A domain decomposition method with fast convergence for the Helmholtz equation

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ABSTRACT

Solving the Helmholtz equation by finite element methods is quite important in acoustics. When the frequency or the size of the problem increase, large meshes are necessary and consequently heavy computations are required. One possibility is to use domain decompositions for which the domain is decomposed into subdomains on which the solutions can be computed more easily. This involves an iterative scheme where data are transmitted between subdomains from the precedent iteration. The main problem is to have a low number of iterations so that the problem can be solved in a reasonable amount of time.

In this work, we present a domain decomposition method based on two main features. The first one is to use extended domains with absorbing boundary conditions. The second feature is to decompose the whole domain into one-dimensional or two-dimensional networks of subdomains so that double sweep preconditioners can be used. Examples are shown where the number of iterations is usually low. This number of iterations is also shown to depend slowly on the number of domains and the frequency.

Keywords: Domain decomposition, Finite Element, Helmholtz equation.

1. INTRODUCTION

We consider here the solution of the Helmholtz equation by finite element methods. When the frequency increases many degrees of freedom are needed. In such cases, it can be interesting to use domain decomposition methods for which the domain is decomposed into several subdomains over which the solutions can be computed more easily by solving independent small size problems. Then an iterative scheme needs to be developed to compute the solutions in the subdomains at a given step from the solutions over the other subdomains at the precedent step. The problem is to get as few iterations as possible and a number of iterations almost independent of the number of subdomains and the frequency.

Many works, such that the development of FETI methods or Schwartz methods, have been done in the past on this subject, see (1-9) for instance. In overlapping Schwartz methods (3), the solutions are computed on overlapping domains and the boundary conditions for a given domain are defined at the boundary of this overlapping domain and so from the solutions at the precedent iteration obtained in the other domains that the present domain overlaps. In non-overlapping Schwartz methods (4, 5), the solutions are computed in each domain and transmission operators are defined at the boundary of the domains to give boundary conditions at a given iteration from the solutions of adjacent domains at the precedent iteration. The number of iterations to achieve the convergence of the algorithm is deeply dependent on the quality of these transmission operators.

Different methods have been proposed to compute these transmission operators. The best one are non-local such as the Dirichlet to Neumann operator but are too complex to be really useful (6). Different local operators have been proposed, see for instance (7) and (8). A good one is the Perfectly Matched Layer (PML) as in (9) but it needs to define special layers around each domain with suitable absorption properties.

Another possibility, which is developed here, is to define the transmission operator from the solutions of the Helmholtz equation in the subdomains completed with surrounding subdomains and with simple absorbing conditions at the exterior boundary. Then the transmission operator is obtained from the solutions in the surrounding subdomains. For being efficient, a preconditionner must be used with these methods such as the double sweep preconditionner found in (10, 11,12). This method is developed for the cases

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where the subdomain decomposition is made of one-dimensional or two-dimensional networks so that the global domain is first decomposed into a sequel of slices at the first level and then each slice can also be decomposed into a second level of subslices.

The paper is organized as follows. In section 2, the method is presented for the Helmholtz equation defined on a two-dimensional domain. Then in section 3, numerical results are given before a conclusion.

2. COMPUTATION BY DOMAIN DECOMPOSITION

2.1 General problem

The objective is to solve by domain decomposition, the Helmholtz equation given by

$$\Delta p + k^2 p = 0 \tag{1}$$

with p the pressure, $k=\omega/c$ the wavenumber, c the sound velocity, $\omega=2\pi f$ and f the frequency. This is defined on a two-dimensional domain Ω with a Neumann boundary condition $\frac{\partial p}{\partial n} = q$ on the exterior boundary Γ of Ω.

