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Upper approximating probabilities of convergence in probabilistic coherence spaces

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Abstract

We develop a theory of probabilistic coherence spaces equipped with an additional extensional structure and apply it to approximating probability of convergence of ground type programs of probabilistic PCF whose free variables are of ground types. To this end we define an adapted version of Krivine Machine which computes polynomial approximations of the semantics of these programs in the model. These polynomials provide approximations from below and from above of probabilities of convergence; this is made possible by extending the language with an error symbol which is extensionally maximal in the model.

Introduction

Various settings are now available for the denotational interpretations of probabilistic programming languages.

- *Game* based models, first proposed in [7] and further developed by various authors (see [3] for an example of this approach). From their deterministic ancestors they typically inherit good definability features.
- Models based on Scott continuous functions on domains endowed with additional probability related structures. Among these models we can mention *Kegelspitzen* [14] (domains equipped with an algebraic convex structure) and *ω -quasi Borel spaces* [16] (domains equipped with a generalized notion of measurability). In contrast with the former, the latter uses an adapted probabilistic powerdomain construction.
- Models based on (a generalization of) Berry stable functions. The first category of this kind was that of *probabilistic coherence spaces* (PCSs) and power series with non-negative coefficients (the Kleisli category of the model of Linear Logic developed in [6]) for which could be proved adequacy and full abstraction with respect to a probabilistic version of PCF [10]. This setting was extended to “continuous data types” (such as \mathbb{R}) by substituting PCSs with *positive cones* and power series with functions featuring an hereditary monotonicity called *stability* [11].

Just as games, probabilistic coherence spaces interpret types using simply defined combinatorial devices called *webs* which are countable sets very similar to game arenas, and even simpler since the order of basic actions is not taken into account. A closed program of type σ is interpreted as a map from the web associated with σ to the non-negative real half-line $\mathbb{R}_{\geq 0}$. Mainly because of the absence of explicit sequencing information in web elements, these functions are not always probability sub-distributions. For instance, a closed term M of type $\iota \Rightarrow 1$ (integer to unit type) is interpreted as a mapping from $\mathcal{M}_{\text{fin}}(\mathbb{N})$ (finite multisets of integers) to $\mathbb{R}_{\geq 0}$, rather seen as an indexed family $S = (\alpha_\mu)_{\mu \in \mathcal{M}_{\text{fin}}(\mathbb{N})}$. Given $u \in \mathbb{R}_{\geq 0}^{\mathbb{N}}$ and

$\mu \in \mathcal{M}_{\text{fin}}(\mathbb{N})$ one sets¹ $u^\mu = \prod_{n \in \mathbb{N}} u_n^{\mu(n)}$ (where $\mu(n)$ is the multiplicity of n in μ) and then we can see S as the function $\widehat{S} : \text{PN} \rightarrow [0, 1]$ defined by $\widehat{S}(u) = \sum_{\mu \in \mathcal{M}_{\text{fin}}(\mathbb{N})} \alpha_\mu u^\mu$ where PN is the set of all $u \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ such that $\sum_{n \in \mathbb{N}} u_n \leq 1$ (subprobability distributions on the natural numbers). It is often convenient to use \widehat{S} for describing S (no information is lost in doing so), what we do now.

Since **Pcoh** (or rather the associated Kleisli CCC **Pcoh_!**) is a model of probabilistic PCF, one has $\widehat{S}(u) \in [0, 1]$ and one can prove that $\widehat{S}(u)$ is exactly the probability that the execution of M converges if we apply it to a random integer distributed along u (such a random integer has also a probability $1 - \sum_{n=0}^{\infty} u_n$ to diverge): we call this property *adequacy* in the sequel. We can consider S as a power series or analytic function which can be infinite “in width” and “in depth”.

- *In width* because the μ 's such that $\alpha_\mu \neq 0$ can contain infinitely many different integers. A typical S which is infinite in width is S_1 such that $\widehat{S}_1(u) = \sum_{n=0}^{\infty} u_n$ (the relevant μ 's are the singleton multisets $[n]$, for all $n \in \mathbb{N}$).
- *In depth* in the sense that a given component of u can be used an unbounded number of time. A typical S which is infinite in depth is S_2 such that $\widehat{S}_2(u) = \sum_{n \in \mathbb{N}} \frac{1}{2^{n+1}} u_0^n$.

More precisely, S is finite in depth if the expression $\widehat{S}(u)$ has finitely non-zero terms when u has a finite support. For instance S_3 such that $\widehat{S}_3 = \sum_{n=0}^{\infty} u_n u_0^n$ is infinite in width but not in depth, in spite of the fact that it has not a bounded degree in u_0 . Notice that this notion of “finiteness in depth” is at the core of the concept of finiteness space introduced in [9]. When S arises as the semantics of a term M of probabilistic PCF (see [10] and Section 3), it is generally not of finite depth because M can contain subterms $\text{fix}(P)$ representing recursive definitions: remember that such a term “reduces” to $(P)\text{fix}(P)$.

Given a closed term M of type $\iota \Rightarrow 1$ (or more generally a closed term M of type $\iota^k \Rightarrow 1$ but we keep $k = 1$ in this introduction for readability) we are interested in approximating effectively the probability that $(M)N$ converges when N is a closed term of type ι which represents a sub-distribution of probabilities u , that is in approximating $\widehat{S}(u)$. To this end, we try to find approximations of S itself: then it will be enough to apply these approximations to the subdistributions u 's we want to consider. Finding approximations from below is not very difficult: it suffices to consider terms M_k obtained from M by unfolding k times all fixpoint operators it contains, that is replacing hereditarily in M each subterm of shape $\text{fix}(P)$ with the term $(P)(P) \cdots (P)\Omega$ (k occurrences of P and Ω is a constant which represents divergence and has semantics 0). The powerseries S_k interpreting such an M_k in **Pcoh** is then of finite depth² and can be computed (e.g. as a lazy data structure) by means of an adapted version \mathcal{K} of the Krivine Machine, see Section 4. This power series S_k can still be an infinite object but since it is of finite depth, by choosing a finite subset J of \mathbb{N} , it is possible to extract effectively from it a finite polynomial S_k^J such that $\widehat{S}_k^J(u) = \widehat{S}_k(u)$ when the support of u is a subset of J (this extraction can be integrated in the Krivine Machine itself, or applied afterwards to the lazy infinite data structure it yields). The sequence S_k is monotone in **Pcoh** and has S as lub hence $\widehat{S}_k(u)$ is a monotone sequence in $[0, 1]$ which converges to $\widehat{S}(u)$. But there is no algorithm which, for any closed term P of type 1 and any $p \in \mathbb{N}$ yields a k such that the semantics of P_k is 2^{-p} -close to that of P : such an algorithm would make deciding almost-sure termination Π_1^0 whereas we know that this problem is Π_2^0 -complete, see [13]. So we cannot systematically know how good the estimate S_k is.

Nevertheless it would be nice to be able to approximate $\widehat{S}(u)$ from above by some $\widehat{S}^k(u)$ where S^k is again a finite depth power series extracted from similar “finite” approximations M^k of M : if we are lucky enough to find k such that $\widehat{S}^k(u) - \widehat{S}(u) \leq \varepsilon$, we are sure that $\widehat{S}^k(u)$ is an ε -approximation of $\widehat{S}(u)$. This is exactly what we do in this paper, developing first a denotational account of these approximations.

The PCS denotational account of approximations from below is based on the fact that any PCS has a least element 0 that we use to interpret Ω . For approximating from above we would need a maximal element

¹In other words, we consider μ as a multiexponent for “variables” indexed by \mathbb{N} and u is considered as a valuation for these variable.

²Because M_k contains no fixpoints. This tree is a kind of PCF Böhm tree very similar to those considered in game semantics, e.g. [12].

that we could use to interpret a constant \mathcal{U} to be added to our PCF: we would then approximate $\text{fix}(P)$ with $(P)(P)\dots(P)\mathcal{U}$. The problem is that, for a PCS X , the associated domain PX (whose order relation is denoted as \leq) has typically no maximal element; the PCS \mathbb{N}_\perp of “flat integers” whose web is \mathbb{N} and PN_\perp is the set of all sub-probability distributions on \mathbb{N} , has no \leq -maximal element since in this PCS $u \leq v$ means $\forall n \in \mathbb{N} u_n \leq v_n$.

So we consider PCSs equipped with an additional (pre)order relation \sqsubseteq for which such a maximal element can exist: an extensional PCS is a tuple $X = (\underline{X}, \sqsubseteq_X, \mathcal{E}_X)$, where \underline{X} is a PCS (the carrier of X) and \sqsubseteq_X is a preorder relation on \mathcal{E}_X , the set of extensional elements of X , which is a subset of $\text{P}\underline{X}$. These objects are a probabilistic analog of Berry’s bidomains [1]. We prove that these objects form again a model of classical linear logic \mathbf{Pcoh}^e whose associated Kleisli category \mathbf{Pcoh}^e has fixpoint operators at all types. The main features of this model are the following.

- At function type, $\mathcal{E}_{X \Rightarrow Y}$ is the set of all $s \in \text{P}(\underline{X} \Rightarrow \underline{Y})$ which are monotone wrt. the extensional preorder, that is $\forall u, v \in \mathcal{E}_X u \sqsubseteq_X v \Rightarrow \widehat{s}(u) \sqsubseteq_Y \widehat{s}(v)$ (where $\widehat{s} : \text{P}\underline{X} \rightarrow \text{P}\underline{Y}$ is the “stable” function associated with s) and, given $s, t \in \mathcal{E}_{X \Rightarrow Y}$, we stipulate that $s \sqsubseteq_{X \Rightarrow Y} t$ if $\forall u \in \mathcal{E}_X \widehat{s}(u) \sqsubseteq_Y \widehat{t}(u)$, that is $\sqsubseteq_{X \Rightarrow Y}$ is the extensional preorder.
- There is an extensional PCS \mathbb{N}_\perp^\top whose web is $\mathbb{N} \cup \{\top\}$ which is an extension of \mathbb{N}_\perp in the sense that³ $\mathbb{N}_\perp^\top \simeq \mathbb{N}_\perp \oplus 1_\top$ and PN_\perp^\top has an \sqsubseteq -maximal element, namely e_\top (the $\mathbb{N} \cup \{\top\}$ -indexed family of scalars which maps $n \in \mathbb{N}$ to 0 and \top to 1).

Accordingly we extend probabilistic PCF from [10] with two new constants Ω, \mathcal{U} of type ι . The operational semantics of this language is defined as a probabilistic rewriting system in the spirit of [10], with new rules for constants Ω and \mathcal{U} which are handled exactly in the same way, as error exceptions. Then we define a syntactic preorder \sqsubseteq on terms such that $\Omega \sqsubseteq M \sqsubseteq \mathcal{U}$ for all term of type ι , and such that $M' \sqsubseteq M \wedge N' \sqsubseteq \text{fix}(M) \Rightarrow (M')N' \sqsubseteq \text{fix}(M)$ and $M \sqsubseteq M' \wedge \text{fix}(M) \sqsubseteq N' \Rightarrow \text{fix}(M) \sqsubseteq (M')N'$. In particular, for any term M we have $M_k \sqsubseteq M \sqsubseteq M^k$ (where M_k and M^k are the “finite” approximations of M obtained by unfolding all fixpoints k times as explained above starting from Ω and \mathcal{U} respectively⁴). We interpret⁵ this language in \mathbf{Pcoh}^e and prove that this interpretation is extensionally monotone: if $\vdash M, N : \sigma$ and $M \sqsubseteq N$ then $\llbracket M \rrbracket \sqsubseteq_{[\sigma]} \llbracket N \rrbracket$ (where $\llbracket M \rrbracket \in \mathcal{E}_{[\sigma]}$ is the interpretation of the term M in the interpretation of its type σ).

We adapt our approximation problem to this extension of PCF, without changing its nature: assuming $x : \iota \vdash M : \iota$ and $u \in \text{PN}_\perp$, approximate from above and below the probability p that $M[N/x]$ reduces to \mathcal{U} , knowing that the probability subdistribution of N is u . To address it, we extend the Krivine Machine \mathcal{K} to handle \mathcal{U} . From a term P without fixpoints and such that $x : \iota \vdash P : \iota$, \mathcal{K} produces a (generally infinite) “Böhm tree” of which we extract a power series S of finite depth which coincides with the denotational interpretation of P in \mathbf{Pcoh}^e , or more precisely with the \top -component of this interpretation. So if $P \sqsubseteq M$ we have $\widehat{S}(u) \leq p$ and if $M \sqsubseteq P$ then $p \leq \widehat{S}(u)$ by the above monotonicity property and adequacy of the semantics⁶. This will hold in particular if $P = M_k$ or $P = M^k$ respectively. Notice that if $J \subseteq \mathbb{N}$ is finite and if the support of u is a subset of J then computing $\widehat{S}(u)$ involves only a finite set of monomials, computable from J .

Notations

If I is a set, we use $\mathcal{M}_{\text{fin}}(I)$ for the set of finite multisets of elements of I , which are functions $\mu : I \rightarrow \mathbb{N}$ such that the set $\text{supp}(\mu) = \{i \in I \mid \mu(i) \neq 0\}$ is finite. We use $[i_1, \dots, i_k]$ for the multiset μ such that $\mu(i_l)$ is the number of indices l such that $i_l = i$. We use \square for the empty multiset and $+$ for the sum of multisets.

³ 1_\top is the PCS $(\{\top\}, \{\top\} \times [0, 1])$.

⁴One can define error terms at all types by simply adding λ -abstractions in front of the ground type Ω and \mathcal{U} .

⁵We omit the proof that the semantics is invariant by reduction and the proof of adequacy as they are simple adaptations of the corresponding proofs in [10].

⁶Which guarantees that p is equal to the interpretation of M applied to u .

We use $\mathbb{R}_{\geq 0}$ for the set of real numbers r such that $r \geq 0$ and we set $\overline{\mathbb{R}_{\geq 0}} = \mathbb{R}_{\geq 0} \cup \{\infty\}$ (the complete half-real line).

If $i \in I$, we use e_i for the element of $(\mathbb{R}_{\geq 0})^I$ such that $(e_i)_j = \delta_{i,j}$ (the Kronecker symbol).

1 Probabilistic coherence spaces (PCS)

For the general theory of PCSs we refer to [6, 10]. We recall briefly the basic definitions for the sake of self-containedness.

Given an at most countable set I and $u, u' \in \overline{\mathbb{R}_{\geq 0}}^I$, we set $\langle u, u' \rangle = \sum_{i \in I} u_i u'_i \in \overline{\mathbb{R}_{\geq 0}}$. Given $\mathcal{P} \subseteq \overline{\mathbb{R}_{\geq 0}}^I$, we define $\mathcal{P}^\perp \subseteq \overline{\mathbb{R}_{\geq 0}}^I$ as

$$\mathcal{P}^\perp = \{u' \in \overline{\mathbb{R}_{\geq 0}}^I \mid \forall u \in \mathcal{P} \langle u, u' \rangle \leq 1\}.$$

Observe that if \mathcal{P} satisfies $\forall a \in I \exists u \in \mathcal{P} u_a > 0$ and $\forall a \in I \exists m \in \mathbb{R}_{\geq 0} \forall u \in \mathcal{P} u_a \leq m$ then $\mathcal{P}^\perp \in (\mathbb{R}_{\geq 0})^I$ and \mathcal{P}^\perp satisfies the same two properties.

A probabilistic pre-coherence space (pre-PCS) is a pair $X = (|X|, \text{PX})$ where $|X|$ is an at most countable set⁷ and $\text{PX} \subseteq \overline{\mathbb{R}_{\geq 0}}^{|X|}$ satisfies $\text{PX}^{\perp\perp} = \text{PX}$. A probabilistic coherence space (PCS) is a pre-PCS X such that $\forall a \in |X| \exists u \in \text{PX} u_a > 0$ and $\forall a \in |X| \exists m \in \mathbb{R}_{\geq 0} \forall u \in \text{PX} u_a \leq m$ or equivalently $\forall a \in |X| \ 0 < \sup_{u \in \text{PX}} u_a < \infty$ so that $\text{PX} \subseteq (\mathbb{R}_{\geq 0})^{|X|}$. We define a "norm" $\| \cdot \|_X : \text{PX} \rightarrow [0, 1]$ by $\|x\|_{\text{PX}} = \inf\{r > 0 \mid x \in r \text{PX}\}$ that we shall use for describing the coproduct of PCSs. Then $X^\perp = (|X|, \text{PX}^\perp)$ is also a PCS and $X^{\perp\perp} = X$.

Equipped with the order relation \leq defined by $u \leq v$ if $\forall a \in |X| u_a \leq v_a$, any PCS X is a complete partial order (all directed lubs exist) with 0 as least element. In general this cpo is not a lattice.

Given $t \in \overline{\mathbb{R}_{\geq 0}}^{I \times J}$ considered as a matrix (where I and J are at most countable sets) and $u \in \overline{\mathbb{R}_{\geq 0}}^I$, we define $t \cdot u \in \overline{\mathbb{R}_{\geq 0}}^J$ by $(t \cdot u)_j = \sum_{i \in I} t_{i,j} u_i$ (usual formula for applying a matrix to a vector), and if $s \in \overline{\mathbb{R}_{\geq 0}}^{J \times K}$ we define the product $st \in \overline{\mathbb{R}_{\geq 0}}^{I \times K}$ of the matrix s and t as usual by $(st)_{i,k} = \sum_{j \in J} t_{i,j} s_{j,k}$. This is an associative operation.

Let X and Y be PCSs, a morphism from X to Y is a matrix $t \in (\mathbb{R}_{\geq 0})^{|X| \times |Y|}$ such that $\forall u \in \text{PX} t \cdot u \in \text{PY}$. It is clear that the identity matrix is a morphism from X to X and that the matrix product of two morphisms is a morphism and therefore, PCS equipped with this notion of morphism form a category **Pcoh**. There is a PCS $X \multimap Y$ such that $|X \multimap Y| = |X| \times |Y|$ and $\text{P}(X \multimap Y)$ is exactly the set of these matrices. Given any a , we define 1_a as the PCS whose web is $\{a\}$ and $\text{P}1_a$ is $[0, 1]$ or, pedantically, $\{a\} \times [0, 1]$. We write 1 instead of 1_a if a is a given element $*$, fixed once and for all.

The condition $t \in \mathbf{Pcoh}(X, Y) = \text{P}(X \multimap Y)$ is equivalent to $\forall u \in \text{PX} \forall v' \in \text{PY}^\perp \langle t \cdot u, v' \rangle \leq 1$ and we have $\langle t \cdot u, v' \rangle = \langle t, u \otimes v' \rangle$ where $(u \otimes v')_{(a,b)} = u_a v'_b$. Given PCS X and Y we define a PCS $X \otimes Y = (X \multimap Y^\perp)^\perp$ such that $\text{P}(X \otimes Y) = \{u \otimes v \mid u \in \text{PX} \text{ and } v \in \text{PY}\}^{\perp\perp}$ where $(u \otimes v)_{(a,b)} = u_a v_b$. Equipped with this operation \otimes and the unit 1, **Pcoh** is a symmetric monoidal category (SMC) with isomorphisms of associativity $\alpha \in \mathbf{Pcoh}((X \otimes Y) \otimes Z, X \otimes (Y \otimes Z))$, symmetry $\gamma \in \mathbf{Pcoh}(X \otimes Y, Y \otimes X)$, neutrality $\lambda \in \mathbf{Pcoh}(1 \otimes X, X)$ and $\rho \in \mathbf{Pcoh}(X \otimes 1, X)$ defined in the obvious way. This SMC **Pcoh** is closed, with internal hom of X and Y the pair $(X \multimap Y, \text{ev})$ where $\text{ev} \in \mathbf{Pcoh}((X \multimap Y) \otimes X, Y)$ is given by $\text{ev}_{((a,b),a'),b'} = \delta_{a,a'} \delta_{b,b'}$ so that $\text{ev} \cdot (t \otimes u) = t \cdot u$. This SMCC is *-autonomous wrt. the dualizing object $\perp = 1$ (essentially because $X \multimap \perp \simeq X^\perp$).

The following property is quite easy and very useful (actually we already used it).

