Berry-Esseen's bound and harmonic moments for supercritical multi-type branching processes in random environments

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Berry-Esseen's bound and harmonic moments for supercritical multi-type branching processes in random environments

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Abstract: . Consider a supercritical multi-type branching process in an independent and identically distributed random environment. We establish a Berry-Esseen type bound for the rate of convergence in the central limit theorem on the population size at time n as n goes to infinity. To this end we first find simple conditions for the existence of harmonic moments of the limit variable of the fundamental martingale.

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1. Introduction

A branching process in a random environment is a natural and important extension of the Galton-Watson process. In such a process, the offspring distributions of particles in n-th generation depend on an environment ξ_n at time n. This process was first introduced by Smith and Wilkinson [27] when the environment sequence (ξ_n) is independent and identically distributed, and by Athreya and Karlin [2, 3] when the environment sequence is stationary and ergodic, where basic results have been established. This process has attracted the attention of many authors in the last two decades, see for example the recent book by Kersting and Vatutin [19] and many references therein. The interest of study of such processes is growing in recent

years, thanks to a large number of applications and interactions to other scientific fields. See for example [1, 16, 17, 5, 33, 10, 11] for the single-type case, and [29, 23, 32, 30, 21, 31] for the multi-type case. The current interest of researchers mainly focuses on the multi-type case, as in this case many important problems are open and challenging. All the papers cited above on multi-type branching processes in random environments (MBPRE) concern the critical or sub-critical cases (mainly on the survival probability), except the paper [21] where asymptotic properties of $\mathbb{P}(Z_n = z)$ are studied for a super-critical MBPRE (Z_n). Very recently, in [13, 14] we obtained a theorem of Kesten-Stigum type and a criterion of L^p- convergence $(p > 1)$ for a suitable norming of the population size, for a super-critical MBPRE (Z_n). In this paper, also for a supercritical MBPRE (Z_n), we will establish a Berry-Esseen type bound for the rate of convergence in a central limit theorem on (Z_n), and prove the existence of harmonic moments of the limit of the normalized population size. These results will play very important role in the study of moderate and large deviations of (Z_n), as we will see in [15], to obtain results similar to those in [6, 22, 16] where the single type case was considered.

Let us give a quick presentation of the model with some preliminary results, and some explanations of the main results with key ideas in the proof. For an integer $d \geqslant 1$, consider a d-type branching process $Z_n = (Z_n(1), \cdots, Z_n(d))$, $n \geqslant 0$, in an independent and identically distributed (i.i.d.) random environment $\xi = (\xi_0, \xi_1, \cdots)$. In the sequel, we will denote by (Z_n^i) the process (Z_n) starting with one initial particle of type i, which means that $Z_0 = e_i$, where e_i is the vector with 1 in the i-th place and 0 elsewhere. Denote by $\mathcal{M}_d(\mathbb{R})$ the set of $d \times d$ matrices. We equip the space $\mathbb{R}^d$ with the L^1-norm $\|\cdot\|$. Let $M_n \in \mathcal{M}_d(\mathbb{R})$ be the random matrix of the conditioned means of the offspring distribution of n-th generation given the environment, that is

$$M_n(i,j) = \mathbb{E}_\xi[Z_{n+1}(j) \mid Z_n = e_i], \quad 1 \leqslant i, j \leqslant d,$$

where $\mathbb{E}_\xi$ denotes the conditional expectation given the environment ξ. Define the product matrix $M_{0,n} = M_0 \cdots M_n$, and the associated Lyapunov exponent

$$\gamma := \lim_{n \to +\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\|,$$

where $\|M_{0,n-1}\|$ is the L_1-norm of the matrix $M_{0,n-1}$.

Recently, the asymptotic behaviour of the MBPRE (Z_n^i) under the supercritical condition $\gamma > 0$ has been studied in [13]. In particular, a strong

law of large number for $\log \|Z_n^i\|$ is proved: under appropriate conditions, it holds that on the explosion event $\{\|Z_n^i\| \to +\infty\}$,

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Z_n^i\| = \gamma \quad \text{a.s.} \tag{1.1}$$

The main objective in this paper is to establish a Berry-Esseen type theorem on the rate of convergence in the central limit theorem for $\log \|Z_n^i\|$: we will show (cf. Theorem 2.5) that

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x \right) - \Phi(x) \right| \leqslant \frac{C}{\sqrt{n}}, \tag{1.2}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\, dt$ is the standard normal distribution function, $\sigma^2 = \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}[(\log \|M_{0,n-1}^T x\| - n\gamma)^2]$ is the asymptotic variance which is independent of x and $C > 0$ is a constant. This result is new for $d \geqslant 2$; for a single type branching process ($d = 1$), Grama, Liu and Miqueu proved (1.2) in [16, Theorem 1.1].

Let us briefly explain our approach for proving (1.2). It is heavily based on the fundamental martingale (W_n^i) associated to the process (Z_n^i) defined in [13]. For each $n, k \geqslant 0$, denote by $\rho_{n,n+k}$ the spectral radius of the matrix $M_{n,n+k} = M_n \cdots M_{n+k}$. By the Perron-Frobenius theorem, $\rho_{n,n+k}$ is an eigenvalue of $M_{n,n+k}$, and there exists a non negative eigenvector $U_{n,n+k}$ associated to $\rho_{n,n+k}$ with $\|U_{n,n+k}\| = 1$. According to Hennion [20, Lemma 3.3 and Theorem 1], under conditions, the limit

$$U_{n,\infty} := \lim_{k\to\infty} U_{n,n+k} \tag{1.3}$$

exists a.s., with $U_{n,\infty} > 0$ and $\|U_{n,\infty}\| = 1$; moreover the sequence $(U_{n,\infty})$ satisfies the relation

$$M_n U_{n+1,\infty} = \lambda_n U_{n,\infty}, \tag{1.4}$$

where $\lambda_n, n \geqslant 0$ are positive random scalars called the pseudo-spectral radii of the random matrices (M_n). Set $\lambda_{0,n} = \lambda_0 \cdots \lambda_n$. By iteration of (1.4), we obtain

$$M_{n,n+k} U_{n+k+1,\infty} = \lambda_{n,n+k} U_{n,\infty}, \quad n, k \geqslant 0. \tag{1.5}$$

Then, we define the martingale (W_n^i) as follows (see [13]):

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, U_{n,\infty}\rangle}{\lambda_{0,n-1} U_{0,\infty}(i)}, \quad n \geqslant 1. \tag{1.6}$$

From (1.6) the following relations between $\log\|Z_n^i\|$ and $\log\|M_{0,n-1}(i,\cdot)\|$ hold:

$$\log\|Z_n^i\| \leqslant \log\|M_{0,n-1}(i,\cdot)\| + \log W_n^i - \min_{1\leqslant j\leqslant d}\log U_{n,\infty}(j), \tag{1.7}$$

$$\log\|Z_n^i\| \geqslant \log\|M_{0,n-1}(i,\cdot)\| + \log W_n^i + \min_{1\leqslant j\leqslant d}\log U_{n,\infty}(j). \tag{1.8}$$

From these relations, since the limit $W^i = \lim_{n\to+\infty} W_n^i$ exists a.s. (as (W_n^i) is a non-negative martingale), and $(U_{n,\infty})$ is a stationary sequence of random variables, these two terms will be negligible in the limit properties that we consider. Actually, using the relations (1.7) and (1.8) we can infer the limiting behaviour of $\log\|Z_n^i\|$ from that of $\log\|M_{0,n-1}(i,\cdot)\|$ by giving a tight control of the quantities $\log W_n^i$ and $\log U_{n,\infty}(j)$. For $\log\|M_{0,n-1}(i,\cdot)\|$ we use the Berry-Esseen bound proved in Xiao, Grama and Liu [34].

An important step in our approach is to establish sufficient conditions for the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$ of the limit $W^i = \lim_{n\to+\infty} W_n^i$. This is the second objective of the paper. Actually the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$ will give us a suitable control of the sequence $(\log W_n^i)$, which will be one of the key arguments to prove (1.2).

Our study of the harmonic moments $\mathbb{E}(W^i)^{-a}$ is composed of two steps.

In the first step, under a strong assumption on the offspring distributions given the environment ξ (see **H2**), we establish a necessary condition and a sufficient condition for the existence of $\mathbb{E}(W^i)^{-a}$ for all $1 \leqslant i \leqslant d$. Set

$$\kappa_1(a) := \lim_{n\to+\infty}\Big(\mathbb{E}\|M_{0,n-1}\|^a \Big\|\prod_{k=0}^{n-1} P_1(\xi_k)\Big\|_\infty\Big)^{1/n}, \quad a>0, \tag{1.9}$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm on $\mathcal{M}_d(\mathbb{R})$, and $P_1(\xi_k)$ is the random matrix whose (i,j)-th component is the probability to produce 1 particle which is of type j by a particle of type i in generation k, given the environment ξ. Then we will prove in Theorem 2.1 the following implications:

$$\kappa_1(a) < 1 \Rightarrow \max_{1\leqslant i\leqslant d}\mathbb{E}(W^i)^{-a} < +\infty \Rightarrow \kappa_1(a) \leqslant 1. \tag{1.10}$$

In particular, the solution $a_0 > 0$ of the equation $\kappa_1(a_0) = 1$ is the critical value for the existence of the harmonic moments, in the sense that $\max_{1\leqslant i\leqslant d}\mathbb{E}(W^i)^{-a} < +\infty$ for $a < a_0$, and $\max_{1\leqslant i\leqslant d}\mathbb{E}(W^i)^{-a} = +\infty$ for $a \geqslant a_0$. For the single type case ($d = 1$), Huang and Liu proved in [22] that $\kappa_1(a) < 1$ is a necessary and sufficient for the existence of the harmonic moment $\mathbb{E}(W^1)^{-a}$. Therefore, our result (1.10) generalizes that of the single

type case, except that we don't know if the harmonic moments exists for the critical exponent a_0.

In the second step, we assume weaker conditions than in the first step (see **H3**), and we prove the existence of a small exponent $a > 0$ such that for all $1 \leqslant i \leqslant d$,

$$\mathbb{E}(W^i)^{-a} < +\infty, \tag{1.11}$$

(cf. Theorem 2.3). Unfortunately, in this case we have no information on the maximal value of the exponent $a > 0$ for which (1.11) holds: we cannot identify the critical exponent.

The outline of the paper is as follows. We introduce some necessary notation and present the main results in Section 2. Section 3 is devoted to the study of the harmonic moments of W^i for any $1 \leqslant i \leqslant d$. We prove in Section 5 the Berry-Esseen type theorem for $\log \|Z_n^i\|$.

2. Notation and main results

For $d \geqslant 1$, let $\mathbb{R}^d$ be the d-dimensional space of vectors. We equip $\mathbb{R}^d$ with the scalar product and the L^1-norm respectively defined by

$$\langle x, y\rangle := \sum_{i=1}^{d} x(i)\, y(i) \quad \text{and} \quad \|x\| := \sum_{i=1}^{d} |x(i)|, \quad x, y \in \mathbb{R}^d.$$

Set $\mathcal{S} = \{x \in \mathbb{R}^d : x \geqslant 0, \|x\| = 1\}$ for the intersection of the unit sphere with the positive quadrant. For each $1 \leqslant i \leqslant d$, e_i will be the d-dimensional vector with 1 in the i-th place and 0 elsewhere. Let $\mathbf{0} = (0, \cdots, 0) \in \mathbb{R}^d$ be the vector with all coordinates equal to 0. Denote by $\mathbb{N} = \{0, 1, \ldots\}$ the set of non-negative integers. Set $\mathbb{1}_A$ for the indicator of an event A.

Denote by $\mathcal{M}_d(\mathbb{R})$ the set of $d \times d$ matrices with entries in $\mathbb{R}$, and define the operator norm with respect to L^1 vectorial norm by

$$\|M\| := \sup_{x \in \mathcal{S}} \|Mx\|, \quad M \in \mathcal{M}_d(\mathbb{R}).$$

In addition we equip $\mathbb{R}^d$ and $\mathcal{M}_d(\mathbb{R})$ with the L^∞-norms:

$$\|x\|_\infty := \max_{1 \leqslant i \leqslant d} |x(i)|, \quad x \in \mathbb{R}^d;$$
$$\|M\|_\infty := \sup_{\|x\|_\infty = 1} \|Mx\|_\infty, \quad M \in \mathcal{M}_d(\mathbb{R}).$$

For a matrix or a vector X, we write $X > 0$ when all the entries of X are strictly positive.

Now we give a precise definition of the multi-type branching process in random environment (MBPRE). The environment $\xi = (\xi_n)_{n\geqslant 0}$ is a sequence of independent and identically distributed (i.i.d.) random variables taking values in an abstract space $\mathbb{X}$. To each realization of ξ_n correspond d probability distributions on $\mathbb{N}^d$ identified by the probability generating functions

$$f_n^r(s) = \sum_{k_1,\cdots,k_d=0}^{\infty} p^r_{k_1,\cdots,k_d}(\xi_n) s_1^{k_1}\cdots s_d^{k_d}, \quad s=(s_1,\ldots,s_d)\in[0,1]^d,$$

where $1 \leqslant r \leqslant d$. The d-type branching process $Z_n = (Z_n(1),\cdots,Z_n(d))$, $n\geqslant 0$, in the random environment ξ is a process with values in $\mathbb{N}^d$ such that $Z_0 \in \mathbb{N}^d$ is a fixed vector, and for all $n \geqslant 0$,

$$Z_{n+1} = \sum_{r=1}^{d} \sum_{l=1}^{Z_n(r)} N^r_{l,n}. \tag{2.1}$$

where $N^r_{l,n} = (N^r_{l,n}(1),\cdots,N^r_{l,n}(d))$ is a random vector whose j-th component $N^r_{l,n}(j)$ represents the offspring of type j at time $n+1$ of the l-th particle of type r in generation n, and $Z_n(j)$ is the total number of particles of type j in generation n. Conditioned on the environment ξ, the random vectors $N^r_{l,n}$ indexed by $l\geqslant 1$, $n\geqslant 0$ and $1\leqslant r\leqslant d$ are independent, each $N^r_{l,n}$ has the same probability generating function f_n^r. In the sequel, the process Z_n will be noted Z_n^i when $Z_0 = e_i$, which corresponds to the MBPRE starting with one initial particle of type i.

Denote by $\mathbb{P}_\xi$ the quenched law, i.e. the probability under which the process is defined when the environment ξ is given. Let τ be the law of ξ. The total probability $\mathbb{P}$ of (Z_n), usually called annealed law, is defined by $\mathbb{P}(dx,d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$. Denote by $\mathbb{E}_\xi$ and $\mathbb{E}$ the corresponding expectation with respect to $\mathbb{P}_\xi$ and $\mathbb{P}$. With our notation,

$$f_n^r(s) = \mathbb{E}_\xi\Big(\prod_{j=1}^{d} s_j^{N^r_{l,n}(j)}\Big), \quad s=(s_1,\ldots,s_d)\in[0,1]^d$$

is the quenched probability generating function of $N^r_{l,n}$. For $n\geqslant 0$, let M_n be the $d\times d$ random matrix whose (i,j)-th entry $M_n(i,j)$ is the conditioned mean of the number of children of type j produced by a particle of type i at time n:

$$M_n(i,j) := \frac{\partial f_n^i}{\partial s_j}(\mathbf{1}) = \mathbb{E}_\xi\big[Z_{n+1}(j)\big|Z_n = e_i\big],$$

where $\frac{\partial f}{\partial s_j}(\mathbf{1})$ denotes the left derivative at $\mathbf{1}$ of a d-dimensional probability generating function f with respect to s_j. Since the sequence of the environments (ξ_n) is i.i.d., the sequence of the mean matrices (M_n) is also i.i.d.. For $0 \leqslant k \leqslant n$, denote by

$$M_{k,n} := M_k \cdots M_n,$$

the product of the mean matrices $M_k, \ldots, M_n$. It follows that, for $n \geqslant 0$ and $1 \leqslant i, j \leqslant d$,

$$\mathbb{E}_\xi Z^i_{n+1}(j) = M_{0,n}(i,j). \tag{2.2}$$

The main objective of this paper is to establish a Berry-Esseen bound for $\log \|Z^i_n\|$. To this end, the key tool will be the fundamental martingale we mentioned in the introduction. Let $\rho_{n,n+k}$ be the spectral radius of $M_{n,n+k}$. We know by the Perron-Frobenius theorem (see e.g. [4]) that $\rho_{n,n+k}$ is a positive eigenvalue of $M_{n,n+k}$, and there exist positive right and left eigenvectors $U_{n,n+k}$ and $V_{n,n+k}$ associated to $\rho_{n,n+k}$ with the normalizations $\|U_{n,n+k}\| = 1$ and $\langle V_{n,n+k}, U_{n,n+k}\rangle = 1$. Denote by $\mathcal{G}^0_+$ the subset of the matrices of $\mathcal{M}_d(\mathbb{R})$ with strictly positive entries. Throughout the paper, we assume that M_0 is allowable (every row and column contains a strictly positive element), and that the following positivity property holds:

$$\mathbb{P}\Big(\bigcup_{n\geqslant 0} \{M_{0,n} \in \mathcal{G}^0_+\} \Big) > 0. \tag{2.3}$$

By the results of Hennion [20, Lemma 3.3 and Theorem 1], under condition (2.3) there exists the random vectors $U_{n,\infty}$ and the random scalars λ_n defined by (1.3) and (1.4), which satisfy the relation (1.5) and that $(U_{n,\infty})$ and (λ_n) are stationary ergodic; moreover, we proved in [13, Theorem 1] that the sequence (W^i_n) defined in (1.6) is a non-negative martingale under the measure $\mathbb{P}_\xi$ and $\mathbb{P}$, w.r.t. the filtration

$$\mathcal{F}_0 = \sigma(\xi), \quad \mathcal{F}_n = \sigma\big(\xi, N^r_{l,k}(j), 0 \leqslant k \leqslant n-1, 1 \leqslant r, j \leqslant d, l \geqslant 1\big) \text{ for } n \geqslant 1.$$

Let $W^i := \lim_{n\to+\infty} W^i_n$ be the a.s. limit of the martingale (W^i_n).

