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# LOCAL WELL-POSEDNESS OF A HAMILTONIAN REGULARISATION OF THE SAINT-VENANT SYSTEM WITH UNEVEN BOTTOM

BILLEL GUELMAME, DIDIER CLAMOND AND STÉPHANE JUNCA

ABSTRACT. We prove in this note the local (in time) well-posedness of a broad class of  $2 \times 2$  symmetrisable hyperbolic system involving additional non-local terms. The latest result implies the local well-posedness of the non dispersive regularisation of the Saint-Venant system with uneven bottom introduced by Clamond et al. [2]. We also prove that, as long as the first derivatives are bounded, singularities cannot appear.

**AMS Classification:** 35Q35; 35L65; 37K05; 35B65; 76B15.

**Key words:** Dispersionless shallow water equations; nonlinear hyperbolic systems; Hamiltonian regularisation; energy conservation.

## 1. INTRODUCTION AND MAIN RESULTS

Clamond and Dutykh [1] have proposed a non-dispersive Hamiltonian regularisation of the Saint-Venant (rSV) system with a constant bottom; this regularisation has been mathematically studied in [11, 12]. Inspired by [1], similar regularisations have been proposed for the inviscid Burgers equation [8], the scalar conservation laws [6] and the barotropic Euler system [7]. A regularisation of the Saint-Venant equations with uneven bottom (rSVub) has also been proposed by Clamond et al. [2]. The latter equations, for the conservation of mass and momentum, can be written in the conservative form

$$h_t + [h u]_x = 0, \quad (1a)$$

$$[h u]_t + [h u^2 + \frac{1}{2} g h^2 + \epsilon \mathcal{R}]_x = \epsilon g h^2 \eta_x d_{xx} + g h d_x, \quad (1b)$$

$$\mathcal{R} \stackrel{\text{def}}{=} 2 h^3 u_x^2 - h^3 [u_t + u u_x + g \eta_x]_x - \frac{1}{2} g h^2 (\eta_x^2 + 2 \eta_x d_x). \quad (1c)$$

Here,  $u = u(t, x)$  is the depth-averaged horizontal velocity,  $h = h(t, x) \stackrel{\text{def}}{=} \eta(t, x) + d(t, x)$  denotes the total water depth,  $\eta$  being the surface elevation from rest and  $d$  being the water depth for the unperturbed free surface. We can assume, without losing generality via a change of frame of reference, that the spacial average of the depth  $\bar{d}$  is constant in time. In that case, the gravity acceleration  $g = g(t)$  may be a function of time. Introducing the Sturm–Liouville operator

$$\mathcal{L}_h \stackrel{\text{def}}{=} h - \epsilon \partial_x h^3 \partial_x, \quad (2)$$

if  $h > 0$ , the operator  $\mathcal{L}_h$  is invertible, then the system (1) can be written on the form

$$h_t + [h u]_x = 0, \quad (3a)$$

$$u_t + u u_x + g \eta_x = \epsilon g \mathcal{L}_h^{-1} \{h^2 \eta_x d_{xx}\} - \epsilon \mathcal{L}_h^{-1} \partial_x \{2 h^3 u_x^2 - \frac{1}{2} g h^2 (\eta_x^2 + 2 \eta_x d_x)\}. \quad (3b)$$

Smooth solutions of (3) satisfy an equation for the conservation of energy

$$\begin{aligned} & \left[ \frac{1}{2} h u^2 + \frac{1}{2} \epsilon h^3 u_x^2 + \frac{1}{2} g \eta^2 + \frac{1}{2} \epsilon g h^2 \eta_x^2 \right]_t \\ & + \left[ \left( \frac{1}{2} h u^2 + g h \eta + \frac{1}{2} \epsilon h^3 u_x^2 + \frac{1}{2} \epsilon g h^2 \eta_x^2 + \epsilon \mathcal{R} \right) u + \epsilon g h^3 \eta_x u_x \right]_x \\ & = \frac{1}{2} \dot{g} (\eta^2 + \epsilon h^2 \eta_x^2) - g \eta d_t - \epsilon g h^2 \eta_x d_{xt}. \end{aligned} \quad (4)$$

