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MAXIMAL-IN-TIME EXISTENCE AND UNIQUENESS OF STRONG SOLUTION OF A 3D FLUID-STRUCTURE INTERACTION MODEL

DEBAYAN MAITY, JEAN-PIERRE RAYMOND, AND ARNAB ROY

Abstract. In this work, we study a system coupling the incompressible Navier-Stokes equations in a cylindrical type domain with an elastic structure, governed by a damped shell equation, located at the lateral boundary of the domain occupied by the fluid. We prove the existence of a unique maximal strong solution.

1. Introduction and main results

We study the interaction between a viscous, Newtonian, incompressible fluid and an elastic structure modelled by a nonlinear damped shell equation. Let us consider a clamped thin cylindrical shell of length L and of reference radius 1. For the fluid, the reference configuration is Ω, and the structure is located at the boundary $\Gamma_s \subset \partial\Omega$, where

$$\Omega = (0,L) \times B(0_{\mathbb{R}^2}, 1), \quad B(0_{\mathbb{R}^2}, 1) = \{(z_2, z_3) \in \mathbb{R}^2 \mid (z_2^2 + z_3^2)^{1/2} < 1\},$$
$$\Gamma_s = \{(z_1, z_2, z_3) \mid z_1 \in (0,L), z_2^2 + z_3^2 = 1\}.$$

We shall also use $\omega = (0,L) \times (0, 2\pi)$ to parametrize Γ_s:

$$\Gamma_s = \{(z_1, \cos\theta, \sin\theta) \mid z_1 \in (0,L), \theta \in (0, 2\pi)\}.$$

When the fluid moves, it deforms the elastic structure which in turn influences the fluid motion. Let $\boldsymbol{\eta}(t,\cdot)$ denote the displacement of the shell from the reference configuration Γ_s at time t. We assume that the displacement $\boldsymbol{\eta}(t,\cdot)$ is only in the radial direction. Thus $\boldsymbol{\eta}(t, z_1, \theta) = \eta(t, z_1, \theta)\mathbf{e}_r(\theta)$ where $\mathbf{e}_r(\theta) = (0, \cos\theta, \sin\theta)$ is the unit vector in the radial direction, $\eta(t, z_1, \theta) \in \mathbb{R}$ and $1 + \eta(t, z_1, \theta) > 0$. Therefore, the domain $\Omega_{\eta(t)}$ occupied by the fluid at time $t > 0$ and the lateral boundary $\Gamma^s_{\eta(t)}$ occupied by the elastic structure at time $t > 0$ are defined by

$$\Omega_{\eta(t)} = \left\{(z_1, x, y) \in \mathbb{R}^3 \mid \sqrt{x^2 + y^2} < 1 + \eta(t, z_1, \theta),\ z_1 \in (0,L), \theta \in (0, 2\pi)\right\},$$
$$\Gamma^s_{\eta(t)} = \left\{(z_1, x, y) \in \mathbb{R}^3 \mid \sqrt{x^2 + y^2} = 1 + \eta(t, z_1, \theta),\ z_1 \in (0,L), \theta \in (0, 2\pi)\right\},$$

or in cylindrical coordinates by

$$\Omega_{\eta(t)} = \left\{(z_1, r, \theta) \in \mathbb{R}^3 \mid 0 \leqslant r < 1 + \eta(t, z_1, \theta),\ z_1 \in (0,L), \theta \in (0, 2\pi)\right\},$$
$$\Gamma^s_{\eta(t)} = \left\{(z_1, r, \theta) \in \mathbb{R}^3 \mid r = 1 + \eta(t, z_1, \theta),\ z_1 \in (0,L), \theta \in (0, 2\pi)\right\}.$$

We set $\Gamma_n = \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$, where the inlet and outlet boundaries, Γ_{in} and Γ_{out}, are defined by

$$\Gamma_{\text{in}} = \left\{(z_1, x, y) \in \mathbb{R}^3 \mid 0 \leqslant \sqrt{x^2 + y^2} < 1,\ z_1 = 0\right\} \quad \text{and}$$
$$\Gamma_{\text{out}} = \left\{(z_1, x, y) \in \mathbb{R}^3 \mid 0 \leqslant \sqrt{x^2 + y^2} < 1,\ z_1 = L\right\}.$$

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Remark 1.1. *In the following, the Cartesian coordinates of points in $\mathbb{R}^3$ will be denoted by (z_1, z_2, z_3) in the reference configuration and by (z_1, x, y) in the deformed configuration. When (z_1, z_2, z_3) belongs to Ω (resp. Γ_s), the corresponding cylindrical coordinates will be denoted by (z_1, r, θ) (resp. (z_1, θ)). A function f in Cartesian coordinates (in the deformed configuration) defines a function $\widetilde{f}$ in cylindrical coordinates via the identity*

$$\widetilde{f}(z_1, r, \theta) = f(z_1, x, y).$$

Since no confusion is possible, to simplify the notation we will omit the superscript $\widetilde{\ }$ and both the functions f and $\widetilde{f}$ will be denoted by f.

Remark 1.2. *Since we write the structure equation in $\omega = (0, L) \times (0, 2\pi)$, any function defined on Γ_s (or $\Gamma^s_{\eta(t)}$) must be viewed as a function defined on ω, 2π-periodic w. r. to θ.*

For $0 < T \leqslant \infty$, we set

$$\widetilde{Q}_T = \cup_{t\in(0,T)}\{t\} \times \Omega_{\eta(t)}, \qquad Q_T = (0, T) \times \Omega, \qquad \omega_T = (0, T) \times \omega,$$
$$\Sigma^s_T = (0, T) \times \Gamma_s, \qquad \Sigma^{\text{in}}_T = (0, T) \times \Gamma_{\text{in}}, \qquad \Sigma^{\text{out}}_T = (0, T) \times \Gamma_{\text{out}}.$$

In this paper, we want to study the following fluid-structure interaction system:

$$\begin{aligned}
&\rho_f(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) - \text{div }\sigma(\mathbf{u}, p) = 0, \quad \text{div }\mathbf{u} = 0 \quad \text{in } \widetilde{Q}_T,\\
&\sigma(\mathbf{u}, p)\mathbf{n} = 0 \text{ on } \Sigma^{\text{in}}_T, \quad \sigma(\mathbf{u}, p)\mathbf{n} = 0 \text{ on } \Sigma^{\text{out}}_T,\\
&\mathbf{u}(t, z_1, 1 + \eta(t, z_1, \theta), \theta) = \eta_t(t, z_1, \theta)\mathbf{e}_r, \quad \text{for } (t, z_1, \theta) \in (0, T) \times \omega,\\
&\rho_s h\eta_{tt} + \mathscr{L}_{mem}\eta + \beta_1\Delta_s^2\eta - \beta_2\Delta_s\eta_t = \mathcal{H}(\mathbf{u}, p, \eta) \quad \text{in } \omega_T,\\
&\eta = \tfrac{\partial\eta}{\partial\mathbf{n}} = 0 \quad \text{on } (0, T) \times \partial\omega = (0, T) \times (\{0\} \cup \{L\}) \times (0, 2\pi),\\
&\eta(0, \cdot) = \eta_1^0, \quad \eta_t(0, \cdot) = \eta_2^0 \quad \text{in } \omega, \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0 \text{ in } \Omega_{\eta_1^0},
\end{aligned} \tag{1.1}$$

where $\rho_f > 0$ denotes the constant density of the fluid, $\mathbf{u}(t, \cdot)$ and $p(t, \cdot)$ respectively denotes the fluid velocity and the fluid pressure in $\Omega_{\eta(t)}$. The fluid stress tensor $\sigma(\mathbf{u}, p)$ is given by

$$\sigma(\mathbf{u}, p) = 2\nu\boldsymbol{\varepsilon}(\mathbf{u}) - pI_{\mathbb{R}^3}, \quad \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \tag{1.2}$$

where $\nu > 0$ is the constant fluid viscosity. In $(1.1)_4$, the structure displacement η is assumed to satisfy a viscoelastic cylindrical nonlinear Koiter shell equation, where $\rho_s > 0$ is the constant structure density, h is the thickness of the structure, $\beta_1 > 0$ is the bending coefficient, $\beta_2 > 0$ is the damping coefficient, and $\mathcal{H}$ is the force exerted by the fluid acting on the structure in the radial direction. When $\beta_2 = 0$, that is to say when there is no damping, this model is introduced in [29] and the differential operator $\mathscr{L}_{mem}$ is defined by

$$\begin{aligned}
\mathscr{L}_{mem}\eta = \frac{h}{2}\Big[&(1+\eta)\Big(\frac{E\sigma_P}{1-\sigma_P^2}\eta_{z_1}^2 + \frac{E}{1-\sigma_P^2}\Big(\eta(\eta+2) + \eta_\theta^2\Big)\Big)\\
&- \frac{\partial}{\partial z_1}\Big(\frac{E}{1-\sigma_P^2}\eta_{z_1}^3 + \frac{E\sigma_P}{1-\sigma_P^2}\eta_{z_1}\Big(\eta(\eta+2) + \eta_\theta^2\Big)\Big)\\
&- \frac{\partial}{\partial\theta}\Big(\frac{E}{1-\sigma_P^2}\eta_{z_1}^2\eta_\theta + \frac{E\sigma_P}{1-\sigma_P^2}\eta_{z_1}\Big(\eta(\eta+2) + \eta_\theta^2\Big)\Big)\\
&- \frac{E}{1+\sigma_P}\Big(\frac{\partial}{\partial\theta}\big(\eta_{z_1}^2\eta_\theta\big) + \frac{\partial}{\partial z_1}\big(\eta_\theta^2\eta_{z_1}\big)\Big)\Big],
\end{aligned} \tag{1.3}$$

where E is the Young's modulus of elasticity and σ_P is the Poisson ratio. In the right hand side of equation $(1.1)_4$, $\mathcal{H}(\mathbf{u}, p, \eta)$ is defined by

$$\mathcal{H}(\mathbf{u}, p, \eta) = -\sqrt{(1+\eta_{z_1}^2)(1+\eta)^2 + \eta_\theta^2}\Big(\sigma(\mathbf{u}, p)\widetilde{\mathbf{n}}\Big)|_{\Gamma^s_{\eta(t)}} \cdot \mathbf{e}_r, \tag{1.4}$$

where

$$\widetilde{\mathbf{n}} = \frac{1}{\sqrt{(1+\eta_{z_1}^2)(1+\eta)^2 + \eta_\theta^2}}\Big(-\eta_{z_1}(1+\eta), (1+\eta)\cos\theta + \eta_\theta\sin\theta, (1+\eta)\sin\theta - \eta_\theta\cos\theta\Big),$$

is the unit normal to $\Gamma^s_{\eta(t)}$ outward $\Omega_{\eta(t)}$. In $(1.1)_2$ homogeneous Neumann boundary conditions are prescribed at the inlet and outlet boundaries of the fluid, but nonhomogeneous boundary conditions of the form

$$\sigma(\mathbf{u},p)\mathbf{n}=\mathbf{g}_{\text{in}} \text{ on } \Sigma_T^{\text{in}}, \quad \sigma(\mathbf{u},p)\mathbf{n}=\mathbf{g}_{\text{out}} \text{ on } \Sigma_T^{\text{out}}, \tag{1.5}$$

with $\mathbf{g}_{\text{in}} \in H^1(0,T;\mathbf{H}^{1/2}(\Gamma_{\text{in}}))$ and $\mathbf{g}_{\text{out}} \in H^1(0,T;\mathbf{H}^{1/2}(\Gamma_{\text{out}}))$, could also be considered. Equation $(1.1)_3$ corresponds to the equality of velocities at the interface $\Gamma^s_{\eta(t)}$ (i.e., no-slip boundary condition).

Remark 1.3. *Let us make the following remarks.*

1. *For simplicity in the writing, we shall take $\rho_f = 1$ and all the structure parameters $h, \rho_s, \beta_1, \beta_2$ are also taken equal to* 1.
2. *In view of Remark 1.2, the boundary condition of the structure stated in $(1.1)_5$ reads as follows*
$$\eta(t,0,\theta)=\eta(t,L,\theta)=\frac{\partial\eta}{\partial z_1}(t,0,\theta)=\frac{\partial\eta}{\partial z_1}(t,L,\theta)=0, \text{ for all } t\in[0,T],$$
$$\text{all } \theta\in[0,2\pi],\ \eta \text{ is } 2\pi\text{-periodic with respect to } \theta.$$
3. *In the rest of this article, to simplify the notation, we do not include the above periodicity condition while writing the systems. For the same reason the notion of periodicity does not appear in the notation of function spaces also.*

Remark 1.4. *In $(1.1)_4$, Δ_s^2 and Δ_s represent respectively the biharmonic and Laplace operators defined on ω. More precisely*

$$D(\Delta_s)=H^2(\omega)\cap H_0^1(\omega), \quad \Delta_s=\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial\theta^2},$$
$$D(\Delta_s^2)=H^4(\omega)\cap H_0^2(\omega), \quad \Delta_s^2=\left(\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial\theta^2}\right)^2.$$

The spaces $H_0^2(\omega)$ and $H_0^1(\omega)$ are introduced in Section 2.

This type of model is motivated by blood flow in human arteries. Actually, the blood is a nonhomogeneous medium, composed of red blood cells, leukocytes (white blood cells) and thrombocytes (platelets) suspended in blood plasma. However, in large vessels, blood can be considered as an incompressible, viscous Newtonian fluid. An artery is a large blood vessel whose thickness is negligible in front of the radius of its section. For more details about the modelling of such systems we refer to [27, 29, 5] and the references therein.

During the last two decades, there has been a considerable interest in fluid-structure interaction problems involving moving interfaces. Generally speaking, these type of models can be classified into two types: either the structure is moving inside the fluid or the structure is located at the boundary of the fluid domain. Since in this article we are interested in studying FSI models where the structure is located at the boundary of the fluid domain, below we mention related works from the literature concerning this case only.

Let us briefly review some existence results for such models. Chambolle et al. [7] proved the existence of weak solutions for a FSI problem coupling a 3D viscous, incompressible fluid with a 2D viscoelastic plate, with homogeneous Dirichlet boundary conditions on the fluid boundaries, as long as the structure does not touch the fixed part of the fluid boundary. Grandmont [14] extended this result to undamped 2D elastic plates. In these studies, the middle surface of the structure is assumed to be flat. Lengeler and Růžička [22] and Lengeler [21] extended the above results to the case when the middle surface of the structure is no longer flat. In [22], the fluid is assumed to be incompressible and Newtonian and the structure is a linear elastic Koiter shell, whereas in [21] the interaction of an incompressible and generalized Newtonian fluid with a linear elastic Koiter shell has been studied.

Recently Muha and Čanić [26] proved the existence of weak solutions for a class of 2D/1D FSI problems. The fluid is viscous, Newtonian and driven by pressure data at the inflow and outflow boundaries, whereas the structure is modelled either by a linear viscoelastic beam or by a linear elastic Koiter shell equation. The proof is based on a splitting time discretization scheme. These results were extended by Muha and Čanić in [27] to the 3D cylindrical domain where the structure is modelled by a linear elastic cylindrical Koiter shell, and in [29], where the structure is described by a nonlinear cylindrical Koiter shell whose displacements are not necessarily

radially symmetric. A FSI problem with two structural layers, a thick layer and a thin layer, was dealt in [28]. Bukač et. al [4] numerically solved the coupled FSI, with a linear viscoelastic cylindrical Koiter shell modeling both radial and longitudinal displacements.

Concerning strong solutions, as far as we know, the first result was obtained by Beirao da Veiga [1]. The author proved local in time existence and uniqueness of solution for small data for an interaction between a $2D$ fluid and a $1D$ viscoelastic beam with periodic boundary conditions in the axial direction of the vessel. This result was extended by Lequeurre in [23] and [24], where the existence of a unique, local in time, strong solution for any data was proved in the case when the structure is modeled by a clamped viscoelastic beam. Local in time existence and uniqueness results, when the structure is purely elastic, have been recently obtained by Grandmont, Hillairet and Lequeurre [16]. Casanova [6] proved local in time existence and uniqueness of solutions with boundary data involving the pressure and when the structure is modeled as a viscoelastic beam. Grandmont and Hillairet [15] proved global in time existence and uniqueness of solutions for any initial data in a 2D/1D setting when the structure equation corresponds to a viscoelastic beam. Both in [15] and [16], periodic boundary conditions in the vessel direction for the fluid were assumed.

In the context of strong solutions, as far as we are aware of, the only articles that deal with more general geometry and nonlinearities in the structure model are Cheng, Coutand and Shkoller [9] and Cheng and Shkoller [10]. In these papers, existence and uniqueness of local in time strong solution is proved for the interaction between a viscous, incompressible fluid and thin nonlinear shells. In [9], the fluid is inside of a deformable elastic structure of Wilmore type in 3D, whereas in [10] the fluid interacts with a nonlinear elastic shell of Koiter type, both in 2D and 3D. However, in both works, whenever the 3D case is considered, the structure has zero inertia. Moreover, in the 3D Koiter shell case, the result is obtained assuming that the thickness of the shell is much smaller than the fluid viscosity. Also, in both works the structure is located on the entire boundary.

In this present work, we prove the existence and uniqueness of a maximal strong solution to the system (1.1). Here we want to emphasize that we consider a shell model with a damping, which is also the case in the references mentioned above concerning the existence of strong solutions in three dimensions. In our case, the structure is located at the lateral boundary of the cylinder (as in [27, 29]), and we prescribe homogeneous Neumann boundary conditions at the inlet and outlet boundaries of the cylindrical domain (the case of nonhomogeneous Neumann boundary conditions can also be considered, see (1.5)).

To state our main result, we need to introduce some notation. We set $\mathbf{L}^2(\Omega) = L^2(\Omega;\mathbb{R}^3)$ and $\mathbf{H}^s(\Omega) = H^s(\Omega;\mathbb{R}^3)$ for $s>0$. We shall use the same notation when Ω is replaced by $\Omega_{\overline{\eta}}$ for some $\overline{\eta}\in H^3(\omega)\cap H_0^2(\omega)$. We also need to introduce function spaces for the fluid velocity and the fluid pressure depending on the displacement η of the structure.

Definition 1.5. *For a given $\eta\in L^2(0,T;H^4(\omega))\cap H^2(0,T;L^2(\omega))$ satisfying $1+\eta(t,z_1,\theta)>0$ on $[0,T]\times\overline{\omega}$, and setting $\overline{\eta}=\eta(0)$, we say that $\mathbf{u}$ belongs to $L^2(0,T;\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\eta(\cdot)}))$ (resp. $H^1(0,T;\mathbf{L}^2(\Omega_{\eta(\cdot)}))$, $C([0,T];\mathbf{H}^1(\Omega_{\eta(\cdot)}))$) when $\mathbf{u}(t,\cdot)$ is a mapping from $\Omega_{\eta(t)}$ into $\mathbb{R}^3$ and when there exists X belonging to $H^2(0,T;L^2(\Omega_{\overline{\eta}}))\cap H^1(0,T;H^2(\Omega_{\overline{\eta}}))\cap L^\infty(0,T;H^3(\Omega_{\overline{\eta}}))$ such that, for all $t\in[0,T]$, $X(t,\cdot)$ is a C^1-diffeomorphism from $\Omega_{\overline{\eta}}$ onto $\Omega_{\eta(t)}$ and $\widehat{\mathbf{u}}$, defined by $\widehat{\mathbf{u}}(t,z_1,z_2,z_3)=\mathbf{u}(t,X(t,z_1,z_2,z_3))$, belongs to $L^2(0,T;\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\overline{\eta}}))$ (resp. $H^1(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))$, $C([0,T];\mathbf{H}^1(\Omega_{\overline{\eta}}))$). Similarly, we say that p belongs to $L^2(0,T;H^{1/2+\varepsilon_0}(\Omega_{\eta(\cdot)}))$ when $p(t,\cdot)$ is a mapping from $\Omega_{\eta(t)}$ into $\mathbb{R}$ and when $\widehat{p}$, defined by $\widehat{p}(t,z_1,z_2,z_3)=p(t,X(t,z_1,z_2,z_3))$, belongs to $L^2(0,T;H^{1/2+\varepsilon_0}(\Omega_{\overline{\eta}}))$ (ε_0 is determined in Theorem 4.2).*

We look for solutions $(\mathbf{u},p,\eta)$ to system (1.1) satisfying

$$\begin{aligned}
&\mathbf{u}\in L^2(0,T;\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\eta(\cdot)}))\cap H^1(0,T;\mathbf{L}^2(\Omega_{\eta(\cdot)}))\cap C([0,T];\mathbf{H}^1(\Omega_{\eta(\cdot)})),\\
&p\in L^2(0,T;H^{1/2+\varepsilon_0}(\Omega_{\eta(\cdot)})),\quad \text{div }\sigma(\mathbf{u},p)\in L^2(\widetilde{Q}_T),\\
&\eta\in L^2(0,T;H^4(\omega))\cap H^2(0,T;L^2(\omega)),\\
&1+\eta(t,z_1,\theta)>0\quad\text{for all }(t,z_1,\theta)\in[0,T]\times\overline{\omega},
\end{aligned}\tag{1.6}$$

for some $\varepsilon_0\in(0,1/2)$.

Definition 1.6. *We say that a triplet* $(\mathbf{u}, p, \eta)$ *is a solution to system* (1.1) *when it satisfies* (1.6)*, equation* $(1.1)_1$ *in the sense of distributions in* $\widetilde{Q}_T$*, equation* $(1.1)_4$ *in the sense of distributions in* ω_T*, equations* $(1.1)_{2,3,5}$ *in the sense of traces, and the initial conditions stated in* $(1.1)_6$*.*

We are now in position to state the first existence and uniqueness result of the paper.

Theorem 1.7. *Let* $(\mathbf{u}_0, \eta_1^0, \eta_2^0)$ *belong to* $\mathbf{H}^1(\Omega_{\eta_1^0}) \times H^{3+\varepsilon_0}(\omega) \times H^1(\omega)$*, for some* $\varepsilon_0 \in (0, 1/2)$*, and satisfy*

$$\begin{aligned}
&\text{div } \mathbf{u}_0 = 0 \text{ in } \Omega_{\eta_1^0},\\
&\eta_1^0(0,\theta) = \eta_1^0(L,\theta) = \tfrac{\partial \eta_1^0}{\partial \mathbf{n}}(0,\theta) = \tfrac{\partial \eta_1^0}{\partial \mathbf{n}}(L,\theta) = 0,\ \theta \in (0,2\pi),\\
&\eta_2^0(0,\theta) = \eta_2^0(L,\theta) = 0,\ \theta \in (0,2\pi),\\
&\mathbf{u}_0(z_1, 1+\eta_1^0(z_1,\theta),\theta) = \eta_2^0(z_1,\theta)\mathbf{e}_r \quad \text{for all } z_1 \in (0,L), \theta \in (0,2\pi)\\
&\min\{1+\eta_1^0(z_1,\theta) \mid (z_1,\theta) \in [0,T]\times\overline{\omega}\} > 0.
\end{aligned} \tag{1.7}$$

Then there exists a $T > 0$*, depending only on the initial data* $(\mathbf{u}_0, \eta_1^0, \eta_2^0)$*, such that the system* (1.1) *admits a strong solution in the sense of Definition 1.6.*

When that solution is maximal-in-time, in the sense introduced in Proposition 3.4, it is unique in the class of functions defined in (3.13)*.*

One of the main difficulties in the analysis of fluid-structure interaction models as in (1.1) is that the fluid equations are written in the deformed configuration (in Eulerian variables), while the structure equations are written in the reference configuration (in Lagrangian variables). Since the fluid domain at time t is one of the unknowns, we first rewrite the system in a fixed spatial domain. This can be achieved either by using a *geometric* change of variables (defined through the displacement of the fluid-structure interface), or a *Lagrangian* change of variables. In our case, it is more convenient to use a geometric change of variables. Next, we associate with the nonlinear problem a linear one, involving non-homogeneous source terms. We show that the operator associated to the linearized problem generates an analytic semigroup in a suitable Hilbert space. Another difficulty in our problem comes from the choice of boundary conditions we have taken. Due to the presence of mixed boundary conditions, we have to look for solutions in some appropriate weighted Sobolev spaces. Moreover, due to the change of variables transforming the system in the deformed configuration to a system in the reference configuration, the velocity in the reference configuration is no longer divergence free. Thus to solve the linear fluid-structure problem, we have to solve problems with non zero divergence conditions. We use the Banach fixed point theorem to prove our main result.

Before ending this review, we would like to mention that, since the end of the nineties, there are many contributions studying the well posedness of different type of fluid-structure interaction problems. We cannot mention all of the different configurations recently studied in the literature. But let us investigate if the method used in the present paper could be used for two typical models. In the case when an elastic structure modeled by the Lamé system of linear elasticity interacts with an incompressible viscous fluid, the existence of strong solutions is obtained for higher regularities than that we consider here (see, e.g., [3, 11, 20, 33]). In that case there is no hope of adapting our results to this type of model. At the opposite, the case of a smooth rigid body immersed in an incompressible viscous fluid, see [35, 17], can be handled more easily because the boundary of the rigid body remains regular. The analytic tools used here could be adapted to obtain the existence and uniqueness of local in time strong solution for those coupled models in three dimensions.

The outline of the paper is as follows. In Section 2, we introduce notations of several functional spaces that we will use later on. We introduce a change of variables and rewrite the system (1.1) in the reference configuration in Section 3. Local in time existence for the system written in reference configuration is stated in Theorem 3.3.

Section 4 is devoted to rewriting the linearized fluid-structure system as an evolution equation. We rewrite the Stokes system as an operator equation in Section 4.1, and the damped shell equation in Section 4.2. The coupled linear fluid-structure system is rewritten as an evolution equation in Section 4.3. The analyticity of the associated semigroup is studied in Section 4.4. Existence and regularity results for the nonhomogeneous linear system are studied in Section 5. Estimates of the nonlinear terms in suitable norms are done in Section 6. Section 7 is devoted to the proof of the main results, Theorem 1.7 and Theorem 3.3. Some technical results are collected in Appendix A.

