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# MONOTONIC COMPUTATION RULES FOR NONASSOCIATIVE CALCULUS

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*Dedicated to Maurice Pouzet on the occasion of his 75th birthday*

## ABSTRACT

In this paper we revisit the so-called computation rules for calculus using a single nonassociative binary operation over possibly infinite sequences of integers. In this paper we focus on the symmetric maximum  $\otimes$  that is an extension of the usual maximum  $\vee$  so that 0 is the neutral element, and  $-x$  is the symmetric (or inverse) of  $x$ , i.e.,  $x \otimes (-x) = 0$ . However, such an extension does not preserve the associativity of  $\vee$ . This fact asks for systematic ways of bracketing terms of a sequence using  $\otimes$ , and which we refer to as *computation rules*.

These computation rules essentially reduce to deleting terms of sequences based on the condition  $x \otimes (-x) = 0$ , and they can be quasi-ordered as follows: say that rule 1 is below rule 2 if for all sequences of numbers, rule 1 deletes more terms in the sequence than rule 2. As it turns out, this quasi-ordered set is extremely complex, e.g., it has infinitely many maximal elements and atoms, and it embeds the powerset of natural numbers by inclusion.

Local properties of computation rules have also been presented by the authors, in particular, concerning their canonical representations. In this paper we address the problem of determining those computation rules that preserve the monotonicity of  $\vee$ , and present an explicit description of monotonic computation rules in terms of their *factorized irredundant form*.

**Keywords** Nonassociative calculus · symmetric maximum · computation rules · monotonic rules

## 1 Motivation

This short contribution is the continuation of the work initiated in [1, 2], and we refer the reader to these references for further motivation. Let  $L$  be a totally ordered set with bottom element 0, and let  $-L := \{-a : a \in L\}$  be its “symmetric” copy endowed with the reversed order. Consider the symmetric ordered structure  $\tilde{L} := L \cup (-L) \setminus \{0\}$ , a bipolar scale analogous to the real line where the zero acts as a neutral element and such that  $a + (-a) = 0$  (symmetry). In particular,  $-(-a) = a$ .

The *symmetric maximum*  $\otimes$  is intended to extend the maximum on  $L$  with 0 as neutral element, while fulfilling symmetry. However, this symmetry requirement immediately entails that any extension  $\otimes$  of the maximum operator  $\vee$  cannot be associative. To illustrate this point, let  $L = \mathbb{N}$  and observe that  $(2 \otimes 3) \otimes (-3) = 3 \otimes (-3) = 0$  whereas  $2 \otimes (3 \otimes (-3)) = 2 \otimes 0 = 2$ .

Nonetheless, Grabisch [3] showed that the “best” definition of  $\otimes$  (see Theorem 1 below) is:

$$a \otimes b = \begin{cases} -(|a| \vee |b|) & \text{if } b \neq -a \text{ and } |a| \vee |b| = -a \text{ or } = -b \\ 0 & \text{if } b = -a \\ |a| \vee |b| & \text{otherwise.} \end{cases} \quad (1)$$

In other words, if  $b \neq -a$ , then  $a \otimes b$  returns the element that is the larger in absolute value among the two elements  $a$  and  $b$ . Moreover, it is not difficult to see that  $\otimes$  satisfies the following properties:

- (C1)  $\otimes$  coincides with the maximum on  $L^2$ ;
- (C2)  $a \otimes (-a) = 0$  for every  $a \in \tilde{L}$ ;
- (C3)  $-(a \otimes b) = (-a) \otimes (-b)$  for every  $a, b \in \tilde{L}$ .

Hence,  $\otimes$  almost behaves like  $+$  on the real line, except for associativity  $a \otimes (b \otimes c) = (a \otimes b) \otimes c$ , for every  $a, b, c \in \tilde{L}$ . For instance, we have:  $(-3 \otimes 3) \otimes 1 = 0 \otimes 1 = 1$  but  $-3 \otimes (3 \otimes 1) = -3 \otimes 3 = 0$ . However, it was shown in [3] that if one requires that (C1), (C2) and (C3) hold, then (1) is the best possible definition for  $\otimes$ .

**Theorem 1.** [3, Prop. 5] No binary operation satisfying (C1), (C2), (C3) is associative on a larger domain than  $\otimes$ .

Further properties of  $\otimes$  were presented in [3, 1, Prop. 5]. In particular, it was shown that  $\otimes$  is associative on an expression involving  $a_1, \dots, a_n \in \tilde{L}$ , with  $|\{i : a_i \neq 0\}| > 2$ , if and only if  $\bigvee_{i=1}^n a_i \neq -\bigwedge_{i=1}^n a_i$ . Sequences fulfilling this condition were referred to as *associative* in [1].

