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Around Sylvester's question in the plane

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Abstract

Pick n points $Z_0, ..., Z_{n-1}$ uniformly and independently at random in a compact convex set H with non empty interior of the plane, and let Q_H^n be the probability that the Z_i's are the vertices of a convex polygon. Blaschke 1917 [1] proved that $Q_T^4 \leq Q_H^4 \leq Q_D^4$, where D is a disk and T a triangle. In the present paper we prove $Q_T^5 \leq Q_H^5 \leq Q_D^5$. One of the main ingredients of our approach is a new formula for Q_H^n of independent interest.

We conjecture that the new formula we provide for Q_H^n will lead in the future to the complete proof that $Q_T^n \leq Q_T^n \leq Q_H^n$, for any n: we provide some partial results in this direction.

Acknowledgments

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1 Introduction

Notations and convention. All the random variables (r.v.) in the paper are assumed to be defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The expectation is denoted by $\mathbb{E}$. For any $n \geq 1$, any generic variable name z, $z[n]$ stands for the n-tuple $(z_0, \ldots, z_{n-1})$. The set of compact convex subsets of $\mathbb{R}^2$ with non empty interior is denoted CCS. For any $H \in \mathsf{CCS}$ and any $n \geq 0$, $\mathbb{P}_H^n$ is the notation for the law of $Z[n]$, a sequence of n i.i.d. points taken under the uniform distribution on H. Last, we denote by $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$.

1.1 The new result

A n-tuple of points $\mathsf{x}[n]$ of the plane is said to be in a convex position if $\{\mathsf{x}_0, \cdots, \mathsf{x}_{n-1}\}$ is the vertex set of a convex polygon. Denote by CP_n the set of n-tuples $x[n]$ in a convex position. Finally let

$$Q_H^n := \mathbb{P}_H^n(\mathsf{CP}_n) = \mathbb{P}(Z[n] \in \mathsf{CP}_n),$$

where $Z[n]$ is $\mathbb{P}_H^n$ distributed. The aim of this paper is to prove the following theorem:

Theorem 1. *For any $H \in \mathsf{CCS}$,*

$$11/36 = Q_T^5 \leq Q_H^5 \leq Q_D^5 = 1 - 305/(48\pi)^2,$$

with equality, at the left, when H is a triangle only, at the right, only when H is an ellipse.

We think that more generally,

Conjecture 1. *For any $n \geq 6$, $Q_T^n \leq Q_H^n \leq Q_D^n$.*

Elements supporting these conjecture are developed in Section 4.

The sequences $(Q_T^n, n \geq 0)$ and $(Q_D^n, n \geq 0)$ are known, see respectively Valtr [14] and Marckert [7].

Of course, when $n \leq 3$, Q_H^n does not depend on H so that the first non trivial results appear when $n = 4$: Blaschke [1] proved in 1917 that for any $H \in \mathsf{CCS}$,

$$2/3 = Q_T^4 \leq Q_H^4 \leq Q_D^4 = 1 - 35/(12\pi)^2,$$

with equality cases being also the triangle and the disk, respectively.

The method of Blaschke relies on two ingredients:

A. there is an "algebraic formula" for Q_H^4 :

$$Q_H^4 = 1 - 4\,\mathbb{E}_H(\mathsf{Area}(Z_0, Z_1, Z_2)) = 1 - 2\,\mathbb{E}_H(|\det(Z_1 - Z_0, Z_2 - Z_0)|) \tag{1}$$

where $\mathbb{E}_H(\mathsf{Area}(Z[3]))$ is the expectation of the area of the triangle $Z[3]$ under $\mathbb{P}_H^3$ (since 4 points are not in a convex position, if one of them lies in the triangle formed by the 3 other ones).

B. Steiner symmetrization and shaking (see definitions below) have the following property:

a. If H^{Sym} (resp. H^{Sha}) are obtained from H by a Steiner symmetrization (resp. a shaking) with respect to the x-axis, then

$$Q_{H^{\mathsf{Sym}}}^4 \geq Q_H^4, \quad Q_{H^{\mathsf{Sha}}}^4 \leq Q_H^4, \tag{2}$$

with equality only for some identified special cases;

b. For any $H_0 \in \mathsf{CCS}$, there exists a sequence of lines $(\Delta_i, i \geq 1)$, so that for H_{i+1} obtained from H_i by Steiner symmetrization (resp. shaking) with respect to Δ_{i+1}, the sequence (H_n) converges to a disk (resp. to a triangle) for the Hausdorff distance (see Klartag [5] and Campi, Colesanti and Gronchi [3] for modern and general treatments).

Remark 2. *Formula (2) is only needed for Steiner symmetrization and shaking with respect to the x-axis, since rotations preserve uniform distributions and convexity, and then can be performed in alternations with symmetrization and shaking.*

In the present paper we use the same methodology, which will lead to another proof of the $n = 4$ case (Blaschke result), and a new proof in the $n = 5$ case : the main contribution is a new "algebraic formula" for Q_n^H. We will prove the following analogous of (2)

Theorem 3. *For $n \in \{4, 5\}$,*

$$Q_{H^{\mathsf{Sym}}}^n \geq Q_H^n, \quad Q_{H^{\mathsf{Sha}}}^n \leq Q_H^n.$$

Theorem 3 together with point **B.b** of the "Blaschke strategy" implies Theorem 1.

The quotation marks around "algebraic formula" are here to mark that the determinant is algebraic in the coordinates of the Z_j's but (1) is somewhat more complex since it involves an absolute value, and an expectation. In the case $n = 4$, this expectation is a triple integral over H that could be as unpleasant as one could imagine.

One of the main advances in the paper is a new "algebraic" formula for Q_H^n (including Q_H^4) which avoids absolute values, and which is given in terms of $n+2$ real integrals of a polynomial. We were able to compare the polynomials appearing when computing $Q_H^n, Q_{H^{\mathsf{Sym}}}^n$ and $Q_{H^{\mathsf{Sha}}}^n$ only for $n \in \{4, 5\}$. We think that the results holds in all generality but despite many attempts we were not able to get a complete proof of the remaining cases.

1.2 Related results

The problem of determining Q_H^n goes back to a question (badly) posed by Sylvester [12]. Finally, the adopted question was to the following: prove that the map $H \mapsto Q_H^4$ takes its maximum on CCS when H is a disk and its minimum when H is a triangle, Theorem finally proved by Blaschke [1] (see Pfeifer [10] for historical notes). Recently some advances have been made on the exact computations of Q_H^n: Valtr [13, 14] showed that if S is a square (or a non flat parallelogram) and if T is a non flat triangle then, for $n \geq 1$,

$$Q_S^n = \left(\frac{\binom{2n-2}{n-1}}{n!} \right)^2, \quad Q_T^n = \frac{2^n (3n-3)!}{(n-1)!^3 (2n)!}.$$

Buchta [2] goes further and gives an expression for $Q_S^{n,m}$ and $Q_T^{n,m}$, the probability that m points exactly among the n random points are on the boundary of the convex hull. The first author of the present paper gives a formula for Q_D^n (and $Q_D^{n,m}$) in the disk case [7].

The literature concerning the question of the number of points on the convex hull for i.i.d. random points taken in a convex domain is huge. We won't make a survey here but rather refer the reader to Reitzner [11] and Hug [4] for an overview of the topic.

As far as we know, Blaschke result has not been extended in the direction we propose here, but rather, in the multidimensional case, where Blaschke [1] proved that

$$Q_K^{d+2} \leq Q_{B_d}^{d+2}, \quad \text{for any } K \in \mathcal{K}_d, \tag{3}$$

where $\mathcal{K}_d$ is the set of compact convex bodies in $\mathbb{R}^d$ with non empty interior, B_d is the unit ball in $\mathbb{R}^d$. The inequality $Q_{\Delta_d}^{d+2} \leq Q_K^{d+2}$ for any $K \in \mathcal{K}_d$, where Δ_d is the simplex in $\mathbb{R}^d$ is still a conjecture. Milman & Pajor [9, Prop. 5.6] established that if it holds, then the hyperplane conjecture (or slicing problem) holds true: there exists a universal constant $c > 0$ such that for every d and for every convex body K of volume one in $\mathbb{R}^d$ there exists an hyperplane H such that $|K \cap H| \geq c$. This connection is another justification for our work since a right understanding of the 2-D case can be a step in the right direction.

1.3 Content of the paper

Most of the paper is devoted to proving Theorem 3. In section 3, we recall what are Steiner symmetrization and shaking with respect to the x-axis.

Take $Z[n]$ under $\mathbb{P}_K^n$ for some $K \in \mathsf{CCS}$, and for every j, let (X_j, Y_j) be the coordinates of Z_j in $\mathbb{R}^2$.

In Section 2, we give a new formula – that we call the bi-comb formula – for

$$\mathbb{P}(Z[n] \in \mathsf{CP}_n \mid X[n] = x[n]), \tag{4}$$

which is the conditional probability that the points of $Z[n]$ are in a convex position, given the abscissa $x[n] = (x_0, \cdots, x_{n-1})$ of these points.

What is somehow remarkable, is that $\mathbb{P}(Z[n] \in \mathsf{CP}_n \mid X[n] = x[n])$ is totally explicit and, more than that, for any fixed $x[n]$, it is a polynomial with degree n in the extreme ordinates $(\overline{y_H}(x_i), \underline{y_H}(x_i))$ of the vertical slices of the convex set H at the abscissas x_i.

Now, the optimisation of these formula can be done : a property of the Steiner symmetrization and shaking with respect to the x-axis is that the distribution of the abscissas $X[n]$ of random points $Z[n]$ taken uniformly, is the same for the convex sets H, H^{Sym} or H^{Sha}. It suffices then to compare the bi-comb formulas for each of these convex sets (with again, $X[n] = x[n]$ fixed) to conclude. This is the intent of Prop. 8 which asserts that Theorem 3 holds when we condition on $X[n] = x[n]$ for any sequence $x[n]$ (Section 3).

2 A general formula for Q_H^{N+2}

A well known property is that inversible affine maps of $\mathbb{R}^2$ conserve convexity and uniformity; hence:

Lemma 4. *For any inversible affine map A of $\mathbb{R}^2$, for any $H \in \mathsf{CCS}$, we have $Q^n_{A(H)} = Q^n_H$.*

In the sequel, we will work with convex bodies with area 1 only.

2.1 Abscissas fibration

As represented on Fig. 1, for any H with area 1, denote by $x_H(\min) = \min\{x : (x,y) \in H\}$ and $x_H(\max) = \max\{x : (x,y) \in H\}$ the minimum and maximum abscissas of H and let,

$$\begin{aligned} \overline{y}_H(x) &:= \sup\{y : (x,y) \in H\}, \quad \text{for } x \in [x_H(\min), x_H(\max)] \\ \underline{y}_H(x) &:= \inf\{y : (x,y) \in H\}, \quad \text{for } x \in [x_H(\min), x_H(\max)]. \end{aligned}$$

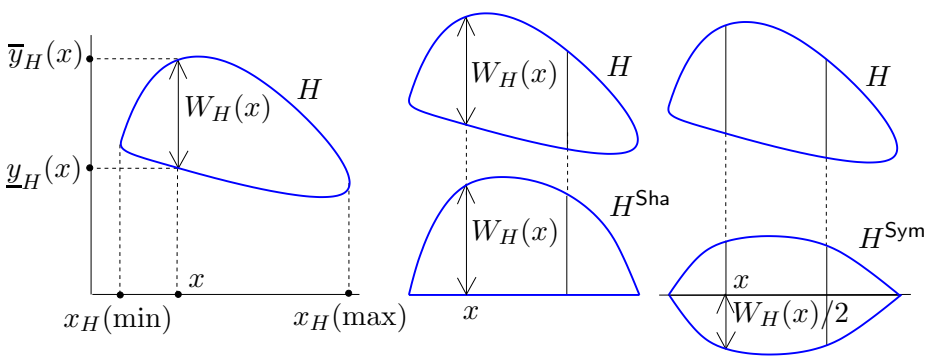

Figure 1: A convex body, and then, its shaking and Steiner symmetrization with respect to the x-axis.

The width function $W_H : \mathbb{R} \to \mathbb{R}$ is defined by

$$W_H(x) = \Big(\overline{y}_H(x) - \underline{y}_H(x)\Big)\, \mathbf{1}_{[x_H(\min), x_H(\max)]}(x).$$

The vertical segment intersecting H at abscissa x is denoted

$$V_H(x) = \Big[(x, \underline{y}_H(x)), (x, \overline{y}_H(x))\Big]. \tag{5}$$

The law of the abscissa X of a uniform point (X,Y) taken in H has density W_H.