We will use the classification of MBPRE's defined in [13]. It is well known that, under the following moment condition

$$\mathbb{E}\log^+ \|M_0\| < +\infty, \tag{2.4}$$

the Lyapunov exponent γ of the sequence of matrices $(M_n)_{n\geqslant 0}$ exists, with

$$\gamma := \lim_{n\to+\infty} \frac{1}{n} \mathbb{E}\log \|M_{0,n-1}\| = \inf_{n\geqslant 1} \frac{1}{n} \mathbb{E}\log \|M_{0,n-1}\|.$$

Moreover, Furstenberg and Kesten established in [12] a strong law of large numbers for $\log \|M_{0,n-1}\|$:

$$\lim_{n\to+\infty} \frac{1}{n} \log \|M_{0,n-1}\| = \gamma \quad \mathbb{P}\text{-a.s.}$$

According to the values of the Lyapunov exponent γ, we have the following classification of MBPRE's: a MBPRE is supercritical if $\gamma > 0$, critical if $\gamma = 0$, and subcritical if $\gamma < 0$. In this article, the process (Z_n) will always be supercritical, i.e. $\gamma > 0$.

Under the supercritical condition $\gamma > 0$, we established in [13, Theorem 2.6 and Corollary 2.8] that the condition

$$\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)}\right) < +\infty \quad \forall 1 \leqslant i,j \leqslant d \tag{2.5}$$

is sufficient for the non-degeneracy of each W^i (in the sense that $\mathbb{P}(W^i > 0) > 0$), which is equivalent to the L^1-convergence of W_n^i to W^i by Sheffé's theorem; moreover, when each W^i is non-degenerate, we have a.s.,

$$\mathbb{E}_\xi W^i = 1 \quad \text{and} \quad \mathbb{P}_\xi(W^i > 0) = \mathbb{P}_\xi(\|Z_n^i\| \underset{n\to\infty}{\to} \infty) = 1 - q^i(\xi) > 0, \tag{2.6}$$

where $q^i(\xi)$ is the quenched probability of extinction of the process (Z_n^i).

First we establish the existence of the harmonic moments of the limits W^i, $1 \leqslant i \leqslant d$. For $n \geqslant 0$, define the vector $p_0(\xi_n)$ and the matrix $P_1(\xi_n)$, whose components are

$$p_0(\xi_n)(i) = f_n^i(\mathbf{0}) \quad \text{and} \quad P_1(\xi_n)(i,j) = \frac{\partial f_n^i}{\partial s_j}(\mathbf{0}), \quad 1 \leqslant i,j \leqslant d.$$

Then, for $1 \leqslant i,j \leqslant d$,

$$p_0(\xi_n)(i) = \mathbb{P}_{T^n\xi}(\|Z_1^i\| = 0) \quad \text{and} \quad P_1(\xi_n)(i,j) = \mathbb{P}_{T^n\xi}(Z_1^i = e_j).$$

Throughout the paper, we will assume the following condition:

H1. *The vector* $p_0(\xi_0) = (f_0^1(\mathbf{0}), \ldots, f_0^d(\mathbf{0}))$ *satisfies*

$$p_0(\xi_0) = \mathbf{0} \quad \mathbb{P}\text{-a.s.} \tag{2.7}$$

The condition **H1** means that each individual of the population gives birth to at least one child, so $q^i(\xi) = 0$ a.s. When (2.6) holds, this implies that $\|Z_n^i\| \to +\infty$ a.s. as $n \to +\infty$.

We introduce the following assumption :

H2. *There exist constants $p \in (1,2]$, $A > A_1 > 1$ and $A_2 > 0$ such that for any $1 \leqslant i,j \leqslant d$, $\mathbb{P}$-a.s.*

$$A_2 \leqslant M_0(i,j), \quad A_1 \leqslant \|M_0(i,\cdot)\| \quad \text{and} \quad \mathbb{E}_\xi(Z_1^i(j)^p) \leqslant A^p.$$

It is clear that **H2** implies the conditions (2.3), (2.4), (2.5) and $\gamma > 0$. From **H2** we have also that for all $1 \leqslant i,j \leqslant d$,

$$M_0(i,j) \leqslant A \quad \mathbb{P}\text{-a.s.}$$

Under condition **H2**, by the sub-multiplicative property of the norms $\|.\|$ and $\|.\|_\infty$ on $\mathcal{M}_d(\mathbb{R})$ and the subadditive ergodic theorem, it follows that, for all $a \geqslant 0$ the limit

$$\kappa_1(a) := \lim_{n\to+\infty} \Big(\mathbb{E}\|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} P_1(\xi_k) \Big\|_\infty\Big)^{1/n} \tag{2.8}$$

exists and is finite, with

$$\kappa_1(a) = \inf_{n\geqslant 1} \Big(\mathbb{E}\|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} P_1(\xi_k) \Big\|_\infty\Big)^{1/n}.$$

We will establish (see Lemma 3.9) that, under **H1** and **H2**, the function κ_1 is continuous and increasing on $\mathbb{R}_+$ and $\kappa_1(0) = \rho(\mathbb{E}P_1(\xi_0))$, where $\rho(M)$ denotes the spectral radius of the matrix M. For a random variable X, set $\|X\|_{L^\infty} := \operatorname{ess\,sup}(X)$ the essential supremum of X. Our first result gives a sufficient and a necessary condition for the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$, $a > 0$.

Theorem 2.1. *Assume conditions **H1**, **H2** and $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty} < 1$. For each fixed $a > 0$, the following implications hold :*

(1) if $\kappa_1(a) < 1$ then $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$;
(2) if $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$, then $\kappa_1(a) \leqslant 1$.

Notice that in Part (2), we can prove more: we will see in the proof that the sequence $\mathbb{E}\|M_{0,n-1}\|^a \,\|\prod_{k=0}^{n-1} P_1(\xi_k)\|_\infty, n \geqslant 1$, is bounded.

From Theorem 2.1 we get the following corollary.

Corollary 2.2. *Under the conditions of Theorem 2.1, it holds:*

(1) $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$ and $a > 0$ if and only if $\mathbb{E}P_1(\xi_0)$ is nilpotent;

(2) if $\mathbb{E}P_1(\xi_0)$ is not nilpotent, then there exists a unique constant $a_0 > 0$ satisfying

$$\kappa_1(a_0) = 1, \tag{2.9}$$

and

$$\max_{1\leqslant i\leqslant d} \mathbb{E}(W^i)^{-a} \begin{cases} < +\infty & \text{if } a \in [0, a_0), \\ = +\infty & \text{if } a \in (a_0, +\infty). \end{cases}$$

Part (1) gives a necessary and sufficient condition to have the existence of all orders of the harmonic moments of each W^i, $1 \leqslant i \leqslant d$. Part (2) reveals that the quantity a_0 is the critical value for the existence of the harmonic moments of all the W^i, $1 \leqslant i \leqslant d$. We believe that at the critical value a_0 the harmonic moments do not exist, i.e., $\mathbb{E}(W^i)^{-a_0} = +\infty$. This is the case when $d = 1$, as shown by Chunmao and Liu [22].

Now we investigate the existence of harmonic moments for W^i, when the boundedness condition **H2** is relaxed to a moment condition. For all $n \geqslant 0$ and $p > 1$ denote by

$$\theta_n(p) := \max_{1\leqslant i,j\leqslant d} \mathbb{E}_\xi \left| \frac{N^i_{1,n}(j)}{M_n(i,j)} - 1 \right|^p.$$

The next result gives a sufficient condition for the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$ of a small order $a > 0$. The single -type case has been considered in [18]. The multi-type case considered here is much more complicated.

To formulate it, we need the following moment condition :

H3. *There exist two constants $p \in (1, 2]$ and $\eta \in (0, 1)$ such that*

$$\mathbb{E}\|M_0\|^\eta < +\infty, \quad \max_{1\leqslant i,j\leqslant d} \mathbb{E}M_0(i,j)^{-\eta} < +\infty \quad \text{and} \quad \mathbb{E}\theta_0(p)^\eta < +\infty.$$

Like **H2**, condition **H3** also implies (2.3) and (2.4) and (2.5). The first two implications are evident; the third will be proved in Section 3.

Theorem 2.3. *Assume conditions **H1**, **H3** and $\gamma > 0$. Then there exists $a > 0$ such that $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$.*

Now we formulate a central limit theorem and a Berry-Esseen type theorem for $\log \|Z^i_n\|$, for all $1 \leqslant i \leqslant d$. We introduce the following assumption:

H4. *The random matrix M_0 satisfies*

$$\mathbb{E}(\log \|M_0\|)^2 < +\infty.$$

Obviously, condition **H4** implies (2.4). Using the central limit theorem due to Hennion [20, Theorem 3] for the norm cocycle $\log \|M_{0,n-1}^T x\|$, where $x \in \mathcal{S}$, we establish the following central limit theorem for $\log \|Z_n^i\|$. Notice that $\|Z_n^i\| = Z_n^i(1) + \cdots + Z_n^i(d)$ represents the population size of generation n.

Theorem 2.4. *Assume conditions (2.3) and **H4**. Assume also **H1**, (2.5) and $\gamma > 0$. Then there exists $\sigma \geqslant 0$ such that for all $1 \leqslant i \leqslant d$, as $n \to \infty$,*

$$\frac{\log \|Z_n^i\| - n\gamma}{\sqrt{n}} \to \mathcal{N}(0, \sigma^2) \quad \textit{in law,}$$

where $\mathcal{N}(0, \sigma^2)$ denotes the normal law with mean 0 and variance σ^2.

Notice that for the single type case $d = 1$, this theorem was established in [22]. By [34, Proposition 3.14], under the condition **H3**, the asymptotic variance σ^2 defined in Theorem 2.4 satisfies

$$\sigma^2 = \lim_{n \to +\infty} \frac{1}{n} \mathbb{E}[(\log \|M_{0,n-1} x\| - n\gamma)^2],$$

uniformly in $x \in \mathcal{S}$. Note that in Theorem 2.4 the limit variance σ^2 can be degenerated: $\sigma^2 = 0$. For the rate of convergence we need the following assumption :

H5. *The asymptotic variance σ^2 satisfies*

$$\sigma^2 > 0.$$

According to [7, Lemma 7.2], a sufficient condition under which **H5** holds is that μ is a non-arithmetic probability measure; the definition of arithmeticity is introduced below.

For $x \in \mathcal{S}$ and $M \in \mathcal{G}_+^0$, define the projective action of M on $\mathcal{S}$ by $M \cdot x := \frac{Mx}{\|Mx\|}$. Let μ be the law of M_0 and $\Gamma_\mu = [\operatorname{supp} \mu]$, the semigroup generated by the support of μ. By the Perron-Frobenius theorem, since any $M \in \Gamma_\mu$ is strictly positive under **H3**, the spectral radius ρ_M of M is the unique eigenvalue with the largest modulus, which is simple. Let $u_M \in \mathcal{S}$ be the associated unique right eigenvector with unit norm. Set $V(\Gamma_\mu) = \overline{\{\pm u_M : M \in \Gamma_\mu\}}$, where $\overline{A}$ denotes the closure of the set A. The measure μ is called arithmetic if there exist $t > 0$, $\theta \in [0, 2\pi)$ and a function $h : \mathcal{S} \to \mathbb{R}$ such that for all $M \in \Gamma_\mu$ and $x \in V(\Gamma_\mu)$, we have

$$\exp\{it \log \|Mx\| - i\theta + ih(M \cdot x) - ih(x)\} = 1.$$

In the following, denote by $\Phi : x \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\, dt$ the standard normal distribution function on $\mathbb{R}$.

Theorem 2.5. *Assume conditions* **H1**, **H3**, **H5** *and* $\gamma > 0$. *Then there exists a constant* $C > 0$ *such that for all* $n \geqslant 1$, $x \in \mathbb{R}$ *and* $1 \leqslant i \leqslant d$,

$$\left|\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x\right) - \Phi(x)\right| \leqslant \frac{C}{\sqrt{n}}.$$

For the single type case $d = 1$, a version of this result exists under different conditions in [16, Theorem 1.1]. Notice that, in this case, we have $\gamma = \mathbb{E}\log m_0$ and $\sigma^2 = \mathbb{E}(\log m_0 - \gamma)^2$ with $m_0 = \mathbb{E}_\xi Z_1$, and the condition **H3** can be simplified to the following: there exist two constants $p \in (1, 2]$ and $\eta \in (0, 1)$ such that

$$\mathbb{E}m_0^\eta < +\infty \quad \text{and} \quad \mathbb{E}\theta_0(p)^\eta < +\infty, \text{ where } \theta_0(p) = \mathbb{E}_\xi\left|\frac{Z_1}{m_0} - 1\right|^p.$$

3. Harmonic moments of W^i

In this section we study the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$ of all the random variables W^i, where $1 \leqslant i \leqslant d$, $a > 0$.

3.1. Auxiliary results

We start with four lemmas that we will need. The first lemma permits to compare the moment of $\phi(W^i)$ with the corresponding one of $\phi(W_n^i)$, with ϕ a positive convex function on $\mathbb{R}_+$.

Lemma 3.1. *Assume condition* (2.5) *and* $\gamma > 0$. *Then for all* $1 \leqslant i \leqslant d$ *and any convex function* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\lim_{n\to+\infty} \mathbb{E}_\xi \phi(W_n^i) = \sup_{n\geqslant 0} \mathbb{E}_\xi \phi(W_n^i) = \mathbb{E}_\xi \phi(W^i), \tag{3.1}$$

and

$$\lim_{n\to+\infty} \mathbb{E}\phi(W_n^i) = \sup_{n\geqslant 0} \mathbb{E}\phi(W_n^i) = \mathbb{E}\phi(W^i). \tag{3.2}$$

Proof. The result is a direct consequence of the fact that, by [13, Theorem 2.6], under the conditions (2.5) and $\gamma > 0$, $(W_n^i, \mathcal{F}_n)$ is a martingale under $\mathbb{P}_\xi$ and $\mathbb{P}$, which converges in L^1. In fact, by Fatou's lemma, $\mathbb{E}\phi(W^i) \leqslant \sup_{n\geqslant 0} \mathbb{E}\phi(W_n^i)$; by the L^1- convergence of (W_n^i) and Jensen's inequality,

$$\mathbb{E}\phi(W^i) = \mathbb{E}[\mathbb{E}[\phi(W^i)|\mathcal{F}_n]] \geqslant \mathbb{E}[\phi(\mathbb{E}[W^i|\mathcal{F}_n])] = \mathbb{E}\phi(W_n^i).$$

Using the fact that $(\phi(W_n^i))$ is a sub-martingale, this concludes the proof of (3.2). The same argument with $\mathbb{E}$ replaced by $\mathbb{E}_\xi$ gives (3.1). □

The second lemma reveals the link between the harmonic moments and the Laplace transform of a positive random variable.

Lemma 3.2. *([24, Lemma 4.4]) Let X be a positive random variable, and $a > 0$. We have the following assertions:*

(1) if $\mathbb{E}X^{-a} < +\infty$ then $\mathbb{E}\,\mathrm{e}^{-tX} = O_{t\to+\infty}(t^{-a})$;
(2) if $\mathbb{E}\,\mathrm{e}^{-tX} = O_{t\to+\infty}(t^{-a})$ then $\mathbb{E}X^{-b} < +\infty$ for all $b \in (0, a)$;
(3) $\mathbb{E}\,\mathrm{e}^{-tX} = O_{t\to+\infty}(t^{-a})$ if and only if $\mathbb{P}(X \leqslant x) = O_{x\to 0}(x^a)$.

For all $1 \leqslant i \leqslant d$, let

$$\phi^i_\xi(t) = \mathbb{E}_\xi\,\mathrm{e}^{-tW^i} \quad \text{and} \quad \phi^i(t) = \mathbb{E}\phi^i_\xi(t) = \mathbb{E}\,\mathrm{e}^{-tW^i}, \qquad t \geqslant 0,$$

be the quenched and annealed Laplace transform of W^i. Denote by

$$\phi_\xi(t) = (\phi^1_\xi(t), \cdots, \phi^d_\xi(t)) \quad \text{and} \quad \phi(t) = (\phi^1(t), \cdots, \phi^d(t)), \quad t \geqslant 0.$$

We will study the decay rate of the Laplace transforms $\phi^i_\xi(t)$ and $\phi^i(t)$ when $t \to +\infty$, and then use Lemma 3.2 to estimate the corresponding harmonic moments. Let T be the shift operator of the environment sequence:

$$T\xi = (\xi_1, \xi_2, \cdots) \quad \text{if} \quad \xi = (\xi_0, \xi_1, \cdots),$$

and let T^n be its n-fold iteration. The third lemma, proved in [13, Theorem 2.4], gives the functional equations that the quenched Laplace transforms ϕ^i_ξ satisfy.

Lemma 3.3. *Assume condition (2.3). Then for all $i = 1, \cdots, d$, the quenched Laplace transform ϕ^i_ξ of W^i satisfies*

$$\phi^i_\xi(t) = f^i_0\left(\phi^1_{T\xi}\left(t\frac{U_{1,\infty}(1)}{\lambda_0 U_{0,\infty}(i)}\right), \cdots, \phi^d_{T\xi}\left(t\frac{U_{1,\infty}(d)}{\lambda_0 U_{0,\infty}(i)}\right)\right), \quad t \geqslant 0.$$

The fourth Lemma will be used to control the L^p-moments of the martingale (W^i_n). It is a direct consequence of the Marcinkiewicz-Zygmund inequality in [8, Theorem 1.5], as stated in [26, Lemma 1.4].

Lemma 3.4. *Let $(X_k)_{k\in\mathbb{N}^*}$ be a sequence of i.i.d. random centered variables. Then for all $n \in \mathbb{N}^*$ and $p > 1$:*

$$\mathbb{E}\left|\sum_{k=1}^n X_k\right|^p \leqslant \begin{cases} (B_p)^p \mathbb{E}|X_k|^p n, & \text{if } 1 < p \leqslant 2, \\ (B_p)^p \mathbb{E}|X_k|^p n^{\frac{p}{2}}, & \text{if } p > 2, \end{cases}$$

where $B_p = 2\min\{k^{1/2} : k \in \mathbb{N}, k \geqslant \frac{p}{2}\}$.