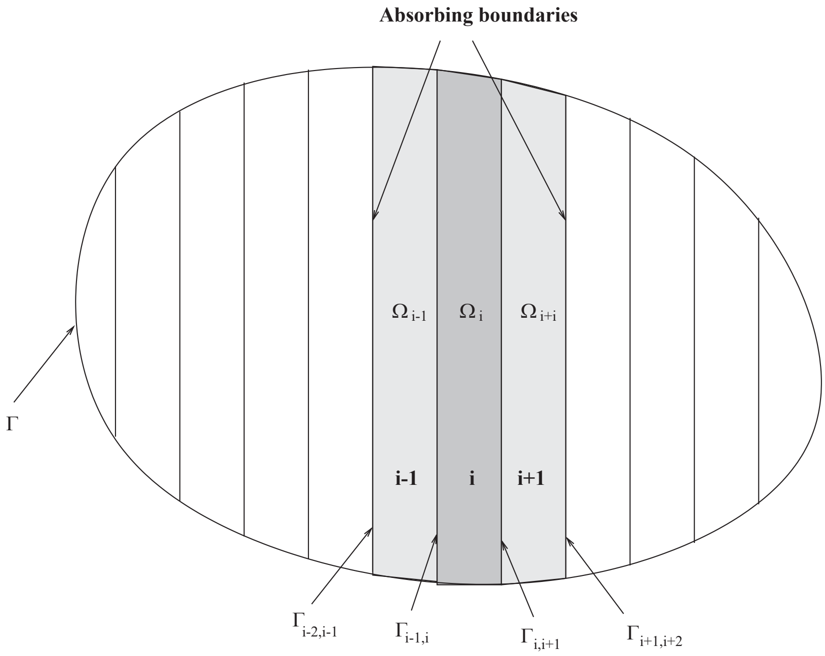

Figure 1: Domain decomposed into one-dimensional slices

The classical way of solving the problem by domain decomposition is to divide the global domain into subdomains, for instance into slices like in Figure 1, and to solve the sequel of following problems indexed by n for each subdomain Ω_i

$$\begin{aligned} &\Delta p_i^n + k^2 p_i^n = 0 \;\; in\, \Omega_i \\ &\frac{\partial p_i^n}{\partial n_i} + S p_i^n = g_{ij}^{n-1} \;\; on\, \Gamma_{ij} \\ &\frac{\partial p_i^n}{\partial n_i} = q_i \quad on\, \Gamma_i \end{aligned} \tag{2}$$

With Γ_{ij} the boundary between the domains i and j and S is an operator that try to simulate the impedance of the domain that extends beyond the boundary. The values of g are updated by

$$\begin{aligned} g_{ij}^n &= -\frac{\partial p_j^n}{\partial n_i} + S p_j^n \;\; on\, \Gamma_{ij} \\ &= -g_{ji}^{n-1} + 2 S p_j^n \end{aligned} \tag{3}$$

Decomposing $p_i^n = v_i^n + w_i$ with v_i^n the solution of problem (1) with $q_i = 0$ and w_i the solution with $g_{ij}^{n-1} = 0$, the problem can be put under the form

$$g^n = Ag^{n-1} + b \tag{4}$$

With the vector b obtained from w_i, the matrix A from v_i^n and g^n is the vector of all the g_{ij}^n for different *i* and *j*. The problem (3) is then solved by the GMRES algorithm associated to the double sweep preconditionner described in (10, 11, 12). The main problem consists in finding a good operator S such that the number of iterations to get the convergence is as low as possible and with a low dependency on the number of subdomains and the frequency.

2.2 Transmission operator

To get this operator *S*, one proposes to solve on extended domains by adding to the domain Ω_i the domains Ω_{i-1} and Ω_{i+1} associated to simple absorbing boundary conditions to the exterior of the global domain, that is on the boundaries $\Gamma_{i-2,i-1}$ and $\Gamma_{i+1,i+2}$, see Figure 1. So the problem (1) is transformed into

$$\begin{aligned}
&\Delta p_i^n + k^2 p_i^n = 0 \;\; in\, \Omega_{i-1} \cup \Omega_i \cup \Omega_{i+1} \\
&\frac{\partial p_i^n}{\partial n_{i-1}} - ikp_i^n = 0 \;\; on\, \Gamma_{i-2,i-1} \\
&\frac{\partial p_i^n}{\partial n_{i+1}} - ikp_i^n = 0 \;\; on\, \Gamma_{i+1,i+2} \\
&\frac{\partial p_i^n}{\partial n_{i-1}} + \frac{\partial p_i^n}{\partial n_i} = g_{i,i-1}^{n-1} \;\; on\, \Gamma_{i-1,i} \\
&\frac{\partial p_i^n}{\partial n_i} + \frac{\partial p_i^n}{\partial n_{i+1}} = g_{i,i+1}^{n-1} \;\; on\, \Gamma_{i,i+1} \\
&\frac{\partial p_i^n}{\partial n_i} = 0 \; on\, \Gamma_{i-1} \cup \Gamma_i \cup \Gamma_{i+1}
\end{aligned} \tag{5}$$