Lemma 1. *Let $t \in \text{P}(X \multimap Y)$, $u \in \text{PX}$ and $v' \in \text{PY}^\perp$. Then $\langle t \cdot x, v' \rangle = \langle t, u \otimes v' \rangle = \langle u, t^\perp \cdot v' \rangle$.*

Pcoh is cartesian: if $(X_i)_{i \in I}$ is an at most countable family of PCSs, then $(\&_{i \in I} X_i, (\pi_i)_{i \in I})$ is the cartesian product of the X_i s, with $|\&_{i \in I} X_i| = \cup_{i \in I} \{i\} \times |X_i|$, $(\pi_i)_{(j,a),a'} = \delta_{i,j} \delta_{a,a'}$, and $u \in \text{P}(\&_{i \in I} X_i)$

⁷This restriction is not technically necessary, but very meaningful from a philosophic point of view; the non countable case should be handled via measurable spaces and then one has to consider more general objects as in [10] for instance.

if $\pi_i \cdot u \in \mathbf{P}X_i$ for each $i \in I$ (for $u \in (\mathbb{R}_{\geq 0})^{|\&i \in I X_i|}$). It is important to observe that $\mathbf{P}(\&i \in I X_i)$ is order isomorphic to $\prod_{i \in I} \mathbf{P}X_i$ by the map $u \mapsto (\pi_i \cdot u)_{i \in I}$. Given $\vec{u} = (u(i))_{i \in I} \in \prod_{i \in I} \mathbf{P}X_i$ we use $\langle \vec{u} \rangle$ for the corresponding element of $\mathbf{P}(\&i \in I X_i)$: $\langle \vec{u} \rangle_{i,a} = u(i)_a$ for $i \in I$ and $a \in |X_i|$. The terminal object (which corresponds to the case $I = \emptyset$) is the PCS $(\emptyset, \{0\})$.

Given $t(i) \in \mathbf{Pcoh}(Y, X_i)$ for each $i \in I$, the unique morphism $t = \langle t(i) \rangle_{i \in I} \in \mathbf{Pcoh}(Y, \&i \in I X_i)$ such that $\pi_i t = t_i$ is simply defined by $t_{b,(i,a)} = (t_i)_{b,a}$. The dual operation $\oplus_{i \in I} X_i$, which is a coproduct, is characterized by $|\oplus_{i \in I} X_i| = \cup_{i \in I} \{i\} \times |X_i|$ and $u \in \mathbf{P}(\oplus_{i \in I} X_i)$ if $u \in \mathbf{P}(\&i \in I X_i)$ and $\sum_{i \in I} \|\pi_i \cdot u\|_{X_i} \leq 1$. A particular case is $\mathbb{N}_\perp = \oplus_{n \in \mathbb{N}} X_n$ where $X_n = 1$ for each n . So that $|\mathbb{N}_\perp| = \mathbb{N}$ and $u \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ belongs to $\mathbf{P}\mathbb{N}_\perp$ if $\sum_{n \in \mathbb{N}} u_n \leq 1$ (that is, u is a sub-probability distribution on \mathbb{N}). There are successor and predecessor morphisms $\overline{\text{succ}}, \overline{\text{pred}} \in \mathbf{Pcoh}(\mathbb{N}_\perp, \mathbb{N}_\perp)$ given by $\overline{\text{succ}}_{n,n'} = \delta_{n+1,n'}$ and $\overline{\text{pred}}_{n,n'} = 1$ if $n = n' = 0$ or $n = n' + 1$ (and $\overline{\text{pred}}_{n,n'} = 0$ in all other cases). An element of $\mathbf{Pcoh}(\mathbb{N}_\perp, \mathbb{N}_\perp)$ is a (sub)stochastic matrix and the very idea of this model is to represent programs as transformations of this kind, and their generalizations.

As to the exponentials, one sets $!|X| = \mathcal{M}_{\text{fin}}(|X|)$ and $\mathbf{P}(!X) = \{u^! \mid u \in \mathbf{P}X\}^{\perp\perp}$ where, given $\mu \in \mathcal{M}_{\text{fin}}(|X|)$, $u_\mu^! = u^\mu = \prod_{a \in |X|} u_a^{\mu(a)}$. Then given $t \in \mathbf{Pcoh}(X, Y)$, one defines $!t \in \mathbf{Pcoh}(!X, !Y)$ in such a way that $!t \cdot x^! = (t \cdot x)^!$ (the precise definition is not relevant here; it is completely determined by this equation). There are natural transformations $\text{der}_X \in \mathbf{Pcoh}(!X, X)$ and $\text{dig}_X \in \mathbf{Pcoh}(!X, !!X)$ which are fully characterized by $\text{der}_X \cdot u^! = u$ and $\text{dig}_X \cdot u^! = u^{!!}$ which equip $!_-$ with a comonad structure. There are also Seely isomorphisms $\mathfrak{m}^0 \in \mathbf{Pcoh}(1, !\top)$ and $\mathfrak{m}_{X,Y}^2 \in \mathbf{Pcoh}(!X \otimes !Y, !(X \& Y))$ which equip this comonad with a strong monoidal structure from de SMC $(\mathbf{Pcoh}, \top, \&)$ to the SMC $(\mathbf{Pcoh}, 1, \otimes)$. They are fully characterized by the equations $\mathfrak{m}^0 \cdot 0^! = e_*$ and $\mathfrak{m}^2 \cdot (u^! \otimes v^!) = \langle u, v \rangle^!$.

Using these structures one can equip any object $!X$ with a commutative comonoid structure consisting of a weakening morphism $w_X \in \mathbf{Pcoh}(!X, 1)$ and a contraction morphism $\text{contr}_X \in \mathbf{Pcoh}(!X, !X \otimes !X)$ characterized by $w_X \cdot u = e_*$ and $\text{contr}_X \cdot u^! = u^! \otimes u^!$.

The resulting cartesian closed category⁸ $\mathbf{Pcoh}!$ can be seen as a category of functions (actually, of stable functions as proved in [5]). Indeed, a morphism $t \in \mathbf{Pcoh}_!(X, Y) = \mathbf{Pcoh}(!X, Y) = \mathbf{P}(!X \multimap Y)$ is completely characterized by the associated function $\hat{t}: \mathbf{P}X \rightarrow \mathbf{P}Y$ such that $\hat{t}(u) = t \cdot u^! = \left(\sum_{\mu \in |X|} t_{\mu,b} u^\mu \right)_{b \in |Y|}$ so that we consider morphisms as power series. They are in particular monotonic and Scott continuous functions $\mathbf{P}X \rightarrow \mathbf{P}Y$. In this cartesian closed category, the product of a family $(X_i)_{i \in I}$ is $\&i \in I X_i$ (written X^I if $X_i = X$ for all i), which is compatible with our viewpoint on morphisms as functions since $\mathbf{P}(\&i \in I X_i) = \prod_{i \in I} \mathbf{P}X_i$ up to trivial iso. The object of morphisms from X to Y is $!X \multimap Y$ with evaluation mapping $(t, u) \in \mathbf{P}(!X \multimap Y) \times \mathbf{P}X$ to $\hat{t}(u)$. The well defined function $\mathbf{P}(!X \multimap X) \rightarrow \mathbf{P}X$ which maps t to $\sup_{n \in \mathbb{N}} t^n(0)$ is a morphism of $\mathbf{Pcoh}!$ (and thus can be described as a power series in $t = (t_{\mu,a})_{\mu \in \mathcal{M}_{\text{fin}}(|X|), a \in |X|}$) by standard categorical considerations using cartesian closeness: it provides us with fixpoint operators at all types.

Given $t \in \mathbf{Pcoh}(X, Y)$, we also use \hat{t} for the associated function $\mathbf{P}X \rightarrow \mathbf{P}Y$. More generally if for instance $t \in \mathbf{P}(!X \otimes Y, Z)$, we use \hat{t} for the associated function $\mathbf{P}X \times \mathbf{P}Y \rightarrow \mathbf{P}Z$, given by $\hat{t}(u, v) = t \cdot (u^! \otimes v)$. This function fully characterizes t (that is the mapping $t \mapsto \hat{t}$ is injective); this can be seen by considering $\text{cur}(t) \in \mathbf{Pcoh}(!X, Y \multimap Z)$.

2 Extensional PCS

Let X be a PCS. A *pre-extensional structure* on X is a pair $\mathcal{U} = (\mathcal{E}, \sqsubseteq)$ where $\mathcal{E} \subseteq \mathbf{P}X$ and \sqsubseteq is a binary relation on \mathcal{E} . We define then the dual pre-extensional structure $\mathcal{U}^\perp = (\mathcal{E}', \sqsubseteq')$ on X^\perp as follows⁹:

- if $u' \in \mathbf{P}X^\perp$, one has $u' \in \mathcal{E}'$ iff $\forall u, v \in \mathcal{E} \ u \sqsubseteq v \Rightarrow \langle u, u' \rangle \leq \langle v, u' \rangle$
- and, given $u', v' \in \mathcal{E}'$, one has $u' \sqsubseteq' v'$ iff $\forall u \in \mathcal{E} \ \langle u, u' \rangle \leq \langle u, v' \rangle$.

⁸This is the Kleisli category of “!” which has actually a comonad structure that we do not make explicit here, again we refer to [6, 10].

⁹Notice the kind of role swapping between \mathcal{E} and \sqsubseteq in this definition; this justifies our choice of presenting these structures as pairs $(\mathcal{E}, \sqsubseteq)$ and not simply as relations \sqsubseteq on $\mathbf{P}X$.

Let $\mathcal{U}_1 = (\mathcal{E}_1, \sqsubseteq_1)$ and $\mathcal{U}_2 = (\mathcal{E}_2, \sqsubseteq_2)$ be pre-extensional structures on X , we write $\mathcal{U}_1 \subseteq \mathcal{U}_2$ if $\mathcal{E}_1 \subseteq \mathcal{E}_2$ and $\sqsubseteq_1 \subseteq \sqsubseteq_2$.

Lemma 2. *If $\mathcal{U}_1 \subseteq \mathcal{U}_2$ then $\mathcal{U}_2^\perp \subseteq \mathcal{U}_1^\perp$. One has $\mathcal{U} \subseteq \mathcal{U}^{\perp\perp}$ and therefore $\mathcal{U}^\perp = \mathcal{U}^{\perp\perp\perp}$.*

Given a pre-extensional structure $(\mathcal{E}, \sqsubseteq)$, when we write $u \sqsubseteq v$, we always assume implicitly that $u, v \in \mathcal{E}$. An *extensional structure* on PX is a pre-extensional structure \mathcal{U} such that $\mathcal{U} = \mathcal{U}^{\perp\perp}$.

Proposition 3. *If $\mathcal{U} = (\mathcal{E}, \sqsubseteq)$ is an extensional structure on the PCS X , then \sqsubseteq is a transitive and reflexive (that is, a preorder) relation on \mathcal{E} . Moreover, $\forall u, v \in \mathcal{E} \ u \leq v \Rightarrow u \sqsubseteq v$, $0 \in \mathcal{E}$, 0 is \sqsubseteq -minimal, \mathcal{E} and \sqsubseteq are sub-convex, closed under multiplication by scalars in $[0, 1]$ and closed under lubs of \leq -increasing ω -chains. Concerning \sqsubseteq , this means that*

- if I is a finite set, $\lambda_i \in \mathbb{R}_{\geq 0}$ for each $i \in I$ with $\sum_{i \in I} \lambda_i \leq 1$, $u(i) \sqsubseteq v(i)$ for each $i \in I$ then $\sum_{i \in I} \lambda_i u(i) \sqsubseteq \sum_{i \in I} \lambda_i v(i)$
- and if $(u(n))_{n \in \mathbb{N}}$ and $(v(n))_{n \in \mathbb{N}}$ are \leq -monotone in PX and such that $u(n) \sqsubseteq v(n)$ for each $n \in \mathbb{N}$, then $\sup_{n \in \mathbb{N}} u(n) \sqsubseteq \sup_{n \in \mathbb{N}} v(n)$ (this can be generalized to directed families).

An *extensional PCS* is a triple $X = (\underline{X}, \mathcal{E}_X, \sqsubseteq_X)$ where \underline{X} is a PCS (the carrier) and $(\mathcal{E}_X, \sqsubseteq_X)$ is an extensional structure on \underline{X} . The dual of X is then $X^\perp = (\underline{X}^\perp, (\mathcal{E}_X, \sqsubseteq_X)^\perp)$, so that $X^{\perp\perp} = X$ by definition. An extensional PCS X is *discrete* if $\mathcal{E}_X = PX$ and $u \sqsubseteq_X v$ iff $u \leq v$ (in PX). Observe that if X is discrete then X^\perp is also discrete. Of course any PCS can be turned into an extensional PCS by endowing it with its discrete extensional structure.

2.1 Extensional PCS as a model of Linear Logic

Lemma 4. *Let X and Y be extensional PCS, we define a pre-extensional structure $(\mathcal{E}, \sqsubseteq)$ on the PCS $\underline{X} \multimap \underline{Y}$ by:*

- given $t \in P(\underline{X} \multimap \underline{Y})$, one has $t \in \mathcal{E}$ if $\forall u \in \mathcal{E}_X \ t \cdot u \in \mathcal{E}_Y$ and $\forall u(1), u(2) \in \mathcal{E}_X \ u(1) \sqsubseteq_X u(2) \Rightarrow t \cdot u(1) \sqsubseteq_Y t \cdot u(2)$
- and given $t(1), t(2) \in P(\underline{X} \multimap \underline{Y})$, one has $t(1) \sqsubseteq t(2)$ if $\forall u \in \mathcal{E}_X \ t(1) \cdot u \sqsubseteq_Y t(2) \cdot u$.

We denote as $X \multimap Y$ the extensional PCS defined by $\underline{X} \multimap \underline{Y} = \underline{X} \multimap \underline{Y}$, $\mathcal{E}_{X \multimap Y} = \mathcal{E}$ and $\sqsubseteq_{X \multimap Y} = \sqsubseteq$.

Proof. Let $(\mathcal{E}', \sqsubseteq')$ be the pre-extensional structure on $(\underline{X} \multimap \underline{Y})^\perp = \underline{X} \otimes \underline{Y}^\perp$ defined by

- $\mathcal{E}' = \{u \otimes v' \mid u \in \mathcal{E}_X \text{ and } v' \in \mathcal{E}_{Y^\perp}\}$
- $\sqsubseteq' = \{(u(1) \otimes v'(1), u(2) \otimes v'(2)) \mid u(1) \sqsubseteq_X u(2) \text{ and } v'(1) \sqsubseteq_{Y^\perp} v'(2)\}$.

We prove that $(\mathcal{E}, \sqsubseteq) = (\mathcal{E}', \sqsubseteq')^\perp$.

Let $(\mathcal{E}'', \sqsubseteq'') = (\mathcal{E}', \sqsubseteq')^\perp$ which is an extensional structure on $\underline{X} \multimap \underline{Y}$.

Let $t \in \mathcal{E}$ and let us prove that $t \in \mathcal{E}''$. So let $u(1) \sqsubseteq_X u(2)$ and $v'(1) \sqsubseteq_{Y^\perp} v'(2)$, we have

$$\langle t, u(1) \otimes v'(1) \rangle = \langle t \cdot u(1), v'(1) \rangle \leq \langle t \cdot u(1), v'(2) \rangle$$

since $t \cdot u(1) \in \mathcal{E}_Y$, and we have $\langle t \cdot u(1), v'(2) \rangle \leq \langle t \cdot u(2), v'(2) \rangle$ because $t \cdot u(1) \sqsubseteq_Y t \cdot u(2)$. Conversely let $t \in \mathcal{E}''$ and let us prove that $t \in \mathcal{E}$. First, let $u \in \mathcal{E}_X$, we must prove that $t \cdot u \in \mathcal{E}_Y$. So let $v'(1) \sqsubseteq_{Y^\perp} v'(2)$, we must prove that $\langle t \cdot u, v'(1) \rangle \leq \langle t \cdot u, v'(2) \rangle$, which results from our assumption on t and from Lemma 1. Next we must prove that, if $u(1) \sqsubseteq_X u(2)$, then $t \cdot u(1) \sqsubseteq_Y t \cdot u(2)$. So let $y' \in \mathcal{E}_{Y^\perp}$, we must show that $\langle t \cdot u(1), y' \rangle \leq \langle t \cdot u(2), y' \rangle$ which results from Lemma 1 and from the assumption that $t \in \mathcal{E}''$.

Let now $t(1) \sqsubseteq t(2)$ and let us prove that $t(1) \sqsubseteq'' t(2)$. So let $u \in \mathcal{E}_X$ and $v' \in \mathcal{E}_{Y^\perp}$, we must prove that $\langle t(1), u \otimes v' \rangle \leq \langle t(2), u \otimes v' \rangle$ which again results from Lemma 1 and from the fact that $t(1) \cdot u \sqsubseteq t(2) \cdot u$. Conversely assume that $t(1) \sqsubseteq'' t(2)$ and let us prove that $t(1) \sqsubseteq t(2)$. So let $u \in \mathcal{E}_X$, we must prove that $t(1) \cdot u \sqsubseteq_Y t(2) \cdot u$. So let $v' \in \mathcal{E}_{Y^\perp}$, we must prove $\langle t(1) \cdot u, v' \rangle \leq \langle t(2) \cdot u, v' \rangle$ which again results from Lemma 1. \square

From this definition it results that if $s \in \mathcal{E}_{X \multimap Y}$ and $t \in \mathcal{E}_{Y \multimap Z}$, one has $ts \in \mathcal{E}_{X \multimap Z}$, and also that $\text{Id}_{\underline{X}} \in \mathcal{E}_{X \multimap X}$. So we have defined a category \mathbf{Pcoh}^e whose objects are the extensional PCS and where $\mathbf{Pcoh}^e(X, Y) = \mathcal{E}_{X \multimap Y}$, identities and composition being defined as in \mathbf{Pcoh} .

Remark. It is very important to notice that a morphism $t \in \mathbf{Pcoh}^e(X, Y)$ acts on *all* the elements of $\mathbf{P}\underline{X}$ and not only on those which belong to \mathcal{E}_X (intuitively, the extensional elements).

Lemma 5. *Let $t \in \mathbf{P}(X \multimap Y)$, we have $t \in \mathbf{Pcoh}^e(X, Y)$ iff $t^\perp \in \mathbf{Pcoh}^e(Y^\perp, X^\perp)$. Let $t(1), t(2) \in \mathbf{Pcoh}^e(X, Y)$, one has $t(1) \sqsubseteq_{X \multimap Y} t(2)$ iff $t(1)^\perp \sqsubseteq_{Y^\perp \multimap X^\perp} t(2)^\perp$.*

2.1.1 Multiplicative structure

As usual we set $X \otimes Y = (X \multimap Y^\perp)^\perp$ so that $\underline{X \otimes Y} = \underline{X} \otimes \underline{Y}$.

Lemma 6. *If $u \in \mathcal{E}_X$ and $v \in \mathcal{E}_Y$, we have $u \otimes v \in \mathcal{E}_{X \otimes Y}$. And if $u(1) \sqsubseteq_X u(2)$ and $v(1) \sqsubseteq_Y v(2)$ then $u(1) \otimes v(1) \sqsubseteq_{X \otimes Y} u(2) \otimes v(2)$.*

For proving the categorical properties of a \otimes operation defined in that way, one always starts with proving a lemma characterizing *bilinear* morphisms.

Lemma 7. *Let $t \in \mathbf{Pcoh}(\underline{X} \otimes \underline{Y}, \underline{Z})$. One has $t \in \mathbf{Pcoh}^e(X \otimes Y, Z)$ iff the following conditions hold:*

1. *if $u \in \mathcal{E}_X$ and $v \in \mathcal{E}_Y$ then $t \cdot (u \otimes v) \in \mathcal{E}_Z$*
2. *if $u(1) \sqsubseteq_X u(2)$ and $v(1) \sqsubseteq_Y v(2)$ then $t \cdot (u(1) \otimes v(1)) \sqsubseteq_Z t \cdot (u(2) \otimes v(2))$.*

Let $t(1), t(2) \in \mathbf{Pcoh}(\underline{X} \otimes \underline{Y}, \underline{Z})$, one has $t(1) \sqsubseteq_{\underline{X \otimes Y} \multimap \underline{Z}} t(2)$ iff for all $u \in \mathcal{E}_X$ and $v \in \mathcal{E}_Y$, one has $t(1) \cdot (u \otimes v) \sqsubseteq_Z t(2) \cdot (u \otimes v)$.