2. Notation

We first introduce the spaces

$$H_0^1(\omega) = \left\{\eta \in H^1(\omega) \mid \eta(0,\theta) = \eta(L,\theta) = 0, \text{ for all } \theta \in (0,2\pi)\right\},$$
$$H_0^2(\omega) = \left\{\eta \in H^2(\omega) \cap H_0^1(\omega) \mid \frac{\partial \eta}{\partial z_1}(0,\theta) = \frac{\partial \eta}{\partial z_1}(L,\theta) = 0, \text{ for all } \theta \in (0,2\pi)\right\}.$$

For any $\overline{\eta}$ belonging to $H^3(\omega) \cap H_0^2(\omega)$, and satisfying

$$\gamma_{\overline{\eta}} = \min\{1 + \overline{\eta}(z_1,\theta) \mid (z_1,\theta) \in (0,L) \times (0,2\pi)\} > 0, \tag{2.1}$$

we set

$$\Omega_{\overline{\eta}} = \left\{(z_1,r,\theta) \in \mathbb{R}^3 \mid 0 \leqslant r < 1 + \overline{\eta}(z_1,\theta),\ z_1 \in (0,L), \theta \in (0,2\pi)\right\},$$
$$\Gamma_{\overline{\eta}}^s = \left\{(z_1,r,\theta) \in \mathbb{R}^3 \mid r = 1 + \overline{\eta}(z_1,\theta),\ z_1 \in (0,L), \theta \in (0,2\pi)\right\}.$$

For $\sigma > 1/2$, $\boldsymbol{\gamma}_{\overline{\eta}}^s$ is the operator belonging to $\mathcal{L}(H^\sigma(\Omega_{\overline{\eta}}), H^{\sigma-1/2}(\omega))$ and defined by

$$\left(\boldsymbol{\gamma}_{\overline{\eta}}^s h\right)(z_1,\theta) = h(z_1, 1 + \overline{\eta}(z_1,\theta), \theta) \quad \text{for all } (z_1,\theta) \in \omega, \tag{2.2}$$

where $h(z_1, 1 + \overline{\eta}(z_1,\theta),\theta)$ is nothing but the trace of $h \in H^\sigma(\Omega_{\overline{\eta}})$ on $\Gamma_{\overline{\eta}}^s$.

We use boldface letters, $\mathbf{L}^2(\Omega_{\overline{\eta}}) = L^2(\Omega_{\overline{\eta}};\mathbb{R}^3)$ and $\mathbf{H}^\sigma(\Omega_{\overline{\eta}}) = H^\sigma(\Omega_{\overline{\eta}};\mathbb{R}^3)$, for functional spaces for the fluid velocity. For $\mathbf{u} \in \mathbf{L}^2(\Omega_{\overline{\eta}})$, we denote by u_1, u_2 and u_3 the components of $\mathbf{u}$.

For an arbitrary integer $\ell \geqslant 0$ and for $-1 < \delta < 1$, we denote by $\mathbf{H}_\delta^\ell(\Omega_{\overline{\eta}})$ and $H_\delta^\ell(\Omega_{\overline{\eta}})$ the weighted Sobolev spaces (see [25, section 6.2.1]), respectively defined as the closure of $C^\infty(\overline{\Omega_{\overline{\eta}}};\mathbb{R}^3)$ and $C^\infty(\overline{\Omega_{\overline{\eta}}})$ for the norms

$$\|\mathbf{u}\|^2_{\mathbf{H}_\delta^\ell(\Omega_{\overline{\eta}})} = \sum_{|\alpha| \leqslant \ell} \sum_{i=1}^3 \int_{\Omega_{\overline{\eta}}} r_0^{2\delta} r_L^{2\delta} |D^\alpha u_i|^2 \, \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3,$$
$$\|p\|^2_{H_\delta^\ell(\Omega_{\overline{\eta}})} = \sum_{|\alpha| \leqslant \ell} \int_{\Omega_{\overline{\eta}}} r_0^{2\delta} r_L^{2\delta} |D^\alpha p|^2 \, \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3,$$

where r_0 (respectively r_L) is the distance to $\partial\Gamma_{\text{in}}$ (respectively $\partial\Gamma_{\text{out}}$), and D^α denotes the partial differential operator associated to the multi-index $\alpha = (\alpha_1,\alpha_2,\alpha_3)$. Here,

$$r_0 = \left[z_1^2 + \left(\sqrt{z_2^2 + z_3^2} - 1\right)^2\right]^{\frac{1}{2}}, \quad r_L = \left[(L - z_1)^2 + \left(\sqrt{z_2^2 + z_3^2} - 1\right)^2\right]^{\frac{1}{2}}.$$

Notice that $r_0 > z_1$ and $r_L > (L - z_1)$.

If $\delta \geqslant 0$, we have $H^\ell(\Omega_{\overline{\eta}}) \subset H_\delta^\ell(\Omega_{\overline{\eta}})$. We also introduce the following spaces

$$\mathbf{V}^0(\Omega_{\overline{\eta}}) = \left\{\mathbf{u} \in \mathbf{L}^2(\Omega_{\overline{\eta}}) \mid \text{div } \mathbf{u} = 0\right\},$$
$$\mathbf{V}^0_{n,\Gamma_{\overline{\eta}}^s}(\Omega_{\overline{\eta}}) = \left\{\mathbf{u} \in \mathbf{L}^2(\Omega_{\overline{\eta}}) \mid \text{div}\, \mathbf{u} = 0 \text{ in } \Omega_{\overline{\eta}},\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma_{\overline{\eta}}^s\right\},$$
$$\mathbf{V}^\sigma_{\Gamma_{\overline{\eta}}^s}(\Omega_{\overline{\eta}}) = \left\{\mathbf{u} \in \mathbf{H}^\sigma(\Omega_{\overline{\eta}}) \cap \mathbf{V}^0(\Omega_{\overline{\eta}}) \mid \mathbf{u} = \mathbf{0} \text{ on } \Gamma_{\overline{\eta}}^s\right\}, \quad \sigma > \frac{1}{2},$$
$$\mathbf{V}^\sigma_{n,\Gamma_{\overline{\eta}}^s}(\Omega_{\overline{\eta}}) = \left\{\mathbf{u} \in \mathbf{H}^1(\Omega_{\overline{\eta}}) \cap \mathbf{V}^0(\Omega_{\overline{\eta}}) \mid \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma_{\overline{\eta}}^s\right\}, \quad \sigma > \frac{1}{2},$$
$$H^\sigma_{\Gamma_n}(\Omega_{\overline{\eta}}) = \left\{f \in H^\sigma(\Omega_{\overline{\eta}}) \mid f = 0 \text{ on } \Gamma_n\right\}, \quad \sigma > \frac{1}{2}.$$

For $-\infty < t_0 < t_1 < +\infty$, we set

$$\mathbf{H}_\delta^{2,1}(Q_{\overline{\eta}}^{t_0,t_1}) = L^2(t_0,t_1;\mathbf{H}_\delta^2(\Omega_{\overline{\eta}})) \cap H^1(t_0,t_1;\mathbf{L}^2(\Omega_{\overline{\eta}})),$$
$$H^{r,s}(\omega \times (t_0,t_1)) = L^2(t_0,t_1;H^r(\omega)) \cap H^s(t_0,t_1;L^2(\omega)), \quad r,s > 0.$$

$(H_\delta^1(\Omega_{\overline{\eta}}))'$ is the dual of $H_\delta^1(\Omega_{\overline{\eta}})$ with $L^2(\Omega_{\overline{\eta}})$ as pivot space.

When $(t_0,t_1) = (0,T)$, we shall simplify the notation by setting $Q_{\overline{\eta}}^T = Q_{\overline{\eta}}^{0,T}$ and $\omega_T = \omega \times (0,T)$.

3. System rewritten in the initial configuration

The domain occupied by the fluid depends on the displacement η of the structure. For $-\infty < t_0 < t_1 < +\infty$, we consider displacement η belonging to

$$E(t_0,t_1) = L^2(t_0,t_1;H^4(\omega)\cap H_0^2(\omega))\cap H^2(t_0,t_1;L^2(\omega)),$$

and satisfying

$$\gamma_\eta(t_0,t_1) = \min\{1+\eta(t,z_1,\theta) \mid (t,z_1,\theta)\in[t_0,t_1]\times\overline{\omega}\} > 0. \tag{3.1}$$

For $\overline{\eta}$ belonging to $H^3(\omega)\cap H_0^2(\omega)$ and satisfying (2.1), we introduce the spaces

$$E_{\overline{\eta}}(t_0,t_1) = \{\eta\in E(t_0,t_1) \mid \eta(t_0)=\overline{\eta}\}$$
$$\text{and for } \ \gamma>0, \quad E_{\overline{\eta}}(t_0,t_1;\gamma) = \{\eta\in E_{\overline{\eta}}(t_0,t_1) \mid \gamma_\eta(t_0,t_1)\geqslant\gamma\}.$$

For $\eta\in E_{\overline{\eta}}(t_0,t_1;\gamma)$ with $\gamma>0$, we introduce the mapping $X_\eta(t,\cdot)\ :\ \Omega\longmapsto\Omega_{\eta(t)}$ defined by

$$X_\eta(t,z_1,z_2,z_3) = (z_1,z_2,z_3)+(0,z_2,z_3)\psi(r)\eta(t,z_1,\theta), \quad \text{with } r=\sqrt{z_2^2+z_3^2}, \tag{3.2}$$

where ψ is a C^∞ nondecreasing function from $[0,1]$ into $[0,1]$ satisfying

$$\psi(r)=0 \ \text{ for all } r\in\big[0,1/4\big] \ \text{ and } \psi(r)=1 \ \text{ for all } r\in[3/4,1].$$

Similarly, if $\overline{\eta}\in H^3(\omega)\cap H_0^2(\omega)$ satisfies (2.1), we set

$$X_{\overline{\eta}}(z_1,z_2,z_3) = (z_1,z_2,z_3)+(0,z_2,z_3)\psi(r)\overline{\eta}(z_1,\theta), \quad \text{with } r=\sqrt{z_2^2+z_3^2}. \tag{3.3}$$

We look for conditions ensuring that $X_\eta(t,\cdot)$ is a C^1-diffeomorphism from Ω onto $\Omega_{\eta(t)}$.

Lemma 3.1. *(i) Let $-\infty<t_0<t_1<+\infty$ and γ be positive. For all η belonging to $E(t_0,t_1)$ and satisfying $\gamma_\eta(t_0,t_1)>0$, the mapping X_η defined in (3.2) satisfies*

- *$X_\eta(t,\Gamma_s)=\Gamma^s_{\eta(t)}$, $X_\eta(t,\Gamma_{\rm in})=\Gamma_{\rm in}$ and $X_\eta(t,\Gamma_{\rm out})=\Gamma_{\rm out}$ for all $t\in[t_0,t_1]$,*
- *$X_\eta(t_0,\Omega)=\Omega_{\eta(t_0)}$,*
- *$X_\eta\in L^2(t_0,t_1;H^4(\Omega))\cap H^2(t_0,t_1;L^2(\Omega))$,*
- *$X_\eta\in H^1(t_0,t_1;H^2(\Omega))\cap L^\infty(t_0,t_1;H^3(\Omega))$.*

Moreover, for all $t\in[t_0,t_1]$, $X_\eta(t,\cdot)$ is a C^1-diffeomorphism from Ω onto $\Omega_{\eta(t)}$.

(ii) If $\overline{\eta}\in H^3(\omega)\cap H_0^2(\omega)$ satisfies (2.1), the mapping $X_{\overline{\eta}}$ defined in (3.3) is a C^1-diffeomorphism from Ω onto $\Omega_{\overline{\eta}}$, and it satisfies

- *$X_{\overline{\eta}}(\Gamma_s)=\Gamma^s_{\overline{\eta}}$, $X_{\overline{\eta}}(\Gamma_{\rm in})=\Gamma_{\rm in}$ and $X_{\overline{\eta}}(\Gamma_{\rm out})=\Gamma_{\rm out}$,*
- *$X_{\overline{\eta}}\in H^3(\Omega)$.*

(iii) With the assumptions stated in (i) and (ii), the mapping $X_{\overline{\eta},\eta}$, defined by

$$X_{\overline{\eta},\eta}(t,\cdot) = X_\eta(t,(X_{\overline{\eta}})^{-1}(\cdot)) \quad \text{for all } t\in(t_0,t_1), \tag{3.4}$$

belongs to $H^2(t_0,t_1;L^2(\Omega_{\overline{\eta}}))\cap H^1(t_0,t_1;H^2(\Omega_{\overline{\eta}}))\cap L^\infty(t_0,t_1;H^3(\Omega_{\overline{\eta}}))$.

Proof. Step 1. The first part of (i) is an obvious consequence of the definition of X_η. Let us prove that $X_\eta(t,\cdot)$ is a C^1-diffeomorphism from Ω onto $\Omega_{\eta(t)}$ for $t\in[t_0,t_1]$. For that we express X_η in cylindrical coordinates. To show that $X_\eta(t,\cdot)$ is a C^1-diffeomorphism from Ω onto $\Omega_{\eta(t)}$, it is sufficient to show that the mapping

$$\mathcal{X}_\eta(t,\cdot)\ :\ (z_1,\theta,r)\mapsto(\zeta_1,\varphi,\rho)=(z_1,\theta,r(1+\psi(r)\eta(t,z_1,\theta)))$$

is a C^1-diffeomorphism from $\Omega=(0,L)\times[0,2\pi)\times[0,1)$ into $\Omega_{\eta(t)}=\{(\zeta_1,\varphi,\rho)\in(0,L)\times[0,2\pi)\times\mathbb{R}^+ \mid 0\leqslant\rho<\eta(t,z_1,\varphi)\}$. It is clear that $\mathcal{X}_\eta(t,\cdot)$ is a map of class C^1 from Ω onto $\Omega_{\eta(t)}$. Moreover, for all $\eta\in\mathbb{R}$ satisfying $1+\eta\geqslant\gamma$, due to the definition of ψ, the mapping

$$r\mapsto\rho=r(1+\psi(r)\eta)$$

is invertible from $[0,1)$ into $[0,1+\eta)$. Let us denote its inverse by $\chi(\rho,\eta)$. It is clear that $(\rho,\eta)\mapsto\chi(\rho,\eta)$ is of class C^1 from $\{(\rho,\eta) \mid 0\leqslant\rho<1+\eta,\ \gamma-1\leqslant\eta<\infty\}$ onto $[0,1]\times[\gamma-1,\infty)$.

Let us set $\mathcal{Y}_\eta(t,\zeta_1,\varphi,\rho)=(t,\zeta_1,\varphi,\chi(\rho,\eta(t,\zeta_1,\varphi)))$. It is clear that, for all $t\in[t_0,t_1]$, $\mathcal{Y}_\eta(t,\cdot)$ is the inverse of $\mathcal{X}_\eta(t,\cdot)$ and is of class C^1.

Step 2. The results stated in (ii) can be proved similarly. The result stated in (iii) is an easy consequence of (i) and (ii). The proof is complete. □

Before rewriting the system (1.1) in the initial configuration $\Omega_{\eta_1^0}$, we first define a family of nonlinear terms indexed by an arbitrary initial displacement $\overline{\eta}$.

For all $\overline{\eta}\in H^3(\omega)\cap H_0^2(\omega)$ satisfying (2.1), and for $-\infty<t_0<t_1<+\infty$, we first introduce the space

$$\begin{aligned}&B(Q_{\overline{\eta}}^{t_0,t_1})\\&=\Big\{(\mathbf{v},p,\eta)\in(\mathbf{H}^{2,1}_{\delta_0}(Q_{\overline{\eta}}^{t_0,t_1})\cap C([t_0,t_1];\mathbf{H}^1(\Omega_{\overline{\eta}})))\times L^2(t_0,t_1;H^1_{\delta_0}(\Omega_{\overline{\eta}}))\times E_{\overline{\eta}}(t_0,t_1)\\&\qquad\mid\mathbf{v}=\eta_t\mathbf{e}_r\quad\text{ on }(t_0,t_1)\times\Gamma^s_{\overline{\eta}}\Big\},\end{aligned}$$

where $\delta_0\in(0,1/2)$ is the parameter introduced in Theorem 4.2. The condition $\mathbf{v}=\eta_t\mathbf{e}_r$ on $(t_0,t_1)\times\Gamma^s_{\overline{\eta}}$, in the definition of $B(Q_{\overline{\eta}}^{t_0,t_1})$, is needed in the proof of Lemma 6.8. We equip $B(Q_{\overline{\eta}}^{t_0,t_1})$ with the norm

$$\begin{aligned}|\!|\!|(\mathbf{v},p,\eta)|\!|\!|_{B(Q_{\overline{\eta}}^{t_0,t_1})}&:=\|\mathbf{v}\|_{\mathbf{H}^{2,1}_{\delta_0}(Q_{\overline{\eta}}^{t_0,t_1})}+\|\mathbf{v}\|_{L^\infty(t_0,t_1;\mathbf{H}^1(\Omega_{\overline{\eta}}))}\\&\quad+\|p\|_{L^2(t_0,t_1;H^1_{\delta_0}(\Omega_{\overline{\eta}}))}+\|\eta\|_{E(t_0,t_1)},\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}&\|\eta\|_{E(t_0,t_1)}\\&=\|\eta\|_{H^{4,2}(\omega\times(t_0,t_1))}+\|\eta\|_{L^\infty(t_0,t_1;H^3(\omega))}+\|\eta_t\|_{H^{2,1}(\omega\times(t_0,t_1))}+\|\eta_t\|_{L^\infty(t_0,t_1;H^1(\omega))}.\end{aligned}$$

We denote by $Y_{\overline{\eta}}$ the inverse of $X_{\overline{\eta}}$. For all $\xi\in E_{\overline{\eta}}(t_0,t_1)$ satisfying $\gamma_\xi(t_0,t_1)>0$, $X_{\overline{\eta},\xi}(t,\cdot)$ is the diffeomorphism from $\Omega_{\overline{\eta}}$ into $\Omega_{\xi(t)}$ introduced in (3.4). We set

$$J_{\overline{\eta},\xi}(t,z)=(J^{j,k}_{\overline{\eta},\xi})_{1\leqslant j,k\leqslant 3}=[\nabla X_{\overline{\eta},\xi}]^{-1}(t,z)\quad\text{ for }(t,z)\in(t_0,t_1)\times\Omega_{\overline{\eta}}.\tag{3.6}$$

If $\overline{\eta}$ and ξ satisfy the conditions mentioned above, and if $(\mathbf{v},q,\xi)$ belongs to $B(Q_{\overline{\eta}}^{t_0,t_1})$, we define the nonlinear terms $\widehat{\mathcal{F}}_{\overline{\eta}}(\mathbf{v},q,\xi)=\Big(\widehat{\mathcal{F}}_{\overline{\eta},i}(\mathbf{v},q,\xi)\Big)_{1\leqslant i\leqslant 3}$ and $\widehat{\mathcal{G}}_{\overline{\eta}}(\mathbf{v},\xi)$, in $Q_{\overline{\eta}}^{t_0,t_1}=(t_0,t_1)\times\Omega_{\overline{\eta}}$, by

$$\begin{aligned}\widehat{\mathcal{F}}_{\overline{\eta},i}(\mathbf{v},q,\xi)=&-(\mathbf{v}-\partial_t X_{\overline{\eta},\xi})\cdot J^\top_{\overline{\eta},\xi}\nabla v_i+\nu\sum_{j,k,l}\frac{\partial^2 v_i}{\partial z_l\partial z_k}\Big(J^{k,j}_{\overline{\eta},\xi}J^{l,j}_{\overline{\eta},\xi}-\delta_{k,j}\delta_{l,j}\Big)\\&+\nu\sum_{j,k,l}\frac{\partial v_i}{\partial z_k}\frac{\partial}{\partial z_l}(J^{k,j}_{\overline{\eta},\xi})J^{l,j}_{\overline{\eta},\xi}+((I_{\mathbb{R}^3}-J^\top_{\overline{\eta},\xi})\nabla q)_i,\quad i=1,2,3,\end{aligned}\tag{3.7}$$

and

$$\widehat{\mathcal{G}}_{\overline{\eta}}(\mathbf{v},\xi)=\nabla\mathbf{v}:(I_{\mathbb{R}^3}-J^\top_{\overline{\eta},\xi}),\tag{3.8}$$

where ∂_t denotes the partial derivative with respect to t. The nonlinear term $\widehat{\mathcal{H}}_{\overline{\eta}}(\mathbf{v},q,\xi)$ is defined in $(t_0,t_1)\times\omega$ by

$$\begin{aligned}\widehat{\mathcal{H}}_{\overline{\eta}}(\mathbf{v},q,\xi)=&-\nu\boldsymbol{\gamma}^s_{\overline{\eta}}\Big(\big(\nabla\mathbf{v}J_{\overline{\eta},\xi}+(\nabla\mathbf{v}J_{\overline{\eta},\xi})^\top\big)\,(\mathrm{cof}(\nabla X_{\overline{\eta},\xi})-I_{\mathbb{R}^3})\,\mathbf{e}_{\overline{\eta}}\Big)\cdot\mathbf{e}_r\\&-\nu\boldsymbol{\gamma}^s_{\overline{\eta}}\Big(\big(\nabla\mathbf{v}(J_{\overline{\eta},\xi}-I_{\mathbb{R}^3})+[\nabla\mathbf{v}(J_{\overline{\eta},\xi}-I_{\mathbb{R}^3})]^\top\big)\mathbf{e}_{\overline{\eta}}\Big)\cdot\mathbf{e}_r\\&+\boldsymbol{\gamma}^s_{\overline{\eta}}\Big(q\big(\mathrm{cof}(\nabla X_{\overline{\eta},\xi})-I_{\mathbb{R}^3}\big)\mathbf{e}_{\overline{\eta}}\Big)\cdot\mathbf{e}_r,\end{aligned}\tag{3.9}$$

where $\mathbf{e}_{\overline{\eta}}$ is the unit normal to $\Gamma^s_{\overline{\eta}}$ exterior to $\Omega_{\overline{\eta}}$, and $\boldsymbol{\gamma}^s_{\overline{\eta}}$ is defined in (2.2).

For any η_1^0 satisfying the assumptions of Theorem 1.7, we consider the following change of unknowns

$$\widehat{\mathbf{u}}(t,\zeta)=\mathbf{u}(t,X_{\eta_1^0,\eta}(t,\zeta)),\quad\widehat{p}(t,\zeta)=p(t,X_{\eta_1^0,\eta}(t,\zeta)).\tag{3.10}$$

The system satisfied by $(\widehat{\mathbf{u}},\widehat{p},\eta)$ is the following

$$\begin{aligned}
&\widehat{\mathbf{u}}_t - \operatorname{div}\sigma(\widehat{\mathbf{u}},\widehat{p}) = \widehat{\mathcal{F}}_{\eta_1^0}(\widehat{\mathbf{u}},\widehat{p},\eta) - \nu\nabla\widehat{\mathcal{G}}_{\eta_1^0}(\widehat{\mathbf{u}},\eta) \quad \text{in } (0,T)\times\Omega_{\eta_1^0},\\
&\operatorname{div}\widehat{\mathbf{u}} = \widehat{\mathcal{G}}_{\eta_1^0}(\widehat{\mathbf{u}},\eta) \quad \text{in } (0,T)\times\Omega_{\eta_1^0},\\
&\widehat{\mathbf{u}}(t,z_1,1+\eta_1^0,\theta) = \eta_t(t,z_1,\theta)\mathbf{e}_r, \quad \text{for } (t,z_1,\theta)\in(0,T)\times\omega,\\
&\sigma(\widehat{\mathbf{u}},\widehat{p})\mathbf{n} = 0 \text{ on } (0,T)\times\Gamma_{\text{in}}, \quad \sigma(\widehat{\mathbf{u}},\widehat{p})\mathbf{n} = 0 \text{ on } (0,T)\times\Gamma_{\text{out}},\\
&\widehat{\mathbf{u}}(0) = \mathbf{u}_0(X_{\eta_1^0}(\cdot)) \text{ in } \Omega_{\eta_1^0},\\
&\eta_{tt} + \mathscr{L}_{mem}\eta + \Delta_s^2\eta - \Delta_s\eta_t =\\
&-\gamma^s_{\eta_1^0}(\sigma(\widehat{\mathbf{u}},\widehat{p})\mathbf{e}_{\eta_1^0})\cdot\mathbf{e}_r + \widehat{\mathcal{H}}_{\eta_1^0}(\widehat{\mathbf{u}},\widehat{p},\eta) \quad \text{in } (0,T)\times\omega,\\
&\eta = \eta_z = 0 \quad \text{on } (0,T)\times\{0,L\}\times(0,2\pi),\\
&\eta(0) = \eta_1^0, \quad \eta_t(0) = \eta_2^0 \quad \text{in } \omega.
\end{aligned} \tag{3.11}$$

Definition 3.2. *We say that a triplet $(\widehat{\mathbf{u}},\widehat{p},\eta)\in B(Q^T_{\eta_1^0})$ is a strong solution to system (3.11), over the time interval $(0,T)$, when η satisfies (3.1), when equation $(3.11)_{1,2}$ is satisfied in the sense of distributions in $Q^T_{\eta_1^0}$, equation $(3.11)_{6,7}$ in the sense of distributions in ω_T, equations $(3.11)_{3,4,8}$ in the sense of traces, and the initial conditions stated in $(3.11)_{5,9}$.*

We say that $(\widehat{\mathbf{u}},\widehat{p},\eta)$ is a maximal strong solution to system (3.11) over the time interval $[0,T_{\text{m}})$ when either $T_{\text{m}}=\infty$, or $T_{\text{m}}<\infty$ and, for all $0<T<T_{\text{m}}$, $(\widehat{\mathbf{u}},\widehat{p},\eta)$ is a strong solution to system (3.11) over the time interval $[0,T]$, and it satisfies

$$\lim_{T\to T_{\text{m}}}\left(\|\!|(\widehat{\mathbf{u}},\widehat{p},\eta)|\!\|_{B(Q^T_{\eta_1^0})} + \max\left\{|1+\eta(T,z_1,\theta)|^{-1} \mid (z_1,\theta)\in\overline{\omega}\right\}\right) = \infty. \tag{3.12}$$

We prove the following existence and uniqueness result for system (3.11).

Theorem 3.3. *Let $(\mathbf{u}_0,\eta_1^0,\eta_2^0)\in\mathbf{H}^1(\Omega_{\eta_1^0})\times H^{3+\varepsilon_0}(\omega)\times H^1(\omega)$, for some $\varepsilon_0>0$, be such that (1.7) is satisfied. Then the system (3.11) admits a unique maximal strong solution. Both the maximal time of existence and the solution are unique.*

To establish the equivalence between solutions to system (1.1) and solutions to system (3.11), we need to introduce the spaces $L^2(0,T;\mathbf{H}^2_{\delta_0}(\Omega_{\eta(\cdot)}))$ and $L^2(0,T;H^1_{\delta_0}(\Omega_{\eta(\cdot)}))$. These spaces can be defined as we did in Definition 1.5. We also need to introduce the following class of functions

$$\begin{aligned}
&\mathbf{u}\in L^2(0,T;\mathbf{H}^2_{\delta_0}(\Omega_{\eta(\cdot)}))\cap H^1(0,T;\mathbf{L}^2(\Omega_{\eta(\cdot)}))\cap C([0,T];\mathbf{H}^1(\Omega_{\eta(\cdot)})),\\
&p\in L^2(0,T;H^1_{\delta_0}(\Omega_{\eta(\cdot)})), \quad \operatorname{div}\sigma(\mathbf{u},p)\in L^2(0,T;L^2(\Omega_{\eta(\cdot)})),\\
&\eta\in L^2(0,T;H^4(\omega))\cap H^2(0,T;L^2(\omega)),\\
&\gamma_\eta(T) = \min\{1+\eta(t,z_1,\theta)\mid(t,z_1,\theta)\in[0,T]\times\overline{\omega}\}>0.
\end{aligned} \tag{3.13}$$

We now state a proposition whose proof is obvious.

