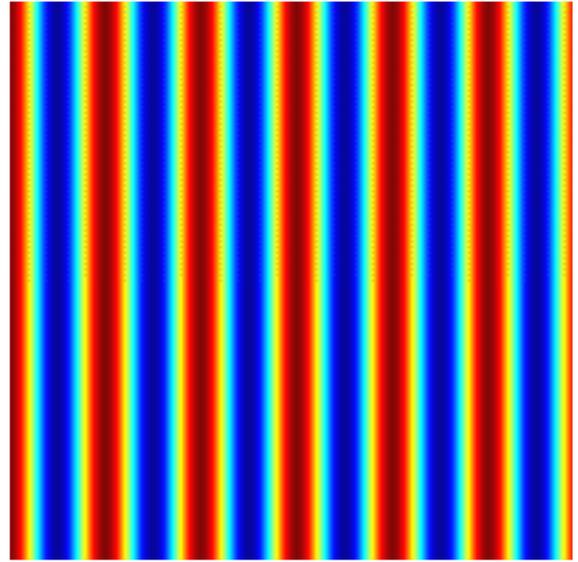


Figure 5: Solution at 2000Hz for 100 subdomains

Other computations are made with the boundary condition given by  $\cos(k_y y)e^{ik_x x}$  with  $k_y = 4\pi/L_y$  and  $k_x = \sqrt{k^2 - k_y^2}$ . The number of iteration is given in table 2 while the solutions for 500 Hz and 2000 Hz are given in figures 6 and 7. In figure 6 the boundary condition leads to an evanescent wave while in figure 7 a propagating wave is clearly visible. This number of iterations is low except for 100 subdomains and low frequencies. This needs further studies to understand this phenomenon as usually the number of iteration is increasing with the frequency.

| Frequency | Number of subdomains |    |     |
|-----------|----------------------|----|-----|
|           | 5                    | 25 | 100 |
| 500 Hz    | 4                    | 4  | 4   |
| 1000 Hz   | 4                    | 4  | 4   |
| 2000 Hz   | 4                    | 4  | 4   |

Table 1: Number of iterations for different frequencies and number of subdomains.

| Frequency | Number of subdomains |    |     |
|-----------|----------------------|----|-----|
|           | 5                    | 25 | 100 |
| 500 Hz    | 10                   | 15 | 73  |
| 1000 Hz   | 13                   | 15 | 41  |
| 2000 Hz   | 17                   | 18 | 20  |

Table 2: Number of iterations for different frequencies and number of subdomains.