Note that, injecting (3b) in (1c), one obtains the alternative definition of  $\mathcal{R}$

$$\mathcal{R} = \left( 1 + \epsilon h^3 \partial_x \mathcal{L}_h^{-1} \partial_x \right) \left\{ 2 h^3 u_x^2 - \frac{1}{2} g h^2 (\eta_x^2 + 2 \eta_x d_x) \right\} - \epsilon h^3 \partial_x \mathcal{L}_h^{-1} \left\{ g h^2 \eta_x d_{xx} \right\}.$$

The rSV and rSVub equations can be compared with the Serre–Green–Naghdi and the two-component Camassa–Holm equations. The local well-posedness of those equations have been studied in the literature (see, e.g., [9, 5]; see also [3] for higher-order Camassa–Holm equations). Liu et al. [11] have proved the local well-posedness of the rSV equations introduced in [1] for constant depth. Liu et al. [11] have constructed some small initial data, such that the corresponding solutions blow-up in finite time. The goal of the present note is to prove the local (in time) well-posedness of the rSVub equations. To this aim, we prove first the local (in time) well-posedness of a general  $2 \times 2$  symmetrisable hyperbolic system. Then, using some estimates of the operator  $\mathcal{L}_h^{-1}$ , we prove that the system (3) is locally well-posed in  $H^s$  for any real number  $s \geq 2$ . We also prove that if the  $L^\infty$ -norm of the first derivatives remain bounded, then the singularities cannot appear in finite time.

In order to state the main results of this note, let  $d$  be a smooth function of  $t$  and  $x$  and

$$h \stackrel{\text{def}}{=} \eta + d, \quad \bar{d} \stackrel{\text{def}}{=} \lim_{|x| \rightarrow +\infty} d(t, x) > 0, \quad \text{and} \quad \inf_{(t,x) \in \mathbb{R}^+ \times \mathbb{R}} d(t, x) > 0, \quad (5)$$

then

**Theorem 1.** *Let  $\tilde{m} \geq s \geq 2$ ,  $0 < g \in C^1([0, +\infty[)$ ,  $d - \bar{d} \in C([0, +\infty], H^{s+1}) \cap C^1([0, +\infty], H^s)$  and let  $W_0 = (\eta_0, u_0)^\top \in H^s$  satisfying  $\inf_{x \in \mathbb{R}} h_0(x) \geq h^* > 0$ , then there exist  $T > 0$  and a unique solution  $W = (\eta, u) \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  of (3) satisfying the non-zero depth condition  $\inf_{(t,x) \in [0, T] \times \mathbb{R}} h(t, x) > 0$ . Moreover, if the maximal time of existence  $T_{max} < +\infty$ , then*

$$\lim_{t \rightarrow T_{max}} \|W\|_{H^s} = +\infty \quad \text{or} \quad \inf_{(t,x) \in [0, T_{max}] \times \mathbb{R}} h(t, x) = 0. \quad (6)$$

Using the energy equation (4) and some estimates, the blow-up criteria (6) can be improved.

**Theorem 2.** *For any interval  $[0, T] \subset [0, T_{max}[$  ( $T_{max}$  is the life span of the smooth solution), there exists  $C > 0$ , such that  $\forall t \in [0, T]$  we have*

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \left[ \frac{1}{2} h u^2 + \frac{1}{2} \epsilon h^3 u_x^2 + \frac{1}{2} g \eta^2 + \frac{1}{2} \epsilon g h^2 \eta_x^2 \right] dx \leq C. \quad (7)$$

Moreover, if  $T_{max} < +\infty$ , then

$$\lim_{t \rightarrow T_{max}} \|W_x\|_{L^\infty} = +\infty. \quad (8)$$

Section 2 is devoted to prove the local well-posedness of a general  $2 \times 2$  system. The proofs of Theorems 1 and 2 are given in Section 3.