Proposition 3.4. *A triplet $(\mathbf{u},p,\eta)$, satisfying (3.13), is solution (respectively a maximal strong solution) to system (1.1) in the sense of Definition 1.6 if and only if the triplet $(\widehat{\mathbf{u}},\widehat{p},\eta)$, where $(\widehat{\mathbf{u}},\widehat{p})$ is defined in (3.10), belongs to $B(Q^T_{\eta_1^0})$ and is a solution (respectively a maximal strong solution) to system (3.11) over the time interval $(0,T)$, in the sense of Definition 3.2.*

Let us notice that the definition of maximal strong solution for system (1.1) is introduced in the above proposition via the equivalence between solutions to system (1.1) and solutions to system (3.11).

4. Study of the linearized system

Throughout this section, we assume that $\overline{\eta}$ satisfies the following conditions

$$\overline{\eta}\in H^{3+\varepsilon_0}(\omega)\cap H^2_0(\omega) \quad \text{and } \gamma_{\overline{\eta}}>0, \tag{4.1}$$

where $\gamma_{\overline{\eta}}$ is defined in (2.1), and $\varepsilon_0\in(0,1/2)$ is the exponent appearing in Theorem 4.1.

To study system (3.11), we are going to prove regularity results for the solutions to the following nonhomogeneous linear system

$$\begin{aligned}
&\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},p) = \mathbf{f} - \nu\nabla g, \quad \operatorname{div}\,\mathbf{v} = g \quad \text{in } (0,T)\times\Omega_{\overline{\eta}},\\
&\mathbf{v} = \zeta_2\mathbf{e}_r \quad \text{on } (0,T)\times\Gamma^s_{\overline{\eta}},\\
&\sigma(\mathbf{v},p)\mathbf{n} = 0 \text{ on } (0,T)\times\Gamma_{\text{in}}, \quad \sigma(\mathbf{v},p)\mathbf{n} = 0 \text{ on } (0,T)\times\Gamma_{\text{out}},\\
&\mathbf{v}(0) = \mathbf{v}_0 \text{ in } \Omega_{\overline{\eta}},\\
&\zeta_{1,t} = \zeta_2 \quad \text{in } (0,T)\times\omega,\\
&\zeta_{2,t} + \Delta_s^2\zeta_1 - \Delta_s\zeta_2 = -\gamma^s_{\overline{\eta}}\Big(\sigma(\mathbf{v},p)\mathbf{e}_{\overline{\eta}}\Big)\cdot\mathbf{e}_r + h \quad \text{in } (0,T)\times\omega,\\
&\zeta_1 = \tfrac{\partial\zeta_1}{\partial z_1} = 0 \quad \text{on } (0,T)\times\{0,L\}\times(0,2\pi),\\
&\zeta_1(0) = \zeta_1^0, \quad \zeta_2(0) = \zeta_2^0 \quad \text{in } \omega.
\end{aligned} \tag{4.2}$$

To rewrite (4.2) when $g = 0$ as an evolution equation and to study the properties of the associated semigroup, we are going to follow the approach in [31, 32]. The idea is to decompose the fluid velocity $\mathbf{v}$, satisfying $(4.2)_{1-4}$ when $g = 0$, into two parts $P\mathbf{v}$ and $(I-P)\mathbf{v}$, where P is the Leray projector introduced in Lemma 4.4. The part $P\mathbf{v}$ satisfies the Stokes equations in a suitable space and $(I-P)\mathbf{v}$ satisfies an algebraic equation. Next we determine an expression for the pressure, which can be broken down into two parts, one which depends on $P\mathbf{v}$ and another one which depends on η_2. This allows us to eliminate the pressure term from the structure equation $(4.2)_6$ and to rewrite the system as an evolution equation for $(P\mathbf{v},\zeta_1,\zeta_2)$.

4.1. **The steady Stokes equation with nonhomogeneous boundary condition.** We consider the following nonhomogenous Stokes equation

$$\begin{aligned}
&-\operatorname{div}\sigma(\mathbf{w},\pi) = \mathbf{f}, \quad \operatorname{div}\,\mathbf{w} = 0 \quad \text{in } \Omega_{\overline{\eta}},\\
&\mathbf{w} = \mathbf{g} \text{ on } \Gamma^s_{\overline{\eta}}, \quad \sigma(\mathbf{w},\pi)\mathbf{n} = 0 \text{ on } \Gamma_n.
\end{aligned} \tag{4.3}$$

Theorem 4.1. *Let us assume that $\overline{\eta}\equiv 0$. If $\mathbf{f}\in\mathbf{L}^2(\Omega)$ and $\mathbf{g}\in\mathbf{H}^{3/2}(\Gamma_s)$. The system (4.3) admits a unique solution $(\mathbf{w},\pi)$ belonging to $\mathbf{H}^2_{\delta_0}(\Omega)\times H^1_{\delta_0}(\Omega)$ satisfying*

$$\|\mathbf{w}\|_{\mathbf{H}^2_{\delta_0}(\Omega)} + \|\pi\|_{H^1_{\delta_0}(\Omega)} \leqslant C(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{3/2}(\Gamma_s)}),$$

for some $0<\delta_0<1/2$. In particular, setting $\varepsilon_0 = \left(\frac{1}{2}-\delta_0\right)\in(0,1/2)$, we have

$$\|\mathbf{w}\|_{\mathbf{H}^{3/2+\varepsilon_0}(\Omega)} + \|\pi\|_{H^{1/2+\varepsilon_0}(\Omega)} \leqslant C(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{3/2}(\Gamma_s)}). \tag{4.4}$$

Proof. This result follows from [25, Theorem 9.1.5]. □

Theorem 4.2. *We assume that $\overline{\eta}$ belongs to $H^{3+\varepsilon_0}(\omega)\cap H^2_0(\omega)$, where $\varepsilon_0 = \left(\frac{1}{2}-\delta_0\right)$ and δ_0 is the exponent introduced in Theorem 4.1, and that (2.1) is satisfied. If $\mathbf{f}\in\mathbf{L}^2(\Omega_{\overline{\eta}})$ and $\mathbf{g}\in\mathbf{H}^{3/2}(\Gamma^s_{\overline{\eta}})$, the system (4.3) admits a unique solution $(\mathbf{w},\pi)$ belonging to $\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}})\times H^1_{\delta_0}(\Omega_{\overline{\eta}})$, and satisfying*

$$\|\mathbf{w}\|_{\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}})} + \|\pi\|_{H^1_{\delta_0}(\Omega_{\overline{\eta}})} \leqslant C(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega_{\overline{\eta}})} + \|\mathbf{g}\|_{\mathbf{H}^{3/2}(\Gamma^s_{\overline{\eta}})}),$$

and

$$\|\mathbf{w}\|_{\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\overline{\eta}})} + \|\pi\|_{H^{1/2+\varepsilon_0}(\Omega_{\overline{\eta}})} \leqslant C(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega_{\overline{\eta}})} + \|\mathbf{g}\|_{\mathbf{H}^{3/2}(\Gamma^s_{\overline{\eta}})}). \tag{4.5}$$

In the above estimates, the constant C depends on $\|\overline{\eta}\|_{H^{3+\varepsilon_0}(\omega)}$ and on $\gamma_{\overline{\eta}}>0$ introduced in (2.1).

Proof. First, from [25, Theorem 9.1.5] it follows that (4.3) admits a unique solution $(\mathbf{w},\pi)\in\mathbf{H}^1(\Omega_{\overline{\eta}})\times L^2(\Omega_{\overline{\eta}})$ satisfying

$$\|\mathbf{w}\|_{\mathbf{H}^1(\Omega_{\overline{\eta}})} + \|\pi\|_{L^2(\Omega_{\overline{\eta}})} \leqslant C(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega_{\overline{\eta}})} + \|\mathbf{g}\|_{\mathbf{H}^{3/2}(\Gamma^s_{\overline{\eta}})}). \tag{4.6}$$

The result in [25, Theorem 9.1.5] is proved for polyhedral domains, but the adaptation to $\Omega_{\overline{\eta}}$ can be easily done.

The proof of (4.5) mainly follows from Theorem 4.1, a localization argument, and (4.6). Let us give the main ideas of the proof. For $\varepsilon > 0$, let us set

$$\Omega_{\overline{\eta},\varepsilon} = \bigcup_{z_1 \in (\varepsilon, L-\varepsilon)} \{z_1\} \times \{(z_2, z_3) \in \mathbb{R}^2 \mid (z_2^2 + z_3^2)^{1/2} < 1 + \overline{\eta}(z_1, \theta))\},$$

with $(\cos\theta, \sin\theta) = \left(\frac{z_2}{(z_2^2+z_3^2)^{1/2}}, \frac{z_3}{(z_2^2+z_3^2)^{1/2}}\right)$. Since $\Gamma^s_{\overline{\eta}}$ is of class C^2, from the classical regularity results for the Stokes equation, it can be shown that

$$\|\mathbf{w}|_{\Omega_{\overline{\eta},\varepsilon}}\|_{\mathbf{H}^2(\Omega_{\overline{\eta},\varepsilon})} + \|\pi|_{\Omega_{\overline{\eta},\varepsilon}}\|_{H^1_{\delta_0}(\Omega_{\overline{\eta},\varepsilon})} \leqslant C(\varepsilon)(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega_{\overline{\eta}})} + \|\mathbf{g}\|_{\mathbf{H}^{3/2}(\Gamma^s_{\overline{\eta}})}).$$

The regularity of $(\mathbf{w}, \pi)$ in $\Omega^c_{\overline{\eta},\varepsilon} = \Omega_{\overline{\eta}} \setminus \Omega_{\overline{\eta},\varepsilon}$, can be deduced from the regularity of $(\widetilde{\mathbf{w}}, \widetilde{\pi}) = (\mathbf{w}, \pi) \circ (X_{\overline{\eta}})^{-1}$ in Ω. We set $\Omega_\varepsilon = (\varepsilon, L-\varepsilon) \times B(0_{\mathbb{R}^2}, 1)$ and $\Omega^c_\varepsilon = \Omega \setminus \Omega_\varepsilon$. Using a localization argument and Theorem 4.1, we can prove that

$$\|\widetilde{\mathbf{w}}|_{\Omega^c_\varepsilon}\|_{\mathbf{H}^2_{\delta_0}(\Omega^c_\varepsilon)} + \|\widetilde{\pi}|_{\Omega^c_\varepsilon}\|_{H^1_{\delta_0}(\Omega^c_\varepsilon)} \leqslant C(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega_{\overline{\eta}})} + \|\mathbf{g}\|_{\mathbf{H}^{3/2}(\Gamma_{\overline{\eta}})^s}).$$

provided that $\varepsilon > 0$ is small enough. For that, we have to notice that $\lim_{\varepsilon\to 0} \|\partial^2_{z_j,z_k} X_{\overline{\eta}} \partial^2_{z_j,z_\ell} X_{\overline{\eta}} - \delta_{k,j}\delta_{\ell,j}\|_{H^1((0,\varepsilon)\cup(L-\varepsilon,L);L^\infty B(0_{\mathbb{R}^2},1))} = 0$, and to use nonlinear estimates as those proved in Section 6. □

Remark 4.3. *From now on, δ_0 and ε_0 are the exponents appearing in Theorem 4.1. Throughout Section 4, we assume that $\overline{\eta}$ satisfies the assumptions of Theorem 4.2.*

We want to rewrite the stationary Stokes equation (4.3) in an operator formulation and to obtain an expression of the pressure π in terms of $\mathbf{f}, \mathbf{g}$ and $\mathbf{w}$. To this aim, we introduce the Leray projector in the case of mixed Dirichlet-Neumann boundary conditions. Let us notice that this projector is different from the classical Leray projector corresponding to the case where only Dirichlet boundary conditions are prescribed on the whole boundary.

Lemma 4.4. *We have the following orthogonal decomposition of the space $\mathbf{L}^2(\Omega_{\overline{\eta}})$:*

$$\begin{aligned}
&\mathbf{L}^2(\Omega_{\overline{\eta}}) = \mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}) \oplus \nabla H^1_{\Gamma_n}(\Omega_{\overline{\eta}}), \quad \text{where} \\
&\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}) = \left\{\mathbf{u} \in \mathbf{L}^2(\Omega_{\overline{\eta}}) \mid \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega_{\overline{\eta}},\ \mathbf{u}\cdot\mathbf{e}_{\overline{\eta}} = 0 \text{ on } \Gamma^s_{\overline{\eta}}\right\} \quad \text{and} \\
&H^1_{\Gamma_n}(\Omega_{\overline{\eta}}) = \left\{p \in H^1(\Omega_{\overline{\eta}}) \mid p = 0 \text{ on } \Gamma_n\right\}.
\end{aligned}$$

The orthogonal projection P from $\mathbf{L}^2(\Omega_{\overline{\eta}})$ onto $\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$ is defined by $P\mathbf{f} = \mathbf{f} - \nabla q_1 - \nabla q_2$, where

$$\begin{aligned}
&q_1 \in H^1_0(\Omega_{\overline{\eta}}), \quad \Delta q_1 = \operatorname{div}\mathbf{f} \text{ in } \Omega_{\overline{\eta}}, \quad \text{and} \\
&q_2 \in H^1_{\Gamma_n}(\Omega_{\overline{\eta}}), \quad \Delta q_2 = 0 \text{ in } \Omega_{\overline{\eta}}, \quad \frac{\partial q_2}{\partial \mathbf{n}} = (\mathbf{f} - \nabla q_1)\cdot\mathbf{e}_{\overline{\eta}} \text{ on } \Gamma^s_{\overline{\eta}}.
\end{aligned} \tag{4.7}$$

In (4.7) $\mathbf{n}$ is the unit normal to $\partial\Omega_{\overline{\eta}}$ exterior to $\Omega_{\overline{\eta}}$, and $\mathbf{n}|_{\Gamma^s_{\overline{\eta}}} = \mathbf{e}_{\overline{\eta}}$. We set

$$N_p\mathbf{f} = q_1 + q_2, \tag{4.8}$$

and we have $(I-P)\mathbf{f} = \nabla N_p(\mathbf{f})$ and $N_p \in \mathcal{L}(\mathbf{L}^2(\Omega_{\overline{\eta}}), H^1(\Omega_{\overline{\eta}}))$.

Moreover the restriction of P to $\mathbf{H}^1_{\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$ belongs to $\mathcal{L}(\mathbf{H}^1_{\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}), \mathbf{V}^1_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})))$.

Proof. For the proof of the above result we refer to [30, Lemma 2.2]. The fact that P belongs to $\mathcal{L}(\mathbf{H}^1_{\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}), \mathbf{V}^1_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})))$ may be proved with Lemma A.1. □

Formally, the pressure π in system (4.3) is the solution of the following elliptic equation

$$\begin{aligned}
&\Delta\pi = \operatorname{div}\mathbf{f} \quad \text{in } \Omega_{\overline{\eta}}, \\
&\frac{\partial\pi}{\partial\mathbf{n}} = \mathbf{f}\cdot\mathbf{e}_{\overline{\eta}} + \nu\Delta\mathbf{w}\cdot\mathbf{e}_{\overline{\eta}} \quad \text{on } \Gamma^s_{\overline{\eta}}, \quad \pi = 2\nu\boldsymbol{\varepsilon}(\mathbf{w})\mathbf{n}\cdot\mathbf{n} \quad \text{on } \Gamma_n.
\end{aligned} \tag{4.9}$$

Since the solution $\mathbf{w}$ of system (4.3) belongs to $\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}})$, and not to $\mathbf{H}^2(\Omega_{\overline{\eta}})$, we cannot define $\Delta\mathbf{w}\cdot\mathbf{n}$ as an element in $H^{-1/2}(\Gamma^s_{\overline{\eta}})$. Moreover $\mathbf{f}\cdot\mathbf{e}_{\overline{\eta}}$ is not defined on $\Gamma^s_{\overline{\eta}}$ when $\mathbf{f} \in \mathbf{L}^2(\Omega_{\overline{\eta}})$. This is why the equation (4.9)

is not well posed. To overcome that difficulty, we write π in the form $\pi = q + \rho$, where $q = N_p(\mathbf{f})$ and ρ is the *formal* solution of the following elliptic equation

$$\Delta\rho = 0 \quad \text{in } \Omega_{\overline{\eta}}, \quad \frac{\partial\rho}{\partial\mathbf{n}} = \nu\Delta\mathbf{w}\cdot\mathbf{e}_{\overline{\eta}} \quad \text{on } \Gamma^s_{\overline{\eta}}, \quad \rho = 2\nu\boldsymbol{\varepsilon}(\mathbf{w})\mathbf{n}\cdot\mathbf{n} \quad \text{on } \Gamma_n.$$

Following [13, Section 4.2], we introduce the operator $N_b \in \mathcal{L}(\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}}), L^2(\Omega_{\overline{\eta}}))$ defined by $N_b\mathbf{w} = \rho$, where ρ is the solution of the variational problem

$$\begin{aligned}&\text{Find } \rho \in L^2(\Omega_{\overline{\eta}}) \text{ such that}\\ &\int_{\Omega_{\overline{\eta}}} \rho\,\xi = 2\nu\int_{\Omega_{\overline{\eta}}} \boldsymbol{\varepsilon}(\mathbf{w}) : \nabla^2\varphi - 2\nu\int_{\Gamma^s_{\overline{\eta}}} \boldsymbol{\varepsilon}(\mathbf{w})\mathbf{e}_{\overline{\eta}}\cdot\nabla\varphi, \text{ for all } \xi \in L^2(\Omega_{\overline{\eta}}),\end{aligned} \tag{4.10}$$

and where $\varphi \in H^1(\Omega_{\overline{\eta}})$ is the solution to the equation

$$\Delta\varphi = \xi \text{ in } \Omega_{\overline{\eta}}, \quad \frac{\partial\varphi}{\partial\mathbf{n}} = 0 \text{ on } \Gamma^s_{\overline{\eta}}, \quad \varphi = 0 \text{ on } \Gamma_n. \tag{4.11}$$

From Theorem A.1, it follows that φ belongs to $H^2(\Omega_{\overline{\eta}})$. Therefore, all the terms in (4.10) are well defined. Actually, using the Lax-Milgram Lemma, we can prove that problem (4.10) admits a unique solution $\rho \in L^2(\Omega_{\overline{\eta}})$, and therefore N_b is a well defined operator belonging to $\mathcal{L}(\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}}), L^2(\Omega_{\overline{\eta}}))$.

We now introduce the Stokes operator $(A_0, \mathcal{D}(A_0))$ in $\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$. We set

$$\begin{aligned}\mathcal{D}(A_0) = \Big\{&\mathbf{u} \in \mathbf{V}^1_{\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}) \cap \mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}}) \mid \operatorname{div}\sigma(\mathbf{u}, N_b\mathbf{u}) \in \mathbf{L}^2(\Omega_{\overline{\eta}}),\\ &\sigma(\mathbf{u}, N_b\mathbf{u})\mathbf{n} = 0 \text{ on } \Gamma_n\Big\}\\ \text{and} \quad A_0\mathbf{u} = P\operatorname{div}\sigma(\mathbf{u}, N_b\mathbf{u}),&\end{aligned}$$

where δ_0 is introduced in Theorem 4.2. Below we prove the analyticity of the semigroup generated by the Stokes operator A_0.

Theorem 4.5. *The operator $(A_0, \mathcal{D}(A_0))$ is the generator of an analytic semigroup on $\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$ and its resolvent is compact.*

Proof. There exists $\lambda_0 > 0$ large enough such that, for all $\mathbf{u} \in \mathcal{D}(A_0)$, we have

$$\langle(\lambda_0 I - A_0)\mathbf{u}, \mathbf{u}\rangle_{\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})} \geqslant \nu\|\mathbf{u}\|^2_{\mathbf{V}^1_{\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})}. \tag{4.12}$$

Then we can use [2, Theorem 2.12, pp. 115] to conclude that $(A_0, \mathcal{D}(A_0))$ generates an analytic semigroup on $\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$.

Now, from Theorem 4.2, it follows that $-A_0^{-1}$ is a bounded operator from $\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$ into $\mathbf{V}^1_{\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$. Since the imbedding from $\mathbf{V}^1_{\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$ into $\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$ is compact, the proof is complete. □

Let us now introduce several operators which are needed to rewrite the system (4.3) in the form of an operator equation. We first introduce the Dirichlet operators $D \in \mathcal{L}(\mathbf{H}^{3/2}(\Gamma^s_{\overline{\eta}}), \mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}}))$ and $D_p \in \mathcal{L}(\mathbf{H}^{3/2}(\Gamma^s_{\overline{\eta}}), H^1_{\delta_0}(\Omega_{\overline{\eta}}))$, defined by

$$(D\mathbf{g}, D_p\mathbf{g}) = (\mathbf{w}, \pi), \tag{4.13}$$

where $(\mathbf{w}, \pi)$ is the solution to system (4.3) when $\mathbf{f} \equiv \mathbf{0}$.

In order to rewrite the stationary Stokes equation (4.3) as an operator equation, we need to use the so-called extrapolation method. We can easily verify that A_0 can be considered as an isomorphism from $\mathcal{D}(A_0)$ into $\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$, and that $(A_0, \mathcal{D}(A_0))$ is a self-adjoint operator in $\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$. Thus, using the so-called transposition method A_0 can also be considered as an isomorphism from $\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$ into $(\mathcal{D}(A_0))'$ (where $(\mathcal{D}(A_0))'$ is the dual of $\mathcal{D}(A_0)$ with $\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$ as pivot space). By this way we can say that A_0 with domain $\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$ is an unbounded operator in $(\mathcal{D}(A_0))'$. That corresponds to a particular use of the extrapolation method.

We now rewrite the stationary Stokes equation (4.3) as an operator equation, thanks to the following proposition.

Proposition 4.6. *Let* $\mathbf{f}$ *belong to* $\mathbf{L}^2(\Omega_{\overline{\eta}})$ *and* $\mathbf{g}$ *belong to* $\mathbf{H}^{3/2}(\Gamma^s_{\overline{\eta}})$. *A pair* $(\mathbf{w},\pi) \in \mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}}) \times H^1_{\delta_0}(\Omega_{\overline{\eta}})$ *is a solution to equation* (4.3) *if and only if*

$$\begin{aligned} &-A_0 P\mathbf{w} + A_0 PD\mathbf{g} = P\mathbf{f} \quad \text{in } (\mathcal{D}(A_0))', \\ &(I-P)\mathbf{w} = (I-P)D\mathbf{g}, \quad \text{and } \pi = N_b(\mathbf{w}) + N_p(\mathbf{f}). \end{aligned} \tag{4.14}$$

Proof. First notice that, since $P\mathbf{w} - PD\mathbf{g}$ belongs to $\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$, the first equation in (4.14) is well defined in $(\mathcal{D}(A_0))'$.

Let $(\mathbf{w},\pi) \in \mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}}) \times H^1_{\delta_0}(\Omega_{\overline{\eta}})$ be the solution to equation (4.3). We set

$$\widehat{\mathbf{w}} = \mathbf{w} - D\mathbf{g} \quad \text{and} \quad \widehat{\pi} = \pi - D_p\mathbf{g}.$$

The pair $(\widehat{\mathbf{w}},\widehat{\pi})$ satisfies

$$-\text{div } \sigma(\widehat{\mathbf{w}},\widehat{\pi}) = \mathbf{f}, \quad \text{div } \widehat{\mathbf{w}} = 0, \text{ in } \Omega_{\overline{\eta}}, \quad \widehat{\mathbf{w}} = \mathbf{0} \text{ on } \Gamma^s_{\overline{\eta}}, \quad \sigma(\widehat{\mathbf{w}},\widehat{\pi})\mathbf{n} = 0 \text{ on } \Gamma_n. \tag{4.15}$$

It implies that $\widehat{\mathbf{w}} \in D(A_0)$, and we have $-A_0P\widehat{\mathbf{w}} = -A_0\widehat{\mathbf{w}} = P\mathbf{f}$. Since $(I-P)\mathbf{w} = (I-P)(P\widehat{\mathbf{w}} + D\mathbf{g}) = (I-P)D\mathbf{g}$, we obtain

$$-A_0P\mathbf{w} + A_0PD\mathbf{g} = P\mathbf{f}, \quad (I-P)\mathbf{w} = (I-P)D\mathbf{g}.$$

Now we want to determine the expression of the pressure π. Let $\xi \in L^2(\Omega_{\overline{\eta}})$ and $\varphi \in H^2(\Omega_{\overline{\eta}})$ be the solution of (4.11). We first notice that the function $\widehat{q} = N_p(\mathbf{f})$ and φ obey

$$\int_{\Omega_{\overline{\eta}}} \mathbf{f}\cdot\nabla\varphi = \int_{\Omega_{\overline{\eta}}} \nabla\widehat{q}\cdot\nabla\varphi.$$

With a Green's formula, we have

$$-\int_{\Omega_{\overline{\eta}}} \nabla\widehat{q}\cdot\nabla\varphi = \int_{\Omega_{\overline{\eta}}} \widehat{q}\,\Delta\varphi = \int_{\Omega_{\overline{\eta}}} N_p(\mathbf{f})\,\xi.$$

Multiplying $(4.3)_1$ by $\nabla\varphi$, where φ is the solution to (4.11), and integrating over $\Omega_{\overline{\eta}}$, we obtain

$$-\int_{\Omega_{\overline{\eta}}} \mathbf{f}\cdot\nabla\varphi - \int_{\Omega_{\overline{\eta}}} \text{div } \sigma(\mathbf{w},\pi)\cdot\nabla\varphi = 0. \tag{4.16}$$

Integrating by parts and using (4.11), we deduce that

$$-\int_{\Omega_{\overline{\eta}}} \mathbf{f}\cdot\nabla\varphi + 2\nu\int_{\Omega_{\overline{\eta}}} \boldsymbol{\varepsilon}(\mathbf{w}) : \nabla^2\varphi - 2\nu\int_{\Gamma^s_{\overline{\eta}}} \boldsymbol{\varepsilon}(\mathbf{w})\mathbf{e}_{\overline{\eta}}\cdot\nabla\varphi - \int_{\Omega_{\overline{\eta}}} \pi\,\xi = 0.$$

By combining the above identities and by using the definition of the operators N_p and N_b, we have

$$\int_{\Omega_{\overline{\eta}}} \pi\,\xi = \int_{\Omega_{\overline{\eta}}} N_b(\mathbf{w})\,\xi + \int_{\Omega_{\overline{\eta}}} N_p(\mathbf{f})\,\xi \quad \text{for all } \xi \in L^2(\Omega_{\overline{\eta}}).$$

Therefore,

$$\pi = N_b(\mathbf{w}) + N_p(\mathbf{f}).$$

To prove the converse statement, we notice that equation (4.14) admits at most one solution. Indeed if $-A_0P\mathbf{w} = 0$ and $(I-P)\mathbf{w} = 0$, then $\mathbf{w} = 0$. Thus equation (4.14) admits a unique solution, which is the solution to (4.3). □

4.2. **Damped shell equation.** In this subsection we study only the structure equation:

$$\begin{aligned} &\eta_{1,t} - \eta_2 = 0 \quad \text{and} \quad \eta_{2,t} + \Delta_s^2\eta_1 - \Delta_s\eta_2 = 0 \text{ in } (0,T)\times\omega, \\ &\eta_1(0) = \eta_1^0, \quad \eta_2(0) = \eta_2^0 \text{ in } \omega, \\ &\eta(t,0,\theta) = \eta(t,L,\theta) = \eta_z(t,0,\theta) = \eta_z(t,L,\theta) = 0,\ \forall t \in (0,T),\ \theta \in [0,2\pi]. \end{aligned} \tag{4.17}$$

Let us introduce the Hilbert space

$$H_s = H^2_0(\omega) \times L^2(\omega), \tag{4.18}$$

equipped with the inner product

$$((\eta_1,\eta_2),(\xi_1,\xi_2))_{H_s} = (-\Delta_s\eta_1,-\Delta_s\xi_1)_{L^2(\omega)} + (\eta_2,\xi_2)_{L^2(\omega)}.$$

We now define the unbounded operator $(A_s,\mathcal{D}(A_s;H_s))$ in H_s by

$$\begin{aligned}&\mathcal{D}(A_s;H_s) = \left(H^4(\omega)\cap H_0^2(\omega)\right)\times H_0^2(\omega) \quad \text{and}\\ &A_s = \begin{pmatrix} 0 & I\\ -\Delta_s^2 & \Delta_s\end{pmatrix}.\end{aligned} \tag{4.19}$$

Proposition 4.7. *The unbounded operator $(A_s,\mathcal{D}(A_s;H_s))$ generates an analytic semigroup of contractions on H_s.*

Proof. See [8, Proposition 3.1]. □

We have the following regularity result.