3.2. Existence of the quenched harmonic moments $\mathbb{E}_\xi(W^i)^{-a}$

In this section, under condition **H2**, we give an estimation of the decay rate of the quenched Laplace transforms $\phi^i_\xi(t)$ as $t \to +\infty$, which implies the uniform boundedness of the quenched harmonic moments $\mathbb{E}_\xi(W^i)^{-a}$, as indicated in the following theorem.

Theorem 3.5. *Assume conditions **H1**, **H2** and $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty} < 1$. Then there exist two constants $C > 0$ and $a > 0$ such that for all $1 \leqslant i \leqslant d$, all $t > 0$ and all $x > 0$,*

$$\phi^i_\xi(t) \leqslant \frac{C}{t^a} \quad \mathbb{P}\text{-}a.s., \tag{3.3}$$

$$\mathbb{P}_\xi(W^i \leqslant x) \leqslant Cx^a \quad \text{and} \quad \mathbb{E}_\xi(W^i)^{-a} \leqslant C. \tag{3.4}$$

For the proof of Theorem 3.5, we will need the following preliminary result about a control of $\phi^i_\xi(t)$, $1 \leqslant i \leqslant d$. For the case $d = 1$, this result was established in [22].

Lemma 3.6. *Assume condition **H2**. Then there exist two constants $\beta \in (0,1)$ and $t_0 > 0$ such that for all $1 \leqslant i \leqslant d$ and $t \geqslant t_0$,*

$$\phi^i_\xi(t) \leqslant \beta \quad \mathbb{P}\text{-}a.s.$$

Proof. We will adapte the approach in [22] where the case $d = 1$ was considered. By (2.1) and (1.6), we have that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} W^i_{n+1} - W^i_n &= \sum_{j=1}^d \frac{U_{n+1,\infty}(j)}{\lambda_{0,n}U_{0,\infty}(i)} \sum_{r=1}^d \sum_{l=1}^{Z^i_n(r)} N^r_{l,n}(j) - W^i_n \\ &= \sum_{r=1}^d \frac{U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)} \sum_{l=1}^{Z^i_n(r)} \sum_{j=1}^d \frac{U_{n+1,\infty}(j)N^r_{l,n}(j)}{\lambda_n U_{n,\infty}(r)} - W^i_n \\ &= \sum_{r=1}^d \frac{U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)} \sum_{l=1}^{Z^i_n(r)} (W^r_{l,n} - 1), \end{aligned} \tag{3.5}$$

where

$$W^r_{l,n} := \frac{\langle N^r_{l,n}, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(r)}.$$

It is clear that, given the environment ξ, the random variables $W^r_{l,n}$, $l \geqslant 1$, are i.i.d., and independent of $\xi_0, \ldots, \xi_{n-1}$ and Z^i_n. Let $p \in (1,2]$ be such that

condition **H2** holds. Notice that, by (1.5), we have $\sum_{r=1}^{d} \frac{M_{0,n-1}(i,r)U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)} = 1$ a.s. for any $n \geqslant 1$ and $1 \leqslant i \leqslant d$. Therefore, applying (3.5), the convexity of the function $x \mapsto x^p$ on $\mathbb{R}_+$ and Lemma 3.4, for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s., we get

$$\begin{aligned}
\mathbb{E}_\xi|W^i_{n+1} - W^i_n|^p &\leqslant \mathbb{E}_\xi\Big(\sum_{r=1}^{d} \frac{U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)}\Big|\sum_{l=1}^{Z^i_n(r)}(W^r_{l,n}-1)\Big|\Big)^p \\
&\leqslant \sum_{r=1}^{d} \frac{M_{0,n-1}(i,r)U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)} \frac{1}{M_{0,n-1}(i,r)^p}\mathbb{E}_\xi\Big|\sum_{l=1}^{Z^i_n(r)}(W^r_{l,n}-1)\Big|^p \\
&\leqslant B_p^p \sum_{r=1}^{d} \frac{M_{0,n-1}(i,r)U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)} \cdot \frac{\mathbb{E}_\xi Z^i_n(r)}{M_{0,n-1}(i,r)^p}\mathbb{E}_\xi|W^r_{1,n}-1|^p \\
&\leqslant B_p^p \max_{1\leqslant r\leqslant d}\Big\{\frac{\mathbb{E}_\xi Z^i_n(r)}{M_{0,n-1}(i,r)^p}\mathbb{E}_\xi|W^r_{1,n}-1|^p\Big\} \\
&= B_p^p \max_{1\leqslant r\leqslant d}\mathbb{E}_\xi|W^r_{1,n}-1|^p \max_{1\leqslant j\leqslant d}(M_{0,n-1}(i,j))^{1-p}. \qquad (3.6)
\end{aligned}$$

Again by the convexity of $x \mapsto x^p$, for all $1 \leqslant r \leqslant d$ and $n \geqslant 0$, $\mathbb{P}$-a.s., we have

$$\begin{aligned}
\mathbb{E}_\xi|W^r_{1,n}-1|^p &= \mathbb{E}_\xi\Big|\frac{\langle N^r_{1,n}, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(r)} - 1\Big|^p \\
&= \mathbb{E}_\xi\Big|\sum_{j=1}^{d}\frac{M_n(r,j)U_{n+1,\infty}(j)}{\lambda_n U_{n,\infty}(r)}\Big(\frac{N^r_{1,n}(j)}{M_n(r,j)}-1\Big)\Big|^p \\
&\leqslant \max_{1\leqslant i,j\leqslant d}\mathbb{E}_\xi\Big|\frac{N^i_{1,n}(j)}{M_n(i,j)}-1\Big|^p = \theta_n(p). \qquad (3.7)
\end{aligned}$$

Combining this with (3.6), we obtain that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\mathbb{E}_\xi|W^i_{n+1} - W^i_n|^p \leqslant B_p^p\theta_n(p)\max_{1\leqslant j\leqslant d}(M_{0,n-1}(i,j))^{1-p}. \qquad (3.8)$$

Using the triangular inequality in L^p under $\mathbb{P}_\xi$ and condition **H2**, for $n \geqslant 0$ we have

$$\theta_n(p) \leqslant \max_{1\leqslant i,j\leqslant d}\Big(\frac{[\mathbb{E}_\xi(N^i_{1,n}(j))^p]^{1/p}}{M_n(i,j)}+1\Big)^p \leqslant \Big(\frac{A}{A_2}+1\Big)^p \quad \mathbb{P}\text{-a.s.} \qquad (3.9)$$

Now we deal with the last factor in (3.8). For $M \in \mathcal{M}_d(\mathbb{R})$, set

$$\iota(M) := \inf_{\|x\|=1}\|Mx\| = \min_{1\leqslant j\leqslant d}\sum_{i=1}^{d} M(i,j). \qquad (3.10)$$

Using **H2** we have $\iota(M_0^T) \geqslant A_1$ a.s. It can be easily seen that the application ι satisfies the inequality $\iota(AB) \geqslant \iota(A)\iota(B)$, for $A, B \in \mathcal{G}_+^0$. Therefore, we deduce that for all $n \geqslant 2$,

$$\begin{aligned} M_{0,n-1}(i,j) &= \sum_{1\leqslant r\leqslant d} M_{0,n-2}(i,r)M_{n-1}(r,j) \\ &\geqslant A_2\iota(M_{0,n-2}^T) \\ &\geqslant A_2\prod_{k=0}^{n-2}\iota(M_k^T) \\ &\geqslant A_2A_1^{n-1}. \end{aligned} \tag{3.11}$$

It is evident that the above inequality remains true for $n = 1$. Combining (3.8), (3.9) and (3.11), we obtain that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}_\xi|W_{n+1}^i - W_n^i|^p \leqslant B_p^p\Big(\frac{A}{A_2}+1\Big)^p A_2^{1-p}(A_1^{1-p})^{n-1} \quad \mathbb{P}\text{-a.s.}, \tag{3.12}$$

with $A_1^{1-p} \in (0,1)$. By (3.7) and (3.9), it is clear that (3.12) holds for $n = 0$. Recall that condition **H2** implies (2.5) and $\gamma > 0$. Then, applying Lemma 3.1 with the convex function $x \mapsto x^p$ on $\mathbb{R}_+$, we have

$$\mathbb{E}_\xi(W^i)^p = \sup_{n\geqslant 0}\mathbb{E}_\xi(W_n^i)^p \quad \mathbb{P}\text{-a.s.}$$

This, together with (3.12) and the triangular inequality in L^p, implies that there exists a constant $C > 0$ such that

$$\mathbb{E}_\xi(W^i)^p = \sup_{n\geqslant 0}\mathbb{E}_\xi(W_n^i)^p \leqslant \Big(1+\sum_{n=0}^{+\infty}\big(\mathbb{E}_\xi|W_{n+1}^i - W_n^i|^p\big)^{1/p}\Big)^p \leqslant C. \tag{3.13}$$

Since $x \to (\mathrm{e}^{-x} - 1 + x)/x^p$ is a positive bounded function on $\mathbb{R}_+^*$, it follows that there exists a constant $C_1 > 0$ such that for all $x > 0$,

$$\mathrm{e}^{-x} \leqslant 1 - x + C_1x^p.$$

Combining this with (3.13), we see that for any $1 \leqslant i \leqslant d$ and $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \phi_\xi^i(t) = \mathbb{E}_\xi\,\mathrm{e}^{-tW^i} &\leqslant \mathbb{E}_\xi(1 - tW^i + C_1t^p(W^i)^p) \\ &= 1 - t + C_1t^p\mathbb{E}_\xi(W^i)^p \\ &\leqslant 1 - t + CC_1t^p. \end{aligned} \tag{3.14}$$

Let $h(t) = 1 - t + CC_1 t^p$, $t \geqslant 0$. We observe that the minimal value of $h(t)$ is $\beta = h(t_0)$, where $t_0 := (pCC_1)^{1/(1-p)}$, and we have

$$\beta = 1 - (pCC_1)^{1/(1-p)} + \frac{1}{p}(pCC_1)^{1/(1-p)} = 1 - \Big(1 - \frac{1}{p}\Big)t_0. \tag{3.15}$$

So $\beta \in (0,1)$. Since the quenched Laplace transform ϕ^i_ξ is decreasing on $\mathbb{R}_+$, we conclude from (3.14) that for all $t \geqslant t_0$,

$$\phi^i_\xi(t) \leqslant \phi^i_\xi(t_0) \leqslant h(t_0) = \beta \quad \mathbb{P}\text{-a.s.} \tag{3.16}$$

This concludes the proof of Lemma 3.6. □

Proof of Theorem 3.5. By Lemma 3.2, we have the implication (3.3) ⇒ (3.4) (but the values of a and C can be changed). Therefore, it remains to prove (3.3). Set

$$\psi^i_\xi(t) = \phi^i_\xi(tU_{0,\infty}(i)), \quad 1 \leqslant i \leqslant d, \quad t \geqslant 0,$$

and $\psi_\xi(t) = (\psi^1_\xi(t), \cdots, \psi^d_\xi(t))$. From Lemma 3.3 we obtain that ψ_ξ satisfies the following equation: for all $1 \leqslant i \leqslant d$ and $t \geqslant 0$,

$$\begin{aligned}
\psi^i_\xi(t) &= f^i_0\Big(\phi^1_{T\xi}\Big(\frac{tU_{1,\infty}(1)}{\lambda_0}\Big), \cdots, \phi^d_{T\xi}\Big(\frac{tU_{1,\infty}(d)}{\lambda_0}\Big)\Big) \\
&= f^i_0\Big(\psi^1_{T\xi}\Big(\frac{t}{\lambda_0}\Big), \cdots, \psi^d_{T\xi}\Big(\frac{t}{\lambda_0}\Big)\Big) \\
&= f^i_0\Big(\psi_{T\xi}\Big(\frac{t}{\lambda_0}\Big)\Big).
\end{aligned} \tag{3.17}$$

For $n \geqslant 0$, denote by $Q_1(\xi_n)$ the positive random matrix whose entries are, for $1 \leqslant i, j \leqslant d$,

$$Q_1(\xi_n)(i,j) = \mathbb{P}_\xi(\|Z_{n+1}\| \geqslant 2, Z_{n+1}(j) \geqslant 1, Z_{n+1}(r) = 0, r < j | Z_n = e_i).$$

It is clear that for $n \geqslant 0$, $Q_1(\xi_n)$ depends only of ξ_n and that the events $\{Z_{n+1}(j) \geqslant 1, Z_{n+1}(r) = 0 \ \forall r < j\}$, $1 \leqslant j \leqslant d$, constitute a partition of $\{\|Z_{n+1}\| \geqslant 1\}$. Hence

$$\|Q_1(\xi_n)(i,\cdot)\| = \sum_{j=1}^d Q_1(\xi_n)(i,j) = \mathbb{P}_\xi(\|Z_{n+1}\| \geqslant 2 | Z_n = e_i) \quad \forall 1 \leqslant i \leqslant d.$$

By **H1** and the fact that $\|P_1(\xi_n)(i,\cdot)\| = \mathbb{P}_\xi(\|Z_{n+1}\| = 1 | Z_n = e_i)$, we get that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\|P_1(\xi_n)(i,\cdot) + Q_1(\xi_n)(i,\cdot)\| = 1, \tag{3.18}$$

which means that $P_1(\xi_n)+Q_1(\xi_n)$ is a stochastic matrix. Then, by definition of the matrix $Q_1(\xi_0)$ and using again **H1** and the partition $\{Z_{n+1}(j) \geqslant 1, Z_{n+1}(r) = 0\ \forall r < j\}$ $(1 \leqslant j \leqslant d)$ of $\{\|Z_{n+1}\| \geqslant 1\}$, we see that for all $s = (s_1, \cdots, s_d) \in [0,1]^d$ and $1 \leqslant i \leqslant d$, we have

$$\begin{aligned} f_0^i(s) &= \sum_{j=1}^d \mathbb{P}_\xi(Z_1^i = e_j)s_j + \sum_{k\in\mathbb{N}^d, \|k\|\geqslant 2} \mathbb{P}_\xi(Z_1^i = k)s_1^{k(1)}\cdots s_d^{k(d)} \\ &\leqslant \sum_{j=1}^d P_1(\xi_0)(i,j)s_j + \|s\|_\infty \sum_{j=1}^d Q_1(\xi_0)(i,j)s_j \\ &= \langle [P_1(\xi_0) + \|s\|_\infty Q_1(\xi_0)]s, e_i\rangle. \end{aligned}$$

This, together with (3.17), implies that for all $t \geqslant 0$,

$$\psi_\xi(t) \leqslant \Big(P_1(\xi_0) + \Big\|\psi_{T\xi}\Big(\frac{t}{\lambda_0}\Big)\Big\|_\infty Q_1(\xi_0)\Big)\psi_{T\xi}\Big(\frac{t}{\lambda_0}\Big) \quad \mathbb{P}\text{-a.s.} \tag{3.19}$$

In particular, by (3.18) we get that for all $t \geqslant 0$,

$$\|\psi_\xi(t)\|_\infty \leqslant \Big\|\psi_{T\xi}\Big(\frac{t}{\lambda_0}\Big)\Big\|_\infty \quad \mathbb{P}\text{-a.s.}$$

By iteration, we see that for all $n \geqslant 1$ and $t \geqslant 0$,

$$\|\psi_\xi(t)\|_\infty \leqslant \Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \quad \mathbb{P}\text{-a.s.} \tag{3.20}$$

By iteration of (3.19) and using inequality (3.20), we obtain that for all $n \geqslant 1$ and $t \geqslant 0$,

$$\psi_\xi(t) \leqslant \prod_{k=0}^{n-1}\Big(P_1(\xi_k) + \Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty Q_1(\xi_k)\Big)\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big) \quad \mathbb{P}\text{-a.s.} \tag{3.21}$$

Notice that by (1.5) and condition **H2**, for any $1 \leqslant i \leqslant d$ we have

$$1 \geqslant U_{0,\infty}(i) = \frac{\langle M_0(i,\cdot), U_{1,\infty}\rangle}{\|M_0U_{1,\infty}\|} \geqslant \frac{\min\limits_{1\leqslant j\leqslant d} M_0(i,j)}{\|M_0\|} \geqslant \frac{A_2}{dA} \quad \mathbb{P}\text{-a.s.} \tag{3.22}$$

Therefore, since ϕ_ξ is a decreasing function on $\mathbb{R}_+$, we obtain that for all $1 \leqslant i \leqslant d$ and $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$\phi_\xi^i(t) \leqslant \psi_\xi^i(t) = \phi_\xi^i(tU_{0,\infty}(i)) \leqslant \phi_\xi^i\Big(t\frac{A_2}{dA}\Big).$$

Combining this with (3.21), it follows that for all $n \geqslant 1$ and $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$\phi_\xi(t) \leqslant \prod_{k=0}^{n-1} \Big(P_1(\xi_k) + \Big\| \phi_{T^n\xi}\Big(\frac{tA_2}{dA\lambda_{0,n-1}}\Big)\Big\|_\infty Q_1(\xi_k)\Big) \phi_{T^n\xi}\Big(\frac{tA_2}{dA\lambda_{0,n-1}}\Big). \tag{3.23}$$

Now, by Lemma 3.6, we know that there exist two constants $\beta \in (0,1)$ and $t_0 > 0$ such that $\|\phi_\xi(t)\|_\infty \leqslant \beta$ a.s. for all $t \geqslant t_0$. Using the inequality $\lambda_{0,n-1} \leqslant \|M_{0,n-1}\| \leqslant (dA)^n$, this implies that for $n \geqslant 1$, $\mathbb{P}$-a.s.,

$$\Big\| \phi_{T^n\xi}\Big(\frac{tA_2}{dA\lambda_{0,n-1}}\Big)\Big\|_\infty \leqslant \beta, \quad t \geqslant t_1 (dA)^n,$$

where $t_1 = \frac{dA}{A_2} t_0$. Combining this with (3.23) and (3.18), we get that for all $n \geqslant 1$ and $t \geqslant t_1(dA)^n$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\|\phi_\xi(t)\|_\infty &\leqslant \beta \prod_{k=0}^{n-1} \|P_1(\xi_k) + \beta Q_1(\xi_k)\|_\infty \\
&= \beta \prod_{k=0}^{n-1} \|\beta[P_1(\xi_k) + Q_1(\xi_k)] + (1-\beta)P_1(\xi_k)\|_\infty \\
&\leqslant \beta \prod_{k=0}^{n-1} (\beta + (1-\beta)\|P_1(\xi_k)\|_\infty) \\
&\leqslant \beta\alpha^n,
\end{aligned} \tag{3.24}$$

where $\alpha = \beta + (1-\beta)\|\|P_1(\xi_0)\|_\infty\|_{L^\infty}$. Clearly we have $\alpha \in (0,1)$, since $\beta \in (0,1)$ and $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty} < 1$ by hypothesis. Set

$$N(t) := \Big\lfloor \frac{\log t - \log t_1}{\log(dA)} \Big\rfloor + 1, \quad t \geqslant t_1.$$

We observe that, if $t \geqslant t_1$, then $N(t) \geqslant 1$, $t \geqslant t_1(dA)^{N(t)}$ and $N(t) \geqslant (\log t - \log t_1)/\log(dA)$. Therefore, using the inequality (3.24) with $n = N(t)$, we deduce that for all $t \geqslant t_1$, $\mathbb{P}$-a.s.,

$$\|\phi_\xi(t)\|_\infty \leqslant \beta\alpha^{N(t)} \leqslant \beta\alpha^{\frac{\log t - \log t_1}{\log(dA)}} = \beta t_1^{-\frac{\log\alpha}{\log(dA)}} t^{\frac{\log\alpha}{\log(dA)}}.$$

Taking $a = -\frac{\log\alpha}{\log(dA)} > 0$ and $C = \beta t_1^a > 0$, we conclude that for all $t \geqslant t_1$,

$$\|\phi_\xi(t)\|_\infty \leqslant \frac{C}{t^a} \quad \mathbb{P}\text{-a.s.},$$

which implies (3.3). This ends the proof of Theorem 3.5. □

3.3. Existence of the annealed harmonic moments $\mathbb{E}(W^i)^{-a}$

The aim of this section is to prove the following theorem which gives the optimal value of a to have $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$, under condition **H2**. (But the proof of the optimality will be done in the next section.)

