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CENTRAL LIMIT THEOREM AND PRECISE LARGE DEVIATIONS FOR BRANCHING RANDOM WALKS WITH PRODUCTS OF RANDOM MATRICES

THI THUY BUI, ION GRAMA, AND QUANSHENG LIU

ABSTRACT. We consider a branching random walk where particles give birth to children as a Galton-Watson process, which moves in \mathbb{R}^d with positions determined by the action of independent and identically distributed random matrices on the position of the parent. We are interested in asymptotic properties of the counting measure Z_n^x which counts the number of particles of generation n situated in a given region, when the process starts with one initial particle located at x . We establish a central limit theorem and a large deviation asymptotic expansion of Bahadur-Rao type for Z_n^x with suitable norming. An integral version of the large deviation result is also established. One of the key points in the proofs is the study of the fundamental martingale related to the spectral gap theory for products of random matrices. As a by-product, we obtain a sufficient and necessary condition for the non-degeneracy of the limit of the fundamental martingale, which extends the Kesten-Stigum type theorem of Biggins.

1. INTRODUCTION

Branching random walk is a system of particles, in which each particle gives birth to new particles of the next generation, whose children move on \mathbb{R} or \mathbb{R}^d according to some probability law. For early fundamental results on this model, see for example [2, 3, 8, 10, 11]. In recent years, this topic has attracted the attention of many authors, see for example, [1, 33, 23, 6, 19, 35, 42]. The model is closely related to various applied probability settings, such as Mandelbrot's cascades (cf. e.g. [36, 44, 5, 16, 49]), perpetuities (see e.g. [51, 15, 34]) and branching Brownian motion (cf. e.g. [37, 18, 7, 48]). For extensions to random environments in space and time, see e.g. [30, 21] and [12, 39, 46, 26, 27]. For other related works and many references, see e.g. the recent books [51, 15, 34]. In the classical branching random walk, a particle whose parent is at position y , moves to position $y + l$ with independent and

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identically distributed (i.i.d.) increments l 's for different particles, so that the moving is a simple random translation. The classical model does not cover the interesting cases occurring in many applications where the movements are determined by linear transformations such as rotations, dilations, shears, reflections, projections etc. In this paper, we deal with the case where the position of a particle is obtained by the action of a matrix A on the position of its parent, where the matrices A 's corresponding to different particles are i.i.d. In other words, the positions of particles are obtained by the action of products of random matrices on the position of one initial particle. This permits us to extend significantly the domains of applications of the theory of branching random walks. However, the study of this model becomes much more involved. One of the fundamental problems in the theory of branching random walks is to give a precise description of the configuration of the process at time n . We will consider this problem by giving precise asymptotics of the counting measure Z_n^x which counts the number of particles of generation n situated in a given region, when the process begins with one initial particle situated at x . More precisely, for the model that we introduce here, we will establish a central limit theorem and a large deviation asymptotic expansion of Bahadur-Rao type for Z_n^x with suitable norming.

To introduce the model we need some notation. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$. Set $\mathbb{U} := \cup_{n=0}^{\infty} (\mathbb{N}^*)^n$, where by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. A particle of generation n will be denoted by a sequence $u = u_1 \cdots u_n = (u_1, \dots, u_n) \in (\mathbb{N}^*)^n$ of length n ; the initial particle will be denoted by the null sequence \emptyset . Assume that on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a set of independent identically distributed random variables $(N_u)_{u \in \mathbb{U}}$ of the same law $p = \{p_k : k \in \mathbb{N}\}$, and a set of independent identically distributed $d \times d$ random matrices $(A_u)_{u \in \mathbb{U}}$ of the same law μ on the set of $d \times d$ matrices $M(d, \mathbb{R})$, where $d \geq 1$. The two families $(N_u)_{u \in \mathbb{U}}$ and $(A_u)_{u \in \mathbb{U}}$ are also assumed to be independent.

A branching random walk with products of random matrices is defined as follows. At time 0, there is one initial particle \emptyset of generation 0, with initial position $Y_\emptyset := x \in \mathbb{R}^d \setminus \{0\}$. At time 1, the initial particle \emptyset is replaced by $N = N_\emptyset$ new particles $i = \emptyset i$ of generation 1, located at $Y_i = A_i Y_\emptyset$, $1 \leq i \leq N$. In general, at time $n + 1$, each particle $u = u_1 \dots u_n$ of generation n , located at $Y_u \in \mathbb{R}^d$, is replaced by N_u new particles ui of generation $n + 1$, located at $Y_{ui} = A_{ui} Y_u$, $1 \leq i \leq N_u$. Namely, the position of the particle ui is obtained from the position Y_u of u by the action of the matrix A_{ui} , so that the position Y_u of a particle u in generation $n \geq 1$ is given by the action of products of random matrices on the initial position x :

$$Y_u = G_u x, \quad \text{where} \quad G_u = A_{u_1 \dots u_n} \dots A_{u_1}. \quad (1.1)$$

Denote by \mathbb{T} the genealogical tree associated to the elements $\{N_u : u \in \mathbb{U}\}$. It is defined by the following properties: 1) $\emptyset \in \mathbb{T}$; 2) when $u \in \mathbb{T}$, then for $i \in \mathbb{N}$, $ui \in \mathbb{T}$ if and only if $1 \leq i \leq N_u$; 3) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$. Let

$$\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$$

be the set of particles of generation n , where $|u|$ denotes the length of the sequence u and represents the number of generation to which u belongs; by convention $|\emptyset| = 0$.

The space \mathbb{R}^d is equipped with the Euclidean norm $|\cdot|$. The position $G_u x$ of the particle u is completely described by two components: its norm $|G_u x|$ and its projection on the unit sphere $\mathbb{S}^{d-1} := \{y \in \mathbb{R}^d, |y| = 1\}$ denoted by

$$X_u^x := \frac{G_u x}{|G_u x|}.$$

Accordingly, we consider the following counting measure of particles of generation n which describes the configuration of the branching random walk at time n : for measurable sets $B \subset \mathbb{S}^{d-1}$ and $C \subset \mathbb{R}$,

$$Z_n^x(B, C) = \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{X_u^x \in B, \log |G_u x| \in C\}}, \quad (1.2)$$

where for a set D , $\mathbb{1}_D$ denotes its indicator function. In particular when $B = \mathbb{S}^{d-1}$ the measure (1.2) reduces to

$$Z_n^x(\mathbb{S}^{d-1}, C) = \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{\log |G_u x| \in C\}}. \quad (1.3)$$

The measure $C \mapsto Z_n^x(\mathbb{S}^{d-1}, C)$ counts the number of particles of generation n with a given distance to the origin; the distributional function $Z_n^x(\mathbb{S}^{d-1}, (-\infty, y])$ counts the number of particles of generation n situated in the ball centered at 0 with radius e^y . This information may be important for example when we consider a model describing the infection by a certain transmittable disease (an infected individual at time n leads to a random number of infected individuals at time $n+1$ who move according to random linear transformations in \mathbb{R}^2 or \mathbb{R}^3), where we would like to know at time n how many infected individuals there are in a region with a given distance from the origin. The measure $(B, C) \mapsto Z_n^x(B, C)$ gives more information. For example, when $d = 2$ and $B = \{e^{i\theta} : \theta \in [\theta_1, \theta_2]\}$ is an arc, $Z_n^x(B, (-\infty, y])$ counts the number of particles of generation n situated in the region $\{re^{i\theta} : \theta \in [\theta_1, \theta_2], r \in [0, e^y]\}$.

When $d = 1$, $x = 1$ and $A_u \neq 0$ for all $u \in \mathbb{T}$, the measure defined by (1.3) is exactly the counting measure considered in the classical model of branching random walk on \mathbb{R} starting from the origin $0 \in \mathbb{R}$, where the position S_u of a particle $u = u_1 \cdots u_n$ is given by $S_u = L_{u_1} + \cdots + L_{u_1 \dots u_n}$, with $L_u = \log |A_u|$. So our model in the one dimensional case $d = 1$

reduces essentially to the classical (additive) branching random walk. For this reason, in the following we will focus on the case $d \geq 2$.

The present work aims to establish asymptotic properties of the counting measure Z_n^x when it is suitably normalized, with $|x| = 1$ and $d \geq 2$. We will consider two cases: when the matrices A_u are nonnegative, and when the matrices A_u are invertible. Our first result is a central limit theorem for the counting measure Z_n^x (see Theorem 2.1). It states that for any fixed $B \subset \mathbb{S}^{d-1}$ and some constants γ, σ defined explicitly, the counting measure $C \mapsto Z_n^x(B, n\gamma + \sigma\sqrt{n}C)$ on \mathbb{R} with a suitable norming converges to the standard normal law. This result extends the corresponding one of Asmussen and Kaplan [3, Theorem 1] on the one dimensional case, which was first conjectured by Harris [31]. Our second result is a precise large deviation result of Kesten-Stugum type (see Theorem 2.6), namely we give an exact asymptotic for $Z_n^x(B, [na, +\infty))$ for fixed $B \subset \mathbb{S}^{d-1}$ and a in a natural range of \mathbb{R} . An extension to an integral version of the large deviation result with target functions on the two components X_u^x and $\log |G_u x|$ is also established (see Theorem 2.4). These results extend the corresponding ones of Biggins [10] on the one dimensional case to the multi-dimensional case.