Proposition 4.8. *If (η_1^0,η_2^0) belongs to $H_0^2(\omega)\times L^2(\omega)$, then system (4.17) admits a unique solution (η_1,η_2) belonging to $H^{3,3/2}(\omega_T)\times H^{1,1/2}(\omega_T)$.*

Proof. Let us set $V = [\mathcal{D}(A_s;H_s),H_s]_{1/2} = \left(H^3(\omega)\cap H_0^2(\omega)\right)\times H_0^1(\omega)$. Let V' be the dual of V with H_s as pivot space, we have

$$V' = H_0^1(\omega)\times H^{-1}(\omega).$$

Since $(A_s,\mathcal{D}(A_s;H_s))$ generates an analytic semigroup on H_s, for $(\eta_1^0,\eta_2^0)\in H_s$, the solution (η_1,η_2) to system (4.17) belongs to $L^2(0,T;V)\cap H^1(0,T;V')$ (see [2, Part II, Chapter 3, Corollary 2.1]). Thus, we have

$$\begin{aligned}&\eta_1 \in L^2(0,T;H^3(\omega))\cap H^1(0,T;H_0^1(\omega)),\\ &\eta_2 = \eta_{1,t} \in L^2(0,T;H_0^1(\omega))\cap H^1(0,T;H^{-1}(\omega)).\end{aligned}$$

Finally, by interpolation we obtain $\eta_1\in H^{3,3/2}(\omega_T)$ and $\eta_2\in H^{1,1/2}(\omega_T)$. □

4.3. **Rewriting system (4.2) as an evolution equation.** In this subsection we rewrite the system (4.2) when $g=0$ as an evolution equation. Let us set

$$D_s\eta_2 = D(\eta_2\mathbf{e}_r), \tag{4.20}$$

where D is defined in (4.13).

We also introduce the operator $N_s\in\mathcal{L}(L^2(\Gamma_{\overline{\eta}}^s),H^1(\Omega_{\overline{\eta}}))$ defined by $N_sh = q$, where $q\in H^1(\Omega_{\overline{\eta}})$ is the solution to the equation

$$\Delta q = 0 \text{ in } \Omega_{\overline{\eta}},\quad \frac{\partial q}{\partial\mathbf{n}} = h \text{ on } \Gamma_{\overline{\eta}}^s,\quad q = 0 \text{ on } \Gamma_n. \tag{4.21}$$

Remark 4.9. *Due to Remark 1.2, the operator D_s defined in (4.20) can be viewed as a bounded operator from $\mathbf{H}^{3/2}(\Omega_{\overline{\eta}})$ onto $\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}})$, i.e. $D_s\in\mathcal{L}(\mathbf{H}^{3/2}(\omega),\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}}))$. Similarly, the operator N_s defined in (4.21) can be viewed as an operator belonging to $\mathcal{L}(L^2(\omega),H^1(\Omega_{\overline{\eta}}))$.*

We have the following proposition.

Proposition 4.10. *Let us assume that $\eta_2\in H^{2,1}(\omega_T)$. A pair $(\mathbf{v},p)\in L^2(0,T;\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}}))\cap H^1(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))\times L^2(0,T;H^1_{\delta_0}(\Omega_{\overline{\eta}}))$ is a solution of system*

$$\begin{aligned}&\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},p) = \mathbf{0},\quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_{\overline{\eta}}^T,\\ &\mathbf{v} = \eta_2\mathbf{e}_r \quad \text{on } \Gamma_{\overline{\eta}}^s\times(0,T),\quad \sigma(\mathbf{v},p)\mathbf{n} = 0 \text{ on } \Sigma_T^{\text{in}}\cup\Sigma_T^{\text{out}},\\ &\mathbf{v}(0) = \mathbf{v}_0 \text{ in } \Omega_{\overline{\eta}},\end{aligned} \tag{4.22}$$

if and only if

$$\begin{aligned}&P\mathbf{v}' = A_0P\mathbf{v} - A_0PD_s\eta_2,\quad P\mathbf{v}(0) = P\mathbf{v}_0 \text{ in } \Omega_{\overline{\eta}},\\ &(I-P)\mathbf{v} = (I-P)D_s\eta_2,\quad \text{and}\quad p = N_b(\mathbf{v}) - N_s\eta_{2,t}.\end{aligned} \tag{4.23}$$

Proof. According to Proposition 4.6, the system (4.22) is equivalent to

$$P\mathbf{v}' = A_0 P\mathbf{v} - A_0 P D_s \eta_2, \quad P\mathbf{v}(0) = P\mathbf{v}_0 \text{ in } \Omega_{\overline{\eta}},$$
$$(I-P)\mathbf{v} = (I-P)D_s\eta_2, \quad p = N_b(\mathbf{v}) + N_p(-\mathbf{v}_t).$$

From the definitions of the operator N_p (defined in (4.8)) and of N_s in (4.21), one can easily verify that

$$N_p(-\mathbf{v}_t) = -N_s\eta_{2,t}.$$

Thus the proof is complete. □

Using the expression of the pressure obtained in Proposition 4.10, we can now rewrite the equation $(4.2)_6$ satisfied by ζ_2 when $g = 0$ as follows

$$(I + \boldsymbol{\gamma}^s_{\overline{\eta}} N_s)\zeta_{2,t} + \Delta_s^2\zeta_1 - \Delta_s\zeta_2 = \boldsymbol{\gamma}^s_{\overline{\eta}} N_b(\mathbf{v}) - \boldsymbol{\gamma}^s_{\overline{\eta}}\Big(2\nu\boldsymbol{\varepsilon}(\mathbf{v})\mathbf{e}_{\overline{\eta}}\Big)\cdot\mathbf{e}_r. \tag{4.24}$$

We want to rewrite the system (4.2) as an evolution equation for $(P\mathbf{v}, \zeta_1, \zeta_2)$ when $g = 0$. Therefore we need to express the right hand side of (4.24) in terms $P\mathbf{v}$, ζ_1 and ζ_2. By writing $\mathbf{v} = P\mathbf{v} + (I-P)\mathbf{v}$ and using the fact that $(I-P)\mathbf{v} = (I-P)D_s\zeta_2$, (4.24) can be written as

$$\begin{aligned}&(I + \boldsymbol{\gamma}^s_{\overline{\eta}} N_s)\zeta_{2,t} + \Delta_s^2\zeta_1 - \Delta_s\zeta_2\\ &= \boldsymbol{\gamma}^s_{\overline{\eta}} N_b(P\mathbf{v} - PD_s\zeta_2 + D_s\zeta_2) - \boldsymbol{\gamma}^s_{\overline{\eta}}\Big(2\nu\boldsymbol{\varepsilon}(P\mathbf{v} - PD_s\zeta_2 + D_s\zeta_2)\mathbf{e}_r\Big)\cdot\mathbf{e}_r.\end{aligned} \tag{4.25}$$

Lemma 4.11. *The operator $K_s = I + \boldsymbol{\gamma}^s_{\overline{\eta}} N_s \in \mathcal{L}(L^2(\omega))$ is an automorphism in $H^{\varepsilon_0}(\omega)$.*

Proof. The proof can be adapted from that of [32, Lemma 3.2]. □

Now we are in a position to write the system satisfied by $(P\mathbf{v}, \eta_1, \eta_2)$ as an evolution equation. Let us recall that the space H_s and the unbounded operator A_s are defined in (4.18) and (4.19). We equip the Hilbert space

$$\mathbf{H} = \mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}) \times H^2_0(\omega) \times L^2(\omega) = \mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}) \times H_s, \tag{4.26}$$

with the inner product

$$((\mathbf{v},\eta_1,\eta_2),(\mathbf{u},\xi_1,\xi_2))_{\mathbf{H}} = (\mathbf{v},\mathbf{u})_{\mathbf{L}^2(\Omega_{\overline{\eta}})} + (-\Delta_s\eta_1, -\Delta_s\xi_1)_{L^2(\omega)} + (\eta_2,\xi_2)_{L^2(\omega)}.$$

We define the unbounded operator $(\mathcal{A}, \mathcal{D}(\mathcal{A};\mathbf{H}))$ in $\mathbf{H}$ by

$$\begin{aligned}\mathcal{D}(\mathcal{A};\mathbf{H}) = \Big\{&(P\mathbf{v},\eta_1,\eta_2) \in \mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}) \times \mathcal{D}(A_s;H_s)\\ &\mid A_0(P\mathbf{v} - PD_s\eta_2) \in \mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})\Big\},\end{aligned}$$

and $\mathcal{A} = \mathcal{A}_1 + \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4$, with

$$\mathcal{A}_1 = \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I \\ 0 & -\Delta_s^2 & \Delta_s \end{pmatrix}, \tag{4.27}$$

$$\mathcal{B}_1\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ K_s^{-1}\gamma^s_{\overline{\eta}} N_b(P\mathbf{v} - PD_s\eta_2 + D_s\eta_2) \end{pmatrix}, \tag{4.28}$$

$$\mathcal{B}_2\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -K_s^{-1}\Big[\gamma^s_{\overline{\eta}}\Big(2\nu\boldsymbol{\varepsilon}(P\mathbf{v} - PD_s\eta_2 + D_s\eta_2)\mathbf{e}_r\Big)\cdot\mathbf{e}_r\Big] \end{pmatrix} \tag{4.29}$$

and

$$\mathcal{B}_3\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(K_s^{-1} - I)\Delta_s^2\eta_1 \end{pmatrix}, \quad \mathcal{B}_4\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (K_s^{-1} - I)\Delta_s\eta_2 \end{pmatrix}. \tag{4.30}$$

Combining Proposition 4.10 and the operators introduced above, we write system (4.2) when $g = 0$ as an evolution equation.

Theorem 4.12. *Let $\mathbf{v} \in L^2(0,T;\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}})) \cap H^1(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))$, $p \in L^2(0,\infty;H^1_{\delta_0}(\Omega_{\overline{\eta}}))$, $\eta_1 \in H^{4,2}(\omega_T)$, $\eta_2 \in H^{2,1}(\omega_T)$. Then $(\mathbf{v},p,\eta_1,\eta_2)$ is a solution of (4.2) if and only if*

$$\begin{aligned}
&\frac{d}{dt}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, \quad \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}_0 \\ 0 \\ \eta_2^0 \end{pmatrix}, \\
&(I-P)\mathbf{v} = (I-P)D_s\eta_2, \\
&p = N_b(P\mathbf{v} - PD_s\eta_2 + D_s\eta_2) - N_s\eta_{2,t}.
\end{aligned} \tag{4.31}$$

In the following subsections we will show that $(\mathcal{A},\mathcal{D}(\mathcal{A};\mathbf{H}))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{H}$.

4.4. **Analyticity of $(e^{t\mathcal{A}})_{t\geqslant 0}$ on $\mathbf{H}$.** In this subsection, we prove the following theorem.

Theorem 4.13. *The operator $(\mathcal{A},\mathcal{D}(\mathcal{A};\mathbf{H}))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{H}$.*

The proof of this theorem is divided into several parts. We first show that the operator $\mathcal{A}_1$, defined in (4.27), generates an analytic semigroup. Then we show that the operators $\mathcal{B}_i$, for $i = 1,2,3,4$, can be treated as perturbations of the operator $\mathcal{A}_1$.

For $\lambda_0 \in \mathbb{R}$ and $\varepsilon \in (0,\pi)$, let us define the sector $S_{\varepsilon,\lambda_0}$ by

$$S_{\varepsilon,\lambda_0} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - \lambda_0)| < \varepsilon\}.$$

Theorem 4.14. *There exists $\lambda_0 \in \mathbb{R}$ and $\varepsilon \in (\pi/2,\pi)$ such that the sector $S_{\varepsilon,\lambda_0}$ is contained in the resolvent set of $\mathcal{A}_1$ and there exists a constant $C > 0$ such that*

$$\left\|(\lambda I - \mathcal{A}_1)^{-1}\right\| \leqslant \frac{C}{|\lambda|}, \quad \text{for all } \lambda \in S_{\varepsilon,\lambda_0}.$$

In particular, the operator $(\mathcal{A}_1,\mathcal{D}(\mathcal{A};\mathbf{H}))$ is infinitesimal generator of an analytic semigroup on $\mathbf{H}$.

Proof. We can prove this result with the help of Theorem 4.5, Proposition 4.7, Proposition 4.8 by following [32, Theorem 3.5] and [32, Theorem 3.6] . □

Lemma 4.15. *The operators $(\mathcal{B}_1,\mathcal{D}(\mathcal{A};\mathbf{H}))$ and $(\mathcal{B}_2,\mathcal{D}(\mathcal{A};\mathbf{H}))$, defined in (4.28) and (4.29), are $\mathcal{A}_1$-bounded with relative bound zero, i.e., for all $\delta > 0$, there exists $C_\delta > 0$ such that*

$$\|\mathcal{B}_1(P\mathbf{v},\eta_1,\eta_2)^\top\|_{\mathbf{H}} + \|\mathcal{B}_2(P\mathbf{v},\eta_1,\eta_2)^\top\|_{\mathbf{H}} \leqslant \delta\|\mathcal{A}_1(P\mathbf{v},\eta_1,\eta_2)^\top\|_{\mathbf{H}} + C_\delta\|(P\mathbf{v},\eta_1,\eta_2)^\top\|_{\mathbf{H}},$$

for all $(P\mathbf{v},\eta_1,\eta_2) \in \mathcal{D}(\mathcal{A};\mathbf{H})$ (see [19]).

Proof. Let us first prove that, for all $(P\mathbf{v},\eta_1,\eta_2) \in \mathcal{D}(\mathcal{A};\mathbf{H})$, $N_b(P\mathbf{v} - PD_s\eta_2 + D_s\eta_2)$ belongs to $H^{1/2+\varepsilon_0}(\Omega_{\overline{\eta}})$. Since $(\mathcal{A}_1,\mathcal{D}(\mathcal{A}_1;\mathbf{H}))$ generates a semigroup on $\mathbf{H}$, for all $(\mathbf{f},g,h)^\top \in \mathbf{H}$, we have a unique solution $(P\mathbf{v},\eta_1,\eta_2) \in \mathcal{D}(\mathcal{A}_1;\mathbf{H})$ to the following system

$$(\lambda I - \mathcal{A}_1)\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ g \\ h \end{pmatrix}, \quad (I-P)\mathbf{v} = PD_s\eta_2,$$

for some $\lambda > 0$. Notice that $(P\mathbf{v},\eta_1,\eta_2)$ satisfies the above system if and only if $(\mathbf{v},p,\eta_1,\eta_2)$ satisfies the following system

$$\begin{aligned}
&\lambda\mathbf{v} - \operatorname{div}\sigma(\mathbf{v},p) = \mathbf{f}, \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega_{\overline{\eta}}, \\
&\mathbf{v} = \eta_2\mathbf{e}_r \text{ on } \Gamma^s_{\overline{\eta}}, \quad \sigma(\mathbf{v},p)\mathbf{n} = 0 \text{ on } \Gamma_n, \\
&\lambda\eta_1 - \eta_2 = g \text{ in } \omega, \\
&\lambda\eta_2 + \Delta_s^2\eta_1 - \Delta_s\eta_2 = h \quad \text{in } \omega, \\
&\eta(0,\theta) = \eta(L,\theta) = \tfrac{\partial\eta}{\partial z_1}(0,\theta) = \tfrac{\partial\eta}{\partial z_1}(L,\theta) = 0, \quad \text{for all } \theta \in (0,2\pi).
\end{aligned} \tag{4.32}$$

Let us note that the solution $(\mathbf{v}, p, \eta_1, \eta_2)$ to the above system can be determined by solving first the system satisfied by (η_1, η_2) and next the system satisfied by $(\mathbf{v}, p)$. For every $(g, h) \in H_0^2(\omega) \times L^2(\omega)$, it is easy to check that (η_1, η_2) belongs to $H^4(\omega) \cap H_0^2(\omega) \times H_0^2(\omega)$. Next, using Theorem 4.2, we obtain $(\mathbf{v}, p) \in \mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\overline{\eta}}) \times H^{1/2+\varepsilon_0}(\Omega_{\overline{\eta}})$. Proceeding as in Proposition 4.10, the pressure p in (4.32) can be expressed as

$$p = N_b(P\mathbf{v} - PD_s\eta_2 + D_s\eta_2) - \lambda N_s\eta_2.$$

Since $N_s \in \mathcal{L}(L^2(\omega), H^1(\Omega_{\overline{\eta}}))$, we obtain that $N_b(P\mathbf{v} - PD_s\eta_2 + D_s\eta_2)$ belongs to $H^{1/2+\varepsilon_0}(\Omega_{\overline{\eta}})$. Thus Lemma 4.11 yields that $K_s^{-1}\gamma_{\overline{\eta}}^s N_b(P\mathbf{v} - PD_s\eta_2 + D_s\eta_2)$ belongs to $H^{\varepsilon_0}(\Omega_{\overline{\eta}})$. Therefore the operator $\mathcal{B}_1 \in \mathcal{L}(\mathcal{D}(\mathcal{A}; \mathbf{H}), \mathbf{H})$ is a compact operator.

Since $P\mathbf{v} - PD_s\eta_2 + D_s\eta_2$ belongs to $\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\overline{\eta}})$, using Lemma 4.11 it is easy to see that

$$K_s^{-1}\Big[\gamma_{\overline{\eta}}^s\Big(2\nu\boldsymbol{\varepsilon}(P\mathbf{v} - PD_s\eta_2 + D_s\eta_2)\mathbf{e}_{\overline{\eta}}\Big) \cdot \mathbf{e}_r\Big] \in H^{\varepsilon_0}(\omega).$$

Thus $\mathcal{B}_2$ is also a compact operator. Therefore $\mathcal{B}_1$ and $\mathcal{B}_2$ are $\mathcal{A}_1$-bounded with relative bound zero (see [12, Section 3.4, Lemma 2.13]). □

Lemma 4.16. *There exists $0 \leqslant \theta < 1$ such that the operators $(\mathcal{B}_3, \mathcal{D}(\mathcal{A}; \mathbf{H}))$ and $(\mathcal{B}_4, \mathcal{D}(\mathcal{A}; \mathbf{H}))$ are bounded from $\mathcal{D}((-\mathcal{A}_1)^\theta)$ into $\mathbf{H}$.*

Proof. The proof may be adapted from that of [32, Lemma 3.9]. □

Proof of Theorem 4.13. The proof of Theorem 4.13 follows from Theorem 4.14, Lemma 4.15 and Lemma 4.16. □

5. Nonhomogeneous linear system

Throughout this section, we assume that $\overline{\eta}$ satisfies (4.1).

To transform the system (4.2) when $g \neq 0$ into an equivalent one in which g will be zero, we consider the following problem

$$\begin{aligned} &-\operatorname{div}\sigma(\mathbf{w}, q) = -\nu\nabla g, \quad \operatorname{div}\mathbf{w} = g \text{ in } \Omega_{\overline{\eta}}, \\ &\mathbf{w} = 0 \text{ on } \Gamma_{\overline{\eta}}^s, \quad \sigma(\mathbf{w}, q)\mathbf{n} = 0 \text{ on } \Gamma_n. \end{aligned} \tag{5.1}$$

We first recall the following regularity result.

Proposition 5.1. *Let g belong to $L^2(0,T; H^1(\Omega_{\overline{\eta}})) \cap H^1(0,T; (H^1_{\delta_0}(\Omega_{\overline{\eta}}))')$ and satisfy $g|_{\Gamma_n} = 0$. The system (5.1) admits a unique solution $(\mathbf{w}, q) \in \mathbf{H}^{2,1}_{\delta_0}(Q_T) \times L^2(0,T; H^1_{\delta_0}(\Omega_{\overline{\eta}}))$ and*

$$\begin{aligned} &\|\mathbf{w}\|_{\mathbf{H}^{2,1}_{\delta_0}(Q^T_{\overline{\eta}})} + \|q\|_{L^2(0,T;H^1_{\delta_0}(\Omega_{\overline{\eta}}))} \leqslant C\|g\|_{L^2(0,T;H^1(\Omega_{\overline{\eta}}))\cap H^1(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}}))')}, \\ &\|\mathbf{w}(0)\|_{\mathbf{H}^1(\Omega_{\overline{\eta}})} \leqslant C\|g(0)\|_{L^2(\Omega_{\overline{\eta}})}, \end{aligned}$$

where the constant C is independent of time T and δ_0 is defined as in Theorem 4.2.

Proof. Using the same arguments as in the proof of Theorem 4.2, the proposition follows from [25, Theorem 9.4.5] and Lemma A.3. □

Theorem 5.2. *For all $(\mathbf{v}_0, \zeta_1^0, \zeta_2^0) \in \mathbf{H}^1(\Omega_{\overline{\eta}}) \times (H^3(\omega) \cap H_0^2(\omega)) \times H_0^1(\omega)$, all $\mathbf{f} \in L^2(0,T; \mathbf{L}^2(\Omega_{\overline{\eta}}))$, $g \in L^2(0,T; H^1(\Omega_{\overline{\eta}})) \cap H^1(0,T; (H^1_{\delta_0}(\Omega_{\overline{\eta}}))')$ with $g|_{\Gamma_n} = 0$, and $h \in L^2(0,T; L^2(\omega))$, satisfying*

$$\operatorname{div}\mathbf{v}_0 = g(0,\cdot) \text{ in } \Omega_{\overline{\eta}}, \quad \mathbf{v}_0(z_1, 1, \theta) = \zeta_2^0(z_1, \theta)\mathbf{e}_r \text{ for all } z_1 \in (0, L), \theta \in (0, 2\pi),$$

the system (4.2) admits a unique solution $(\mathbf{v}, p, \zeta_1, \zeta_2) \in \mathbf{H}^{2,1}_{\delta_0}(\Omega_{\overline{\eta}} \times (0,T)) \times L^2(0,T; H^1_{\delta_0}(\Omega_{\overline{\eta}})) \times H^{4,2}(\omega \times (0,T)) \times H^{2,1}(\omega \times (0,T))$. Moreover, there exists a constant $C_{\overline{\eta}}$, depending on $\overline{\eta}$ but independent of $T > 0$, such

that

$$\begin{aligned}\|\mathbf{v}\|_{\mathbf{H}^{2,1}_{\delta_0}(\Omega_{\overline{\eta}}\times(0,T))} &+ \|\mathbf{v}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_{\overline{\eta}}))} + \|p\|_{L^2(0,T;H^1_{\delta_0}(\Omega_{\overline{\eta}}))} + \|\zeta_1\|_{H^{4,2}(\omega\times(0,T))}\\ &+ \|\zeta_1\|_{L^\infty(0,T;H^3(\omega))} + \|\zeta_2\|_{H^{2,1}(\omega\times(0,T))} + \|\zeta_2\|_{L^\infty(0,T;H^1(\omega))}\\ &\leqslant C_{\overline{\eta}}\Big(\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega_{\overline{\eta}})} + \|\zeta_1^0\|_{H^3(\omega)} + \|\zeta_2^0\|_{H^1_0(\omega)} + \|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))}\\ &\qquad + \|g\|_{L^2(0,T;H^1(\Omega_{\overline{\eta}}))} + \|g\|_{H^1(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}}))')} + \|h\|_{L^2(0,T;L^2(\omega))}\Big). \end{aligned} \tag{5.2}$$

If, in addition, h belongs to $L^2(0,T;H^{\varepsilon_0}(\omega))$, then the function $t\mapsto \min(t,1)\zeta_1(t)$ satisfies

$$\begin{aligned}\|t\mapsto \min(t,1)\zeta_1(t)\|_{C([0,T];H^{3+\varepsilon_0}(\omega))} &\\ \leqslant C_{\overline{\eta}}\Big(\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega_{\overline{\eta}})} &+ \|\zeta_1^0\|_{H^3(\omega)} + \|\zeta_2^0\|_{H^1_0(\omega)} + \|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))}\\ &+ \|g\|_{L^2(0,T;H^1(\Omega_{\overline{\eta}}))} + \|g\|_{H^1(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}}))')} + \|h\|_{L^2(0,T;H^{\varepsilon_0}(\omega))}\Big). \end{aligned} \tag{5.3}$$

Remark 5.3. *The estimate in (5.3) yields that, for all $t\in(0,\min(T,1)]$, we have*

$$\begin{aligned}&\|\zeta_1(t)\|_{H^{3+\varepsilon_0}(\omega)}\\ &\leqslant \frac{C_{\overline{\eta}}}{t}\Big(\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega_{\overline{\eta}})} + \|\zeta_1^0\|_{H^3(\omega)} + \|\zeta_2^0\|_{H^1_0(\omega)} + \|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))}\\ &+\|g\|_{L^2(0,T;H^1(\Omega_{\overline{\eta}}))} + \|g\|_{H^1(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}}))')} + \|h\|_{L^2(0,T;H^{\varepsilon_0}(\omega))}\Big).\end{aligned}$$

Since the regularity results used to prove (5.2) and (5.3) are based on the regularity results obtained in Section 4, the constant $C_{\overline{\eta}}$ depends on $\|\overline{\eta}\|_{H^{3+\varepsilon_0}(\omega)}$ and on $\gamma_{\overline{\eta}}>0$.