To remove the ambiguity when evaluating  $\otimes$  on nonassociative sequences, Grabisch [3] suggested ways of making  $\otimes$  associative. The solution proposed was to define a *rule of computation*, that is, a systematic way of putting parentheses so that the result is no longer ambiguous. Let us present here informally three of these rules<sup>1</sup> that are rather natural:

- (i) aggregate separately positive and negative terms, then compute their symmetric maximum. Taking the sequence 3, 2, -3, 1, -3, -2, 1, we obtain

$$\begin{aligned} \otimes(3, 2, -3, 1, -3, -2, 1) &= (3 \otimes 2 \otimes 1 \otimes 1) \otimes ((-3) \otimes (-3) \otimes (-2)) \\ &= 3 \otimes (-3) = 0. \end{aligned}$$

- (ii) aggregate first extremal opposite terms to cancel them, till there is no more extremal opposite terms. This gives:

$$\begin{aligned} \otimes(3, 2, -3, 1, -3, -2, 1) &= (3 \otimes (-3)) \otimes (2 \otimes 1 \otimes (-3) \otimes (-2) \otimes 1) \\ &= 0 \otimes (-3) = -3. \end{aligned}$$

- (iii) the same as above, but first aggregate these extremal opposite terms. This gives:

$$\begin{aligned} \otimes(3, 2, -3, 1, -3, -2, 1) &= (3 \otimes ((-3) \otimes (-3))) \otimes (2 \otimes (-2)) \otimes (1 \otimes 1) \\ &= (3 \otimes (-3)) \otimes 0 \otimes 1 = 0 \otimes 1 = 1. \end{aligned}$$

One sees that all results differ, and that many other rules can be created. In fact, it is more convenient to define a rule as a systematic way of *deleting* terms in a sequence of numbers, so as to make it associative, provided the way of deleting terms corresponds to some arrangement of parentheses. Indeed, the first rule consists in deleting all terms whenever the sequence does not fulfill the condition of associativity. The second rule consists in deleting recursively all pairs of extremal opposite elements, and the third rule deletes recursively all occurrences of extremal opposite elements. However, one has to be careful that any systematic way of deleting elements making any sequence associative does not necessarily correspond to an arrangement of parentheses. For example, deleting the maximal element 3 in the above sequence makes it associative, however no arrangement of parentheses can produce this.

This framework based on rules of computation was formalized in [1], and we will recall it in the next section. We will also recall equivalent, yet semantically rather different, quasi-orderings of rules, and briefly survey the main characteristics of the resulting partially ordered set of (equivalent classes) of computation rules.

Denoting a computation rule by  $R$ ,  $\otimes_R$  is an unambiguously defined operator acting on any sequence of  $\tilde{L}$ , by first making the sequence associative by means of  $R$ , and then computing the result by  $\otimes$ . Then, to any given computation rule  $R$  corresponds an *aggregation operator*  $\otimes_R$ , aggregating all "numbers" of a sequence into a single number in  $\tilde{L}$ . In the sequel, we only deal with countable sets  $L$ , so that  $\tilde{L}$  can be thought to be  $\mathbb{Z}$ . It follows that such a study is related to the aggregation of integers, in particular to the so-called integer means or  $\mathbb{Z}$ -means, see [4]. In the latter work, it is shown that the decomposability property introduced by Kolmogoroff [5] imposes a very limitative form of integer means, namely that the output depends only on the smallest and greatest entries. In [2], we have weakened the decomposability property and shown that a whole family of operators  $\otimes_R$  can serve as integer means.

The main objective of this paper is to study monotonic computation rules  $R$ , that is, leading to an aggregation operator  $\otimes_R$  which is monotonically nondecreasing w.r.t. all terms of the sequence. This property is a basic requirement in most fields of application, and this is why aggregation operators, defined on either real numbers or integers, are always required to be nondecreasing (see, e.g., any kind of means, median, order statistics, etc.). As it will be shown, not all computation rules are monotonic. The main result of this paper, shown in Section 3, is to give a characterization of the set of monotonic computation rules.

<sup>1</sup>These will be revisited in Section 2 and formally defined in the proposed language formalism of [1].

## 2 Rules of Computation

We now recall the formalism of [1]. As we will only consider countable sequences of elements of  $\tilde{L}$ , without loss of generality, we may assume that  $\tilde{L} = \mathbb{Z}$ . In this way, elements of  $\tilde{L}^*$  are (finite) sequences of integers, denoted by  $\sigma = (\lambda_i)_{i \in I}$  for some finite index set  $I$ , including the empty sequence  $\varepsilon$ , i.e.,

$$\tilde{L}^* = \left( \bigcup_{n \in \mathbb{N}} (\tilde{L})^n \right) \cup \{\varepsilon\}.$$

This convention will simplify our exposition and establish connections to the theory of integer means.