The starting point in the proofs of our results is a decomposition formula which permits to express the counting measure as the sum of conditionally independent random variables, using the branching property like in the one dimensional case for which we may refer to [3, 10]. However, there is much to do to arrive to the conclusions in the multidimensional case, due to the appearance of products of random matrices. In particular, for the proof of Theorem 2.1 about the central limit theorem and Theorem 2.4 about the precise large deviation with target functions, we use respectively the central limit theorem and the recent progress on the spectral gap theory and precise large deviations for products of random matrices. Another step forward in the proof of Theorem 2.4 concerns the extension of Biggins' martingale to the case of branching products of random matrices, for which we prove a criterion for the non-degeneracy of the limit of the fundamental martingale (see Theorem 2.2) which completes a result of Mentemeier [49] obtained in the context of the multivariate smoothing transform, and extends the Kesten-Stigum type theorem of Biggins [8] on the classical branching random walk.

The outline of the paper is as follows. The main results will be stated in Section 2. Theorem 2.1 on the asymptotic normality of the counting measure is proved in Section 3. The necessary and sufficient condition for the non-degeneracy of the limit of the fundamental martingale is given in Theorem 2.2 and Corollary 2.3, which are proved in Section 4. Theorem 2.4 on the precise asymptotic of large deviations, which implies Theorem 2.6, is established in Section 5.

2. MAIN RESULTS

In this section, we introduce necessary notation and assumptions, and present the main results.

2.1. Notation and assumptions on products of random matrices.

Note that in our model, along each branch we encounter a product of random matrices. In this section, we introduce some notation and the necessary assumptions on products of random matrices in order to formulate our main results. We shall consider two cases, the case when the matrices are non-negative and the case when the matrices are invertible.

Let $M(d, \mathbb{R})$ be equipped with the operator norm: for any $\mathbf{a} \in M(d, \mathbb{R})$ we set $\|\mathbf{a}\| = \sup_{x \in \mathbb{S}^{d-1}} |\mathbf{a}x|$, where $|\cdot|$ is a given vectorial norm on \mathbb{R}^d , and $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere in \mathbb{R}^d . Denote by $\Gamma_\mu := [\text{supp } \mu]$ the smallest closed semigroup of $M(d, \mathbb{R})$ generated by the support of μ . Let us recall some definitions in matrix theory. A matrix $\mathbf{a} \in M(d, \mathbb{R})$ is said to be *proximal* if it has an algebraic simple dominant eigenvalue. Denote by \mathcal{M}_+ the set of matrices with nonnegative entries. A nonnegative matrix $\mathbf{a} \in \mathcal{M}_+$ is said to be *allowable* if every row and every column has a positive entry.

We say that the measure μ is *arithmetic* if there is $t > 0$ together with $\theta \in [0, 2\pi)$ and a function $\vartheta : \mathbb{S}_+^{d-1} \rightarrow \mathbb{R}$ such that

$$\forall \mathbf{a} \in \Gamma, \forall x \in V(\Gamma) : \exp[it \log |\mathbf{a}x| - i\theta + i(\vartheta(\mathbf{a} \cdot x) - \vartheta(x))] = 1,$$

where $\mathbb{S}_+^{d-1} = \{x \geq 0 : |x| = 1\}$ is the intersection of the unit sphere with the positive quadrant. Notice when $d = 1$, we have $\mathbb{S}_+^{d-1} = \{1\}$, and the above arithmetic condition reduces to the following more usual form: $\log a$ is almost surely concentrated on an arithmetic progression $a_0 + a_1\mathbb{N}$ for some $a_0, a_1 > 0$.

We will need the following assumptions on the law μ .

C1.

(1) *For invertible matrices:*

- (a) *(Strong irreducibility) There is no finite union $\mathcal{W} = \bigcup_{i=1}^n W_i$ of proper subspaces $0 \neq W_i \subsetneq \mathbb{R}^d$ which is Γ_μ -invariant (in the sense that $a\mathcal{W} = \mathcal{W}$ for each $a \in \Gamma_\mu$)*
- (b) *(Proximality) Γ_μ contains at least one proximal matrix.*

(2) *For nonnegative matrices:*

- (a) *(Allowability) Every $\mathbf{a} \in \Gamma_\mu$ is allowable.*
- (b) *(Positivity) Γ_μ contains at least one matrix belonging to $\text{int}(\mathcal{M}_+)$.*
- (c) *(Non-arithmeticity) The measure μ is non-arithmetic.*

For both invertible matrices and nonnegative matrices, we will need a moment condition. For $\mathbf{a} \in M(d, \mathbb{R})$, set

$$\iota(\mathbf{a}) := \inf_{x \in \mathcal{S}} |\mathbf{a}x|, \quad \text{and} \quad \mathbf{a} \cdot x := \frac{\mathbf{a}x}{|\mathbf{a}x|} \quad \text{when } \mathbf{a}x \neq 0,$$

where $\mathbf{a} \cdot x$ is called the projective action of the matrix \mathbf{a} on the vector $x \in \mathbb{S}^{d-1}$. Then $\iota(\mathbf{a}) > 0$ for both invertible matrices and allowable nonnegative matrices. Set, for an invertible or nonnegative matrix \mathbf{a} ,

$$N(\mathbf{a}) = \max\{\|\mathbf{a}\|, \iota(\mathbf{a})^{-1}\}.$$

For invertible matrices we have $\iota(\mathbf{a}) = \|\mathbf{a}^{-1}\|^{-1}$ and $N(\mathbf{a}) = \max\{\|\mathbf{a}\|, \|\mathbf{a}^{-1}\|\}$.

C2. (*Moment condition*) *There exists $\eta_0 \in (0, 1)$ such that*

$$\mathbb{E}[N(A_1)^{\eta_0}] < \infty.$$

We will consider the action of invertible matrices on the projective space \mathbb{P}^{d-1} which is obtained from \mathbb{S}^{d-1} by identifying x and $-x$, and the action of nonnegative matrices on \mathbb{S}_+^{d-1} . For convenience, we identify $x \in \mathbb{P}^{d-1}$ with one of its representants in \mathbb{S}^{d-1} . To unify the exposition, we use the symbol \mathcal{S} to denote \mathbb{P}^{d-1} for invertible matrices, and \mathbb{S}_+^{d-1} for nonnegative matrices. The space \mathcal{S} will be equipped with the metric \mathbf{d} , which is the angular distance (see [13]) for invertible matrices, and the Hilbert cross-ratio metric (see [32]) for nonnegative matrices. Moreover, \mathcal{S} is a separable metric space equipped with Borel σ -field.

Let $G_n = A_n \dots A_2 A_1$ be the product of i.i.d. $d \times d$ real random matrices A_i , defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with common law μ . Let $x \in \mathcal{S}$ be a starting point. As mentioned in the introduction, the random walk $G_n x$ is completely determined by its log norm and its projection on \mathcal{S} , denoted respectively by

$$S_n^x := \log |G_n x|, \quad X_n^x := G_n \cdot x = \frac{G_n x}{|G_n x|}, \quad n \geq 0,$$

with the convention that $G_0 x = x$. Since $S_n^x = \log |A_n X_{n-1}^x| + S_{n-1}^x$ and $X_n^x = A_n \cdot X_{n-1}^x$, the sequence $(S_n^x, X_n^x)_{n \geq 0}$ is a Markov chain.

Denote by \mathbb{E} the expectation corresponding to \mathbb{P} . By the law of large numbers of Furstenberg [25], under conditions **C1** and **C2**, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} S_n^x = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[S_n^x] = \gamma \quad \mathbb{P}\text{-a.s.},$$

where $\gamma = \inf_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E} \log \|G_n\|$ is the upper Lyapunov exponent associated with the product sequence (G_n) . Le Page [40] and Henion [32] showed that

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} (S_n^x - n\gamma)^2 \quad (2.1)$$

exists and is independent of x for invertible matrices and nonnegative matrices, respectively. Moreover, there exists a unique μ -stationary probability measure ν on \mathcal{S} (see [29, 16]); the μ -stationarity of ν means that $\mu * \nu = \nu$, that is, for any continuous function φ on \mathcal{S} ,

$$(\mu * \nu)(\varphi) := \int_{\mathcal{S}} \int_{\Gamma_{\mu}} \varphi(\mathbf{a} \cdot x) \mu(d\mathbf{a}) \nu(dx) = \nu(\varphi).$$

where $\nu(\varphi) = \int_{\mathcal{S}} \varphi(x) \nu(dx)$. This notation for the integral will be used for any function and any measure. Set

$$I_{\mu} = \{s \geq 0 : \mathbb{E} \|A_1\|^s < \infty\}.$$

Note that I_{μ} is an interval of \mathbb{R}_+ . Let $s_{\infty} = \sup I_{\mu}$. Define the transfer operator on the set $\mathcal{C}(\mathcal{S})$ of continuous functions on \mathcal{S} as follows: for any $s \in (-\eta_0, s_{\infty})$, and $f \in \mathcal{C}(\mathcal{S})$,

$$P_s f(x) = \mathbb{E}[|A_1 x|^s f(A_1 \cdot x)], \quad \text{for all } x \in \mathcal{S}. \quad (2.2)$$