Proof. Step 1. Proof of (5.2), except the $L^\infty(0,T;\mathbf{H}^1(\Omega_{\overline{\eta}}))$-estimate. We set $\widetilde{\mathbf{v}}=\mathbf{v}-\mathbf{w}$ and $\widetilde{p}=p-q$, where $(\mathbf{w},q)$ is the solution of the equation (5.1). Proceeding as in Theorem 4.12, it is easy to see that $(\widetilde{\mathbf{v}},\widetilde{p},\zeta_1,\zeta_2)$ is the solution to

$$\begin{aligned}&\frac{d}{dt}\begin{pmatrix}P\widetilde{\mathbf{v}}\\ \zeta_1\\ \zeta_2\end{pmatrix} = \mathcal{A}\begin{pmatrix}P\widetilde{\mathbf{v}}\\ \zeta_1\\ \zeta_2\end{pmatrix} + \begin{pmatrix}P\mathbf{F}\\ 0\\ H\end{pmatrix}, \quad \begin{pmatrix}P\widetilde{\mathbf{v}}(0)\\ \zeta_1(0)\\ \zeta_2(0)\end{pmatrix} = \begin{pmatrix}P(\mathbf{v}_0-\mathbf{w}(0))\\ \zeta_1^0\\ \zeta_2^0\end{pmatrix},\\ &(I-P)\widetilde{\mathbf{v}} = (I-P)D_s\zeta_2,\\ &\widetilde{p} = N_b(P\widetilde{\mathbf{v}}-PD_s\zeta_2+D_s\zeta_2) - N_s\zeta_{2,t} + q_1+q_2,\end{aligned} \tag{5.4}$$

where

$$\mathbf{F}=\mathbf{f}-\mathbf{w}_t, \quad H = K_s^{-1}\left(-\gamma^s_{\overline{\eta}}\Big(\sigma(\mathbf{w},q)\mathbf{e}_{\overline{\eta}}\Big)\cdot\mathbf{e}_r + h + \gamma_s(q_1+q_2)\right), \tag{5.5}$$

with

$$\begin{aligned}&q_1\in H^1_0(\Omega_{\overline{\eta}}), \quad \Delta q_1 = \operatorname{div}\mathbf{F} \text{ in } \Omega_{\overline{\eta}}, \quad \text{and}\\ &q_2\in H^1_{\Gamma_n}(\Omega_{\overline{\eta}}), \quad \Delta q_2 = 0 \text{ in } \Omega_{\overline{\eta}}, \quad \frac{\partial q_2}{\partial\mathbf{n}} = (\mathbf{F}-\nabla q_1)\cdot\mathbf{n} \text{ on } \Gamma^s_{\overline{\eta}}.\end{aligned}$$

We can easily verify that $P(\mathbf{v}_0-\mathbf{w}(0)-D_s\zeta_2)=\mathbf{v}_0-\mathbf{w}(0)-D_s\zeta_2\in\mathbf{V}^1_{\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})$. From the expression of $\mathbf{F}$ and H in (5.5) and using Proposition 5.1, it is easy to see that

$$\begin{aligned}&\|P\mathbf{F}\|_{L^2(0,T;\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}))} + \|H\|_{L^2(0,T;L^2(\omega))}\\ &\leqslant C\big(\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))} + \|g\|_{L^2(0,T;H^1(\Omega_{\overline{\eta}}))\cap H^1(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}}))')} + \|h\|_{L^2(0,T;L^2(\omega))}\big),\end{aligned} \tag{5.6}$$

where the constant C is independent of T. To obtain the regularity of $(P\widetilde{\mathbf{v}},\zeta_1,\zeta_2)$, we will use a maximal regularity result for analytic semigroups (see [2, Theorem 3.1, Chapter 1]). In order to obtain estimates with

continuity constant independent of time T, we have to distinguish the cases where $T \leqslant 1$ and $T > 1$. For that, we set $\widehat{T} = \max(1, T)$. If $T \geqslant 1$, we set $\overline{\mathbf{F}} = \mathbf{F}$ and $\overline{H} = H$. If $T < 1$, we set

$$\overline{\mathbf{F}} = \begin{cases} \mathbf{F} & \text{if } 0 < t \leqslant T \\ 0 & \text{if } T < t \leqslant \widehat{T} \end{cases}, \qquad \overline{H} = \begin{cases} H & \text{if } 0 < t \leqslant T \\ 0 & \text{if } T < t \leqslant \widehat{T} \end{cases}.$$

Obviously, $P\overline{\mathbf{F}} \in L^2(0,\widehat{T}; \mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}))$ and $\overline{H} \in L^2(0,\widehat{T}; L^2(\omega))$. Instead of (5.4), we consider the following problem

$$\frac{d}{dt}\begin{pmatrix} P\overline{\mathbf{v}} \\ \overline{\zeta}_1 \\ \overline{\zeta}_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} P\overline{\mathbf{v}} \\ \overline{\zeta}_1 \\ \overline{\zeta}_2 \end{pmatrix} + \begin{pmatrix} P\overline{\mathbf{F}} \\ 0 \\ \overline{H} \end{pmatrix} \quad t \in (0,\widehat{T}), \quad \begin{pmatrix} P\overline{\mathbf{v}}(0) \\ \overline{\zeta}_1(0) \\ \overline{\zeta}_2(0) \end{pmatrix} = \begin{pmatrix} P(\mathbf{v}_0 - \mathbf{w}(0)) \\ \zeta_1^0 \\ \zeta_2^0 \end{pmatrix}.$$

Let us recall that $\mathcal{A}$ generates an analytic semigroup on $\mathbf{H}$ (Theorem 4.13). As in [13, Proposition 6.14], we can show that $[\mathcal{D}(\mathcal{A};\mathbf{H}),\mathbf{H}]_{1/2} = \{(P\mathbf{v},\eta_1,\eta_2) \in \mathbf{V}^{3/4+\varepsilon_0/2}_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}) \times H^3(\omega) \cap H^2_0(\omega) \times H^1_0(\omega)) \mid (P\mathbf{v},\eta_1,\eta_2) - PD_s\eta_2 \in \mathbf{V}^1_{\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})\}$. Thus $(P(\mathbf{v}_0 - \mathbf{w}(0)), \zeta_1^0, \zeta_2^0)$ belongs to $[\mathcal{D}(\mathcal{A};\mathbf{H}),\mathbf{H}]_{1/2}$, and from [2, Theorem 3.1,Chapter 1], it follows that $(P\overline{\mathbf{v}}, \overline{\zeta}_1, \overline{\zeta}_2)$ belongs to $L^2(0,\widehat{T}; \mathcal{D}(\mathcal{A};\mathbf{H})) \cap H^1(0,\widehat{T};\mathbf{H})$ and

$$\begin{aligned} &\|P\overline{\mathbf{v}}\|_{H^1(0,\widehat{T};\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}))} + \|\overline{\zeta}_1\|_{H^{4,2}(\omega_{\widehat{T}})} + \|\overline{\zeta}_2\|_{H^{2,1}(\omega_{\widehat{T}})} \\ &\leqslant C\Big(\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega_{\overline{\eta}})} + \|\zeta_1^0\|_{H^3(\omega)} + \|\zeta_2^0\|_{H^1_0(\omega)} \\ &\qquad + \|P\mathbf{F}\|_{L^2(0,T;\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}))} + \|H\|_{L^2(0,T;L^2(\omega))}\Big), \end{aligned} \tag{5.7}$$

where the positive constant C is independent of T. Next, using the continuous embeddings $H^1(0,\widehat{T};\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})) \hookrightarrow L^\infty(0,\widehat{T};\mathbf{L}^2(\Omega_{\overline{\eta}}))$, $H^{4,2}(\omega_{\widehat{T}}) \hookrightarrow L^\infty(0,\widehat{T};H^3(\omega))$ and $H^{2,1}(\omega_{\widehat{T}}) \hookrightarrow L^\infty(0,\widehat{T};H^1(\omega))$, we obtain $P\overline{\mathbf{v}} \in L^\infty(0,\widehat{T};L^2(\Omega_{\overline{\eta}}))$, $\overline{\zeta}_1 \in L^\infty(0,\widehat{T};H^3(\omega))$ and $\overline{\zeta}_2 \in L^\infty(0,\widehat{T};H^1(\omega))$. Therefore

$$\begin{aligned} &\|P\overline{\mathbf{v}}\|_{L^\infty(0,\widehat{T};L^2(\Omega_{\overline{\eta}}))} + \|\overline{\zeta}_1\|_{L^\infty(0,\widehat{T};H^3(\omega))} + \|\overline{\zeta}_2\|_{L^\infty(0,\widehat{T};H^1(\omega))} \\ &\leqslant C(\|P\overline{\mathbf{v}}\|_{H^1(0,\widehat{T};\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}))} + \|\overline{\zeta}_1\|_{H^{4,2}(\omega_{\widehat{T}})} + \|\overline{\zeta}_2\|_{H^{2,1}(\omega_{\widehat{T}})}). \end{aligned} \tag{5.8}$$

Notice that, by construction, $(P\widetilde{\mathbf{v}}, \zeta_1, \zeta_2) = (P\overline{\mathbf{v}}, \overline{\zeta}_1, \overline{\zeta}_2)$ in $[0,T]$. Thus $(P\widetilde{\mathbf{v}}, \zeta_1, \zeta_2)$ belongs to $H^1(0,T;\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})) \cap L^\infty(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}})) \times H^{4,2}(\omega_T) \cap L^\infty(0,T;H^3(\omega)) \times H^{2,1}(\omega_T) \cap L^\infty(0,T;H^1(\omega))$ and using (5.7) - (5.8), we obtain

$$\begin{aligned} &\|P\widetilde{\mathbf{v}}\|_{H^1(0,T;\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})) \cap L^\infty(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))} + \|\zeta_1\|_{H^{4,2}(\omega_T) \cap L^\infty(0,T;H^3(\omega))} \\ &\qquad + \|\zeta_2\|_{H^{2,1}(\omega_T) \cap L^\infty(0,T;H^1(\omega))} \\ &\leqslant \|P\overline{\mathbf{v}}\|_{H^1(0,\widehat{T};\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}})) \cap L^\infty(0,\widehat{T};L^2(\Omega_{\overline{\eta}}))} + \|\overline{\zeta}_1\|_{H^{4,2}(\omega_{\widehat{T}}) \cap L^\infty(0,\widehat{T};H^3(\omega))} \\ &\qquad + \|\overline{\zeta}_2\|_{H^{2,1}(\omega_{\widehat{T}}) \cap L^\infty(0,\widehat{T};H^1(\omega))} \\ &\leqslant C\Big(\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega_{\overline{\eta}})} + \|\zeta_1^0\|_{H^3(\omega)} + \|\zeta_2^0\|_{H^1_0(\omega)} \\ &\qquad + \|P\mathbf{F}\|_{L^2(0,T;\mathbf{V}^0_{n,\Gamma^s_{\overline{\eta}}}(\Omega_{\overline{\eta}}))} + \|H\|_{L^2(0,T;L^2(\omega))}\Big). \end{aligned} \tag{5.9}$$

Next using the fact that $D_s \in \mathcal{L}(L^2(\omega), \mathbf{L}^2(\Omega_{\overline{\eta}}))$ (see Lemma A.2), we have

$$\begin{aligned} &\|(I-P)\widetilde{\mathbf{v}}\|_{H^1(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}})) \cap L^\infty(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))} \\ &= \|(I-P)D_s\zeta_2\|_{H^1(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}})) \cap L^\infty(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))} \\ &\leqslant C\|\zeta_2\|_{H^{2,1}(\omega_T) \cap L^\infty(0,T;L^2(\omega))}, \end{aligned} \tag{5.10}$$

where the constant C is independent of T. To estimate the space regularity of $\widetilde{\mathbf{v}}$ and $\widetilde{p}$ we consider the following system

$$\begin{aligned}&-\operatorname{div}\sigma(\widetilde{\mathbf{v}},\widetilde{p})=\mathbf{f}-\mathbf{w}_t-\widetilde{\mathbf{v}}_t,\quad \operatorname{div}\widetilde{\mathbf{v}}=0,\quad \text{in } \Omega_{\overline{\eta}},\\ &\widetilde{\mathbf{v}}=\zeta_2\mathbf{e}_r,\quad \text{on } \Gamma^s_{\overline{\eta}},\quad \sigma(\widetilde{\mathbf{v}},\widetilde{p})\mathbf{n}=0 \text{ on } \Gamma_{\text{in}},\quad \sigma(\widetilde{\mathbf{v}},\widetilde{p})\mathbf{n}=0 \text{ on } \Gamma_{\text{out}}.\end{aligned}$$

As $\mathbf{f}-\mathbf{w}_t-\widetilde{\mathbf{v}}_t\in L^2(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))$ and $\zeta_2\in L^2(0,T;H^2_0(\omega))$, Theorem 4.2 yields that $(\widetilde{\mathbf{v}},\widetilde{p})$ belongs to $L^2(0,T;\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}}))\times L^2(0,T;H^1_{\delta_0}(\Omega_{\overline{\eta}}))$ and

$$\|\widetilde{\mathbf{v}}\|_{L^2(0,T;\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}}))}+\|\widetilde{p}\|_{L^2(0,T;H^1_{\delta_0}(\Omega_{\overline{\eta}}))}\leqslant C\Big(\|\mathbf{f}-\mathbf{w}_t-\widetilde{\mathbf{v}}_t\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))}+\|\zeta_2\|_{L^2(0,T;H^2_0(\omega))}\Big),\tag{5.11}$$

where the constant C is independent of T. Since $\mathbf{v}=\widetilde{\mathbf{v}}+\mathbf{w}$ and $p=\widetilde{p}+q$, Proposition 5.1 and estimates (5.6), (5.9) - (5.11) yields

$$\begin{aligned}&\|\mathbf{v}\|_{\mathbf{H}^{2,1}_{\delta_0}(Q^T_{\overline{\eta}})}+\|\mathbf{v}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))}+\|p\|_{L^2(0,T;H^1_{\delta_0}(\Omega_{\overline{\eta}}))}+\|\zeta_1\|_{H^{4,2}(\omega_T)}\\ &+\|\zeta_1\|_{L^\infty(0,T;H^3(\omega))}+\|\zeta_2\|_{H^{2,1}(\omega_T)}+\|\zeta_2\|_{L^\infty(0,T;H^1(\omega))}\\ &\leqslant C\Big(\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega_{\overline{\eta}})}+\|\zeta_3\|_{H^1_0(\omega)}+\|\zeta_2^0\|_{H^1_0(\omega)}+\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))}\\ &+\|g\|_{L^2(0,T;H^1(\Omega_{\overline{\eta}}))}+\|g\|_{H^1(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}})'))}+\|h\|_{L^2(0,T;L^2(\omega))}\Big),\end{aligned}\tag{5.12}$$

where the constant C is independent of T.

Step 2. Proof of the $L^\infty(0,T;\mathbf{H}^1(\Omega_{\overline{\eta}}))$-estimate. We have to show that $\mathbf{v}\in L^\infty(0,T;\mathbf{H}^1(\Omega_{\overline{\eta}}))$. Multiplying $(4.2)_1$ by $\mathbf{v}_t$ and integrating by parts over $\Omega_{\overline{\eta}}$, we obtain

$$\begin{aligned}\|\mathbf{v}_t(t,\cdot)\|^2_{\mathbf{L}^2(\Omega_{\overline{\eta}})}+2\nu\int_{\Omega_{\overline{\eta}}}\boldsymbol{\varepsilon}(\mathbf{v}):\nabla\mathbf{v}_t&\\ \leqslant\frac{1}{2}\|\mathbf{v}_t(t,\cdot)\|^2_{\mathbf{L}^2(\Omega_{\overline{\eta}})}+\frac{1}{2}\|\mathbf{f}(t,\cdot)-\nu\nabla g(t,\cdot)\|^2_{\mathbf{L}^2(\Omega_{\overline{\eta}})}&\\ +\frac{1}{2}\|p(t,\cdot)\|^2_{H^1_{\delta_0}(\Omega_{\overline{\eta}})}+\frac{1}{2}\|g_t(t,\cdot)\|^2_{(H^1_{\delta_0}(\Omega_{\overline{\eta}}))'}+\frac{1}{2}\|\sigma(\mathbf{v},p)(t,\cdot)\mathbf{n}\|^2_{L^2(\Gamma^s_{\overline{\eta}})}&\\ +\frac{1}{2}\|\zeta_{2,t}(t,\cdot)\mathbf{e}_\mathbf{r}\|^2_{L^2(\Gamma^s_{\overline{\eta}})}\Big).&\end{aligned}$$

The above estimate yields

$$\begin{aligned}\frac{d}{dt}\sum_{i,j=1}^3\int_{\Omega_{\overline{\eta}}}\left|\frac{\partial v_i}{\partial z_j}+\frac{\partial v_j}{\partial z_i}\right|^2\leqslant C\Big(\|\mathbf{f}(t,\cdot)\|^2_{\mathbf{L}^2(\Omega_{\overline{\eta}})}+\|g(t,\cdot)\|^2_{H^1(\Omega_{\overline{\eta}})}+\|g_t(t,\cdot)\|^2_{(H^1_{\delta_0}(\Omega_{\overline{\eta}}))'}&\\ +\|p(t,\cdot)\|^2_{H^1_{\delta_0}(\Omega_{\overline{\eta}})}+\|\mathbf{v}(t,\cdot)\|^2_{\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}})}+\|\zeta_{2,t}(t,\cdot)\|^2_{L^2(\omega)}\Big).&\end{aligned}$$

Integrating the above inequality over $[0,T]$ and applying Korn's inequality, one can easily check that

$$\begin{aligned}\|\mathbf{v}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_{\overline{\eta}}))}\leqslant C\Big(\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega_{\overline{\eta}})}+\|\mathbf{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))}+\|g\|_{L^2(0,T;H^1(\Omega_{\overline{\eta}}))}&\\ +\|g\|_{H^1(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}}))')}+\|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}}))}+\|\mathbf{v}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))}&\\ +\|p\|_{L^2(0,T;H^1_{\delta_0}(\Omega_{\overline{\eta}}))}+\|\zeta_2\|_{H^{2,1}(\omega_T)}\Big),&\end{aligned}$$

where C is independent of T. The above estimate together with (5.12) gives (5.2).

Step 3. Proof of (5.3). Let us set $H^{\varepsilon_0}_s=H^{2+\varepsilon_0}_0(\omega)\times H^{\varepsilon_0}(\omega)$ and $\mathcal{D}(A_s;H^{\varepsilon_0}_s)=\big(H^{4+\varepsilon_0}(\omega)\cap H^2_0(\omega)\big)\times H^{2+\varepsilon_0}_0(\omega)$. From [8, Theorem 1.2], we know that $(A_s,\mathcal{D}(A_s;H^{\varepsilon_0}_s))$ is the infinitesimal generator of an analytic semigroup of contractions on $H^{\varepsilon_0}_s$. Since $-\gamma^s_{\overline{\eta}}\Big(\sigma(\mathbf{v},p)\mathbf{e}_{\overline{\eta}}\Big)\cdot\mathbf{e}_r+h\in L^2(0,T;H^{\varepsilon_0}(\omega))$, if (ζ^0_1,ζ^0_2) belonged to $H^{3+\varepsilon_0}(\omega)\cap H^2_0(\omega)\times H^{1+\varepsilon_0}_0(\omega)$, then ζ_1 would be in $C([0,T];H^{3+\varepsilon_0}(\omega))$. We know that ζ^0_1 belongs to $H^{3+\varepsilon_0}(\omega)\cap H^2_0(\omega)$, but ζ^0_2 only belongs to $H^1_0(\omega)$ and not to $H^{1+\varepsilon_0}_0(\omega)$. But, due to the regularizing effect of the semigroup $(e^{tA_s})_{t\geqslant0}$, we can expect that $\zeta_1(t)$ belongs to $H^{3+\varepsilon_0}(\omega)$ for $t>0$. To obtain an estimate for $\zeta_1(t)$ in

$H^{3+\varepsilon_0}(\omega)$, and for $\zeta_2(t)$ in $H^{1+\varepsilon_0}(\omega)$, for $t \in (0,1)$, it is sufficient to write the system satisfied by $(t\mathbf{v}, tp, t\zeta_1, t\zeta_2)$ over the time interval $[0,1]$, and to use the fact that $(A_s, \mathcal{D}(A_s; H_s^{\varepsilon_0}))$ is the infinitesimal generator of an analytic semigroup on $H_s^{\varepsilon_0}$. When $t > 1$, the estimate of ζ_1 in $C([1,T]; H^{3+\varepsilon_0}(\omega))$ can be deduced from the regularity of $(\zeta_1(1), \zeta_2(1)) \in H^{3+\varepsilon_0}(\omega) \cap H_0^2(\omega) \times H_0^{1+\varepsilon_0}(\omega)$, and from the fact that $(A_s, \mathcal{D}(A_s; H_s^{\varepsilon_0}))$ is the infinitesimal generator of an analytic semigroup on $H_s^{\varepsilon_0}$. □

6. Estimates of the nonlinear terms

Throughout this section, we assume that $\overline{\eta}$ satisfies (4.1).

We are going to obtain different estimates for the nonlinear terms $\widehat{\mathcal{F}}_{\overline{\eta}}$ and $\widehat{\mathcal{G}}_{\overline{\eta}}$ defined in $Q_{\overline{\eta}}^{t_0,t_1} = \Omega_{\overline{\eta}} \times (t_0, t_1)$, and $\widehat{\mathcal{H}}_{\overline{\eta}}$ and $\mathscr{L}_{mem}$ defined in $\omega \times (t_0, t_1)$, introduced in (3.7) - (3.9) and in (1.3). For notational simplicity we shall only treat the case where $(t_0, t_1) = (0, T)$.

6.1. Preliminary results.

Let us first recall some lemmas which we shall use later on. The first lemma is a direct consequence of [17, Lemma 1]:

Lemma 6.1. *Let Z be a Banach space and $1/2 < s \leqslant 1$. Then there is a bounded extension operator from $\{f \in H^s(0,T;Z) \mid f(0) = 0\}$ to $H^s(0,\infty;Z)$ with a norm bounded independently of T.*

Lemma 6.2. *Let Z be a Banach space space and $1/2 < s \leqslant 1$. Then there exist a constant $C > 0$ and $\alpha > 0$, depending only on s but independent of T, such that*

$$\|f\|_{C([0,T];Z)} \leqslant CT^{\alpha}\|f\|_{H^s(0,T;Z)}, \quad \text{for all } f \in H^s(0,T;Z) \text{ satisfying } f(0) = 0.$$

By using $C([0,T];Z)$ rather than $L^\infty(0,T;Z)$ we avoid talking about weak and strong measurability.

Proof. Let us fix $1/2 < s_1 < s$. With Lemma 6.1 and by interpolation, we have

$$\|f\|_{C([0,T];Z)} \leqslant C\|f\|_{H^{s_1}(0,T;Z)} \leqslant C\|f\|_{H^s(0,T;Z)}^{s_1/s}\|f\|_{L^2(0,T;Z)}^{(s-s_1)/s},$$

with C independent of T since $f(0) = 0$. Hölder's inequality implies that

$$\|f\|_{C([0,T];Z)} \leqslant CT^{(s-s_1)/(2s)}\|f\|_{H^s(0,T;Z)}^{s_1/s}\|f\|_{C([0,T];Z)}^{(s-s_1)/s}.$$

Simplifying the above estimate by $\|f\|_{C([0,T];Z)}^{(s-s_1)/s}$, we obtain

$$\|f\|_{C([0,T];Z)} \leqslant CT^{(s-s_1)/(2s_1)}\|f\|_{H^s(0,T;Z)}.$$

□

Now we prove several lemmas used to estimate the nonlinear terms. We set

$$B(0_{\mathbb{R}^2}, 1 + \overline{\eta}(z_1, \cdot)) = \{(z_2, z_3) \in \mathbb{R}^2 \mid (z_2^2 + z_3^2)^{1/2} < 1 + \overline{\eta}(z_1, \theta))\},$$
$$\text{with} \quad (\cos\theta, \sin\theta) = \left(\tfrac{z_2}{(z_2^2+z_3^2)^{1/2}}, \tfrac{z_3}{(z_2^2+z_3^2)^{1/2}}\right).$$

Thus $B(0_{\mathbb{R}^2}, 1+\overline{\eta}(z_1,\cdot))$ depends on z_1. To simplify the notation, we shall simply write $B(0, 1+\overline{\eta})$ for $B(0_{\mathbb{R}^2}, 1+\overline{\eta}(z_1,\cdot))$. We say that a function g belongs to $H^1(0,L;L^\infty(B(0,1+\overline{\eta})))$ if and only if $g \circ X_{\overline{\eta}}$ belongs to $H^1(0,L;L^\infty(B(0_{\mathbb{R}^2},1)))$.