Theorem 3.7. *Assume conditions **H1**, **H2** and $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty} < 1$. Let $a > 0$ be such that $\kappa_1(a) < 1$. Then there exists a constant $C > 0$ such that for all $1 \leqslant i \leqslant d$ and $t > 0$,*

$$\phi^i(t) \leqslant \frac{C}{t^a}, \tag{3.25}$$

and that for all $1 \leqslant i \leqslant d$, $0 < b < a$ and $x > 0$,

$$\mathbb{P}(W^i \leqslant x) \leqslant Cx^a \quad \text{and} \quad \mathbb{E}(W^i)^{-b} \leqslant C. \tag{3.26}$$

For the proof of Theorem 3.7, we shall need the following technical lemma about the decay late of a function which satisfies a functional inequality.

Lemma 3.8. *([24, Lemma 4.1]) Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded function. Assume there exists a random variable $A > 0$ and constants $\alpha \in (0, 1)$, $C \geqslant 0$, $a > 0$ and $t_0 \geqslant 0$ such that*

$$\phi(t) \leqslant \alpha \mathbb{E}\phi(At) + \frac{C}{t^a}, \quad t > t_0. \tag{3.27}$$

If $\alpha \mathbb{E}A^{-a} < 1$, then $\phi(t) = O_{t\to+\infty}(t^{-a})$.

Proof of Theorem 3.7. First, by Lemma 3.2, we have the implication (3.25) $\Rightarrow$ (3.26). So, it remains to prove (3.25).

Let $\varepsilon \in (0, 1)$ and $a > 0$ be such that $\kappa_1(a) < 1$. By (3.3) in Theorem 3.5, we get that there exists a constant $t_\varepsilon > 0$ such that for all $t \geqslant t_\varepsilon$,

$$\|\phi_\xi(t)\|_\infty \leqslant \varepsilon \quad \mathbb{P}\text{-a.s.} \tag{3.28}$$

Recall that, by **H2**, we have $\lambda_{0,n-1} \leqslant \|M_{0,n-1}\| \leqslant (dA)^n$ a.s., $n \geqslant 1$. Then, combining (3.28) and (3.23), we obtain that for all $n \geqslant 1$ and $t \geqslant \frac{(dA)^{n+1}}{A_2} t_\varepsilon$, $\mathbb{P}$-a.s.,

$$\phi_\xi(t) \leqslant \prod_{k=0}^{n-1} (P_1(\xi_k) + \varepsilon Q_1(\xi_k)) \phi_{T^n\xi}\Big(\frac{tA_2}{dA\|M_{0,n-1}\|}\Big). \tag{3.29}$$

Taking expectation in (3.29) and using the independence of the environments ξ_n, it follows that for all $n \geqslant 1$ and $t \geqslant \frac{(dA)^{n+1}}{A_2} t_\varepsilon$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\phi(t) &\leqslant \mathbb{E}\Big(\prod_{k=0}^{n-1} (P_1(\xi_k) + \varepsilon Q_1(\xi_k)) \mathbb{E}\Big[\phi_{T^n\xi}\Big(\frac{tA_2}{dA\|M_{0,n-1}\|} \Big) \Big| \xi_k, 0 \leqslant k < n \Big] \Big) \\ &= \mathbb{E}\Big[\prod_{k=0}^{n-1} (P_1(\xi_k) + \varepsilon Q_1(\xi_k)) \phi\Big(\frac{tA_2}{dA\|M_{0,n-1}\|} \Big) \Big].\end{aligned}$$

This implies that for all $n \geqslant 1$ and $t \geqslant \frac{(dA)^{n+1}}{A_2} t_\varepsilon$,

$$\begin{aligned}\|\phi(t)\|_\infty &\leqslant \mathbb{E}\Big[\Big\| \prod_{k=0}^{n-1} (P_1(\xi_k) + \varepsilon Q_1(\xi_k)) \Big\|_\infty \Big\| \phi\Big(\frac{tA_2}{dA\|M_{0,n-1}\|} \Big) \Big\|_\infty \Big] \\ &= \alpha_{n,\varepsilon} \mathbb{E}\|\phi(\tilde{A}_{n,\varepsilon} t)\|_\infty, \end{aligned} \tag{3.30}$$

where $\alpha_{n,\varepsilon} = \mathbb{E}\| \prod_{k=0}^{n-1} (P_1(\xi_k) + \varepsilon Q_1(\xi_k)) \|_\infty > 0$ and $\tilde{A}_{n,\varepsilon}$ is a positive random variable whose distribution is determined by

$$\mathbb{E}h(\tilde{A}_{n,\varepsilon}) = \frac{1}{\alpha_{n,\varepsilon}} \mathbb{E}\Big[\Big\| \prod_{k=0}^{n-1} (P_1(\xi_k) + \varepsilon Q_1(\xi_k)) \Big\|_\infty h\Big(\frac{A_2}{dA\|M_{0,n-1}\|} \Big) \Big]$$

for all bounded function h on $\mathbb{R}_+$. Now we prove that there exist $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$ satisfying the two following conditions:

$$\alpha_{n,\varepsilon} < 1, \tag{3.31}$$

and

$$\alpha_{n,\varepsilon} \mathbb{E}\tilde{A}_{n,\varepsilon}^{-a} < 1. \tag{3.32}$$

If (3.31) and (3.32) hold, then by Lemma 3.8,

$$\|\phi(t)\|_\infty = O_{t\to+\infty}(t^{-a}), \tag{3.33}$$

which is equivalent to (3.25). So it remains to prove (3.31) and (3.32).

First we show (3.31). Notice that for any positive random matrix $M \in \mathcal{M}_d(\mathbb{R})$,

$$\begin{aligned}\|\mathbb{E}M\|_\infty &= \max_{1\leqslant i\leqslant d} \mathbb{E}\Big(\sum_{j=1}^d M(i,j) \Big) \\ &\leqslant \mathbb{E}\|M\|_\infty \leqslant \sum_{i,j=1}^d \mathbb{E}M(i,j) \leqslant d\|\mathbb{E}M\|_\infty. \end{aligned} \tag{3.34}$$

Using this and the fact that $P_1(\xi_k)+Q_1(\xi_k)$ are stochastic matrices (so that $\|\prod_{k=0}^{n-1}(P_1(\xi_k)+Q_1(\xi_k))\|_\infty = 1$), we get for all $n \geqslant 1$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}\alpha_{n,\varepsilon} &\leqslant \mathbb{E}\Big\| \prod_{k=0}^{n-1} P_1(\xi_k) + \varepsilon\Big(\prod_{k=0}^{n-1} (P_1(\xi_k)+Q_1(\xi_k)) - \prod_{k=0}^{n-1} P_1(\xi_k)\Big)\Big\|_\infty \\ &\leqslant (1-\varepsilon)\mathbb{E}\Big\| \prod_{k=0}^{n-1} P_1(\xi_k)\Big\|_\infty + \varepsilon\mathbb{E}\Big\| \prod_{k=0}^{n-1} (P_1(\xi_k)+Q_1(\xi_k))\Big\|_\infty \\ &\leqslant d(1-\varepsilon)\|(\mathbb{E}P_1(\xi_0))^n\|_\infty + \varepsilon.\end{aligned}$$

The hypothesis $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty} < 1$ implies that $\rho(\mathbb{E}P_1(\xi_0)) < 1$, so that $\lim_{n\to+\infty}\|(\mathbb{E}P_1(\xi_0))^n\|_\infty = 0$. Therefore, we obtain that for all $\varepsilon \in (0,1)$,

$$\limsup_{n\to+\infty} \alpha_{n,\varepsilon} \leqslant \varepsilon < 1. \tag{3.35}$$

Therefore (3.31) holds for all $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$ sufficiently large.

Now we prove (3.32). By definition of $\tilde{A}_{n,\varepsilon}$, for all $n \geqslant 1$ and $\varepsilon \in (0,1)$ we have

$$\alpha_{n,\varepsilon}\mathbb{E}\tilde{A}_{n,\varepsilon}^{-a} = \Big(\frac{dA}{A_2}\Big)^a \mathbb{E}\Big[\|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} (P_1(\xi_k)+\varepsilon Q_1(\xi_k))\Big\|_\infty\Big]. \tag{3.36}$$

Notice that $\|M_{0,n-1}\|^a \| \prod_{k=0}^{n-1} (P_1(\xi_k)+\varepsilon Q_1(\xi_k))\|_\infty \leqslant \|M_{0,n-1}\|^a$ a.s., with $\mathbb{E}\|M_{0,n-1}\|^a < +\infty$. Therefore, by the Lebesgue dominated convergence theorem, letting $\varepsilon \to 0$ in (3.36), we get that for all $n \geqslant 1$,

$$\alpha_{n,\varepsilon}\mathbb{E}\tilde{A}_{n,\varepsilon}^{-a} \underset{\varepsilon\to 0}{\to} \Big(\frac{dA}{A_2}\Big)^a \mathbb{E}\Big[\|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} P_1(\xi_k)\Big\|_\infty\Big].$$

This implies

$$\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\Big(\alpha_{n,\varepsilon}\mathbb{E}\tilde{A}_{n,\varepsilon}^{-a}\Big)^{1/n} = \kappa_1(a) < 1,$$

so (3.32) holds for $n \in \mathbb{N}$ sufficiently large and $\varepsilon > 0$ small enough.

We have therefore proved (3.31) and (3.32), which implies (3.33). This concludes the proof of Theorem 3.7. □

3.4. *Proofs of Theorem 2.1 and Corollary 2.2*

In this section, using the sufficient condition for the existence of harmonic moments established in Theorem 3.7 of the preceding section, we prove Theorem 2.1 and Corollary 2.2 which give a necessary and a sufficient condition

for the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$. To this end, we need the following lemma about the behaviour of the function κ_1 on $\mathbb{R}^+$. Recall that a matrix M is nilpotent if there is an integer $n \geqslant 1$ such that $M^n = 0$.

Lemma 3.9. *Assume condition **H2**. Then the function κ_1 satisfies the following properties:*

(1) $\kappa_1(0) = \rho(\mathbb{E}P_1(\xi_0))$;
(2) The following assertions are equivalent: (i) $\kappa_1(0) = 0$; (ii) $\kappa_1(a) = 0$ for all $a \geqslant 0$; (iii) $\mathbb{E}P_1(\xi_0)$ is nilpotent;
(3) if $\mathbb{E}P_1(\xi_0)$ is not nilpotent, then κ_1 is a strictly increasing continuous function on $\mathbb{R}_+$, with $\lim_{a\to+\infty}\kappa_1(a) = +\infty$.

Proof. (1) Part (1) follows because, using inequalities (3.34), we have

$$\begin{aligned}\kappa_1(0) &= \lim_{n\to+\infty}\Big(\mathbb{E}\Big\|\prod_{k=0}^{n-1}P_1(\xi_k)\Big\|_\infty\Big)^{1/n}\\ &= \lim_{n\to+\infty}\Big(\Big\|\mathbb{E}\Big[\prod_{k=0}^{n-1}P_1(\xi_k)\Big]\Big\|_\infty\Big)^{1/n}\\ &= \lim_{n\to+\infty}\big(\|[\mathbb{E}P_1(\xi_0)]^n\|_\infty\big)^{1/n}\\ &= \rho(\mathbb{E}P_1(\xi_0)).\end{aligned} \tag{3.37}$$

(2) We next prove part (2). Notice that the matrix $\mathbb{E}P_1(\xi_0)$ is nilpotent if and only if $\rho(\mathbb{E}P_1(\xi_0)) = 0$. So by part (1), we only need to prove the equivalence between (i) and (ii). To this end we will prove that for all $a, b \geqslant 0$, there exist constants $c_1, c_2 > 0$ such that $\kappa_1(a+b) \leqslant c_1\kappa(a) \leqslant c_2\kappa_1(a+b)$.

By **H2** we have $\|M_{0,n-1}\| \leqslant (dA)^n$ a.s., so for all $a, b \geqslant 0$,

$$\begin{aligned}\kappa_1(a+b) &= \lim_{n\to+\infty}\Big(\mathbb{E}\|M_{0,n-1}\|^{a+b}\Big\|\prod_{k=0}^{n-1}P_1(\xi_k)\Big\|_\infty\Big)^{1/n}\\ &\leqslant \lim_{n\to+\infty}\Big((dA)^{nb}\mathbb{E}\|M_{0,n-1}\|^{a}\Big\|\prod_{k=0}^{n-1}P_1(\xi_k)\Big\|_\infty\Big)^{1/n}\\ &= (dA)^b\kappa_1(a).\end{aligned} \tag{3.38}$$

On the other hand, by (3.11) we get that for all $n \geqslant 0$,

$$\|M_{0,n-1}\| \geqslant \min_{1\leqslant i\leqslant d} M_{0,n-1}(i,j) \geqslant A_2A_1^{n-1} \quad \mathbb{P}\text{-a.s.}$$

Therefore, we obtain that for all $a, b \geqslant 0$, with $A_1 > 1$,

$$\begin{aligned}\kappa_1(a+b) &\geqslant \lim_{n\to+\infty} \Big((A_2 A_1^{n-1})^b \mathbb{E}\|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} P_1(\xi_k)\Big\|_\infty\Big)^{1/n} \\ &= A_1^b \kappa_1(a). \end{aligned} \tag{3.39}$$

From (3.38) and (3.39) we see that $\kappa_1(a) = 0$ for all $a \geqslant 0$ if and only if $\kappa_1(0) = 0$. This ends the proof of part (2).

(3) We finally prove part (3). Assume that $\mathbb{E}P_1(\xi_0)$ is not nilpotent. Then, by parts (1) and (2), we have $\kappa_1(0) = \rho(\mathbb{E}P_1(\xi_0)) > 0$. It follows from (3.38) and (3.39) that κ_1 is a strictly increasing continuous function on $\mathbb{R}_+$ with $\lim_{a\to+\infty} \kappa_1(a) = +\infty$. This concludes part (3).

The proof of Lemma 3.9 is complete. □

Proof of Theorem 2.1. First we assume that $\kappa_1(a) < 1$, where $a > 0$. By Lemma 3.9, κ_1 is a continuous function on $\mathbb{R}_+$, so there exists $\varepsilon > 0$ such that $\kappa_1(a+\varepsilon) < 1$. Using Theorem 3.7 (with a replaced by $a+\varepsilon$), this implies that $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$.