Also, as  $\otimes$  is commutative, the order of symbols in the word does not matter, and we can consider the decreasing order of the absolute values of the elements in the sequence (e.g., 5, 5, -5, -3, 2, -2, 1, 0). Since sequences are ordered, we can consider the following convenient formalism for representing sequences. For an arbitrary sequence

$$\sigma = \underbrace{(n_1, \dots, n_1)}_{p_1 \text{ times}}, \underbrace{(-n_1, \dots, -n_1)}_{m_1 \text{ times}}, \dots, \underbrace{(n_q, \dots, n_q)}_{p_q \text{ times}}, \underbrace{(-n_q, \dots, -n_q)}_{m_q \text{ times}}$$

with  $n_1 \geq \dots \geq n_q$ , let  $\theta(\sigma) = (n_1, \dots, n_q)$  be the sequence of absolute values (magnitudes) of integers in  $\sigma$ , and let  $\psi(\sigma) = ((p_1, m_1), \dots, (p_q, m_q))$  be the sequence of pairs of numbers of occurrence of these integers. For instance, if  $\sigma = (3, 3, -3, 2, -2, -2, 1, 1, 1, 1)$ , then

$$\theta(\sigma) = (3, 2, 1); \quad \psi(\sigma) = ((2, 1), (1, 2)(4, 0)).$$

Let  $\mathfrak{S}$  denote the set of all integer sequences in this formalism, including the empty sequence, and let  $\mathfrak{S}_0$  be the subset of all nonassociative sequences.

To facilitate the precise definition of rules of computation, we proposed [1] a language formalism over a 5-element alphabet made of 5 elementary rules  $\rho_i : \mathfrak{S} \rightarrow \mathfrak{S}$  that act on  $\sigma$  in the following way:

- (i) Elementary rule  $\rho_1$ : if  $p_1 > 1$  and  $m_1 > 0$ , then  $p_1$  is changed to  $p_1 = 1$ ;
- (ii) Elementary rule  $\rho_2$ : same as in (i) with  $p_1, m_1$  exchanged;
- (iii) Elementary rule  $\rho_3$ : if  $p_1 > 0, m_1 > 0$ , the pair  $(p_1, m_1)$  is changed into  $(p_1 - c, m_1 - c)$ , where  $c = p_1 \wedge m_1$ ;
- (iv) Elementary rule  $\rho_4$ : if  $p_1 > 0, m_1 > 0$ , and if  $p_2 > 0$ , then  $p_2$  is changed into  $p_2 = 0$ ;
- (v) Elementary rule  $\rho_5$ : same as in (iv) with  $m_2$  replacing  $p_2$ .

Hence, elementary rules delete terms only in nonassociative sequences, and leave the associative ones invariant.

A (well-formed) computation rule  $R$  is a word built with the alphabet  $\{\rho_1, \dots, \rho_5\}$ , i.e.,  $R \in \mathcal{L}(\rho_1, \dots, \rho_5)$ , such that  $R(\sigma) \in \mathfrak{S} \setminus \mathfrak{S}_0$  for all  $\sigma \in \mathfrak{S}$ . The set of (well-formed) computation rules is denoted by  $\mathfrak{R}$ . Examples of rules are (words are read from left to right)

- (i)  $\langle \cdot \rangle_{-}^{+} = (\rho_4 \rho_5)^* \rho_1 \rho_2 \rho_3$ , that corresponds to first putting parentheses around all positive terms and all negative terms, and then computing the symmetric maximum of the two results.
- (ii)  $\langle \cdot \rangle_0 = \rho_3^*$ , that corresponds to putting parentheses around each pair of maximal symmetric terms.
- (iii)  $\langle \cdot \rangle_{=} = (\rho_1 \rho_2 \rho_3)^*$ , that corresponds to putting parentheses around terms with the same absolute value and sign, and then to putting parentheses around each each pair of maximal symmetric resulting terms.

It is shown in [1] that each computation rule  $R \in \mathfrak{R}$  corresponds to an arrangement of parentheses together with a permutation on the terms of sequences. Thus each  $R \in \mathfrak{R}$  turns the symmetric maximum into an associative operation  $\otimes_R : \tilde{L}^* \rightarrow \tilde{L}$  defined by  $\otimes_R = \otimes \circ R$ , since  $R(\sigma) \in \mathfrak{S} \setminus \mathfrak{S}_0$  for all  $\sigma \in \mathfrak{S}$ <sup>2</sup>. Moreover, each computation rule has the form  $R = T_1 T_2 \dots$ , where each  $T_i$  has the form  $\omega \rho_1^\alpha \rho_2^\beta \rho_3$ , with  $\omega \in \mathcal{L}(\rho_4, \rho_5)$  and  $\alpha, \beta \in \{0, 1\}$  (factorization scheme)<sup>3</sup>

Now, to compute  $\otimes_R(\sigma)$  one needs to delete symbols in the sequence  $\theta(\sigma)$  exactly as they are deleted in  $\psi(\sigma)$ . This entails an ordering of  $\mathfrak{R}$  that is discussed below.