Where $g_{i,i-1}^{n-1}$ and $g_{i,i+1}^{n-1}$ are values obtained from the precedent iteration by relation (3) and the operator S is obtained as the normal derivative of the solution of the pressure on the extended domains Ω_{i-1} and Ω_{i+1}. More precisely on Ω_{i-1} one has $Su = \frac{\partial u}{\partial n_{i-1}}$ on $\Gamma_{i-1,i}$ with u the solution of the following problem

$$\begin{aligned}
&\Delta u + k^2 u = 0 \;\; in\, \Omega_{i-1} \\
&\frac{\partial u}{\partial n_{i-1}} - iku = 0 \;\; on\, \Gamma_{i-2,i-1} \\
&\frac{\partial u}{\partial n_{i-1}} = 0 \;\; on\, \Gamma_{i-1} \\
&u = p_i^n \;\; on\, \Gamma_{i-1,i}
\end{aligned} \tag{6}$$

And a similar definition for the domain Ω_{i+1}.

2.3 Two-dimensional network

The problem can be solved as in the precedent subsection or can be further decomposed into a two-dimensional network of subdomains as in Figure 2. In this latter case, the sequel of following problems indexed by *m* is solved.

$$
\begin{aligned}
\Delta p_{ij}^{n,m} + k^2 p_{ij}^{n,m} &= 0 \ in \ \widetilde{\Omega}_{i,j} \\
\frac{\partial p_{ij}^{n,m}}{\partial n} - ikp_{ij}^{n,m} &= 0 \ on \ \widetilde{\Gamma}_{i,j} \\
\frac{\partial p_{ij}^{n,m}}{\partial n_{j-1}} + \frac{\partial p_{ij}^{n,m}}{\partial n_j} &= r_{j,j-1}^{m-1} \ \ on \ \Gamma_{j-1,j} \\
\frac{\partial p_{ij}^{n,m}}{\partial n_j} + \frac{\partial p_{ij}^{n,m}}{\partial n_{j+1}} &= r_{j,j+1}^{m-1} \ \ on \ \Gamma_{j,j+1}
\end{aligned} \tag{7}
$$

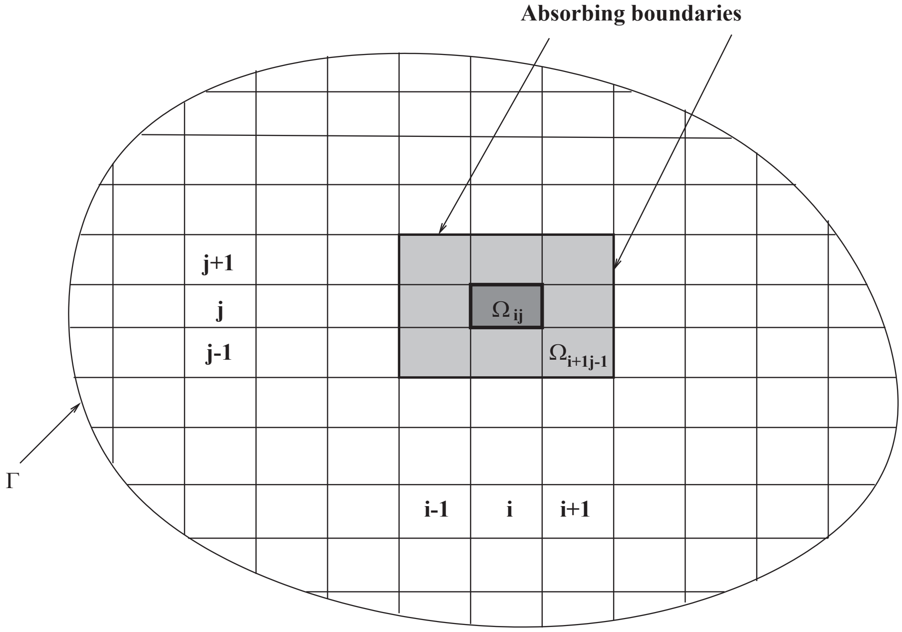

Figure 2: Two-dimensional network of subdomains

Where $\widetilde{\Omega}_{\boldsymbol{i,j}}$ is made of the domain $\Omega_{i,j}$ and its surrounding domains such that

$$
\widetilde{\Omega}_{i,j} = \Omega_{i,j} \cup \Omega_{i-1,j} \cup \Omega_{i+1,j} \cup \Omega_{i-1,j-1} \cup \Omega_{i,j-1} \cup \Omega_{i+1,j-1} \cup \Omega_{i-1,j+1} \cup \Omega_{i,j+1} \cup \Omega_{i+1,j+1} \tag{8}
$$