Now we suppose that $a > 0$ is such that $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$. We will prove that $\kappa_1(a) \leqslant 1$. Let $Z^r_{l,n,k} = (Z^r_{l,n,k}(1), \cdots, Z^r_{l,n,k}(d))$, where $Z^r_{l,n,k}(j)$ denotes the offspring of type j at time $n+k$ of the l-th particle of type r in the generation n. By iteration of (2.1), it is easy to see that the MBPRE $(Z^i_n)_{n\geqslant 0}$ satisfies the relation

$$Z^i_{n+k} = \sum_{j=1}^{d} \sum_{l=1}^{Z^i_n(j)} Z^j_{l,n,k}, \quad n, k \geqslant 0.$$

It follows that for all $n, k \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} W^i_{n+k} &= \sum_{j=1}^{d} \sum_{l=1}^{Z^i_n(j)} \frac{\langle Z^{rj}_{l,n,k}, U_{n+k,\infty}\rangle}{\lambda_{0,n+k-1} U_{0,\infty}(i)} \\ &= \sum_{j=1}^{d} \frac{U_{n,\infty}(j)}{\lambda_{0,n-1} U_{0,\infty}(i)} \sum_{l=1}^{Z^i_n(j)} W^j_{l,n,k}, \end{aligned} \tag{3.40}$$

where

$$W^j_{l,n,k} := \frac{\langle Z^j_{l,n,k}, U_{n+k,\infty}\rangle}{\lambda_{n,n+k-1} U_{n,\infty}(j)}.$$

Clearly $(W^j_{l,n,k})_{k\geqslant 0}$ is the fundamental martingale associated to the MBPRE $(Z^j_{l,n,k})_{k\geqslant 0}$ in the shifted random environment $T^n\xi$, starting with the l-th

particle of type j in the generation n. Hence $(W^j_{l,n,k})_{k\geqslant 0}$ converges a.s. to a random variable $W^j_{l,n}$. Letting $k\to+\infty$ in (3.40), we get the following distributional equation on the limit variables W^i: for all $1\leqslant i\leqslant d$ and $n\geqslant 0$,

$$W^i=\sum_{r=1}^{d}\frac{U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)}\sum_{l=1}^{Z^i_n(r)}W^r_{l,n}, \tag{3.41}$$

where for each $n\in\mathbb{N}$, under $\mathbb{P}_\xi$, the random variables $W^r_{l,n}$, indexed by $l\geqslant 0$ and $1\leqslant r\leqslant d$, are independent of each other and independent of Z^i_n, each with the distribution

$$\mathbb{P}_\xi(W^r_{l,n}\in\cdot)=\mathbb{P}_{T^n\xi}(W^r\in\cdot).$$

Therefore, we get from (3.41) that for all $1\leqslant i\leqslant d$ and $n\geqslant 1$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\mathbb{E}_\xi(W^i)^{-a}&=\mathbb{E}_\xi\Big(\sum_{r=1}^{d}\frac{U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)}\sum_{l=1}^{Z^i_n(r)}W^r_{l,n}\Big)^{-a}\\&\geqslant\sum_{r=1}^{d}\Big(\frac{U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)}\Big)^{-a}\mathbb{E}_\xi\Big[\Big(\sum_{l=1}^{Z^i_n(r)}W^r_{l,n}\Big)^{-a}\mathbb{1}_{\{Z^i_n=e_r\}}\Big]\\&=(\lambda_{0,n-1}U_{0,\infty}(i))^a\sum_{r=1}^{d}U_{n,\infty}(r)^{-a}\mathbb{E}_{T^n\xi}(W^r)^{-a}\mathbb{P}_\xi(Z^i_n=e_r).\end{aligned} \tag{3.42}$$

For $n\geqslant 0$, let X_n be the vector in $\mathbb{R}^d$ whose i-th entry is

$$X_n(i)=U_{n,\infty}(i)^{-a}\mathbb{E}_{T^n\xi}(W^i)^{-a},\quad 1\leqslant i\leqslant d. \tag{3.43}$$

By **H1** and the definition of the random matrices $P_1(\xi_n)$, we have that for all $n\geqslant 1$ and $1\leqslant i,r\leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\mathbb{P}_\xi(Z^i_{n+1}=e_r)&=\sum_{j=1}^{d}\mathbb{P}_\xi(Z^i_n=e_j)\mathbb{P}_\xi(Z^i_{n+1}=e_r|Z^i_n=e_j)\\&=\sum_{j=1}^{d}\mathbb{P}_\xi(Z^i_n=e_j)P_1(\xi_n)(j,r).\end{aligned}$$

By iterating this relation, we get that for all $n\geqslant 1$ and $1\leqslant i,r\leqslant d$, $\mathbb{P}$-a.s.,

$$\mathbb{P}_\xi(Z^i_n=e_r)=\Big[\prod_{k=0}^{n-1}P_1(\xi_k)\Big](i,r). \tag{3.44}$$

Using this and the notation X_n (cf. (3.43)), we can re-write (3.42) as follows: for all $n \geqslant 1$, $\mathbb{P}$-a.s.,

$$X_0 \geqslant \lambda_{0,n-1}^a \prod_{k=0}^{n-1} P_1(\xi_k)\, X_n. \tag{3.45}$$

By (1.5) and (3.22), we get that for all $n \geqslant 1$,

$$\lambda_{0,n-1} = \|M_{0,n-1}U_{n,\infty}\| \geqslant \frac{A_2}{dA}\|M_{0,n-1}\mathbf{1}\| \geqslant \frac{A_2}{dA}\|M_{0,n-1}\| \quad \mathbb{P}\text{-a.s.}$$

This, together with (3.45), implies that for all $n \geqslant 1$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\|X_0\|_\infty &\geqslant \Big(\frac{A_2}{dA}\Big)^a \|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} P_1(\xi_k)\, X_n\Big\|_\infty \\ &\geqslant \Big(\frac{A_2}{dA}\Big)^a \|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} P_1(\xi_k)\Big\|_\infty \min_{1\leqslant i\leqslant d} X_n(i).\end{aligned} \tag{3.46}$$

Notice that $(X_n)_{n\geqslant 0}$ is a stationary ergodic sequence of strictly positive random vectors, and X_n is independent of $\xi_0, \cdots, \xi_{n-1}$. Therefore, taking expectation in (3.46), we get that for all $n \geqslant 1$,

$$\mathbb{E}\|X_0\|_\infty \geqslant \Big(\frac{A_2}{dA}\Big)^a \mathbb{E}\Big[\|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} P_1(\xi_k)\Big\|_\infty\Big]\, \mathbb{E}\big(\min_{1\leqslant i\leqslant d} X_0(i)\big). \tag{3.47}$$

Using again (3.22), for any $1 \leqslant i \leqslant d$ we have

$$0 < \mathbb{E}X_0(i) = \mathbb{E}[U_{0,\infty}(i)^{-a}\mathbb{E}_\xi(W^i)^{-a}] \leqslant \Big(\frac{A_2}{dA}\Big)^{-a}\mathbb{E}(W^i)^{-a} < +\infty.$$

Therefore, by (3.47) we obtain for all $n \geqslant 1$,

$$\mathbb{E}\|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} P_1(\xi_k)\Big\|_\infty \leqslant \Big(\frac{dA}{A_2}\Big)^a \frac{\mathbb{E}\|X_0\|_\infty}{\mathbb{E}\big(\min_{1\leqslant i\leqslant d} X_0(i)\big)}.$$

This implies that

$$\kappa_1(a) = \lim_{n\to+\infty}\Big(\mathbb{E}\|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} P_1(\xi_k)\Big\|_\infty\Big)^{1/n} \leqslant 1,$$

which is the desired result. This concludes the proof of Theorem 2.1 and the remark following it. □

Proof of Corollary 2.2. First we suppose that $\mathbb{E}P_1(\xi_0)$ is nilpotent. By Lemma 3.9, we know that $\kappa_1(a) = 0 < 1$ for all $a \geqslant 0$. Therefore, using Theorem 2.1, we get that $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$ and $a > 0$.

Now, we assume that $\mathbb{E}P_1(\xi_0)$ is not nilpotent. By Lemma 3.9 and the condition $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty} < 1$, the function κ_1 is continuous and strictly increasing on $\mathbb{R}_+$ with $\kappa_1(0) = \rho(\mathbb{E}P_1(\xi_0)) < 1$ and $\lim_{a\to+\infty} \kappa_1(a) = +\infty$. Therefore, there exists a unique constant $a_0 > 0$ such that $\kappa_1(a_0) = 1$, and we have $\kappa_1(a) < 1$ if $a < a_0$, and $\kappa_1(a) > 1$ if $a > a_0$. Using again Theorem 2.1, this implies that

$$\max_{1\leqslant i\leqslant d} \mathbb{E}(W^i)^{-a} \begin{cases} < +\infty & \text{if } a \in [0, a_0), \\ = +\infty & \text{if } a \in (a_0, +\infty). \end{cases}$$

The proof of Corollary 2.2 is complete. □

3.5. *Proof of Theorem 2.3*

In this section, we prove Theorem 2.3 which gives the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$ for a small $a > 0$, under the moment condition **H3**, instead of the boundedness condition **H2** considered in the preceding section. We will prove the following theorem.

Theorem 3.10. *Assume conditions **H1**, **H3** and $\gamma > 0$. Then there exist two constants $a > 0$ and $C > 0$ such that for all $1 \leqslant i \leqslant d$, all $t > 0$, and all $x > 0$,*

$$\phi^i(t) \leqslant \frac{C}{t^a}, \tag{3.48}$$

$$\mathbb{P}(W^i \leqslant x) \leqslant Cx^a \quad \text{and} \quad \mathbb{E}(W^i)^{-a} \leqslant C. \tag{3.49}$$

Clearly Theorem 2.3 follows from Theorem 3.10. We need some previous results to prove Theorem 3.10. The first one gives the non-degeneracy of the limits W^i under the conditions of Theorem 3.10.

Lemma 3.11. *The following implication holds: **H3** $\Rightarrow$ (2.5). Moreover, under the conditions **H3** and $\gamma > 0$, the limit W^i is non-degenerate for each $1 \leqslant i \leqslant d$, and (2.6) hold.*

Proof. We first prove the implication **H3** $\Rightarrow$ (2.5). Assume condition **H3**. Then, by Hölder's inequality $\mathbb{E}_\xi|XY| \leqslant (\mathbb{E}_\xi|X|^\alpha)^{1/\alpha}(\mathbb{E}_\xi|Y|^\beta)^{1/\beta}$ with $\alpha =$

$1/(1-\eta)$ and $\beta = 1/\eta$, we see that for all $1 \leqslant i, j \leqslant d$ we have, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\mathbb{E}_\xi\Big(\frac{Z_1^i(j)}{M_0(i,j)}\log^+\frac{Z_1^i(j)}{M_0(i,j)}\Big)\\
=\;&\mathbb{E}_\xi\Big[\Big(\frac{Z_1^i(j)}{M_0(i,j)}\Big)^{1-\eta}\Big(\frac{Z_1^i(j)}{M_0(i,j)}\Big)^{\eta}\log^+\frac{Z_1^i(j)}{M_0(i,j)}\Big]\\
\leqslant\;&\Big[\mathbb{E}_\xi\Big(\frac{Z_1^i(j)}{M_0(i,j)}\Big)\Big]^{1-\eta}\Big\{\mathbb{E}_\xi\Big[\frac{Z_1^i(j)}{M_0(i,j)}\Big(\log^+\frac{Z_1^i(j)}{M_0(i,j)}\Big)^{1/\eta}\Big]\Big\}^{\eta}\\
=\;&\Big\{\mathbb{E}_\xi\Big[\frac{Z_1^i(j)}{M_0(i,j)}\Big(\log^+\frac{Z_1^i(j)}{M_0(i,j)}\Big)^{1/\eta}\Big]\Big\}^{\eta}. \qquad (3.50)
\end{aligned}$$

We know that there exists a constant $C > 0$ such that $\log^+ x \leqslant Cx^{\eta(p-1)}$ for all $x > 0$. Taking expectation in (3.50), we obtain that for all $1 \leqslant i, j \leqslant d$,

$$\mathbb{E}\Big(\frac{Z_1^i(j)}{M_0(i,j)}\log^+\frac{Z_1^i(j)}{M_0(i,j)}\Big) \leqslant C\mathbb{E}\Big[\mathbb{E}_\xi\Big(\frac{Z_1^i(j)}{M_0(i,j)}\Big)^{p}\Big]^{\eta} < +\infty, \qquad (3.51)$$

since $\mathbb{E}\theta_0(p)^\eta < +\infty$ by condition **H3**. Hence the condition (2.5) holds.

Recall that by [13, Theorem 2.6 and Corollary 2.8], condition (2.5) together with $\gamma > 0$ implies that the limit W^i is non-degenerate for each $1 \leqslant i \leqslant d$, and (2.6) hold. Combining this with the implication **H3** $\Rightarrow$ (2.5), we conclude the proof of Lemma 3.11. □

By the sub-additive ergodic theorem and the fact that $\|M_{0,n}(i,\cdot)\| \geqslant 1$ a.s. for any $1 \leqslant i \leqslant d$ (which follows from **H1**), for all $s < 0$ the limit

$$\kappa(s) := \lim_{n\to+\infty}\big(\mathbb{E}\|M_{0,n-1}\|^s\big)^{1/n} \qquad (3.52)$$

exists, with $\kappa(s) = \sup_{n\geqslant 1}\big(\mathbb{E}\|M_{0,n-1}\|^s\big)^{1/n} \leqslant 1$. The proof of Theorem 3.10 is based on the following result about the properties of the function κ.

Lemma 3.12. *Assume conditions **H1**, **H3** and $\gamma > 0$. Then for all $s < 0$,*

$$\kappa(s) < 1.$$

*Moreover, if $\eta > 0$ satisfies **H3**, then for all $0 \leqslant a < s \leqslant \eta/2$,*

$$\Big(\mathbb{E}\big[\|M_{0,n-1}\|^a \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^{-s}\big]\Big)^{1/n} \underset{n\to+\infty}{\to} \kappa(a-s) < 1. \qquad (3.53)$$

Proof. First we prove that $\kappa(s) < 1$ for all $s < 0$. Recall that $\iota(M) = \inf_{\|x\|=1}\|Mx\|$, for $M \in \mathcal{M}_d(\mathbb{R})$, and $\iota(M_{0,n}^T) \geqslant 1$ a.s. by (3.10) and **H1**.

Since ι is super-multiplicative, the sequence $\mathbb{E}\,\iota(M_{0,n-1}^T)^s$ is sub-multiplicative. So using the sub-additive ergodic theorem we get

$$\lim_{n\to+\infty}\left(\mathbb{E}\,\iota(M_{0,n-1}^T)^s\right)^{1/n}=\inf_{n\geqslant 1}\left(\mathbb{E}\,\iota(M_{0,n-1}^T)^s\right)^{1/n}\leqslant 1,\quad \forall s<0.$$

By the inequalities $\iota(M^T)\leqslant\|M^T\|\leqslant d\|M\|$, $M\in\mathcal{M}_d(\mathbb{R})$, this implies that for all $s<0$,

$$\kappa(s)\leqslant\lim_{n\to+\infty}\left(\mathbb{E}\,\iota(M_{0,n-1}^T)^s\right)^{1/n}=\inf_{n\geqslant 1}\left(\mathbb{E}\,\iota(M_{0,n-1}^T)^s\right)^{1/n}.$$

Therefore, for all $s<0$ and $n\geqslant 1$, we have

$$\kappa(s)^n\leqslant\mathbb{E}\,\iota(M_{0,n-1}^T)^s\leqslant 1. \tag{3.54}$$

Clearly by (3.54), if $\kappa(s)=1$, then $\mathbb{E}\,\iota(M_{0,n-1}^T)^s=1$ for all $n\geqslant 1$, which is equivalent to $\iota(M_{0,n-1}^T)=1$ a.s. for all $n\geqslant 1$. Consequently, if we show that

$$\lim_{n\to+\infty}\mathbb{P}(\iota(M_{0,n-1}^T)=1)=0, \tag{3.55}$$

then we will have $\kappa(s)<1$ for all $s<0$. So, it remains to prove (3.55). Using **H1** and the identity $\iota(M)=\min\limits_{1\leqslant j\leqslant d}\|M(\cdot,j)\|$, $M\in\mathcal{M}_d(\mathbb{R})$, for all $n\geqslant 1$ we have

$$\mathbb{P}(\iota(M_{0,n-1}^T)=1)\leqslant\sum_{i=1}^d\mathbb{P}(\|M_{0,n-1}(i,\cdot)\|=1)\leqslant\sum_{i=1}^d\mathbb{P}(\|Z_n^i\|=1). \tag{3.56}$$

By Lemma 3.11, the conditions **H3** and $\gamma>0$ imply that the limits W^i, $1\leqslant i\leqslant d$, are non-degenerate, and (2.6) holds. Combining this with condition **H1**, we obtain that $\lim_{n\to+\infty}\|Z_n^i\|=+\infty$ a.s. Therefore, we obtain that for all $1\leqslant i\leqslant d$,

$$\lim_{n\to+\infty}\mathbb{P}(\|Z_n^i\|=1)=0, \tag{3.57}$$

and (3.55) follows from (3.56). So $\kappa(s)<1$ for all $s<0$, which is the desired result.

Now we prove (3.53). Let $\eta>0$ be such that condition **H3** holds, and set $0<a<s\leqslant\eta/2$. First, for all $n\geqslant 1$ we have

$$\mathbb{E}\Big[\|M_{0,n-1}\|^a\max_{1\leqslant i,j\leqslant d}M_{0,n-1}(i,j)^{-s}\Big]\geqslant\mathbb{E}\|M_{0,n-1}\|^{a-s}. \tag{3.58}$$

Next, for all $n \geqslant 2$,

$$\begin{aligned}
& \mathbb{E}\Big[\|M_{0,n-1}\|^a \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^{-s}\Big] \\
= \;& \mathbb{E}\Big[\|M_{0,n-1}\|^a \max_{1\leqslant i,j\leqslant d} \Big(\sum_{r=1}^{d} M_0(i,r)M_{1,n-1}(r,j)\Big)^{-s}\Big] \\
\leqslant \;& \mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,r\leqslant d} M_0(i,r)^{-s}\Big]\mathbb{E}\Big[\|M_{1,n-1}\|^a \max_{1\leqslant j\leqslant d} \|M_{1,n-1}(\cdot,j)\|^{-s}\Big] \\
= \;& \mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,r\leqslant d} M_0(i,r)^{-s}\Big]\mathbb{E}\Big[\|M_{0,n-2}\|^a \max_{1\leqslant j\leqslant d} \|M_{0,n-2}(\cdot,j)\|^{-s}\Big]
\end{aligned}$$

It follows that for all $n \geqslant 3$,

$$\begin{aligned}
& \mathbb{E}\Big[\|M_{0,n-1}\|^a \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^{-s}\Big] \\
\leqslant \;& \mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{-s}\Big] \times \\
& \mathbb{E}\Big[\|M_{0,n-2}\|^a \max_{1\leqslant j\leqslant d} \Big(\sum_{r=1}^{d} \|M_{0,n-3}(\cdot,r)\|M_{n-2}(r,j)\Big)^{-s}\Big] \\
\leqslant \;& \mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{-s}\Big] \times \\
& \mathbb{E}\Big[\|M_{n-2}\|^a \max_{1\leqslant i,j\leqslant d} M_{n-2}(i,j)^{-s}\Big]\mathbb{E}\Big[\|M_{0,n-3}\|^a (d\|M_{0,n-3}\|)^{-s}\Big] \\
= \;& d^{-s}\Big(\mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{-s}\Big]\Big)^2 \mathbb{E}\|M_{0,n-3}\|^{a-s}. \qquad (3.59)
\end{aligned}$$

Then, by Hölder's inequality and condition **H3**, since $2a < 2s \leqslant \eta$, we have

$$\begin{aligned}
\mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{-s}\Big] &\leqslant \big(\mathbb{E}\|M_0\|^{2a}\big)^{1/2}\Big(\mathbb{E}\Big[\max_{1\leqslant i,j\leqslant d} M_0(i,j)^{-2s}\Big]\Big)^{1/2} \\
&< +\infty.
\end{aligned}$$

Combining this with the inequalities (3.58) and (3.59), we obtain that for all $n \geqslant 3$,

$$\mathbb{E}\|M_{0,n-1}\|^{a-s} \leqslant \mathbb{E}\Big[\|M_{0,n-1}\|^a \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^{-s}\Big] \leqslant C\,\mathbb{E}\|M_{0,n-3}\|^{a-s},$$

where $C = d^{-s}\big(\mathbb{E}[\|M_0\|^a \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{-s}]\big)^2 < +\infty$. This, together with (3.52), implies (3.53), which concludes the proof of Lemma 3.12. □

Proof of Theorem 3.10. The implication (3.48) $\Rightarrow$ (3.49) holds by Lemma 3.2. Therefore, it remains to show (3.48) under the given conditions.