Lemma 8. *Let $t(i) \in \mathbf{Pcoh}^e(X_i, Y_i)$ for $i = 1, 2$, then $t(1) \otimes t(2)$, which is an element of $\mathbf{Pcoh}(\underline{X_1} \otimes \underline{X_2}, \underline{Y_1} \otimes \underline{Y_2})$, satisfies $t(1) \otimes t(2) \in \mathbf{Pcoh}^e(X_1 \otimes X_2, Y_1 \otimes Y_2)$.*

Proof. We apply Lemma 7, so let first $u(i) \in \mathcal{E}_{X_i}$ for $i = 1, 2$. We have $(t(1) \otimes t(2)) \cdot (u(1) \otimes u(2)) = (t(1) \cdot u(1)) \otimes (t(2) \cdot u(2)) \in \mathcal{E}_{Y_1 \otimes Y_2}$ by Lemma 6 since we have $t(i) \cdot u(i) \in \mathcal{E}_{Y_i}$ for $i = 1, 2$. Next assume that $u^1(i) \sqsubseteq_{X_i} u^2(i)$ for $i = 1, 2$. We have $t(i) \cdot u^1(i) \sqsubseteq_{X_i} t(i) \cdot u^2(i)$ for $i = 1, 2$ and hence $(t(1) \cdot u^1(1)) \otimes (t(2) \cdot u^2(1)) \sqsubseteq_{Y_1 \otimes Y_2} (t(1) \cdot u^1(2)) \otimes (t(2) \cdot u^2(2))$ by Lemma 6. \square

Let X, Y and Z be extensional PCS, we have an isomorphism

$$\alpha : \mathbf{Pcoh}((\underline{X} \otimes \underline{Y}) \otimes \underline{Z}, \underline{X} \otimes (\underline{Y} \otimes \underline{Z}))$$

defined in the obvious way: $\alpha_{((a,b),c),(a',b',c')} = \delta_{a,a'} \delta_{b,b'} \delta_{c,c'}$. We prove first that $\alpha \in \mathbf{Pcoh}^e((X \otimes Y) \otimes Z, X \otimes (Y \otimes Z))$. For this, by Lemma 5, it suffices to prove that $\alpha^\perp \in \mathbf{Pcoh}^e((X \otimes (Y \otimes Z))^\perp, ((X \otimes Y) \otimes Z)^\perp)$, that is

$$\alpha^\perp \in \mathbf{Pcoh}^e(X \multimap (Y \multimap Z^\perp), X \otimes Y \multimap Z^\perp).$$

Let $t \in \mathcal{E}_{X \multimap (Y \multimap Z^\perp)}$, we prove that $\alpha^\perp \cdot t \in \mathcal{E}_{X \otimes Y \multimap Z^\perp}$ applying Lemma 7. Let first $u \in \mathcal{E}_X$ and $v \in \mathcal{E}_Y$, we have $(\alpha^\perp \cdot t) \cdot (u \otimes v) = (t \cdot u) \cdot v \in \mathcal{E}_{Z^\perp}$. Next let $u(1) \sqsubseteq_X u(2)$ and $v(1) \sqsubseteq_Y v(2)$, we have $t \cdot u(1) \sqsubseteq_{Y \multimap Z^\perp} t \cdot u(2)$, hence $(t \cdot u(1)) \cdot v(1) \sqsubseteq_{Z^\perp} (t \cdot u(2)) \cdot v(1)$ and since $t \cdot u(2) \in \mathcal{E}_{Y \multimap Z^\perp}$, we have

$$(t \cdot u(2)) \cdot v(1) \sqsubseteq_{Z^\perp} (t \cdot u(2)) \cdot v(1) \sqsubseteq_{Z^\perp} (t \cdot u(2)) \cdot v(2)$$

so that $(t \cdot u(2)) \cdot v(1) \sqsubseteq_{Z^\perp} (t \cdot u(2)) \cdot v(2)$ by transitivity of \sqsubseteq_{Z^\perp} (Proposition 3).

Next, given $t(1) \sqsubseteq_{X \multimap (Y \multimap Z^\perp)} t(2)$, we prove that $\alpha^\perp \cdot t(1) \sqsubseteq_{X \otimes Y \multimap Z^\perp} \alpha^\perp \cdot t(2)$. By Lemma 7 it suffices to prove that, given $u \in \mathcal{E}_X$ and $v \in \mathcal{E}_Y$, one has $(\alpha^\perp \cdot t(1)) \cdot (u \otimes v) \sqsubseteq_{Z^\perp} (\alpha^\perp \cdot t(2)) \cdot (u \otimes v)$ which results from the definition of α which yields $(\alpha^\perp \cdot t(i)) \cdot (u \otimes v) = (t(i) \cdot u) \cdot v$ and from our assumption on the $t(i)$'s.

Let $\beta = \alpha^{-1}$, we prove that $\beta \in \mathbf{Pcoh}^e(X \otimes (Y \otimes Z), (X \otimes Y) \otimes Z)$, that is

$$\beta^\perp \in \mathbf{Pcoh}^e(X \otimes Y \multimap Z^\perp, X \multimap (Y \multimap Z^\perp)).$$

So let $t \in \mathcal{E}_{X \otimes Y \multimap Z^\perp}$, we prove first that $\beta^\perp \cdot t \in \mathcal{E}_{X \multimap (Y \multimap Z^\perp)}$. Let $u \in \mathcal{E}_X$ and $v \in \mathcal{E}_Y$, we have $((\beta^\perp \cdot t) \cdot u) \cdot v = t \cdot (u \otimes v) \in \mathcal{E}_{Z^\perp}$. This shows that $(\beta^\perp \cdot t) \cdot u \in \mathcal{E}_{Y \multimap Z^\perp}$. Next let $u(1) \sqsubseteq_X u(2)$ and $v \in \mathcal{E}_Y$, we have $((\beta^\perp \cdot t) \cdot u(i)) \cdot v = t(u(i) \otimes v)$ and hence $((\beta^\perp \cdot t) \cdot u(1)) \cdot v \sqsubseteq_{Z^\perp} ((\beta^\perp \cdot t) \cdot u(2)) \cdot v$ since $u(1) \otimes v \sqsubseteq_{X \otimes Y} u(2) \otimes v$. This shows that $(\beta^\perp \cdot t) \cdot u(1) \sqsubseteq_{Y \multimap Z^\perp} (\beta^\perp \cdot t) \cdot u(2)$. Now let $u(1) \sqsubseteq_X u(2)$ and $v \in \mathcal{E}_Y$, we prove similarly $((\beta^\perp \cdot t) \cdot u(1)) \cdot v \sqsubseteq_{Z^\perp} ((\beta^\perp \cdot t) \cdot u(1)) \cdot v$ which shows that $(\beta^\perp \cdot t) \cdot u(1) \sqsubseteq_{Y \multimap Z^\perp} (\beta^\perp \cdot t) \cdot u(2)$.

Last let $t(1) \sqsubseteq_{X \otimes Y \multimap Z} t(2)$, $u \in \mathcal{E}_X$ and $v \in \mathcal{E}_Y$, we have $u \otimes v \in \mathcal{E}_{X \otimes Y}$ and hence $t(1) \cdot (u \otimes v) \sqsubseteq_{Z^\perp} t(2) \cdot (u \otimes v)$ that is $((\beta^\perp \cdot t(1)) \cdot u) \cdot v \sqsubseteq_{Z^\perp} ((\beta^\perp \cdot t(2)) \cdot u) \cdot v$, so $(\beta^\perp \cdot t(1)) \cdot u \sqsubseteq_{Y \multimap Z^\perp} (\beta^\perp \cdot t(2)) \cdot v$ and hence $\beta^\perp \cdot t(1) \sqsubseteq_{X \multimap (Y \multimap Z^\perp)} \beta^\perp \cdot t(2)$.

The fact that we have a symmetry isomorphism $\gamma \in \mathbf{Pcoh}^e(X \otimes Y, Y \otimes X)$ such that $\gamma_{(a,b),(b',a')} = \delta_{a,a'} \delta_{b,b'}$ is a direct consequence of Lemmas 7 and 6.

The tensor unit is 1, equipped with the discrete extensional structure. It is straightforward to check that λ and ρ are isos in \mathbf{Pcoh}^e . To summarize, we have proven the first part of the following result.

Theorem 9. *Equipped with $(\otimes, 1, \lambda, \rho, \alpha, \gamma)$, the category \mathbf{Pcoh}^e is symmetric monoidal. Moreover, this symmetric monoidal category is closed (SMCC), the object of linear morphisms $X \rightarrow Y$ being $(X \multimap Y, \text{ev})$ ¹⁰. This SMCC has a *-autonomous structure, with 1 as dualizing object.*

Given $t \in \mathbf{Pcoh}^e(Z \otimes X, Y)$, we use $\text{cur}(t)$ for the curried version of t which belongs to $\mathbf{Pcoh}^e(Z, X \multimap Y)$ and is defined exactly as in \mathbf{Pcoh} (that is $\text{cur}(t)_{c,(a,b)} = t_{(c,a),b}$).

2.1.2 The additives

The additive structure is quite simple. Given an at most countable family $(X_i)_{i \in I}$ of extensional PCS, we define a pre-extensional PCS $X = \&_{i \in I} X_i$ as follows:

- $\underline{X} = \&_{i \in I} \underline{X}_i$ (in \mathbf{Pcoh} of course)
- $\mathcal{E}_X = \{u \in \text{PX} \mid \forall i \in I \pi_i \cdot u \in \mathcal{E}_{X_i}\}$
- $u \sqsubseteq_X v$ if $\forall i \in I \pi_i \cdot u \sqsubseteq_{X_i} \pi_i \cdot v$.

Lemma 10. *Let $X = \&_{i \in I} X_i$. Then the extensional PCS X^\perp is characterized by*

- $\underline{X}^\perp = \oplus_{i \in I} \underline{X}_i^\perp$
- $\mathcal{E}_{X^\perp} = \{u' \in \text{PX}^\perp \mid \forall i \in I \pi_i \cdot u' \in \mathcal{E}_{X_i^\perp}\}$
- $u' \sqsubseteq_{X^\perp} v'$ if $\forall i \in I \pi_i \cdot u' \sqsubseteq_{X_i^\perp} \pi_i \cdot v'$.

Remember that the elements of PX^\perp are the $u' \in \prod_{i \in I} \text{PX}_i^\perp$ such that $\sum_{i \in I} \|\pi_i \cdot u'\|_{X_i^\perp} \leq 1$.

Lemma 11. *The pre-extensional PCS $X = \&_{i \in I} X_i$ is an extensional PCS. For each $i \in I$, the projection $\pi_j \in \mathbf{Pcoh}(\underline{X}, \underline{X}_i)$ belongs to $\mathbf{Pcoh}^e(X, X_i)$ and, equipped with these projections, X is the cartesian product of the X_i 's in \mathbf{Pcoh}^e .*

If all X_i 's are the same extensional PCS X , we use the notation X^I for the extensional PCS $\&_{i \in I} X_i$. Notice that $|\underline{X}^I| = I \times |\underline{X}|$.

Remark. One should observe that the constructions introduced so far preserve discreteness. More precisely, if X and Y are discrete extensional PCSs, so are $X \multimap Y$, $X \otimes Y$, X^\perp etc, and if $(X_i)_{i \in I}$ is an at most countable family of discrete extensional PCSs, so are $\&_{i \in I} X_i$ and $\oplus_{i \in I} X_i$. This is not the case of the exponentials.

¹⁰Where $\text{ev} \in \mathbf{Pcoh}((X \multimap Y) \otimes X, Y)$ is the evaluation morphism of \mathbf{Pcoh} , see Section 1.

2.1.3 The exponentials

Let X be an extensional PCS. We define $!X = (!\underline{X}, (\mathcal{E}_{!X}^0, \sqsubseteq_{!X}^0)^{\perp\perp})$ where the pre-extensional structure $(\mathcal{E}_{!X}^0, \sqsubseteq_{!X}^0)$ on $!X$ is defined by:

$$\begin{aligned}\mathcal{E}_{!X}^0 &= \{u^! \mid u \in \mathcal{E}_X\} \\ \sqsubseteq_{!X}^0 &= \{(u(1)^!, u(2)^!) \mid u(1) \sqsubseteq_X u(2)\}.\end{aligned}$$

The main consequence of this definition is the following.

Theorem 12. *Let X and Y be extensional PCSs.*

- *Given $t \in \mathbf{P}(!X \multimap Y)$, one has $t \in \mathcal{E}_{!X \multimap Y}$ iff $\forall u \in \mathcal{E}_X \ t \cdot u^! \in \mathcal{E}_Y$ and for any $u(1) \sqsubseteq_X u(2)$, one has $t \cdot u(1)^! \sqsubseteq_Y t \cdot u(2)^!$.*
- *Given $t(1), t(2) \in \mathcal{E}_{!X \multimap Y}$, one has $t(1) \sqsubseteq_{!X \multimap Y} t(2)$ iff for all $u \in \mathcal{E}_X$, one has $t(1) \cdot u^! \sqsubseteq_Y t(2) \cdot u^!$.*

Now we can derive important consequences of that result.

Theorem 13. *Given $t \in \mathbf{Pcoh}^e(X, Y)$, one has $!t \in \mathbf{Pcoh}^e(!X, !Y)$. Moreover, $\text{der}_{\underline{X}} \in \mathbf{Pcoh}^e(!X, X)$ and $\text{dig}_{\underline{X}} \in \mathbf{Pcoh}^e(!X, !!X)$.*

Of course the diagram commutations which hold in \mathbf{Pcoh} relative to this constructs still hold in \mathbf{Pcoh}^e (composition is identical in both categories) and so $!_-$ equipped with these two natural transformation is a comonad.

The Seely isomorphisms of \mathbf{Pcoh} are easily checked to be morphisms in \mathbf{Pcoh}^e . The case of \mathbf{m}^0 is straightforward so we deal only with $\mathbf{m}^2 \in \mathbf{Pcoh}(!\underline{X} \otimes !\underline{Y}, !(X \& Y))$, which is the isomorphism in \mathbf{Pcoh} given by

$$\mathbf{m}_{([a_1, \dots, a_m], [b_1, \dots, b_n]), \rho}^2 = \begin{cases} 1 & \text{if } \rho = [(1, a_1), \dots, (1, a_m), (2, b_1), \dots, (2, b_n)] \\ 0 & \text{otherwise} \end{cases}$$

which satisfies $\mathbf{m}^2(u^! \otimes v^!) = \langle u, v \rangle^!$. We know that $\text{cur } \mathbf{m}^2 \in \mathbf{Pcoh}(!\underline{X}, !\underline{Y} \multimap !(X \& Y))$ and we prove that actually $t = \text{cur } \mathbf{m}^2 \in \mathbf{Pcoh}^e(!X, !Y \multimap !(X \& Y))$, applying again Theorem 12.

Let $u \in \mathcal{E}_X$, we prove that $t \cdot u^! \in \mathcal{E}_{!Y \multimap !(X \& Y)}$. So let $v \in \mathcal{E}_Y$, we have $(t \cdot u^!) \cdot v^! = \langle u, v \rangle^! \in \mathcal{E}_{!(X \& Y)}$ because $\langle u, v \rangle \in \mathcal{E}_{X \& Y}$. Let $v(1) \sqsubseteq_Y v(2)$, we have $(t \cdot u^!) \cdot v(1)^! = \langle u, v(1) \rangle^! \sqsubseteq_{!(X \& Y)} \langle u, v(2) \rangle^! = (t \cdot u^!) \cdot v(2)^!$ because $\langle u, v(1) \rangle \sqsubseteq_{X \& Y} \langle u, v(2) \rangle$ by definition of $X \& Y$. Next let $u(1) \sqsubseteq_X u(2)$ and let us check that $t \cdot u(1)^! \sqsubseteq_{!Y \multimap !(X \& Y)} t \cdot u(2)^!$. So let $v \in \mathcal{E}_Y$, it suffices to prove that $(t \cdot u(1)^!) \cdot v^! \sqsubseteq_{!(X \& Y)} (t \cdot u(2)^!) \cdot v^!$ which is obtained exactly as above.

Last we have also to prove that $(\mathbf{m}^2)^{-1} \in \mathbf{Pcoh}^e(!(X \& Y), !X \otimes !Y)$; this is an easy consequence of Theorem 12, of the fact that $\mathcal{E}_{X \& Y} = \mathcal{E}_X \times \mathcal{E}_Y$ (up to the isomorphism between $\mathbf{P}(X \& Y)$ and $\mathbf{P}X \times \mathbf{P}Y$) and similarly, $\sqsubseteq_{X \& Y}$ is the product preorder and $(\mathbf{m}^2)^{-1} \cdot \langle u, v \rangle^! = u^! \otimes v^!$.

This ends the proof that \mathbf{Pcoh}^e is a (new) Seely category (see [15]), that is, a categorical model of classical linear logic.

As a consequence, the Kleisli category $\mathbf{Pcoh}^e_!$ of $!_-$ over \mathbf{Pcoh}^e is cartesian closed.

Given $t \in \mathbf{Pcoh}_!(\underline{X}, \underline{Y}) = \mathbf{Pcoh}(!\underline{X}, \underline{Y})$, we use \hat{t} for the corresponding function $\mathbf{P}\underline{X} \rightarrow \mathbf{P}\underline{Y}$ defined by $\hat{t}(u) = t u^!$. Such a t is a morphism of $\mathbf{Pcoh}^e_!$ iff $\forall u \in \mathcal{E}_X \ \hat{t}(u) \in \mathcal{E}_Y$. Moreover such a t is \sqsubseteq -monotone, that is

$$u \sqsubseteq_X v \Rightarrow \hat{t}(u) \sqsubseteq_Y \hat{t}(v)$$

and also, given $s, t \in \mathbf{Pcoh}^e_!(X, Y)$, one has

$$s \sqsubseteq_{!X \multimap Y} t \Leftrightarrow \forall u \in \mathcal{E}_X \ \hat{s}(u) \sqsubseteq_Y \hat{t}(u)$$

that is, $\sqsubseteq_{!X \multimap Y}$ is the pointwise (pre)order on functions.

We use $X \Rightarrow Y$ for the object of morphisms from X to Y in that category, which is $!X \multimap Y$, and is equipped with the evaluation morphism $\text{Ev} \in \mathbf{Pcoh}^e_!(X \Rightarrow Y) \& X, Y$ characterized of course by $\widehat{\text{Ev}}(s, u) = \widehat{s}(u)$.

Example 14. Let X be the extensional PCS $1 \Rightarrow 1$. Up to a trivial iso, an element of \mathbf{PX} is a family $s \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ such that $\sum_{n \in \mathbb{N}} s_n \leq 1$ and the associated function $\widehat{s} : [0, 1] \rightarrow [0, 1]$ is given by $\widehat{s}(u) = \sum_{n \in \mathbb{N}} s_n u^n$. Since 1 is discrete, one has $\mathcal{E}_X = \mathbf{PX}$, and given $s, t \in \mathcal{E}_X$, one has

$$s \sqsubseteq_X t \Leftrightarrow \forall u \in [0, 1] \widehat{s}(u) \leq \widehat{t}(u).$$

For each $n \in \mathbb{N}$, let $e(n)$ be the element of \mathbf{PX} such that $e(n)_i = \delta_{n,i}$, that is $\widehat{e(n)}(u) = u^n$, one has $\forall n \in \mathbb{N} e(n+1) \sqsubseteq_X e(n)$ since $\forall u \in [0, 1] u^{n+1} \leq u^n$. So $(e(n))_{n \in \mathbb{N}}$ is an \sqsubseteq_X -decreasing ω -chain which has 0 as glb. Notice that this glb is not the pointwise glb of the $e(n)$'s considered as functions since $\inf_{n \in \mathbb{N}} \widehat{e(n)}(1) = 1$ (glb. computed in $[0, 1]$). Observe also that the $e(n)$'s are pairwise unbounded for the standard \leq order on \mathbf{PX} (each of them is \leq -maximal in this PCS).

2.1.4 \mathbf{Pcoh}^e is an enriched category over preorders

Given objects X and Y of \mathbf{Pcoh}^e , we can equip $\mathbf{Pcoh}^e(X, Y)$ with the preorder relation $\sqsubseteq_{X \multimap Y}$ that we denote as $\sqsubseteq_{X, Y}$. In other words, given $s, t \in \mathbf{Pcoh}^e(X, Y)$, one has $s \sqsubseteq_{X, Y} t$ iff $\forall u \in \mathcal{E}_X s \cdot u \sqsubseteq_Y t \cdot u$. This turns \mathbf{Pcoh}^e into an enriched category over the monoidal category of partial order (with the usual product of preorders as monoidal product) and actually into an ‘‘enriched Seely category’’ in the sense that all the constructions involved in the definition of \mathbf{Pcoh}^e as a Seely category are \sqsubseteq -monotone. For instance, if $s, t \in \mathbf{Pcoh}^e(X, Y)$ satisfy $s \sqsubseteq_{X, Y} t$, then $!s \sqsubseteq_{!X, !Y} !t$ as an easy consequence of Theorem 12.