Note 5. *Instead of taking n points at random in H, in the sequel we will take $N+2$ points at random! Of course for $n = N+2$ this is equivalent, but it will be useful in our decompositions/recurrences to have a point with rank 0, and one with rank $N+1$, to provide simpler recurrence, with respect to the involved indices.*

Let $Z[N+2]$ be taken under P_H^{N+2} and let (X_j, Y_j) be the coordinates of Z_j in the plane. Consider τ the a.s. well defined permutation in the symmetric group $\mathcal{S}([\![0, N+1]\!])$ such that

$$X_{\tau(0)} \leq \cdots \leq X_{\tau(N+1)}.$$

By symmetry, the permutation τ is uniform in $\mathcal{S}([\![0, N+1]\!])$ and independent from the set of values $\{X_j, 0 \leq j \leq N+1\}$. The density of $X_\tau := \big(X_{\tau(j)}, 0 \leq j \leq N+1\big)$ on $\mathbb{R}^{N+2}$ is

$$f_H(x[N+2]) = (N+2)! \left(\prod_{j=0}^{N+1} W_H(x_j) \right) \mathbf{1}_{\mathsf{ND}_{N+2}}(x[N+2]) \tag{6}$$

where, for any $n \geq 1$, $\mathsf{ND}_n = \{x[n] \ : x_0 < \cdots < x_{n-1}\}$ is the set of non decreasing sequences with n elements. Conditional on $(X_{\tau(j)}, 0 \leq j \leq N+1) = x[N+2]$, the variables $Z_{\tau(0)}, \cdots, Z_{\tau(N+1)}$ are independent, and $Z_{\tau(j)}$ is uniform on $V_H(x_j)$. We introduce a crucial object of the paper:

Definition 6. *Consider $N+2$ (vertical or not) segments $S[N+2] = (S_0, \cdots, S_{N+1})$ of the plane and $U[N+2]$ a $N+2$-tuple of independent r.v. where U_j is uniform on S_j. We denote by*

$$\big\langle S[N+2] \big\rangle := P(U[N+2] \in \mathsf{CP}_{N+2}),$$

the probability that the U_j's are in a convex position.

To compute Q_H^{N+2}, one can condition on the value of X_τ and get

$$Q_H^{N+2} \quad = \quad \int_{\mathsf{ND}_{N+2}} \big\langle V_H(x_0), \cdots, V_H(x_{N+1}) \big\rangle \, f_H(x[N+2]) \, dx_0 \cdots dx_{N+1}. \tag{7}$$

In the next subsection, we will see that there exists a close formula for $\big\langle V_H(x_0), \cdots, V_H(x_{N+1}) \big\rangle$, and we will call this formula the bi-comb formula.

For the optimisation purpose, one may say that the only thing that really matters is to understand $\big\langle V_H(x_0), \cdots, V_H(x_{N+1}) \big\rangle$. To see this, recall the following :

Definition 7. *The convex bodies obtained from H by Steiner symmetrization and shaking with respect to the x-axis are respectively (see Fig. 3) :*

$$\begin{aligned} H^{\mathsf{Sym}} &= \{(x,y) \ : x_H(\min) \leq x \leq x_H(\max), |y| \leq W_H(x)/2\}, \\ H^{\mathsf{Sha}} &= \{(x,y) \ : x_H(\min) \leq x \leq x_H(\max), 0 \leq y \leq W_H(x)\}. \end{aligned}$$

In other words, $x_H(\min) = x_{H^{\mathsf{Sha}}}(\min) = x_{H^{\mathsf{Sym}}}(\min)$ and $x_H(\max) = x_{H^{\mathsf{Sha}}}(\max) = x_{H^{\mathsf{Sym}}}(\max)$, and

$$\begin{aligned} \overline{y}_{H^{\mathsf{Sym}}}(x) &= -\underline{y}_{H^{\mathsf{Sym}}}(x) = W_H(x)/2, \\ \overline{y}_{H^{\mathsf{Sha}}}(x) &= W_H(x), \quad \underline{y}_{H^{\mathsf{Sha}}}(x) = 0. \end{aligned}$$

Since the width functions $W_H, W_{H^{\mathsf{Sha}}}$ and $W_{H^{\mathsf{Sym}}}$ coincide, by (6)

$$f_H(x[N+2]) = f_{H^{\mathsf{Sha}}}(x[N+2]) = f_{H^{\mathsf{Sym}}}(x[N+2]).$$

Hence, Theorem 1 appears to be a consequence of the following proposition:

Proposition 8. *For $H \in \mathsf{CCS}$, $N \in \{2,3\}$, and $x[N+2] \in \mathsf{ND}_{N+2} \cap [x_H(\min), x_H(\max)]^{N+2}$,*

$$\big\langle V_H^{\mathsf{Sha}}(x_0), \cdots, V_H^{\mathsf{Sha}}(x_{N+1}) \big\rangle \leq \big\langle V_H(x_0), \cdots, V_H(x_{N+1}) \big\rangle \leq \big\langle V_H^{\mathsf{Sym}}(x_0), \cdots, V_H^{\mathsf{Sym}}(x_{N+1}) \big\rangle. \tag{8}$$

It is important to stop here, and have a look to the three sequences of vertical lines appearing in (8). Each of these sequences will be called "bi-comb" in the sequel. They are successions of vertical lines at the same abscissas, and having the same lengths (see Fig. 2).

Remark 9. *The exact position in the plane of these segments is not important : applying an inversible affine map to a comb preserves convexity and the independence and uniform distribution of the random points picked on the segments, so that it is possible to normalise the three combs appearing in the previous Proposition.*

A procedure to fix the same extremal segments (see Fig. 2)

Lemma 10 (Normalization Lemma)**.** *Consider a bi-comb $C = (W(x_i), 0 \le i \le N+1)$. By an inversible affine map which preserves abscissas it is possible to send C onto $C' = (W'(x_i), 0 \le i \le N+1)$ with same abscissa, same successive lengths, but having now the middle of the two extreme segments $W'(x_0)$ and $W'(x_{N+1})$ on the x-axis.*

Proof. Inversible affine map which preserves abscissas are maps $(x, y) \to (x, y - ax - b)$ for some constant a, b. □

Using the normalisation Lemma, each of the three bi-combs $V_{\mathsf{Sha}} := (V_H^{\mathsf{Sha}}(x_i), 0 \le i \le N+1)$, $V_H := (V_H(x_i), 0 \le i \le N+1)$ and $V_{\mathsf{Sym}} := (V_H^{\mathsf{Sym}}(x_i), 0 \le i \le N+1)$ can be sent on $V'_{\mathsf{Sha}}, V'_H, V'_{\mathsf{Sym}}$ where as represented on Fig. 2, $V'_{\mathsf{Sym}} = V_{\mathsf{Sym}}$ and the trapezoid with extreme segments $V_H^{\mathsf{Sym}}(x_0)$ and $V_H^{\mathsf{Sym}}(x_{N+1})$ is common to the three new bi-combs. The bottom-most position of the segments of V'_{Sha} lies on the bottom-line of the trapezoid.

We can also suppose that $x_0 = 0 < x_1 < \cdots < x_{N+1} = 1$ since the inversible affine map $(x, y) \mapsto (\frac{x - x_0}{x_{N+1} - x_0}, y)$

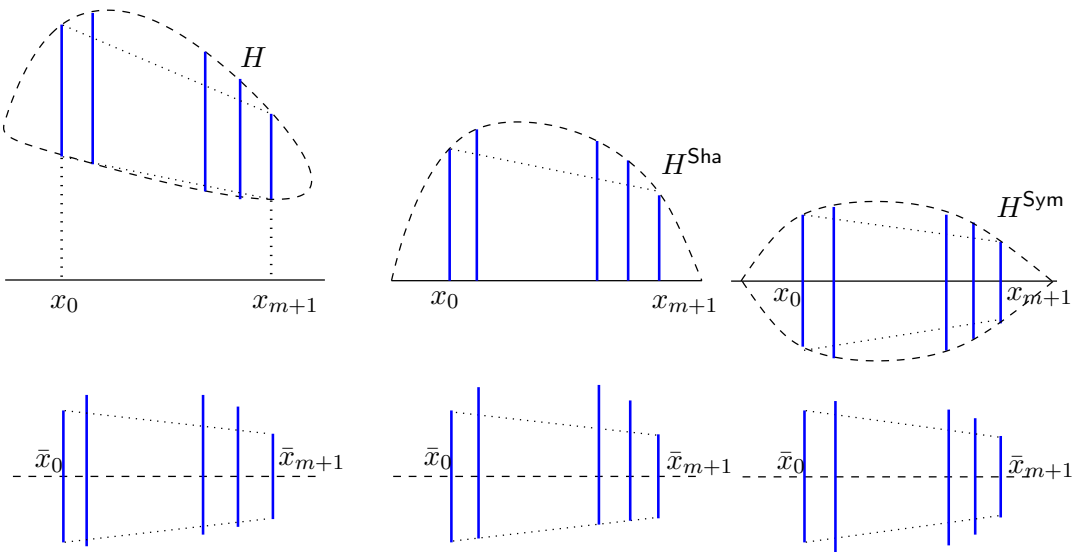

Figure 2: Shaking and Steiner symmetrization with respect to the x-axis of the vertical blue segments, followed by the centering of the extreme segments, for H and H^{Sha} by a map preserving verticality. For the three figures on the second line, the trapezoids with sides the two extreme vertical segments coincide. In the shaking case, the segments raise at the bottom line of the trapezoid.

sends $(x_0, 0)$ and $(x_{N+1}, 0)$ to $(0, 0)$ and $(1, 0)$, and keeps the other properties of the segments.

By Remark 9, (8) is equivalent to the same statement in which we replace $(V_{\mathsf{Sha}}, V_H, V_{\mathsf{Sym}})$ by $(V'_{\mathsf{Sha}}, V'_H, V'_{\mathsf{Sym}})$. But then the three sequences V'_{Sha}, V', V_H^{Sym} may be represented thanks to some "default of symmetries as follows".

■ First, encode a generic trapezoid: set, for $i = 0$ to $N+1$,

$$L_i = A(1 - x_i) + Bx_i \tag{9}$$

so that $L_0 = A \ge 0$ and $L_{N+1} = B \ge 0$ (hence A plays the role of $W_H^{\mathsf{Sym}}(x_0)/2$ and B that of $W_H^{\mathsf{Sym}}(x_{N+1})/2$, but the abstraction is here to make disappear the relation with a given convex set H). The points (x_i, L_i)

are aligned and correspond to the above line of the trapezoid with vertical sides $V_H^{\mathsf{Sym}}(0)$ and $V_H^{\mathsf{Sym}}(1)$.
■ For $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$, denote by

$$\mathsf{CV}^+(x[N+2]) = \{(\lambda_0 = 0, \lambda_1, \cdots, \lambda_n, \lambda_{N+1} = 0) : \lambda_i \geq 0, [(x_i, \lambda_i), 0 \leq i \leq N+1] \in \mathsf{CP}_{N+2}\} \tag{10}$$

in other words, this is the set of sequences $\lambda[N+2]$ such that $\{(x_i, \lambda_i), i \in [\![0, N+2]\!]\}$ is in a convex position, bove the x-axis, and have their "two extreme points" at $(0,0)$ and $(1,0)$.
■ Denote by $\mathsf{Compa}(\lambda[N+2])$ the set of sequences $\beta[N+2]$ of real numbers such that $\beta_0 = \beta_{N+1} = 0$ and such that $|\beta_i| \leq \lambda_i$, and such that the points $((x_i, \lambda_i + \beta_i), 0 \leq i \leq N+1)$ are non negative and in a convex position as well as the points $((x_i, \lambda_i - \beta_i), 0 \leq i \leq N+1)$. Denote by

$$V_{L,\lambda}^{\beta}(x_i) = [\ (x_i,\ \ -(L_i + \lambda_i - \beta_i)),\ (x_i,\ (L_i + \lambda_i + \beta_i))\ ]$$

so that in the segment $(V^{\beta}(x_i), 0 \leq x_i \leq N+1)$ the two extreme segments have their center on the x axis, the set formed by the up-most points of each segment, and the set formed by the bottom-most points of each segment are in a convex position.
– The case $\beta \equiv 0$ corresponds to the segment V_H^{Sym} in which the segments are symmetric with respect to the x-axis,
– The case $\beta \equiv \lambda$ corresponds to the segment V'_{Sha} (obtained from shaking followed by a normalisation, so that the bottom points of the segments lies on the bottom line of the trapezoid, see Fig. 2),
– and the general case is covered by all the $\beta \in \mathsf{Compa}(\lambda)$.

Instead of proving Proposition 8, we will prove the following proposition

Proposition 11. *For any $x[N+2], \lambda[N+2], L$ fixed as above, The map*

$$\begin{array}{rccl} \Theta : & \mathsf{Compa}(\lambda[N+2]) & \longrightarrow & \mathbb{R}^+ \\ & \beta & \longmapsto & \langle V_{L,\lambda}^{\beta}(x_0), \cdots, V_{L,\lambda}^{\beta}(x_{N+1}) \rangle \end{array},$$

reaches its maximum for $\beta = (0, \cdots, 0)$, and his minimum for $\beta = \lambda$.