Lemma 6.3. *Let $f \in L^2_{\delta_0}(\Omega_{\overline{\eta}})$ and $g \in H^1(0,L;L^\infty(B(0,1+\overline{\eta}))$ with $g = 0$ on Γ_n. Then fg belongs to $L^2(\Omega_{\overline{\eta}})$ and*

$$\|fg\|_{L^2(\Omega_{\overline{\eta}})} \leqslant C\|f\|_{L^2_{\delta_0}(\Omega_{\overline{\eta}})}\|g\|_{H^1(0,L;L^\infty(B(0,1+\overline{\eta})))}.$$

Proof. Since $g = 0$ on Γ_n, we get

$$|g(z_1,z_2,z_3)| = \left|\int_0^{z_1} g_{z_1}(s,z_2,z_3)\,\mathrm{d}s\right| \leqslant z_1^{1/2}\left(\int_0^L g_{z_1}^2(s,z_2,z_3)\,\mathrm{d}s\right)^{1/2}.$$

In a similar manner we obtain

$$|g(z_1,z_2,z_3)| \leqslant (L-z_1)^{1/2}\left(\int_0^L g_{z_1}^2(s,z_2,z_3)\ \mathrm{d}s\right)^{1/2}.$$

Using the above estimates we obtain

$$\begin{aligned}
&\int_0^{L/2}\int_{B(0,1+\overline{\eta})} f^2g^2\ \mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_1\\
&\leqslant \int_0^{L/2}\int_{B(0,1+\overline{\eta})} f^2\ z_1\left(\int_0^L g_{z_1}^2(s,z_2,z_3)\ \mathrm{d}s\right)\ \mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_1\\
&\leqslant \|g_{z_1}\|^2_{L^2(0,L;L^\infty(B(0,1+\overline{\eta})))}\int_0^{L/2}\int_{B(0,1+\overline{\eta})} z_1^{2\delta_0}(L-z_1)^{2\delta_0}f^2\ \frac{z_1^{1-2\delta_0}}{(L-z_1)^{2\delta_0}}\ \mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_1\\
&\leqslant (L/2)^{1-4\delta_0}\|g\|^2_{H^1(0,L;L^\infty(B(0,1+\overline{\eta})))}\int_0^{L/2}\int_{B(0,1+\overline{\eta})} z_1^{2\delta_0}(L-z_1)^{2\delta_0}f^2\ \mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_1,
\end{aligned}$$

where in the last inequality, we crucially use the fact that $\delta_0 < 1/2$. Also observe that, $z_1 < r_0$ and $(L-z_1) < r_L$. Hence, the above inequality becomes

$$\begin{aligned}
&\int_0^{L/2}\int_{B(0,1+\overline{\eta})} f^2g^2\ \mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_1\\
&\qquad\leqslant (L/2)^{1-4\delta_0}\|g\|^2_{H^1(0,L;L^\infty(B(0,1+\overline{\eta})))}\int_0^{L/2}\int_{B(0,1+\overline{\eta})} r_0^{2\delta_0}r_L^{2\delta_0}f^2\ \mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_1. \quad (6.1)
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
&\int_{L/2}^{L}\int_{B(0,1+\overline{\eta})} f^2g^2\ \mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_1\\
&\qquad\leqslant (L/2)^{1-4\delta_0}\|g\|^2_{H^1(0,L;L^\infty(B(0,1+\overline{\eta})))}\int_{L/2}^{L}\int_{B(0,1+\overline{\eta})} r_0^{2\delta_0}r_L^{2\delta_0}f^2\ \mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_1. \quad (6.2)
\end{aligned}$$

Combining estimates (6.1)-(6.2), we complete the proof of the lemma. □

Lemma 6.4. *Let s belong to $(1/2, 1/2+\varepsilon_0)$ and $\mu > 0$. Then there exists a constant $C_s > 0$, depending only on $\overline{\eta}$, μ and s, such that, for all $0 < T < 1$ and all $\mathbf{u}$ satisfying $\|\mathbf{u}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))\cap L^2(0,T;\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\overline{\eta}}))} \leqslant \mu$, we have*

$$\|\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{1+s}(\Omega_{\overline{\eta}}))} \leqslant C_s\, T^{\frac{1+2\varepsilon_0-2s}{2(3+2\varepsilon_0)}}.$$

Proof. By interpolation we have

$$\|\mathbf{u}(t,\cdot)\|_{\mathbf{H}^{1+s}(\Omega_{\overline{\eta}})} \leqslant C(\Omega_{\overline{\eta}},s)\|\mathbf{u}(t,\cdot)\|^{\frac{2(1+s)}{3+2\varepsilon_0}}_{\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\overline{\eta}})}\|\mathbf{u}(t,\cdot)\|^{\frac{1+2\varepsilon_0-2s}{3+2\varepsilon_0}}_{\mathbf{L}^2(\Omega_{\overline{\eta}})}.$$

Therefore a simple application of Hölder's inequality yields

$$\begin{aligned}
&\|\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{1+s}(\Omega_{\overline{\eta}}))}\\
&\leqslant C\|\mathbf{u}\|^{\frac{1+2\varepsilon_0-2s}{3+2\varepsilon_0}}_{L^\infty(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))}\left(\int_0^T \|\mathbf{u}(t,\cdot)\|^{\frac{4(1+s)}{3+2\varepsilon_0}}_{\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\overline{\eta}})}\ \mathrm{d}t\right)^{1/2}\\
&\leqslant CT^{\frac{1+2\varepsilon_0-2s}{2(3+2\varepsilon_0)}}\|\mathbf{u}\|^{\frac{1+2\varepsilon_0-2s}{3+2\varepsilon_0}}_{L^\infty(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))}\|\mathbf{u}\|^{\frac{2(1+s)}{3+2\varepsilon_0}}_{L^2(0,T;\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\overline{\eta}}))}\\
&\leqslant CT^{\frac{1+2\varepsilon_0-2s}{2(3+2\varepsilon_0)}}.
\end{aligned}$$

□

Lemma 6.5. *There exist positive constants C_s and C_{s_1}, depending only on $\mu > 0$, $\overline{\eta}$, $s \in (0, 1/2)$, and $s_1 \in [0, 1)$, such that, for all $0 < T < 1$ and all $\eta \in E_{\overline{\eta}}(0, T)$ satisfying $\|\eta\|_{E(0,T)} \leqslant \mu$, we have*

$$\begin{aligned}
&\|\eta - \overline{\eta}\|_{L^\infty(0,T;H^{5/2+s}(\omega))} \leqslant C_s\, T^\alpha,\\
&\|\nabla\eta\|_{H^{3/4-s/2}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))\cap L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))} \leqslant C_s,\\
&\|\nabla\eta\|_{H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))} \leqslant C_s,\\
&\|\partial_t\eta\|_{L^\infty(0,T;H^1(\Omega_{\overline{\eta}}))} + \|\partial_t\nabla\eta\|_{L^2(0,T;H^s(\Omega_{\overline{\eta}}))} \leqslant C_s,\\
&\|\eta\|_{L^2(0,T;H^{3+s_1}(\omega))} + \|\nabla\eta_t\|_{L^2(0,T;H^{s_1}(\omega))} \leqslant C_{s_1}\, T^{\alpha_1},
\end{aligned} \tag{6.3}$$

where α only depends on $s \in (0, 1/2)$ and α_1 on $s_1 \in [0, 1)$.

Proof. By interpolation, we have

$$\|\eta - \overline{\eta}\|_{H^{3/4-s/2}(0,T;H^{5/2+s}(\omega))} \leqslant C\|\eta - \overline{\eta}\|^{(1+2s)/4}_{L^2(0,T;H^4(\omega))}\|\eta - \overline{\eta}\|^{(3-2s)/4}_{H^1(0,T;H^2(\omega))} \leqslant C\mu,$$

where C independent of T since $\eta(0) = \overline{\eta}$ (see Lemma 6.1). Since $(3/4 - s/2) \in (1/2, 3/4)$, applying Lemma 6.2 we conclude that

$$\|\eta - \overline{\eta}\|_{L^\infty(0,T;H^{5/2+s}(\omega))} \leqslant CT^\alpha,$$

with constant C independent of T.

The estimates $(6.3)_{2-4}$ follow from the bound $\|\eta\|_{E(0,T)} \leqslant \mu$. The estimates of η in $L^2(0, T; H^{3+s_1}(\omega))$ and $\nabla\eta_t$ in $L^2(0, T; H^{s_1}(\omega))$ are similar to that in the proof of Lemma 6.4. □

Estimates of $X_{\overline{\eta},\eta}$ and $J_{\overline{\eta},\eta}$.

Lemma 6.6. *There exists a positive constant C, depending only on $\mu > 0$, $\gamma > 0$, $\overline{\eta}$ and $s \in (0, 1/2)$, such that, for all $0 < T < 1$, and all $\eta \in E_{\overline{\eta}}(0, T; \gamma)$ satisfying $\|\eta\|_{E(0,T)} \leqslant \mu$, $X_{\overline{\eta},\eta}$ and $J_{\overline{\eta},\eta}$ defined in (3.4) and (3.6) satisfy*

$$\begin{aligned}
&\|\nabla X_{\overline{\eta},\eta}\|_{H^{3/4-s/2}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))\cap L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))} \leqslant C,\\
&\|\nabla X_{\overline{\eta},\eta}\|_{H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))} \leqslant C,\\
&\|\partial_t X_{\overline{\eta},\eta}\|_{L^\infty(0,T;H^1(\Omega_{\overline{\eta}}))} + \|\partial_t\nabla X_{\overline{\eta},\eta}\|_{L^2(0,T;H^s(\Omega_{\overline{\eta}}))} \leqslant C,\\
&\|\mathrm{cof}(\nabla X_{\overline{\eta},\eta})\|_{H^{3/4-s/2}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))\cap L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))} \leqslant C,\\
&\|\mathrm{cof}(\nabla X_{\overline{\eta},\eta})\|_{H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))} \leqslant C,\\
&\|\det(\nabla X_{\overline{\eta},\eta})\|_{H^{3/4-s/2}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))\cap L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))} \leqslant C,\\
&\|\det(\nabla X_{\overline{\eta},\eta})\|_{H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))} \leqslant C,\\
&\|J_{\overline{\eta},\eta}\|_{H^{3/4-s/2}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))\cap L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))} \leqslant C,\\
&\|J_{\overline{\eta},\eta}\|_{H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))} \leqslant C.
\end{aligned} \tag{6.4}$$

Moreover, for all $0 < s < 1/2$ and $0 < \alpha < (1 - 2s)/(6 - 4s)$, we have

$$\begin{aligned}
&\|J_{\overline{\eta},\eta} - I\|_{L^\infty_w(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))}\\
&\quad\leqslant CT^\alpha\|J_{\overline{\eta},\eta} - I\|_{H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))},\\
&\|J_{\overline{\eta},\eta}\otimes J_{\overline{\eta},\eta} - I\otimes I\|_{L^\infty_w(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))}\\
&\quad\leqslant CT^\alpha\|J_{\overline{\eta},\eta}\otimes J_{\overline{\eta},\eta} - I\otimes I\|_{H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))},
\end{aligned} \tag{6.5}$$

where $(J_{\overline{\eta},\eta}\otimes J_{\overline{\eta},\eta} - I\otimes I)_{k,l} = \sum_{j=1}^3(J^{k,j}_{\overline{\eta},\eta}J^{l,j}_{\overline{\eta},\eta} - \delta_{k,j}\delta_{l,j})$, and $L^\infty_w(0, T; H^1(0, L; L^\infty(B(0, 1 + \overline{\eta}))))$ is the space of bounded and weakly measurable functions from $(0, T)$ to $H^1(0, L; L^\infty(B(0, 1 + \overline{\eta})))$. For all $0 < s < 1/2$ and $0 < \alpha < (1 - 2s)/(6 - 4s)$, we have

$$\begin{aligned}
&\|J_{\overline{\eta},\eta} - I\|_{L^\infty(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))} \leqslant CT^\alpha,\\
&\|\mathrm{cof}(\nabla X_{\overline{\eta},\eta}) - I\|_{L^\infty(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))} \leqslant CT^\alpha.
\end{aligned} \tag{6.6}$$

Proof. Step 1. Proof of (6.4). From the definition of $X_{\overline{\eta},\eta}$, it follows that $\|\nabla X_{\overline{\eta},\eta}\|_Z \leqslant C\|\nabla\eta\|_Z$, $\|\partial_t X_{\overline{\eta},\eta}\|_Z \leqslant C\|\partial_t\eta\|_Z$, and $\|\partial_t\nabla X_{\overline{\eta},\eta}\|_Z \leqslant C\|\partial_t\nabla\eta\|_Z$, for the various spaces Z involved in the estimates $(6.4)_{1-3}$. Thus $(6.4)_{1-3}$ follow from $(6.3)_{2-4}$.

To estimate the norms of $\mathrm{cof}(\nabla X_{\overline{\eta},\eta})$ and $\det(\nabla X_{\overline{\eta},\eta})$, we note that the spaces $H^{3/4-s/2}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))$, $L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))$ and $H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))$ are algebras for $s\in(0,1/2)$. The estimates with constants independent of T follows from [33, Lemma A.1].

Observe that, if $\eta\in E_{\overline{\eta}}(0,T;\gamma)$ then $\det(\nabla X_{\overline{\eta},\eta})\geqslant m>0$ in $\Omega_{\overline{\eta}}\times(0,T)$. Finally, we estimate $J_{\overline{\eta},\eta}$ by using Lemma A.4 and the following relation

$$J_{\overline{\eta},\eta}=\frac{1}{\det(\nabla X_{\overline{\eta},\eta})}\mathrm{cof}(\nabla X_{\overline{\eta},\eta}).$$

Step 2. Proof of (6.5) *and* (6.6). To prove $(6.5)_1$ it is enough to notice that $(J_{\overline{\eta},\eta}-I)\,|_{t=0}=0$, and to use Lemma 6.2. The estimate $(6.5)_2$ can be deduced from $(6.5)_1$ and $(6.4)_9$ and the fact that the space $H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))$ is an algebra for $s\in(0,1/2)$.

Estimate (6.6) can be proved with Lemma 6.2 and $(6.4)_8$. □

6.2. **Lipschitz estimates.** We introduce the space

$$B(Q_{\overline{\eta}}^{t_0,t_1};\gamma)=\Big\{(\mathbf{v},p,\eta)\in B(Q_{\overline{\eta}}^{t_0,t_1})\mid \eta\in E_{\overline{\eta}}(0,T;\gamma)\Big\}.$$

We define the ball $B(Q_{\overline{\eta}}^{t_0,t_1};\gamma,\mu)$ in $B(Q_{\overline{\eta}}^{t_0,t_1};\gamma)$ as follows

$$B(Q_{\overline{\eta}}^{t_0,t_1};\gamma,\mu)=\Big\{(\mathbf{v},p,\eta)\in B(Q_{\overline{\eta}}^{t_0,t_1};\gamma)\mid |\!|\!|(\mathbf{v},p,\eta)|\!|\!|_{B(Q_{\overline{\eta}}^{t_0,t_1})}\leqslant\mu\Big\}. \tag{6.7}$$

As mentioned at the begining of the section, for notational simplicity, we only treat the case where $(t_0,t_1)=(0,T)$. But all what follows is valid for an arbitrary interval (t_0,t_1).

Estimate of $\widehat{\mathcal{F}}_{\overline{\eta}}$.

Lemma 6.7. *There exist positive constants $C_{\widehat{\mathcal{F}}}$ and α, depending on $\mu>0$, $\gamma>0$, and $\overline{\eta}$ such that, for all $0<T<1$, all $(\widehat{\mathbf{u}},\widehat{p},\eta)\in B(Q_{\overline{\eta}}^T;\gamma,\mu)$, $(\widehat{\mathbf{u}}^1,\widehat{p}^1,\eta^1)\in B(Q_{\overline{\eta}}^T;\gamma,\mu)$ and $(\widehat{\mathbf{u}}^2,\widehat{p}^2,\eta^2)\in B(Q_{\overline{\eta}}^T;\gamma,\mu)$, we have*

$$\begin{aligned}
&\|\widehat{\mathcal{F}}_{\overline{\eta}}(\widehat{\mathbf{u}},\widehat{p},\eta)\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))}\leqslant C_{\widehat{\mathcal{F}}}T^\alpha,\\
&\|\widehat{\mathcal{F}}_{\overline{\eta}}(\widehat{\mathbf{u}}^1,\widehat{p}^1,\eta^1)-\widehat{\mathcal{F}}_{\overline{\eta}}(\widehat{\mathbf{u}}^2,\widehat{p}^2,\eta^2)\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\overline{\eta}}))}\leqslant C_{\widehat{\mathcal{F}}}T^\alpha\|(\widehat{\mathbf{u}}^1,\widehat{p}^1,\eta^1)-(\widehat{\mathbf{u}}^2,\widehat{p}^2,\eta^2)\|_{B(Q_{\overline{\eta}}^T)}.
\end{aligned}\tag{6.8}$$

Proof. Step 1. Let us prove $(6.8)_1$. Let us recall that $\widehat{\mathcal{F}}_{\overline{\eta},i}$ is defined in (3.7). With [18, Proposition B1], Lemma 6.4, and Lemma 6.6, we estimate the first term of $\widehat{\mathcal{F}}_{\overline{\eta},i}$ as follows: For s fixed in $(1/2,1/2+\varepsilon_0)$, we have

$$\begin{aligned}
&\big\|(\widehat{\mathbf{u}}-\partial_t X_{\overline{\eta},\eta})\cdot J_{\overline{\eta},\eta}^\top\nabla\widehat{u}_i\big\|_{L^2(0,T;L^2(\Omega_{\overline{\eta}}))}\\
&\leqslant C\Big(\|\widehat{\mathbf{u}}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_{\overline{\eta}}))}+\|\partial_t X_{\overline{\eta},\eta}\|_{L^\infty(0,T;H^1(\Omega_{\overline{\eta}}))}\Big)\times\|\nabla\widehat{u}_i\|_{L^2(0,T;H^s(\Omega_{\overline{\eta}}))}\|J_{\overline{\eta},\eta}\|_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))}\\
&\leqslant CT^\alpha.
\end{aligned}$$

From $(6.5)_2$ and Lemma 6.3, we estimate the second term of $\widehat{\mathcal{F}}_{\overline{\eta},i}$ as follows

$$\begin{aligned}
&\left\|\nu\sum_{j,k,l}\frac{\partial^2\widehat{u}_i}{\partial z_l\partial z_k}\left(J_{\overline{\eta},\eta}^{k,j}J_{\overline{\eta},\eta}^{l,j}-\delta_{k,j}\delta_{l,j}\right)\right\|_{L^2(0,T;L^2(\Omega_{\overline{\eta}}))}\\
&\leqslant\nu\sum_{j,k,l}\left\|\frac{\partial^2\widehat{u}_i}{\partial z_l\partial z_k}\right\|_{L^2(0,T;L^2_{\delta_0}(\Omega_{\overline{\eta}}))}\|J_{\overline{\eta},\eta}^{k,j}J_{\overline{\eta},\eta}^{l,j}-\delta_{k,j}\delta_{l,j}\|_{L^\infty_w(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))}\\
&\leqslant CT^\alpha.
\end{aligned}$$

To estimate the third term of $\widehat{\mathcal{F}}_{\overline{\eta},i}$, we use [18, Proposition B1], Lemma 6.4 and Lemma 6.6. We obtain

$$\begin{aligned}
&\left\| \nu \sum_{j,k,l} \frac{\partial \widehat{u}_i}{\partial z_k} \frac{\partial}{\partial z_l} (J^{k,j}_{\overline{\eta},\eta}) J^{l,j}_{\overline{\eta},\eta} \right\|_{L^2(0,T;L^2(\Omega_{\overline{\eta}}))} \\
&\leqslant \nu \sum_{j,k,l} \left\| \frac{\partial \widehat{u}_i}{\partial z_k} \right\|_{L^2(0,T;H^s(\Omega_{\overline{\eta}}))} \left\| \frac{\partial}{\partial z_l} (J^{k,j}_{\overline{\eta},\eta}) \right\|_{L^\infty(0,T;H^1(\Omega_{\overline{\eta}}))} \left\| J^{l,j}_{\overline{\eta},\eta} \right\|_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))} \\
&\leqslant CT^\alpha, \quad \text{for all } s \in (1/2, 1/2+\varepsilon_0).
\end{aligned}$$

With $(6.5)_1$ and Lemma 6.3, we obtain the following estimate of the last term of $\widehat{\mathcal{F}}_{\overline{\eta},i}$

$$\begin{aligned}
&\left\| (I - J^\top_{\overline{\eta},\eta}) \nabla \widehat{p} \right\|_{L^2(0,T;L^2(\Omega_{\overline{\eta}}))} \\
&\leqslant \| I - J^\top_{\overline{\eta},\eta} \|_{L^\infty_w(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))} \| \nabla p \|_{L^2(0,T;L^2_{\delta_0}(\Omega_{\overline{\eta}}))} \leqslant CT^\alpha.
\end{aligned}$$

This completes the proof of $(6.8)_1$.

Step 2. Let us prove $(6.8)_2$. That Lipschitz estimate can be proved as in Step 1. However, for clarity, let us explain how the Lipschitz estimate can be proved for the second term in $\widehat{\mathcal{F}}_{\overline{\eta}}$. We have

$$\begin{aligned}
&\sum_{j,k,l} \frac{\partial^2 \widehat{u}^1_i}{\partial z_l \partial z_k} \left(J^{k,j}_{\overline{\eta},\eta^1} J^{l,j}_{\overline{\eta},\eta^1} - \delta_{k,j}\delta_{l,j} \right) - \sum_{j,k,l} \frac{\partial^2 \widehat{u}^2_i}{\partial z_l \partial z_k} \left(J^{k,j}_{\overline{\eta},\eta^2} J^{l,j}_{\overline{\eta},\eta^2} - \delta_{k,j}\delta_{l,j} \right) \\
&= \sum_{j,k,l} \frac{\partial^2 \widehat{u}^1_i}{\partial z_l \partial z_k} J^{k,j}_{\overline{\eta},\eta^1} \left(J^{l,j}_{\overline{\eta},\eta^1} - J^{l,j}_{\overline{\eta},\eta^2} \right) + \sum_{j,k,l} \frac{\partial^2 \widehat{u}^1_i}{\partial z_l \partial z_k} \left(J^{k,j}_{\overline{\eta},\eta^1} - J^{k,j}_{\overline{\eta},\eta^2} \right) J^{l,j}_{\overline{\eta},\eta^2} \\
&\quad + \sum_{j,k,l} \left(\frac{\partial^2 \widehat{u}^1_i}{\partial z_l \partial z_k} - \frac{\partial^2 \widehat{u}^2_i}{\partial z_l \partial z_k} \right) \left(J^{k,j}_{\overline{\eta},\eta^2} J^{l,j}_{\overline{\eta},\eta^2} - \delta_{k,j}\delta_{l,j} \right).
\end{aligned}$$

All terms can be estimated as in Step 1, but we notice that $\left(J^{k,j}_{\overline{\eta},\eta^2} J^{l,j}_{\overline{\eta},\eta^2} - \delta_{k,j}\delta_{l,j} \right)|_{t=0} = 0$ is used in an essential way. □

Estimate of $\widehat{\mathcal{G}}_{\overline{\eta}}$.

Lemma 6.8. *There exist positive constants $C_{\widehat{\mathcal{G}}}$ and α, depending on $\mu > 0$, $\gamma > 0$, and $\overline{\eta}$ such that, for all $0 < T < 1$, all $(\widehat{\mathbf{u}}, \widehat{p}, \eta) \in B(Q^T_{\overline{\eta}}; \gamma, \mu)$, $(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1) \in B(Q^T_{\overline{\eta}}; \gamma, \mu)$ and $(\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2) \in B(Q^T_{\overline{\eta}}; \gamma, \mu)$, we have*

$$\begin{aligned}
&\| \widehat{\mathcal{G}}_{\overline{\eta}}(\widehat{\mathbf{u}}, \eta) \|_{L^2(0,T;H^1(\Omega_{\overline{\eta}})) \cap H^1(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}}))')} \leqslant C_{\widehat{\mathcal{G}}} T^\alpha, \\
&\| \widehat{\mathcal{G}}_{\overline{\eta}}(\widehat{\mathbf{u}}^1, \eta^1) - \widehat{\mathcal{G}}_{\overline{\eta}}(\widehat{\mathbf{u}}^2, \eta^2) \|_{L^2(0,T;H^1(\Omega_{\overline{\eta}})) \cap H^1(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}}))')} \\
&\leqslant C_{\widehat{\mathcal{G}}} T^\alpha \| (\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1) - (\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2) \|_{B(Q^T_{\overline{\eta}})}.
\end{aligned} \tag{6.9}$$

Proof. Let us recall $\widehat{\mathcal{G}}_{\overline{\eta}}$ is defined in (3.8). Let us note that $J_{\overline{\eta},\eta}(0,\cdot) = I$ and $J_{\overline{\eta},\eta} = I$ on Γ_n. Thus

$$\widehat{\mathcal{G}}_{\overline{\eta}}(0,\cdot) = 0 \text{ in } \Omega_{\overline{\eta}} \quad \text{and} \quad \widehat{\mathcal{G}}_{\overline{\eta}} = 0 \text{ on } \Gamma_n.$$

First we estimate the $L^2(0,T;H^1(\Omega_{\overline{\eta}}))$-norm of $\widehat{\mathcal{G}}_{\overline{\eta}}$. The terms

$$\frac{\partial}{\partial z_i} \widehat{\mathcal{G}}_{\overline{\eta}}(\widehat{\mathbf{u}}, \eta) = \sum_{j,k} \frac{\partial^2 \widehat{u}_k}{\partial z_j \partial z_i} (\delta_{j,k} - J^{k,j}_{\overline{\eta},\eta}) - \frac{\partial \widehat{u}_k}{\partial z_j} \frac{\partial}{\partial z_i} J^{k,j}_{\overline{\eta},\eta}$$

are similar to those in $\widehat{\mathcal{F}}_{\overline{\eta}}$. Thus, proceeding as in the proof of Lemma 6.7, we obtain

$$\| \widehat{\mathcal{G}}_{\overline{\eta}}(\widehat{\mathbf{u}}, \eta) \|_{L^2(0,T;H^1(\Omega_{\overline{\eta}}))} \leqslant CT^\alpha. \tag{6.10}$$

Next we estimate the $H^1(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}}))')$-norm of $\widehat{\mathcal{G}}_{\overline{\eta}}$. We calculate

$$\frac{\partial}{\partial t}\widehat{\mathcal{G}}_{\overline{\eta}}(\widehat{\mathbf{u}},\eta)=\sum_{j,k}\frac{\partial}{\partial t}\frac{\partial\widehat{u}_k}{\partial z_j}(\delta_{j,k}-J^{k,j}_{\overline{\eta},\eta})-\frac{\partial\widehat{u}_k}{\partial z_j}\frac{\partial}{\partial t}J^{k,j}_{\overline{\eta},\eta}.$$

Let $\varphi\in H^1_{\delta_0}(\Omega_{\overline{\eta}})$. For all $1\leqslant j,k\leqslant 3$, and all $t\in(0,T)$, we have

$$\begin{aligned}
&\left\langle\frac{\partial}{\partial t}\frac{\partial\widehat{u}_k}{\partial z_j}(\delta_{j,k}-J^{k,j}_{\overline{\eta},\eta}),\varphi\right\rangle_{(H^1_{\delta_0}(\Omega_{\overline{\eta}}))',H^1_{\delta_0}(\Omega_{\overline{\eta}})}\\
&=-\int_{\Omega_{\overline{\eta}}}\frac{\partial\widehat{u}_k}{\partial t}(\delta_{j,k}-J^{k,j}_{\overline{\eta},\eta})\frac{\partial\varphi}{\partial z_j}+\int_{\Omega_{\overline{\eta}}}\frac{\partial\widehat{u}_k}{\partial t}\frac{\partial}{\partial z_j}J^{k,j}_{\overline{\eta},\eta}\varphi+\int_{\Gamma^s_{\overline{\eta}}}(\eta_{tt}\mathbf{e}_r)_k(\delta_{j,k}-J^{k,j}_{\overline{\eta},\eta})|_{\Gamma^s_{\overline{\eta}}}\varphi|_{\Gamma^s_{\overline{\eta}}}\\
&\leqslant C\Big(\|\widehat{u}_{k,t}\|_{L^2(\Omega_{\overline{\eta}})}\|I-J_{\overline{\eta},\eta}\|_{H^1(0,L;L^\infty(B(0,1+\overline{\eta})))}\\
&+\|\widehat{u}_{k,t}\|_{L^2(\Omega_{\overline{\eta}})}\left\|\frac{\partial}{\partial z_j}J^{k,j}_{\overline{\eta},\eta}\right\|_{H^{1/2+s}(\Omega_{\overline{\eta}})}+\|\eta_{tt}\|_{L^2(\omega)}\|I-J_{\overline{\eta},\eta}\|_{H^{3/2+s}(\Omega_{\overline{\eta}})}\Big)\|\varphi\|_{H^1_{\delta_0}(\Omega_{\overline{\eta}})},
\end{aligned}$$

where $s\in(0,1/2)$ such that $1+s+\varepsilon_0\geqslant 3/2$. Therefore, with the above estimate and with $(6.5)_1$, we obtain

$$\begin{aligned}
&\left\|\frac{\partial}{\partial t}\frac{\partial\widehat{u}_k}{\partial z_j}(\delta_{j,k}-J^{k,j}_{\overline{\eta},\eta})\right\|_{L^2(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}}))')}\\
&\leqslant C\|\widehat{u}_{k,t}\|_{L^2(0,T;L^2(\Omega_{\overline{\eta}}))}\Big(\|\delta_{k,j}-J^{k,j}_{\overline{\eta},\eta}\|_{L^\infty_w(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))}\\
&+\left\|\frac{\partial}{\partial z_j}J^{k,j}_{\overline{\eta},\eta}\right\|_{L^\infty(0,T;H^{1/2+s}(\Omega_{\overline{\eta}}))}\Big)+\|\eta_{tt}\|_{L^2(0,T;L^2(\omega))}\|\delta_{j,k}-J^{k,j}_{\overline{\eta},\eta}\|_{L^\infty(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))}\\
&\leqslant CT^\alpha.
\end{aligned}$$