And $\widetilde{\Gamma}_{i,j}$ is the exterior boundary of $\widetilde{\Omega}_{i,j}$. The iteration continues on *m* until convergence in slice *i* to get the solution of problem (5). Other points such as the definition of the transmission operator in slice *i* are defined in a similar way to relations (2) and (6).

3. NUMERICAL EXAMPLES

3.1 One-dimensional decomposition

We consider the Helmholtz equation on a square domain of size $1m\times1m$. The sound velocity is *c=340m/s.* The domain is broken down into vertical slices as in Figure 3. First the boundary condition is defined by a plane wave such that the normal derivative of the pressure is $q = ikn_x e^{ikx}$. The number of iterations is given in Table 1 and an example of solutions for the frequency 1000Hz is presented in Figure 4.

Table 1: Number of iterations for different frequencies *f* and number of subdomains *n* for a plane wave

f	$n = 5$	$n = 25$	$n = 50$	$n = 100$
500Hz	5	5	5	5
1000Hz	5	5	5	5
2000Hz	5	9	5	5

In this example, one can see that the number of iterations is low and almost independent of the frequency and the number of subdomains. Figure 4 shows the plane wave on the whole domain with full continuity at the boundaries of the subdomains.

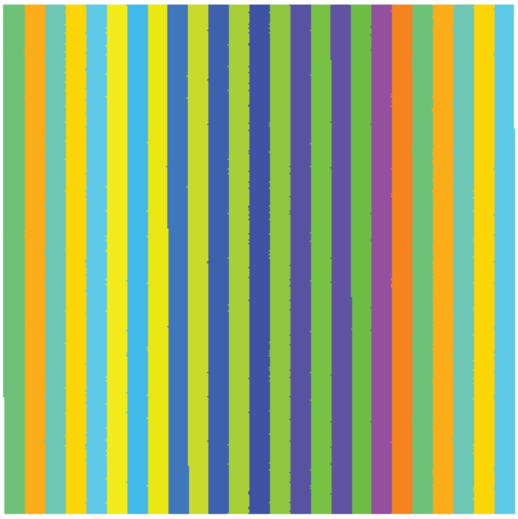

Figure 3: Domain decomposition into 25 subdomains

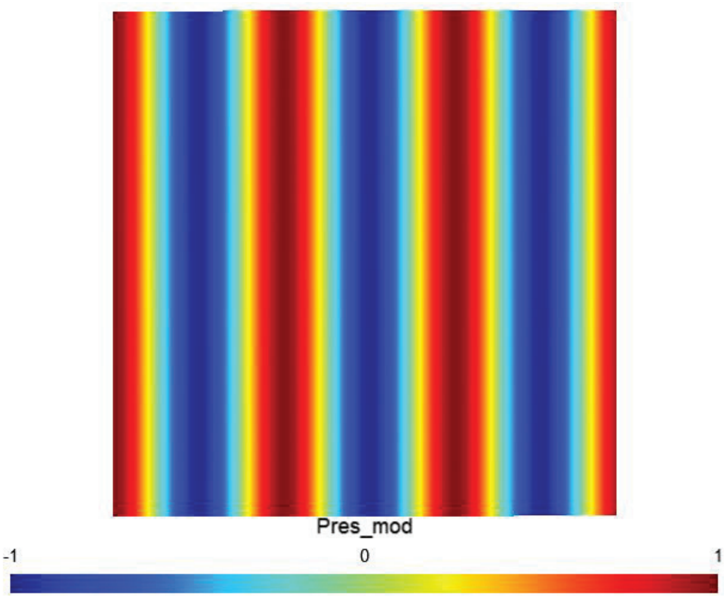

Figure 4: Pressure for a plane wave at 1000Hz

Next the case of a higher order wave defined by $q(x,y) = ik_x n_x \cos\left(\frac{4\pi y}{L}\right) e^{ik_x x} - \frac{4\pi}{L} n_y \sin\left(\frac{4\pi y}{L}\right) e^{ik_x x}$ with $k^2 = k_x^2 + \left(\frac{4\pi}{L}\right)^2$ and L=1m is presented. The number of iterations is given in Table 2 and an example of results for the frequency 1000Hz is presented in Figure 4.