2.1.5 General recursion, fixpoints

Theorem 15. Let $t \in \mathbf{Pcoh}^e_!(X, X)$, the least fixpoint of t in \mathbf{PX} , which is $\sup_{n \in \mathbb{N}} \widehat{t}^n(0)$, belongs to \mathcal{E}_X . As a consequence, the \mathbf{Pcoh} least fixpoint operator $\text{Fix}_{\underline{Y}} \in \mathbf{Pcoh}(\underline{Y} \Rightarrow \underline{Y}, \underline{Y})$ (characterized by $\widehat{\text{Fix}_{\underline{Y}}}(t) = \sup_{n \in \mathbb{N}} \widehat{t}^n(0)$) actually belongs to $\mathbf{Pcoh}^e_!(Y \Rightarrow Y, Y)$.

Proof. For the first statement, remember that $0 \in \mathcal{E}_X$ by Proposition 3 and hence by a straightforward induction $\forall n \in \mathbb{N} \widehat{t}^n(0) \in \mathcal{E}_X$. Therefore, by Proposition 3 again, we have $\sup_{n \in \mathbb{N}} \widehat{t}^n(0) \in \mathcal{E}_X$.

The second part of the theorem is proven by applying the first part in the following special case: $X = ((Y \Rightarrow Y) \Rightarrow Y)$ and $t \in \mathbf{Pcoh}^e_!(X, X)$ is characterized by $\widehat{t}(F)(s) = \widehat{s}(\widehat{F}(\widehat{s}))$ for $F \in \mathbf{PX}$ and $s \in \mathbf{P}(Y \Rightarrow Y)$. The existence of t results from the cartesian closeness of $\mathbf{Pcoh}^e_!$. Then the least fixed point of \widehat{t} is $\text{Fix}_{\underline{Y}}$ and this prove our contention by the first part of the theorem. \square

Example 16. Let again X be the extensional PCS $1 \Rightarrow 1$. If we are given $F \in \mathcal{E}_{X \Rightarrow X}$, we know that \widehat{F} has a least fixpoint $t \in \mathcal{E}_X$ given by $t = \sup_{n \in \mathbb{N}} \widehat{F}^n(0)$. Given $u \in [0, 1]$, we know that $\widehat{t}(u) \in [0, 1]$, and if we set $t(n) = \widehat{F}^n(0)$, we have $t(n) \leq t$ and hence $\widehat{t}(u) \geq \widehat{t(n)}(u)$ and hence $\widehat{t(n)}(u)$ gives us a lower approximation of $\widehat{t}(u)$. We even know that $\widehat{t}(u)$ is the lub of these approximations but this gives us no clue on how good a given approximation $\widehat{t(n)}(u)$ is (how far it is from the target value $\widehat{t}(u)$).

One main feature of the \sqsubseteq_X relation is that it has a maximal element, namely $e(0)$ (notations of Example 14) that we simply denote as 1 , since it represents the constant function 1 , in sharp contrast with the \leq relation on \mathbf{PX} . Since $F \in \mathcal{E}_{X \Rightarrow X}$, the function \widehat{F} is \sqsubseteq_X -monotonic, and hence $(s(n) = \widehat{F}^n(1))_{n \in \mathbb{N}}$ is an \sqsubseteq_X -decreasing sequence. Therefore the sequence $(\widehat{s(n)}(u))_{n \in \mathbb{N}}$ is decreasing in $[0, 1]$ (for the usual order relation which coincides with \sqsubseteq_1). Moreover, since $0 \sqsubseteq_1 1$, we have $\forall n \in \mathbb{N} \widehat{F}^n(0) \sqsubseteq_X \widehat{F}^n(1)$ by induction on n and hence for all $n \in \mathbb{N}$, $\widehat{t}(u) \leq \widehat{s(n)}(u)$. In particular, given $\varepsilon > 0$, if we find $n \in \mathbb{N}$ such that

$\widehat{t(n)}(u) - \widehat{s(n)}(u) \leq \varepsilon$, we are certain that $\widehat{t(n)}(u)$ and $\widehat{s(n)}(u)$ are at most at distance ε of the probability $\widehat{t}(u)$ we are interested in.

It may happen that for some $\varepsilon > 0$ the condition $\widehat{t(n)}(u) - \widehat{s(n)}(u) \leq \varepsilon$ never holds. Take for instance F to be the identity and $u = 1$: in that case $\forall n \in \mathbb{N} \widehat{t(n)}(1) = 0$ and $\widehat{s(n)} = 1$. But if we manage to fulfill this condition by taking n big enough, we are certain to get an ε -approximation whereas having only the lower approximations $t(n)$ we could never know, whatever be the value of n . Moreover we can expect that some reasonable syntactic guardedness restrictions on the program defining F will guarantee that these ε -approximations always exist (such restrictions certainly already exist in the rich literature on abstract interpretation).

The remainder of the paper is essentially devoted to extending this idea to more useful datatypes.

2.2 Flat types with errors

Another crucial feature of extensional PCSs is that they allow to build basic data types extended with an “error” or “escape” element which is maximal for the \sqsubseteq preorder but not for the \leq preorder, just as 0 is minimal, thus allowing to extend the observation of Example 16 to languages having datatypes like booleans or integers and not just the poorly expressive unit type.

Given an at most countable set I , we defined the ordinary PCS I_\perp as follows: $|I_\perp| = I$ and $\mathsf{P}(I_\perp) = \{u \in (\mathbb{R}_{\geq 0})^I \mid \sum_{i \in I} u_i \leq 1\}$. As an extensional PCS, it is equipped with the discrete structure.

2.2.1 General definitions and basic properties

Now we introduce another object I_\perp^\top , with a non-trivial extensional structure. First we take $\underline{I_\perp^\top} = I_\perp \oplus 1_\top$.

We take $\mathcal{E}_{I_\perp^\top} = \mathsf{P}(\underline{I_\perp^\top})$ and we are left with defining $\sqsubseteq_{I_\perp^\top}$. Given $u, v \in \mathsf{P}(\underline{I_\perp^\top})$ we set

$$\mathsf{inv}(u, v) = \{i \in I \mid u_i > v_i\}$$

that is, $\mathsf{inv}(u, v)$ is the set of all $i \in I$ “where it is not true that u is less than v ” (the inversion indices) and we stipulate that $u \sqsubseteq_{I_\perp^\top} v$ if

$$\sum_{i \in \mathsf{inv}(u, v)} (u_i - v_i) \leq v_\top - u_\top, \quad \text{that is} \quad u_\top + \sum_{i \in \mathsf{inv}(u, v)} u_i \leq v_\top + \sum_{i \in \mathsf{inv}(u, v)} v_i.$$

Notice that this condition implies that $u_\top \leq v_\top$.

In other words $u \sqsubseteq_{I_\perp^\top} v$ means that the difference of probabilities of the “error” \top compensates the sum of all probability inversions from u to v . In that way we have equipped the PCS $\underline{I_\perp^\top}$ with an extensional structure, as we prove now.

Proposition 17. *We have $(\mathcal{E}_{I_\perp^\top}, \sqsubseteq_{I_\perp^\top}) = \mathcal{U}'^\perp$ where $\mathcal{U}' = (\mathcal{E}', \sqsubseteq')$ is the pre-extensional structure on $\underline{I_\perp^\top}^\perp$ defined as follows:*

- $\mathcal{E}' = \{u' \in \mathsf{P}(\underline{I_\perp^\top}^\perp) \mid \forall i \in I \ u'_i \leq u'_\top\}$
- and $u' \sqsubseteq' v'$ if $u' \leq v'$.

Therefore $I_\perp^\top = (\underline{I_\perp^\top}, \sqsubseteq_{I_\perp^\top})$ is an extensional PCS. One has $\forall u \in \mathcal{E}_{I_\perp^\top} \ u \sqsubseteq_{I_\perp^\top} e_\top$ and it is also true that $(I_\perp^\top)^\perp = (\underline{I_\perp^\top}^\perp, \mathcal{U}')$.

Proof. Using these notations, let $(\mathcal{E}, \sqsubseteq) = \mathcal{U}^\perp$. We have $\mathcal{E} = \mathcal{E}_{I_\perp} = \mathbf{P}I_\perp^\top$ since, if $u \in \mathbf{P}I_\perp^\top$ and $u'(1) \sqsubseteq' u'(2)$ one has $u'(1) \leq u'(2)$ and hence $\langle u, u'(1) \rangle \leq \langle u, u'(2) \rangle$. Assume now that $u(1) \sqsubseteq u(2)$ and let us prove that $u(1) \sqsubseteq_{I_\perp} u(2)$. Let $u' \in \mathbf{P}(\mathbb{N}_\perp^\top)^\perp$ be defined by

$$u'_a = \begin{cases} 1 & \text{if } a \in I \text{ and } a \in \text{inv}(u(1), u(2)) \\ 1 & \text{if } a = \top \\ 0 & \text{otherwise} \end{cases}$$

then we have $u' \in \mathcal{E}'$ and hence $\langle u(1), u' \rangle \leq \langle u(2), u' \rangle$ which means exactly that $u(1) \sqsubseteq_{I_\perp} u(2)$. Conversely assume that $u(1) \sqsubseteq_{I_\perp} u(2)$ and let $u' \in \mathcal{E}'$, we have

$$\begin{aligned} \langle u(1), u' \rangle &= \sum_{i \in I} u(1)_i u'_i + u(1)_\top u'_\top = \sum_{\substack{i \in I \\ i \notin \text{inv}(u(1), u(2))}} u(1)_i u'_i + \sum_{i \in \text{inv}(u(1), u(2))} u(1)_i u'_i + u(1)_\top u'_\top \\ &\leq \sum_{\substack{i \in I \\ i \notin \text{inv}(u(1), u(2))}} u(2)_i u'_i + \sum_{i \in \text{inv}(u(1), u(2))} u(1)_i u'_i + u(1)_\top u'_\top \\ &= \sum_{i \in \mathbb{N}} u(2)_i u'_i + \sum_{i \in \text{inv}(u(1), u(2))} (u(1)_i - u(2)_i) u'_i + u(1)_\top u'_\top \\ &= \langle u(2), u' \rangle + \sum_{\substack{i \in I \\ i \in \text{inv}(u(1), u(2))}} (u(1)_i - u(2)_i) u'_i + (u(1)_\top - u(2)_\top) u'_\top \\ &\leq \langle u(2), u' \rangle + \sum_{\substack{i \in I \\ i \in \text{inv}(u(1), u(2))}} (u(1)_i - u(2)_i) u'_\top + (u(1)_\top - u(2)_\top) u'_\top \end{aligned}$$

because $u(1)_i - u(2)_i \geq 0$ when $i \in \text{inv}(u(1), u(2))$ and $u'_i \leq u'_\top$ for each $i \in I$, which shows that $\langle u(1), u' \rangle \leq \langle u(2), u' \rangle$ since $u(1) \sqsubseteq_{I_\perp} u(2)$ which implies that

$$\sum_{\substack{i \in I \\ i \in \text{inv}(u(1), u(2))}} (u(1)_i - u(2)_i) u'_\top + (u(1)_\top - u(2)_\top) u'_\top \leq 0.$$

Next we observe¹¹ that for all $u \in \mathbf{P}I_\perp^\top$, one has $u \sqsubseteq_{I_\perp} e_\top$: we have

$$\sum_{i \in \text{inv}(x, e_\top)} (u_i - (e_\top)_i) = \sum_{i \in I} u_i \leq 1 - u_\top$$

by definition of $\mathbf{P}I_\perp^\top$ and this is exactly the definition of $u \sqsubseteq_{I_\perp} e_\top$.

Last let us prove that $\mathcal{U}' = (\mathcal{E}_{I_\perp}, \sqsubseteq_{I_\perp})^\perp$. The direction $\mathcal{U}' \subseteq (\mathcal{E}_{I_\perp}, \sqsubseteq_{I_\perp})^\perp$ results from what we have proven so far so let us prove the converse, and let us introduce the notation $(\mathcal{E}^\bullet, \sqsubseteq^\bullet)$ for $(\mathcal{E}_{I_\perp}, \sqsubseteq_{I_\perp})^\perp$.

Let $u' \in \mathcal{E}^\bullet$ and $i \in I$, we have $e_i \sqsubseteq_{I_\perp} e_\top$ and hence $\langle e_i, u' \rangle \leq \langle e_\top, u' \rangle$ which proves that $\forall i \in I \ u'_i \leq u'_\top$, that is $u' \in \mathcal{E}'$. Assume next that $u(1)' \sqsubseteq^\bullet u(2)'$ and let $a \in |I_\perp^\top|$, then since $e_a \in \mathcal{E}_{I_\perp}$ we have $\langle e_a, u'(1) \rangle \leq \langle e_a, u'(2) \rangle$, that is $u'(1)_a \leq u'(2)_a$ so that $u'(1) \leq u'(2)$. \square

Example 18. *The extensional PCS \emptyset_\perp^\top coincides with 1 equipped with its discrete extensional structure.*

The elements of \mathcal{E}_{1_\perp} are all pairs $u = (u_, u_\top) \in (\mathbb{R}_{\geq 0})^2$ such that $u_* + u_\top \leq 1$, and $u \sqsubseteq_{1_\perp} v$ if $u_\top \leq v_\top$ and $u_* + u_\top \leq v_* + v_\top$; in this case we don't need to mention $\text{inv}(u, v)$.*

¹¹This was actually the goal of all this construction!

Now let $X = \{0, 1\}_\perp^\top$ which represents the type of booleans, so an element of \mathcal{E}_X is a triple $u = (u_0, u_1, u_\top) \in (\mathbb{R}_{\geq 0})^3$ such that $u_0 + u_1 + u_\top \leq 1$. We have for instance $u = (1, 0, 0) \sqsubseteq_X v = (0, 0, 1)$ because in this case $\text{inv}(u, v) = \{0\}$. Notice that we do not have for instance $u = (1, 0, 0) \sqsubseteq_X v = (0, 1, 0)$ in spite of the fact that $u_\top \leq v_\top$ and $u_0 + u_1 + u_\top \leq v_0 + v_1 + v_\top$. In this case we need to use the sets $\text{inv}(u, v) = \{0\}$ to characterize \sqsubseteq_X : we do not have $u_0 + u_\top \leq v_0 + v_\top$ in this specific example.

We shall use the morphism $\bar{\varepsilon}_I \in \mathbf{Pcoh}^e(I_\perp^\top, \perp)$ given by $(\bar{\varepsilon}_I)_{a,*} = \delta_{a,\top}$, in other words $\widehat{\bar{\varepsilon}}_I(u) = u_\top$.

2.2.2 Case construct with error

Given an extensional PCS X , remember that we use X^I for the extensional PCS $\&_{i \in I} X_i$ where $X_i = X$ for each $i \in I$. Let J be another at most countable set. We define

$$\widetilde{\text{case}}^{I,J} \in (\mathbb{R}_{\geq 0})^{|I_\perp^\top \otimes !((J_\perp^\top)^I) \multimap J_\perp^\top|}$$

as follows:

$$\widetilde{\text{case}}_{a,\mu,b}^{I,J} = \begin{cases} 1 & \text{if } a = i \in I \text{ and } b \in |J_\perp^\top| \text{ and } \mu = [(i, b)] \\ 1 & \text{if } a = \top \text{ and } \mu = [] \text{ and } b = \top \\ 0 & \text{otherwise} \end{cases}$$

Given $u \in \mathbf{P}(I_\perp^\top)$ and $\vec{v} = (v(i))_{i \in I} \in \prod_{i \in I} \mathbf{P}(J_\perp^\top)$, let $w = \sum_{i \in I} u_i v(i) + u_\top e_\top \in (\mathbb{R}_{\geq 0})^{|J_\perp^\top|}$. We have

$$\begin{aligned} \sum_{j \in J} w_j + w_\top &= \sum_{j \in J} \sum_{i \in I} u_i v(i)_j + \sum_{i \in I} u_i v(i)_\top + u_\top = \sum_{i \in I} u_i \left(\sum_{j \in J} v(i)_j + v(i)_\top \right) + u_\top \\ &\leq \sum_{i \in I} u_i + u_\top \leq 1. \end{aligned}$$

This shows that $\widetilde{\text{case}}^{I,J} \in \mathbf{P}(I_\perp^\top \otimes !((J_\perp^\top)^I) \multimap J_\perp^\top)$, the associated function $\widehat{\text{case}}^{I,J} : \mathbf{P}(I_\perp^\top) \times \prod_{i \in I} \mathbf{P}(J_\perp^\top) \rightarrow \mathbf{P}(J_\perp^\top)$ being given by

$$\widehat{\text{case}}^{I,J}(u, \vec{v}) = \sum_{i \in I} u_i v(i) + u_\top e_\top$$

where $\vec{v} = (v(i))_{i \in I}$.

Lemma 19. *The morphism $\widetilde{\text{case}}^{I,J}$ is extensional, that is, it belongs to $\mathbf{Pcoh}^e(I_\perp^\top \otimes !((J_\perp^\top)^I), J_\perp^\top)$.*

Proof. Let $u^1, u^2 \in \mathcal{E}_{I_\perp^\top}$ be such that $u^1 \sqsubseteq_{I_\perp^\top} u^2$ and let $\vec{v}^1, \vec{v}^2 \in \prod_{i \in I} \mathcal{E}_{J_\perp^\top}$ be such that $v^1(i) \sqsubseteq_{J_\perp^\top} v^2(i)$ for each $i \in I$, we must prove that $\widehat{\text{case}}^{I,J}(u^1, \vec{v}^1) \sqsubseteq_{J_\perp^\top} \widehat{\text{case}}^{I,J}(u^2, \vec{v}^2)$. So let $w' \in \mathcal{E}_{(J_\perp^\top)^\perp}$ which simply means that $w' \in [0, 1]^{J \cup \{\top\}}$ and $\forall j \in J w'_j \leq w'_\top$. We have

$$\begin{aligned} \langle \widehat{\text{case}}^{I,J}(u^1, \vec{v}^1), w' \rangle &= \sum_{j \in J} \widehat{\text{case}}^{I,J}(u^1, \vec{v}^1)_j w'_j + \widehat{\text{case}}^{I,J}(u^1, \vec{v}^1)_\top w'_\top \\ &= \sum_{j \in J} \sum_{i \in I} u_i^1 v^1(i)_j w'_j + \sum_{i \in I} u_i^1 v^1(i)_\top w'_\top + u_\top^1 w'_\top \\ &= \sum_{i \in I} u_i^1 \langle v^1(i), w' \rangle + u_\top^1 w'_\top \\ &\leq \sum_{i \in I} u_i^1 \langle v^2(i), w' \rangle + u_\top^1 w'_\top \\ &= \langle u^1, u' \rangle \end{aligned}$$

where $u' \in [0, 1]^{I \cup \{\top\}}$ is defined by $u'_i = \langle v^2(i), w' \rangle$ and $u'_\top = w'_\top$. We have

$$u'_i = \sum_{j \in J} v^2(i)_j w'_j + v^2(i)_\top w'_\top \leq \sum_{j \in J} v^2(i)_j w'_\top + v^2(i)_\top w'_\top \quad \text{since } \forall j \ w'_j \leq w'_\top \leq w'_\top = u'_\top$$

hence $u'_i \leq u'_\top$ since $\sum_{j \in J} v^2(i)_j + v^2(i)_\top \leq 1$; this shows that $u' \in \mathcal{E}_{(I_\perp^\top)^\perp}$. It follows that $\langle u^1, u' \rangle \leq \langle u^2, u' \rangle$ since $u^1 \sqsubseteq_{I_\perp^\top} u^2$. Therefore

$$\langle \widehat{\text{case}}^{I,J}(u^1, \overrightarrow{v^1}), w' \rangle \leq \langle u^1, u' \rangle \leq \langle u^2, u' \rangle = \langle \widehat{\text{case}}^{I,J}(u^2, \overrightarrow{v^2}), w' \rangle$$

thus proving our contention. \square

2.2.3 The “let” with error

Let X be an extensional PCS and let I and J be two sets which are at most countable. Given $t \in \mathbf{Pcoh}^e(!X \otimes !I_\perp^\top, J_\perp^\top)$, we define $\widetilde{\text{let}}(t) \in (\mathbb{R}_{\geq 0})^{!X \otimes !I_\perp^\top \rightarrow J_\perp^\top}$ by

$$\widetilde{\text{let}}(t)_{\rho, a, b} = \begin{cases} 1 & \text{if } \rho = [] \text{ and } a = b = \top \\ \sum_{\mu \in \mathcal{M}_{\text{fin}}(\{i\})} t_{\rho, \mu, b} & \text{if } a = i \in I \end{cases}$$

The associated function $\widehat{s} : \mathbf{P}\underline{X} \times \mathbf{P}\left(\underline{I}_\perp^\top\right) \rightarrow \overline{\mathbb{R}}_{\geq 0}^{!J_\perp^\top}$ is characterized by $\widehat{s}(u, v) = \sum_{i \in I} v_i \widehat{t}(u, e_i) + v_\top e_\top$. We have

$$\begin{aligned} \sum_{j \in J} \widehat{s}(u, v)_j + \widehat{s}(u, v)_\top &= \sum_{j \in J} \sum_{i \in I} v_i \widehat{t}(u, e_i)_j + \sum_{i \in I} v_i \widehat{t}(u, e_i)_\top + v_\top \\ &= \sum_{i \in I} v_i \left(\sum_{j \in J} \widehat{t}(u, e_i)_j + \widehat{t}(u, e_i)_\top \right) + v_\top \\ &\leq \sum_{i \in I} v_i + v_\top \leq 1 \end{aligned}$$

which shows that $s \in \mathbf{Pcoh}^e(!X \otimes !I_\perp^\top, J_\perp^\top)$.