β has to be understood as a symmetry defect, L describes the "centered" extreme segments, λ describes the symmetric case, $L + \lambda + \beta$ and $-(L + \lambda - \beta)$ describe the topmost and bottom-most polygons.

We now provide our new close formula for $\langle V(x_0), \cdots, V(x_{N+1}) \rangle$, "the bi-comb formula".

2.2 The comb and the bi-comb formula

Technically, a comb is a sequence of teeth plugged on a shaft: the bottom of the teeth extremities are aligned, and the top extremities are in a convex position.

A geometrical comb, and the comb formula: Consider a pair of sequences $[x[N+2], \ell[N+2]]$ each being of size $N+2$, where the sequence (x_i) is increasing, and where:
• all the points (x_i, ℓ_i) lies (in the weak sense) above the line passing by (x_0, ℓ_0) and (x_{N+1}, ℓ_{N+1}),
• the points $\{(x_i, \ell_i), 0 \leq i \leq N+1\}$ are in a convex position (as on Fig. 3).

The ordinate of the "bottom" line passing by (x_0, ℓ_0) and (x_{N+1}, ℓ_{N+1}) at abscissa x_i is

$$b_i := \ell_0 + \frac{x_i - x_0}{x_{N+1} - x_0}(\ell_{N+1} - \ell_0).$$

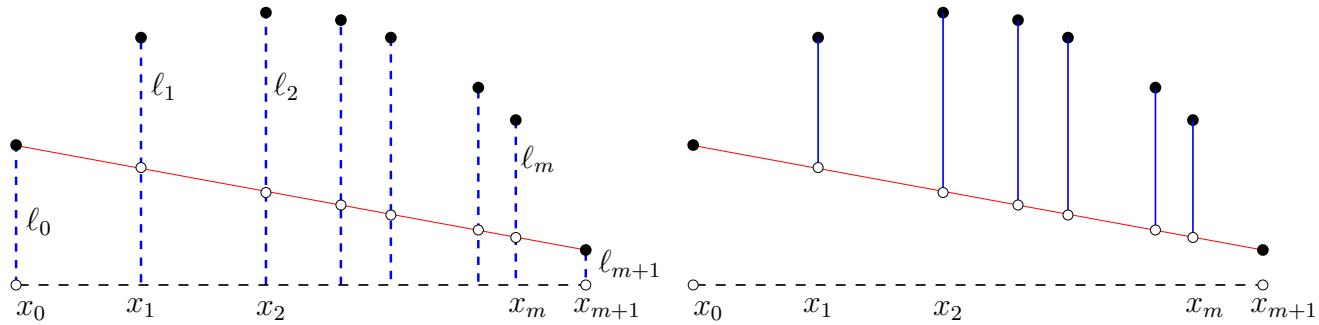

Figure 3: A comb $\mathsf{Cb}[x[m+2], \ell[m+2]]$: the shaft is the red segment.

We call comb

$$\mathsf{Cb}[x[N+2], \ell[N+2]] = S[N+2]$$

the sequence of $N+2$ segments $(S_0, \cdots, S_{N+1})$ where S_i is the segment

$$S_i = [\ (x_i, b_i)\,,\ (x_i, \ell_i)\ ]$$

so that its two extreme segments have zero length, and such that all the first extremities of the segments belong to the "shaft-segment" $[(x_0, \ell_0), (x_{N+1}, \ell_{N+1})]$. We will sometimes refer to the segments as the teeth of the comb. We let again

$$\langle \mathsf{Cb}[x[N+2], \ell[N+2]\rangle$$

the probability that a set of $N+2$ points taken uniformly and independently, one per vertical segments are in a convex position.

Proposition 12. *For any comb with $N+2$ segments (for any $N \geq 0$),*

$$\langle \mathsf{Cb}\,(x[N+2], \ell[N+2])\rangle = \frac{\mathbf{K}\,(x[N+2], \ell[N+2])}{\prod_{i=1}^{N} \ell_i}, \tag{11}$$

where $\mathbf{K}$ is a function of two lists of same number m of elements, for some $m \geq 2$, defined recursively:

$$\mathbf{K}\,([x_0, x_1], [\ell_0, \ell_1]) = 1 \tag{12}$$

and when $N+2 \geq 3$,

$$\mathbf{K}\,[x[N+2], \ell[N+2]] = \sum_{j=1}^{N} q_{0,j,N+1}(x[N+2], \ell[N+2])\,\mathbf{K}\,(x[\![0,j]\!], \ell[\![0,j]\!])\ \mathbf{K}\,(x[\![j,N+1]\!], \ell[\![j,N+1]\!]) \tag{13}$$

for

$$q_{n_1,n_2,n_3}(x[n+2], \ell[n+2]) \quad = \quad \frac{1}{n_3 - n_1 - 1}\left[\ell_{n_2} - \left(\ell_{n_1} + \frac{x_{n_2} - x_{n_1}}{x_{n_3} - x_{n_1}}(\ell_{n_3} - \ell_{n_1})\right)\right]. \tag{14}$$

The proof will be given in Section 2.4.

Remark 13. ■ *The first message is that* $\langle \mathsf{Cb}\,(x[N+2], \ell[N+2])\rangle$ *is explicit, and* $\mathbf{K}$ *is a polynomial in the variables* ℓ_i *!*

■ *Comb are not bi-comb but they are quite similar except that the extreme segment of combs are points, and the shaft coincides with the bottom-most line.*

■ *The factor* $q_{0,j,N+1}(x[N+2], \ell[N+2])$ *appearing in the recursive description of* $\mathbf{K}$ *has a probabilistic and geometric interpretation: we define more generally* q_{n_1,n_2,n_3} *for integers* $n_1 < n_2 < n_3$ *for the expanded formula. The denominator* $n_3 - n_1 - 1$ *is the number of integers lying in* $[\![1+n_1, n_3-1]\!]$*, so that* $1/(n_3-n_1-1)$ *can be thought as the probability a random element chosen in this interval is* n_2*. Now, the bracket in (14) is a length between a vertex and the point obtained by the vertical projection on the opposite side (see Fig. 4).*

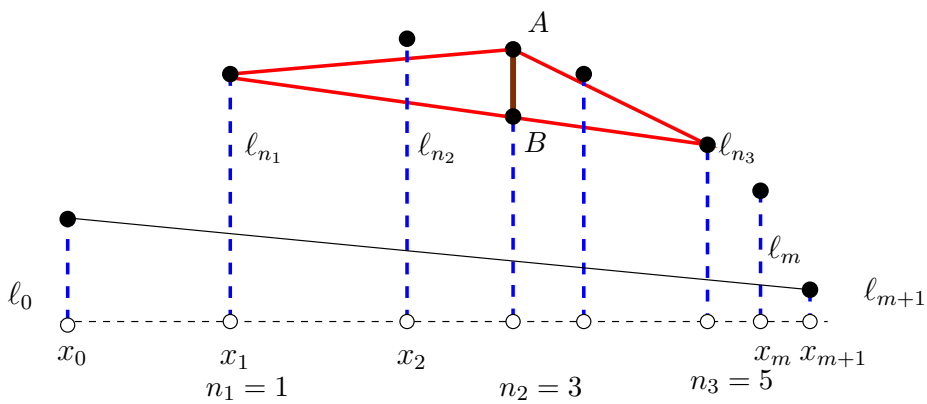

Figure 4: Geometrical interpretation of $\ell_{n_2} - \Big(\ell_{n_1} + \frac{x_{n_2}-x_{n_1}}{x_{n_3}-x_{n_1}}(\ell_{n_3} - \ell_{n_1})\Big)$ in the case $(n_1, n_2, n_3) = (1, 3, 5)$: it corresponds to the length AB.

Hence, (13) can be thought has follows (see Fig.5): choose equally likely a random a vertex with index lying in $[\![1, N]\!]$, split the domain below the convex hull of the comb in three parts : in the middle, a triangle "whose height" and number of segments which crosses it allows to compute $q_{0,j,N+1}(x[N+2], \ell[N+2])$. At each side of the superior apex of this triangle, one can find one comb.

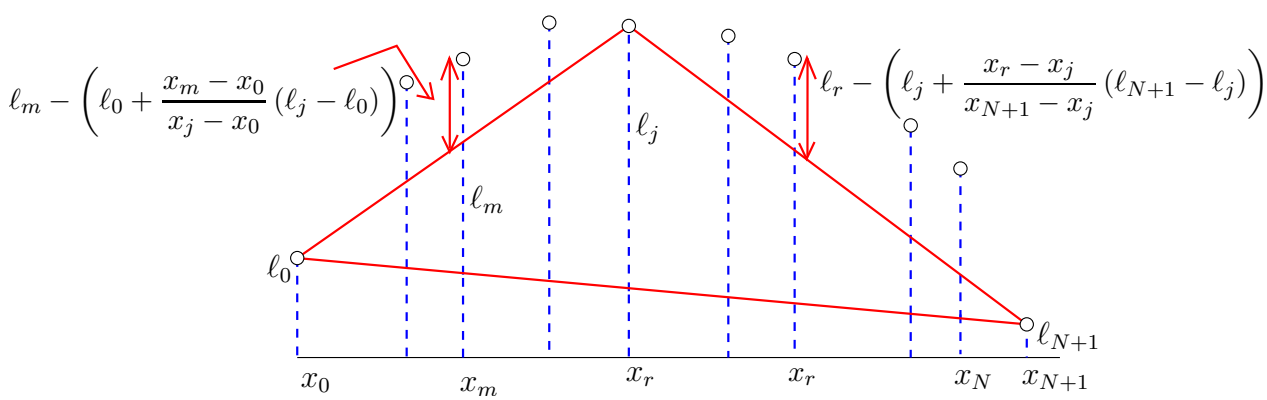

Figure 5: Illustration of a term in the decomposition of $\langle \mathsf{Cb}\big(x[N+1], \ell[N+1]\big)\rangle$.

Formula (13) implies that $\mathbf{K}\big(x[m+2], \ell[m+2]\big)$ possesses a binary structure, and then, can be expanded

using the set of triangulations of the polygon $[(x_i, \ell_i), 0 \le i \le m+2]$.

For $m \ge 1$, call *triangle* of $[\![0, m+1]\!]$ three integers $t = (j_1, j_2, j_3)$ with $0 \le j_1 < j_2 < j_3 \le m+1$. We call *triangulation* of $[\![0, m+1]\!]$ a set T of triangles satisfying the following conditions (see Fig . 6):

• $\#T = m$ (this is the number of different triangles in T),

• the triangles are non crossing : if $t = (j_1, j_2, j_3)$ and $t' = (j'_1, j'_2, j'_3)$ are two different triangles in T then either:

– $\{j_1, j_2, j_3\}$ is included in one of the intervals $[\![0, j'_1]\!]$, $[\![j'_1, j'_2]\!]$, $[\![j'_2, j'_3]\!]$, $[\![j'_3, m+1]\!]$

– or $\{j'_1, j'_2, j'_3\}$ is included in one of the intervals $[\![0, j_1]\!]$, $[\![j_1, j_2]\!]$, $[\![j_2, j_3]\!]$, $[\![j_3, m+1]\!]$.

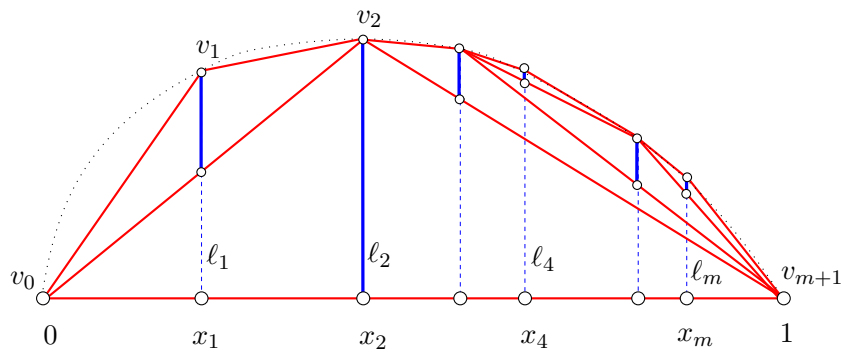

Figure 6: Illustration of a triangulation t.

Denote by $\mathsf{Tri}(m+2)$ the set of triangulations of $[\![0, m+1]\!]$. Each triangulation $T \in \mathsf{Tri}(m+2)$ is a set of m triangles $\{t_1, \cdots, t_m\}$ whose m central points are distinct and their set coincides with $[\![1, m]\!]$.

The expansion of Formula (13) gives

Theorem 14. *(i) For $m = 0$, $\mathbf{K}(x[m+2], \ell[m+2]) = 1$ and for $m \ge 1$,*

$$\mathbf{K}(x[m+2], \ell[m+2]) = \sum_{T \in \mathsf{Tri}(m+2)} \prod_{t \in T} q_t(x[m+2], \ell[m+2]). \tag{15}$$

(ii) In view of (14), $\mathbf{K}(x[m+2], \ell[m+2])$ is an homogeneous polynomial of degree m in the variables $\ell_0, \cdots, \ell_{m+1}$, whose coefficients are rational fractions in the variables $(x_0, \cdots, x_{m+1})$.