Let  $R, R' \in \mathfrak{R}$  and, for each sequence  $\sigma = (a_i)_{i \in I}$ , let  $J_\sigma \subseteq I$  and  $J'_\sigma \subseteq I$ , be the sets of indices of the terms in  $\sigma$  deleted by  $R$  and  $R'$ , respectively. Then, we write  $R \leq R'$  if for all sequences  $\sigma \in \mathfrak{S}$  we have  $J_\sigma \supseteq J'_\sigma$ . Clearly, it is

<sup>2</sup>For convenience, we assume that  $\otimes_R(\varepsilon) = 0$  and  $\otimes_R(a) = a$ , for every  $a \in \tilde{L}$

<sup>3</sup>Here,  $\rho^0 = \varepsilon$  and  $\rho^1 = \rho$ .

reflexive and transitive, and thus it is a preorder. This induces an equivalence relation  $\sim$  defined as follows:  $R \sim R'$  if  $R \leq R'$  and  $R' \leq R$ . The following proposition reassembles several results in [1], and provides equivalent definitions of  $\sim$ .

**Proposition 1.** Let  $R, R' \in \mathfrak{R}$ . Then the following assertions are equivalent.

- (i)  $R \sim R'$ .
- (ii)  $\mathbb{O}_R = \mathbb{O}_{R'}$ .
- (iii)  $\text{Ker}(\mathbb{O}_R) = \text{Ker}(\mathbb{O}_{R'})$ , where  $\text{Ker}(\mathbb{O}_R)$  denotes the *kernel* of  $\mathbb{O}_R$  that is defined by

$$\text{Ker}(\mathbb{O}_R) = \{\sigma \in \mathfrak{S} \mid \mathbb{O}_R(\sigma) = 0\}.$$

Furthermore, any two equivalent rules have exactly the same ‘‘factorized irredundant form’’. Recall that a rule  $R \in \mathfrak{R}$  is considered in *factorized irredundant form (FIF)* if the two following conditions are verified:

- (i) **Factorization:**  $R$  can be factorized into a composition

$$R = T_1 T_2 \cdots T_i \cdots \quad (2)$$

where each term has the form  $T_i = \omega_i \rho_1^{a_i} \rho_2^{b_i} \rho_3$ , with  $\omega_i \in \mathcal{L}(\{\rho_4, \rho_5\})$  (possibly empty), and  $a_i, b_i \in \{0, 1\}$ .

- (ii) **Simplification:** Suppose that in (2) there exists  $j \in \mathbb{N}$  such that  $\omega_j = \omega \rho_4^* \text{ or } \omega \rho_5^*$  for some  $\omega \in \mathcal{L}(\{\rho_4, \rho_5\})$ , or that  $\rho_4$  and  $\rho_5$  alternate infinitely many times in  $\omega_j$ . Let

$$\begin{aligned} k_1 &= \min\{j : \omega_j = \omega \rho_4^* \text{ or } \omega \rho_5^*\}, \quad \text{and} \\ k_2 &= \min\{j : \rho_4 \text{ and } \rho_5 \text{ alternate infinitely many times in } \omega_j\}. \end{aligned}$$

- If  $k_1 < k_2$ , then  $R \sim T_1 \cdots T_{k_1}$ .
- Otherwise,  $k_2 \leq k_1$ , and  $R \sim T_1 \cdots T'_{k_2}$ , where  $T'_{k_2} = (\rho_4 \rho_5)^* \rho_1^{a_{k_2}} \rho_2^{b_{k_2}} \rho_3$ .

Observe that every non-terminal term  $T_j$  (i.e., of the form  $\omega \rho_1^a \rho_2^b \rho_3$ ) in a rule in FIF has a ‘‘certificate’’.

Certificates can be defined recursively as follows. A *certificate*  $\gamma$  of non-terminal term  $T = \omega \rho_1^a \rho_2^b \rho_3$  is an element of  $\text{Ker}(T)$  such that no letter of  $T$  is left unused (unread or without deleting an element of  $\gamma$ ) when  $\gamma$  is deleted. For instance, consider  $T = \rho_4 \rho_5^2 \rho_4 \rho_3$ . Then  $\sigma = (1, 1)(2, 1)(0, 1)$  is a kernel element but not a certificate, while  $\gamma = (1, 1)(2, 1)(1, 1)$  is a certificate.<sup>4</sup> The definition is then recursively extended to rules in  $\mathfrak{R}/\sim$  using factorization.