## 2. LOCAL WELL-POSEDNESS OF A GENERAL $2 \times 2$ SYSTEM

We prove here the local well-posedness of a class of systems with non-local operators in the  $H^s$  space with  $s > 3/2$ . Let  $d$  be a smooth function such that (5) holds. Let also  $N \geq 1$  be a natural number and  $G \stackrel{\text{def}}{=} (g_1, \dots, g_N)$  be a smooth function of  $t$  and  $x$ , possibly depending on  $d$ , such that

$$g_\infty(t) \stackrel{\text{def}}{=} g_1(t, \infty) = \lim_{|x| \rightarrow \infty} g_1(t, x) > 0 \quad \text{and} \quad g_{\text{inf}} \stackrel{\text{def}}{=} \inf_{(t,x) \in \mathbb{R}^+ \times \mathbb{R}} g_1(t, x) > 0. \quad (9)$$

Let  $f(d, h)$  be a positive function and let  $f_1, f_2$  be functions of  $d, h, u, \eta_x, u_x$  and  $G$ . Let also  $a, b, c, f_3, f_4$  be functions of  $d, h, u$  and  $G$ . We consider the symmetrisable hyperbolic system

$$\eta_t + a(d, h, u, G) \eta_x + b(d, h, u, G) u_x = \mathfrak{A}_1 f_1 + \mathfrak{A}_3 f_3, \quad (10a)$$

$$u_t + g_1 f(d, h) b(d, h, u, G) \eta_x + c(d, h, u, G) u_x = \mathfrak{A}_2 f_2 + \mathfrak{A}_4 f_4, \quad (10b)$$

where the  $\mathfrak{A}_j$  are linear operators depending on  $h$  and  $u$ . In order to obtain the well-posedness of the system (10) in  $H^s$  with  $s > 3/2$ , we define  $W \stackrel{\text{def}}{=} (\eta, u)^T$ ,  $G_0 \stackrel{\text{def}}{=} (g_\infty, 0, \dots, 0)$  and

$$B(W) \stackrel{\text{def}}{=} \begin{pmatrix} a(d, h, u, G) & b(d, h, u, G) \\ g_1 f(d, h) b(d, h, u, G) & c(d, h, u, G) \end{pmatrix},$$

$$F(W) \stackrel{\text{def}}{=} \begin{pmatrix} \mathfrak{A}_1 f_1(d, h, u, h_x, u_x, G) + \mathfrak{A}_3 f_3(d, h, u, G) \\ \mathfrak{A}_2 f_2(d, h, u, h_x, u_x, G) + \mathfrak{A}_4 f_4(d, h, u, G) \end{pmatrix},$$

the system (10) can be written as

$$W_t + B(W) W_x = F(W), \quad W(0, x) = W_0(x). \quad (11)$$

We assume that:

(A1) For  $s \leq \tilde{m} \in \mathbb{N}$ , we have

- $d - \bar{d}, g_1 - g_\infty, g_2, g_3, \dots, g_N \in C(\mathbb{R}^+, H^s)$  and  $d - \bar{d}, g_1 - g_\infty \in C^1(\mathbb{R}^+, H^{s-1})$ ;
- $f \in C^{\tilde{m}+2}([0, +\infty[^2])$  and for all  $h_1, h_2 > 0$  we have  $f(h_1, h_2) > 0$ ;
- $f_1, f_2 \in C^{\tilde{m}+2}([0, +\infty[^2 \times \mathbb{R}^3 \times ]0, +\infty[ \times \mathbb{R}^{N-1})$ ;
- $a, b, c, f_3, f_4 \in C^{\tilde{m}+2}([0, +\infty[^2 \times \mathbb{R} \times ]0, +\infty[ \times \mathbb{R}^{N-1})$ ;
- $f_1(\bar{d}, \bar{d}, 0, 0, 0, G_0) = f_2(\bar{d}, \bar{d}, 0, 0, 0, G_0) = f_3(\bar{d}, \bar{d}, 0, G_0) = f_4(\bar{d}, \bar{d}, 0, G_0) = 0$ .