(iii) For any t, the maps $\ell \mapsto q_t(x[m+2], \ell[m+2])$ is linear so that $q_t(x[m+2], \ell[m+2] + \ell'[m+2]) = q_t(x[m+2], \ell[m+2]) + q_t(x[m+2], \ell'[m+2])$.

(iv) If L is affine ($L(x) = Ax + B$, for all x) then $q_t(x[m+2], [L(x_i), 0 \le i \le m+1]) = 0$, so that

$$\mathbf{K}(x[m+2], \ell[m+2]) = \mathbf{K}(x[m+2], \ell[m+2] + [L(x_i), 0 \le i \le m+1])$$

The map $\mathbf{K}$ is invariant by translation: for any constant $c, c' \in \mathbb{R}$,

$$\mathbf{K}(x[m+2], \ell[m+2]) = \mathbf{K}([c + x_i, 0 \le i \le m+1], [c' + \ell_i, 0 \le i \le m+1])$$

These last properties are consequence of the geometrical interpretation of $\mathbf{K}$, see e.g. Fig. 4.

The view of the first values of $\mathbf{K}$ may help the reader to build its intuition on these objects: we have

$$\begin{aligned}
\mathbf{K}(x[3],\ell[3]) &= \ell_1-\ell_0-\frac{(\ell_2-\ell_0)(x_1-x_0)}{x_2-x_0},\\
\mathbf{K}(x[4],\ell[4]) &= 1/2\left(\ell_2-\ell_1-\frac{(\ell_3-\ell_1)(x_2-x_1)}{x_3-x_1}\right)\left(\ell_1-\ell_0-\frac{(\ell_3-\ell_0)(x_1-x_0)}{x_3-x_0}\right)\\
&\quad +1/2\left(\ell_1-\ell_0-\frac{(\ell_2-\ell_0)(x_1-x_0)}{x_2-x_0}\right)\left(\ell_2-\ell_0-\frac{(\ell_3-\ell_0)(x_2-x_0)}{x_3-x_0}\right).
\end{aligned}$$

Of course, $\mathbf{K}(x[n],\ell[n])$ "complexity" grows fast as a function of n.

The bi-comb formula : We call bi-comb a sequence of vertical lines of the type $[V_0,\cdots,V_{N+1}]$ where

$$V_i=[(x_i,-f_i^-),\ (x_i,f_i^+)]$$

where the two extreme segments are centered:

$$A=f_0^+=f_0^-,\quad B=f_{N+1}^+=f_{N+1}^-$$

Again, $0=x_0<\cdots<x_{N+1}=1$, the (x_i,f_i^+) are above the line between (x_0,f_0^+) and (x_{N+1},f_{N+1}^+) and in a convex position, and the same holds for (x_i,f_i^-) instead (here f_i^- is non negative), but the bottom-most position of V_i is $(x_i,-f_i^-)$.

Theorem 15. *For any bi-comb* $[V_H(x_0),\cdots,V_H(x_{N+1})]$

$$\begin{aligned}
\langle V_0,\cdots,V_{N+1}\rangle &= \frac{1}{\prod_{j=0}^{N+1}(f_j^++f_j^-)}\int_{-A}^{A}\int_{-B}^{B}\sum_{D\subset[\![1,N]\!]}\mathbf{K}\left[[x_0,(x_j,j\in D),x_{n+1}],[a,(f_j^+,j\in D^c),b]\right] &(16)\\
&\quad\times\mathbf{K}\left[[x_0,(x_j,j\in D^c),x_{N+1}],[-a,(f_j^-,j\in D^c),-b]\right]dadb, &(17)\\
&= \frac{1}{\prod_{j=0}^{N+1}(f_j^++f_j^-)}\int_0^A\int_0^B g_N^{\mathsf{refi.}}(a,b,\varepsilon,\varepsilon',f^+,f_-)\,da\,db &(18)
\end{aligned}$$

(here refi. *stands for the word "refined"), and where*

$$\begin{aligned}
g_N[a,b,(f_j^+),(f_j^-)] &= \sum_{\substack{\varepsilon\in\{-1,1\}\\ \varepsilon'\in\{-1,1\}}} g_N^{\mathsf{refi.}}[a,b,\varepsilon,\varepsilon',(f_j^+),(f_j^-)] &(19)\\
g_N^{\mathsf{refi.}}[a,b,\varepsilon,\varepsilon',(f_j^+),(f_j^-)] &= \sum_{D\subset[\![1,N]\!]}\mathbf{K}\left[[x_0,(x_j,j\in D),x_{n+1}],[\varepsilon a,(f_j^+,j\in D^c),\varepsilon' b]\right] &(20)\\
&\quad\times\mathbf{K}\left[[x_0,(x_j,j\in D^c),x_{N+1}],[-\varepsilon a,(f_j^-,j\in D^c),-\varepsilon' b]\right] &(21)
\end{aligned}$$

Remark 16. *(i) By the properties given in Theorem 14, ignoring the pre-factor* $\frac{1}{\prod_{j=0}^{N+1}(f_j^++f_j^-)}$ *in (16), the double integral is a polynomial of degree* $N+2$ *in the variables* f_j^+ *and* f_j^- *with coefficients in the field of rational fractions in the variables* $(x_i,0\le i\le N+1)$.

(ii) Notice that in the definition of g_N*,* a *and* b *are the ordinates of the random points at the boundary segments, when* (f_j^+) *and* (f_j^-) *are sequences indexed by the integer* 1 *to* N*, and then they encode the "inner" segments, somehow.*

The next Corollary of this Theorem and of (7) allows to get the value of Q_H^{N+2} under an integral form

Corollary 17. *For any $H \in \mathsf{CCS}$ and any $n \geq 0$, for $N+2=n$,*

$$Q_H^{N+2} \;=\; \int_{\mathsf{ND}_{N+2}} \int_{\underline{y}_H(x_0)}^{\overline{y}_H(x_0)} \int_{\underline{y}_H(x_{N+1})}^{\overline{y}_H(x_{N+1})} g_N\left[a,b,(\overline{y}_H(x_j)),(\underline{y}_H(x_j))\right] da\, db\, d_{x_0} \cdots d_{x_{N+1}}.$$

Hence, Q_H^{N+2} is represented here as $N+4$ real integrals of a polynomial.

2.3 Proof of Theorem 15

Till the end of the proof of Theorem 15 we will assume that Proposition 12 holds true.

Take some independent random variables $(U_i, 0 \leq i \leq N+1)$, such that U_i is uniform on V_i . Hence

$$U_i = (x_i, Y_i), \quad \text{for } Y_i \text{ uniform on } [-f_i^-, f_i^+].$$

Let us condition by $(U_0, U_{N+1}) = ((0,a),(1,b))$, the position of the two extremes points U_0 and U_{N+1}. The line passing by U_0 and U_{N+1} cuts each segment V_i in two parts (see Fig. 7). Let

$$u_j := a + \frac{x_j - x_0}{x_{N+1} - x_0}(b-a) = a\,(1-x_j) + x_j\, b, \quad j \in [\![0, N+1]\!] \tag{22}$$

the ordinate of the line (U_0, U_{N+1}) at abscissa x_j (where it intersects V_j). Let

$$\mathsf{Abo} := \{j \in [\![1,N]\!] \;: Y_j \geq u_j\},$$

the set of indices corresponding to the U_j's above the line (U_0, U_{N+1}). Since, the U_i are above (or below) the line (U_0, U_{N+1}) independently, and with a probability proportional to the fraction of the segment that lies above (or below) (U_0, U_{N+1}), we have that for any $D \subset [\![1,N]\!]$ (and for $D^c := [\![1,N]\!] \setminus D$),

$$\mathbb{P}(\mathsf{Abo} = D \mid Y_0 = a, Y_{N+1} = b) = \prod_{j \in D} \frac{f_j^+ - u_j}{f^+ + f_j^-} \prod_{j \in D^c} \frac{u_j + f_j^-}{f_j^+ + f_j^-}, \tag{23}$$

The proof of the following lemma is immediate (see Fig. 7):

Lemma 18. *Conditionally on $(Y_0, Y_{N+1}, \mathsf{Abo}) = (a,b,D)$:*

(a) The ordinates $(Y_i, 1 \leq i \leq N)$ are independent,

(b) if $j \in D$, then U_j is uniform in $[u_j, f_j^+]$, and if $j \in D^c$, then U_j is uniform in $[-f_j^-, u_j]$

(c) the sequence $U[N+2]$ is in convex position iff both sets $\{U_i, i \in \{0, N+1\} \cup D\}$ and $\{U_i, i \in \{0, N+1\} \cup D^c\}$ are in convex position. This happens, conditionally on $(Y_0, Y_{N+1}, \mathsf{Abo}) = (a,b,D)$, with probability

$$Q_{D,a,b} \;=\; \langle (x_0,a), [\ (x_i,u_i),\ (x_i,f_i^+)\ ]_{i \in D}, (x_{N+1},b)\rangle \tag{24}$$

$$\times\; \langle (x_0,-a), [\ (x_i,-u_i),\ (x_i,f_i^-)\ ]_{i \in D^c}, (x_{N+1},-b)\rangle. \tag{25}$$

In other words, in the last formula,

■ the first comb has $|D|+2$ segments: the two extreme ones being reduced to the single points (x_0,a) and (x_{N+1},b), and the other ones are $[(x_i,u_i),(x_i,f_i^+)]$ for $i \in D$,

■ the second comb is obtained by making the symmetry of the "comb" below (U_0, U_{N+1}) composed by the segments indexed by D^c with respect to the x-axis: it has $|D^c|+2$ segments. The two extreme ones being reduced to the single points $(x_0,-a)$ and $(x_{N+1},-b)$, and the other ones are $[(x_i,-u_i),(x_i,f_i^-)]$ for $i \in D^c$.

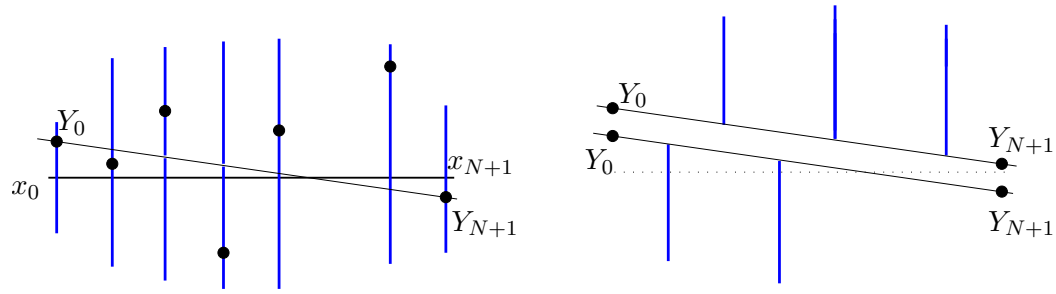

Figure 7: On the left picture: the two extreme points being fixed $(Z_0, Z_{N+1}) = ((x_0, a), (x_{N+1}, b))$ being fixed, the set of indices of the points above the line is $\mathsf{Abo} = \{2, 4, 5\}$. Now, on the right one: Conditional on $\mathsf{Abo} = \{2, 4, 5\}$, there are some uniform points above the line in each of the part of the segments $2, 4$ and 5, and some uniform points under the line in the part of the segments 1 and 3. Appears, the two inclined combs above and below the line. The $N + 2 = 7$ points are in a convex positions, if in the superior comb, the 5 points are in a convex positions, and below, the 4 points also.

Proof of 18. The second point only needs a small explanation: a random variable with a uniform distribution on a segment $[A, B]$ conditioned to be in $[C, D]$ with $[C, D] \subset [A, B]$ is uniform in $[C, D]$. □

We can now conclude the proof of Theorem 15: To compute $\langle V_0, \cdots, V_{N+1}\rangle$, we first condition by $(U_0, U_{N+1}, \mathsf{Abo}) = (a, b, D)$ and using (23) and (24), we then get

$$\langle V_0, \cdots, V_{N+1}\rangle = \int_{-f_0^+}^{f_0^+} \int_{-f_{N+1}^+}^{f_{N+1}^+} \sum_{D \subset [\![1,N]\!]} \mathbb{P}(\mathsf{Abo} = D \mid Y_0 = a, Y_{N+1} = b)\, Q_{D,a,b}\, da\, db$$

by (11), we conclude.

2.4 Proof of Proposition 12

There are two parts in the Proposition: first (11) relates $\langle \mathsf{Cb}\,(x[N+2], \ell[N+2])\rangle$ and $\mathbf{K}\,(x[N+2], \ell[N+2])$, and, there is the decomposition formula for $\mathbf{K}\,(x[N+2], \ell[N+2])$.