It is known that under conditions **C1**, and **C2**, there exists a small constant $0 < \eta_1 < \eta_0$ such that for any $s \in (-\eta_1, s_{\infty})$, there are a unique probability measure ν_s and a unique Hölder continuous normalized function r_s (under the normalizing condition $\nu_s(r_s) = 1$) on \mathcal{S} satisfying

$$\nu_s P_s = \kappa(s) \nu_s \quad \text{and} \quad P_s r_s = \kappa(s) r_s, \quad (2.3)$$

where $\kappa(s)$ is the unique dominant eigenvalue of P_s , $\nu_s P_s$ is the measure on \mathcal{S} such that $(\nu_s P_s)(f) = \nu_s(P_s f)$ for all $f \in \mathcal{C}(\mathcal{S})$. For $s \in [0, s_{\infty})$, the property (2.3) is proved in [16, Proposition 3.1] and [17, Corollary 7.3] for positive matrices, and in [29, Theorem 2.6 and Corollary 3.20] for invertible matrices. For both positive matrices and invertible matrices, the existence of $\eta_1 > 0$ and the property (2.3) for complex-valued s with $|s| < \eta_1$ are proved in [55, Proposition 3.1], where the following properties are also established: the functions $s \mapsto \kappa(s)$ and $s \mapsto r_s(x)$ are strictly positive and analytic in $(-\eta_1, s_{\infty})$, for $x \in \mathcal{S}$. Below we shall make use of normalized function r_s , i.e. $r_s(x) \leq 1$ for all $x \in \mathcal{S}$, $s \in (-\eta_1, s_{\infty})$. Moreover, it is proved (see [29, Lemma 3.5], [16, Lemma 6.2], [55, Propositions 3.12 and 3.14]) that, under conditions **C1** and **C2**, the function

$$\Lambda(s) = \log \kappa(s)$$

is finite and analytic on $(-\eta_1, s_{\infty})$, and satisfies

$$\Lambda(0) = 0, \quad \Lambda'(0) = \gamma, \quad \Lambda''(0) = \sigma^2 > 0, \quad \text{and} \quad \Lambda''(s) > 0 \quad \forall s \in (-\eta_1, s_{\infty}).$$

2.2. Main results. Note that the population size at time n is $Z_n = Z_n^x(S, \mathbb{R})$, which does not depend on the starting point x and forms a Galton-Watson process with $Z_0 = 1$ and $Z_1 = N$. Denote by $m = \mathbb{E}N$ the expected value of the offspring distribution. Throughout the paper, we shall assume that $1 < m < \infty$, which ensures that the branching process (Z_n) is supercritical,

so that $Z_n \rightarrow \infty$ as $n \rightarrow \infty$ with positive probability. It is well known that $\mathbb{E}Z_n = m^n$. Let

$$W_n = \frac{Z_n}{m^n} \quad \text{for } n \geq 0, \quad \text{and} \quad W = \lim_{n \rightarrow \infty} W_n.$$

The sequence $\{W_n\}$ is the fundamental martingale for the Galton-Watson process (Z_n) ; the limit above exists almost surely by the martingale convergence theorem. The famous Kesten-Stigum theorem states that W is non-degenerate if and only if $\mathbb{E}N \log_+ N < \infty$ (see [4]), where and through this paper $\log_+ x = \max\{0, \log x\}$ denotes the positive part of $\log x$. We will need the following slightly stronger condition.

C3. *There exists a constant $\eta > 1$ such that*

$$\mathbb{E}N \log_+^{\eta+1} N < \infty. \quad (2.4)$$

We start with a central limit theorem for the normalized counting measure (1.2). For $t \in \mathbb{R}$, let

$$Z_n^x(B, t) = Z_n^x\left(B, (-\infty, n\gamma + t\sigma\sqrt{n}]\right) = \sum_{u \in \mathbb{T}_n} \mathbf{1}_{\{X_u^x \in B, \frac{\log |G_{ux}| - n\gamma}{\sigma\sqrt{n}} \leq t\}}.$$

Theorem 2.1. *Assume that the law μ of the random matrices satisfies conditions C1 and C2. Assume also that the offspring distribution satisfies condition C3. Then, for any $x \in \mathcal{S}$, any measurable set $B \subseteq \mathcal{S}$ with $\nu(\partial B) = 0$ and any $t \in \mathbb{R}$, we have, as $n \rightarrow \infty$,*

$$\frac{Z_n^x(B, t)}{m^n} \rightarrow \nu(B)\Phi(t)W \quad \mathbb{P}\text{-a.s.}, \quad (2.5)$$

where $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-x^2/2} dx$ is the distribution function of the standard normal law.

For the one dimensional case (where $d = 1$), the result is due to Asmussen and Kaplan [3, Theorem 1], which was first conjectured by Harris [31, p.75] but with convergence in probability instead of the a.s. convergence in (2.5). Harris' conjecture was first solved by Stam [52], then improved by Asmussen and Kaplan [2, 3] to L^2 -convergence and a.s. convergence. More general cases have been considered by Klebaner [38] and Biggins [11], who studied respectively the varying environment case and the general branching random walk where the displacements of particles with the same parent may have different laws. The random environment case has been considered by Gao, Liu and Wang [28]. The exact convergence rate in (2.5) has been considered by Chen [18] and Gao and Liu [26]. Asymptotic expansions have been obtained in [27]. Theorem 2.5 opens ways for extending some results in [2, 18, 28, 26, 27] to the multi-dimensional case where the moving of particles is determined by products of random matrices.

Our second main result is on the large deviation for the counting measure Z_n^x . To study the large deviation of the measure Z_n^x , a natural way would be to consider its Laplace transform defined by, for $(s_1, s_2) \in \mathbb{R}^2$,

$$\tilde{Z}_n^x(s_1, s_2) = \int_{\mathbb{R}^2} e^{s_1 y_1 + s_2 y_2} Z_n^x(dy_1, dy_2) = \sum_{u \in \mathbb{T}_n} e^{s_1 X_u^x + s_2 S_u^x}.$$

In the one dimensional case, when $x = 1$ and $A_n > 0$, we have $X_u^x = 1$, so that $\tilde{Z}_n^x(s_1, s_2)/\mathbb{E}\tilde{Z}_n^x(s_1, s_2)$ reduces to Biggins' fundamental martingale of the branching random walk:

$$\frac{\sum_{u \in \mathbb{T}_n} e^{s_2 S_u^x}}{\mathbb{E} [\sum_{u \in \mathbb{T}_n} e^{s_2 S_u^x}]}, \quad n \geq 0, \quad (2.6)$$

which has been well studied (see [8], for example), and which plays an essential role in many problems. However, in the multidimensional case, in general the sequence (2.6) is no longer a martingale, nor the sequence

$$\frac{\tilde{Z}_n^x(s_1, s_2)}{\mathbb{E}\tilde{Z}_n^x(s_1, s_2)} = \frac{\sum_{u \in \mathbb{T}_n} e^{s S_u^x + s' X_u^x}}{\mathbb{E} [\sum_{u \in \mathbb{T}_n} e^{s S_u^x + s' X_u^x}]}, \quad n \geq 0, \quad (2.7)$$

for $(s_1, s_2) \in \mathbb{R}^2$. So an important difficulty arises when we mimic Cramér's change of measure for random walks by use of the Laplace transform of Z_n^x .

However, there is still a natural martingale in the present setting. By the spectral gap property (2.3), it is easy to verify that (see Section 4 for more details), for $s \in (-\eta_1, s_\infty)$ and $x \in \mathcal{S}$, the sequence

$$W_{s,n}^x := \frac{\sum_{u \in \mathbb{T}_n} e^{s S_u^x} r_s(X_u^x)}{m^n \kappa(s)^n r_s(x)}, \quad n \geq 0, \quad (2.8)$$

constitutes a positive martingale with respect to the natural filtration

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \text{ and } \mathcal{F}_n = \sigma(N_u, A_{ui} : i \geq 1, |u| < n) \text{ for } n \geq 1,$$

as observed by Mentemeier [49] in the study of the multivariate smoothing transform. By the martingale convergence theorem, the limit

$$W_s^x := \lim_{n \rightarrow \infty} W_{s,n}^x \text{ exists in } \mathbb{R} \text{ } \mathbb{P}\text{-a.s.}$$

It turns out that the martingale $(W_{s,n}^x)$ in the multidimensional case plays the same role as Biggins' fundamental martingale for one dimensional case, for large deviations.

Just as in the case of Biggins' martingale, it is crucial to know when the limit variable W_s^x of the fundamental martingale $W_{s,n}^x$ is non-degenerate. When the matrices A_u are nonnegative and $s > 0$, Mentemeier [49, Proposition 4.4] gave a sufficient condition for W_s^x to be non-degenerate. In the following we complete his result by considering the necessary and sufficient conditions, and by treating meanwhile the case $s < 0$ and the case of invertible matrices.