Assume condition **H1**, **H3** and $\gamma > 0$. Set $\psi_\xi(t) = (\psi_\xi^1(t), \cdots, \psi_\xi^d(t))$, where for all $t \geqslant 0$, and $1 \leqslant i \leqslant d$,

$$\psi_\xi^i(t) = \phi_\xi^i(tU_{0,\infty}(i)).$$

Let ψ be the function on $\mathbb{R}_+$ defined by

$$\psi(t) = \mathbb{E}\|\psi_\xi(t)\|_\infty, \quad t \geqslant 0.$$

Since ϕ_ξ^i is a decreasing function on $\mathbb{R}_+$ and $0 < U_{0,\infty}(i) < 1$ a.s., for all $t \geqslant 0$ we have

$$\|\phi(t)\|_\infty = \|\mathbb{E}\phi_\xi(t)\|_\infty \leqslant \mathbb{E}\|\phi_\xi(t)\|_\infty \leqslant \mathbb{E}\|\psi_\xi(t)\|_\infty = \psi(t).$$

Therefore, if we prove that there exists a constant $a > 0$ such that

$$\psi(t) = O_{t\to+\infty}(t^{-a}), \tag{3.60}$$

then (3.48) holds.

Now we prove (3.60). As in the proof of Theorem 2.1, the argument will still be based on (3.21). The idea is to take expectation at both sides of this inequality, and to get an inequality on ψ in order to use Lemma 3.8 to conclude. The difficult point is to have a bound in terms of ψ while taking expectation on the right hand side of (3.21). This will be done by truncation and iteration.

Recall that $\lambda_{0,n-1} = \|M_{0,n-1}U_{n,\infty}\| \leqslant \|M_{0,n-1}\|$ for all $n \geqslant 1$, and ψ_ξ^i is decreasing on $\mathbb{R}_+$. Therefore, we get that for all $n \geqslant 1$ and $t \geqslant 0$,

$$\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big) \leqslant \psi_{T^n\xi}\Big(\frac{t}{\|M_{0,n-1}\|}\Big) \quad \mathbb{P}\text{-a.s.} \tag{3.61}$$

Moreover, by similar arguments as in (3.14)-(3.16), we have that for all $K > 0$, there exist two constants $t_K > 0$ and $\beta_K \in (0,1)$ such that for all $t \geqslant t_K$, $\|\phi_\xi(t)\|_\infty \leqslant \beta_K$ $\mathbb{P}$-a.s. on the event $\{\max_{1\leqslant i\leqslant d}\mathbb{E}_\xi(W^i)^p \leqslant K^p\}$. Therefore when $t \geqslant \frac{t_K\lambda_{0,n-1}}{\min\limits_{1\leqslant i\leqslant d} U_{n,\infty}(i)}$ and $\max\limits_{1\leqslant i\leqslant d}\mathbb{E}_{T^n\xi}(W^i)^p \leqslant K^p$, we have a.s.

$$\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \leqslant \beta_K.$$

Combining this with (3.21) and (3.61), we obtain that for all $K>0$, $n\geqslant 1$ and $t\geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\mathbb{E}\Big[\|\psi_\xi(t)\|_\infty \mathbb{1}_{\{t\geqslant \frac{t_K\lambda_{0,n-1}}{\min\limits_{1\leqslant i\leqslant d} U_{n,\infty}(i)}, \max\limits_{1\leqslant i\leqslant d}\mathbb{E}_{T^n\xi}(W^i)^p\leqslant K^p\}}\Big]\\
\leqslant\ & \mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}(P_1(\xi_k)+\beta_K Q_1(\xi_k))\psi_{T^n\xi}\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big\|_\infty\Big]\\
\leqslant\ & \mathbb{E}\Big\{\Big\|\prod_{k=0}^{n-1}(P_1(\xi_k)+\beta_K Q_1(\xi_k))\Big\|_\infty \mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big\|_\infty\Big|\xi_0,\cdots,\xi_{n-1}\Big]\Big\}\\
=\ & \mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}(P_1(\xi_k)+\beta_K Q_1(\xi_k))\Big\|_\infty \psi\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big].
\end{aligned}$$

This, together with (3.20), implies that for all $K>0$, $n\geqslant 1$ and $t\geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\psi(t)\\
\leqslant\ & \mathbb{E}\Big[\|\psi_\xi(t)\|_\infty \mathbb{1}_{\{t\geqslant \frac{t_K\lambda_{0,n-1}}{\min_{1\leqslant i\leqslant d} U_{n,\infty}(i)}, \max\limits_{1\leqslant i\leqslant d}\mathbb{E}_{T^n\xi}(W^i)^p\leqslant K^p\}}\Big]\\
&+\mathbb{E}\Big[\|\psi_\xi(t)\|_\infty \mathbb{1}_{\{\max\limits_{1\leqslant i\leqslant d}\mathbb{E}_{T^n\xi}(W^i)^p> K^p\}}\Big]+\mathbb{E}\Big[\|\psi_\xi(t)\|_\infty \mathbb{1}_{\{t<\frac{t_K\lambda_{0,n-1}}{\min_{1\leqslant i\leqslant d} U_{n,\infty}(i)}\}}\Big]\\
\leqslant\ & \mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}(P_1(\xi_k)+\beta_K Q_1(\xi_k))\Big\|_\infty \psi\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big]\\
&+\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \mathbb{1}_{\{\max\limits_{1\leqslant i\leqslant d}\mathbb{E}_{T^n\xi}(W^i)^p> K^p\}}\Big]\\
&+\mathbb{P}\Big(\frac{\|M_{0,n-1}\|}{\min_{1\leqslant i\leqslant d} U_{n,\infty}(i)}>\frac{t}{t_K}\Big).
\end{aligned}\tag{3.62}$$

Now we control the second and last terms in (3.62).

We first find a bound for the second term in terms of ψ. Using the triangular inequality in L^p and (3.8), we get that for all $1\leqslant i\leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\big(\mathbb{E}_\xi(W^i)^p\big)^{1/p}&\leqslant 1+\sum_{n=0}^{+\infty}\big(\mathbb{E}_\xi|W^i_{n+1}-W^i_n|^p\big)^{1/p}\\
&\leqslant 1+B_p\sum_{n=0}^{+\infty}\theta_n(p)^{\frac{1}{p}}\max_{1\leqslant j\leqslant d}M_{0,n-1}(i,j)^{\frac{1-p}{p}}.
\end{aligned}$$

By the sub-additivity of the function $x\mapsto x^{\frac{\eta}{2}}$ on $\mathbb{R}_+$ (since $0<\eta<1$), this

implies that for all $K>0$, $n\geqslant 1$ and $t\geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \mathbb{1}_{\{\max\limits_{1\leqslant i\leqslant d}\mathbb{E}_{T^n\xi}(W^i)^p>K^p\}}\Big]\\
\leqslant\ & \frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \max_{1\leqslant i\leqslant d}\big(\mathbb{E}_{T^n\xi}(W^i)^p\big)^{\frac{\eta}{2p}}\Big]\\
\leqslant\ & \frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty\Big] + \Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}}\times\\
&\sum_{k=0}^{+\infty}\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \theta_{n+k}(p)^{\frac{\eta}{2p}}\max_{1\leqslant i,j\leqslant d}M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big].
\end{aligned}\tag{3.63}$$

By (3.20) and (3.61), we get that for $n,k\geqslant 0$ and $t\geqslant 0$, $\mathbb{P}$-a.s.,

$$\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \leqslant \Big\|\psi_{T^{n+k+1}\xi}\Big(\frac{t}{\lambda_{0,n+k}}\Big)\Big\|_\infty \leqslant \Big\|\psi_{T^{n+k+1}\xi}\Big(\frac{t}{\|M_{0,n+k}\|}\Big)\Big\|_\infty.$$

Combining this with (3.63), we obtain that for all $K>0$, $n\geqslant 1$ and $t\geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \mathbb{1}_{\{\max\limits_{1\leqslant i\leqslant d}\mathbb{E}_{T^n\xi}(W^i)^p>K^p\}}\Big]\\
\leqslant\ & \frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big\|_\infty\Big] + \Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}}\times\\
&\sum_{k=0}^{+\infty}\mathbb{E}\Big[\Big\|\psi_{T^{n+k+1}\xi}\Big(\frac{t}{\|M_{0,n+k}\|}\Big)\Big\|_\infty \theta_{n+k}(p)^{\frac{\eta}{2p}}\max_{1\leqslant i,j\leqslant d}M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big]\\
\leqslant\ & \frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\Big[\psi\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big] + \Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}}\times\\
&\sum_{k=0}^{+\infty}\mathbb{E}\Big[\psi\Big(\frac{t}{\|M_{0,n+k}\|}\Big)\theta_{n+k}(p)^{\frac{\eta}{2p}}\max_{1\leqslant i,j\leqslant d}M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big],
\end{aligned}\tag{3.64}$$

where, for the last inequality, we have used the fact that $M_{0,n+k}$ and $\theta_{n+k}(p)$ are independent of $T^{n+k+1}\xi$. We have therefore obtained a bound of the second term in (3.62) in terms of ψ.

For the last term in (3.62), by Markov's inequality we get that for all $K>0$, $n\geqslant 1$ and $t>0$,

$$\begin{aligned}
\mathbb{P}\Big(\frac{\|M_{0,n-1}\|}{\min_{1\leqslant i\leqslant d}U_{n,\infty}(i)}>\frac{t}{t_K}\Big) &\leqslant \Big(\frac{t_K}{t}\Big)^{\frac{\eta}{2}}\mathbb{E}\Big(\frac{\|M_{0,n-1}\|^{\frac{\eta}{2}}}{\min_{1\leqslant i\leqslant d}U_{n,\infty}(i)^{\frac{\eta}{2}}}\Big)\\
&\leqslant \frac{C(n,K)}{t^{\eta/2}},
\end{aligned}\tag{3.65}$$

where $C(n,K) := t_K^{\frac{\eta}{2}}(\mathbb{E}\|M_0\|^{\frac{\eta}{2}})^n\mathbb{E}(\max_{1\leqslant i\leqslant d}U_{0,\infty}(i)^{-\frac{\eta}{2}})$. We claim that $C(n,K)<\infty$ because, by (1.4), Cauchy-Schwarz's inequality and condition **H3**, we have for all $1\leqslant i\leqslant d$

$$\begin{aligned}\mathbb{E}(U_{0,\infty}(i)^{-\frac{\eta}{2}}) &= \mathbb{E}\Big(\frac{\langle M_0(i,\cdot),U_{1,\infty}\rangle^{-\frac{\eta}{2}}}{\|M_0U_{1,\infty}\|^{-\frac{\eta}{2}}}\Big)\\ &\leqslant \mathbb{E}\Big(\frac{\max_{1\leqslant j\leqslant d}M_0(i,j)^{-\frac{\eta}{2}}}{\|M_0\|^{-\frac{\eta}{2}}}\Big)\\ &\leqslant \Big(\mathbb{E}\Big[\max_{1\leqslant j\leqslant d}M_0(i,j)^{-\eta}\Big]\Big)^{1/2}(\mathbb{E}\|M_0\|^{\eta})^{1/2}<+\infty. \qquad (3.66)\end{aligned}$$

Putting together the inequalities (3.62), (3.63), (3.64) and (3.65), we obtain the following inequality on ψ: for all $K>0$, $n\geqslant 1$ and $t>0$,

$$\begin{aligned}&\psi(t)\\ \leqslant\ &\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}(P_1(\xi_k)+\beta_KQ_1(\xi_k))\Big\|_\infty\psi\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big]\\ &+\Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}}\sum_{k=0}^{+\infty}\mathbb{E}\Big[\psi\Big(\frac{t}{\|M_{0,n+k}\|}\Big)\theta_{n+k}(p)^{\frac{\eta}{2p}}\max_{1\leqslant i,j\leqslant d}M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big]\\ &+\frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\Big[\psi\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big]+\frac{C(n,K)}{t^{\frac{\eta}{2}}}. \qquad (3.67)\end{aligned}$$

We will write this inequality in the form (3.27) in order to use Lemma 3.8. For $K>0$ and $n\geqslant 1$, set

$$\begin{aligned}\alpha_{n,K} :=\ &\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}(P_1(\xi_k)+\beta_KQ_1(\xi_k))\Big\|_\infty\Big]\\ &+\Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}}\mathbb{E}\Big[\theta_0(p)^{\frac{\eta}{2p}}\Big]\sum_{k=0}^{+\infty}\mathbb{E}\Big[\max_{1\leqslant i,j\leqslant d}M_{0,k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big]+\frac{1}{K^{\frac{\eta}{2}}}.\end{aligned}$$

By condition **H3** and equality (3.18), and using (3.53) in Lemma 3.12 with $a=0$ and $s=\frac{\eta(p-1)}{2p}\in(0,\frac{\eta}{2})$, we get for all $K>0$ and $n\geqslant 1$,

$$\alpha_{n,K}<+\infty.$$

For any $K>0$ and $n\geqslant 1$, let $\tilde{A}_{n,K}$ be a positive random variable whose distribution is determined by the following expectation: for all bounded and

mesurable function h on $\mathbb{R}_+$,

$$\begin{aligned}
&\mathbb{E}h(\tilde{A}_{n,K})\\
=\ &\frac{1}{\alpha_{n,K}}\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\big(P_1(\xi_k)+\beta_K Q_1(\xi_k)\big)\Big\|_\infty h\Big(\frac{1}{\|M_{0,n-1}\|}\Big)\Big]\\
&+\frac{B_p^{\frac{\eta}{2}}}{K^{\frac{\eta}{2}}\alpha_{n,K}}\sum_{k=0}^{+\infty}\mathbb{E}\Big[h\Big(\frac{1}{\|M_{0,n+k}\|}\Big)\theta_{n+k}(p)^{\frac{\eta}{2p}}\max_{1\leqslant i,j\leqslant d}M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big]\\
&+\frac{1}{K^{\frac{\eta}{2}}\alpha_{n,K}}\mathbb{E}\Big[h\Big(\frac{1}{\|M_{0,n-1}\|}\Big)\Big].
\end{aligned}$$

So we can rewrite (3.67) as in the form (3.27): for all $K>0$, $n\geqslant 1$ and $t>0$,

$$\psi(t)\leqslant \alpha_{n,K}\mathbb{E}[\psi(\tilde{A}_{n,K}t)]+\frac{C(n,K)}{t^{\eta/2}}. \tag{3.68}$$

We will prove that there exist $K>0$, $n\geqslant 1$ and $0<a<\eta/2$ such that

$$\alpha_{n,K}<1, \tag{3.69}$$

and

$$\alpha_{n,K}\mathbb{E}\tilde{A}_{n,K}^{-a}<1. \tag{3.70}$$

By Lemma 3.8, inequality (3.68), together with (3.69) and (3.70), implies (3.60), which is the desired result. So it remains to prove (3.69) and (3.70).

First, we prove (3.69). By (3.44), for all $1\leqslant i,j\leqslant d$ we have

$$\mathbb{P}(Z_n^i=e_j)=\mathbb{E}\Big[\prod_{k=0}^{n-1}P_1(\xi_k)\Big](i,j)=\big(\mathbb{E}P_1(\xi_0)\big)^n(i,j).$$

This, together with (3.57), implies that

$$\rho(\mathbb{E}P_1(\xi_0))<1. \tag{3.71}$$

Notice that the relation (3.35) can be proved by using (3.71) instead of the condition $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty}<1$. Therefore, (3.71) implies that for all $K>0$,

$$\limsup_{n\to+\infty}\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\big(P_1(\xi_k)+\beta_K Q_1(\xi_k)\big)\Big\|_\infty\Big]\leq\beta_K. \tag{3.72}$$

By the sub-additivity we have

$$\lim_{n\to+\infty}\Big(\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\left(P_1(\xi_k)+\beta_K Q_1(\xi_k)\right)\Big\|_\infty\Big]\Big)^{1/n} = \inf_{n\geqslant 1}\Big(\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\left(P_1(\xi_k)+\beta_K Q_1(\xi_k)\right)\Big\|_\infty\Big]\Big)^{1/n}.$$

Together with (3.72), this implies that for all $K>0$,

$$\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\left(P_1(\xi_k)+\beta_K Q_1(\xi_k)\right)\Big\|_\infty\Big] \underset{n\to+\infty}{\to} 0.$$

It follows that for all $K>0$,

$$\alpha_{n,K} \underset{n\to+\infty}{\to} \Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}}\mathbb{E}\Big[\theta_0(p)^{\frac{\eta}{2p}}\Big]\sum_{k=0}^{+\infty}\mathbb{E}\Big[\max_{1\leqslant i,j\leqslant d} M_{0,k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big]+\frac{1}{K^{\frac{\eta}{2}}}. \tag{3.73}$$

Letting $K\to+\infty$ in (3.73), we deduce

$$\lim_{K\to+\infty}\lim_{n\to+\infty}\alpha_{n,K}=0,$$

so that (3.69) holds for K and n sufficiently large.