The proof will be simpler to follow if the shaft is on the x-axis, and the teeth of the comb are orthogonal to it. As already discussed if two combs C and C' such that C' is image of C by an inversible affine transformation, then $\langle C\rangle = \langle C'\rangle$.

Consider the inversible affine transformation $\Psi\ :\ (x, y) \to (x, y+\alpha x+\beta)$ which preserves the abscissa (and the up-down direction) where (α, β) is chosen such that $\Psi(x_0, \ell_0) = (x_0, 0)$ and $\Psi(x_{N+1}, \ell_{N+1}) = (x_{N+1}, 0)$. The image of the comb $(x[N+1], \ell[N+1])$ by Ψ is a new comb $(x[N+1], \ell'[N+1])$ with same teeth length, whose teeth directions are parallel to the y-axis, but with a shaft lying on the x-axis. Since, for any two combs C and C' that can be sent on each other by inversible affine transformations, we have $\langle C\rangle = \langle C'\rangle$, then $\langle x[N+1], \ell[N+1]\rangle = \langle x[N+1], \ell'[N+1]\rangle$. By Theorem 14, $\mathbf{K}(x[N+1], \ell[N+1]) = \mathbf{K}(x[N+1], \ell'[N+1])$ is preserved by this affine transformation. Hence, it suffices to prove (11) for this kinds of comb. We then assume that $\ell_0 = \ell_{N+1} = 0$.

In order to complete the proof of Proposition 12 it suffices to prove the next proposition which provides a decomposition formula for $\langle \mathsf{Cb}[x[N+2], \ell[N+2]\rangle$ (the function $\mathbf{K}(.)$ satisfies a simpler decomposition formula than $\langle \mathsf{Cb}(.)\rangle$ and this is why we introduced it).

Proposition 19. *For any $x_0 < x_1$, ℓ_0, ℓ_1, we have $\langle \mathsf{Cb}[[x_0, x_1], [\ell_0, \ell_1]] \rangle = 1$, and for $N \geq 0$*

$$\left\langle \mathsf{Cb}\Big[x[N+2], \ell[N+2]\Big] \right\rangle = \frac{1}{N} \sum_{j=1}^{N} \left(\prod_{1 \leq k < j} \frac{\ell_k - \frac{(x_k - x_0)}{(x_j - x_0)} \ell_j}{\ell_k} \right) \left(\prod_{j < k \leq m} \frac{\ell_k - \frac{x_{N+1} - x_k}{x_{N+1} - x_j} \ell_j}{\ell_k} \right) \tag{26}$$

$$\times \left\langle \mathsf{Cb} \left[x[\![0, j]\!], \left[\ell_k - \frac{(x_k - x_0)}{(x_j - x_0)} \ell_j, 0 \leq k \leq j \right] \right] \right\rangle \tag{27}$$

$$\times \left\langle \mathsf{Cb} \left[x[\![j, N+1]\!], \left[\ell_k - \frac{x_{N+1} - x_k}{x_{N+1} - x_j} \ell_j, j \leq k \leq N+1 \right] \right] \right\rangle, \tag{28}$$

these two last combs probability are those of the combs are the right and at the left of the initial comb deprived of the triangle $(x_0, \ell_0), (x_j, \ell_j), (x_{N+1}, \ell_{N+1})$ (see Fig. 8).

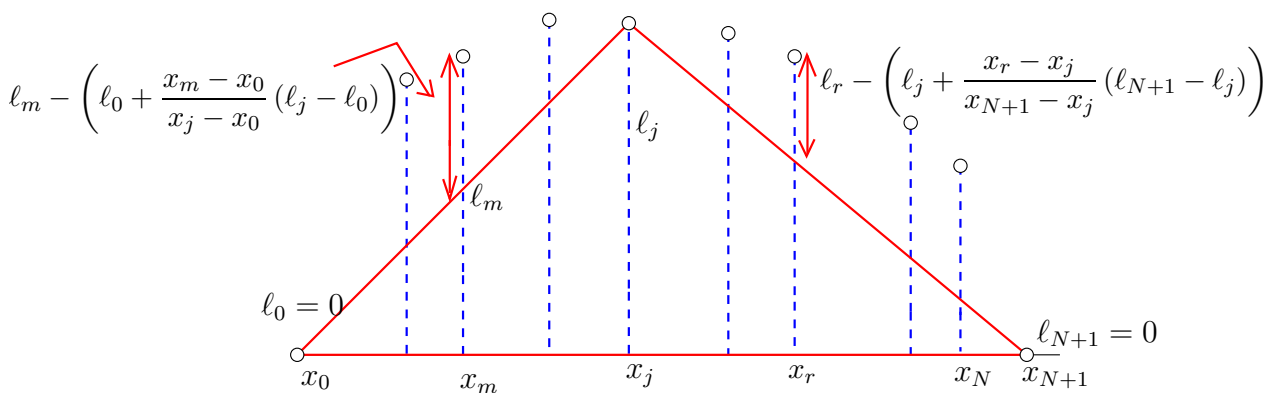

Figure 8: Illustration of the decomposition of $\langle \mathsf{Cb}\big(x[N+1], \ell[N+1]\big) \rangle$ on Proposition 19.

Proof. This is a proof by induction: the case $N = 0$ is trivial since in this case the comb is reduced to the two extreme points, so that $\langle \mathsf{Cb}(x[N+2], \ell[N+2]) \rangle = 1$ which is also the r.h.s. of (11) by definition (12).

Assume $N > 0$, denote by U_i the uniform random variable on the ith segment $S_i = [(x_i, 0), (x_i, \ell_i)]$, where the $(U_i, 0 \leq i \leq N+1)$ are independent, and $U_0 = (x_0, 0)$, $U_{N+1} = (x_{N+1}, 0)$, and for any $i \in [\![1, N]\!]$, $U_i = (x_i, U^{(i)} \times \ell_i)$ for $U^{(1)}, \cdots, U^{(N)}$ i.i.d. uniform random variables on $[0, 1]$. Denote by $U^\star = \max\{U^{(i)}, 1 \leq i \leq N\}$ and $\mathbf{j}^\star$ the almost surely unique index realizing the max : $U^{(\mathbf{j}^\star)} = U^\star$. Conditional on $(\mathbf{j}^\star, U^\star) = (j, u^\star)$, all the other $U^{(j)}$ are uniform on $[0, u^\star]$ (since a random variable uniform on $[0, 1]$ conditioned to be smaller than $u^\star$ is uniform on $[0, u^\star]$). Conditionally on $(\mathbf{j}^\star, U^\star) = (j, u^\star)$, each of the random variables U_i are then uniform on the "reduced tooth"

$$S_i(u^\star) = [(x_i, 0), (x_i, u^\star \ell_i)].$$

In other words, if we remove the topmost $(1 - u^\star)$ fraction of each segment, then the random point $U_{\mathbf{j}^\star}$ of the segment $\mathbf{j}^\star$, is at its top, and the other points are uniform in the remaining fraction. Now, remember that the problem we are studying, is invariant by affine transformation: hence,

$$\mathbb{P}\left(U[N+2] \in \mathsf{CP}_{N+2} \mid (\mathbf{j}^\star, U^\star) = (j, u^\star)\right) = \mathbb{P}\left(U[N+2] \in \mathsf{CP}_{N+2} \mid U_j = (x_j, \ell_j)\right)$$

that is, this coincides with the case where the chosen point in the jth segment is at its top. Now, have a look at picture 8. When the jth point U_j is at its top, taking into account that $U_0 = (x_0, 0)$ and $U_{N+1} = (x_{N+1}, 0)$,

the sequence of points $(U_i, i \in [\![0, N+1]\!])$ are in a convex position iff:
- the $(U_i, 0 \le i \le j)$ are above the line going from $(x_0, 0)$ to (x_j, ℓ_j) and are in a convex position,
- the $(U_i, j \le i \le N+1)$ are above the line going from (x_j, ℓ_j) to (x_{N+1}, ℓ_{N+1}) and are in a convex position.

It remains to interpret Formula (26). The first factor $1/N$ and the sum over j, stands for the distribution of $\mathbf{j}^\star$: its value is j with probability $1/N$ (since all $U^{(k)}$ are equally likely the maximum one), and the value of j runs from 1 to N. Above, in the proof, at a moment, we conditioned by the value of $U^{(\mathbf{j}^\star)}$, but then observe that the reduction to the case where $U^{(\mathbf{j}^\star)} = 1$ was possible independently from $u^\star$: the distribution of $U^\star$ plays no role at the end. The factor at the r.h.s. of (26) is the probability that each the variables U_i are above the lines described before. When the U_i are indeed above these lines, in the interval $[\![0, j]\!]$, the sequence of segments above the line is formed by $j+1$ segments, the two extremes having length 0, and the kth, the length $\ell_k - \frac{(x_k - x_0)}{(x_j - x_0)}\ell_j$ as specified in (27). These segments form a comb. The last factor given in (28) can be interpreted in the same way. □

3 Proof of Theorem 1

There are several steps in the proof. We were unable to prove a full proof of Conjecture 1, but we will see that many intermediate results we prove are valid for all $n \ge 4$.

As explained in the Previous sections, it suffices to prove Proposition 11 to conclude.

Let us examine a bit the possible symmetry defects. For $\lambda[N+2] \in \mathsf{CV}^+(x[N+2])$, and $\beta[N+2] \in \mathsf{Compa}(\lambda[N+2])$. Consider the slope differences, or *second discrete derivatives*,

$$r_j(\lambda) := \frac{\Delta\lambda_j}{\Delta x_j} - \frac{\Delta\lambda_{j+1}}{\Delta x_{j+1}}, \quad j \in [\![1, N]\!] \tag{29}$$

$$s_j(\beta) := \frac{\Delta\beta_j}{\Delta x_j} - \frac{\Delta\beta_{j+1}}{\Delta x_{j+1}}, \quad j \in [\![1, N]\!] \tag{30}$$

(where $\Delta y_j := y_j - y_{j-1}$). They always satisfy

$$r_j(\lambda) \ge 0, \quad |s_j(\beta)| \le r_j(\lambda), \quad \text{for any } j \in [\![1, N]\!]. \tag{31}$$

Notice that both $r(\lambda)$ and $s(\beta)$ and defined by the same formula using respectively λ and β, but the $r_j(\lambda)$ are non negative, when the $s_j(\beta)$ can be negative. The set of symmetry defects compatible with $\lambda[N+2]$ is exactly

$$\mathsf{Compa}(\lambda[N+2]) = \{\beta[N+2] : |s_j(\beta)| \le r_j(\lambda), \beta_0 = \beta_{N+1} = 0, |\beta_j| \le \lambda_j\}. \tag{32}$$

Since given some elements $r = (r_j, j \in [\![1, N]\!]) \in [0, +\infty)$, one can find $\lambda[N+2] \in \mathsf{CV}^+(x[N+2])$ so that $(r_i, 1 \le i \le N) = (r_i(\lambda), 1 \le i \le N)$. Take $\lambda_0 = \lambda_{N+1} = 0$, and for $m \in [\![1, N]\!]$, take

$$\lambda_m = \frac{\sum_{j=m}^{N} r_j (x_j - x_{N+1})(x_0 - x_m) + \sum_{j=1}^{m-1} r_j (x_m - x_{N+1})(x_0 - x_j)}{x_{N+1} - x_0}. \tag{33}$$

This implies that

$$\mathsf{CV}^+(x[N+2]) = \left\{\lambda[N+2] \ : \lambda_0 = \lambda_{N+1} = 0, (r_j(\lambda), 1 \le j \le N) \in [0, +\infty)^N\right\}.$$

The same formula as (33) holds for β in terms of $s(\beta)$.

Proof of Proposition 11 Recall that $L_i = A(1-x_i) + B\,x_i$, for $A \geq 0$, $B \geq 0$. For $0 \leq a \leq A$,$0 \leq b \leq B$, $\lambda[N+2] \in \mathsf{CV}^+(x[N+2])$, $\beta \in \mathsf{Compa}(\lambda[N+2])$, set

$$\left\{ \begin{array}{rcl} \overline{\Gamma}_N[a,b,L,\lambda,\beta] & = & g_N[a,b,L+\lambda,L+\lambda] - g_N[a,b,L+\lambda-\beta,L+\lambda-\beta], \\ \underline{\Gamma}_N[a,b,L,\lambda,\beta] & = & g_N[a,b,L+\lambda+\beta,L+\lambda-\beta] - g_N[a,b,L+2\lambda,L]. \end{array} \right. \tag{34}$$

It suffices to prove the positivity of these two functions. We are able to do that when $N=2$ and $N=3$ (which corresponds to the case $n=4$ and $n=5$).

A part of the analysis needs explicit computations, and a part can be treated using what we know on the functions $\mathbf{K}$. Indeed, the map $\mathbf{K}$ can be expanded over triangulations as explained below Proposition 12. Each triangle in the triangulation corresponds to a factor q as expressed in (13) which is linear in the variables into play L, λ, β, a or b. For $N=4$, the analysis is immediate, and we recover Blaschke result.