The next lemma uses the notations above.

Lemma 20. *Let $w' \in \mathcal{E}_{(J_\perp^\top)^\perp}$. Then $\langle \widehat{s}(u, v), w' \rangle = \langle v, v'(t, u, w') \rangle$ where $v'(t, u, w') \in \mathcal{E}_{(I_\perp^\top)^\perp}$ is given by: $v'(t, u, w')_i = \langle \widehat{t}(u, e_i), w' \rangle$ for $i \in I$ and $v'(t, u, w')_\top = w'_\top$. If $t(1) \sqsubseteq_{!X \otimes !I_\perp^\top, J_\perp^\top} t(2)$ and $u(1) \sqsubseteq_X u(2)$ then $v'(t(1), u(1), w') \sqsubseteq_{(J_\perp^\top)^\perp} v'(t(2), u(2), w')$.*

Now we prove that $s \in \mathbf{Pcoh}^e(!X \otimes !I_\perp^\top, J_\perp^\top)$. Since $\mathcal{E}_{J_\perp^\top} = \mathbf{P}\left(\underline{J}_\perp^\top\right)$ we only have to prove \sqsubseteq -monotonicity. So let $u(1), u(2) \in \mathcal{E}_X$ with $u(1) \sqsubseteq_X u(2)$ and let $v(1), v(2) \in \mathbf{P}\left(\underline{I}_\perp^\top\right)$ with $v(1) \sqsubseteq_{I_\perp^\top} v(2)$. For each $i \in I$ we have $\widehat{t}(u(1), e_i) \sqsubseteq_{J_\perp^\top} \widehat{t}(u(2), e_i)$. Let $w' \in \mathcal{E}_{(J_\perp^\top)^\perp}$, we have by Lemma 20

$$\langle \widehat{s}(u(1), v(1)), w' \rangle = \langle v(1), v'(t, u(1), w') \rangle \leq \langle v(2), v'(t, u(2), w') \rangle = \langle \widehat{s}(u(2), v(2)), w' \rangle$$

which proves our contention.

Lemma 21. *Let $s(1), s(2) \in \mathbf{Pcoh}^e(!X \otimes !I_\perp^\top, J_\perp^\top)$. If $s(1) \sqsubseteq_{!X \otimes !I_\perp^\top, J_\perp^\top} s(2)$, then*

$$t(1) = \widetilde{\text{let}}(s(1)) \sqsubseteq_{!X \otimes !I_\perp^\top, J_\perp^\top} t(2) = \widetilde{\text{let}}(s(2)).$$

Proof. Let $u \in \mathcal{E}_X$, $v \in \mathcal{E}_{I_\perp^\top} = \mathbf{P}\left(\underline{I}_\perp^\top\right)$ and $w' \in \mathcal{E}_{(J_\perp^\top)^\perp}$. We have by Lemma 20

$$\langle \widehat{t(1)}(u, v), w' \rangle = \langle v, v'(s(1), u, w') \rangle \leq \langle v, v'(s(2), u, w') \rangle = \langle \widehat{t(2)}(u, v), w' \rangle.$$

\square

2.2.4 Basic functions with error

With the same notations as above, let $f : I \rightarrow J$ be a partial function of domain $D \subseteq I$. Then we define $\tilde{f} \in (\mathbb{R}_{\geq 0})^{|I_{\perp}^{\top} \rightarrow J_{\perp}^{\top}|}$ as follows:

$$\tilde{f}_{a,b} = \begin{cases} 1 & \text{if } a \in D \text{ and } b = f(a) \in J \\ 1 & \text{if } a = b = \top \\ 0 & \text{otherwise.} \end{cases}$$

We have $\tilde{f} \in \mathbf{Pcoh}^e(I_{\perp}^{\top}, J_{\perp}^{\top})$. Indeed let first $u \in \mathcal{E}_{I_{\perp}^{\top}} = \mathbf{P}I_{\perp}^{\top}$, we have

$$\sum_{b \in |\mathcal{E}_{J_{\perp}^{\top}}|} (\tilde{f} \cdot u)_b = \sum_{j \in J} (\tilde{f} \cdot u)_j + (\tilde{f} \cdot u)_{\top} = \sum_{j \in J} \sum_{\substack{i \in D \\ f(i)=j}} u_j + x_{\top} = \sum_{i \in D} u_i + u_{\top} \leq 1$$

and hence $\tilde{f} \cdot u \in \mathbf{PE}_{J_{\perp}^{\top}}$. Assume now that $v(1) \sqsubseteq_{I_{\perp}^{\top}} v(2)$ and let $w' \in \mathcal{E}_{(J_{\perp}^{\top})^{\perp}}$, we have

$$\langle \tilde{f} \cdot v(1), w' \rangle = \sum_{j \in J} (\tilde{f} \cdot x(1))_j w'_j + u_{\top} w'_{\top} = \langle v(1), v' \rangle$$

where $v' \in (\mathbb{R}_{\geq 0})^{|I_{\perp}^{\top}|}$ is defined by $v'_i = w'_{f(i)}$ if $i \in D$, $v'_i = 0$ if $i \in I \setminus D$ and $v'_{\top} = w'_{\top}$ so that obviously $v' \in \mathcal{E}_{(I_{\perp}^{\top})^{\perp}}$. Therefore $\langle \tilde{f} \cdot v(1), w' \rangle \leq \langle \tilde{f} \cdot v(2), w' \rangle$.

2.2.5 Upper and lower approximating the identity at ground types.

Let $J \subseteq I$, we define $\text{ld}_{J,I}^{\sqsubseteq}, \text{ld}_{J,I}^{\supseteq} \in (\mathbb{R}_{\geq 0})^{|I_{\perp}^{\top} \rightarrow I_{\perp}^{\top}|}$ as follows:

$$(\text{ld}_{J,I}^{\sqsubseteq})_{a,b} = \begin{cases} 1 & \text{if } a = b \in J \cup \{\top\} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (\text{ld}_{J,I}^{\supseteq})_{a,b} = \begin{cases} 1 & \text{if } a = b \in J \cup \{\top\} \\ 1 & \text{if } a \in I \setminus J \text{ and } b = \top \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\text{ld}_{J,I}^{\sqsubseteq}, \text{ld}_{J,I}^{\supseteq} \in \mathbf{P}(I_{\perp}^{\top} \rightarrow I_{\perp}^{\top})$. Notice that, if $v \in \mathbf{P}(I_{\perp}^{\top})$, we have

$$\widehat{\text{ld}_{J,I}^{\sqsubseteq}}(v) = \sum_{j \in J} v_j e_j + v_{\top} e_{\top} \quad \text{and} \quad \widehat{\text{ld}_{J,I}^{\supseteq}}(v) = \sum_{j \in J} v_j e_j + \left(\sum_{i \in I \setminus J} v_i + x_{\top} \right) e_{\top}$$

The fact that $\text{ld}_{J,I}^{\sqsubseteq} \in \mathbf{Pcoh}^e(I_{\perp}^{\top}, I_{\perp}^{\top})$ is a consequence of Section 2.2.4 applied to the restriction of the identity function to I . Next we obviously have $\text{ld}_{J,I}^{\sqsubseteq} \leq \text{ld}$ and hence $\text{ld}_{J,I}^{\sqsubseteq} \sqsubseteq_{I_{\perp}^{\top}, I_{\perp}^{\top}} \text{ld}$.

Lemma 22. *One has $\text{ld}_{J,I}^{\supseteq} \in \mathbf{Pcoh}^e(I_{\perp}^{\top}, J_{\perp}^{\top})$ and $\text{ld} \sqsubseteq_{I_{\perp}^{\top}, I_{\perp}^{\top}} \text{ld}_{J,I}^{\supseteq}$.*

3 Probabilistic PCF with errors

For the sake of simplicity and readability our language of interest is a simple and nonetheless very expressive probabilistic extension of the well known Scott-Plotkin PCF language, extended with one uncatchable exception \mathcal{U} of ground type called *convergence*. For convenience we also add a constant Ω for representing *divergence* (it could be defined using the $\text{fix}(M)$ construct). As in [10] the language has a $\text{let}(x, M, N)$ construct allowing to deal with the ground data type ι in a call-by-value manner; this is essential to implement

$$\begin{array}{c}
\frac{n \in \mathbb{N}}{\Gamma \vdash \underline{n} : \iota} \quad \frac{E \in \{\Omega, \mathcal{U}\}}{\Gamma \vdash E : \iota} \quad \frac{}{x_1 : \sigma_1, \dots, x_k : \sigma_k \vdash x_i : \sigma_i} \quad \frac{r \in [0, 1] \cap \mathbb{Q}}{\Gamma \vdash \text{coin}(r) : \iota} \quad \frac{\Gamma \vdash M : \iota}{\Gamma \vdash \text{succ}(M) : \iota} \\
\frac{\Gamma \vdash M : \iota}{\Gamma \vdash \text{pred}(M) : \iota} \quad \frac{\Gamma \vdash M : \iota \quad \Gamma \vdash P : \iota \quad \Gamma \vdash Q : \iota}{\Gamma \vdash \text{if}(M, P, Q) : \iota} \quad \frac{\Gamma \vdash M : \iota \quad \Gamma, x : \iota \vdash P : \iota}{\Gamma \vdash \text{let}(x, M, P) : \iota} \\
\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x^\sigma M : \sigma \Rightarrow \tau} \quad \frac{\Gamma \vdash M : \sigma \Rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash (M)N : \tau} \quad \frac{\Gamma \vdash M : \sigma \Rightarrow \sigma}{\Gamma \vdash \text{fix}(M) : \sigma}
\end{array}$$

Figure 1: Typing rule for our PCF dialect

$$\begin{array}{c}
\frac{}{\text{succ}(\underline{n}) \beta_{\text{wh}} \underline{n+1}} \quad \frac{}{\text{pred}(\underline{0}) \beta_{\text{wh}} \underline{0}} \quad \frac{}{\text{pred}(\underline{n+1}) \beta_{\text{wh}} \underline{n}} \quad \frac{E \in \{\Omega, \mathcal{U}\}}{\text{succ}(E) \beta_{\text{wh}} E} \quad \frac{E \in \{\Omega, \mathcal{U}\}}{\text{pred}(E) \beta_{\text{wh}} E} \\
\frac{}{\text{if}(\underline{0}, P, Q) \beta_{\text{wh}} P} \quad \frac{}{\text{if}(\underline{n+1}, P, Q) \beta_{\text{wh}} Q} \quad \frac{E \in \{\Omega, \mathcal{U}\}}{\text{if}(E, P, Q) \beta_{\text{wh}} E} \quad \frac{}{\text{let}(x, \underline{n}, M) \beta_{\text{wh}} M [\underline{n}/x]} \\
\frac{E \in \{\Omega, \mathcal{U}\}}{\text{let}(x, E, M) \beta_{\text{wh}} E} \quad \frac{}{(\lambda x^\sigma M)N \beta_{\text{wh}} M [N/x]} \quad \frac{}{\text{fix}(M) \beta_{\text{wh}} (M)\text{fix}(M)} \\
\frac{M \beta_{\text{wh}} M'}{M \beta_{\text{wh}}^1 M'} \quad \frac{r \in [0, 1] \cap \mathbb{Q}}{\text{coin}(r) \beta_{\text{wh}}^r \underline{0}} \quad \frac{r \in [0, 1] \cap \mathbb{Q}}{\text{coin}(r) \beta_{\text{wh}}^{1-r} \underline{1}} \quad \frac{M \beta_{\text{wh}}^p M'}{\text{succ}(M) \beta_{\text{wh}}^p \text{succ}(M')} \quad \frac{M \beta_{\text{wh}}^p M'}{\text{pred}(M) \beta_{\text{wh}}^p \text{pred}(M')} \\
\frac{M \beta_{\text{wh}}^p M'}{\text{if}(M, P, Q) \beta_{\text{wh}}^p \text{if}(M', P, Q)} \quad \frac{M \beta_{\text{wh}}^p M'}{\text{let}(x, M, P) \beta_{\text{wh}}^p \text{let}(x, M', P)}
\end{array}$$

Figure 2: Weak head reduction for our PCF dialect

meaningful probabilistic algorithms in this language which is call-by-name globally.

$$\begin{aligned}
&\sigma, \tau, \dots := \iota \mid \sigma \Rightarrow \tau \\
&M, N, P, \dots := x \mid \underline{n} \mid \text{succ}(M) \mid \text{pred}(M) \mid \text{if}(M, N, P) \mid \text{let}(x, M, N) \\
&\quad \mid (M)N \mid \lambda x^\sigma M \mid \text{fix}(M) \mid \text{coin}(r) \mid \Omega \mid \mathcal{U}
\end{aligned}$$

The typing rules are given in Figure 1. Notice the strong typing restrictions on the conditional and let constructs, due to the fact that our convergence and divergence constants are of ground types. Given a typing context Γ and a type σ , we use $\langle \Gamma \vdash \sigma \rangle$ for the set of terms M such that $\Gamma \vdash M : \sigma$.

Remark. This is not a restriction in term of expressiveness since more general versions of these constructs can be defined using λ -abstractions. Nevertheless this is clearly not satisfactory, and the solution is to develop a monadic description of the error \mathcal{U} in **Pcoh**^e. This is not completely straightforward because this description has to be compatible with the extensional structures.

3.1 Operational semantics

We equip this language with an operational semantics which is a probabilistic extension of the usual deterministic weak head reduction. More precisely, $M \beta_{\text{wh}} M'$ means that M reduces to M' deterministically and $M \beta_{\text{wh}}^p M'$ means that M reduces to M' with probability $p \in [0, 1]$. The reduction rules are given in Figure 2.

Lemma 23. *If $M \beta_{\text{wh}}^p M'$ then M and M' are distinct terms, and p can be recovered from M and M' .*

This results from a simple inspection of the rule. A convergence path is a sequence $\gamma = (M_0, \dots, M_n)$ with $n \in \mathbb{N}$ such that there are $p_0, \dots, p_{n-1} \in [0, 1]$ such that $\forall i \in \{0, \dots, n-1\} M_i \beta_{\text{wh}}^{p_i} M_{i+1}$ and M_n is β_{wh}^p -normal for all p (we simply say that M_n is wh-normal). By Lemma 23, the sequence (p_0, \dots, p_{n-1}) is determined by γ . We use the following notations: $\text{len}(\gamma) = n$ (length), $\text{s}(\gamma) = M_0$ and $\text{t}(\gamma) = M_n$ (source and target), and $\text{pr}(\gamma) = \prod_{i=0}^{n-1} p_i$ (probability). Given $M, P \in \langle \Gamma \vdash \sigma \rangle$ with P wh-normal, we use $\text{cp}(M, P)$ for the set of all convergence paths γ such that $\text{s}(\gamma) = M$ and $\text{t}(\gamma) = P$. Then

$$\mathbb{P}(M \downarrow P) = \sum_{\gamma \in \text{cp}(M, P)} \text{pr}(\gamma)$$

$$\begin{array}{c}
\frac{M \in \langle \Gamma \vdash \iota \rangle}{\Omega \sqsubseteq M} \quad \frac{M \in \langle \Gamma \vdash \iota \rangle}{M \sqsubseteq \mathcal{U}} \quad \frac{}{x \sqsubseteq x} \quad \frac{}{\underline{n} \sqsubseteq \underline{n}} \quad \frac{r \in [0, 1] \cap \mathbb{Q}}{\text{coin}(r) \sqsubseteq \text{coin}(r)} \quad \frac{M \sqsubseteq M'}{\text{succ}(M) \sqsubseteq \text{succ}(M')} \\
\frac{M \sqsubseteq M'}{\text{pred}(M) \sqsubseteq \text{pred}(M')} \quad \frac{M \sqsubseteq M' \quad P \sqsubseteq P' \quad Q \sqsubseteq Q'}{\text{if}(M, P, Q) \sqsubseteq \text{if}(M', P', Q')} \quad \frac{M \sqsubseteq M' \quad P \sqsubseteq P'}{\text{let}(x, M, P) \sqsubseteq \text{let}(x, M', P')} \\
\frac{M \sqsubseteq M' \quad N \sqsubseteq N'}{(M)N \sqsubseteq (M')N'} \quad \frac{M \sqsubseteq M'}{\lambda x^\sigma M \sqsubseteq \lambda x^\sigma M'} \quad \frac{M \sqsubseteq M'}{\text{fix}(M) \sqsubseteq \text{fix}(M')} \\
\frac{M \sqsubseteq M' \quad \text{fix}(M) \sqsubseteq N'}{\text{fix}(M) \sqsubseteq (M')N'} \quad \frac{M' \sqsubseteq M \quad N' \sqsubseteq \text{fix}(M)}{(M')N' \sqsubseteq \text{fix}(M)}
\end{array}$$

Figure 3: Extensional preorder on terms

is the probability that M reduces to P . See [10] for a discrete Markov chain interpretation of this definition¹², showing in particular that this number is actually a probability.

Example 24. Consider the term $M = \text{if}(\text{coin}(\frac{1}{2}), \underline{0}, \underline{0})$. Then we have two distinct convergence paths from M , with the same target $\underline{0}$, namely $(M, \text{if}(\underline{0}, \underline{0}, \underline{0}), \underline{0})$ and $(M, \text{if}(\underline{1}, \underline{0}, \underline{0}), \underline{0})$. Both have probability $\frac{1}{2}$ and we have $\mathbb{P}(M \downarrow \underline{0}) = 1$.

A term $P \in \langle \vdash \iota \rangle$ is wh-normal iff $P = \underline{n}$ for some $n \in \mathbb{N}$ or $P \in \{\Omega, \mathcal{U}\}$. For $M \in \langle \vdash \iota \rangle$, we are mainly interested in evaluating $\mathbb{P}(M \downarrow \mathcal{U})$, that we also denote as $\mathbb{P}_{\mathcal{U}}(P)$.

3.2 A preorder relation on terms

We define a binary relation \sqsubseteq by the deduction rules of Figure 3.

Lemma 25. If $\Gamma \vdash M : \sigma$ and $M' \sqsubseteq M$ or $M \sqsubseteq M'$, then $\Gamma \vdash M' : \sigma$.

Proof. Easy induction on the derivation of $M' \sqsubseteq M$ or $M \sqsubseteq M'$. □

The following property is natural and worth being noticed, though it plays no technical role in the sequel.

Proposition 26. The relation \sqsubseteq is a preorder relation on $\langle \Gamma \vdash \sigma \rangle$, for each context Γ and type σ .

3.3 Denotational semantics in \mathbf{Pcoh}^e

We use the functions $\mathbf{s}, \mathbf{p} : \mathbb{N} \rightarrow \mathbb{N}$: $\mathbf{s}(n) = n + 1$, $\mathbf{p}(0) = 0$ and $\mathbf{p}(n + 1) = n$.

