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Mathematical analysis of a three-tiered microbial food-web model with new input substrates

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Résumé : In this work, we generalise a reduced anaerobic digestion model describing the anaerobic mineralisation of chlorophenol in a three-tiered food-web. The aim of the study is to take into account, the phenol and the hydrogen inflowing concentrations $S_{ph,in}$ and $S_{H_2,in}$, in addition of the chlorophenol input substrate. This case present interesting behaviours. We prove that the system can have eight types of steady-states, instead of only three types of steady-states as in the case $S_{ph,in} = S_{H_2,in} = 0$. We give conditions for the existence and stability of the steady-states of the system without mortality terms. By means of operating diagrams, we illustrate the asymptotic behavior of the model.

Mots-Clefs : Anaerobic digestion, Microbial ecology, Local stability, Operating diagram.

1 Introduction

The anaerobic digestion (AD) is a natural process in which organic material is converted into biogas in an environment without oxygen by the action of a microbial ecosystem. It is used for the treatment of waste or wastewater and has the advantage of producing methane or hydrogen under appropriate conditions. The three-tiered model analysed here is based on Anaerobic Digestion Model No.1(ADM1) and has three substrates and three biomass variables. In this model, a chlorophenol-dechlorinating bacterium is introduced in a two-tiered model involving two other organisms which are a phenol degrader and a hydrogenotrophic methanogen, see [3] and [2]. The chlorophenol degrader uses both chlorophenol and hydrogen for growth, giving phenol as a product. Phenol is consumed by the phenol degrader forming hydrogen, which inhibits its own growth. The methanogen scavenges this hydrogen and acts as the primary syntroph. The aim of this study is to give a comprehensive analysis of the three-tiered model of [3], taking into account the three input substrates. We present a description of the model to be investigated and we give assumptions on the growth functions. We describe all steady states and their stability properties, in the case where the mortality terms are neglected. To describe the qualitative behavior of the system, we determine the operating diagrams of the model according to the operating parameters. The operating diagrams can be useful to interpret experimental results.

2 The model

We are interested in the mathematical analysis of the following system, obtained from the three-tiered food-web model of [3], by using change variables and simplified notations:

$$\frac{dx_0}{dt} = -Dx_0 + \mu_0(s_0, s_2)x_0 \quad (1)$$

$$\frac{dx_1}{dt} = -Dx_1 + \mu_1(s_1, s_2)x_1 \quad (2)$$

$$\frac{dx_2}{dt} = -Dx_2 + \mu_2(s_2)x_2 \quad (3)$$

$$\frac{ds_0}{dt} = D(s_0^{\text{in}} - s_0) - \mu_0(s_0, s_2)x_0 \quad (4)$$

$$\frac{ds_1}{dt} = D(s_1^{\text{in}} - s_1) + \mu_0(s_0, s_2)x_0 - \mu_1(s_1, s_2)x_1 \quad (5)$$

$$\frac{ds_2}{dt} = D(s_2^{\text{in}} - s_2) + \mu_1(s_1, s_2)x_1 - \omega\mu_0(s_0, s_2)x_0 - \mu_2(s_2)x_2 \quad (6)$$

where s_0, x_0, s_1, x_1, s_2 and x_2 represent the chlorophenol, the phenol and the hydrogen substrate and degrader concentrations, respectively. $s_0^{\text{in}}, s_1^{\text{in}}$ and s_2^{in} are the inflowing concentrations and $\mu_0(\cdot, \cdot), \mu_1(\cdot, \cdot)$ and $\mu_2(\cdot)$ are the growth functions.

Under general hypotheses on the growth functions, we first prove the positivity and the boundedness of the solution of system (1)-(6).

3 Analysis of the steady-states

The analysis of the steady-states of model (1)-(6) prove the existence of eight steady-states: the washout steady-state which always exists, a positive steady-state where all degrader populations are maintained and six other steady-states corresponding to the extinction of one or two degrader populations. We give the existence conditions of these steady-states as functions of the operating parameters. The local stability conditions of all identified steady-states are obtained using the Jacobian matrix. Without mortality terms, the latter matrix has a block triangular form. The stability of the system is determined by calculating eigen-values of three-order matrices and using the Routh-Hurwitz criterion.

4 Operating diagrams

The operating diagrams define regions of existence and local stability of equilibria. They show how the system behaves when we vary the four operating parameters $s_0^{\text{in}}, s_1^{\text{in}}, s_2^{\text{in}}$ and D , the biological parameters being fixed. To plot the operating diagrams in the plane, we must fix two of the four operating parameters.

5 Conclusion

We have considered a mathematical model of three microbial species competing on three resources. We prove that the system can have eight types of steady-states. We give sufficient and necessary conditions for their existence and stability. We present numerical simulations which illustrate the demonstrated mathematical results.

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