3.1 Case $n=4$ (that is $N=2$)

[Already known, by Blaschke by a different method] The polynomial $\overline{\Gamma}_N[a,b,L,\lambda,\beta]$ is even in β (for every N), of total degree N in the variables A,B,λ, β,a, b, even in a and b. The difference stated in (34) provides many simplifications. A complete computation gives:

$$\left\{ \begin{array}{rcl} \overline{\Gamma}_2[a,b,L,\lambda,\beta] & = & 4\,\dfrac{1-x_2}{1-x_1}\beta_1^2 + 4\,\dfrac{x_1}{x_2}\beta_2^2, \\ \underline{\Gamma}_2[a,b,L,\lambda,\beta] & = & 4\,\dfrac{1-x_2}{1-x_1}(\lambda_1^2-\beta_1^2) + 4\,\dfrac{x_1}{x_2}(\lambda_2^2-\beta_2^2) \end{array} \right. \tag{35}$$

which are non negative. That's all!

The similitude of these two values can be explained (this will be done in Section 3.2.1 for the case $N=5$ where the same property appear).
The simplicity of these formula may be understood without computations: recall (19): since $g_2(a,b,f^+,f^-) = \sum_{\substack{\varepsilon\in\{-1,+1\}\\ \varepsilon'\in\{-1,+1\}}} g_2^{\mathsf{refi.}}(a,b,\varepsilon,\varepsilon',f^+,f^-)$ is a polynomial of degree 2 in the $f_1^+, f_2^+, f_1^-, f_2^-$, symmetric in the sense $g_2(a,b,f^+,f^-) = g_2(a,b,f^-,f^+)$ so that in (35) the l.h.s. is even and quadratic in β_1 and β_2 since the other terms cancel out.

3.2 Case $n=5$ (that is $N=3$)

Observe that for $N=3$, $g_3(a,b,L,\lambda,\beta)$ is a polynomial of degree 3 in the $L_i, \lambda_i, \beta_i, a, b$. Since $\overline{\Gamma_3}[a,b,L,\lambda,\beta]$ and $\underline{\Gamma_3}[a,b,L,\lambda,\beta]$ are both even in β, and as polynoms in β have non constant term, they are quadratic in the variables $(\beta_1,\beta_2,\beta_3)$. Hence, seen as polynomials in β their coefficients are linear in the λ_i and in the L_i. Something which can seem strange, is that

$$\left\{ \begin{array}{rcl} \overline{\Gamma_3}[a,b,L,\lambda,\beta] & = & \overline{\Gamma_3}[0,0,L,\lambda,\beta], \\ \underline{\Gamma_3}[a,b,L,\lambda,\beta] & = & \underline{\Gamma_3}[0,0,L,\lambda,\beta]. \end{array} \right.$$

The reason again, is that since for any N, $g_N(a,b,L,\lambda,\beta) = \displaystyle\sum_{\substack{\varepsilon\in\{-1,1\}\\ \varepsilon'\in\{-1,1\}}} g_N^{\mathsf{refi.}}(a,b,\varepsilon,\varepsilon',L,\lambda,\beta)$, it turns out that $g_N(a,b,L,\lambda,\beta)$ is even in a and in b. The fact that it is even in β implies that for $N=2$ or $N=3$ all contributions with degree ≥ 2 in a or b and ≥ 2 in the β_i vanish. So the factor in front of the the quadratic terms of a and b is the same in $g_N(a,b,L,\lambda,\beta)$ and in $g_N(a,b,L,\lambda,0)$ which is 0. It is worth mentioning that

this property holds only for $N = 2$ and $N = 3$, and this is one of the reasons our method does not apply directly for $N > 3$.

In this section, often, we will write r_i instead of $r_i(\lambda)$ and s_i instead of $s_i(\beta)$.

Lemma 20. *$\overline{\Gamma}_3(a, b, L, \lambda, \beta)$ can be rewritten under the form*

$$\overline{\Gamma}_3(a, b, L, \lambda, \beta) = \sum_{j=1}^{3} R_j \times P_j[s_1, s_2, s_3]$$

where $R_j = r_j(\lambda) + 1_{j=1}\frac{A}{x_1} + 1_{j=3}\frac{B}{1-x_3}$, and where P_j is an homogeneous polynomial of degree 2 in the variables (s_1, s_2, s_3).

Proof. The sequence of second derivatives of the sequence $(L_i, 1 \le i \le N)$ is $1_{j=1}\frac{A}{x_1} + 1_{j=3}\frac{B}{1-x_3}$. Recall that by (29), the second derivative $r_i(\lambda)$ are linear with respect to λ, and the $s_i(\beta)$ linear in β, so that the degree analysis exposed above is still valid: $\overline{\Gamma}_3(a, b, L, \lambda, \beta)$ is quadratic in β (or in the s_i), and linear in the $r_i(\lambda)$ and in the L_i. In fact, because of the very similar role of $r_1(\lambda)$ and of $1_{j=1}\frac{A}{x_1}$ (resp. of $r_3(\lambda)$ and $1_{j=3}\frac{B}{1-x_3}$), the representation given in Lemma 20 follows. □

We have packed r_j and A and B, to reduce the number of terms to be studied. What we will prove now, is that the three polynomial P_1, P_2 and P_3 are non negative, and for this, it is sufficient to prove that they are definite positive quadratic forms. By symmetry of the problem with respect to the symmetry $(x, y) \mapsto (1 - x, y)$, we can just prove this for P_1 and P_2 only. We make the following change of variables, $x_0 = 0$, and

$$x_k = y_1 + \cdots + y_k \text{ and for } k \in [\![1, 4]\!],$$

which will help to prove the non negativity of some quantities below.

One then finds that for i from 1 to 2 one has $P_i(s_1, s_2, s_3) = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} M^{(i)} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$ and

$$M^{(1)} = \frac{2y_1y_4}{y_1 + y_2 + y_3} \begin{bmatrix} (y_3 + y_4)(y_2 + y_3){y_1}^2 & (1/2)W_1y_1 & (y_2 + y_3)(y_1 + y_2)y_1y_4 \\ (1/2)W_1y_1 & \frac{W_6}{y_2+y_3+y_4} & \frac{W_5y_4}{y_2+y_3+y_4} \\ (y_2 + y_3)(y_1 + y_2)y_1y_4 & \frac{W_5y_4}{y_2+y_3+y_4} & \frac{(y_1+y_2+y_3)W_4y_4}{(y_2+y_3+y_4)(y_3+y_4)} \end{bmatrix}$$

$$M^{(2)} = \frac{y_1y_4}{y_2 + y_3 + y_4} \begin{bmatrix} \frac{y_1(y_2+y_3+y_4)W_3}{y_1+y_2+y_3} & 2\frac{(y_3+y_4)W_2y_1}{y_1+y_2+y_3} & 2\frac{W_2y_1y_4}{y_1+y_2+y_3} \\ 2\frac{(y_3+y_4)W_2y_1}{y_1+y_2+y_3} & \frac{W_1W_2}{(y_1+y_2+y_3)(y_2+y_3)} & 2\frac{(y_1+y_2)W_2y_4}{y_1+y_2+y_3} \\ 2\frac{W_2y_1y_4}{y_1+y_2+y_3} & 2\frac{(y_1+y_2)W_2y_4}{y_1+y_2+y_3} & W_3y_4 \end{bmatrix}$$

for

$$\begin{aligned}
c_1 &= y_1y_2y_4 + y_1{y_3}^2 + y_1y_3y_4 + {y_2}^2y_4 \\
c_2 &= y_1y_2y_3 + {y_2}^2y_3 + y_2{y_3}^2 + y_2y_3y_4 \\
W_1 &= c_2 + 2c_1 \\
W_2 &= c_1 + 2c_2 \\
W_3 &= 2c_1 + 3c_2 \\
W_4 &= y_2c_2 + (2y_2 + y_3 + y_4)c_1 + y_2(y_2 + y_4)^2(y_2 + y_3) \\
W_5 &= y_1c_1 + 2y_2(y_2 + y_3)(y_1 + y_2) \\
W_6 &= y_2c_2 + (y_1 + y_2)c_1(y_2 + y_3 + y_4) + y_2^2y_4(y_1 + y_2)
\end{aligned}$$

The computation of the principal minors of these matrices give non negative coefficients: they are rational fractions with positive coefficients in the y_i's. We print them here for the sake of completness: the following computations are immediate using the representations of the matrices $M^{(1)}$ and $M^{(2)}$ given above (see if needed [8]). The first minor of $M^{(1)}$ and of $M^{(2)}$ are clearly of this type . The second minors of these matrices are respectively :
$m_2^{(1)} := (3\,y_1y_2{y_3}^2 + 8\,y_1y_2y_3y_4 + 4\,y_1y_2{y_4}^2 + 3\,y_1{y_3}^3 + 7\,y_1{y_3}^2y_4 + 4\,y_1y_3{y_4}^2 + 3\,{y_2}^2{y_3}^2 + 8\,{y_2}^2y_3y_4 + 4\,{y_2}^2{y_4}^2 + 6\,y_2{y_3}^3 + 14\,y_2{y_3}^2y_4 + 8\,y_2y_3{y_4}^2 + 3\,{y_3}^4 + 6\,{y_3}^3y_4 + 3\,{y_3}^2{y_4}^2){y_1}^4{y_2}^2{y_4}^2/((y_2 + y_3 + y_4)\,(y_1 + y_2 + y_3)^2)$ and

$$m_2^{(2)} \quad := \quad m_2^{(1)} \times \frac{2\,y_1y_2y_3 + y_1y_2y_4 + y_1{y_3}^2 + y_1y_3y_4 + 2\,{y_2}^2y_3 + {y_2}^2y_4 + 2\,y_2{y_3}^2 + 2\,y_2y_3y_4}{y_1\,(y_2 + y_3 + y_4)\,(y_2 + y_3)}$$

and for the third minors
$m_3^{(1)} = 2\,{y_1}^5{y_4}^4{y_3}^2{y_2}^2(3\,y_1{y_2}^2y_3 + 2\,y_1{y_2}^2y_4 + 6\,y_1y_2{y_3}^2 + 9\,y_1y_2y_3y_4 + 3\,y_1y_2{y_4}^2 + 3\,y_1{y_3}^3 + 6\,y_1{y_3}^2y_4 + 3\,y_1y_3{y_4}^2 + 3\,{y_2}^3y_3 + 2\,{y_2}^3y_4 + 6\,{y_2}^2{y_3}^2 + 8\,{y_2}^2y_3y_4 + 2\,{y_2}^2{y_4}^2 + 3\,y_2{y_3}^3 + 6\,y_2{y_3}^2y_4 + 3\,y_2y_3{y_4}^2)/((y_3 + y_4)(y_2 + y_3 + y_4)(y_1 + y_2 + y_3)^2)$ and
$m_3^{(2)} = 3\,{y_4}^4{y_3}^2{y_2}^2{y_1}^4(2\,y_1y_2y_3 + y_1y_2y_4 + y_1{y_3}^2 + y_1y_3y_4 + 2\,{y_2}^2y_3 + {y_2}^2y_4 + 2\,y_2{y_3}^2 + 2\,y_2y_3y_4)(3\,y_1y_2y_3 + 2\,y_1y_2y_4 + 2\,y_1{y_3}^2 + 2\,y_1y_3y_4 + 3\,{y_2}^2y_3 + 2\,{y_2}^2y_4 + 3\,y_2{y_3}^2 + 3\,y_2y_3y_4)/((y_2 + y_3 + y_4)^2(y_1 + y_2 + y_3)^2(y_2 + y_3))$ (these computations are direct from the matrices above, see if needed [8]).

Since finally,

$$\overline{\Gamma}(a, b, L, \lambda, \beta) = \sum_{j=1}^{3} \left(r_j(\lambda) + 1_{j=1}\frac{A}{x_1} + 1_{j=3}\frac{B}{1 - x_3} \right) \times [s_1, s_2, s_3]M^{(j)} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \tag{36}$$

we deduce that $\overline{\Gamma}(a, b, L, \lambda, \beta) \geq 0$ for all values of a, b, L, λ, β satisfying the hypothesis.

These computations can be done by hand, and be checked by a small program if needed (see [8]).