Given a type σ , we define an extensional PCS $\llbracket \sigma \rrbracket$ by induction: $\llbracket \iota \rrbracket = \mathbb{N}_\perp^\top$ and $\llbracket \sigma \Rightarrow \tau \rrbracket = \llbracket \sigma \rrbracket \Rightarrow \llbracket \tau \rrbracket$. Given a context $\Gamma = (x_1 : \sigma_1, \dots, x_k : \sigma_k)$, we define $\llbracket \Gamma \rrbracket = \&_{i=1}^k \llbracket \sigma_i \rrbracket$ which is an object of \mathbf{Pcoh}^e .

Next given $M \in \langle \Gamma \vdash \tau \rangle$ we define $\llbracket M \rrbracket_\Gamma \in \mathbf{Pcoh}^e!(\llbracket \Gamma \rrbracket, \llbracket \tau \rrbracket)$ by induction on M . We know that this morphism is fully characterized by the associated function $f = \widehat{\llbracket M \rrbracket}_\Gamma : \mathbf{P}(\&_{i=1}^k \llbracket \sigma_i \rrbracket) \rightarrow \mathbf{P}\llbracket \tau \rrbracket$. Let

$C = \&_{i=1}^k \llbracket \sigma_i \rrbracket$.

► If $M = x_i$ then $\llbracket M \rrbracket_\Gamma$ is defined as the following composition of morphisms in \mathbf{Pcoh}^e :

$!C \xrightarrow{\text{der}_C} C \xrightarrow{\pi_i} \llbracket \sigma_i \rrbracket$, so that $f(\vec{u}) = u_i$.

► If $M = \underline{n}$ then $\llbracket M \rrbracket_\Gamma$ is defined as the following composition of morphisms in \mathbf{Pcoh}^e :

$!C \xrightarrow{\text{w}_C} 1 \xrightarrow{e_n} \mathbb{N}_\perp^\top$ so that $f(\vec{u}) = e_n$.

► If $M = \Omega$ then $\llbracket M \rrbracket_\Gamma = 0$ so that $f(\vec{u}) = 0$.

► If $M = \mathcal{U}$ then $\llbracket M \rrbracket_\Gamma$ is defined as the following composition of morphisms in \mathbf{Pcoh}^e :

$!C \xrightarrow{\text{w}_C} 1 \xrightarrow{e_\top} \mathbb{N}_\perp^\top$ so that $f(\vec{u}) = e_\top$.

¹²For a version of PCF without the exceptions \mathcal{U} and Ω , but their addition to the language does not change the proofs and results.

► If $M = \text{succ}(N)$ with $N \in \langle \Gamma \vdash \iota \rangle$ then $\llbracket M \rrbracket_\Gamma$ is defined as the following composition of morphisms in \mathbf{Pcoh}^e : $!C \xrightarrow{\llbracket N \rrbracket_\Gamma} \mathbb{N}_\perp^\top \xrightarrow{\tilde{s}} \mathbb{N}_\perp^\top$ so that $f(\vec{u}) = \sum_{n \in \mathbb{N}} g(\vec{u})_n e_{n+1} + g(\vec{u})_\top e_\top$ where $g = \widehat{\llbracket N \rrbracket}_\Gamma$.

► If $M = \text{pred}(N)$ with $N \in \langle \Gamma \vdash \iota \rangle$ then $\llbracket M \rrbracket_\Gamma$ is defined as the following composition of morphisms in \mathbf{Pcoh}^e : $!C \xrightarrow{\llbracket N \rrbracket_\Gamma} \mathbb{N}_\perp^\top \xrightarrow{\tilde{p}} \mathbb{N}_\perp^\top$ so that $f(\vec{u}) = g(\vec{u})_0 e_0 + \sum_{n \in \mathbb{N}} g(\vec{u})_{n+1} e_n + g(\vec{u})_\top e_\top$ where $g = \widehat{\llbracket N \rrbracket}_\Gamma$.

► Assume that $M = \text{if}(N, P_1, P_2)$ with $N, P_1, P_2 \in \langle \Gamma \vdash \iota \rangle$. Let $s = \llbracket N \rrbracket_\Gamma$ and, for $n \in \mathbb{N}$, let $t_n \in \mathbf{Pcoh}^e_!(C, \mathbb{N}_\perp^\top)$ be defined by $t_0 = \llbracket P_1 \rrbracket_\Gamma$ and $t_{n+1} = \llbracket P_2 \rrbracket_\Gamma$ for each $n \in \mathbb{N}$. Let $t = \langle t_n \rangle_{n \in \mathbb{N}} \in \mathbf{Pcoh}^e_!(C, (\mathbb{N}_\perp^\top)^\mathbb{N})$. Then $\llbracket M \rrbracket_\Gamma$ is defined as the following composition of morphisms in \mathbf{Pcoh}^e :

$$!C \xrightarrow{\text{contr}_C} !C \otimes !C \xrightarrow{s \otimes t^!} \mathbb{N}_\perp^\top \otimes !((\mathbb{N}_\perp^\top)^\mathbb{N}) \xrightarrow{\widehat{\text{case}}^{\mathbb{N}, \mathbb{N}}} \mathbb{N}_\perp^\top$$

so that $f(\vec{u}) = g(\vec{u})_0 h_1(\vec{u}) + (\sum_{n=0}^\infty g(\vec{u})_{n+1}) h_2(\vec{u}) + g(\vec{u})_\top e_\top$ where $g = \widehat{\llbracket N \rrbracket}_\Gamma$ and $h_i = \widehat{\llbracket P_i \rrbracket}_\Gamma$ for $i = 1, 2$.

► Assume that $M = \text{let}(x, N, P)$ with $N \in \langle \Gamma \vdash \iota \rangle$ and $P \in \langle \Gamma, x : \iota \vdash \iota \rangle$. Let $s = \llbracket N \rrbracket_\Gamma \in \mathbf{Pcoh}^e_!(C, \mathbb{N}_\perp^\top)$ and $t = \llbracket P \rrbracket_{\Gamma, x : \iota} \in \mathbf{Pcoh}^e_!(C \& \mathbb{N}_\perp^\top, \mathbb{N}_\perp^\top)$ so that $t \mathbf{m}^2 \in \mathbf{Pcoh}^e(!C \otimes !\mathbb{N}_\perp^\top, \mathbb{N}_\perp^\top)$. Then $\llbracket M \rrbracket_\Gamma$ is defined as the following composition of morphisms in \mathbf{Pcoh}^e :

$$!C \xrightarrow{\text{contr}_C} !C \otimes !C \xrightarrow{!C \otimes s} !C \otimes \mathbb{N}_\perp^\top \xrightarrow{\widehat{\text{let}}(t \mathbf{m}^2)} \mathbb{N}_\perp^\top$$

so that $f(\vec{u}) = \sum_{n \in \mathbb{N}} g(\vec{u})_n h(\vec{u}, e_n) + g(\vec{u})_\top e_\top$ where $g = \widehat{\llbracket N \rrbracket}_\Gamma$ and $h = \widehat{\llbracket P \rrbracket}_{\Gamma, x : \iota}$.

► Assume that $M = (N)P$ with $N \in \langle \Gamma \vdash \sigma \Rightarrow \tau \rangle$ and $P \in \langle \Gamma \vdash \sigma \rangle$. Let $s = \llbracket N \rrbracket_\Gamma \in \mathbf{Pcoh}^e_!(C, \llbracket \sigma \Rightarrow \tau \rrbracket)$ and $t = \llbracket P \rrbracket_\Gamma \in \mathbf{Pcoh}^e_!(C, \llbracket \sigma \rrbracket)$. Then $\llbracket M \rrbracket_\Gamma$ is defined as the following composition of morphisms in \mathbf{Pcoh}^e :

$$!C \xrightarrow{\text{contr}_C} !C \otimes !C \xrightarrow{s \otimes t^!} (!\llbracket \sigma \rrbracket \multimap \llbracket \tau \rrbracket) \otimes !\llbracket \sigma \rrbracket \xrightarrow{\text{ev}} \llbracket \tau \rrbracket$$

so that $f(\vec{u}) = \widehat{g(\vec{u})}(h(\vec{u}))$ where $g = \widehat{\llbracket N \rrbracket}_\Gamma$ and $h = \widehat{\llbracket t \rrbracket}_\Gamma$.

► If $M = \lambda x^\sigma N$ with $N \in \langle \Gamma, x : \sigma \vdash \varphi \rangle$ (so that $\tau = (\sigma \Rightarrow \varphi)$), let $s = \llbracket N \rrbracket_{\Gamma, x : \sigma} \in \mathbf{Pcoh}^e_!(C \& \llbracket \sigma \rrbracket, \llbracket \varphi \rrbracket)$, so that $s \mathbf{m}^2 \in \mathbf{Pcoh}^e(!C \otimes !\llbracket \sigma \rrbracket, \llbracket \varphi \rrbracket)$. Then $\llbracket M \rrbracket_\Gamma = \text{cur}(s \mathbf{m}^2) \in \mathbf{Pcoh}^e_!(C, \llbracket \tau \rrbracket)$ so that $f(\vec{u})$ is the element of $\mathcal{E}_{\llbracket \sigma \Rightarrow \varphi \rrbracket}$ characterized by $\forall u \in \mathbb{P}\llbracket \sigma \rrbracket \ f(\vec{u}) = g(\vec{u}, u)$ where $g = \widehat{\llbracket N \rrbracket}_{\Gamma, x : \sigma}$.

► If $M = \text{fix}(N)$ with $N \in \langle \Gamma \vdash \tau \Rightarrow \tau \rangle$, let $s = \llbracket N \rrbracket_\Gamma \in \mathbf{Pcoh}^e_!(C, \llbracket \tau \rrbracket \Rightarrow \llbracket \tau \rrbracket)$. Then $\llbracket M \rrbracket_\Gamma = \text{Fix}_{\llbracket \tau \rrbracket} s$ which belongs to $\mathbf{Pcoh}^e_!(C, \llbracket \tau \rrbracket)$ by Theorem 15 so that $f(\vec{u}) = \sup_{n \in \mathbb{N}} h^n(0)$ where $h = \widehat{g(\vec{u})}$, where $g = \widehat{\llbracket N \rrbracket}_\Gamma$. Notice that $t = \llbracket M \rrbracket_\Gamma$ satisfies the following commutation in \mathbf{Pcoh}^e :

$$\begin{array}{ccc} !C \xrightarrow{\text{contr}_C} !C \otimes !C \xrightarrow{s \otimes t^!} (!\llbracket \tau \rrbracket \multimap \llbracket \tau \rrbracket) \otimes !\llbracket \tau \rrbracket & & \\ & \searrow t & \downarrow \text{ev} \\ & & \llbracket \tau \rrbracket \end{array} \quad (1)$$

Lemma 27 (Substitution). *If $\Gamma, x : \sigma \vdash M : \tau$ and $\Gamma \vdash N : \sigma$ then $\llbracket M[N/x] \rrbracket_\Gamma$ coincides with the composition of morphisms (setting $C = \llbracket \Gamma \rrbracket$) $!C \xrightarrow{\text{contr}_C} !C \otimes !C \xrightarrow{\text{Id} \otimes \llbracket N \rrbracket_\Gamma^!} !C \otimes !\llbracket \sigma \rrbracket \xrightarrow{\llbracket M \rrbracket_\Gamma} \llbracket \tau \rrbracket$ where we keep implicit the Seely isomorphism.*

Proof. Simple induction on M . □

Theorem 28. *If $M \in \langle \vdash \iota \rangle$ then $\llbracket M \rrbracket_\top = \mathbb{P}_\top(M)$.*

Proof. Follows exactly the same pattern as the proof of adequacy in [10]. □

Theorem 29. *Let $\Gamma = (x_1 : \sigma_1, \dots, x_k : \sigma_k)$ be a typing context and assume that $M_1, M_2 \in \langle \vdash \tau \rangle$ and $M_1 \sqsubseteq M_2$. Then for all $\vec{u} \in \prod_{i=1}^k \mathcal{E}_{\llbracket \sigma_i \rrbracket}$ one has $\widehat{\llbracket M_1 \rrbracket}_\Gamma(\vec{u}) \sqsubseteq_{\llbracket \tau \rrbracket} \widehat{\llbracket M_2 \rrbracket}_\Gamma(\vec{u})$.*

Proof. By induction on the height of the proof of $M \sqsubseteq N$ in the deduction system of Figure 3.

If $M = \Omega$ and N is any term such that $\Gamma \vdash N : \iota$ then we use the fact that \emptyset is $\sqsubseteq_{\mathbb{N}_\perp^\top}$ -minimal.

Assume $M = x_i$, then $N = x_i$ or $N = \mathcal{U}$ (if $\sigma_i = \iota$). In the first case we use reflexivity of $\sqsubseteq_{[\tau_i]}$ (see Proposition 3) and in the first case we use the fact that e_\top is $\sqsubseteq_{\mathbb{N}_\perp^\top}$ -maximal. The cases $M = \underline{n}$ and $M = \text{coin}(r)$ are similar.

Assume that $M = \text{succ}(M_0)$ so that $\tau = \iota$ and $\Gamma \vdash M_0 : \iota$. Then we have either $N = \mathcal{U}$ or $N = \text{succ}(N_0)$ with $M_0 \sqsubseteq N_0$. In the first case we use as above the $\sqsubseteq_{\mathbb{N}_\perp^\top}$ -maximality of e_\top . In the second case, we use the inductive hypothesis, the definition of $\llbracket M \rrbracket_\Gamma$ and the $\sqsubseteq_{\mathbb{N}_\perp^\top}$ -monotonicity of $\tilde{\omega}$, see Section 2.2.4. The cases $M = \text{pred}(M_0)$ and $\text{if}(M_0, M_1, M_2)$ are similar, using Section 2.2.2 for the conditional.

Assume now that $M = \text{let}(x, M_0, M_1)$ with $\Gamma \vdash M_0 : \iota$ and $\Gamma, x : \iota \vdash M_1 : \iota$. The case $N = \top$ is dealt with as above so assume that $N = \text{let}(x, N_0, N_1)$ with $M_i \sqsubseteq N_i$ for $i = 0, 1$. By inductive hypothesis we have $\llbracket M_1 \rrbracket_{\Gamma, x : \iota} \sqsubseteq_{![\Gamma, x : \iota], [\iota]} \llbracket N_1 \rrbracket_{\Gamma, x : \iota}$ and hence $\llbracket M_1 \rrbracket_{\Gamma, x : \iota} \mathfrak{m}^2 \sqsubseteq_{![\Gamma, x : \iota] \otimes !\mathbb{N}_\perp^\top, \mathbb{N}_\perp^\top} \llbracket N_1 \rrbracket_{\Gamma, x : \iota} \mathfrak{m}^2$. It follows that $\widetilde{\text{let}}(\llbracket M_1 \rrbracket_{\Gamma, x : \iota} \mathfrak{m}^2) \sqsubseteq_{![\Gamma, x : \iota] \otimes \mathbb{N}_\perp^\top, \mathbb{N}_\perp^\top} \widetilde{\text{let}}(\llbracket N_1 \rrbracket_{\Gamma, x : \iota} \mathfrak{m}^2)$ by Section 2.2.3. By inductive hypothesis we have also $\llbracket M_1 \rrbracket_\Gamma \sqsubseteq_{![\Gamma], \mathbb{N}_\perp^\top} \llbracket N_1 \rrbracket_\Gamma$ and hence $\llbracket M \rrbracket_\Gamma \sqsubseteq_{![\Gamma], \mathbb{N}_\perp^\top} \llbracket N \rrbracket_\Gamma$ by definition of these interpretations and \sqsubseteq -monotonicity of composition. The case $M = \lambda x^\sigma M_0$ with $\Gamma, x : \sigma \vdash M_0 : \varphi$ (and hence $\tau = \sigma \Rightarrow \varphi$) is similar.

Assume that $M = (M_0)M_1$ with $\Gamma \vdash M_0 : \sigma \Rightarrow \tau$ and $\Gamma \vdash M_1 : \sigma$. The case $N = \mathcal{U}$ (and hence $\tau = \iota$) is dealt with as above. Assume that $N = (N_0)N_1$ with $M_i \sqsubseteq N_i$ for $i = 0, 1$. By inductive hypothesis $\llbracket M_1 \rrbracket_\Gamma \sqsubseteq_{![\Gamma], [\sigma]} \llbracket N_1 \rrbracket_\Gamma$ and hence $\llbracket M_1 \rrbracket_\Gamma^! \sqsubseteq_{![\Gamma], ![\sigma]} \llbracket N_1 \rrbracket_\Gamma^!$ (see Section 2.1.4). We also have $\llbracket M_0 \rrbracket_\Gamma \sqsubseteq_{![\Gamma], ![\sigma] \rightarrow ![\tau]} \llbracket N_0 \rrbracket_\Gamma$ and hence $\llbracket M \rrbracket_\Gamma \sqsubseteq_{![\Gamma], ![\tau]} \llbracket N \rrbracket_\Gamma$ by definition of these interpretations and \sqsubseteq -monotonicity of ev . Assume last that $N = \text{fix}(N_0)$ with $M_0 \sqsubseteq N_0$ and $M_1 \sqsubseteq N$ (so that $\sigma = \tau$ and $\Gamma \vdash N_0 : \tau \Rightarrow \tau$), by derivations shorter than that of $M \sqsubseteq N$. Setting $C = ![\Gamma]$, $s = \llbracket N_0 \rrbracket_\Gamma$ and $t = \llbracket N \rrbracket_\Gamma$, we know that Diagram (1) commutes. By inductive hypothesis, we have $\llbracket M_0 \rrbracket_\Gamma \sqsubseteq_{!C, ![\tau] \rightarrow ![\tau]} \llbracket N_0 \rrbracket_\Gamma$ and $\llbracket M_1 \rrbracket_\Gamma \sqsubseteq_{!C, ![\tau]} \llbracket N \rrbracket_\Gamma$ and hence

$$\llbracket M \rrbracket_\Gamma = \text{ev}(\llbracket M_0 \rrbracket_\Gamma \otimes \llbracket M_1 \rrbracket_\Gamma^!) \text{contr}_C \sqsubseteq_{!C, ![\tau]} \text{ev}(\llbracket N_0 \rrbracket_\Gamma \otimes \llbracket N \rrbracket_\Gamma^!) \text{contr}_C = \llbracket N \rrbracket_\Gamma.$$

Assume last that $M = \text{fix}(M_0)$ with $\Gamma \vdash M_0 : \tau \Rightarrow \tau$. The case $N = \mathcal{U}$ (and hence $\tau = \iota$) is dealt with as above. Assume $N = \text{fix}(N_0)$ with $M_0 \sqsubseteq N_0$. By inductive hypothesis we have $\llbracket M_0 \rrbracket_\Gamma \sqsubseteq_{!C, ![\tau] \rightarrow ![\tau]} \llbracket N_0 \rrbracket_\Gamma$ and hence

$$\llbracket M \rrbracket_\Gamma = \text{Fix}_{![\tau]} \llbracket M_0 \rrbracket_\Gamma^! \sqsubseteq_{!C, ![\tau]} \text{Fix}_{![\tau]} \llbracket N_0 \rrbracket_\Gamma^! = \llbracket N \rrbracket_\Gamma$$

by Section 2.1.4 and Theorem 15. The last case is $N = (N_0)N_1$ with $M_0 \sqsubseteq N_0$, $M_0 \sqsubseteq N_0$ and $M \sqsubseteq N_1$, which is dealt with as the case $M = (M_0)M_1$ and $N = \text{fix}(N_0)$ above. \square

Combining the two previous theorems we get:

Theorem 30. *If $M, N \in \langle \vdash \iota \rangle$ satisfy $M \sqsubseteq N$, one has $\mathbb{P}_{\mathcal{U}}(M) = \llbracket M \rrbracket_\top \leq \llbracket N \rrbracket_\top = \mathbb{P}_{\mathcal{U}}(N)$.*

4 Approximating probabilities of convergence with a Krivine machine

4.1 A Krivine machine computing polynomials

Here we present the outputs of the machine which are trees representing some kind of polynomials, and its inputs which, not surprisingly are pairs made of a term and a stack. The main peculiarities to keep in mind is that these states are not closed but only “almost closed” in the sense that all free variables have ground type.