We first establish the following theorem, whose proof is deferred to Section 4. To state the result, we need some notation. For $s \in (-\eta_1, s_\infty)$, set $\Lambda^*(q_s) = sq_s - \Lambda(s)$ with $q_s = \Lambda'(s)$. Since $\Lambda''(s) > 0$ and $\frac{\partial}{\partial s}\Lambda^*(q_s) = s\Lambda''(s)$, $\Lambda^*(q_s)$ attains its minimum at $s = 0$, so that $\Lambda^*(q_s) \geq \Lambda^*(q_0) = -\Lambda(0) = 0$ for all $s \in (-\eta_1, s_\infty)$.

Theorem 2.2. *Assume conditions **C1**, **C2**. If*

$$\Lambda^*(q_s) - \log m < 0 \quad (2.9)$$

and

$$\mathbb{E}[\max_{x \in \mathcal{S}} W_{s,1}^x \log_+ \max_{x \in \mathcal{S}} W_{s,1}^x] < \infty, \quad (2.10)$$

then for all $x \in \mathcal{S}$,

$$\mathbb{E}[W_s^x] = 1. \quad (2.11)$$

Conversely, if

$$\mathbb{E}[W_s^x] > 0 \quad (2.12)$$

for some $x \in \mathcal{S}$, then (2.9) holds, and

$$\mathbb{E}[\min_{x \in \mathcal{S}} W_{s,1}^x \log_+ \min_{x \in \mathcal{S}} W_{s,1}^x] < \infty. \quad (2.13)$$

Corollary 2.3. *Suppose the conditions **C1**, **C2**.*

(1) *Assume (2.9) together with*

$$\mathbb{E}N \log_+ N < \infty \quad \text{and} \quad \mathbb{E}\|A_1\|^s \log_+ \|A_1\| < \infty. \quad (2.14)$$

Then (2.11) holds for all $x \in \mathcal{S}$.

(2) *Assume that the random matrix $A_1 = (A_1(i, j))$ satisfies the Furstenberg-Kesten condition: there exists a constant $C > 1$ such that*

$$\frac{\max_{1 \leq i, j \leq d} |A_1(i, j)|}{\min_{1 \leq i, j \leq d} |A_1(i, j)|} \leq C \quad \text{a.s.} \quad (2.15)$$

Then the three conditions (2.10), (2.13) and (2.14) are equivalent, and (2.12) holds for all $x \in \mathcal{S}$ if and only if (2.9) and (2.14) hold. Moreover, if (2.12) holds for some $x \in \mathcal{S}$, then (2.11) holds for all $x \in \mathcal{S}$.

Notice that by Sheffé's theorem, for each $x \in \mathcal{S}$, if (2.12) holds, then $W_{s,n}^x \rightarrow W_s^x$ in L^1 . So the martingale $(W_{s,n}^x)$ converges in L^1 for some $x \in \mathcal{S}$ if and only if (2.9) and (2.14) hold; moreover, when the martingale converges in L^1 for some $x \in \mathcal{S}$, then it converges in L^1 for all $x \in \mathcal{S}$.

When the matrices A_u are nonnegative and $s > 0$, Part (1) has been established by Mentemeier [49, Proposition 4.4]. When $d = 1$, Part (2) is essentially the well-known Kesten-Stigum type theorem for the classical branching random walk on the real line, due to Biggins [8]; see also [36] for

Mandelbrot's cascades and [47, 43] for versions which are slightly different to the initial result of Biggins [8].

Now we consider the precise large deviations for Z_n^x with target functions f and g on the components $X_u^x = G_u \cdot x$ and $S_u^x = \log |G_u x|$. More precisely, we shall study the asymptotic of the large deviations of the following integral:

$$\int_{\mathcal{S} \times \mathbb{R}} f(y)g(z - nq_s)Z_n^x(dy, dz) = \sum_{u \in \mathbb{T}_n} f(X_u^x)g(S_u^x - nq_s). \quad (2.16)$$

Our result will be stated under the very general assumption that $e^{-sz}g(z)$, $z \in \mathbb{R}$ is directly Riemann integrable, see Feller [24], Chapter XI.

Theorem 2.4. *Assume conditions C1 and C2, and let $s \in (-\eta_1, s_\infty)$ be fixed such that $\Lambda^*(q_s) - \log m < 0$ and that*

$$\mathbb{E} \left[\max_{x \in \mathcal{S}} W_1^x(s) \log_+^{\delta+1} \max_{x \in \mathcal{S}} W_1^x(s) \right] < \infty \quad \text{for some } \delta > 3/2. \quad (2.17)$$

Then for any continuous function f on \mathcal{S} and any measurable function g on \mathbb{R} such that $z \mapsto e^{-sz}g(z)$ is directly Riemann integrable, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt{2\pi n} \sigma_s e^{n\Lambda^*(q_s)}}{m^n} \int_{\mathcal{S} \times \mathbb{R}} f(y)g(z - nq_s)Z_n^x(dy, dz) \\ = W_s^x r_s(x) \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} e^{-sz}g(z)dz, \quad \mathbb{P}\text{-a.s.}, \end{aligned} \quad (2.18)$$

where $\pi_s \left(\frac{f}{r_s} \right) = \frac{\nu_s(f)}{\nu_s(r_s)}$, and $\sigma_s^2 = \Lambda''(s)$.

When $s = 0$ this result reduces to the following local limit theorem for the counting measure Z_n^x :

Corollary 2.5. *Assume conditions C1 and C2. Assume also that (2.17) holds with $s = 0$. Then*

$$\lim_{n \rightarrow \infty} \frac{\sigma \sqrt{2\pi n}}{m^n} \int_{\mathcal{S} \times \mathbb{R}} f(y)g(z - n\gamma)Z_n^x(dy, dz) = W\nu(f) \int_{\mathbb{R}} g(z)dz.$$

When $f = 1$ and $g = \mathbb{1}_{[a,b]}$ with $-\infty < a < b < \infty$, it gives the precise asymptotic of $Z_n^x(\mathcal{S}, n\gamma + [a, b])$ as $n \rightarrow \infty$.

The following theorem describes the asymptotic size of the number of particles in n -th generation situated in the regions $(B, [e^{nq_s}, +\infty))$ for $s > 0$, and $(B, (0, e^{nq_s}))$ for $s < 0$, where $B \subseteq \mathcal{S}$.

Theorem 2.6. *Assume the conditions of Theorem 2.4. Then, for any $x \in \mathcal{S}$, any measurable set $B \subseteq \mathcal{S}$ with $\nu(\partial B) = 0$, we have, \mathbb{P} -a.s., for $s > 0$,*

$$\lim_{n \rightarrow \infty} \frac{\sqrt{2\pi n} \sigma_s e^{n\Lambda^*(q_s)} Z_n^x(B, [nq_s, +\infty))}{m^n} = \frac{1}{s} W_s^x r_s(x) \int_B \frac{1}{r_s(y)} \pi_s(dy),$$

and for $s < 0$,

$$\lim_{n \rightarrow \infty} \sqrt{2\pi n} \sigma_s e^{n\Lambda^*(q_s)} \frac{Z_n^x(B, (-\infty, nq_s])}{m^n} = \frac{1}{s} W_s^x r_s(x) \int_B \frac{1}{r_s(y)} \pi_s(dy).$$

This theorem is obtained from Theorem 2.4 by taking $g = \mathbb{1}_{[0, +\infty)}$ when $s > 0$, and $g = \mathbb{1}_{(-\infty, 0]}$ when $s < 0$, and by using a smooth approximation of indicator function (see [14, Lemma 4.1]).

In the one dimensional case (where $d = 1$), Theorems 2.4 and 2.6 reduce to the Bahadur-Rao type results of Biggins [10]. The large deviation principle was established earlier by Biggins in [9].

3. PROOF OF THEOREM 2.1

This section is devoted to prove Theorem 2.1, the central limit theorem on the counting measure Z_n^x .

3.1. Basic decomposition. For all $u \in \mathbb{U}$, let $\mathbb{T}(u)$ be the shifted tree of \mathbb{T} at u associated to the elements $\{N_{uv}\}$. It is defined by the following properties: 1) $\emptyset \in \mathbb{T}(u)$, 2) $vi \in T(u)$ implies $v \in \mathbb{T}(u)$ and 3) if $v \in T(u)$, then $vi \in \mathbb{T}(u)$ if and only if $1 \leq i \leq N_{uv}$. Define $T_n(u) = \{v \in \mathbb{T}(u) : |v| = n\}$. Then $\mathbb{T} = \mathbb{T}(\emptyset)$ and $\mathbb{T}_n = \mathbb{T}_n(\emptyset)$.