(A2) For all  $r \in [s-1, s]$ , if  $\phi \in H^r$  and  $\psi \in H^{r-1}$ , then

$$\|\mathfrak{A}_1 \psi\|_{H^r} + \|\mathfrak{A}_2 \psi\|_{H^r} \leq C(s, r, d, \|W\|_{H^r}) \|\psi\|_{H^{r-1}},$$

$$\|\mathfrak{A}_3 \phi\|_{H^r} + \|\mathfrak{A}_4 \phi\|_{H^r} \leq C(s, r, d, \|W\|_{H^r}) \|\phi\|_{H^r}.$$

(A3) If  $\phi, W, \tilde{W} \in H^s$  and  $\psi \in H^{s-1}$ , then

$$\|(\mathfrak{A}_1(W) - \mathfrak{A}_1(\tilde{W})) \psi\|_{H^{s-1}} + \|(\mathfrak{A}_2(W) - \mathfrak{A}_2(\tilde{W})) \psi\|_{H^{s-1}} \leq C \|W - \tilde{W}\|_{H^{s-1}},$$

$$\|(\mathfrak{A}_3(W) - \mathfrak{A}_3(\tilde{W})) \phi\|_{H^{s-1}} + \|(\mathfrak{A}_4(W) - \mathfrak{A}_4(\tilde{W})) \phi\|_{H^{s-1}} \leq C \|W - \tilde{W}\|_{H^{s-1}},$$

$$\text{where } C = C\left(s, d, \|W\|_{H^s}, \|\tilde{W}\|_{H^s}, \|\phi\|_{H^s}, \|\psi\|_{H^{s-1}}\right).$$

Note that if  $h$  is far from zero (i.e.,  $\inf h > 0$ ), then  $g_1 f(d, h)$  is positive and far from zero. Then, the system (10) is symmetrisable and hyperbolic. The main result of this section is the following theorem:

**Theorem 3.** For  $s > 3/2$  and under the assumptions (A1), (A2) and (A3), if  $W_0 \in H^s$  satisfy the non-emptiness condition

$$\inf_{x \in \mathbb{R}} h_0(x) = \inf_{x \in \mathbb{R}} (\eta_0(x) + d(0, x)) \geq h^* > 0, \quad (12)$$

then there exist  $T > 0$  and a unique solution  $W \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  of the system (11). Moreover, if the maximal existence time  $T_{max} < +\infty$ , then

$$\inf_{(t,x) \in [0, T_{max}] \times \mathbb{R}} h(t, x) = 0 \quad \text{or} \quad \lim_{t \rightarrow T_{max}} \|W\|_{H^s} = +\infty. \quad (13)$$

**Remarks:** (i) Theorem 3 holds also for periodic domains; (ii) The right-hand side of (11) can be replaced by a finite sum on the form

$$F(W) = \left( \mathfrak{A}_1 f_1 + \mathfrak{A}_3 f_3 \right) + \left( \mathfrak{B}_1 k_1 + \mathfrak{B}_3 k_3 \right) + \dots, \quad (14)$$

where the additional terms satisfy also the conditions (A1), (A2) and (A3); (iii) Under some additional assumptions, the blow-up criteria (13) can be improved (see Theorem 2, for example); (iv) If for some  $2 \leq i \leq N$ , the function  $g_i$  appears only on  $f_1$  and  $f_2$ , then, due to (A2), the assumption  $g_i \in C(\mathbb{R}^+, H^s)$  can be replaced by  $g_i \in C(\mathbb{R}^+, H^{s-1})$ .

In order to prove the local well-posedness of (11), we consider

$$\partial_t W^{n+1} + B(W^n) \partial_x W^{n+1} = F(W^n), \quad W^n(0, x) = (\eta_0(x), u_0(x))^T, \quad (15)$$

where  $n \geq 0$  and  $W^0(t, x) = (\eta_0(x), u_0(x))^T$ . The idea of the proof is to solve the linear system (15), then, taking the limit  $n \rightarrow \infty$ , we obtain a solution of (11). Note that we have assumed that  $g_1$  and  $f$  are positive, so  $g_1 f > 0$ , then the system (15) is hyperbolic; it is an important point to solve each iteration in (15). Note that a symmetriser of the matrix  $B(W)$  is  $A(W) \stackrel{\text{def}}{=} \begin{pmatrix} g_1 f(d, h) & 0 \\ 0 & 1 \end{pmatrix}$ .