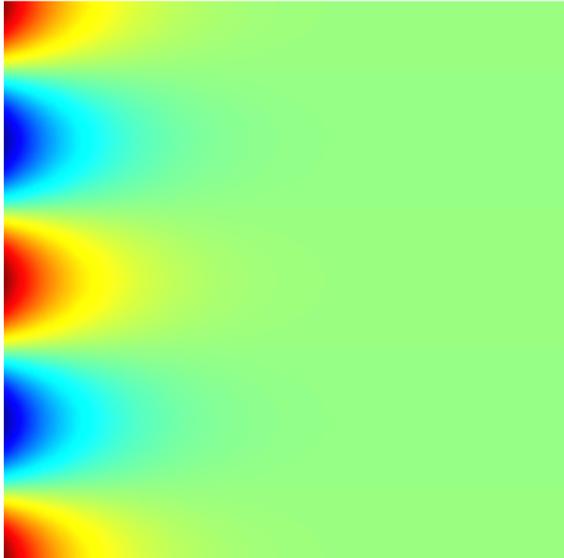


Figure 6: Solution at 500Hz for 100 subdomains

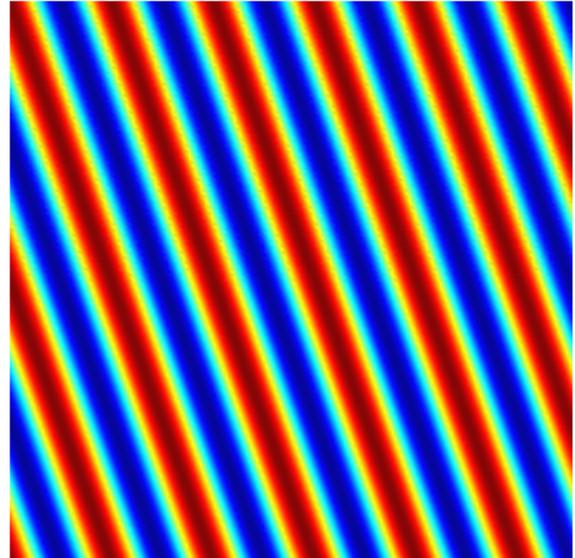


Figure 7: Solution at 2000Hz for 100 subdomains

### 3.2 Disk

The domain is now a disk of radius 1m whose mesh and decomposition into 100 subdomains is shown in figure 8. The solution for the plane wave excitation is shown in figure 9. In this case the number of iterations is shown in table 3. There is a slow increase in the number of iteration with the frequency and the number of subdomains.

| Frequency | Number of subdomains |    |     |
|-----------|----------------------|----|-----|
|           | 5                    | 25 | 100 |
| 500 Hz    | 10                   | 14 | 26  |
| 1000 Hz   | 15                   | 17 | 23  |
| 2000 Hz   | 28                   | 34 | 38  |

Table 3: Number of iterations for different frequencies and number of subdomains.

## 4 CONCLUSION

We have presented transmission conditions which are intermediate between the local conditions on the boundary and domain conditions as provided by a PML layer. So it gives an equilibrium between the ease of implementation and the efficiency. It involves only modified

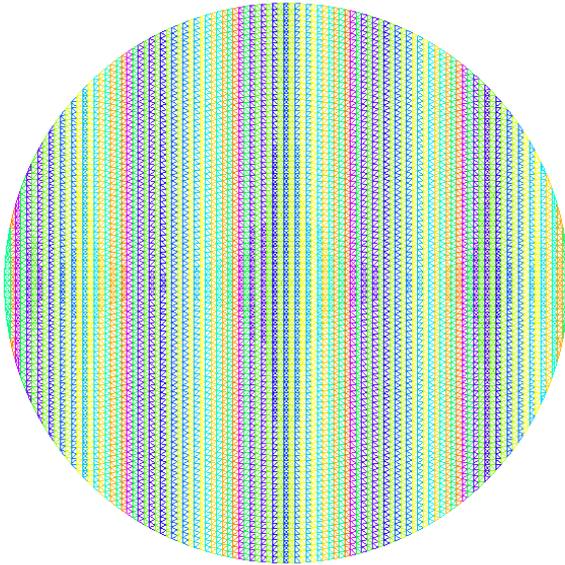


Figure 8: Disk domain with 100 subdomains

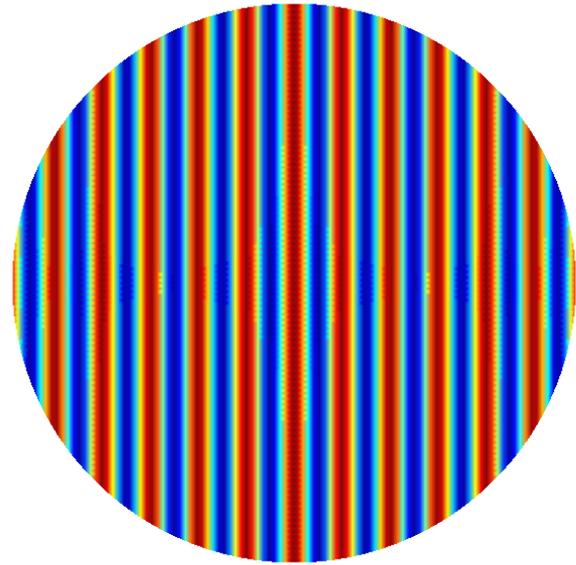


Figure 9: Solution at 2000Hz for 100 subdomains

matrices on the boundaries but no new matrix in the domains. Subsequent research should focus on the improvement of the boundary condition  $\frac{\partial p}{\partial n} = ikp - \frac{i}{4k} \frac{\partial^2 p}{\partial \tau^2}$ .

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