The structure of the poset  $\mathfrak{R}/\sim$  of equivalence classes endowed with the partial order induced by  $\leq$  was investigated in [1] and shown to be highly complex. To give an idea, the subposet  $\mathfrak{R}_{123}/\sim$  of equivalence classes of rules  $R \in \mathcal{L}(\rho_1, \rho_2, \rho_3)$  has infinitely many maximal elements, and  $(\mathfrak{R}_{123}/\sim, \leq)$  (and thus  $(\mathfrak{R}/\sim, \leq)$ ) embeds the powerset  $(2^{\mathbb{N}}, \subseteq)$  of natural numbers, and hence it is of continuum cardinality. For further results on  $\mathfrak{R}/\sim$ , see [1].

The complex structure of  $(\mathfrak{R}/\sim, \leq)$  gives little hope to obtain a complete description of this poset. In addition to considering restrictions on the syntax of computation rules, another approach to provide local descriptions is to consider computation rules with certain desirable properties. One of such properties is monotonicity which is particularly relevant in applied mathematics, especially, in decision making and aggregation theory. In the next section we provide the explicit description of monotonic computation rules in terms of their factorized irredundant form (FIF).

### 3 Monotonic computation rules

In this section we aim to describe those computation rules that are monotonic. Recall that a rule  $R \in \mathfrak{R}$  is *monotonic* if  $\mathbb{O}_R(a_1, \dots, a_n) \leq \mathbb{O}_R(a'_1, \dots, a'_n)$ , whenever  $a_i \leq a'_i$  for every  $n \in \mathbb{N}$  and  $i = 1, \dots, n$ . For instance, it is not difficult to see that both  $\langle \cdot \rangle_0$  and  $\langle \cdot \rangle_+$  are monotonic, however,  $\langle \cdot \rangle_+$  is not:

$$\mathbb{O}_{\langle \cdot \rangle_+}(5, -5, -5, 4, 3) = 4 \quad \text{whereas} \quad \mathbb{O}_{\langle \cdot \rangle_+}(5, -5, -4, 4, 3) = 3.$$

In order to study monotonicity, first observe the following facts.

- (i)  $\mathbb{O}_R$  is monotonic for every rule  $R$  on  $\mathfrak{S} \setminus \mathfrak{S}_0$ . Hence, we can consider only sequences in  $\mathfrak{S}_0$ .

<sup>4</sup>Note that a certificate exists if and only if  $\omega$  neither contains  $\rho_4^*$ ,  $\rho_5^*$  nor  $(\rho_4 \rho_5)^*$ .

- (ii) It is sufficient to study the effect of increasing one element of the sequence  $\sigma$ . If we increase  $n_k$  to  $n > n_1$ , then the sequence becomes associative, and the value of  $\mathbb{Q}_R$  is  $n$ . Hence, it is sufficient to consider an increase to any value at most  $n_1$ .

**Lemma 1.** Let  $\sigma \in \mathfrak{S}_0$ . Then  $\mathbb{Q}_R$  is monotonic w.r.t. any element  $n_1$  or  $-n_1$  of the sequence, for any rule  $R = T^1 T^2 \dots$  with  $T^1 = \omega \rho_3$ .

*Proof.* Suppose that an element  $n_1$  is changed to  $n'_1 > n_1$ . Then the new sequence  $\sigma'$  becomes associative and  $\mathbb{Q}_R(\sigma') = n'_1 > \mathbb{Q}_R(\sigma)$ . Suppose now that an element  $-n_1$  is changed to  $-n_1 + \epsilon \leq -n_2$ . Then  $(p_1, m_1)$  is changed to  $(p_1, m_1 - 1)$ , which can only increase the result of  $\mathbb{Q}_R$ , as  $\rho_1, \rho_2$  are not present in  $T^1$ .  $\square$

Let us start with computation rules with a single term.

**Lemma 2.** If  $R$  has the form  $(\rho_4 \rho_5)^* \rho_1^a \rho_2^b \rho_3$  then  $\mathbb{Q}_R$  is monotonic.

*Proof.* Let  $\sigma \in \mathfrak{S}_0$ . After the application of  $(\rho_4 \rho_5)^*$  only the first term  $(p_1, m_1)$  remains, so that it is enough to study the effect of increasing  $\pm n_1$ . If  $n_1$  is increased to  $n'_1$ , then  $\mathbb{Q}_R(\sigma') = n'_1$ , and if  $-n_1$  is increased, this can only increase the result of  $\mathbb{Q}_R$ .  $\square$

**Lemma 3.** Let  $R \in \mathfrak{R}$  be in FIF.

- (i) Suppose that  $R$  has the form  $\omega \rho_1^a \rho_2^b \rho_3$  for  $\omega = \omega' \rho_4^*$  with  $\omega' \in \mathcal{L}(\rho_4, \rho_5)$ . Then  $R$  is monotonic if and only if  $(a, b) = (a, 0)$ , for  $a \in \{0, 1\}$ , and  $\omega' = \varepsilon$ .
- (ii) Suppose that  $R$  has the form  $\omega \rho_1^a \rho_2^b \rho_3$  for  $\omega = \omega' \rho_5^*$  with  $\omega' \in \mathcal{L}(\rho_4, \rho_5)$ . Then  $R$  is monotonic if and only if  $(a, b) = (0, b)$ , for  $b \in \{0, 1\}$ , and  $\omega' = \varepsilon$ .