Table 2: Number of iterations for different frequencies *f* and number of subdomains *n* for a higher mode

f	$n = 5$	$n = 25$	$n = 50$	$n = 100$
500Hz	4	10	12	13
1000Hz	7	10	10	10

2000Hz	6	13	7	13

In this case, the number of iterations depends on the frequency and the number of subdomains. However, it seems to reach rapidly a maximum value still independent of the frequency and the number of subdomains. Figure 5 presents the pressure for the frequency 1000Hz. A good continuous solution can be seen.

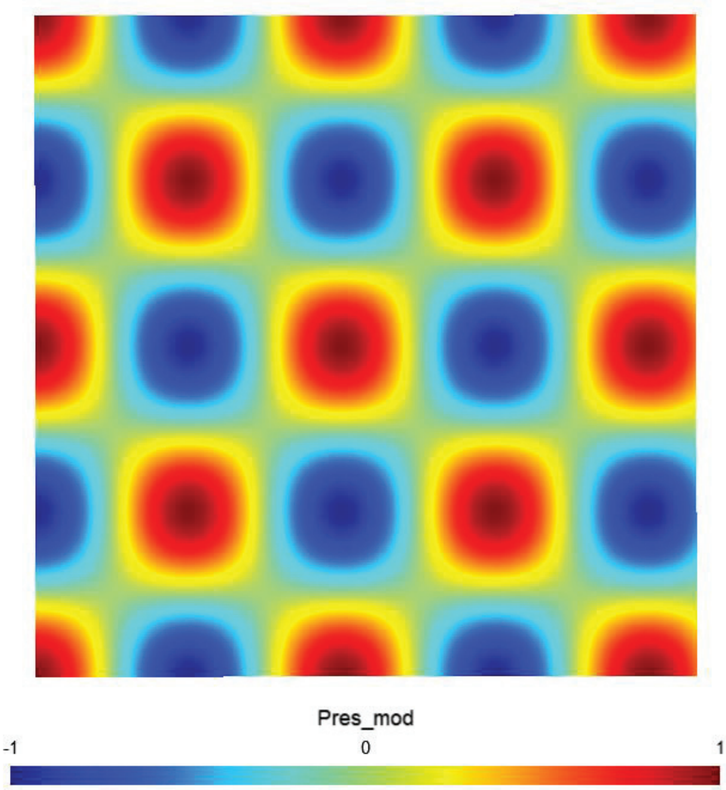

Figure 5: Pressure for the fourth mode at 1000Hz

3.2 Two-dimensional decomposition

We first consider the case of a rectangular domain of size $1m \times 1m$ with a boundary condition given by a plane wave e^{ikx} on the left and right boundaries as in the precedent subsections. The domain is now divided into $n \times n$ subdomains and each subdomain is mesh with triangular elements of degree 1. An example of decomposition is presented in Figure 6. Each subdomain has approximately 400 elements so that the total number of elements will depend on the number of subdomains. Table 3 presents the number of iterations for different frequencies and number of subdomains. The number inside the parenthesis is the average number of subiterations for solving problem (7) into a slice.

Table 3: Number of iterations for different frequencies and number of subdomains

f	5×5	10×10	20×20	40×40
500Hz	5 (8.3)	4 (8.2)	4 (9.9)	4 (11.0)
1000Hz	5 (8.5)	4 (8.2)	4 (8.9)	5 (9.4)
2000Hz	9 (12.6)	5 (8.4)	6 (8.0)	4 (8.2)

One can see that the numbers of iterations is low and with a low dependency on the frequency or the number of subdomains. Almost the same observation can be made for the average number of subiterations in a slice. Figure 7 presents the equivalent of figure 5 for the same boundary conditions. The solution is the same even if in the present case the computation is made on the two-dimensional network.