In the above estimate we have used the fact that $\dfrac{\partial}{\partial z_j}J^{k,j}_{\overline{\eta},\eta}\Big|_{t=0}=0$. Similarly, for all $1\leqslant j,k\leqslant 3$,

$$\begin{aligned}
\left\langle\frac{\partial\widehat{u}_k}{\partial z_j}\frac{\partial}{\partial t}J^{k,j}_{\overline{\eta},\eta},\varphi\right\rangle_{(H^1_{\delta_0}(\Omega_{\overline{\eta}}))',H^1_{\delta_0}(\Omega_{\overline{\eta}})}&=\int_{\Omega_{\overline{\eta}}}\frac{\partial\widehat{u}_k}{\partial z_j}\left(\frac{\partial}{\partial t}J^{k,j}_{\overline{\eta},\eta}\varphi\right)\\
&\leqslant C\left\|\frac{\partial\widehat{u}_k}{\partial z_j}(t,\cdot)\right\|_{L^2(\Omega_{\overline{\eta}})}\left\|\frac{\partial}{\partial t}J^{k,j}_{\overline{\eta},\eta}(t,\cdot)\right\|_{H^s(\Omega_{\overline{\eta}})}\|\varphi\|_{H^1_{\delta_0}(\Omega_{\overline{\eta}})},
\end{aligned}$$

where $s\in(0,1)$ such that $s+1/2+\varepsilon_0\geqslant 3/2$. Let us fix such s. We want to estimate $L^2(0,T;H^s(\Omega_{\overline{\eta}}))$ norm of $\partial_t J_{\overline{\eta},\eta}$. We write

$$\frac{\partial}{\partial t}J_{\overline{\eta},\eta}=-\nabla X^{-1}_{\overline{\eta},\eta}\left(\frac{\partial}{\partial t}\nabla X_{\overline{\eta},\eta}\right)\nabla X^{-1}_{\overline{\eta},\eta}=-J_{\overline{\eta},\eta}\left(\frac{\partial}{\partial t}\nabla X_{\overline{\eta},\eta}\right)J_{\overline{\eta},\eta}.$$

Applying [18, Proposition B1], Lemma 6.6 and Lemma 6.5, we obtain

$$\begin{aligned}
\|J_{\overline{\eta},\eta,t}\|_{L^2(0,T;H^s(\Omega_{\overline{\eta}}))}&\leqslant C\|J_{\overline{\eta},\eta}\|^2_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))}\|\partial_t\nabla X_{\overline{\eta},\eta}\|_{L^2(0,T;H^s(\Omega_{\overline{\eta}}))}\\
&\leqslant C\|\nabla\eta_t\|_{L^2(0,T;H^s(\Omega_{\overline{\eta}}))}\leqslant CT^\alpha.
\end{aligned}$$

Combining the above two estimates, we get

$$\begin{aligned}
&\left\|\frac{\partial\widehat{u}_k}{\partial z_j}\frac{\partial}{\partial t}J^{k,j}_{\overline{\eta},\eta}\right\|_{L^2(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}}))')}\\
&\leqslant C\left\|\frac{\partial\widehat{u}_k}{\partial z_j}\right\|_{L^\infty(0,T;L^2(\Omega_{\overline{\eta}}))}\left\|\frac{\partial}{\partial t}J^{k,j}_{\overline{\eta},\eta}\right\|_{L^2(0,T;H^s(\Omega_{\overline{\eta}}))}\leqslant CT^\alpha.
\end{aligned}$$

Thus we have proved

$$\left\| \frac{\partial}{\partial t} \widehat{\mathcal{G}}_{\overline{\eta}}(\widehat{\mathbf{u}}, \eta) \right\|_{L^2(0,T;(H^1_{\delta_0}(\Omega_{\overline{\eta}}))')} \leqslant CT^{\alpha}. \tag{6.11}$$

We obtain $(6.9)_1$, by combining (6.10)-(6.11). We can prove the Lipschitz estimate in a similar manner. □

Estimate of $\widehat{\mathcal{H}}_{\overline{\eta}}$.

Lemma 6.9. *There exist positive constant $C_{\widehat{\mathcal{H}}}$ and α, depending on $\mu > 0$, $\gamma > 0$, and $\Omega_{\overline{\eta}}$ such that, for all $0 < T < 1$, all $(\widehat{\mathbf{u}}, \widehat{p}, \eta) \in B(Q^T_{\overline{\eta}}; \gamma, \mu)$, $(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1) \in B(Q^T_{\overline{\eta}}; \gamma, \mu)$ and $(\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2) \in B(Q^T_{\overline{\eta}}; \gamma, \mu)$, we have*

$$\begin{aligned} &\|\widehat{\mathcal{H}}_{\overline{\eta}}(\widehat{\mathbf{u}}, \widehat{p}, \eta)\|_{L^2(0,T;H^{\varepsilon_0}(\omega))} \leqslant C_{\widehat{\mathcal{H}}} T^{\alpha}, \\ &\|\widehat{\mathcal{H}}_{\overline{\eta}}(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1) - \widehat{\mathcal{H}}_{\overline{\eta}}(\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2)\|_{L^2(0,T;L^2(\omega))} \\ &\leqslant C_{\widehat{\mathcal{H}}} T^{\alpha} \|(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1) - (\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2)\|_{B(Q^T_{\overline{\eta}})}. \end{aligned} \tag{6.12}$$

Proof. Let us recall $\widehat{\mathcal{H}}_{\overline{\eta}}$ is defined in (3.9). With $(6.6)_1$, we have

$$\begin{aligned} &\left\| \nu \boldsymbol{\gamma}^s_{\overline{\eta}} \Big(\big(\nabla \widehat{\mathbf{u}} (J_{\overline{\eta},\eta} - I_{\mathbb{R}^3}) + [\nabla \widehat{\mathbf{u}} (J_{\overline{\eta},\eta} - I_{\mathbb{R}^3})]^{\top} \big) \mathbf{e}_{\overline{\eta}} \Big) \cdot \mathbf{e}_r \right\|_{L^2(0,T;H^{\varepsilon_0}(\omega))} \\ &\leqslant C \left\| \nu \Big(\nabla \widehat{\mathbf{u}} (J_{\overline{\eta},\eta} - I_{\mathbb{R}^3}) + [\nabla \widehat{\mathbf{u}} (J_{\overline{\eta},\eta} - I_{\mathbb{R}^3})]^{\top} \Big) \right\|_{L^2(0,T;H^{1/2+\varepsilon_0}(\Omega_{\overline{\eta}}))} \\ &\leqslant C \|\nabla \widehat{\mathbf{u}}\|_{L^2(0,T;\mathbf{H}^{1/2+\varepsilon_0}(\Omega_{\overline{\eta}}))} \|J_{\overline{\eta},\eta} - I_{\mathbb{R}^3}\|_{L^{\infty}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))} \leqslant CT^{\alpha}, \quad s \in (0, 1/2). \end{aligned}$$

In a similar manner, with (6.6), we obtain the following estimate of the last term of $\widehat{\mathcal{H}}_{\overline{\eta}}$:

$$\begin{aligned} &\left\| \boldsymbol{\gamma}^s_{\overline{\eta}} \big(\widehat{p} \, (\operatorname{cof}(\nabla X_{\overline{\eta},\eta}) - I_{\mathbb{R}^3}) \, \mathbf{e}_{\overline{\eta}} \big) \cdot \mathbf{e}_r \right\|_{L^2(0,T;H^{\varepsilon_0}(\omega))} \\ &\leqslant C \, \|\widehat{p} \, (\operatorname{cof}(\nabla X_{\overline{\eta},\eta}) - I_{\mathbb{R}^3})\|_{L^2(0,T;H^{1/2+\varepsilon_0}(\Omega_{\overline{\eta}}))} \\ &\leqslant C \|\widehat{p}\|_{L^2(0,T;H^{1/2+\varepsilon_0}(\Omega_{\overline{\eta}}))} \, \|\operatorname{cof}(\nabla X_{\overline{\eta},\eta}) - I_{\mathbb{R}^3}\|_{L^{\infty}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))} \leqslant CT^{\alpha}. \end{aligned}$$

Finally, using Lemma 6.4, Lemma 6.5 and [18, Proposition B1], we estimate the second term of $\widehat{\mathcal{H}}$ as follows:

$$\begin{aligned} &\left\| \nu \boldsymbol{\gamma}^s_{\overline{\eta}} \big(\big(\nabla \widehat{\mathbf{u}} (Z - I_{\mathbb{R}^3}) + [\nabla \widehat{\mathbf{u}} (J_{\overline{\eta},\eta} - I_{\mathbb{R}^3})]^{\top} \big) \mathbf{e}_{\overline{\eta}} \big) \cdot \mathbf{e}_r \right\|_{L^2(0,T;H^{\varepsilon_0}(\omega))} \\ &\leqslant C \left\| \nu \Big(\nabla \widehat{\mathbf{u}} (J_{\overline{\eta},\eta} - I_{\mathbb{R}^3}) + [\nabla \widehat{\mathbf{u}} (J_{\overline{\eta},\eta} - I_{\mathbb{R}^3})]^{\top} \Big) \right\|_{L^2(0,T;H^{1/2+\varepsilon_0}(\Omega))} \\ &\leqslant C \|\widehat{\mathbf{u}}\|_{L^2(0,T;\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\overline{\eta}}))} \|J_{\overline{\eta},\eta} - I_{\mathbb{R}^3}\|_{L^{\infty}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))} \leqslant CT^{\alpha}. \end{aligned}$$

Thus $(6.12)_1$ is proved. The Lipschitz estimate can be proved in a similar manner. □

Estimate of $\mathscr{L}_{mem}$.

Lemma 6.10. *There exist positive constants $C_{\mathscr{L}}$ and α, depending on $\mu > 0$, $\gamma > 0$, and $\Omega_{\overline{\eta}}$ such that, for all $0 < T < 1$, all $(\widehat{\mathbf{u}}, \widehat{p}, \eta) \in B(Q^T_{\overline{\eta}}; \gamma, \mu)$, $(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1) \in B(Q^T_{\overline{\eta}}; \gamma, \mu)$ and $(\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2) \in B(Q^T_{\overline{\eta}}; \gamma, \mu)$, we have*

$$\begin{aligned} &\|\mathscr{L}_{mem}\eta\|_{L^2(0,T;H^{\varepsilon_0}(\omega))} \leqslant C_{\mathscr{L}} T^{\alpha}, \\ &\|\mathscr{L}_{mem}\eta^1 - \mathscr{L}_{mem}\eta^2\|_{L^2(0,T;L^2(\omega))} \\ &\quad \leqslant C_{\mathscr{L}} T^{\alpha} \|(\widehat{\mathbf{u}}^1, \widehat{p}^1, \eta^1) - (\widehat{\mathbf{u}}^2, \widehat{p}^2, \eta^2)\|_{B(Q^T_{\overline{\eta}})}. \end{aligned} \tag{6.13}$$

Proof. Using Lemma 6.5, we can estimate the several terms of $\mathscr{L}_{mem}\eta$ as follows

$$\begin{aligned} &\left\| (1+\eta) \frac{E\sigma_P}{1-\sigma_P^2} \eta^2_{z_1} \right\|_{L^2(0,T;H^{\varepsilon_0}(\omega))} \leqslant C \|\eta_{z_1}(1+\eta)\|_{L^{\infty}(0,T;H^2(\omega))} \|\eta\|_{L^2(0,T;H^{1+\varepsilon_0}(\omega))} \\ &\leqslant CT^{\alpha}, \end{aligned}$$

and

$$\left\| \frac{E}{1+\sigma_P} \frac{\partial}{\partial \theta} (\eta^2_{z_1} \eta_{\theta}) \right\|_{L^2(0,T;H^{\varepsilon_0}(\omega))} \leqslant CT^{1/2} \|\eta\|^3_{L^{\infty}(0,T;H^3(\omega))}.$$

The other estimates can be obtained similarly. Thus $(6.13)_1$ is proved and $(6.13)_2$ can be deduced with the same arguments. □

7. Proofs of Theorem 3.3 and Theorem 1.7

Proof of Theorem 3.3. Step 1. Local in time existence. We choose $\mu > 0$ and $M > 0$ such that

$$\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega_{\eta_1^0})} + \|\eta_1^0\|_{H^{3+\varepsilon_0}(\omega)} + \|\eta_2^0\|_{H^1(\omega)} \leqslant M \qquad \mu = 2C_{\eta_1^0} M, \tag{7.1}$$

where $C_{\eta_1^0}$ is the continuity constant in (5.2) corresponding to $\overline{\eta} = \eta_1^0$. Let us set

$$\gamma = \gamma_{\eta_1^0}/2, \quad \text{with} \quad \gamma_{\eta_1^0} = \min\{1 + \eta_1^0(z_1,\theta) \mid (z_1,\theta) \in \overline{\omega}\} > 0.$$

We recall that the norm $|\!|\!|\cdot|\!|\!|_{B(Q^T_{\eta_1^0})}$ and the ball $B(Q^T_{\eta_1^0};\gamma,\mu)$ are introduced in (3.5) and (6.7) respectively. We consider the following system

$$\begin{aligned}
&\widehat{\mathbf{u}}_t - \operatorname{div}\,\sigma(\widehat{\mathbf{u}},\widehat{p}) = \widehat{\mathcal{F}}_{\eta_1^0}(\mathbf{\Phi},\psi,k) - \nu\nabla\widehat{\mathcal{G}}_{\eta_1^0}(\mathbf{\Phi},k), \quad \operatorname{div}\,\widehat{\mathbf{u}} = \widehat{\mathcal{G}}_{\eta_1^0}(\mathbf{\Phi},k) \quad \text{in } Q^T_{\eta_1^0},\\
&\widehat{\mathbf{u}} = \eta_t \mathbf{e}_r \quad \text{on } (0,T)\times\Gamma^s_{\eta_1^0},\\
&\sigma(\widehat{\mathbf{u}},\widehat{p})\mathbf{n} = 0 \text{ on } \Sigma_T^{\text{in}}, \quad \sigma(\widehat{\mathbf{u}},\widehat{p})\mathbf{n} = 0 \text{ on } \Sigma_T^{\text{out}},\\
&\widehat{\mathbf{u}}(0) = \mathbf{u}_0 \circ X_{\eta_1^0} \text{ in } \Omega_{\eta_1^0},\\
&\eta_{tt} + \Delta_s^2\eta - \Delta_s\eta_t = -\boldsymbol{\gamma}^s_{\eta_1^0}\Big(\sigma(\widehat{\mathbf{u}},\widehat{p})\mathbf{e}_{\eta_1^0}\Big)\cdot\mathbf{e}_r\\
&\qquad\qquad + \widehat{\mathcal{H}}_{\eta_1^0}(\mathbf{\Phi},\psi,k) - \mathscr{L}_{mem}k \quad \text{in } (0,T)\times\omega,\\
&\eta = \frac{\partial\eta}{\partial z_1} = 0 \quad \text{on } (0,T)\times\{0,L\}\times(0,2\pi),\\
&\eta(0) = \eta_1^0, \quad \eta_t(0) = \eta_2^0 \quad \text{in } (0,L),
\end{aligned} \tag{7.2}$$

where the nonlinear terms are defined as in (3.7)-(3.9).

To prove the existence of a strong solution to system (3.11), we are going to show that there exists $0 < T < 1$ such that the mapping

$$\mathcal{N} : (\mathbf{\Phi},\psi,k) \mapsto (\widehat{\mathbf{u}},\widehat{p},\eta),$$

where $(\widehat{\mathbf{u}},\widehat{p},\eta)$ is the solution to system (7.2), is a strict contraction in $B(Q^T_{\eta_1^0};\gamma,\mu)$.

Applying Theorem 5.2 to system (7.2), we obtain

$$\begin{aligned}
|\!|\!|(\widehat{\mathbf{u}},\widehat{p},\eta)|\!|\!|_{B(Q^T_{\eta_1^0})} \leqslant C_{\eta_1^0}\Big(&\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega_{\eta_1^0})} + \|\eta_1^0\|_{H^3(\omega)} + \|\eta_2^0\|_{H^1_0(\omega)}\\
&+ \|\widehat{\mathcal{F}}_{\eta_1^0}(\mathbf{\Phi},\psi,k)\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\eta_1^0}))} + \|\widehat{\mathcal{G}}_{\eta_1^0}(\mathbf{\Phi},k)\|_{L^2(0,T;H^1(\Omega_{\eta_1^0}))\cap H^1(0,T;(H^1_{\delta_0}(\Omega_{\eta_1^0}))')}\\
&+ \|\widehat{\mathcal{H}}_{\eta_1^0}(\mathbf{\Phi},\psi,k)\|_{L^2(0,T;L^2(\omega))} + \|\mathscr{L}_{mem}k\|_{L^2(0,T;L^2(\omega))}\Big).
\end{aligned} \tag{7.3}$$

where $C_{\eta_1^0}$ is the constant appearing in (5.2) corresponding to $\overline{\eta} = \eta_1^0$. Since $(\mathbf{\Phi},\psi,k) \in B(Q^T_{\eta_1^0};\gamma,\mu)$, applying Lemma 6.7 - Lemma 6.10, estimate (7.3) becomes

$$|\!|\!|(\widehat{\mathbf{u}},\widehat{p},\eta)|\!|\!|_{B(Q^T_{\eta_1^0})} \leqslant C_{\eta_1^0} M + C_{\eta_1^0}(C_{\widehat{\mathcal{F}}} + C_{\widehat{\mathcal{G}}} + C_{\widehat{\mathcal{H}}} + C_{\mathscr{L}})T^\alpha.$$

Therefore, with the choice of μ in (7.1), we can choose $T > 0$ small enough to have

$$|\!|\!|\mathcal{N}(\mathbf{\Phi},\psi,k)|\!|\!|_{B(Q^T_{\eta_1^0})} \leqslant \mu.$$

Using the continuous embedding $L^\infty(0,T;H^{5/2+s_1}(\omega)) \hookrightarrow L^\infty(\omega_T)$, with Lemma 6.5, we obtain

$$\|\eta - \eta_1^0\|_{L^\infty(\omega_T)} \leqslant CT^{\alpha_1},$$

where the constant C is independent of T. By choosing $T > 0$ small enough, we get $1 + \eta(t,z_1,\theta) \geqslant \gamma$ for all $(t,z_1,\theta) \in (0,T)\times\omega$. Therefore, $\mathcal{N}$ maps $B(Q^T_{\eta_1^0};\gamma,\mu)$ into itself. Now we will show that $\mathcal{N}$ is a contraction

in $B(Q^T_{\eta_1^0};\gamma,\mu)$. Let $(\mathbf{\Phi}^1,\psi^1,k^1)$ and $(\mathbf{\Phi}^2,\psi^2,k^2)$ belong to $B(Q^T_{\eta_1^0};\gamma,\mu)$. For $j=1,2$, we set $\mathcal{N}(\mathbf{\Phi}^j,\psi^j,k^j):=(\widehat{\mathbf{u}}^j,p^j,\eta^j)$. Using Theorem 5.2, Lemma 6.7, Lemma 6.8 and Lemma 6.9, we obtain

$$\begin{aligned}&\left|\!\left|\!\left|(\widehat{\mathbf{u}}^1,p^1,\eta^1)-(\widehat{\mathbf{u}}^2,p^2,\eta^2)\right|\!\right|\!\right|_{B(Q^T_{\eta_1^0})}\\&\qquad\leqslant C_{\eta_1^0}(C_{\widehat{\mathcal{F}}}+C_{\widehat{\mathcal{G}}}+C_{\widehat{\mathcal{H}}}+C_{\mathscr{L}})T^\alpha\left|\!\left|\!\left|(\mathbf{\Phi}^1,\psi^1,k^1)-(\mathbf{\Phi}^2,\psi^2,k^2)\right|\!\right|\!\right|_{B(Q^T_{\eta_1^0})}.\end{aligned}$$

Thus $\mathcal{N}$ is a contraction in $B(Q^T_{\eta_1^0};\gamma,\mu)$ for $T>0$ small enough. The proof of the local in time existence is complete.

Step 2. We prove that any local-in-time strong solution $(\widehat{\mathbf{u}},\widehat{p},\eta)$, over $[0,T_1]$, to the nonlinear system (3.11) can be extended to a maximal strong solution. We look for functions $(\widehat{\mathbf{v}},\widehat{q},\xi)\in B(Q^T_{\eta_1^0})$, with $T\geqslant T_1$, such that

$$\begin{aligned}&(\widehat{\mathbf{v}},\widehat{q},\xi)(t)=(\widehat{\mathbf{u}},\widehat{p},\eta)(t)\quad\text{for all } t\in[0,T_1],\\&(\widehat{\mathbf{v}},\widehat{q},\xi)\text{ is a strong solution to (3.11) over } [0,T].\end{aligned}\tag{7.4}$$

For a given quadruplet $(\widehat{\mathbf{v}},\widehat{q},\xi,T)$ satisfying (7.4), we set

$$\widehat{T}=\sup\{T\geqslant T_1\mid(\widehat{\mathbf{v}},\widehat{q},\xi,T)\text{ satisfies (7.4)}\}.$$

If there exists $(\widehat{\mathbf{v}},\widehat{q},\xi,T)$ satisfying (7.4) for all $T<\widehat{T}=\infty$, the proof is complete.

We assume that $(\widehat{\mathbf{v}},\widehat{q},\xi,T)$ satisfies (7.4) for all $T<\widehat{T}<\infty$. We have to show that (3.12) holds for $(\widehat{\mathbf{u}},\widehat{p},\eta)=(\widehat{\mathbf{v}},\widehat{q},\xi)$ and $T_{\rm m}=\widehat{T}$. We argue by contradiction. We assume that

$$\lim_{T\to\widehat{T}}\left(\|(\widehat{\mathbf{v}},\widehat{q},\xi)\|_{B(Q^T_{\eta_1^0})}+\max\left\{|1+\xi(T,z_1,\theta)|^{-1}\mid(z_1,\theta)\in\overline{\omega}\right\}\right)<\infty.\tag{7.5}$$

Thus, we have

$$\|(\widehat{\mathbf{v}},\widehat{q},\xi)\|_{B(Q^{\widehat{T}}_{\eta_1^0})}=\mu_{\widehat{T}}<\infty,\quad\min\{1+\xi(t,z_1,\theta)\mid(z_1,\theta)\in\overline{\omega},\ t\in[0,\widehat{T}]\}=\gamma_{\widehat{T}}>0.$$

We are going to show that the solution $(\widehat{\mathbf{v}},\widehat{q},\xi)$ can be extended to $[\widehat{T},\widehat{T}+\varepsilon]$, for some $\varepsilon>0$, so that $(\widehat{\mathbf{v}},\widehat{q},\xi)$ is solution to system (3.11) over the time interval $[0,\widehat{T}+\varepsilon]$.

Rather than considering the system (3.11) over the time interval $[\widehat{T},\tau]$, we are going to rewrite it in the configuration $\Omega_{\xi(\widehat{T})}$.