*Proof.* We show that (i) holds; the proof of (ii) is analogous. To see that the condition is necessary, suppose that  $(a, b) = (a, 1)$ , where  $a \in \{0, 1\}$ . Consider the sequence

$$\sigma_1 = (1, 2) \sigma_{\omega'} \sigma',$$

where  $\sigma_{\omega'}$  is a certificate of  $\omega'$  and  $\sigma'$  a sequence such that the difference between the smallest absolute value of  $\sigma_{\omega'}$  and the greatest absolute value of  $\sigma'$  is at least 2. Then  $\mathbb{Q}_R(\sigma_1) = -n$  for some  $-n$  in  $\sigma'$  if it exists, or  $\mathbb{Q}_R(\sigma_1) = 0$ . Consider now the sequence

$$\sigma_2 = (1, 1) \sigma_{\omega'} (0, 1) \sigma',$$

obtained from  $\sigma$  by increasing  $-n_1$  to  $-n'$  greater than all  $-n$  in  $\sigma_{\omega'}$  and smaller than all  $-n$  in  $\sigma'$ . Clearly,  $\sigma_1 < \sigma_2$  but  $\mathbb{Q}_R(\sigma_2) = -n' < \mathbb{Q}_R(\sigma_1)$ .

To see that we must have  $\omega' = \varepsilon$ , suppose to the contrary that  $\omega' \neq \varepsilon$ . Hence,  $\omega'$  has the form  $\omega' = \omega'' \rho_5$ , otherwise we would have  $\omega' \rho_4^* = \rho_4^*$ . Consider the sequences

$$\sigma = (1, 1) \sigma_{\omega''} (0, 1) (1, 0) < (1, 1) \sigma_{\omega''} (1, 0) (0, 1) = \sigma'$$

where  $\sigma'$  has been obtained from  $\sigma$  by increasing the last but one element  $-n$  to  $-n'$  s.t.  $n' < n''$ , with  $n''$  the last element in  $\sigma$ . Then  $\mathbb{Q}_R(\sigma) = 0 > \mathbb{Q}_R(\sigma') = -n'$ , which contradicts the fact that  $R$  is monotonic. Hence,  $\omega' = \varepsilon$ .

To prove sufficiency, consider the case  $(a, b) = (0, 0)$  (the case  $(a, b) = (1, 0)$  is similar). Any sequence  $\sigma$  has the form  $\sigma = (p_1, m_1)(p_2, 0) \dots (p_t, 0) \sigma'$  with  $t \geq 0$ . Note that (a) if  $p_1 > m_1$ , then  $\mathbb{Q}_R(\sigma) = n_1$ , (b) if  $p_1 < m_1$ , then  $\mathbb{Q}_R(\sigma) = -n_1$  (smallest value), and (c) if  $p_1 = m_1$ ,  $\mathbb{Q}_R(\sigma) = -n$  if it exists in  $\sigma'$ , or  $\mathbb{Q}_R(\sigma) = 0$ . It is not difficult to check that, in each case, any increase in  $\sigma$  can only result in an increase of  $\mathbb{Q}_R(\sigma)$ .  $\square$

We now extend our study to rules made of several terms, and we will make use of the two following auxiliary results to simplify our search for nonmonotonic rules.

**Lemma 4.** Suppose that  $\mathbb{Q}_R$  is not monotonic, and let  $T \in \mathcal{L}(\rho_1, \dots, \rho_5)$  such that  $TR \in \mathfrak{R}$  be in FIF. Then  $\mathbb{Q}_{TR}$  also is not monotonic.

*Proof.* Since  $TR \in \mathfrak{R}$  is in FIF,  $T = T_1 T_2 \dots T_i \dots$  is finite and each term  $T_j$  has a certificate  $\sigma_j$ . Hence, the composition  $\sigma = \sigma_1 \sigma_2 \dots \sigma_i \dots$  is a certificate of  $T$ .

Suppose that  $\mathbb{Q}_R$  is not monotonic, and let  $\sigma'$  and  $\sigma''$  be sequences such that  $\sigma' < \sigma''$  and  $\mathbb{Q}_R(\sigma') > \mathbb{Q}_R(\sigma'')$ . Consider the two composite sequences  $\sigma \sigma'$  and  $\sigma \sigma''$ . Clearly,  $\sigma \sigma' < \sigma \sigma''$  but

$$\mathbb{Q}_{TR}(\sigma \sigma') = \mathbb{Q}_R(\sigma') > \mathbb{Q}_R(\sigma'') = \mathbb{Q}_{TR}(\sigma \sigma'').$$

In other words,  $\mathbb{Q}_{TR}$  is not monotone.  $\square$

**Lemma 5.** Let  $R \in \mathfrak{R}$  be in FIF, and let  $T = \rho_3^k R$  with  $k \geq 1$ . Then,  $\mathbb{V}_R$  is monotonic if and only if  $\mathbb{V}_T$  is monotonic.