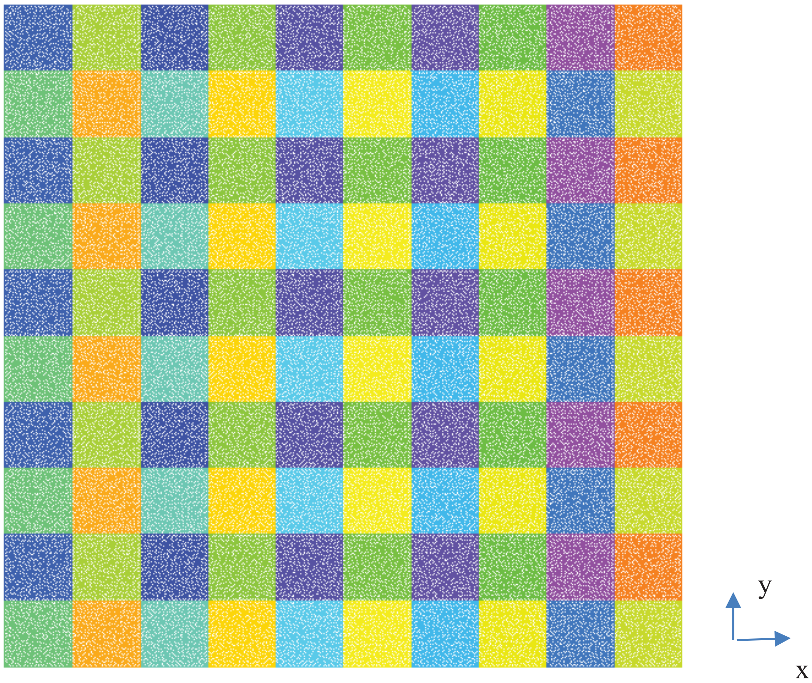

Figure 6: Domain decomposition into 10x10 subdomains

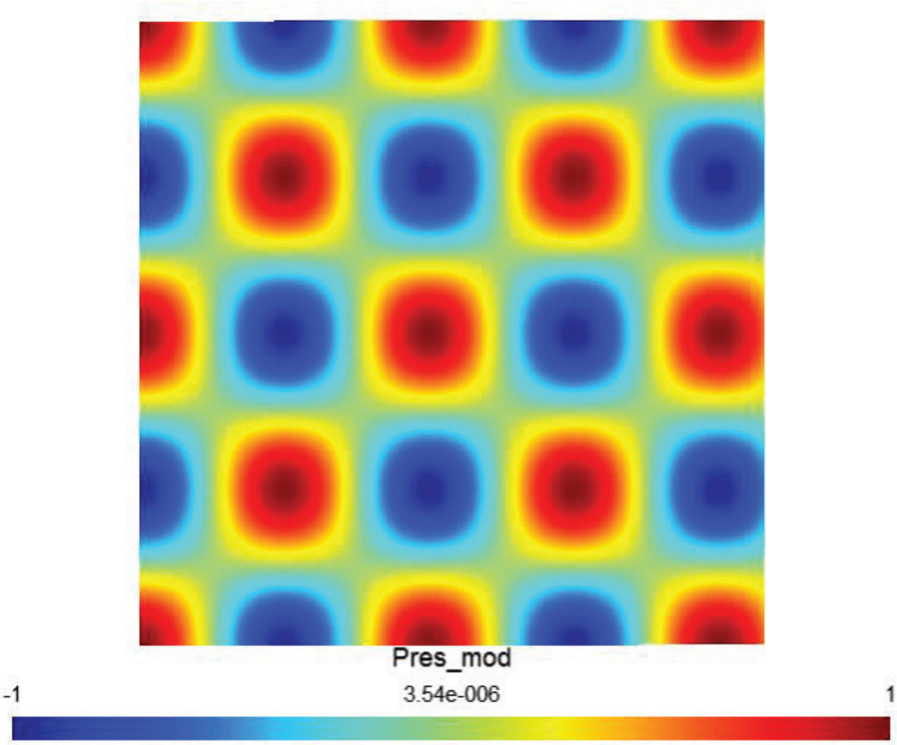

Figure 7: Pressure for the fourth mode at 1000Hz for 10x10 subdomains

4. CONCLUSION

We have presented transmission conditions which are intermediate between the local conditions on the boundary and domain conditions as provided by a PML layer. So it gives an equilibrium between the ease of implementation and the efficiency. It involves only modified matrices on the boundaries but no new matrix in the domains. This was shown to converge in a low number of iterations both for one-dimensional and two-dimensional networks of subdomains. Subsequent research should focus on the improvement of the boundary condition and the consideration of domains with more complex shapes.

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