4.1.1 Infinite polynomials.

Let I be an at most countable set and V be a finite set of variables. Intuitively, a variable $x \in V$ represents an I -indexed family of scalars taken in some fixed semiring \mathbb{K} (in the sequel, \mathbb{K} will be $\mathbb{R}_{\geq 0}$). So for each $x \in V$

and $i \in I$ we introduce the notation $x(i)$ representing intuitively the i th component of x . A multi-exponent is a family $\vec{\mu} = (\mu_x)_{x \in V} \in \mathcal{M}_{\text{fin}}(I)^V$. The support of $\vec{\mu}$ is $\text{supp}(\vec{\mu}) = \bigcup_{x \in V} \{i \in I \mid \mu_x(i) \neq 0\}$, which is a finite subset of I . The associated monomial is the formal commutative product $V^{\vec{\mu}} = \prod_{x \in V} \prod_{i \in I} x(i)^{\mu_x(i)}$. As an example, take $I = \mathbb{N}$ and $V = \{x, y, z\}$. A typical multi-exponent is $\vec{\mu} = (\mu_x, \mu_y, \mu_z)$ such that $\mu_x = [1, 1, 3]$, $\mu_y = [2, 4, 4]$ and $\mu_z = [1, 1, 1, 1]$. The associated monomial is $V^{\vec{\mu}} = x(1)^2 x(3) y(2) y(4)^2 z(1)^4$. Formally, a polynomial is a family $(\alpha_{\vec{\mu}})$ of scalars $\in \mathbb{K}$, indexed by monomials, such that for any finite subset I_0 of I , the set $\{\vec{\mu} \mid \alpha_{\vec{\mu}} \neq 0 \text{ and } \text{supp}(\vec{\mu}) \subseteq I_0\}$ is finite; this condition will be called the *finiteness condition on polynomials* (this corresponds to what was called finite depth in the Introduction). In standard algebraic notations, such a polynomial is written as the formal sum $\sum_{\vec{\mu}} \alpha_{\vec{\mu}} V^{\vec{\mu}}$.

Notice that if I is finite, a polynomial is just a finite linear combination of monomials, but this is no more true when I is infinite. Here is a typical example of (infinite) polynomial: $\sum_{n \in \mathbb{N}} x(n)^n$. On the other hand, the expression $\sum_{n \in \mathbb{N}} x(1)^n$ is not a polynomial.

We use $\mathbb{K}[V, I]$ for the set of these polynomials. Equipped with usual addition and multiplication this set is a \mathbb{K} -algebra. Let indeed $A = \sum_{\vec{\mu}} \alpha_{\vec{\mu}} V^{\vec{\mu}}$ and $B = \sum_{\vec{\mu}} \beta_{\vec{\mu}} V^{\vec{\mu}}$ be polynomials. Then $A + B = \sum_{\vec{\mu}} (\alpha_{\vec{\mu}} + \beta_{\vec{\mu}}) V^{\vec{\mu}}$ is a polynomial because if $\alpha_{\vec{\mu}} + \beta_{\vec{\mu}} \neq 0$ then $\alpha_{\vec{\mu}} \neq 0$ or $\beta_{\vec{\mu}} \neq 0$ and hence $A + B$ satisfies the finiteness condition on polynomials. As to products, observe first that, given a multi-exponent $\vec{\mu}$, that there are only finitely many pairs of multi-exponents $(\vec{\lambda}, \vec{\rho})$ such that $\vec{\lambda} + \vec{\rho} = \vec{\mu}$. So we can set $AB = \sum_{\vec{\mu}} (\sum_{\vec{\lambda} + \vec{\rho} = \vec{\mu}} \alpha_{\vec{\lambda}} \beta_{\vec{\rho}}) V^{\vec{\mu}}$. Let us check that AB satisfies the finiteness condition on polynomials, so let I_0 be a finite subset of I . If $\vec{\mu}$ is such that $\sum_{\vec{\lambda} + \vec{\rho} = \vec{\mu}} \alpha_{\vec{\lambda}} \beta_{\vec{\rho}} \neq 0$ then there is a pair $(\vec{\lambda}, \vec{\rho})$ such that $\alpha_{\vec{\lambda}} \neq 0$ and $\beta_{\vec{\rho}} \neq 0$ and $\vec{\lambda} + \vec{\rho} = \vec{\mu}$. If $\text{supp}(\vec{\mu}) \subseteq I_0$ we also have $\text{supp}(\vec{\lambda}), \text{supp}(\vec{\rho}) \subseteq I_0$. Since A and B are polynomials, there are only finitely many such pairs $(\vec{\lambda}, \vec{\rho})$ and hence there are only finitely many $\vec{\mu}$ such that $\sum_{\vec{\lambda} + \vec{\rho} = \vec{\mu}} \alpha_{\vec{\lambda}} \beta_{\vec{\rho}} \neq 0$ and $\text{supp}(\vec{\mu}) \subseteq I_0$.

Let \vec{v} be a valuation for variables $x \in V$ in \mathbb{K}^I , that is, for each $x \in V$, v_x is an I -indexed family $(v_x(i))_{i \in I}$ of elements of \mathbb{K} which vanishes almost everywhere. We set $v^{\vec{v}} = \prod_{x \in V} \prod_{i \in I} v_x(i)^{\mu_x(i)} \in \mathbb{K}$; this is a finite product in \mathbb{K} (it would be the case even for $v_x \in \mathbb{K}^I$). Then by the finiteness property the sum $A(\vec{v}) = \sum_{\vec{\mu}} \alpha_{\vec{\mu}} v^{\vec{\mu}}$ has only finitely many non-vanishing terms and hence $A(\vec{v}) \in \mathbb{K}$ is well-defined. This is the main motivation for the finiteness condition in the definition of infinite polynomials.

Lemma 31. *If, for each $i \in I$, $A_i \in \mathbb{K}[V, I]$ and $x \in V$, then $A = \sum_{i \in I} x(i) A_i \in \mathbb{K}[V, I]$.*

Proof. If $i \in I$, we use $i \cdot x$ for the multiexponent $\vec{\mu}$ such that $\mu_x = [i]$ and $\mu_y = []$ for $y \neq x$, in other words $V^{i \cdot x} = x(i)$. Let us write $A_i = \sum_{\vec{\mu}} \alpha(i)_{\vec{\mu}} V^{\vec{\mu}}$. We have

$$A = \sum_{i \in I} \sum_{\vec{\mu}} \alpha(i)_{\vec{\mu}} V^{\vec{\mu} + i \cdot x} = \sum_{\vec{v}} \left(\sum_{i \in \text{supp}(v_x)} \alpha(i)_{\vec{v} - i \cdot x} \right) V^{\vec{v}}.$$

Since $\text{supp}(v_x)$ is a finite set, each coefficient $\beta_{\vec{v}} = \sum_{i \in \text{supp}(v_x)} \alpha(i)_{\vec{v} - i \cdot x}$ in this expression is a finite sum. Let $I_0 \subseteq I$ be finite. Let M be the set of all \vec{v} 's such that $\text{supp}(\vec{v}) \subseteq I_0$ and $\beta_{\vec{v}} \neq 0$. For each $\vec{v} \in M$ there must be $i(\vec{v}) \in I$ such that $i(\vec{v}) \in \text{supp}(v_x)$ and $\alpha(i(\vec{v}))_{\vec{v} - i(\vec{v}) \cdot x} \neq 0$. Assume M is infinite. We have $\forall \vec{v} \in M$ $i(\vec{v}) \in I_0$ and hence, since I_0 is finite, there must be $i \in I_0$ such that $i(\vec{v}) = i$ for all $\vec{v} \in M'$ where M' is an infinite subset of M (pigeonhole principle). Since the map $\vec{v} \mapsto \vec{v} - i \cdot x$ is injective on M' , since $\text{supp}(\vec{v} - i \cdot x) \subseteq I_0$ and since $\alpha(i)_{\vec{v} - i \cdot x} \neq 0$ for all $\vec{v} \in M'$, we have a contradiction with the finiteness condition on polynomials satisfied by A_i . Hence M is finite and A satisfies the finiteness condition on polynomials. \square

4.1.2 Well-founded trees and polynomials.

Our Krivine machine will produce polynomials presented as some kind of well-founded Böhm trees that we define now, we call them polynomial trees. They are generated by the following syntax:

$$\begin{array}{c}
\frac{}{\Gamma, \varepsilon : \iota \vdash} \quad \frac{\Gamma, \pi : \iota \vdash}{\Gamma, \text{succ} \cdot \pi : \iota \vdash} \quad \frac{\Gamma, \pi : \iota \vdash}{\Gamma, \text{pred} \cdot \pi : \iota \vdash} \quad \frac{\Gamma \vdash P : \iota \quad \Gamma \vdash Q : \iota \quad \Gamma, \pi : \iota \vdash}{\Gamma, \text{if}(P, Q) \cdot \pi : \iota \vdash} \\
\frac{\Gamma, x : \iota \vdash M : \iota \quad \Gamma, \pi : \iota \vdash}{\Gamma, \text{let}(x, M) \cdot \pi : \iota \vdash} \quad \frac{\Gamma \vdash M : \sigma \quad \Gamma, \pi : \tau \vdash}{\Gamma, \text{arg}(M) \cdot \pi : \sigma \Rightarrow \tau \vdash} \quad \frac{\Gamma \vdash M : \sigma \quad \Gamma, \pi : \sigma \vdash}{\Gamma \vdash \langle M, \pi \rangle}
\end{array}$$

Figure 4: Typing rules for stacks and states of the machine

- 1 and 0 are polynomial trees;
- if $x \in V$ and \mathcal{S} is a function from I to polynomial trees then $x \triangleleft \mathcal{S}$ is a polynomial tree;
- if $\alpha_1, \alpha_2 \in \mathbb{K}$ and S_1, S_2 are polynomial trees then $[\alpha_1 \cdot S_1, \alpha_2 \cdot S_2]$ is a polynomial tree.

We use $\mathbb{K}[V, I]_{\text{wf}}$ for the set of these trees. We define a map $\text{pol} : \mathbb{K}[V, I]_{\text{wf}} \rightarrow \mathbb{K}[V, I]$ by well-founded induction: $\text{pol}(1) = 1$, $\text{pol}([\alpha_1 \cdot S_1, \alpha_2 \cdot S_2]) = \alpha_1 \text{pol}(S_1) + \alpha_2 \text{pol}(S_2)$ and $\text{pol}(x \triangleleft \mathcal{S}) = \sum_{i \in I} x(i) \text{pol}(\mathcal{S}(i))$. The fact that $\text{pol}(S) \in \mathbb{K}[V, I]$ results from Lemma 31.

4.1.3 Stacks, states and their denotational semantics

Our machine is restricted to almost free and fix-free PCF terms: a term, a stack or a state is fix-free if it does not contain the $\text{fix}(_)$ construct and almost closed if its only variables are of type ι . The machine computes polynomial trees $\in \mathbb{R}_{\geq 0}[V, \mathbb{N}^{\top}]_{\text{wf}}$ where $\mathbb{N}^{\top} = \mathbb{N} \cup \{\top\}$.

However it is more natural to define stacks, states and their semantics without these additional restrictions which will be useful only for the machine itself. General stacks are given by the following grammar

$$\pi, \rho \cdots := \varepsilon \mid \text{succ} \cdot \pi \mid \text{pred} \cdot \pi \mid \text{if}(P, Q) \cdot \pi \mid \text{let}(x, M) \cdot \pi \mid \text{arg}(M) \cdot \pi$$

where M, P and Q are terms. Stacks and states are typed by the rules of Figure 4.

The judgment $\Gamma, \pi : \sigma \vdash$ means that π is a continuation which expects an argument of type σ in context Γ . The judgment $\Gamma \vdash q$ means that the state q is well typed in context Γ , its “type” is left implicit (it would be the formula \perp of linear logic). The empty stack ε should be understood as a continuation which catches the \top “exception”.

4.1.4 Denotational semantics of stacks and states

If $\Gamma, \pi : \sigma \vdash$ (with $\Gamma = (x_1 : \sigma_1, \dots, x_k : \sigma_k)$) then, setting $C = \&_{i=1}^k \llbracket \sigma_i \rrbracket$, we define $\llbracket \pi \rrbracket_{\Gamma} \in \mathbf{Pcoh}^e(!C \otimes \llbracket \sigma \rrbracket, \perp)$ and if $\Gamma \vdash q$ then $\llbracket q \rrbracket_{\Gamma} \in \mathbf{Pcoh}^e(!C, \perp)$. We know that $\llbracket \pi \rrbracket_{\Gamma}$ is fully characterized by the associated function $\widehat{\llbracket \pi \rrbracket}_{\Gamma} : PC \times P\llbracket \sigma \rrbracket \rightarrow [0, 1]$ (non-linear in the first parameter and linear in the second one). And $\llbracket q \rrbracket_{\Gamma}$ is fully characterized by the associated function $\widehat{\llbracket q \rrbracket}_{\Gamma} : PC \rightarrow [0, 1]$.

► If $\pi = \varepsilon$, so that $\sigma = \iota$ then $\llbracket \pi \rrbracket_{\Gamma}$ is the following composition of morphisms ($\bar{\varepsilon}_{\mathbb{N}}$ is defined at the end of Section 2.2.1): $!C \otimes \mathbb{N}_{\perp}^{\top} \xrightarrow{w_C} \mathbb{N}_{\perp}^{\top} \xrightarrow{\bar{\varepsilon}_{\mathbb{N}}} \perp$. The associated function $f : PC \times P(\mathbb{N}_{\perp}^{\top}) \rightarrow [0, 1]$ is given by $f(\vec{u}, x) = x_{\top}$.

► If $\pi = \text{succ} \cdot \rho$ so that $\sigma = \iota$ and $\Gamma, \rho : \mathbb{N} \vdash$ then $\llbracket \pi \rrbracket_{\Gamma}$ is defined as the following composition of morphisms: $!C \otimes \mathbb{N}_{\perp}^{\top} \xrightarrow{!C \otimes \tilde{s}} !C \otimes \mathbb{N}_{\perp}^{\top} \xrightarrow{\llbracket \rho \rrbracket_{\Gamma}} \perp$ and the semantics of $\text{pred} \cdot \rho$ is defined similarly, using \mathbf{p} instead of \mathbf{s} .

The associated function $f : PC \times P(\mathbb{N}_{\perp}^{\top}) \rightarrow [0, 1]$ is given by $f(\vec{u}, x) = g(\vec{u}, \tilde{s} \cdot x)$ where $g = \widehat{\llbracket \rho \rrbracket}_{\Gamma}$ (remember that $\tilde{s} \cdot x = x_{\top} e_{\top} + \sum_{n \in \mathbb{N}} x_n e_{n+1}$). The case $\pi = \text{pred} \cdot \rho$ is similar, replacing \mathbf{s} with \mathbf{p} .

► Assume that $\pi = \text{if}(P_1, P_2) \cdot \rho$ so that $\sigma = \iota$, $\Gamma, \rho : \iota \vdash$ and $\Gamma \vdash P_i : \iota$ for $i = 1, 2$. For $n \in \mathbb{N}$, let $t_n \in \mathbf{Pcoh}^e_1(C, \mathbb{N}_{\perp}^{\top})$ be defined by $t_0 = \llbracket P_1 \rrbracket_{\Gamma}$ and $t_{n+1} = \llbracket P_2 \rrbracket_{\Gamma}$ for each $n \in \mathbb{N}$. Let $t = \langle t_n \rangle_{n \in \mathbb{N}} \in \mathbf{Pcoh}^e_1(C, (\mathbb{N}_{\perp}^{\top})^{\mathbb{N}})$. Then $\llbracket \pi \rrbracket_{\Gamma}$ is interpreted as the following composition of morphisms:

| | |
|---|--|
| $\mathcal{K}\langle \perp, \pi \rangle = 1$ | $\mathcal{K}\langle \Omega, \pi \rangle = 0$ |
| $\mathcal{K}\langle \text{coin}(r), \pi \rangle = [r \cdot \mathcal{K}\langle \underline{0}, \pi \rangle, (1-r) \cdot \mathcal{K}\langle \underline{1}, \pi \rangle]$ | $\mathcal{K}\langle x, \pi \rangle = x \triangleleft \mathcal{S}$ where $\mathcal{S}(\top) = 1$ and $\mathcal{S}(n) = \mathcal{K}\langle \underline{n}, \pi \rangle$ |
| $\mathcal{K}\langle \underline{n}, \varepsilon \rangle = 0$ | $\mathcal{K}\langle \underline{n}, \text{succ} \cdot \pi \rangle = \mathcal{K}\langle \underline{n+1}, \pi \rangle$ |
| $\mathcal{K}\langle \underline{0}, \text{pred} \cdot \pi \rangle = \mathcal{K}\langle \underline{0}, \pi \rangle$ | $\mathcal{K}\langle \underline{n+1}, \text{pred} \cdot \pi \rangle = \mathcal{K}\langle \underline{n}, \pi \rangle$ |
| $\mathcal{K}\langle \underline{0}, \text{if}(P, Q) \cdot \pi \rangle = \mathcal{K}\langle P, \pi \rangle$ | $\mathcal{K}\langle \underline{n+1}, \text{if}(P, Q) \cdot \pi \rangle = \mathcal{K}\langle Q, \pi \rangle$ |
| $\mathcal{K}\langle \text{let}(x, M, N), \pi \rangle = \mathcal{K}\langle M, \text{let}(x, N) \cdot \pi \rangle$ | $\mathcal{K}\langle \underline{n}, \text{let}(x, N) \cdot \pi \rangle = \mathcal{K}\langle N[\underline{n}/x], \pi \rangle$ |
| $\mathcal{K}\langle (M)N, \pi \rangle = \mathcal{K}\langle M, \text{arg}(N) \cdot \pi \rangle$ | $\mathcal{K}\langle \lambda x^\sigma M, \text{arg}(N) \cdot \pi \rangle = \mathcal{K}\langle M[N/x], \pi \rangle$ |
| $\mathcal{K}\langle \text{fix}(M), \pi \rangle = \mathcal{K}\langle M, \text{arg}(\text{fix}(M)) \cdot \pi \rangle$ | |

Figure 5: A Krivine function from almost closed states to polynomials

$$\begin{array}{c}
!C \otimes \mathbb{N}_\perp^\top \xrightarrow{\text{contr}_C \otimes \text{Id}} !C \otimes !C \otimes \mathbb{N}_\perp^\top \xrightarrow{!C \otimes \gamma} !C \otimes \mathbb{N}_\perp^\top \otimes !C \xrightarrow{\text{Id} \otimes t} !C \otimes \mathbb{N}_\perp^\top \otimes !((\mathbb{N}_\perp^\top)^\mathbb{N}) \\
\downarrow \text{Id} \otimes \widehat{\text{case}}^{\mathbb{N}, \mathbb{N}} \\
\perp \longleftarrow \llbracket \rho \rrbracket_\Gamma \longleftarrow !C \otimes \mathbb{N}_\perp
\end{array}$$

Let $h_i = \llbracket P_i \rrbracket_\Gamma : PC \rightarrow \mathbb{P}(\mathbb{N}_\perp^\top)$ for $i = 1, 2$. The function associated with $\llbracket \pi \rrbracket_\Gamma$ is $f : PC \times \mathbb{P}(\mathbb{N}_\perp^\top) \rightarrow [0, 1]$ given by $f(\vec{u}, x) = g(\vec{u}, x_\top e_\top + x_0 h_1(\vec{u}) + (\sum_{n=1}^\infty x_n) h_2(\vec{u}))$ where $g = \llbracket \rho \rrbracket_\Gamma$.