3.2.1 The minimum is reached for the maximum symmetry defect $\beta = \lambda$

We write

$$\begin{aligned}
\underline{\Gamma_3}[a, b, L, \lambda, \beta] &= g_N[a, b, L + \lambda + \beta, L + \lambda - \beta] - g_N[a, b, L + 2\lambda, L] \\
&= \overline{\Gamma_3}[a, b, L, \lambda, \lambda] - \overline{\Gamma_3}[a, b, L, \lambda, \beta]
\end{aligned}$$

For any square and symmetric matrix M and vector v and w with same size, ${}^tvMv - {}^twMw = {}^t(v+w)M(v-w)$ By (36)

$$\underline{\Gamma}_3(\beta) = \sum_{j=1}^{3} \left(r_j(\lambda) + 1_{j=1}\frac{A}{x_1} + 1_{j=3}\frac{B}{1-x_3} \right) \left[r_1+s_1, r_2+s_2, r_3+s_3 \right] M^{(j)} \begin{bmatrix} r_1-s_1 \\ r_2-s_2 \\ r_3-s_3 \end{bmatrix}$$

Since the matrices $M^{(j)}$ have positive coefficients, since for each i, $r_i+s_i \geq 0$ and $r_i - s_i \geq 0$, we can conclude that $\underline{\Gamma_3}[a,b,L,\lambda,\beta] \geq 0$.

4 Toward the general case, what remains to be done ?

We tried many things to prove that in all generality $\underline{\Gamma_N}[a,b,L,\lambda,\beta] \geq 0$ and $\overline{\Gamma_N}[a,b,L,\lambda,\beta] \geq 0$. Taking into account that the $\mathbf{K}$ are polynomials, and $\beta \in \mathsf{Compa}(\lambda[N+2])$ (see (32)), and since $\mathsf{Compa}(\lambda[N+2])$ is a convex subset of $\mathbb{R}^{N+2}$ and also a semi-algebraic set, it is natural to try to prove the non-negativity of $\underline{\Gamma_N}$ and $\overline{\Gamma_N}$ using convexity arguments or using polynomial theory : the so-called Positivstellensätze and Nichtnegativstellensätze aims to prove positivity or non negativity of polynomials on semi-algebraic sets (see e.g. Lasserre [6]). To use these powerful tools, "a polynomial certificate" has to be guessed, which despite important effort, we were not able to do.

However, we assembled in a Proposition some results that allow to support the conjecture 1.

Theorem 21. *Let $N \geq 2$ (that is $n \geq 4$)*

(i) $\underline{\Gamma_N}[0,0,0,\lambda,\beta] \geq 0$, which is equivalent to $g_N(0,0,\lambda+\eta,0) \leq g_N(0,0,\lambda,\eta)$ holds for all every $N > 0$, any $\lambda,\eta \in \mathsf{CV}^+(x[N+2])$ and all $\beta \in \mathsf{Compa}(\lambda)$ (this is the case where the two extreme segments of the bi-combs have length 0).

(ii) For any fixed a,b,L,λ, if for each i, $r_i(\lambda) > 0$, the map $\beta \mapsto \underline{\Gamma_N}[a,b,L,\lambda,\beta]$ reaches its minimum for β in the boundary of $\mathsf{Compa}(\lambda)$, and has no strict local minimum in the interior of $\mathsf{Compa}(\lambda)$.

In (ii), we say that β is in the interior of $\mathsf{Compa}(\lambda)$ if for any $\nu[N+2] \in \mathbb{R}^{N+2}$ with $\nu_0 = \nu_{N+1} = 0$, for y small enough $\beta + y\nu$ in $\mathsf{Compa}(\lambda)$. Since $(r_i(\lambda), 1 \leq i \leq N) \in (0,+\infty)$, notice that for any fixed $\beta[N+2] \in \mathbb{R}^{N+2}$ with $\beta_0 = \beta_{N+1} = 0$, if α is small enough then $\alpha\beta \in \mathsf{Compa}(\lambda)$.

Proof of Theorem 21 (i) This is the only bound by recurrence we were able to proved.

The first lemma provides a bound for $\mathbf{K}$ when an additional inner tooth is removed. For this Lemma we adopt the following shorter notation for $\mathbf{K}$ which is more suitable for the insertion purpose. For any set of indices I,

$$\mathbf{K}_{x(I)}(\lambda) = \mathbf{K}\left[(x_i, i \in I), (\lambda_i, i \in I)\right].$$

For any sequence $y[N+2]$, denote by $y[N+2]_{(-j)} = (y_0, \cdots, y_{j-1}, y_{j+1}, \cdots, y_{N+1})$.

Lemma 22. *Let $N \geq 1$, $x_0 < \cdots < x_{N+1}$ given and $\kappa \in [\![1,N]\!]$ an inner segment index. Consider $\lambda[N+2] \in \mathsf{CV}^+(N+2)$. We have*

$$\mathbf{K}_{x[N+2]_{(-\kappa)}}(\lambda) \geq \mathbf{K}_{x[\![0,\kappa]\!]}(\lambda) \times \mathbf{K}_{x[\![\kappa,N+1]\!]}(\lambda)$$

Proof. Recall the definition of q_{n_1,n_2,n_3} in (14) : at the pre-factor denominator lies $n_3 - n_1 + 1$ which is the number of inner segments between n_1 and n_3.

This is a proof by induction. We will use the dissection formula (13) to expand the inner N segments formula of $\mathbf{K}\left[x[N+2]_{(-\kappa)}, \lambda[N+2]_{(-\kappa)}\right]$

$$\begin{aligned}\mathbf{K}_{x[N+2]_{(-\kappa)}}(\lambda) &= \sum_{j=1}^{\kappa-1} q_{x_0,x_j,x_{N+1}}(x[N+2]_{(-\kappa)}, \lambda[N+2]_{(-\kappa)})\mathbf{K}_{x[\![0,j]\!]}(\lambda)\mathbf{K}_{x[\![j,N+2]\!]_{(-\kappa)}}(\lambda) \\ &+ \sum_{j=\kappa+1}^{N} q_{x_0,x_j,x_{N+1}}(x[N+2]_{(-\kappa)}, \lambda[N+2]_{(-\kappa)})\mathbf{K}_{x[\![0,j]\!]_{(-\kappa)}}(\lambda)\mathbf{K}_{x[\![j,N+1]\!]}(\lambda).\end{aligned}$$

By recurrence, in the first sum $\mathbf{K}_{x[\![j,N+2]\!]_{(-\kappa)}} \geq \mathbf{K}_{x[\![j,\kappa]\!]}\mathbf{K}_{x[\![\kappa,N+2]\!]}$ and in the second sum $\mathbf{K}_{x[\![0,j]\!]_{(-\kappa)}} \geq \mathbf{K}_{x[\![0,\kappa]\!]}\mathbf{K}_{x[\![\kappa,j]\!]}$. Taking into account the pre-factors in q, just comparing the distance to the projection, in the first sum we have

$$q_{x_0,x_j,x_{N+1}}(x[N+2]_{(-\kappa)}, \lambda[N+2]_{(-\kappa)}) \geq \frac{\kappa-1}{N} q_{x_0,x_j,x_\kappa}(x[\![0,\kappa]\!], \lambda[\![0,\kappa]\!])$$

and in the second one

$$q_{x_0,x_j,\kappa}(x[N+2]_{(-\kappa)}, \lambda[N+2]_{(-\kappa)}) \geq \frac{N+1-\kappa}{N} q_{x_\kappa,x_j,x_{N+2}}(x[\![\kappa,N+1]\!], \lambda[\![\kappa,N+1]\!]).$$

Hence,

$$\begin{aligned}\mathbf{K}_{x[N+2]_{(-\kappa)}}(\lambda) &\geq \frac{\kappa-1}{N}\mathbf{K}_{x[\![\kappa,N+1]\!]}(\lambda)\sum_{j=1}^{\kappa-1} q_{x_0,x_j,x_\kappa}(x[\![0,\kappa]\!], \lambda[\![0,\kappa]\!])\mathbf{K}_{x[\![0,j]\!]}\mathbf{K}_{x[\![j,\kappa]\!]}(\lambda) \\ &+ \frac{N-\kappa+1}{N}\mathbf{K}_{x[\![0,\kappa]\!]}(\lambda)\sum_{j=\kappa+1}^{N} q_{x_\kappa,x_j,x_{N+1}}(x[\![\kappa,N+1]\!], \lambda[\![\kappa,N+1]\!])\mathbf{K}_{x[\![\kappa,j]\!]}\mathbf{K}_{x[\![j,N+1]\!]}(\lambda) \\ &\geq \left(\frac{\kappa-1}{N} + \frac{N-\kappa+1}{N}\right)\mathbf{K}_{x[\![0,\kappa]\!]}(\lambda)\mathbf{K}_{x[\![\kappa,N+1]\!]}(\lambda) = \mathbf{K}_{x[\![0,\kappa]\!]}(\lambda)\mathbf{K}_{x[\![\kappa,N+1]\!]}(\lambda).\end{aligned}$$

□

End of the poof of Theorem 21 (i) This is a proof by induction. The formula holds when $N = 0$. Assume that the formula holds for $n \leq N$ for some $N \geq 0$, and let us establish it for $n = N+1$. Write

$$\mathbf{K}_{x[\![0,N+2]\!]}[\lambda+\eta] = \sum_{1\leq j\leq N+1} \mathbf{K}_{x[\![0,j]\!]}(\lambda+\eta)\,\mathbf{K}_{x[\![j,N+2]\!]}(\lambda+\eta)\,q_{x_0,x_j,x_{N+2}}(x[\![0,N+2]\!], \lambda+\eta).$$

Expanding $q_{x_0,x_j,x_{N+2}}(x[\![0,N+2]\!], \lambda+\eta) = q_{x_0,x_j,x_{N+2}}(x[\![0,N+2]\!], \lambda) + q_{x_0,x_j,x_{N+2}}(x[\![0,N+2]\!], \eta)$, and applying the recurrence hypothesis, we get for $E_j = \{x_1, \cdots, x_{j-1}\}$, $E'_j = \{x_{j+1}, \cdots, x_{N+1}\}$,

$$\begin{aligned}\mathbf{K}_{x[\![0,N+2]\!]}[\lambda+\eta] &\leq \sum_{1\leq j\leq N+1}\sum_{\substack{A\subset E_j\\ B\subset E'_j}} \mathbf{K}_{x_0Ax_j}(\lambda)\mathbf{K}_{x_0(E_j\setminus A)x_j}(\eta)\mathbf{K}_{x_jBx_{N+2}}(\lambda)\mathbf{K}_{x_j(E'\setminus B)x_{N+2}}(\eta) &(37)\\ &\times q_{x_0,x_j,x_{N+2}}(x[\![0,N+2]\!], \lambda) &(38)\\ &+ \sum_{1\leq j\leq N+1}\sum_{\substack{A\subset E_j\\ B\subset E'_j}} \mathbf{K}_{x_0Ax_j}(\lambda)\mathbf{K}_{x_0(E_j\setminus A)x_j}(\eta)\mathbf{K}_{x_jBx_{N+2}}(\lambda)\mathbf{K}_{x_j(E'\setminus B)x_{N+2}}(\eta) &(39)\\ &\times q_{x_0,x_j,x_{N+2}}(x[\![0,N+2]\!], \eta)) &(40)\end{aligned}$$

to conclude: in the last sum, use by Lemma 22

$$\mathbf{K}_{x_0(E_j\setminus A)x_j}(\eta)\mathbf{K}_{x_j(E'\setminus B)x_{N+2}}(\eta) \le \mathbf{K}_{x_0,(E_j\setminus A)\cup(E'_j\setminus B),x_{n+2}}(\eta)$$

and for any C,

$$\begin{aligned}
&\sum_{j\in C}\mathbf{K}_{x_0C_{<j}x_j}(\lambda)\mathbf{K}_{x_jC_{>j}x_{N+2}}(\lambda)q_{x_0,x_j,x_{N+2}}(x[\![0,N+2]\!],\lambda)\\
=&\sum_{j\in C}\mathbf{K}_{x_0C_{<j}x_j}(\lambda)\mathbf{K}_{x_jC_{>j}x_{N+2}}(\lambda)q_{x_0,x_j,x_{N+2}}(x_0,(x_i,i\in C),x_{N+2},\lambda)\frac{|C|}{N+1}\\
=&\mathbf{K}_{x_0Cx_{N+2}}(\lambda)\frac{C}{N+1}
\end{aligned}$$

and then finally the two sums in the RHS of (37) are bounded by

$$\sum_C \frac{|C|+(N+1-|C|)}{N+1}\mathbf{K}_{x_0Cx_{N+2}}(\lambda)\mathbf{K}_{x_0x_C^cx_{N+2}}(\eta) \le g_{N+1}(0,0,\lambda,\eta).$$

Proof of Theorem 21 (ii) For any $n>j$, denote by e_j the jth vector of the canonical basis of $\mathbb{R}^n$, starting from e_0. A vector $\nu=(\nu_0,\dots,\nu_{N+1})$ is then identified with $\sum_{j=0}^{N+1}\nu_je_j$. In the following, adding vectors with the mean of the e_j is used to introduced some additional symmetry defects to the segments. The main underlying idea of what follows is that we are not really able to compare directly $g_N(0,0,f^+,f^-)$ with $g_N(0,0,f^++xe_j,f^--xe_j)$, but just with $(\sum_{\varepsilon\in\{-1,1\}}g_N(0,0,f^++x\varepsilon e_j,f^--x\varepsilon e_j))/2$.