We set $\overline{\eta}=\xi(\widehat{T})$. From estimates (5.3), $(6.12)_1$, and $(6.13)_1$, knowing that $\widehat{T}>0$, it follows that $\overline{\eta}\in H^{3+\varepsilon_0}(\omega)\cap H^2_0(\omega)$. With (7.5), we have $1+\overline{\eta}(z_1,\theta)\geqslant\overline{\gamma}$ for some $\overline{\gamma}>0$. We look for displacements $\widetilde{\xi}$ satisfying

$$\widetilde{\xi}\in H^{4,2}(\omega\times(\widehat{T},\tau))\quad\text{and}\quad\widetilde{\xi}(\widehat{T})=\xi(\widehat{T})=\overline{\eta}.\tag{7.6}$$

We look for a solution $(\widetilde{\mathbf{v}},\widetilde{q},\widetilde{\xi})$, over the time interval $(\widehat{T},\tau)$ with $\tau>\widehat{T}$, to the system

$$\begin{aligned}&\widetilde{\mathbf{v}}_t-\operatorname{div}\sigma(\widetilde{\mathbf{v}},\widetilde{q})=\widehat{\mathcal{F}}_{\overline{\eta}}(\widetilde{\mathbf{v}},\widetilde{q},\widetilde{\xi})-\nu\nabla\widehat{\mathcal{G}}_{\overline{\eta}}(\widetilde{\mathbf{v}},\widetilde{\xi})\quad\text{in }(\widehat{T},\tau)\times\Omega_{\overline{\eta}},\\&\operatorname{div}\widetilde{\mathbf{v}}=\widehat{\mathcal{G}}_{\overline{\eta}}(\widetilde{\mathbf{v}},\widetilde{\xi})\quad\text{in }(\widehat{T},\tau)\times\Omega_{\overline{\eta}},\\&\widetilde{\mathbf{v}}=\widetilde{\xi}_t\mathbf{e}_r\quad\text{on }(\widehat{T},\tau)\times\Gamma^s_{\overline{\eta}},\\&\sigma(\widetilde{\mathbf{v}},\widetilde{p})\mathbf{n}=0\text{ on }(\widehat{T},\tau)\times\Gamma_{\rm in},\quad\sigma(\widetilde{\mathbf{v}},\widetilde{p})\mathbf{n}=0\text{ on }(\widehat{T},\tau)\times\Gamma_{\rm out},\\&\widetilde{\mathbf{v}}(\widehat{T})=\widehat{\mathbf{v}}(\widehat{T},X_{\overline{\eta},\eta_1^0})\text{ in }\Omega_{\overline{\eta}},\\&\widetilde{\xi}_{tt}+\mathscr{L}_{mem}\widetilde{\xi}+\Delta_s^2\widetilde{\xi}-\Delta_s\widetilde{\xi}_t=\\&-\gamma^s_{\overline{\eta}}\big(\sigma(\widetilde{\mathbf{v}},\widetilde{q})\mathbf{e}_{\overline{\eta}}\big)\cdot\mathbf{e}_r+\widehat{\mathcal{H}}_{\overline{\eta}}(\widetilde{\mathbf{v}},\widetilde{p},\xi)\quad\text{in }(\widehat{T},\tau)\times\omega,\\&\widetilde{\xi}=\frac{\partial\widetilde{\xi}}{\partial z_1}=0\quad\text{on }(\widehat{T},\tau)\times\{0,L\}\times(0,2\pi),\\&\widetilde{\xi}(\widehat{T})=\xi(\widehat{T}),\quad\widetilde{\xi}_t(\widehat{T})=\xi_t(\widehat{T})\quad\text{in }\omega,\end{aligned}\tag{7.7}$$

where $X_{\overline{\eta},\eta_1^0} = X_{\eta_1^0} \circ Y_{\overline{\eta}}$. We consider the system

$$
\begin{aligned}
&\widetilde{\mathbf{v}}_t - \operatorname{div} \sigma(\widetilde{\mathbf{v}}, \widetilde{q}) = \widehat{\mathcal{F}}_{\overline{\eta}}(\mathbf{v}, q, \zeta) - \nu \nabla \widehat{\mathcal{G}}_{\overline{\eta}}(\mathbf{v}, \zeta) \quad \text{in } (\widehat{T}, \tau) \times \Omega_{\overline{\eta}},\\
&\operatorname{div} \widetilde{\mathbf{v}} = \widehat{\mathcal{G}}_{\overline{\eta}}(\mathbf{v}, \zeta) \quad \text{in } (\widehat{T}, \tau) \times \Omega_{\overline{\eta}},\\
&\widetilde{\mathbf{v}} = \widetilde{\zeta}_t \mathbf{e}_r \quad \text{on } (\widehat{T}, \tau) \times \Gamma^s_{\overline{\eta}},\\
&\sigma(\widetilde{\mathbf{v}}, \widetilde{p})\mathbf{n} = 0 \text{ on } (\widehat{T}, \tau) \times \Gamma_{\text{in}}, \quad \sigma(\widetilde{\mathbf{v}}, \widetilde{p})\mathbf{n} = 0 \text{ on } (\widehat{T}, \tau) \times \Gamma_{\text{out}},\\
&\widetilde{\mathbf{v}}(\widehat{T}) = \widehat{\mathbf{v}}(\widehat{T}, X_{\overline{\eta},\eta_1^0}) \text{ in } \Omega_{\overline{\eta}},\\
&\widetilde{\xi}_{tt} + \mathscr{L}_{mem}\zeta + \Delta_s^2 \widetilde{\xi} - \Delta_s \widetilde{\xi}_t =\\
&- \boldsymbol{\gamma}^s_{\overline{\eta}}\big(\sigma(\widetilde{\mathbf{v}}, \widetilde{q})\mathbf{e}_{\overline{\eta}}\big) \cdot \mathbf{e}_r + \widehat{\mathcal{H}}_{\overline{\eta}}(\mathbf{v}, p, \zeta) \quad \text{in } (\widehat{T}, \tau) \times \omega,\\
&\widetilde{\xi} = \frac{\partial \widetilde{\xi}}{\partial z_1} = 0 \quad \text{on } (\widehat{T}, \tau) \times \{0, L\} \times (0, 2\pi),\\
&\widetilde{\xi}(\widehat{T}) = \xi(\widehat{T}), \quad \widetilde{\xi}_t(\widehat{T}) = \xi_t(\widehat{T}) \quad \text{in } \omega.
\end{aligned}
\tag{7.8}
$$

We set

$$\widetilde{\gamma} = \gamma_{\widehat{T}}/2, \quad \text{and} \quad \widetilde{\mu} = C_{\overline{\eta}}\mu_{\widehat{T}} + \mu_{\widehat{T}}.$$

We introduce the mapping $\widetilde{\mathcal{N}}_\tau$ from $B(Q_{\overline{\eta}}^{\widehat{T},\tau}; \widetilde{\gamma}, \widetilde{\mu})$ into $B(Q_{\overline{\eta}}^{\widehat{T},\tau})$, defined by

$$\widetilde{\mathcal{N}}_\tau : (\mathbf{v}, q, \zeta) \mapsto (\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\xi}),$$

where $(\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\xi})$ is the solution to system (7.8) over $[\widehat{T}, \tau]$.

Applying Theorem 5.2 to system (7.8), as in Step 1, we can prove that $\widetilde{\mathcal{N}}_\tau$ is a contraction in $B(Q_{\overline{\eta}}^{\widehat{T},\tau}; \widetilde{\gamma}, \widetilde{\mu})$ for $\tau - \widehat{T} > 0$ small enough. Thus the system (7.7) admits a solution over the time interval $(\widehat{T}, \tau)$ in $\widetilde{B}(Q_{\overline{\eta}}^{\widehat{T},\tau}; \widetilde{\gamma}, \widetilde{\mu})$ for $\tau - \widehat{T} > 0$ small enough. We can extend the triplet $(\widehat{\mathbf{v}}, \widehat{q}, \xi)$, defined over $(0, \widehat{T})$, to $(\widehat{T}, \tau)$ by setting

$$\widehat{\mathbf{v}}(t, z) = \widetilde{\mathbf{v}}(t, Y_{\overline{\eta}}(t, z)), \quad \widehat{q}(t, z) = \widetilde{q}(t, Y_{\overline{\eta}}(t, z)), \quad \xi(t, z_1, \theta) = \widetilde{\xi}(t, z_1, \theta).$$

We can show that $(\widehat{\mathbf{v}}, \widehat{q}, \xi)$ is solution to (3.11) over $[0, \tau]$. We have a contradiction with the definition of $\widehat{T}$. Thus (7.5) is false and (3.12) is proved. We have proved that any local-in-time strong solution may be extended to a maximal strong solution.

Step 3. Uniqueness of maximal solution. Let us prove that system (3.11) admits a unique maximal solution. Let $(\widehat{\mathbf{u}}, \widehat{p}, \eta)$ be a maximal solution to system (3.11) over $[0, T_{\text{m}})$, and let $(\widehat{\mathbf{v}}, \widehat{q}, \xi)$ be another maximal solution to system (3.11) over $[0, \widetilde{T}_{\text{m}})$. Let us assume that $T_{\text{m}} \leqslant \widetilde{T}_{\text{m}}$. Let us set

$$\widehat{T} = \sup\Big\{t \in [0, T_{\text{m}}) \mid (\widehat{\mathbf{u}}, \widehat{p}, \eta)(\tau) = (\widehat{\mathbf{v}}, \widehat{q}, \xi)(\tau) \text{ for all } \tau \in [0, t]\Big\}.$$

If $\widehat{T} = T_{\text{m}} = \widetilde{T}_{\text{m}}$, then the two maximal solutions are identical and the proof is complete.

If $\widehat{T} = T_{\text{m}} < \widetilde{T}_{\text{m}}$, then

$$
\begin{aligned}
&\lim_{T \to T_{\text{m}}} \left(\|(\widehat{\mathbf{u}}, \widehat{p}, \eta)\|_{B(Q^T_{\eta_1^0})} + \max\Big\{|1 + \eta(z_1, \theta, T)|^{-1} \mid (z_1, \theta) \in \overline{\omega}\Big\}\right)\\
&= \left(\|(\widehat{\mathbf{v}}, \widehat{q}, \xi)\|_{B(Q^{T_{\text{m}}}_{\eta_1^0})} + \max\Big\{|1 + \xi(z_1, \theta, T_{\text{m}})|^{-1} \mid (z_1, \theta) \in \overline{\omega}\Big\}\right) < \infty,
\end{aligned}
$$

which is in contradiction with the fact that $(\widehat{\mathbf{u}}, \widehat{p}, \eta)$ is a maximal solution to system (3.11) over $[0, T_{\text{m}})$. Thus the proof is complete in that case too.

Let us examine the last case $\widehat{T} < T_{\text{m}}$. We have to treat separately the cases when $\widehat{T} > 0$ and the case when $\widehat{T} = 0$. The case when $\widehat{T} = 0$ can be treated with the arguments in Step 1. Let us treat the case when $\widehat{T} > 0$.

For $\widehat{T} < \tau < T_{\rm m}$, we consider the system (3.11) over the time interval $(0,\tau)$. We set

$$\mu_1 = \|(\widehat{\mathbf{u}},\widehat{p},\eta)\|_{B(Q^{\widehat{T}}_{\eta_1^0})}, \quad \gamma_1 = \min\{1+\eta(z_1,\theta,t) \mid (z_1,\theta)\in\overline{\omega},\ t\in[0,\widehat{T}]\},$$
$$\mu_2 = \|(\widehat{\mathbf{v}},\widehat{q},\xi)\|_{B(Q^{\widehat{T}}_{\eta_1^0})}, \quad \gamma_2 = \min\{1+\xi(z_1,\theta,t) \mid (z_1,\theta)\in\overline{\omega},\ t\in[0,\widehat{T}]\},$$
$$\widehat{\mu} = 2\mu_1 + 2\mu_2, \quad \text{and} \quad \widehat{\gamma} = \min\{\gamma_1/2, \gamma_2/2\}.$$

We set $\overline{\eta} = \eta(\widehat{T})$. From (5.3), we know that $\overline{\eta}\in H^{3+\varepsilon_0}(\Gamma_s)$ and that $1+\overline{\eta}(z_1,\theta) \geqslant \overline{\gamma}$ for some $\overline{\gamma}>0$.

Let $(\mathbf{u},p,\eta)$ (resp. $(\mathbf{v},q,\xi)$) be a solution of (1.1) associated with the solution $(\widehat{\mathbf{u}},\widehat{p},\eta)$ (resp. $(\widehat{\mathbf{v}},\widehat{q},\xi)$) of (3.11). We set

$$\widetilde{\mathbf{u}}(t,\zeta) = \mathbf{u}(t, X_{\overline{\eta},\eta}(t,\zeta)), \quad \widetilde{p}(t,\zeta) = p(t, X_{\overline{\eta},\eta}(t,\zeta)),$$
$$\widetilde{\mathbf{v}}(t,\zeta) = \widehat{\mathbf{v}}(t, X_{\overline{\eta},\eta}(t,\zeta)) \quad \text{and} \quad \widetilde{q}(t,\zeta) = q(t, X_{\overline{\eta},\eta}(t,\zeta)) \quad \text{for all } t\in[\widehat{T},T_{\rm m}).$$

We can easily check that the function $(\mathbf{v},q,\zeta) = (\widetilde{\mathbf{u}},\widetilde{p},\eta) - (\widetilde{\mathbf{v}},\widetilde{q},\xi)$ satisfies the system (4.2), over the interval $(\widehat{T},\tau)$, with

$$\mathbf{v}(\widehat{T}) = 0, \quad \eta_1(\widehat{T}) = 0, \quad \eta_2(\widehat{T}) = 0,$$
$$\mathbf{f} = \widehat{\mathcal{F}}_{\overline{\eta}}(\widetilde{\mathbf{u}},\widetilde{p},\eta) - \widehat{\mathcal{F}}_{\overline{\eta}}(\widetilde{\mathbf{v}},\widetilde{q},\xi), \quad g = \widehat{\mathcal{G}}_{\overline{\eta}}(\widetilde{\mathbf{u}},\eta) - \widehat{\mathcal{G}}_{\overline{\eta}}(\widetilde{\mathbf{v}},\xi),$$
$$h = \widehat{\mathcal{H}}_{\overline{\eta}}(\widetilde{\mathbf{u}},\widetilde{p},\eta) - \widehat{\mathcal{H}}_{\overline{\eta}}(\widetilde{\mathbf{v}},\widetilde{p},\xi) - \mathscr{L}_{mem}\eta + \mathscr{L}_{mem}\xi.$$

Using the Lipschitz estimates for $\widehat{\mathcal{F}}_{\overline{\eta}}$, $\widehat{\mathcal{G}}_{\overline{\eta}}$, $\widehat{\mathcal{H}}_{\overline{\eta}}$, and $\mathscr{L}_{mem}$ in $B(Q^{\widehat{T},\tau}_{\overline{\eta}};\widehat{\gamma},\widehat{\mu})$, with Theorem 5.2, we can show that

$$\|(\mathbf{v},q,\zeta)\|_{B(Q^{\widehat{T},\tau}_{\overline{\eta}})} \leqslant C_{\overline{\eta}}(C_{\widehat{\mathcal{F}}} + C_{\widehat{\mathcal{G}}} + C_{\widehat{\mathcal{H}}} + C_{\mathscr{L}})(\tau-\widehat{T})^{\alpha}\,\|(\mathbf{v},q,\zeta)\|_{B(Q^{\widehat{T},\tau}_{\overline{\eta}})}.$$

Notice that the Lipschitz estimates of Lemmas 6.7, 6.8, 6.9, and 6.10 can be used, over the interval $(\widehat{T},\tau)$, because $J_{\overline{\eta},\eta}\big|_{t=\widehat{T}} = J_{\overline{\eta},\xi}\big|_{t=\widehat{T}} = I$.

Thus, if $\tau > \widehat{T}$ is such that

$$C_{\overline{\eta}}(C_{\widehat{\mathcal{F}}} + C_{\widehat{\mathcal{G}}} + C_{\widehat{\mathcal{H}}} + C_{\mathscr{L}})(\tau-\widehat{T})^{\alpha} < 1,$$

we have

$$(\mathbf{v},q,\zeta)(t) = 0 \quad \text{for } t\in[\widehat{T},\tau].$$

Thus $(\widetilde{\mathbf{u}},\widetilde{p},\eta)(t) = (\widetilde{\mathbf{v}},\widetilde{p},\xi)(t)$ for all $t\in[\widehat{T},\tau]$, from which we deduce that $(\widehat{\mathbf{u}},\widehat{p},\eta)(t) = (\widehat{\mathbf{v}},\widehat{p},\xi)(t)$ for all $t\in[\widehat{T},\tau]$, and the proof is complete. □

Finally, we prove our main result:

Proof of Theorem 1.7. Let us assume that $(\mathbf{u}_0,\eta_1^0,\eta_2^0) \in \mathbf{H}^1(\Omega)\times H^{3+\varepsilon_0}(\omega)\times H^1(\omega)$ satisfies (1.7). From Theorem 3.3, it follows that the system (3.11) admits a unique maximal strong solution $(\widehat{\mathbf{u}},\widehat{p},\eta)$ over the interval $(0,T_{\rm m})$, and $(\widehat{\mathbf{u}},\widehat{p},\eta)\in B(Q^T_{\eta_1^0})$ for all $0<T<T_{\rm m}$. Since $\gamma_\eta(T)>0$, $X(t,\cdot)$ is a C^1-diffeomorphism from Ω into $\Omega_{\eta(t)}$ (Lemma 3.1). Thus there is a unique $Y(t,\cdot)$ from $\Omega_{\eta(t)}$ onto Ω such that $Y(t,\cdot) = X(t,\cdot)^{-1}$. We set

$$\mathbf{u}(t,z_1,x,y) = \widehat{\mathbf{u}}(t,Y(t,z_1,x,y)),$$
$$p(t,z_1,x,y) = \widehat{p}(t,Y(t,z_1,x,y)) \text{ for all } (z_1,x,y)\in\Omega_{\eta(t)}.$$

One can easily check that $(\mathbf{u},p,\eta)$ satisfy the original system (1.1) over the time interval $(0,T)$ for all $0<T<T_{\rm m}$, and that

$$\mathbf{u}\in L^2(0,T;\mathbf{H}^{3/2+\varepsilon_0}(\Omega_{\eta(\cdot)}))\cap H^1(0,T;\mathbf{L}^2(\Omega_{\eta(\cdot)}))\cap C([0,T];\mathbf{H}^1(\Omega_{\eta(\cdot)})),$$
$$p\in L^2(0,T;H^{1/2+\varepsilon_0}(\Omega_{\eta(\cdot)})), \quad \text{div }\sigma(\mathbf{u},p)\in L^2(\widetilde{Q}_T),$$
$$\eta\in H^{4,2}(\omega_T).$$

for all $0<T<T_{\rm m}$. Due to Proposition 3.4, the uniqueness of solution $(\mathbf{u},p,\eta)$ to system (1.1), in the class of functions satisfying (3.13), follows from the uniqueness of solution to system (3.11). □

Appendix A. Technical results

Throughout this section, we assume that $\overline{\eta}$ satisfies (4.1).

Theorem A.1. *Let $\xi \in L^2(\Omega_{\overline{\eta}})$. Then the solution φ to the problem*

$$-\Delta\varphi = \xi \text{ in } \Omega_{\overline{\eta}}, \quad \frac{\partial\varphi}{\partial\mathbf{n}} = 0 \text{ on } \Gamma^s_{\overline{\eta}}, \quad \varphi = 0 \text{ on } \Gamma_n,$$

belongs to $H^2(\Omega_{\overline{\eta}})$.

Proof. There may be a loss of regularity for φ at the junctions $\{(0,z_2,z_3) : z_2^2+z_3^2=1\}$ and $\{(L,z_2,z_3) : z_2^2+z_3^2=1\}$. At first we want to study the regularity at the junction $\{(0,z_2,z_3) : z_2^2+z_3^2=1\}$, the other junction can be studied in the similar way. To study the regularity of φ, we extend the equation satisfied by φ by using a symmetry argument. Let us set

$$\begin{aligned}
\widetilde{\eta}(z_1,\theta) &= \begin{cases} \overline{\eta}(-z_1,\theta), & z_1 \in (-L,0) \\ \overline{\eta}(z_1,\theta), & z_1 \in (0,L), \end{cases} \\
\widetilde{\Omega}_{\overline{\eta}} &= (-L,L)\times B(0,1+\widetilde{\eta}), \\
\widetilde{\Gamma}^s_{\overline{\eta}} &= \{(z_1, r\cos\theta, r\sin\theta) \mid r = 1+\widetilde{\eta}(z_1,\theta),\ z_1\in(-L,L),\ \theta\in(0,2\pi)\}, \\
\widetilde{\varphi}(z_1,z_2,z_3) &= \begin{cases} -\varphi(-z_1,z_2,z_3), & z_1 \in (-L,0) \\ \varphi(z_1,z_2,z_3), & z_1 \in (0,L) \end{cases} \\
\widetilde{\xi}(z_1,z_2,z_3) &= \begin{cases} -\xi(-z_1,z_2,z_3), & z_1 \in (-L,0) \\ \xi(z_1,z_2,z_3), & z_1 \in (0,L). \end{cases}
\end{aligned}$$

Thus $\widetilde{\Gamma}^s_{\overline{\eta}}$ is of class C^2, because $\widetilde{\eta}$ belongs to $H^{3+\varepsilon_0}((-L,L)\times(0,2\pi))$, and $\widetilde{\varphi}$ is solution to the equation

$$-\Delta\widetilde{\varphi} = \widetilde{\xi} \text{ in } \widetilde{\Omega}_{\overline{\eta}}, \quad \frac{\partial\widetilde{\varphi}}{\partial\mathbf{n}} = 0 \text{ on } \widetilde{\Gamma}^s_{\overline{\eta}}, \quad \widetilde{\varphi} = 0 \text{ on } (\{-L\}\cup\{L\})\times B(0,1+\widetilde{\eta}),$$

and $\widetilde{\varphi}$ belongs to $H^2(\widetilde{\Omega}_{\overline{\eta},\varepsilon})$, where $\widetilde{\Omega}_{\overline{\eta},\varepsilon} = \bigcup_{z_1\in(-L+\varepsilon,L-\varepsilon)}\{z_1\}\times B(0,1+\widetilde{\eta}(z_1,\cdot))$ and $\varepsilon>0$. By similar argument, we can deal with the other junction $\{(L,z_2,z_3) : z_2^2+z_3^2<1\}$. □

Lemma A.2. *The operator $D_s \in \mathcal{L}(H^{3/2}(\omega);\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}}))$, defined in (4.20), may be extended as a bounded operator from $L^2(\omega)$ to $\mathbf{L}^2(\Omega_{\overline{\eta}})$.*

Proof. We recall that $D_s\eta_2 = D(\eta_2\mathbf{e}_r) = D\mathbf{g}$. We consider the following equation

$$-\text{div }\sigma(\mathbf{v},q) = \mathbf{\Phi}, \quad \text{div }\mathbf{v} = 0 \text{ in } \Omega_{\overline{\eta}}, \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma^s_{\overline{\eta}}, \quad \sigma(\mathbf{v},q)\mathbf{n} = 0 \text{ on } \Gamma_n. \tag{A.1}$$

From [25, Theorem 9.4.5], it follows that $(\mathbf{v},q)$ satisfies the estimate

$$\|\mathbf{v}\|_{\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}})} + \|q\|_{H^1_{\delta_0}(\Omega_{\overline{\eta}})} \leqslant C\|\mathbf{\Phi}\|_{\mathbf{L^2(\Omega_{\overline{\eta}})}}.$$

Using the mixed variational formulations satisfied by $(\mathbf{w},\pi)$ and $(\mathbf{v},q)$ (as introduced in [25, page 384]), we have

$$\int_{\Omega_{\overline{\eta}}}\mathbf{\Phi}\cdot\mathbf{w} = \int_{\Omega_{\overline{\eta}}}\mathbf{f}\cdot\mathbf{v} - \int_{\Gamma^s_{\overline{\eta}}}\sigma(\mathbf{v},q)\mathbf{n}\cdot\mathbf{g}.$$

Hence, we have

$$\|\mathbf{w}\|_{\mathbf{L}^2(\Omega_{\overline{\eta}})} \leqslant C\left(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega_{\overline{\eta}})} + \|\mathbf{g}\|_{\mathbf{L}^2(\Gamma^s_{\overline{\eta}})}\right).$$

□

Lemma A.3. *Let us assume that $g\in H^1(\Omega_{\overline{\eta}})$ and $g=0$ on Γ_n. Then the solution $(\mathbf{w},\pi)\in\mathbf{H}^2_{\delta_0}(\Omega_{\overline{\eta}})\times H^1_{\delta_0}(\Omega_{\overline{\eta}})$ to the equation*

$$-\text{div }\sigma(\mathbf{w},\pi) = -\nu\nabla g, \ \text{div }\mathbf{w} = g \text{ in } \Omega_{\overline{\eta}}, \ \mathbf{w} = 0 \text{ on } \Gamma^s_{\overline{\eta}}, \ \sigma(\mathbf{w},\pi)\mathbf{n} = 0 \text{ on } \Gamma_n, \tag{A.2}$$

satisfies the following estimate

$$\|\mathbf{w}\|_{\mathbf{L}^2(\Omega_{\overline{\eta}})} \leqslant C\|g\|_{(H^1_{\delta_0}(\Omega_{\overline{\eta}}))'}.$$

Proof. Let $\mathbf{\Phi}$ belong to $\mathbf{L}^2(\Omega_{\overline{\eta}})$ and $(\mathbf{v}, q)$ be the solution of (A.1). Using the mixed variational formulations satisfied by $(\mathbf{w}, \pi)$ and $(\mathbf{v}, q)$, we can verify that

$$\int_{\Omega_{\overline{\eta}}} \mathbf{\Phi}\cdot\mathbf{w} = -\int_{\Omega_{\overline{\eta}}} q\ g.$$

Thus

$$\begin{aligned}\|\mathbf{w}\|_{\mathbf{L}^2(\Omega_{\overline{\eta}})} &= \sup_{\|\mathbf{\Phi}\|_{\mathbf{L}^2(\Omega_{\overline{\eta}})}=1}\left|\int_{\Omega_{\overline{\eta}}}\mathbf{\Phi}\cdot\mathbf{w}\right| = \sup_{\|\mathbf{\Phi}\|_{\mathbf{L}^2(\Omega_{\overline{\eta}})}=1}\left|\int_{\Omega_{\overline{\eta}}} q\ g\right| \\ &\leqslant \sup_{\|\mathbf{\Phi}\|_{\mathbf{L}^2(\Omega_{\overline{\eta}})}=1}\|g\|_{(H^1_{\delta_0}(\Omega_{\overline{\eta}}))'}\|q\|_{H^1_{\delta_0}(\Omega_{\overline{\eta}})} \leqslant C\|g\|_{(H^1_{\delta_0}(\Omega_{\overline{\eta}}))'}.\end{aligned}$$

□

Lemma A.4. *Let $s\in(0,1/2)$. If $f\in H^{3/4-s/2}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))\cap L^\infty(0,T;H^2(\Omega_{\overline{\eta}})\cap H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))$ and $0<m\leqslant f(x,t)$ in $\Omega_{\overline{\eta}}\times(0,T)$, then $1/f$ belongs to $H^{3/4-s/2}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))\cap L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))\cap H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))$, and it satisfies the following estimate:*

$$\begin{aligned}&\|1/f\|_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))} \leqslant C(1+\|f\|_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))})\|f\|_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))},\\ &\|1/f\|_{H^{3/4-s/2}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))} \leqslant \frac{C}{\|f\|^2_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))}}\|f\|_{H^{3/4-s/2}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))},\\ &\|1/f\|_{H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))}\\ &\quad\leqslant \frac{C}{\|f\|^2_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))}}\|f\|_{H^{3/4-s/2}(0,T;H^1(0,L;L^\infty(B(0,1+\overline{\eta}))))},\end{aligned}$$

with C is independent of T.

Proof. Let G be a C^∞, nonnegative function such that $G(0)=0$ and $G(r)=1/r$ for $r\geqslant m$. From [34, Theorem 2, pp. 336], for all $t>0$, we get

$$\begin{aligned}\|G(f(t))\|^2_{H^2(\Omega_{\overline{\eta}})} &\leqslant C\left(1+\|f(t)\|_{L^\infty(\Omega_{\overline{\eta}})}\right)^2\|f(t)\|^2_{H^2(\Omega_{\overline{\eta}})}\\ &\leqslant C\left(1+\|f\|_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))}\right)^2\|f(t)\|^2_{H^2(\Omega_{\overline{\eta}})}.\end{aligned}$$

By taking the supremum with respect to t in the above estimate, we have

$$\|G(f(t))\|_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))} \leqslant C\left(1+\|f\|_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))}\right)\|f\|_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))}.$$

With [18, Proposition B1], we can estimate

$$\begin{aligned}\|1/f\|_{H^{3/4-s/2}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))} &= \int_0^T\int_0^T \frac{\|1/f(t)-1/f(\tau)\|_{H^{3/2+s}(\Omega_{\overline{\eta}})}}{|t-\tau|^{5/2-s}}\\ &\leqslant \frac{C}{\|f\|^2_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))}}\int_0^T\int_0^T\frac{\|f(t)-f(\tau)\|_{H^{3/2+s}(\Omega_{\overline{\eta}})}}{|t-\tau|^{5/2-s}}\\ &\leqslant C\frac{\|f\|_{H^{3/4-s/2}(0,T;H^{3/2+s}(\Omega_{\overline{\eta}}))}}{\|f\|^2_{L^\infty(0,T;H^2(\Omega_{\overline{\eta}}))}}.\end{aligned}$$

The last estimate can be proved similarly. □

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