It follows from the additive property of the branching process that, for $k \leq n$, any measurable set $B \subseteq \mathcal{S}$ and any Borel set C in \mathbb{R} ,

$$\begin{aligned} Z_n^x(B, C) &= \sum_{u \in \mathbb{T}_k} \sum_{v \in \mathbb{T}_{n-k}(u)} \mathbb{1}_{\{X_{uv}^x \in B, S_{uv}^x \in C\}} \\ &= \sum_{u \in \mathbb{T}_k} Z_{n-k}^{X_u^x}(B, C - S_u^x), \end{aligned} \quad (3.1)$$

where

$$Z_{n-k}^{X_u^x}(B, C - S_u^x) = \sum_{v \in \mathbb{T}_{n-k}(u)} \mathbb{1}_{\{X_v^{X_u^x} \in B, S_v^{X_u^x} \in C - S_u^x\}}$$

represents the number of descendants of u at time n in the region characterized by $(B, C - S_u^x)$, and $C - S_u^x = \{y - S_u^x : y \in C\}$. In this section, we consider $C = (-\infty, n\gamma + t\sigma\sqrt{n}]$, $t \in \mathbb{R}$. For simplicity, we will use the

following notation:

$$\begin{aligned}
Z_n^x(B, t) &= \sum_{u \in \mathbb{T}_n} \mathbb{1}_{\{X_u^x \in B, \frac{\log |G_{ux}| - n\gamma}{\sigma\sqrt{n}} \leq t\}}, \\
Z_{n-k}^{X_u^x}(B, t) &= \sum_{v \in \mathbb{T}_{n-k}(u)} \mathbb{1}_{\{X_v^{X_u^x} \in B, \frac{S_v^{X_u^x} - (n-k)\gamma}{\sigma\sqrt{n-k}} \leq t\}}, \\
Z_{n-k}^{X_u^x}(B, \mathbb{R}) &= \sum_{v \in \mathbb{T}_{n-k}(u)} \mathbb{1}_{\{X_v^{X_u^x} \in B\}}, \\
W_{n-k}^{X_u^x}(B, t) &= \frac{Z_{n-k}^{X_u^x}(B, t)}{m^{n-k}}, \\
W_{n-k}^{X_u^x} &= W_{n-k}^{X_u^x}(\mathcal{S}, \mathbb{R}) = \frac{Z_{n-k}^{X_u^x}(\mathcal{S}, \mathbb{R})}{m^{n-k}}.
\end{aligned}$$

Notice that $Z_{n-k}^{X_u^x}(\mathcal{S}, \mathbb{R})$ is the population size of generation $n - k$ of the Galton-Watson process beginning from the particle u (whose genealogical tree is the shifted tree of \mathbb{T} at u). So $Z_{n-k}^{X_u^x}(\mathcal{S}, \mathbb{R})$ and $W_{n-k}^{X_u^x}$ do not depend on the position of u .

For conditional probabilities and expectations, we write

$$\mathbb{P}_n(\cdot) = \mathbb{P}(\cdot | \mathcal{F}_n), \quad \mathbb{E}_n(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_n).$$

We obtain the following decomposition from (3.1), which will play a key role in our approach:

$$\begin{aligned}
& \frac{1}{m^n} Z_n^x(B, t) - \nu(B)\Phi(t)W \\
&= \frac{1}{m^n} \sum_{u \in \mathbb{T}_{k_n}} Z_{n-k_n}^{X_u^x} \left(B, \frac{t\sigma\sqrt{n} - S_u^x + k\gamma}{\sigma\sqrt{n-k}} \right) - \nu(B)\Phi(t)W \\
&= A_n + B_n + C_n,
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
A_n &= \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left\{ W_{n-k_n}^{X_u^x} \left(B, \frac{t\sigma\sqrt{n} - S_u^x + k\gamma}{\sigma\sqrt{n-k}} \right) \right. \\
&\quad \left. - \mathbb{E}_{k_n} \left[W_{n-k_n}^{X_u^x} \left(B, \frac{t\sigma\sqrt{n} - S_u^x + k\gamma}{\sigma\sqrt{n-k}} \right) \right] \right\}, \\
B_n &= \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left\{ \mathbb{E}_{k_n} \left[W_{n-k_n}^{X_u^x} \left(B, \frac{t\sigma\sqrt{n} - S_u^x + k\gamma}{\sigma\sqrt{n-k}} \right) \right] - \nu(B)\Phi(t) \right\}, \\
C_n &= (W_{k_n} - W) \nu(B)\Phi(t).
\end{aligned}$$

The idea is now to choose suitable k_n with $k_n \rightarrow \infty$ such that $A_n, B_n, C_n \rightarrow 0$ a.s.

3.2. Proof of Theorem 2.1. We choose β with $\frac{1}{\eta} < \beta < 1$ and $\alpha > \frac{2}{\beta-1-1}$. For each n , let $j = j(n) \in \mathbb{N}$ be such that $j^{\alpha/\beta} \leq n < (j+1)^{\alpha/\beta}$; set $k_n = a_j = \lfloor j^\alpha \rfloor$, the integer part of j^α . We will prove that with this choice of (k_n) , we have $A_n, B_n, C_n \rightarrow 0$ a.s. By the decomposition (3.2), this will imply Theorem 2.1.

By the convergence of the martingale W_n to W , we have clearly $C_n \rightarrow 0$, \mathbb{P} -a.s. It remains to show the following two lemmas.

Lemma 3.1. *Under the hypothesis of Theorem 2.1, we have*

$$A_n \xrightarrow{n \rightarrow \infty} 0 \quad \mathbb{P}\text{-a.s.} \quad (3.3)$$

Lemma 3.2. *Under the hypothesis of Theorem 2.1, then*

$$B_n \xrightarrow{n \rightarrow \infty} 0 \quad \mathbb{P}\text{-a.s.} \quad (3.4)$$

For the proof of Lemma 3.1, We shall use the following result on the weighted moments of $W^* := \sup_n \{W_n\}$, which is proved in Liang and Liu [41, Theorem 1.2].

Lemma 3.3. *Under the hypothesis of Theorem 2.1, then*

$$\mathbb{E}(W^* + 1) \log^\eta(W^* + 1) < \infty. \quad (3.5)$$

Proof of Lemma 3.1. By definition, $A_n = \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} Y_{n,u}$, where

$$Y_{n,u} = W_{n-k_n}^{X_u^x} \left(B, \frac{t\sigma\sqrt{n} - S_u^x + k\gamma}{\sigma\sqrt{n-k}} \right) - \mathbb{E}_{k_n} \left[W_{n-k_n}^{X_u^x} \left(B, \frac{t\sigma\sqrt{n} - S_u^x + k\gamma}{\sigma\sqrt{n-k}} \right) \right].$$

We see that for any $u \in \mathbb{T}_{k_n}$,

$$\begin{aligned} |Y_{n,u}| &\leq W_{n-k_n}^{X_u^x}(\mathcal{S}, \mathbb{R}) + \mathbb{E}_{k_n} W_{n-k_n}^{X_u^x}(\mathcal{S}, \mathbb{R}) \\ &= W_{n-k_n}^{X_u^x} + 1, \end{aligned} \quad (3.6)$$

where the last equality holds because $W_{n-k_n}^{X_u^x} = W_{n-k_n}^{X_u^x}(\mathcal{S}, \mathbb{R})$ represents the fundamental martingale of the Galton-Watson process beginning with the particle $u \in \mathbb{T}_{k_n}$. Let

$$\bar{Y}_{n,u} = Y_{n,u} \mathbf{1}_{\{|Y_{n,u}| < m^{k_n}\}} \quad \text{and} \quad \bar{A}_n = \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \bar{Y}_{n,u}.$$

We will use the decomposition

$$A_n = (A_n - \bar{A}_n) + (\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n]) + \mathbb{E}_{k_n}[\bar{A}_n],$$

and prove that each of the three terms on the right side of this identity tends to zero as $n \rightarrow \infty$. We divide the proof into 3 steps.

Step 1. We first prove that $A_n - \bar{A}_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$, as a consequence of

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n \neq \bar{A}_n) < \infty. \quad (3.7)$$

In fact, by the Lemma of Borel-Cantelli, (3.7) implies that a.s. $A_n - \bar{A}_n = 0$ when n is large enough. By the definition of $\bar{Y}_{n,u}$ and the inequality (3.6), we have:

$$\begin{aligned} \mathbb{P}_{k_n}(A_n \neq \bar{A}_n) &\leq \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n}(Y_{n,u} \neq \bar{Y}_{n,u}) = \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n}(|Y_{n,u}| \geq m^{k_n}) \\ &\leq \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n}(W_{n-k_n}^{X_u^x} + 1 \geq m^{k_n}). \end{aligned}$$

Since the law of $W_{n-k_n}^{X_u^x}$ conditioned upon \mathcal{F}_{k_n} is that of W_{n-k_n} , it follows that

$$\begin{aligned} \mathbb{P}_{k_n}(A_n \neq \bar{A}_n) &\leq Z_{k_n} \mathbb{P}(W_{n-k_n} + 1 \geq m^{k_n}) \leq W_{k_n} m^{k_n} \mathbb{P}(W^* + 1 \geq m^{k_n}) \\ &\leq W_{k_n} \mathbb{E}[(W^* + 1) \mathbf{1}_{\{W^* + 1 \geq m^{k_n}\}}] \\ &\leq \frac{W_{k_n}}{(\log m^{k_n})^\eta} \mathbb{E}[(W^* + 1) \log^\eta(W^* + 1)]. \end{aligned}$$

Taking expectation and denoting $C = \mathbb{E}[(W^* + 1) \log^\eta(W^* + 1)]$ (which is finite by Lemma 3.3), we get $\mathbb{P}(A_n \neq \bar{A}_n) \leq \frac{C}{(\log m)^\eta k_n^\eta}$. Since $k_n^\eta \sim j^{\alpha\eta} \sim n^{\beta\eta}$ and $\beta\eta > 1$, (3.7) is proved.