Now, we prove (3.70). For all $K>0$, $n\geqslant 1$ and $a>0$, we have

$$\begin{aligned}
&\alpha_{n,K}\mathbb{E}\tilde{A}_{n,K}^{-a}\\
=\;&\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\left(P_1(\xi_k)+\beta_K Q_1(\xi_k)\right)\Big\|_\infty\|M_{0,n-1}\|^a\Big]+\frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\|M_{0,n-1}\|^a\\
&+\Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}}\sum_{k=0}^{+\infty}\mathbb{E}\Big[\|M_{0,n+k}\|^a\theta_{n+k}(p)^{\frac{\eta}{2p}}\max_{1\leqslant i,j\leqslant d}M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big].
\end{aligned}$$

By the independence of the environments ξ_n, $n\geqslant 0$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\|M_{0,n+k}\|^a\theta_{n+k}(p)^{\frac{\eta}{2p}}\max_{1\leqslant i,j\leqslant d}M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big]\\
&\qquad=\mathbb{E}\big[\|M_{n+k}\|^a\theta_{n+k}(p)^{\frac{\eta}{2p}}\big]\,\mathbb{E}\big[\|M_{0,n-1}\|^a\big]\times\\
&\qquad\quad\mathbb{E}\Big[\|M_{n,n+k-1}\|^a\max_{1\leqslant i,j\leqslant d}M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big]
\end{aligned}$$

Therefore, by the stationarity, we get that for all $K>0$, $n\geqslant 1$ and $a>0$,

$$\begin{aligned}
&\alpha_{n,K}\mathbb{E}\tilde{A}_{n,K}^{-a}\\
=\;&\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\left(P_1(\xi_k)+\beta_K Q_1(\xi_k)\right)\Big\|_\infty \|M_{0,n-1}\|^a\Big]+\frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\|M_{0,n-1}\|^a\\
&+\Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}}\mathbb{E}[\|M_0\|^a\theta_0(p)^{\frac{\eta}{2p}}]\times\\
&\mathbb{E}\|M_{0,n-1}\|^a\sum_{k=0}^{+\infty}\mathbb{E}\Big[\|M_{0,k-1}\|^a\max_{1\leqslant i,j\leqslant d}M_{0,k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big].
\end{aligned}\tag{3.74}$$

For any fixed $n\geqslant 1$, we have $\|M_{0,n-1}\|^a\to 1$ $\mathbb{P}$-a.s. as $a\to 0$, and $\|M_{0,n-1}\|^a$ is dominated by $\|M_{0,n-1}\|^{\eta(p-1)/4p}$ when $a\in(0,\frac{\eta(p-1)}{4p})$. We will apply the dominated convergence theorem in (3.74) as $a\to 0$. Notice that, by condition **H3** and (3.18) , for all $K>0$ and $n\geqslant 1$,

$$\begin{aligned}
\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\left(P_1(\xi_k)+\beta_K Q_1(\xi_k)\right)\Big\|_\infty \|M_{0,n-1}\|^{\frac{\eta(p-1)}{4p}}\Big]&\leqslant \mathbb{E}\|M_{0,n-1}\|^{\frac{\eta(p-1)}{4p}}\\
&\leqslant\Big(\mathbb{E}\|M_0\|^{\frac{\eta(p-1)}{4p}}\Big)^n<+\infty.
\end{aligned}$$

Using (3.53) in Lemma 3.12, we obtain

$$\sum_{k=0}^{+\infty}\mathbb{E}\Big[\|M_{0,k-1}\|^{\frac{\eta(p-1)}{4p}}\max_{1\leqslant i,j\leqslant d}M_{0,k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big]<+\infty.$$

By Hölder's inequality and condition **H3** we have

$$\mathbb{E}\Big[\|M_0\|^{\frac{\eta(p-1)}{4p}}\theta_0(p)^{\frac{\eta}{2p}}\Big]\leqslant\big(\mathbb{E}\|M_0\|^{\frac{\eta}{4}}\big)^{\frac{p}{p-1}}\big(\mathbb{E}\theta_0(p)^{\frac{\eta}{2}}\big)^{\frac{1}{p}}<+\infty.$$

Therefore, applying the Lebesgue dominated convergence theorem, by letting $a\to 0$ in (3.74), we obtain that for all $K>0$ and $n\geqslant 1$,

$$\begin{aligned}
\lim_{a\to 0}\left(\alpha_{n,K}\mathbb{E}\tilde{A}_{n,K}^{-a}\right)=\;&\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\left(P_1(\xi_k)+\beta_K Q_1(\xi_k)\right)\Big\|_\infty\Big]+\frac{1}{K^{\frac{\eta}{2}}}\\
&+\Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}}\mathbb{E}[\theta_0(p)^{\frac{\eta}{2p}}]\sum_{k=0}^{+\infty}\mathbb{E}\Big[\max_{1\leqslant i,j\leqslant d}M_{0,k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big].
\end{aligned}\tag{3.75}$$

Letting $n \to +\infty$ in (3.75), by (3.73) it holds that for all $K > 0$,

$$\lim_{n\to+\infty} \lim_{a\to 0} \left(\alpha_{n,K} \mathbb{E}\tilde{A}_{n,K}^{-a}\right) = \frac{1}{K^{\frac{\eta}{2}}} + \left(\frac{B_p}{K}\right)^{\frac{\eta}{2}} \mathbb{E}\left[\theta_0(p)^{\frac{\eta}{2p}}\right] \sum_{k=0}^{+\infty} \mathbb{E}\Big[\max_{1\leqslant i,j\leqslant d} M_{0,k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big].$$

Then, letting $K \to +\infty$, we conclude that

$$\lim_{K\to+\infty} \lim_{n\to+\infty} \lim_{a\to 0} \left(\alpha_{n,K} \mathbb{E}\tilde{A}_{n,K}^{-a}\right) = 0.$$

This implies (3.70) for $K > 0$ and $n \geqslant 1$ sufficiently large, and $a > 0$ small enough. Therefore (3.60) holds, and this concludes the proof of Theorem 3.10. □

4. Central limit theorem for $\log \|Z_n^i\|$

In this section, we prove Theorem 2.4, that is, a central limit theorem for the logarithm of the population size $\log \|Z_n^i\|$ for each $1 \leqslant i \leqslant d$. To this end, we will use the following central limit theorem for the norm cocycle $\log \|M_{0,n-1}^T x\|$ established by Hennion [20, Theorem 3].

Lemma 4.1. *Assume conditions (2.3) and **H4**. Then there exists $\sigma \geqslant 0$ such that for all $x \in \mathcal{S}$, as $n \to \infty$,*

$$\frac{\log \|M_{0,n-1}^T x\| - n\gamma}{\sqrt{n}} \to \mathcal{N}(0, \sigma^2) \quad \textit{in law.}$$

Proof of Theorem 2.4. By definition of the martingale (W_n^i), for any $n \geqslant 1$ and $1 \leqslant i \leqslant d$ we have, $\mathbb{P}$-a.s.,

$$\frac{\|Z_n^i\|}{\|M_{0,n-1}(i,\cdot)\|} \min_{1\leqslant j\leqslant d} U_{n,\infty}(j) \leqslant W_n^i \leqslant \frac{\|Z_n^i\|}{\|M_{0,n-1}(i,\cdot)\|} \frac{1}{\min_{1\leqslant j\leqslant d} U_{n,\infty}(j)}.$$

From this, we obtain the two following inequalities: for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\log \|Z_n^i\| \leqslant \log \|M_{0,n-1}(i,\cdot)\| + \log W_n^i - \min_{1\leqslant j\leqslant d} \log U_{n,\infty}(j), \tag{4.1}$$

$$\log \|Z_n^i\| \geqslant \log \|M_{0,n-1}(i,\cdot)\| + \log W_n^i + \min_{1\leqslant j\leqslant d} \log U_{n,\infty}(j). \tag{4.2}$$

We know that $(U_{n,\infty})$ is a stationary and ergodic sequence of positive random variables, so for all $1 \leqslant j \leqslant d$ it holds

$$\frac{\log U_{n,\infty}(j)}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0, \tag{4.3}$$

where $\xrightarrow{\mathbb{P}}$ denotes the convergence in probability with respect to $\mathbb{P}$. On the other hand, by [13, Theorem 2.6 and Corollary 2.8] we know that under conditions (2.5) and $\gamma > 0$ the limit W^i of the martingale (W_n^i) is non-degenerate, and (2.6) holds. This, combining with condition **H1**, implies that $W^i > 0$ $\mathbb{P}$-a.s. for each $1 \leqslant i \leqslant d$. Therefore, we obtain that for all $1 \leqslant i \leqslant d$,

$$\frac{\log W_n^i}{\sqrt{n}} \underset{n\to+\infty}{\to} 0 \quad \mathbb{P}\text{-a.s.} \tag{4.4}$$

Putting together the relations (4.1)-(4.4), we get that for all $1 \leqslant i \leqslant d$,

$$\left| \frac{\log \|Z_n^i\| - n\gamma}{\sqrt{n}} - \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sqrt{n}} \right| \xrightarrow{\mathbb{P}} 0. \tag{4.5}$$

Using Lemma 4.1 for $x = e_i$, we see that

$$\frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sqrt{n}} \to \mathcal{N}(0,\sigma^2) \quad \text{in law.}$$

Combining this with (4.5), we conclude the proof of Theorem 2.4. □

5. Berry-Esseen bound for $\log \|Z_n^i\|$

In this section, we prove Theorem 2.5 which gives a Berry-Esseen bound for the logarithm of the population size $\log \|Z_n^i\|$, for any $1 \leqslant i \leqslant d$.

First, we formulate the following lemma giving the convergence in L^1 of $\log W_n^i$ to $\log W^i$ with an exponential rate, for all $1 \leqslant i \leqslant d$.

Lemma 5.1. *Assume conditions **H1**, **H3** and $\gamma > 0$. Then there exist two constants $C > 0$ and $\delta \in (0,1)$ such that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,*

$$\mathbb{E}|\log W_n^i - \log W^i| \leqslant C\delta^n.$$

Proof. For any $n \geqslant 0$ and $1 \leqslant i \leqslant d$, set

$$R_n^i := \frac{W_{n+1}^i}{W_n^i} - 1.$$

Then, for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$ we have

$$\log W_{n+1}^i - \log W_n^i = \log(1 + R_n^i). \tag{5.1}$$

Let $K \in (0,1)$ be a constant. From (5.1) we get that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}\mathbb{E}|\log W_{n+1}^i - \log W_n^i| &= \mathbb{E}|\log(1+R_n^i)\mathbb{1}_{\{R_n^i \geqslant -K\}}| + \mathbb{E}|\log(1+R_n^i)\mathbb{1}_{\{R_n^i < -K\}}| \\ &=: I_1(n) + I_2(n). \end{aligned} \tag{5.2}$$

In the following, $C > 0$ will be a constant which may depend on K, p and η, and which may differ from line to line. Now we control the two terms $I_1(n)$ and $I_2(n)$.

Control of $I_1(n)$. Since $\frac{\eta p}{4} \leqslant 1$, the function $x \mapsto |x|^{-\frac{\eta p}{4}}\log(1+x)$ is bounded on $[-K, +\infty)$, so for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$I_1(n) = \mathbb{E}|\log(1+R_n^i)\mathbb{1}_{\{R_n^i \geqslant -K\}}| \leqslant C\mathbb{E}|R_n^i|^{\frac{\eta p}{4}}. \tag{5.3}$$

Control of $I_2(n)$. Applying Theorem 2.3, we get that there exists a constant $a > 0$ such that $\mathbb{E}(W^i)^{-a} < +\infty$ for any $1 \leqslant i \leqslant d$. Recall that by Lemma 3.11 we have the implication **H3** $\Rightarrow$ (2.5). So (2.5) holds, and using Lemma 3.1 with the convex function $x \mapsto x^{-a}$, this implies that for all $1 \leqslant i \leqslant d$,

$$\sup_{n \geqslant 0} \mathbb{E}(W_n^i)^{-a} = \mathbb{E}(W^i)^{-a} < +\infty. \tag{5.4}$$

Combining (5.1), (5.4) and the inequality $|\log(x)|^2 \leqslant C(x + x^{-a})$ for $x > 0$, we obtain that for all $1 \leqslant i \leqslant d$,

$$\begin{aligned}\sup_{n \geqslant 0} \left(\mathbb{E}|\log(1+R_n^i)|^2\right)^{\frac{1}{2}} &\leqslant 2 \sup_{n \geqslant 0} \left(\mathbb{E}|\log W_n^i|^2\right)^{\frac{1}{2}} \\ &\leqslant C \sup_{n \geqslant 0} \left(\mathbb{E}W_n^i + \mathbb{E}(W_n^i)^{-a}\right)^{\frac{1}{2}} \\ &\leqslant C\left(1 + \mathbb{E}(W^i)^{-a}\right)^{\frac{1}{2}} < +\infty. \end{aligned} \tag{5.5}$$

Applying Cauchy-Schwarz's inequality, (5.5) and Markov's inequality, we get that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} I_2(n) &= \mathbb{E}|\log(1+R_n^i)\mathbb{1}_{\{R_n^i < -K\}}| \\ &\leqslant \left(\mathbb{E}|\log(1+R_n^i)|^2\right)^{\frac{1}{2}} \left(\mathbb{E}\mathbb{1}_{\{R_n^i < -K\}}\right)^{\frac{1}{2}} \\ &\leqslant \left[\sup_{k \geqslant 0} \left(\mathbb{E}|\log(1+R_k^i)|^2\right)^{\frac{1}{2}}\right] \mathbb{P}(|R_n^i| > K)^{\frac{1}{2}} \\ &\leqslant C(\mathbb{E}|R_n^i|^{\frac{\eta p}{4}})^{\frac{1}{2}}. \end{aligned} \tag{5.6}$$

Together with (5.2), (5.3) and (5.6), this implies that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}\mathbb{E}|\log W_{n+1}^i - \log W_n^i| &\leqslant C\mathbb{E}|R_n^i|^{\frac{\eta p}{4}} + C(\mathbb{E}|R_n^i|^{\frac{\eta p}{4}})^{\frac{1}{2}} \\ &\leqslant C\mathbb{E}\big[(\mathbb{E}_\xi|R_n^i|^p)^{\frac{\eta}{4}}\big] + C\big\{\mathbb{E}\big[(\mathbb{E}_\xi|R_n^i|^p)^{\frac{\eta}{4}}\big]\big\}^{\frac{1}{2}}, \qquad (5.7)\end{aligned}$$

since $\frac{\eta}{4} < 1$. Notice that inequality (5.7) holds for any $p \in (1,2]$ satisfying condition **H3**. In the following, we take $p \in (1,2]$ sufficiently close to 1 such that p verifies **H3** and $p < 1 + a$.

Now we show that there exists a constant $\delta \in (0,1)$ such that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}\big[(\mathbb{E}_\xi|R_n^i|^p)^{\frac{\eta}{4}}\big] \leqslant C\delta^{2n}. \qquad (5.8)$$

By (3.5), for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}R_n^i &= \frac{1}{W_n^i}\sum_{r=1}^d \frac{U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)} \sum_{l=1}^{Z_n^i(r)} (W_{l,n}^r - 1) \\ &= \sum_{r=1}^d \frac{U_{n,\infty}(r)}{\langle Z_n^i, U_{n,\infty}\rangle} \sum_{l=1}^{Z_n^i(r)} (W_{l,n}^r - 1). \qquad (5.9)\end{aligned}$$

Recall that, given the environment ξ, for each $1 \leqslant r \leqslant d$ the random variables $W_{l,n}^r$ indexed by $l \geqslant 1$ are i.i.d., and independent of $\xi_0, \ldots, \xi_{n-1}$ and Z_n^i. Using (5.9), the convexity of the function $x \mapsto x^p$ on $\mathbb{R}_+$ (together with the fact that $\sum_{r=1}^d U_{n,\infty}(r) = 1$), and Lemma 3.4, for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s., we obtain

$$\begin{aligned}\mathbb{E}_\xi|R_n^i|^p &\leqslant \mathbb{E}_\xi\Big(\sum_{r=1}^d \frac{U_{n,\infty}(r)}{\langle Z_n^i, U_{n,\infty}\rangle}\Big|\sum_{l=1}^{Z_n^i(r)} (W_{l,n}^r - 1)\Big|\Big)^p \\ &\leqslant \sum_{r=1}^d U_{n,\infty}(r)\mathbb{E}_\xi\Big(\frac{1}{\langle Z_n^i, U_{n,\infty}\rangle^p}\Big|\sum_{l=1}^{Z_n^i(r)} (W_{l,n}^r - 1)\Big|^p\Big) \\ &\leqslant B_p^p \sum_{r=1}^d \mathbb{E}_\xi\Big[\frac{U_{n,\infty}(r)Z_n^i(r)}{\langle Z_n^i, U_{n,\infty}\rangle^p}\Big]\mathbb{E}_\xi|W_{1,n}^r - 1|^p \\ &\leqslant B_p^p\mathbb{E}_\xi\langle Z_n^i, U_{n,\infty}\rangle^{1-p} \max_{1\leqslant r\leqslant d} \mathbb{E}_\xi|W_{1,n}^r - 1|^p.\end{aligned}$$

Combining this with (3.7) and the convexity of $x \mapsto x^{1-p}$, we get that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\mathbb{E}_\xi|R_n^i|^p &\leqslant B_p^p\mathbb{E}_\xi(W_n^i)^{1-p} \cdot \langle M_{0,n-1}(i,\cdot), U_{n,\infty}\rangle^{1-p} \cdot \theta_n(p) \\ &\leqslant B_p^p\mathbb{E}_\xi(W_n^i)^{1-p}\,\theta_n(p) \max_{1\leqslant r,j\leqslant d} M_{0,n-1}(r,j)^{1-p}. \qquad (5.10)\end{aligned}$$

Therefore, by Cauchy-Schwarz's inequality and the independence between $\theta_n(p)$ and $M_{0,n-1}$, we deduce from (5.10) that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
&\mathbb{E}\big[(\mathbb{E}_\xi|R_n^i|^p)^{\frac{\eta}{4}}\big] \\
\leqslant\ & B_p^{\frac{\eta p}{4}} \Big(\mathbb{E}\big[\mathbb{E}_\xi(W_n^i)^{1-p}\big]^{\frac{\eta}{2}}\Big)^{\frac{1}{2}} \Big(\mathbb{E}\Big[\theta_n(p)^{\frac{\eta}{2}} \max_{1\leqslant r,j\leqslant d} M_{0,n-1}(r,j)^{\frac{\eta(1-p)}{2}}\Big]\Big)^{\frac{1}{2}} \\
\leqslant\ & B_p^{\frac{\eta p}{4}} \big(\mathbb{E}(W_n^i)^{1-p}\big)^{\frac{\eta}{4}} \Big(\mathbb{E}\Big[\theta_0(p)^{\frac{\eta}{2}}\Big]\mathbb{E}\Big[\max_{1\leqslant r,j\leqslant d} M_{0,n-1}(r,j)^{\frac{\eta(1-p)}{2}}\Big]\Big)^{\frac{1}{2}}, \quad (5.11)
\end{aligned}$$

since $\frac{\eta}{2} < 1$. Clearly condition **H3** implies that $\mathbb{E}\big[\theta_0(p)^{\frac{\eta}{2}}\big] < +\infty$. Since $0 < p-1 < a$, by Hölder's inequality and (5.4), for any $1 \leqslant i \leqslant d$ we have

$$\sup_{n\geqslant 0} \mathbb{E}(W_n^i)^{1-p} \leqslant \sup_{n\geqslant 0} \big(\mathbb{E}(W_n^i)^{-a}\big)^{\frac{p-1}{a}} = \big(\mathbb{E}(W^i)^{-a}\big)^{\frac{p-1}{a}} < +\infty. \quad (5.12)$$

Moreover, by (3.53) (with $a = 0$) it holds that for all $n \geqslant 0$,

$$\mathbb{E}\Big[\max_{1\leqslant r,j\leqslant d} M_{0,n-1}(r,j)^{\frac{\eta(1-p)}{2}}\Big] \leqslant C\delta^{4n}, \quad (5.13)$$

where $\delta > 0$ is a constant such that $\kappa(\frac{\eta(1-p)}{2}) < \delta^4 < 1$. Combining inequalities (5.11)-(5.13), we get (5.8).