Let  $(\cdot, \cdot)$  be the scalar product in  $L^2$  and let the energy of (15) be defined as

$$E^{n+1}(t) \stackrel{\text{def}}{=} (\Lambda^s W^{n+1}, A^n \Lambda^s W^{n+1}) \quad \forall t \geq 0.$$

If  $g_1 f$  is bounded and far from 0, then  $E^n(t)$  is equivalent to  $\|W^n\|_{H^s}$ . In order to prove Theorem 3, the following results are needed.

**Theorem 4.** Let  $s > 3/2$ ,  $h^* > 0$  and  $R > 0$ , then there exist  $K, T > 0$  such that: if the initial data  $(\eta_0, h_0) \in H^s$  satisfy

$$\inf_{x \in \mathbb{R}} h_0(x) \geq 2h^*, \quad E^n(0) < R, \quad (16)$$

and  $W^n \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$ , satisfying for all  $t \in [0, T]$

$$h^n \geq h^*, \quad \|(W^n)_t\|_{H^{s-1}} \leq K, \quad E^n(t) \leq R, \quad (17)$$

then there exists a unique  $W^{n+1} \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  a solution of (15) such that

$$h^{n+1} \geq h^*, \quad \|(W^{n+1})_t\|_{H^{s-1}} \leq K, \quad E^{n+1}(t) \leq R. \quad (18)$$

The proof of Theorem 4 is classic (it can be done following Guelmame et al. [7], Israwi [9], Liu et al. [11] and using the following lemmas).

Let  $\Lambda$  be defined such that  $\widehat{\Lambda f} = (1 + \xi^2)^{\frac{1}{2}} \widehat{f}$ , and let  $[A, B] \stackrel{\text{def}}{=} AB - BA$  be the commutator of the operators  $A$  and  $B$ . We have the following lemma.

**Lemma 1.** (Kato and Ponce [10]) If  $r \geq 0$ , then

$$\|fg\|_{H^r} \lesssim \|f\|_{L^\infty} \|g\|_{H^r} + \|f\|_{H^r} \|g\|_{L^\infty}, \quad (19)$$

$$\|[\Lambda^r, f]g\|_{L^2} \lesssim \|f_x\|_{L^\infty} \|g\|_{H^{r-1}} + \|f\|_{H^r} \|g\|_{L^\infty}. \quad (20)$$

**Lemma 2.** Let  $k \in \mathbb{N}^*$ ,  $F \in C^{m+2}(\mathbb{R}^k)$  with  $F(0, \dots, 0) = 0$  and  $0 \leq s \leq m$ , then there exists a continuous function  $\tilde{F}$ , such that for all  $f = (f_1, \dots, f_2) \in H^s \cap W^{1,\infty}$  we have

$$\|F(f)\|_{H^s} \leq \tilde{F}(\|f\|_{W^{1,\infty}}) \|f\|_{H^s}. \quad (21)$$

*Proof.* The case  $k = 1$  has been proved in [4]. Here, we prove the inequality (21) by induction (on  $s$ ). Note that

$$\begin{aligned} F(f_1, \dots, f_k) &= F(0, f_2, \dots, f_k) + \int_0^{f_1} F_{f_1}(g_1, f_2, \dots, f_k) dg_1 \\ &= F(0, 0, f_3, \dots, f_k) + \int_0^{f_1} F_{f_1}(g_1, f_2, \dots, f_k) dg_1 + \int_0^{f_2} F_{f_2}(0, g_2, f_3, \dots, f_k) dg_2 + \dots \\ &= \int_0^{f_1} F_{f_1}(g_1, f_2, \dots, f_k) dg_1 + \dots + \int_0^{f_k} F_{f_k}(0, \dots, 0, g_k) dg_k. \end{aligned}$$