► Assume that $\pi = \text{let}(x, P) \cdot \rho$ so that $\sigma = \iota$, $\Gamma, \rho : \iota \vdash$ and $\Gamma, x : \iota \vdash P : \iota$. Then we have $\llbracket P \rrbracket_{\Gamma, x : \iota} \mathbf{m}^2 : !C \otimes !\mathbb{N}_\perp^\top \rightarrow \mathbb{N}_\perp^\top$ and hence $t = \widehat{\text{let}}(\llbracket P \rrbracket_{\Gamma, x : \iota} \mathbf{m}^2) \in \mathbf{Pcoh}^e(!C \otimes \mathbb{N}_\perp^\top, \mathbb{N}_\perp^\top)$. Then $\llbracket \pi \rrbracket_\Gamma$ is defined as the following composition of morphisms:

$$!C \otimes \mathbb{N}_\perp^\top \xrightarrow{\text{contr}_C \otimes \text{Id}} !C \otimes !C \otimes \mathbb{N}_\perp^\top \xrightarrow{\text{Id} \otimes t} !C \otimes \mathbb{N}_\perp^\top \xrightarrow{\llbracket \rho \rrbracket_\Gamma} \perp .$$

Let $h = \llbracket P \rrbracket_{\Gamma, x : \iota} : PC \times \mathbb{P}(\mathbb{N}_\perp^\top) \rightarrow \mathbb{P}(\mathbb{N}_\perp^\top)$. The function associated with $\llbracket \pi \rrbracket_\Gamma$ is $f : PC \times \mathbb{P}(\mathbb{N}_\perp^\top) \rightarrow [0, 1]$ given by $f(\vec{u}, x) = g(\vec{u}, x_\top e_\top + \sum_{n=0}^\infty x_n h(\vec{u}, e_n))$ where $g = \llbracket \rho \rrbracket_\Gamma$.

► We end with the semantics of states so assume that $q = \langle M, \pi \rangle$ with $\Gamma \vdash M : \sigma$ and $\Gamma, \pi : \sigma \vdash$ so that $\Gamma \vdash q$. Then we have $\llbracket M \rrbracket_\Gamma : !C \rightarrow \llbracket \sigma \rrbracket$ and $\llbracket \pi \rrbracket : !C \otimes \llbracket \sigma \rrbracket \rightarrow \perp$, and $\llbracket q \rrbracket_\Gamma : !C \rightarrow \perp$ is the following composition of morphisms $!C \xrightarrow{\text{contr}_C} !C \otimes !C \xrightarrow{\text{Id} \otimes \llbracket M \rrbracket_\Gamma} !C \otimes \llbracket \sigma \rrbracket \xrightarrow{\llbracket \pi \rrbracket_\Gamma} \perp$. The function $f : PC \rightarrow [0, 1]$ associated with $\llbracket q \rrbracket_\Gamma$ is given by $f(\vec{u}) = g(\vec{u}, h(\vec{u}))$ where $g = \llbracket \pi \rrbracket_\Gamma$ and $h = \llbracket M \rrbracket_\Gamma$.

Lemma 32. *If $\Gamma \vdash \pi : \iota$ then the associated function $g = \llbracket \pi \rrbracket_\Gamma : PC \times \mathbb{P}(\mathbb{N}_\perp^\top) \rightarrow [0, 1]$ is Scott-continuous, and linear in its last argument $\in \mathbb{P}(\mathbb{N}_\perp^\top)$ and satisfies $g(\vec{u}, e_\top) = 1$.*

Proof. Linearity and continuity results from the fact that $\llbracket \pi \rrbracket_\Gamma \in \mathbf{Pcoh}^e(!\llbracket \Gamma \rrbracket \otimes \mathbb{N}_\perp^\top, \perp)$. The property $g(\vec{u}, e_\top) = 1$ results from a simple inspection of the description above of the semantics of stacks. \square

4.1.5 The machine.

Remember that a term M is *almost closed* if $\Gamma \vdash M : \sigma$ where Γ is a ground context (all types appearing in Γ are ι); similarly a stack π is almost closed if $\Gamma, \pi : \sigma \vdash$ where Γ is ground. Notice that a ground context can be identified with a finite set V of variables (the variables which are assigned the type ι by Γ).

We define a function \mathcal{K} (a priori partial) from almost closed typed states $\mathbb{R}_{\geq 0}[V, \mathbb{N}^\top]_{\text{wf}}$. The definition is given in Figure 5. Notice that there is no rule for terms of shape $\text{fix}(M)$ and hence our machine is restricted to fix-free states; the corresponding rule should be $\mathcal{K}\langle \text{fix}(M), \pi \rangle = \mathcal{K}\langle M, \text{arg}(\text{fix}(M)) \cdot \pi \rangle$. The main reason is that, with this additional rule, Lemma 35 does not hold anymore.

4.2 Main property of the machine

We prove now that this function \mathcal{K} is total on almost closed fix-free states and yields elements of $\mathbb{R}_{\geq 0} [V, \mathbb{N}^\top]_{\text{wf}}$. The proof is by reducibility, following the general format used by Jean-Louis Krivine in his work on classical realizability.

Let V be a finite set of variables considered as a ground context¹³, we define first a *pole* $\perp\!\!\!\perp_V$ which is the set of all fix-free states q such that

- $V \vdash q$
- $\mathcal{K}(q)$ is a well-defined tree S which belongs to $\mathbb{R}_{\geq 0} [V, \mathbb{N}^\top]_{\text{wf}}$
- and the associated polynomial $\text{pol}(S)$ belongs to $\mathbf{Pcoh}^e((\mathbb{N}_\perp^\top)^V, \perp)$ and satisfies $\text{pol}(S) = \llbracket q \rrbracket_V$.

With each type σ we associate a set $\|\sigma\|$ of stacks π such that $V, \pi : \sigma \vdash$ by:

- $\|\iota\|$ is the set of all fix-free stacks π such that $V, \pi : \iota \vdash$ and $\forall n \in \mathbb{N} \langle \underline{n}, \pi \rangle \in \perp\!\!\!\perp_V$. Notice that in particular $\varepsilon \in \|\iota\|$.
- and $\|\sigma \Rightarrow \tau\| = \{\text{arg}(M). \pi \mid M \in |\sigma| \text{ and } \pi \in \|\tau\|\}$ where $|\sigma| = \{M \in \langle V \vdash \sigma \rangle_{\text{fin}} \mid \forall \pi \in \|\sigma\| \langle M, \pi \rangle \in \perp\!\!\!\perp_V\}$. Here $\langle V \vdash \sigma \rangle_{\text{fin}}$ is the set of all fix-free terms M such that $V \vdash M : \sigma$.

Lemma 33. *If $\pi \in \|\iota\|$ then $\text{succ} \cdot \pi, \text{pred} \cdot \pi \in \|\iota\|$. If $P, Q \in |\iota|$ then $\text{if}(P, Q) \cdot \pi \in \|\iota\|$. If $V, y : \iota \vdash P : \iota$ and $\forall n \in \mathbb{N} P[\underline{n}/y] \in |\iota|$, then $\text{let}(y, P) \cdot \pi \in \|\iota\|$.*

Lemma 34. *We have $V \subseteq |\iota|$ and $\forall n \in \mathbb{N} \underline{n} \in |\iota|$. Last $\Omega, \mathcal{U} \in |\iota|$ and for all $r \in [0, 1] \cap \mathbb{Q}$ one has $\text{coin}(r) \in |\iota|$.*

Proof. Let $y \in V$ and $\pi \in \|\iota\|$, we have to prove that $q = \langle y, \pi \rangle \in \perp\!\!\!\perp_V$. We have $\mathcal{K}(q) = y \triangleleft \mathcal{S}$ where $\mathcal{S}(n) = \mathcal{K}(\underline{n}, \pi)$ for each $n \in \mathbb{N}$ and $\mathcal{S}(\top) = 1$. By definition of $\perp\!\!\!\perp_V$ we have $\forall n \in \mathbb{N} \mathcal{S}(n) \in \mathbb{R}_{\geq 0} [V, \mathbb{N}^\top]_{\text{wf}}$ and hence $\mathcal{K}(q)$ is well-defined and belongs to $\mathbb{R}_{\geq 0} [V, \mathbb{N}^\top]_{\text{wf}}$. Setting $C = \widehat{\mathbb{N}_\perp^\top}^V$, the map $f = \llbracket q \rrbracket_V : PC \rightarrow [0, 1]$ is characterized by $f(\vec{u}) = g(\vec{u}, u(y))$ (remember that \vec{u} is a vector $(u(z))_{z \in V}$ of elements of $\mathbf{P}(\widehat{\mathbb{N}_\perp^\top})$) where $g = \widehat{\llbracket \pi \rrbracket}_V$. On the other hand the polynomial associated with $\mathcal{K}(q)$ is $\text{pol}(\mathcal{K}(q)) = y_\top + \sum_{n \in \mathbb{N}} y_n \text{pol}(\mathcal{S}(n))$ which defines the function $f' : PC \rightarrow \overline{\mathbb{R}_{\geq 0}}$ by $f'(\vec{u}) = u(y)_\top + \sum_{n \in \mathbb{N}} u(y)_n f_n(\vec{u})$ where we know that $f_n(\vec{u}) = \llbracket \langle \underline{n}, \pi \rangle \rrbracket_V(\vec{u})$ by our assumption that $\pi \in \|\iota\|$, that is $f_n(\vec{u}) = g(\vec{u}, e_n)$ by definition of the interpretation of states. This proves $f'(\vec{u}) = g(\vec{u}, u(y)) = f(\vec{u})$ by Lemma 32.

Let $n \in \mathbb{N}$ and $\pi \in \|\iota\|$, by definition of $|\iota|$ we have $\langle \underline{n}, \pi \rangle \in \perp\!\!\!\perp_V$ and hence $\underline{n} \in |\iota|$.

Let $\pi \in \|\iota\|$, then $\mathcal{K}(\Omega, \pi) = 0$ and $\mathcal{K}(\mathcal{U}, \pi) = 1$ are well-defined and belong to $\mathbb{R}_{\geq 0} [V, \mathbb{N}^\top]_{\text{wf}}$. The map $g = \widehat{\llbracket \pi \rrbracket}_V : PC \times \mathbf{P}(\widehat{\mathbb{N}_\perp^\top}) \rightarrow [0, 1]$ satisfies $g(\vec{u}, 0) = 0$ by Lemma 32 and since $\llbracket \Omega \rrbracket_V(\vec{u}) = 0$ we have $\llbracket \langle \Omega, \pi \rangle \rrbracket_V(\vec{u}) = 0$ and hence $\llbracket \langle \Omega, \pi \rangle \rrbracket_V = \text{pol}(\mathcal{K}(\Omega, \pi))$. We deal similarly with \mathcal{U} since $\llbracket \mathcal{U} \rrbracket_V(\vec{u}) = e_\top$ and $g(\vec{u}, e_\top) = 1$ by Lemma 32. Last we have $\mathcal{K}(\text{coin}(r), \pi) = [r \cdot \mathcal{K}(\underline{0}, \pi), (1-r) \cdot \mathcal{K}(\underline{1}, \pi)]$ and by our assumption about π we have $\langle \underline{0}, \pi \rangle, \langle \underline{1}, \pi \rangle \in \perp\!\!\!\perp_V$. It follows that $S = \mathcal{K}(\text{coin}(r), \pi)$ is well-defined and belongs to $\mathbb{R}_{\geq 0} [V, \mathbb{N}^\top]_{\text{wf}}$. Moreover $\text{pol}(S) = r \text{pol}(S_0) + (1-r) \text{pol}(S_1)$ where $S_n = \mathcal{K}(\underline{n}, \pi)$ for $n = 0, 1$. We also know that $\text{pol}(S_n) = \llbracket \langle \underline{n}, \pi \rangle \rrbracket_V$ for $n = 0, 1$. With the same notations as above we have $\llbracket \langle \text{coin}(r), \pi \rangle \rrbracket_V(\vec{u}) = g(\vec{u}, r e_0 + (1-r) e_1) = r g(\vec{u}, e_0) + (1-r) g(\vec{u}, e_1)$ by Lemma 32. This ends the proof that $\text{pol}(S) = \llbracket \langle \text{coin}(r), \pi \rangle \rrbracket_V$ since $\llbracket \langle \underline{n}, \pi \rangle \rrbracket_V(\vec{u}) = g(\vec{u}, e_n)$. \square

Lemma 35. *If M is fix-free and if $V, x_1 : \sigma_1, \dots, x_k : \sigma_k \vdash M : \tau$ and if $N_j \in |\sigma_j|$ for $j = 1, \dots, k$, then $M[\vec{N}/\vec{x}] \in |\tau|$.*

¹³This requires a total ordering on the elements of V , we keep this further information implicit identifying V with such a sequence.

Proof. By induction on the proof of the typing judgment $V, \Gamma \vdash M : \tau$ where $\Gamma = (x_1 : \sigma_1, \dots, x_k : \sigma_k)$. We use M' for $M \left[\vec{N} / \vec{x} \right]$ to increase readability.

In cases $M = \underline{n}$, $M = \text{coin}(r)$, $M = \Omega$, $M = \cup$ and $M = y \in V$ we have $M' = M$ and $M \in |\iota|$ by Lemma 34.

Assume that $M = x_i$ for some $i \in \{1, \dots, k\}$, we have $\tau = \sigma_i$ and $M' = N_i$ so that $M' \in |\tau|$ by our assumption about N_i .

Assume that $M = \text{if}(R, P, Q)$, so that $\tau = \iota$, $M' = \text{if}(R', P', Q')$ and, by inductive hypothesis, $R', P', Q' \in |\iota|$. Let $\pi \in \|\iota\|$, we have $\mathcal{K}\langle M', \pi \rangle = \mathcal{K}\langle R', \text{if}(P', Q') \cdot \pi \rangle$ which is well-defined and belongs to $\mathbb{R}_{\geq 0} [V, \mathbb{N}^\top]_{\text{wf}}$ by inductive hypothesis since $\text{if}(P', Q') \cdot \pi \in \|\iota\|$ by Lemma 33. We end the proof that $\langle M', \pi \rangle \in \perp_V$ by observing that $\llbracket \langle M', \pi \rangle \rrbracket_V = \llbracket \langle R', \text{if}(P', Q') \cdot \pi \rangle \rrbracket_V$.

The cases $M = \text{succ}(N)$ and $M = \text{pred}(N)$ are similar and simpler.

Assume that $M = \text{let}(x, R, P)$ so that $\tau = \iota$ and $M' = \text{let}(x, R', P')$. Let $\pi \in \|\iota\|$, we have $\mathcal{K}\langle M', \pi \rangle = \mathcal{K}\langle R', \text{let}(x, P') \cdot \pi \rangle$. By inductive hypothesis we have $R' \in |\iota|$ and $P' [\underline{n}/x] \in |\iota|$ since $\underline{n} \in |\iota|$ for all $n \in \mathbb{N}$. Therefore $\text{let}(x, P') \cdot \pi \in \|\iota\|$ by Lemma 33. It follows that $\mathcal{K}\langle M', \pi \rangle$ is well-defined and belongs to $\mathbb{R}_{\geq 0} [V, \mathbb{N}^\top]_{\text{wf}}$. We end the proof that $\langle M', \pi \rangle \in \perp_V$ by observing that $\llbracket \langle M', \pi \rangle \rrbracket_V = \llbracket \langle R', \text{let}(x, P') \cdot \pi \rangle \rrbracket_V$.

Assume that $M = (R)P$ with $V, \Gamma \vdash R : \sigma \Rightarrow \tau$ and $V, \Gamma \vdash P : \sigma$, we have $M' = (R')P'$ and, by inductive hypothesis, $R' \in |\sigma \Rightarrow \tau|$ and $P' \in |\sigma|$. Let $\pi \in \|\tau\|$, we have $\text{arg}(P') \cdot \pi \in \|\sigma \Rightarrow \tau\|$ by definition of this latter set and hence $\mathcal{K}\langle M', \pi \rangle = \mathcal{K}\langle R', \text{arg}(P') \cdot \pi \rangle$ is well-defined and belongs to $\mathbb{R}_{\geq 0} [V, \mathbb{N}^\top]_{\text{wf}}$. We end the proof that $\langle M', \pi \rangle \in \perp_V$ by observing that $\llbracket \langle M', \pi \rangle \rrbracket_V = \llbracket \langle R', \text{arg}(P') \cdot \pi \rangle \rrbracket_V$.

Assume last that $M = \lambda x^\sigma P$ with $V, \Gamma, x : \sigma \vdash P : \varphi$ and $\tau = (\sigma \Rightarrow \varphi)$. Let $\rho \in \|\sigma \Rightarrow \varphi\|$, that is $\rho = \text{arg}(N) \cdot \pi$ with $N \in |\sigma|$ and $\pi \in \|\varphi\|$. By inductive hypothesis applied to P , we have that $P' [N/x] \in |\varphi|$ and hence $\mathcal{K}\langle M', \rho \rangle = \mathcal{K}\langle P' [N/x], \pi \rangle \in \mathbb{R}_{\geq 0} [V, \mathbb{N}^\top]_{\text{wf}}$. We end the proof that $\langle M', \pi \rangle \in \perp_V$ by observing that $\llbracket \langle M', \text{arg}(N) \cdot \pi \rangle \rrbracket_V = \llbracket \langle P' [N/x], \pi \rangle \rrbracket_V$. \square

Theorem 36. *Assume that $V \vdash M : \iota$ and that M is fix-free. Then $\mathcal{K}\langle M, \varepsilon \rangle$ is a well-defined element S of $\mathbb{R}_{\geq 0} [V, \mathbb{N}^\top]_{\text{wf}}$ which satisfies $\text{pol}(S) = \bar{\varepsilon} \llbracket M \rrbracket_V$, that is, for all $\vec{u} \in \mathbb{P} \llbracket V \rrbracket$ one has $\widehat{\text{pol}(S)}(\vec{u}) = \widehat{\llbracket M \rrbracket_V}(\vec{u})_\tau$.*

This is the special case of the above lemma when Γ is the empty context.

4.3 Application

Let M be such that $V \vdash M : \iota$ (which typically can contain fixpoint constructs). Then if $V \vdash M_0, M^0 : \iota$ are fix-free and satisfy $M_0 \sqsubseteq M \sqsubseteq M^0$ then $S_0 = \mathcal{K}\langle M_0, \varepsilon \rangle$ and $S^0 = \mathcal{K}\langle M^0, \varepsilon \rangle$ are elements of $\mathbb{R}_{\geq 0} [V, \mathbb{N}^\top]_{\text{wf}}$ which satisfy $\text{pol}(S^0) = \bar{\varepsilon} \llbracket M^0 \rrbracket_V$ and $\text{pol}(S_0) = \bar{\varepsilon} \llbracket M_0 \rrbracket_V$ and hence $\text{pol}(S_0) \sqsubseteq_{!(\mathbb{N}^\top)^\vee \rightarrow \perp} \bar{\varepsilon} \llbracket M \rrbracket_V \sqsubseteq_{!(\mathbb{N}^\top)^\vee \rightarrow \perp} \text{pol}(S^0)$ by Theorem 29.

Of course the polynomials $\text{pol}(S_0)$ and $\text{pol}(S^0)$ are usually infinite but for any finite subset J of \mathbb{N} we can precompose the former with $(\text{Id}_{J, \mathbb{N}}^\leftarrow)^V$ and the latter with $(\text{Id}_{J, \mathbb{N}}^\rightarrow)^V$ in \mathbf{Pcoh}^e , and one obtains by Lemma 22 in that way two finite polynomials t_0 and t^0 such that $t_0 \sqsubseteq_{!(\mathbb{N}^\top)^\vee \rightarrow \perp} \bar{\varepsilon} \llbracket M \rrbracket_V \sqsubseteq_{!(\mathbb{N}^\top)^\vee \rightarrow \perp} t^0$. Notice that these restrictions by J can be performed on the fly during the run of \mathcal{K} which will then return finite polynomials.

Remark. This reducibility proof would also work for terms containing some restricted form of recursion such as the higher order primitive recursion of Gödel System T. It turns out that, for such terms, the support of the interpretation of type σ in \mathbf{Pcoh} is a finitary set in the sense of the Finiteness Space semantics [9]. In that case we can apply the method above to the term M itself, without taking before syntactic approximations M_0 and M^0 . We just need to precompose $(\text{Id}_{J, \mathbb{N}}^\leftarrow)^V$ and $(\text{Id}_{J, \mathbb{N}}^\rightarrow)^V$ for getting finite polynomial approximations.

Related work and conclusion

This work takes place in a general trend trying to extract formal tools from denotational models, much in the spirit of Abstract Interpretation. Typical developments of this kind are the various intersection typing systems dating back to the early work of Coppo and Dezani [4] which are often deeply related with

denotational models such as Scott semantics of the relational model of LL and of the λ -calculus. Among these contributions one of the most relevant to the present work is [2] where an intersection typing system is designed for approximating probabilities of convergence. It is still an open problem to understand the connection between this type-based approximations and those, based on PCS, that we develop here. Another interesting connection might be found in the work of Salvati and Walukiewicz on Higher Order Recursion Schemes where Krivine machines play a essential role, in connection with denotational properties of λ -terms with fixpoints.

More examples and practical computations will be the object of a further paper.

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