Step 2. We next prove that $\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n] \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$, as a consequence of

$$\sum_{n=1}^{\infty} \mathbb{P}(|\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon) < \infty \quad \forall \varepsilon > 0 \quad (3.8)$$

(by the Lemma of Borel-Cantelli). By Chebyshev's inequality we have

$$\mathbb{P}_{k_n}(|\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon) \leq \frac{1}{\varepsilon^2} \mathbb{E}_{k_n}(\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n])^2. \quad (3.9)$$

By the definition of \bar{A}_n and \mathcal{F}_{k_n} , and the fact that $(\bar{Y}_{n,u})_{u \in \mathbb{T}_{k_n}}$ is a sequence of independent random variables under \mathbb{E}_{k_n} , we have

$$\begin{aligned} \mathbb{E}_{k_n}(\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n])^2 &= \frac{1}{m^{2k_n}} \mathbb{E}_{k_n} \left[\sum_{u \in \mathbb{T}_{k_n}} (\bar{Y}_{n,u} - \mathbb{E}_{k_n}[\bar{Y}_{n,u}]) \right]^2 \\ &= \frac{1}{m^{2k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left(\mathbb{E}_{k_n}[\bar{Y}_{n,u}^2] - [\mathbb{E}_{k_n} \bar{Y}_{n,u}]^2 \right). \quad (3.10) \end{aligned}$$

By the definition of $\bar{Y}_{n,u}$ and Fubini's theorem,

$$\begin{aligned}
\mathbb{E}_{k_n}[\bar{Y}_{n,u}^2] &= \int_0^\infty 2x\mathbb{P}_{k_n}(|\bar{Y}_{n,u}| > x)dx \\
&= 2\int_0^\infty x\mathbb{P}_{k_n}(|Y_{n,u}|\mathbb{1}_{\{|Y_{n,u}| < m^{k_n}\}} > x)dx \\
&= 2\mathbb{E}_{k_n}\int_0^\infty x\mathbb{1}_{\{|Y_{n,u}|\mathbb{1}_{\{|Y_{n,u}| < m^{k_n}\}} > x\}}dx \\
&\leq 2\mathbb{E}_{k_n}\int_0^{m^{k_n}} x\mathbb{1}_{\{|Y_{n,u}| > x\}}dx = \int_0^{m^{k_n}} x\mathbb{P}_{k_n}(|Y_{n,u}| > x)dx.
\end{aligned}$$

Using (3.6), we obtain

$$\begin{aligned}
\mathbb{P}_{k_n}(|Y_{n,u}| > x) &\leq \mathbb{P}_{k_n}(W_{n-k_n}^{X_u^x} + 1 > x) = \mathbb{P}(W_{n-k_n} + 1 > x) \\
&\leq \mathbb{P}(W^* + 1 > x).
\end{aligned}$$

Therefore,

$$\mathbb{E}_{k_n}[\bar{Y}_{n,u}^2] \leq 2\int_0^{m^{k_n}} x\mathbb{P}(W^* + 1 > x)dx. \quad (3.11)$$

Using (3.9), (3.10) and (3.11) and then taking expectation, we obtain

$$\mathbb{P}(|\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon) \leq \frac{2}{\varepsilon^2 m^{k_n}} \int_0^{m^{k_n}} x\mathbb{P}(W^* + 1 > x)dx. \quad (3.12)$$

We split the above integral according to $x \in [0, e]$ and $x \in (e, m^{k_n}]$. Using $\mathbb{P}(W^* + 1 > x) \leq 1$, we see that

$$\int_0^e x\mathbb{P}(W^* + 1 > x)dx \leq \frac{e^2}{2}. \quad (3.13)$$

For the integral over $(e, m^{k_n}]$, using $x\mathbb{1}_{\{W^*+1>x\}} \leq (W^* + 1)\frac{\log^\eta(W^* + 1)}{\ln^\eta x}$, we have

$$\int_e^{m^{k_n}} x\mathbb{P}(W^* + 1 > x)dx \leq \mathbb{E}[(W^* + 1)\log^\eta(W^* + 1)] \int_e^{m^{k_n}} \frac{1}{(\ln x)^\eta} dx. \quad (3.14)$$

Taking a constant $b \in]1, m[$, we get

$$\begin{aligned}
\int_e^{m^{k_n}} \frac{1}{(\ln x)^\eta} dx &= \int_e^{b^{k_n}} \frac{1}{(\ln x)^\eta} dx + \int_{b^{k_n}}^{m^{k_n}} \frac{1}{(\ln x)^\eta} dx \\
&\leq b^{k_n} + \frac{(m^{k_n} - b^{k_n})}{(k_n \ln b)^\eta}.
\end{aligned}$$

From (3.12), (3.13) and (3.14), we obtain

$$\mathbb{P}(|\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon) \leq \frac{1}{\varepsilon^2 m^{k_n}} \left(e^2 + 2C \left(b^{k_n} + \frac{m^{k_n} - b^{k_n}}{(k_n \log b)^\eta} \right) \right),$$

where $C = \mathbb{E}[(W^* + 1) \log^\eta(W^* + 1)] < \infty$ by Lemma 3.3. Therefore,

$$\begin{aligned} & \sum_{n=1}^{\infty} \mathbb{P}(|\bar{A}_n - \mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon) \\ & \leq \frac{e^2}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{m^{k_n}} + \frac{2C}{\varepsilon^2} \left(\sum_{n=1}^{\infty} \frac{b^{k_n}}{m^{k_n}} + \sum_{n=1}^{\infty} \frac{m^{k_n} - b^{k_n}}{m^{k_n} (k_n \log b)^\eta} \right). \end{aligned} \quad (3.15)$$

Since $k_n \sim j^\alpha \sim n^\beta$, $1 < b < m$ and $\beta\eta > 1$, the three series $\sum_{n=1}^{\infty} \frac{1}{m^{k_n}}$, $\sum_{n=1}^{\infty} \left(\frac{b}{m}\right)^{k_n}$ and $\sum_{n=1}^{\infty} \frac{1}{(k_n \log b)^\eta}$ converge. Therefore from (3.15), we get (3.8).

Step 3. We finally prove that $\mathbb{E}_{k_n}[\bar{A}_n] \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$, as a consequence of

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|\mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon\right) < \infty, \quad \forall \varepsilon > 0 \quad (3.16)$$

(again by the Lemma of Borel-Cantelli). By Markov's inequality, the fact that $0 = \mathbb{E}_{k_n}[A_n] = \mathbb{E}_{k_n}[\bar{A}_n] + \frac{1}{m^{k_n}} \mathbb{E}_{k_n} \left[\sum_{u \in \mathbb{T}_{k_n}} Y_{n,u} \mathbf{1}_{\{|Y_{n,u}| \geq m^{k_n}\}} \right]$, and the inequality (3.6), we obtain:

$$\begin{aligned} \mathbb{P}(|\mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon) & \leq \frac{1}{\varepsilon} \mathbb{E} \left[|\mathbb{E}_{k_n}[\bar{A}_n]| \right] \\ & = \frac{1}{\varepsilon m^{k_n}} \mathbb{E} \left\{ \left| \mathbb{E}_{k_n} \left[\sum_{u \in \mathbb{T}_{k_n}} (-Y_{n,u}) \mathbf{1}_{\{|Y_{n,u}| \geq m^{k_n}\}} \right] \right| \right\} \\ & \leq \frac{1}{\varepsilon m^{k_n}} \mathbb{E} \left\{ \mathbb{E}_{k_n} \left[\sum_{u \in \mathbb{T}_{k_n}} (W_{n-k_n}^{X_u^x} + 1) \mathbf{1}_{\{W_{n-k_n}^{X_u^x} + 1 \geq m^{k_n}\}} \right] \right\} \\ & = \frac{1}{\varepsilon} \mathbb{E} \left[(W_{n-k_n} + 1) \mathbf{1}_{\{W_{n-k_n} + 1 \geq m^{k_n}\}} \right]. \end{aligned}$$

It follows that

$$\mathbb{P}\left(|\mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon\right) \leq \frac{1}{\varepsilon (\log m^{k_n})^\eta} \mathbb{E}[(W^* + 1) \log^\eta(W^* + 1)].$$

Therefore, with $C = \mathbb{E}[(W^* + 1) \log^\eta(W^* + 1)] < \infty$ (by Lemma 3.3),

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|\mathbb{E}_{k_n}[\bar{A}_n]| > \varepsilon\right) \leq \frac{C}{\varepsilon \log^\eta m} \sum_{n=1}^{\infty} \frac{1}{k_n^\eta} < \infty$$

since $k_n^\eta \sim j^{\alpha\eta} \sim n^{\beta\eta}$ and $\beta\eta > 1$. So (3.16) is proved.

So the proof of Lemma 3.1 is finished. \square

The proof of Lemma 3.2 will be based on the following central limit theorem on the couple (X_n^x, S_n^x) (see Theorem 2.1, part (2) in [14]).