This implies that

$$\|F(f_1, \dots, f_k)\|_{L^2} \lesssim \|f\|_{L^2}, \quad (22)$$

which is (21) for  $s = 0$ . For  $s \in ]0, 1[$ , let

$$\begin{aligned} &|F(f_1(x+y), \dots, f_k(x+y)) - F(f_1(x), \dots, f_k(x))| \\ &\leq |F(f_1(x+y), \dots, f_k(x+y)) - F(f_1(x), f_2(x+y), \dots, f_k(x+y))| \\ &\quad + |F(f_1(x), f_2(x+y), \dots, f_k(x+y)) - F(f_1(x), f_2(x), f_3(x+y), \dots, f_k(x+y))| \\ &\quad + \dots + |F(f_1(x), \dots, f_{k-1}(x), f_k(x+y)) - F(f_1(x), \dots, f_k(x))| \\ &\leq \sum_{i=1}^k |f_i(x+y) - f_i(x)| \|F_{f_i}\|_{L^\infty}. \end{aligned}$$

The last inequality, with the definition  $H^s \stackrel{\text{def}}{=} \left\{ f \in L^2, \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x+y) - f(x)|^2}{|y|^{1+2s}} dx dy < +\infty \right\}$ , implies (21). For  $s \geq 1$ , the proof is done by induction. Using (19) and (22), we obtain  $\|F(f)\|_{H^s} \lesssim \left\| \sum_{i=1}^k F_{f_i}(f) \partial_x f_i \right\|_{H^{s-1}} + \|F(f)\|_{L^2} \lesssim \|f\|_{H^s} + \sum_{i=1}^k \|F_{f_i}(f)\|_{H^{s-1}}$ . Using the induction and the last inequality, we obtain (21) for all  $s \geq 0$ .  $\square$

**Proof of Theorem 3.** Using Theorem 4, one obtains that  $(W^n)$  is uniformly bounded in  $C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  and satisfies  $h^n \geq h^*$ . Defining

$$\tilde{E}^{n+1}(t) \stackrel{\text{def}}{=} (\Lambda^{s-1}(W^{n+1} - W^n), A^n \Lambda^{s-1}(W^{n+1} - W^n)), \quad (23)$$

and using (15), one obtains

$$\begin{aligned} \tilde{E}_t^{n+1} &= 2 (\Lambda^{s-1}(F^n - F^{n-1} + (B^{n-1} - B^n)W_x^n), A^n \Lambda^{s-1}(W^{n+1} - W^n)) \\ &\quad - 2 ([\Lambda^{s-1}, B^n](W^{n+1} - W^n)_x, A^n \Lambda^{s-1}(W^{n+1} - W^n)) \\ &\quad + (\Lambda^{s-1}((A^n B^n)_x(W^{n+1} - W^n)), \Lambda^{s-1}(W^{n+1} - W^n)) \\ &\quad + (\Lambda^{s-1}(W^{n+1} - W^n), (A^n)_t \Lambda^{s-1}(W^{n+1} - W^n)). \end{aligned} \quad (24)$$

With (A2), (A3), (19) and (21), one obtains

$$\|F^n - F^{n-1}\|_{H^{s-1}} + \|(B^{n-1} - B^n)\partial_x W^n\|_{H^{s-1}} \lesssim \|W^n - W^{n-1}\|_{H^{s-1}} \lesssim \sqrt{\tilde{E}^n}, \quad (25)$$

and using (20) and (21), we obtain

$$\|[\Lambda^{s-1}, B^n](W^{n+1} - W^n)_x\|_{L^2} \lesssim \|W^{n+1} - W^n\|_{H^{s-1}} \lesssim \sqrt{\tilde{E}^{n+1}}. \quad (26)$$

From (19) and (21), it follows that

$$\|(A^n B^n)_x(W^{n+1} - W^n)\|_{H^{s-1}} \lesssim \|W^{n+1} - W^n\|_{H^{s-1}} \lesssim \sqrt{\tilde{E}^{n+1}}. \quad (27)$$