*Proof.* From Lemma 4 it follows that the condition is sufficient. Conversely, suppose that  $\mathbb{V}_R$  is monotonic. It suffices to prove that  $\mathbb{V}_{\rho_3 R}$  is monotonic and apply  $k$  times the result. Consider any sequence  $\sigma = (p_1, m_1)\sigma_1 \in \mathfrak{S}_0$ . By Lemma 1,  $\mathbb{V}_{\rho_3 R}$  is monotonic w.r.t.  $\pm n_1$ . Now, consider  $\sigma'$  obtained by increasing any element in  $\sigma_1$ . Then

$$\sigma' = \begin{cases} (p_1, m_1)\sigma'_1 & \text{with } \sigma'_1 > \sigma_1 \\ (p_1 + 1, m_1)\sigma'_1 & \end{cases}$$

In the first case,  $\mathbb{V}_{\rho_3 R}(\sigma') = \mathbb{V}_{\rho_3 R}(\sigma)$  when  $p_1 \neq m_1$ , otherwise  $\mathbb{V}_{\rho_3 R}(\sigma') = \mathbb{V}_R(\sigma'_1) \geq \mathbb{V}_R(\sigma_1)$  since  $R$  is monotonic.

In the second case, we have  $\mathbb{V}_{\rho_3 R}(\sigma') > \mathbb{V}_{\rho_3 R}(\sigma)$  if  $p_1 = m_1$  or  $p_1 = m_1 - 1$ , otherwise  $\mathbb{V}_{\rho_3 R}(\sigma') = \mathbb{V}_{\rho_3 R}(\sigma)$ .  $\square$

**Lemma 6.** Let  $R = T^1 T^2 \dots$  be in FIF where  $T^i = \omega_i \rho_1^{a_i} \rho_2^{b_i} \rho_3$ . If there exists  $k$  such that

- $(a_k, b_k) = (1, 0)$  and  $\omega_k \neq \rho_4^*, (\rho_4 \rho_5)^*$ ,
- $(a_k, b_k) = (0, 1)$  and  $\omega_k \neq \rho_5^*, (\rho_4 \rho_5)^*$ , or
- $(a_k, b_k) = (1, 1)$  and  $\omega_k \neq \rho_4^*, \rho_5^*, (\rho_4 \rho_5)^*$ ,

then  $\mathbb{V}_R$  is not monotonic.

*Proof.* Assuming that  $R$  is in FIF, by Lemma 4, we may assume that  $k = 1$ .

- Suppose that  $(a_1, b_1) = (1, 0)$  and  $\omega_1 \neq \rho_4^*, (\rho_4 \rho_5)^*$ . Consider the sequences

$$\sigma_1 = (1, 1)(1, 0)^{|\omega_1|_{\rho_4}}(1, 0) \quad \text{and} \quad \sigma_2 = (2, 1)(1, 0)^{|\omega_1|_{\rho_4}}$$

obtained from  $\sigma_1$  by augmenting  $n_{|\omega_1|_{\rho_4}+2}$  to  $n_1$ , where  $|\omega_1|_{\rho_4}$  indicates the number of occurrences of  $\rho_4$  in  $\omega_1$ . Although  $\sigma_1 < \sigma_2$  we have

$$\mathbb{V}_R(\sigma_1) = n_{|\omega_1|_{\rho_4}+2} > 0 = \mathbb{V}_R(\sigma_2),$$

and thus  $\mathbb{V}_R$  is not monotonic.

- Suppose that  $(a_1, b_1) = (0, 1)$  and  $\omega_1 \neq \rho_5^*, (\rho_4 \rho_5)^*$ . Consider the sequences

$$\sigma_1 = (1, 2)(0, 1)^{|\omega_1|_{\rho_5}} \quad \text{and} \quad \sigma_2 = (1, 1)(0, 1)^{|\omega_1|_{\rho_5}}(0, 1),$$

obtained from  $\sigma_1$  by increasing the value  $-n_1$  to  $-n_{|\omega_1|_{\rho_5}+2}$ . Clearly,  $\sigma_1 < \sigma_2$  but  $\mathbb{V}_R(\sigma_1) = 0 > -n_{|\omega_1|_{\rho_5}+2} = \mathbb{V}_R(\sigma_2)$ , and thus  $\mathbb{V}_R$  is not monotonic.