Now, from (5.7) and (5.8) it follows that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}|\log W_{n+1}^i - \log W_n^i| \leqslant C\delta^n.$$

This implies that for all $n \geqslant 0$, $k \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
\mathbb{E}|\log W_{n+k}^i - \log W_n^i| &\leqslant \sum_{r=n}^{n+k-1} \mathbb{E}|\log W_{r+1}^i - \log W_r^i| \\
&\leqslant C \sum_{r=n}^{n+k-1} \delta^r \\
&\leqslant C\delta^n. \quad (5.14)
\end{aligned}$$

So $(\log W_n^i)_{n\geqslant 0}$ is a Cauchy sequence in L^1, hence it converges in L^1 to $\log W^i$, for all $1 \leqslant i \leqslant d$. By letting $k \to +\infty$ in (5.14), we obtain that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}|\log W^i - \log W_n^i| \leqslant C\delta^n.$$

This concludes the proof of Lemma 5.1. □

Now we formulate the Berry-Esseen bound for $\log\|M_{0,n-1}^T y\|$, for any $y \in \mathcal{S}$. This result was established in [34, Theorem 2.1]; it plays a crucial role in proving the Berry-Esseen theorem for $\log\|Z_n^i\|$.

Lemma 5.2. *Assume conditions **H3** and **H5**. Then there exists a constant $C > 0$ such that for all $n \geqslant 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$,*

$$\left|\mathbb{P}\left(\frac{\log\|M_{0,n-1}^T y\| - n\gamma}{\sigma\sqrt{n}} \leqslant x\right) - \Phi(x)\right| \leqslant \frac{C}{\sqrt{n}}.$$

The next lemma gives inequalities about the concentration of the joint law $\left(\log\|Z_n^i\|, \log\|M_{0,n-1}(i,\cdot)\|\right)$ for any $1 \leqslant i \leqslant d$. It reveals that $\log\|Z_n^i\|$ and $\log\|M_{0,n-1}(i,\cdot)\|$ behaves similarly with large probability.

Lemma 5.3. *Assume conditions **H1**, **H3**, **H5** and $\gamma > 0$. Then there exists a constant $C > 0$ such that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,*

$$\mathbb{P}\left(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x\right) \leqslant \frac{C}{\sqrt{n}}, \tag{5.15}$$

and

$$\mathbb{P}\left(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} \leqslant x\right) \leqslant \frac{C}{\sqrt{n}}. \tag{5.16}$$

Proof. We will only give a proof of (5.15), since the other inequality (5.16) can be proved by similar arguments. For $n \geqslant 1$, $0 \leqslant m \leqslant n$, $y \in \mathcal{S}$ and $1 \leqslant i \leqslant d$, set

$$S_{m,n}^y := \frac{\log\|M_{m,n-1}^T y\| - (n-m)\gamma}{\sigma\sqrt{n}} \quad \text{and} \quad L_{m,n}^i := \frac{\log W_m^i}{\sigma\sqrt{n}},$$

where by convention $M_{m,n-1}$ denotes the identity matrix when $m = n$. By (4.2) we have that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}\mathbb{P}&\left(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x\right)\\ &\leqslant \mathbb{P}\left(S_{0,n}^{e_i} + L_{n,n}^i + \min_{1\leqslant j\leqslant d}\frac{\log U_{n,\infty}(j)}{\sigma\sqrt{n}} \leqslant x, S_{0,n}^{e_i} > x\right).\end{aligned} \tag{5.17}$$

In the following we take $m := m(n) = \lfloor\sqrt{n}\rfloor$, where $\lfloor x\rfloor$ is the integer part of x; $C > 0$ will be a constant which may depend on p and η, and which

may differ from line to line. By Markov's inequality and Lemma 5.1, we get that there exists a constant $\delta \in (0,1)$ such that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}\mathbb{P}\Big(|L^i_{n,n} - L^i_{m,n}| > \frac{1}{\sqrt{n}}\Big) &\leqslant \sqrt{n}\,\mathbb{E}|L^i_{n,n} - L^i_{m,n}| \\ &= \frac{1}{\sigma}\,\mathbb{E}|\log W^i_n - \log W^i_m| \\ &\leqslant \frac{1}{\sigma}\,\mathbb{E}|\log W^i_n - \log W^i| + \frac{1}{\sigma}\,\mathbb{E}|\log W^i_m - \log W^i| \\ &\leqslant C(\delta^n + \delta^m).\end{aligned}$$

Since $\delta^n + \delta^m = o(\frac{1}{\sqrt{n}})$ as $n \to +\infty$, this implies that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\mathbb{P}\Big(|L^i_{n,n} - L^i_{m,n}| > \frac{1}{\sqrt{n}}\Big) \leqslant \frac{C}{\sqrt{n}}.$$

Combining this with (5.17), we obtain that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}&\mathbb{P}\Big(\frac{\log\|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x\Big) \\ \leqslant\ &\mathbb{P}\Big(S^{e_i}_{0,n} + L^i_{m,n} + \min_{1\leqslant j\leqslant d}\frac{\log U_{n,\infty}(j)}{\sigma\sqrt{n}} \leqslant x + \frac{1}{\sqrt{n}}, S^{e_i}_{0,n} > x\Big) \\ &+\mathbb{P}\Big(|L^i_{n,n} - L^i_{m,n}| > \frac{1}{\sqrt{n}}\Big) \\ \leqslant\ &\mathbb{P}\Big(S^{e_i}_{0,n} + L^i_{m,n} + \min_{1\leqslant j\leqslant d}\frac{\log U_{n,\infty}(j)}{\sigma\sqrt{n}} \leqslant x + \frac{1}{\sqrt{n}}, S^{e_i}_{0,n} > x\Big) + \frac{C}{\sqrt{n}}.\end{aligned} \tag{5.18}$$

Recall that for $y \in \mathcal{S}$ and $M \in \mathcal{G}_+$, we denote by $M\cdot y := \frac{My}{\|My\|}$ the projective action of M on $\mathcal{G}_+$. Then, for $y \in \mathcal{S}$ the process

$$X^y_0 = y, \quad \text{and} \quad X^y_n = M^T_{0,n-1}\cdot y, \quad n \geqslant 1,$$

is a Markov chain on $\mathcal{S}$. Notice that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$, we have the decomposition

$$\begin{aligned}S^{e_i}_{0,n} &= \frac{\log\|M^T_{m+1,n-1}(M^T_{0,m}e_i)\| - n\gamma}{\sigma\sqrt{n}} \\ &= \frac{\log\|M^T_{0,m}e_i\| + \log\|M^T_{m+1,n-1}(M^T_{0,m}\cdot e_i)\| - n\gamma}{\sigma\sqrt{n}} \\ &= \sqrt{\frac{m+1}{n}}S^{e_i}_{0,m+1} + S^{X^{e_i}_{m+1}}_{m+1,n}.\end{aligned} \tag{5.19}$$

Moreover, by (1.4) we have that for all $n \geqslant 0$,

$$1 \geqslant \min_{1\leqslant r\leqslant d} U_{n,\infty}(r) = \min_{1\leqslant r\leqslant d} \frac{\langle M_n(r,\cdot), U_{n+1,\infty}\rangle}{\|M_n U_{n+1,\infty}\|} \geqslant \min_{1\leqslant r,j\leqslant d} \frac{M_n(r,j)}{\|M_n\|}. \tag{5.20}$$

Therefore, putting together the relations (4.1) and (5.18)-(5.20), we obtain that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
&\mathbb{P}\Big(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x\Big) \\
\leqslant\ &\mathbb{P}\Big(\sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} + B_{m,n}^i \leqslant x + \frac{1}{\sqrt{n}}, \\
&\sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} > x\Big) + \frac{C}{\sqrt{n}},
\end{aligned} \tag{5.21}$$

with

$$\begin{aligned}
B_{m,n}^i := &\frac{1}{\sigma\sqrt{n}} \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|} + \frac{1}{\sigma\sqrt{n}} \min_{1\leqslant r,j\leqslant d} \log \frac{M_m(r,j)}{\|M_m\|} \\
&+ \min_{1\leqslant j\leqslant d} \frac{\log U_{n,\infty}(j)}{\sigma\sqrt{n}}.
\end{aligned}$$

Denote by $\nu_{m,n}^i$ the joint law of $\Big(X_{m+1}^{e_i}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i}, B_{m,n}^i\Big)$ on $\mathcal{S}\times\mathbb{R}\times\mathbb{R}$. For $y \in \mathcal{S}$ and $x \in \mathbb{R}$, set

$$G_{m,n}^y(x) = \mathbb{P}(S_{m,n}^y \leqslant x).$$

Since $S_{m+1,n}^y$ is independent of $X_{m+1}^{e_i}$, $S_{0,m+1}^{e_i}$ and $B_{m,n}^i$ for any $y \in \mathcal{S}$, we obtain from (5.21) that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
&\mathbb{P}\Big(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x\Big) \\
\leqslant\ &\int \mathbb{P}\Big(S_{m+1,n}^y + s + t \leqslant x + \frac{1}{\sqrt{n}}, S_{m+1,n}^y + s > x\Big)\nu_{m,n}^i(dy,ds,dt) + \frac{C}{\sqrt{n}} \\
=\ &\int \mathbb{1}_{\{t\leqslant\frac{1}{\sqrt{n}}\}}\Big[G_{m+1,n}^y\Big(x - s - t + \frac{1}{\sqrt{n}}\Big) - G_{m+1,n}^y(x-s)\Big]\nu_{m,n}^i(dy,ds,dt) \\
&+\frac{C}{\sqrt{n}}.
\end{aligned} \tag{5.22}$$

The random matrices M_n, $n \geqslant 0$, are i.i.d., so for $n \geqslant 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$ we have

$$G_{m+1,n}^y(x) = \mathbb{P}\Big(\frac{\log\|M_{0,n-m-1}^T y\| - (n-m-1)\gamma}{\sigma\sqrt{n}} \leqslant x\Big) = G_{0,n-m-1}^y(a_n x),$$

with $a_n = \sqrt{\frac{n}{n-m-1}}$. Notice that $a_n = (1-\frac{m+1}{n})^{-1/2} = 1+O(\frac{m}{n}) = 1+O(\frac{1}{\sqrt{n}})$ as $n \to +\infty$. Therefore, applying the Berry-Esseen bound of Lemma 5.2, we get that for all $n \geqslant 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$,

$$\begin{aligned} |G^y_{m+1,n}(x) - \Phi(a_n x)| &= |G^y_{0,n-m-1}(a_n x) - \Phi(a_n x)| \\ &\leqslant \frac{C}{\sqrt{n-m-1}} \\ &= \frac{Ca_n}{\sqrt{n}} \leqslant \frac{C}{\sqrt{n}}. \end{aligned} \tag{5.23}$$

Moreover, using the mean value theorem on the function $t \mapsto \Phi(tx)$ with $t \geqslant 1$, we obtain that for all $n \geqslant 1$ and $x \in \mathbb{R}$,

$$|\Phi(a_n x) - \Phi(x)| \leqslant |a_n - 1| \sup_{t \geqslant 1} |x\Phi'(tx)| \leqslant \frac{C}{\sqrt{n}} \sup_{z \in \mathbb{R}} |z\Phi'(z)|. \tag{5.24}$$

It is clear that $z \mapsto |z\Phi'(z)|$ is a bounded function on $\mathbb{R}$. Combining this with the inequalities (5.22)-(5.24), we deduce that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} &\mathbb{P}\Big(\frac{\log \|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x\Big) \\ \leqslant\ & \int \mathbb{1}_{\{t \leqslant \frac{1}{\sqrt{n}}\}} \Big|\Phi\Big(x - s - t + \frac{1}{\sqrt{n}}\Big) - \Phi(x-s)\Big| \nu^i_{m,n}(dy, ds, dt) + \frac{C}{\sqrt{n}}. \end{aligned} \tag{5.25}$$

By the mean value theorem and the fact that $\sup_{x \in \mathbb{R}} |\Phi'(x)| \leqslant 1$, for all $x, z \in \mathbb{R}$ we have

$$|\Phi(x+z) - \Phi(x)| \leqslant |z|.$$

This, together with (5.25), implies that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} &\mathbb{P}\Big(\frac{\log \|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x\Big) \\ \leqslant\ & \int \mathbb{1}_{\{t \leqslant \frac{1}{\sqrt{n}}\}} \Big|\frac{1}{\sqrt{n}} - t\Big| \nu^i_{m,n}(dy, ds, dt) + \frac{C}{\sqrt{n}} \\ \leqslant\ & \int |t| \nu^i_{m,n}(dy, ds, dt) + \frac{C}{\sqrt{n}} \\ =\ & \mathbb{E}|B^i_{m,n}| + \frac{C}{\sqrt{n}}. \end{aligned} \tag{5.26}$$

By definition of $B^i_{m,n}$, combining with (4.1), (4.2) and (5.20), we get that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\sigma\sqrt{n}|B^i_{m,n}| &\leqslant \left|\log \frac{\|Z^i_m\|}{\|M_{0,m-1}(i,\cdot)\|}\right| + \min_{1\leqslant r,j\leqslant d} |\log M_m(r,j)| + |\log\|M_m\||\\ &\quad + \min_{1\leqslant j\leqslant d} |\log U_{n,\infty}(j)|\\ &\leqslant |\log W^i_m| + |\log\|M_m\||\\ &\quad + \max_{1\leqslant r,j\leqslant d}\Big(|\log U_{m,\infty}(j)| + |\log M_m(r,j)| + |\log U_{n,\infty}(j)|\Big)\\ &\leqslant |\log W^i_m| + 2|\log\|M_m\||\| + |\log\|M_n\||\\ &\quad + \max_{1\leqslant r,j\leqslant d}\Big(2|\log M_m(r,j)| + |\log M_n(r,j)||\Big). \end{aligned} \tag{5.27}$$

By Lemma 5.1 we have $\sup_{n\geqslant 0}\mathbb{E}|\log W^i_n| < +\infty$ for any $1 \leqslant i \leqslant d$. Moreover, from condition **H3** and the inequality $|\log x| \leqslant C(x^\eta + x^{-\eta})$ for $x > 0$, it holds that $\mathbb{E}|\log\|M_0\|| < +\infty$ and $\mathbb{E}|\log M_0(r,j)| < +\infty$, $1 \leqslant r,j \leqslant d$. Therefore, taking expectation in (5.27), this implies that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}|B^i_{m,n}| \leqslant \frac{C}{\sqrt{n}}. \tag{5.28}$$

Hence, (5.15) follows from (5.26) and (5.28). This concludes the proof of Lemma 5.3. □

Proof of Theorem 2.5. For $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$, we write

$$\begin{aligned}&\mathbb{P}\Big(\frac{\log\|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x\Big)\\ =\ &\mathbb{P}\Big(\frac{\log\|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} \leqslant x\Big)\\ &+\mathbb{P}\Big(\frac{\log\|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x\Big)\\ =\ &\mathbb{P}\Big(\frac{\log\|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} \leqslant x\Big)\\ &-\mathbb{P}\Big(\frac{\log\|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} > x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} \leqslant x\Big)\\ &+\mathbb{P}\Big(\frac{\log\|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x\Big).\end{aligned}$$

By Lemma 5.3, we get that there exists a constant $C > 0$ such that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\left|\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x\right) - \mathbb{P}\left(\frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} \leqslant x\right)\right| \leqslant \frac{C}{\sqrt{n}}. \quad (5.29)$$

Combining (5.29) with Lemma 5.2, we obtain the Berry-Esseen bound for $\log ||Z_n^i||$, for any $1 \leqslant i \leqslant d$. This concludes the proof of Theorem 2.5. □

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