Combining the estimates above, we obtain that  $\tilde{E}_t^{n+1} \lesssim \tilde{E}^{n+1} + \tilde{E}^n$ , and using that  $\tilde{E}^n(0) = 0$ , we obtain  $\tilde{E}^{n+1} \leq (e^{Ct} - 1)\tilde{E}^n$ . Taking  $T > 0$  small enough, it follows that

$$\|W^{n+1} - W^n\|_{H^{s-1}} \lesssim \tilde{E}^{n+1} \leq \frac{1}{2}\tilde{E}^n \leq \frac{1}{2^n}\tilde{E}^1. \quad (28)$$

Finally, taking the limit  $n \rightarrow \infty$  in the weak formulation of (15) and using (A3), we obtain a solutions of the system (10). This completes the proof of Theorem 3.  $\square$

### 3. PROOF OF THEOREMS 1 AND 2

The system (3) is written in the form (10) by replacing the right-hand side of (10), as in (14), taking  $N = 4$  and  $G(t, x) = (g, d_x, d_{xx}, d_t)$ ,  $a(d, h, u, g, d_x, d_{xx}, d_t) = c(d, h, u, g, d_x, d_{xx}, d_t) = u$ ,  $b(d, h, u, g, d_x, d_{xx}, d_t) = h$ ,  $f(d, h) = h^{-1}$ ,  $f_1 = f_4 = k_1 = k_3 = k_4 = 0$ ,  $f_2(d, h, u, h_x, u_x, g, d_x, d_{xx}, d_t) = 2h^3 u_x^2 - \frac{1}{2}gh^2(\eta_x^2 + 2\eta_x d_x)$ ,  $f_3(d, h, u, g, d_x, d_{xx}, d_t) = -d_t - u d_x$ ,  $k_2(d, h, u, h_x, u_x, g, d_x, d_{xx}, d_t) = gh^2 \eta_x d_{xx}$ ,  $\mathfrak{A}_1 = \mathfrak{A}_4 = \mathfrak{B}_1 = \mathfrak{B}_3 = \mathfrak{B}_4 = 0$ ,  $\mathfrak{A}_2 = -\varepsilon \mathcal{L}_h^{-1} \partial_x$  and  $\mathfrak{A}_3 = 1$ ,  $\mathfrak{B}_2 = \varepsilon \mathcal{L}_h^{-1}$ . Then, in order to prove Theorem 1, the following lemma is needed:

**Lemma 3.** (*Liu et al. [11]*) *Let  $0 < h_{\inf} \leq h \in W^{1,\infty}$ , then the operator  $\mathcal{L}_h$  is an isomorphism from  $H^2$  to  $L^2$  and if  $0 \leq s \leq \tilde{m} \in \mathbb{N}$ , then*

$$\|\mathcal{L}_h^{-1} \psi\|_{H^{s+1}} + \|\mathcal{L}_h^{-1} \partial_x \psi\|_{H^{s+1}} \leq C \|\psi\|_{H^s} (1 + \|h - \bar{d}\|_{H^s}), \quad (29)$$

where  $C$  depends on  $s, \varepsilon, h_{\inf}, \|h - \bar{d}\|_{W^{1,\infty}}$  and not on  $\|h - \bar{d}\|_{H^s}$ .

**Proof of Theorem 1.** In order to prove Theorem 1, it suffices to verify (A1)–(A3). The assumption (A1) is obviously satisfied and (A2) follows from Lemma 3. In order to prove (A3), let  $W, \tilde{W}, \psi \in H^s$ . Using Lemma 3 and (19), we obtain

$$\begin{aligned} & \left\| (\mathcal{L}_h^{-1} - \mathcal{L}_{\tilde{h}}^{-1}) \psi \right\|_{H^{s-1}} = \left\| \mathcal{L}_h^{-1} (\mathcal{L}_{\tilde{h}} - \mathcal{L}_h) \mathcal{L}_{\tilde{h}}^{-1} \psi \right\|_{H^{s-1}} \\ & \lesssim \left\| (\mathcal{L}_{\tilde{h}} - \mathcal{L}_h) \mathcal{L}_{\tilde{h}}^{-1} \psi \right\|_{H^{s-2}} \lesssim \|h - \tilde{h}\|_{H^{s-1}} \leq \|W - \tilde{W}\|_{H^{s-1}}. \end{aligned}$$

where the constants depend on  $s, d, \|W\|_{H^s}, \|\tilde{W}\|_{H^s}, \|\psi\|_{H^{s-1}}$ . The same proof can be used with the operator  $\mathfrak{A}_2$ .  $\square$