Proposition 23. *Let $\lambda\in\mathsf{CV}^+(x[N+2])$ such that the $r_i(\lambda)>0$ for all $i\in[\![1,N]\!]$, for L as usual (see (9)), for any $(\beta_i,0\le i\le N+1)\in\{0\}\times\mathbb{R}^N\times\{0\}$ in the interior of $\mathsf{Compa}(\lambda)$, for any $\nu[N+2]\in\{0\}\times\mathbb{R}^n\times\{0\}$, for y small enough*

$$2^{-N}\sum_{(\varepsilon_i,1\le i\le N)\in\{1,-1\}^N} g_N\Big(a,b,L+\lambda+\beta+y\sum_{j=1}^N\varepsilon_j\nu_je_j,L+\lambda-\beta-y\sum_{j=1}^N\varepsilon_j\nu_je_j\Big) \tag{41}$$

$$\le g_N(a,b,L+\lambda+\beta,L+\lambda-\beta) \tag{42}$$

If its hypothesis is satisfied, this proposition implies Theorem 21 (ii) : at the l.h.s. observe that a mean of 2^N different perturbations of $(\lambda+\beta,\lambda-\beta)$. Since the mean of these values is non larger than $g_N(a,b,L+\lambda+\beta,L+\lambda-\beta)$ then either all these values are equal to $g_N(a,b,L+\lambda+\beta,L+\lambda-\beta)$, or one of them at least is smaller than $g_N(a,b,L+\lambda+\beta,L+\lambda-\beta)$. In any case, since this holds for all y small enough, this implies that $\beta\mapsto g_N(a,b,L+\lambda+\beta,L+\lambda-\beta)$ does not possess a strict local minima in the interior of $\mathsf{Compa}(\lambda)$.

We start by proving some properties satisfied by $\mathbf{K}$.

Proposition 24. *For any $\lambda[N+2]\in\mathsf{CV}^+(x[N+2])$, any $1\le i\le N$, any $\alpha\in\mathbb{R}$ small enough,*

(i) $\displaystyle\sum_{\varepsilon\in\{-1,1\}}\mathbf{K}[x[N+2],\lambda[N+2]+\varepsilon\,\alpha e_i]\le 2\,\mathbf{K}[x[N+2],\lambda[N+2]],$

(ii) If $\lambda\in\mathsf{Compa}(\lambda)$ and α is small enough such that $\beta\pm\alpha e_i\in\mathsf{Compa}(\lambda[N+2])$,

$$\sum_{\varepsilon\in\{-1,1\}}g_N(0,0,\lambda+\beta+\varepsilon\alpha e_i,\lambda-\beta-\varepsilon\alpha e_i)\le 2g_N(0,0,\lambda+\beta,\lambda-\beta). \tag{43}$$

Proof. (i) The l.h.s. in (i) is even in α: take some $\alpha \geq 0$. Recall (15): $\mathbf{K}$ is up to the product of the teeth lengths, the probability for a point on each tooth of a comb to be in a convex position. Rewrite (i) as

$$\sum_{\varepsilon\in\{-1,1\}} \frac{p(\varepsilon\alpha)}{p(|\alpha|)} \frac{\mathbf{K}[x[N+2],\lambda[N+2]+\varepsilon\alpha e_i]}{p(\varepsilon\alpha)} \leq 2\frac{p(0)}{p(|\alpha|)} \frac{\mathbf{K}\left[x[N+2],\lambda[N+2]\right]}{p(0)}, \tag{44}$$

for $p(y) := \prod_{j=1}^n (\lambda_j + y\mathbf{1}_{i=j})$. For $a(\alpha) := \dfrac{p(\alpha)}{p(|\alpha|)} \left\langle \mathsf{Cb}\Big[x[N+2],\lambda[N+2]+\alpha e_i\Big]\right\rangle$, by (15), (44) is equivalent to

$$a(\alpha) - a(0) \leq a(0) - a(-\alpha). \tag{45}$$

Each of $a(\alpha), a(0)$ and $a(-\alpha)$ are probabilities of some events. Indeed, consider $\mathsf{Cb}[x[\![N+2]\!],\lambda[N+2]+|\alpha|e_i]$, where the ith tooth is $|\alpha|$ longer. And now, take independently some points $(Y_i, 1 \leq i \leq N)$, where Y_j is uniform on the jth tooth $[(x_j,0),(x_j,\lambda_j+|\alpha|1_{i=j})]$, and as usual, the two boundary points are $(x_0,Y_0)=(0,0)$ and $(x_{N+1},Y_{N+1})=(1,0)$. Denote by

$$\begin{aligned} E &= \{Y[N+2]\in \mathsf{CP}_{N+2}, Y_i \leq \lambda_i - |\alpha|\},\\ E' &= \{Y[N+2]\in \mathsf{CP}_{N+2}, Y_i \leq \lambda_i\},\\ E'' &= \{Y[N+2]\in \mathsf{CP}_{N+2}, Y_i \leq \lambda_i + |\alpha|\}, \end{aligned}$$

in words, these events encode the fact that the points (Y_j) are in a convex position taking into account the position of the ith point with respect to the "deformation" αe_i of the ith tooth.
It is immediately checked that $a(\alpha)=\mathbb{P}(E'')$, $a(-\alpha)=\mathbb{P}(E)$ and $a(0)=\mathbb{P}(E')$, so that (45) is equivalent to

$$\mathbb{P}(Y[N+2]\in \mathsf{CP}_{N+2}, \lambda_i \leq Y_i \leq \lambda_i+\alpha) \leq \mathbb{P}(Y[N+2]\in \mathsf{CP}_{N+2}, \lambda_i-\alpha \leq Y_i \leq \lambda_i). \tag{46}$$

We claim that this property holds for α small enough, and that it can be seen on a simple figure as Fig. 9: the two events $A := \{\lambda_i \leq Y_i \leq \lambda_i+\alpha\}$ and $B := \{\lambda_i-\alpha \leq Y_i \leq \lambda_i\}$ are equally likely, and since Y_i is uniform on $[0,\lambda_i+\alpha]$, Conditional on A (resp. on B), Y_i is uniform on $[\lambda,\lambda_i+\alpha]$ (resp. $[\lambda_i-\alpha,\lambda_i]$). We can then just compare the situation where Y_i has the form $\lambda_i-\alpha+u$ with that when it is λ_i+u for some $u\in[0,\alpha]$. Conditional on these respecting values $\lambda_i-\alpha+u$ (resp. λ_i+u), $Y[N+2]\in\mathsf{CP}_{N+2}$ if the rest of the points avoid a region "above $\lambda_i-\alpha+u$" (resp. λ_i+u), and those at the left of x_i and those at the right are in a convex position. In the case where a deformation of the i line (above and below λ_i) gives a comb in a convex position, for α small enough, the regions above do not contains any points of the other teeth: it appears now clearly that for α small, the other points at the left and at the right of x_i (including (X_i,Y_i) for both case), must be in a convex position, and avoid the the forbidden region as represented in Fig. 9: excluding the regions above, the forbidden region is smaller when $\lambda_i-\alpha+u$ compared to the case where it is λ_i+u.
(ii) For a set $A\subset[\![1,N]\!]$, set $\mathbf{K}_A(\nu)=\mathbf{K}[(x_0,(x_i,i\in A),x_{N+1}),(0,(\nu_j,j\in A),0)]$.
Notice that if $i\notin A$, $\mathbf{K}_A(\nu+\alpha e_i)=\mathbf{K}_A(\nu)$ since the expansion (15) does not use the ith tooth. Now, write

$$\begin{aligned} \sum_{\varepsilon\in\{-1,1\}} g_N(0,0,\lambda+\beta+\varepsilon\alpha e_i,\lambda-\beta-\varepsilon\alpha e_i) &= \sum_{\varepsilon\in\{-1,1\}}\sum_{A\subset[\![1,n]\!]:i\in A} \mathbf{K}_A(\lambda+\beta+\varepsilon\alpha e_i)\mathbf{K}_{A^c}(\lambda-\beta) \quad (47)\\ &+ \sum_{\varepsilon\in\{-1,1\}}\sum_{A\subset[\![1,n]\!]:i\in A^c} \mathbf{K}_A(\lambda+\beta)\mathbf{K}_{A^c}(\lambda-\beta-\alpha e_i) \quad (48) \end{aligned}$$

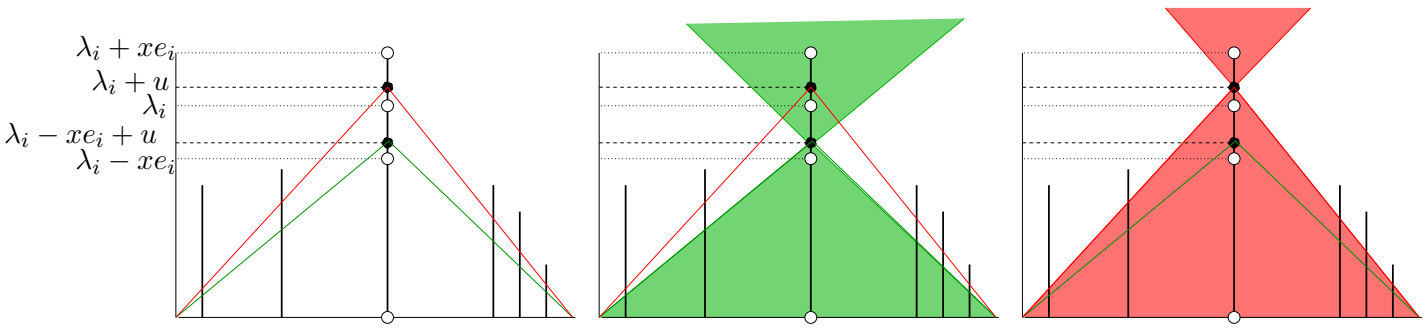

Figure 9: Comparison of the two quantities appearing in (46). On the second picture, respective forbidden region for the other points when $Y_i = \lambda_i - \alpha + u$, on the third one, when $Y_i = \lambda_i + u$.

By exchanging the sums over ε and over A, by (i), this is smaller than

$$2\Big(\sum_{A\subset[\![1,n]\!]:i\in A}\mathbf{K}_A(\lambda+\beta)\mathbf{K}_{A^c}(\lambda-\beta)+\sum_{A\subset[\![1,n]\!]:i\in A^c}\mathbf{K}_A(\lambda+\beta)\mathbf{K}_{A^c}(\lambda-\beta)\Big)$$

which is exactly $2g_N(0,0,\lambda+\beta,\lambda-\beta)$. □

Proof of Proposition 23. For two points u,v, consider the line from (x_0,u) to (x_{N+1},v) denote by $T_i(u,v) := v(1-x_i)+ux_i$ the ordinate of this line at position x_i. Since the line $T(\varepsilon a,\varepsilon' b)$ serves as a basis (the shafts) for the two combs, one has :

$$\begin{aligned} g_N(a,b,L+\lambda,\beta) &= \sum_{\substack{\varepsilon\in\{-1,1\}\\ \varepsilon'\in\{-1,1\}}} g_N^{\mathsf{refi.}}(a,b,\varepsilon,\varepsilon',L+\lambda+\beta,L+\lambda-\beta)\\ &= \sum_{\substack{\varepsilon\in\{-1,1\}\\ \varepsilon'\in\{-1,1\}}} g_N^{\mathsf{refi.}}(0,0,0,0,L+\lambda+\beta-T(\varepsilon a,\varepsilon' b),L+\lambda+\beta+T(\varepsilon a,\varepsilon' b)) \end{aligned}$$

From this property, the l.h.s. of (41) rewrites

$$2^{-|J|}\sum_{\substack{(\varepsilon_i,i\in J)\in\{1,-1\}^J\\ \varepsilon\in\{-1,1\}\\ \varepsilon'\in\{-1,1\}}} g_N^{\mathsf{refi.}}\Big(0,0,0,0,L+\lambda+\beta+y\sum_{j\in J}\varepsilon_j\nu_je_j-T(\varepsilon a,\varepsilon' b),L+\lambda-\beta-y\sum_{j\in J}\varepsilon_j\nu_je_j+T(\varepsilon a,\varepsilon' b)\Big)$$

which is by applying Prop. 24 repeatedly (by adding one by one the terms $\varepsilon_j y_j e_j$ for $\varepsilon_j\in\{-1,+1\}$, as a fifth and sixth entry of g_N), this is not larger than

$$2^{-|J|}\sum_{(\varepsilon_i,i\in J)\in\{1,-1\}^J}\sum_{\substack{\varepsilon\in\{-1,1\}\\ \varepsilon'\in\{-1,1\}}} g_N^{\mathsf{refi.}}\left(0,0,L+\lambda+\beta-T(\varepsilon a,\varepsilon' b),L+\lambda-\beta+T(\varepsilon a,\varepsilon' b)\right) \tag{49}$$

the pre-factor and the first sum produces "an extra factor 1", and then this last quantity reduces to the r.h.s. of (41). □

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