- The remaining case  $(a_1, b_1) = (1, 1)$  and  $\omega_1 \neq \rho_4^*, \rho_5^*, (\rho_4 \rho_5)^*$  is dealt with similarly.  $\square$

We now consider the case where  $(a_i, b_i) = (0, 0)$  in each term  $T^i = \omega_i \rho_1^{a_i} \rho_2^{b_i} \rho_3$ .

**Lemma 7.** Suppose  $R = T^1 T^2 \dots$  is in FIF, and that no term contains  $\rho_1$  nor  $\rho_2$ . If there is  $k \geq 1$  such that  $\omega_k$  in  $T^k$  is of the AFT type or equal to  $\rho_4^\alpha$  or  $\rho_5^\beta$ , then  $\mathbb{V}_R$  is not monotonic.

*Proof.* By Lemma 4, it suffices to consider the case  $k = 1$ . Suppose first that  $R = \omega \rho_3 R'$  with  $\omega$  of the AFT type, say  $\omega = \rho_4^{\alpha_1} \rho_5^{\beta_1} \dots \rho_4^{\alpha_t} \rho_5^{\beta_t}$ . Consider the sequence

$$\sigma = (1, 1)(1, 0)^{\chi_1}(0, 1)^{\xi_1} \dots (1, 0)^{\chi_t}(0, 1)^{\xi_t}(1, 0)(1, 0).^5$$

Clearly,  $\mathbb{V}_R(\sigma) = n_{t+2}$ . Now let us increase the term with value  $n_{t+2}$  to  $n_j$ , where  $j$  is the first index such that  $\chi_j = 1$ , so that we obtain the sequence  $\sigma'$ . Clearly, we have  $\sigma < \sigma'$  but  $\mathbb{V}_R(\sigma) = n_{t+2} > n_{t+3} = \mathbb{V}_R(\sigma')$ .

Now, w.l.o.g. suppose that  $\omega = \rho_4^\alpha$  with  $\omega \neq \rho_4^*$ ; the other case  $\omega = \rho_5^\beta$  with  $\omega \neq \rho_5^*$  is dealt with similarly. Consider  $\sigma = (1, 1)(1, 0)^\alpha(1, 0)(1, 0)$  and  $\sigma'$  obtained from  $\sigma$  by increasing the value of  $n_{\alpha+2}$  to  $n_2$ , i.e.,

$$\sigma = (1, 1)(2, 0)(1, 0)^{\alpha-1}(1, 0).$$

In this case, we get  $\mathbb{V}_R(\sigma) = n_{\alpha+2} > n_{\alpha+3} = \mathbb{V}_R(\sigma')$ .

In both cases, we get that  $\mathbb{V}_R$  is not monotonic.  $\square$

<sup>5</sup>Here  $\chi_i = 1$  if  $\alpha_i > 0$ , otherwise  $\chi_i = 0$ . Similarly,  $\xi_i = 1$  if  $\beta_i > 0$ , otherwise  $\xi_i = 0$ .

We can now provide a complete description of monotonic rules.

**Theorem 2.** Let  $R \in \mathfrak{R}$  be in FIF. Then  $\mathcal{O}_R$  is monotonic if and only if either

- (i)  $R = \rho_3^*$ , or
- (ii)  $R = \rho_3^k T$ , where  $T = \omega \rho_1^a \rho_2^b \rho_3$  satisfies the following conditions
  - if  $(a, b) = (1, 0)$ , then  $\omega = \rho_4^*$  or  $(\rho_4 \rho_5)^*$ ,
  - if  $(a, b) = (0, 1)$ , then  $\omega = \rho_5^*$  or  $(\rho_4 \rho_5)^*$ ,
  - if  $(a, b) = (1, 1)$ , then  $\omega = (\rho_4 \rho_5)^*$ ,
  - if  $(a, b) = (0, 0)$ , then  $\omega = \rho_4^*, \rho_5^*, (\rho_4, \rho_5)^*$ .

*Proof.* Let us prove that all rules in (i) and (ii) are monotonic. It was already established that  $\mathcal{O}_{\rho_3^*} = \langle \cdot \rangle_0$  is monotonic. As for (ii), by using Lemma 5, it suffices to prove monotonicity for  $R = T$ , which is obtained by Lemmas 2 and 3.

It remains to prove that no other rule is monotonic. As rules are in FIF, no term can exist after  $T$ . Moreover, by Lemmas 6 and 7, no term of the form  $T' = \omega' \rho_1^{a'} \rho_2^{b'} \rho_3$  with  $\omega' \in \mathcal{L}(\rho_4, \rho_5)$  finite can occur before  $T$  or before  $\rho_3^k$ . Furthermore, by Lemma 3, it is not possible to add a finite  $\omega' \in \mathcal{L}(\rho_4, \rho_5)$  before  $T$ . Thus, every monotonic rule must be of one of the stated forms, and the proof of Theorem 2 is now complete.  $\square$

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