**Proof of Theorem 2.** Using the characteristics  $\chi(0, x) = x$  and  $\chi_t(t, x) = u(t, \chi(t, x))$ , the conservation of the mass (3a) becomes

$$dh/dt + u_x h = 0, \quad \implies \quad h_0(x) e^{-t\|u_x\|_{L^\infty}} \leq h(t, x) \leq h_0(x) e^{t\|u_x\|_{L^\infty}}. \quad (30)$$

The energy equation (4) implies that

$$\mathcal{E}'(t) \leq (|\dot{g}|/g + 1) \mathcal{E}(t) + \frac{1}{2}g \int_{\mathbb{R}} (d_t^2 + \varepsilon h^2 d_{xt}^2) dx, \quad (31)$$

since  $h$  is bounded, the inequality (7) follows by Gronwall lemma.

In order to prove the blow-up criterion, we first suppose that  $\|W_x\|_{L^\infty}$  is bounded and we show that the scenario (6) is impossible. The equation (30) implies that  $h$  is bounded and far from 0. Using  $\|W\|_{L^\infty} \leq \|W\|_{H^1} \lesssim \mathcal{E}(t)$ , one obtains that  $\|W\|_{W^{1,\infty}}$  is bounded on any interval  $[0, T]$ . Using Lemma 3 and doing some classical energy estimates (see [7, 9, 11]), we can prove that  $\|W\|_{H^s}$  is also bounded. This ends the proof of Theorem 2.  $\square$

#### REFERENCES

- [1] D. Clamond and D. Dutykh. Non-dispersive conservative regularisation of nonlinear shallow water (and isentropic Euler) equations. *Comm. Nonlin. Sci. Numer. Simul.*, 55:237–247, 2018.
- [2] D. Clamond, D. Dutykh, and D. Mitsotakis. Hamiltonian regularisation of shallow water equations with uneven bottom. *Journal of Physics A: Mathematical and Theoretical*, 52(42):42LT01, 2019.
- [3] G. M. Coclite, H. Holden and K. H. Karlsen. Well-posedness of higher-order Camassa–Holm equations. *Journal of Differential Equations.*, 246(3):929–963, 2009.
- [4] A. Constantin and L. Molinet. The initial value problem for a generalized Boussinesq equation. *Differential and Integral equations.*, 15(9):1061–1072, 2002.
- [5] C. Guan, K. H. Karlsen and Z. Yin. Well-posedness and blow-up phenomena for a modified two-component Camassa–Holm equation. *Contemp. Math.* 526:199–220, 2010.
- [6] B. Guelmame. On a Hamiltonian regularization of scalar conservation laws. *hal-02512810*, 2020.
- [7] B. Guelmame, D. Clamond, and S. Junca. Hamiltonian regularisation of the unidimensional barotropic Euler equations. *Preprint*, 2020.
- [8] B. Guelmame, S. Junca, D. Clamond, and R. Pego. Global weak solutions of a Hamiltonian regularised Burgers equation. *hal-02478872*, 2020.
- [9] S. Israwi. Large time existence for 1D Green–Naghdi equations. *Nonlinear Analysis*, 74(1):81–93, 2011.
- [10] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier–Stokes equations. *Communications on Pure and Applied Mathematics.*, 41(7):891–907, 1988.
- [11] J.-L. Liu, R. L. Pego, and Y. Pu. Well-posedness and derivative blow-up for a dispersionless regularized shallow water system. *Nonlinearity.*, 32(11):4346, 2019.
- [12] Y. Pu, R. L. Pego, D. Dutykh, and D. Clamond. Weakly singular shock profiles for a non-dispersive regularization of shallow-water equations. *Comm. Math. Sci.*, 16(5):1361–1378, 2018.

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