Lemma 3.4. *Under the hypothesis of Theorem 2.1, for any measurable set $B \subseteq \mathcal{S}$ with $\nu(\partial B) = 0$, we have*

$$\lim_{n \rightarrow \infty} \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{P} \left(X_n^x \in B, \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right) - \nu(B)\Phi(t) \right| = 0.$$

Proof of Lemma 3.2. We first calculate the conditional expectation in the definition of B_n . Denoting $t_n := \frac{t\sigma\sqrt{n} - S_u^x + k_n\gamma}{\sigma\sqrt{n - k_n}}$ and using the branching property, we have

$$\begin{aligned} \mathbb{E}_{k_n} \left[W_{n-k_n}^{X_u^x}(B, t_n) \right] &= \frac{1}{m^{k_n}} \mathbb{E}_{k_n} \left[\sum_{v \in \mathbb{T}_{n-k_n}(u)} \mathbb{1}_{\left\{ X_v^{X_u^x} \in B, \frac{S_v^{X_u^x} - (n-k_n)\gamma}{\sigma\sqrt{n-k_n}} \leq t_n \right\}} \right] \\ &= \mathbb{P}_{k_n} \left(X_{n-k_n}^{X_u^x} \in B, \frac{S_{n-k_n}^{X_u^x} - (n-k_n)\gamma}{\sigma\sqrt{n-k_n}} \leq t_n \right). \end{aligned}$$

Therefore, by the definition of B_n ,

$$\begin{aligned} B_n &= \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left\{ \mathbb{P}_{k_n} \left(X_{n-k_n}^{X_u^x} \in B, \frac{S_{n-k_n}^{X_u^x} - (n-k_n)\gamma}{\sigma\sqrt{n-k_n}} \leq t_n \right) - \nu(B)\Phi(t) \right\} \\ &= \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left\{ \mathbb{P}_{k_n} \left(X_{n-k_n}^{X_u^x} \in B, \frac{S_{n-k_n}^{X_u^x} - (n-k_n)\gamma}{\sigma\sqrt{n-k_n}} \leq t_n \right) - \nu(B)\Phi(t_n) \right\} \\ &\quad + \frac{\nu(B)}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} [\Phi(t_n) - \Phi(t)]. \end{aligned}$$

Hence

$$|B_n| \leq W_{k_n} D_n + \frac{\nu(B)}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} |\Phi(t_n) - \Phi(t)| \quad (3.17)$$

where

$$D_n = \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{P} \left(X_{n-k_n}^x \in B, \frac{S_{n-k_n}^x - (n-k_n)\gamma}{\sigma\sqrt{n-k_n}} \leq t \right) - \nu(B)\Phi(t) \right|.$$

The first term in the right hand side of (3.17) tends to 0 a.s. because, by Lemma 3.4, we have

$$D_n \xrightarrow{n \rightarrow \infty} 0 \quad \mathbb{P}\text{-a.s.}$$

We now prove that the second term in the right hand side of (3.17) also tends to 0 a.s. Remarking that

$$\begin{aligned} |\Phi(t_n) - \Phi(t)| &\leq \left| \Phi\left(\frac{t\sigma\sqrt{n}}{\sigma\sqrt{n-k_n}}\right) - \Phi(t) \right| + \left| \Phi\left(\frac{t\sigma\sqrt{n}}{\sigma\sqrt{n-k_n}}\right) - \Phi(t_n) \right| \\ &\leq \left| \Phi\left(\frac{t\sigma\sqrt{n}}{\sigma\sqrt{n-k_n}}\right) - \Phi(t) \right| + \left| \frac{t\sigma\sqrt{n}}{\sigma\sqrt{n-k_n}} - t_n \right| \end{aligned}$$

(since $|\Phi(x+h) - \Phi(x)| \leq |h|$ for any $x, h \in \mathbb{R}$), we obtain

$$\begin{aligned} &\frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} |\Phi(t_n) - \Phi(t)| \\ &\leq W_{k_n} \left| \Phi\left(\frac{t\sigma\sqrt{n}}{\sigma\sqrt{n-k_n}}\right) - \Phi(t) \right| + \frac{1}{m^{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \frac{|S_u^x - k_n\gamma|}{\sigma\sqrt{n-k_n}}. \end{aligned} \quad (3.18)$$

It is clear that the first term in the above display tends to 0 a.s. as $n \rightarrow \infty$. So we need only to prove that the second term also tends to 0 a.s. Recall that $a_j = k_n$ and notice that $n - k_n \sim n \sim k_n^{1/\beta} = j^{\alpha/\beta}$. So it suffices to show that

$$M_j := \frac{1}{m^{a_j}} \sum_{u \in \mathbb{T}_{a_j}} j^{-\frac{\alpha}{2\beta}} |S_u^x - a_j\gamma| \rightarrow 0 \quad \mathbb{P}\text{-a.s. as } j \rightarrow +\infty. \quad (3.19)$$

Notice that

$$\begin{aligned} \sum_{j=1}^{\infty} \mathbb{E}[M_j] &= \sum_{j=1}^{\infty} j^{-\frac{\alpha}{2\beta}} \mathbb{E}[|S_{a_j}^x - a_j\gamma|] \\ &\leq \sum_{j=1}^{\infty} j^{-\frac{\alpha}{2\beta}} \sqrt{\mathbb{E}[(S_{a_j}^x - a_j\gamma)^2]} \\ &= \sum_{j=1}^{\infty} j^{-\frac{\alpha}{2\beta}} a_j^{1/2} \sqrt{\frac{\mathbb{E}[(S_{a_j}^x - a_j\gamma)^2]}{a_j}} < \infty, \end{aligned}$$

where the last series converges by the expression of σ^2 (cf. (2.1)) and the fact that $j^{-\frac{\alpha}{2\beta}} a_j^{1/2} \sim j^{-\frac{\alpha}{2}(\frac{1}{\beta}-1)}$ with $\frac{\alpha}{2}(\frac{1}{\beta}-1) > 1$. Thus $\sum_{j=1}^{\infty} M_j < \infty$ a.s., which implies (3.19). So, by (3.18), the second term in the right hand side of (3.17) tends to 0 a.s. This ends the proof of Lemma 3.2. \square

4. PROOF OF THEOREM 2.2 AND COROLLARY 2.3

In this section we establish Theorem 2.2 and Corollary 2.3 about the non-degeneracy of the limit variable W_s^x of the fundamental martingale $(W_{s,n}^x)$.

Let $s \in (-\eta_1, s_\infty)$ and $x \in \mathcal{S}$ be fixed. Consider the positive function

$$H(n, y) = \frac{e^{s \log |y|} r_s\left(\frac{y}{|y|}\right)}{[m \kappa(s)]^n r_s(x)}, \quad n \geq 0, y \in \mathbb{R}^*.$$

Since r_s is the eigenfunction of the operator P_s with respect to the eigenvalue $\kappa(s)$ (see (2.3)), we see that H is a mean-harmonic function (see [12]) in the sense that for each $n \geq 0$ and $u \in \mathbb{T}_n$,

$$\mathbb{E}\left[\sum_{i=1}^{N_u} H(n+1, A_{ui} G_u x) \mid \mathcal{F}_n\right] = H(n, G_u x).$$

Indeed, we have:

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^{N_u} H(n+1, A_{ui} G_u x) \mid \mathcal{F}_n\right] &= \mathbb{E}\left[\sum_{i=1}^{N_u} \frac{e^{s \log |A_{ui} G_u x|} r_s(A_{ui} \cdot X_u^x)}{[m \kappa(s)]^{n+1} r_s(x)} \mid \mathcal{F}_n\right] \\ &= \frac{e^{s \log |G_u x|}}{[m \kappa(s)]^{n+1} r_s(x)} \mathbb{E}\left[\sum_{i=1}^{N_u} e^{s \log(|A_{ui} X_u^x|)} r_s(A_{ui} \cdot X_u^x) \mid \mathcal{F}_n\right] \\ &= \frac{e^{s \log |G_u x|} m P_s r_s(X_u^x)}{[m \kappa(s)]^{n+1} r_s(x)} \\ &= \frac{e^{s \log |G_u x|} r_s(X_u^x)}{[m \kappa(s)]^n r_s(x)}. \end{aligned}$$

Therefore

$$W_{s,n}^x = \sum_{u \in \mathbb{T}_n} H(n, G_u x) = \sum_{u \in \mathbb{T}_n} \frac{e^{s \log |G_u x|} r_s(X_u^x)}{[m \kappa(s)]^n r_s(x)}, \quad n \geq 0$$

is a positive martingale, so that the limit

$$W_s^x = \lim_{n \rightarrow \infty} W_{s,n}^x$$

exists a.s. with values in \mathbb{R}_+ . For $u \in \mathbb{U}$, denote

$$W_{s,1}^{X_u^x} = \sum_{i=1}^{N_u} \frac{e^{s \log |A_{ui} X_u^x|} r_s(A_{ui} \cdot X_u^x)}{m \kappa(s) r_s(X_u^x)},$$

which represents the first term of the fundamental martingale corresponding to the branching process starting from the particle u ; in particular for $u = \emptyset$, $W_{s,1}^{X_\emptyset^x}(\emptyset) = W_{s,1}^x$ with the usual convention that $X_\emptyset^x = x$.

For fixed $s \in (-\eta_1, s_\infty)$ and $x \in \mathcal{S}$, the spectral gap property (2.3) allows to define a probability measure \mathbb{Q}_s^x on (Ω, \mathcal{F}) such that for any $n \in \mathbb{N}$ and

any bounded and measurable function h on $(\mathcal{S} \times \mathbb{R})^{n+1}$,

$$\begin{aligned} & \mathbb{E} \left[\frac{e^{sS_n^x} r_s(X_n^x)}{\kappa^n(s) r_s(x)} h(X_0^x, S_0^x, \dots, X_n^x, S_n^x) \right] \\ &= \mathbb{E}_{\mathbb{Q}_s^x} [h(X_0^x, S_0^x, \dots, X_n^x, S_n^x)], \end{aligned} \quad (4.1)$$

where $\mathbb{E}_{\mathbb{Q}_s^x}$ denotes the expectation with respect to \mathbb{Q}_s^x . See [16, 17, 29] for $s \geq 0$, and [55] for $s < 0$.

With the mean-harmonic function H specified above and the probability measure \mathbb{Q}_s^x introduced here, from [12, Theorem 2.1] we obtain the following result for the non-degeneracy of the limit W_s^x . We use the usual notation that $x \wedge y = \min\{x, y\}$, and we denote by $1_n = (1, \dots, 1) \in \mathbb{N}^{*n}$ the sequence of length n whose components are all equal to 1.

Lemma 4.1. *For $s \in (-\eta_1, s_\infty)$ and $x \in \mathcal{S}$, we have:*

(i) $\mathbb{E}[W_s^x] = 1$ if

$$\sum_{n=1}^{\infty} \mathbb{E} \left[W_{s,1}^{X_{1_n}^x} \left(H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} \wedge 1 \right) \middle| \mathcal{F}_n \right] < \infty \quad \mathbb{Q}_s^x\text{-a.s.} \quad (4.2)$$

(ii) $\mathbb{E}[W_s^x] = 0$ if either

$$\limsup_{n \rightarrow \infty} H(n, G_{1_n} x) = \infty \quad \mathbb{Q}_s^x\text{-a.s.} \quad (4.3)$$

or for all $y > 0$,

$$\sum_{n=1}^{\infty} \mathbb{E} \left[W_{s,1}^{X_{1_n}^x} \mathbf{1}_{\{H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} > y\}} \middle| \mathcal{F}_n \right] = \infty \quad \mathbb{Q}_s^x\text{-a.s.} \quad (4.4)$$

4.1. Proof of Theorem 2.2.

Sufficient condition. We assume conditions (2.9) and (2.10), together with **C1** and **C2**, and we will prove (4.2) which, by Lemma 4.1, will imply that $\mathbb{E}[W_s^x] = 1$. By the definition of \mathbb{Q}_s^x (cf. (4.1)), we have

$$\begin{aligned} & \mathbb{E} \left[W_{s,1}^{X_{1_n}^x} \left((H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x}) \wedge 1 \right) \middle| \mathcal{F}_n \right] \\ &= \mathbb{E} \left[\sum_{i=1}^{N_{1_n}} \frac{e^{s \log |A_{1_n i} X_{1_n}^x|} r_s(A_{1_n i} \cdot X_{1_n}^x)}{m \kappa(s) r_s(X_{1_n}^x)} \left((H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x}) \wedge 1 \right) \middle| \mathcal{F}_n \right] \\ &= \mathbb{E}_{\mathbb{Q}_s^x} \left[(H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x}) \wedge 1 \middle| \mathcal{F}_n \right]. \end{aligned}$$

By the extended Borel-Cantelli Lemma, we have

$$\begin{aligned} & \left\{ \sum_{n=1}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[(H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x}) \wedge 1 \mid \mathcal{F}_n \right] < \infty \right\} \\ & \stackrel{\mathbb{Q}_s^x\text{-a.s.}}{=} \left\{ \sum_{n=1}^{\infty} \left(H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} \wedge 1 \right) < \infty \right\}. \end{aligned} \quad (4.5)$$

We shall prove that \mathbb{Q}_s^x -a.s. the term $H(n, G_{1_n} x) \rightarrow 0$ exponentially and $W_{s,1}^{X_{1_n}^x} \rightarrow \infty$ subexponentially. This will imply that the two series in (4.5) converge \mathbb{Q}_s^x -a.s., and thus conclude the proof of (4.2).

We first prove that $H(n, G_{1_n} x) \rightarrow 0$ \mathbb{Q}_s^x -a.s. with an exponential rate. We start by rewriting $H(n, G_{1_n} x)$ in form

$$H(n, G_{1_n} x) = \frac{r_s(X_{1_n}^x)}{r_s(x)} \exp \left\{ n \left[s \left(\frac{S_{1_n}^x}{n} - q_s \right) + (sq_s - \log[m \kappa(s)]) \right] \right\}. \quad (4.6)$$

Recall that the function $r_s(\cdot)$ is strictly positive and Hölder continuous on the compact set \mathcal{S} . It is therefore bounded from above and from below by two positive constants. By the strong law of large numbers for $S_{1_n}^x$ under \mathbb{Q}_s^x (see [55, Proposition 3.12], [16, Theorem 6.1], [29, Theorem 3.10]),

$$\lim_{n \rightarrow \infty} \frac{S_{1_n}^x}{n} = q_s \quad \mathbb{Q}_s^x\text{-a.s.} \quad (4.7)$$

Therefore

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log H(n, G_{1_n} x) = sq_s - \log[m \kappa(s)] \quad \mathbb{Q}_s^x\text{-a.s.} \quad (4.8)$$

By hypothesis $sq_s - \log[m \kappa(s)] < 0$, so $H(n, G_{1_n} x) \rightarrow 0$ exponentially \mathbb{Q}_s^x -a.s.

We next prove that $W_{s,1}^{X_{1_n}^x}$ grows to infinity subexponentially \mathbb{Q}_s^x -a.s., in the sense that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log_+ W_{s,1}^{X_{1_n}^x} = 0 \quad \mathbb{Q}_s^x\text{-a.s.}$$

By the lemma of Borel-Cantelli, it is enough to prove that

$$\sum_{n=0}^{\infty} \mathbb{Q}_s^x \left(\log_+ W_{s,1}^{X_{1_n}^x} > \varepsilon n \right) < \infty \quad \forall \varepsilon > 0. \quad (4.9)$$

By the definition of \mathbb{Q}_s^x and Fubini's Theorem, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{Q}_s^x \left(\log_+ W_{s,1}^{X_{1n}^x} > \varepsilon n \right) &= \sum_{n=0}^{\infty} \mathbb{E} \left[W_{s,1}^{X_{1n}^x} \mathbf{1}_{\{\log_+ W_{s,1}^{X_{1n}^x} > \varepsilon n\}} \right] \\ &\leq \mathbb{E} \left[\sum_{n=0}^{\infty} \max_{x \in \mathcal{S}} W_{s,1}^x \mathbf{1}_{\{\log_+ \max_{x \in \mathcal{S}} W_{s,1}^x > \varepsilon n\}} \right] \\ &\leq \frac{1}{\varepsilon} \mathbb{E} \left[\max_{x \in \mathcal{S}} W_{s,1}^x \log_+ \max_{x \in \mathcal{S}} W_{s,1}^x + 1 \right], \end{aligned}$$

which is finite by hypothesis (2.10). Therefore, the property (4.9) is proved.

Necessary condition. It suffices to prove that if either $\Lambda^*(q_s) - \log m \geq 0$ or $\mathbb{E} \left[\min_{x \in \mathcal{S}} W_{s,1}^x \log_+ \min_{x \in \mathcal{S}} W_{s,1}^x \right] = \infty$, then $\mathbb{E} W_s^x = 0$ for all $x \in \mathcal{S}$. In the following we consider three cases.

Case 1. Suppose that $\Lambda^*(q_s) - \log m > 0$. Then by (4.6) and (4.7) we see that

$$\lim_{n \rightarrow \infty} H(n, G_{1n} x) = \infty, \quad \mathbb{Q}_s^x\text{-a.s.} \quad (4.10)$$

This implies $\mathbb{E}[W_s^x] = 0$ by Lemma 4.1.

Case 2. Suppose that $\Lambda^*(q_s) - \log m = 0$. Then by (4.6),

$$H(n, G_{1n} x) = \frac{r_s(X_{1n}^x)}{r_s(x)} e^{s(S_{1n}^x - nq_s)}.$$

Since r_s is bounded from below and from above by two positive constants, using Lemma 4.2 below, we see that (4.10) still holds, which implies $\mathbb{E}[W_s^x] = 0$ by Lemma 4.1, just as in the first case.

Here we have used the following law of iterated logarithm for products of random matrices. For $s = 0$, it was established in [40, Theorem 5].

Lemma 4.2. *Let $s \in (-\eta_1, s_\infty)$ and $x \in \mathcal{S}$. Under conditions C1 and C2,*

$$\limsup_{n \rightarrow \infty} \frac{S_n^x - nq_s}{\sigma_s \sqrt{2n \log \log n}} = 1 \quad \mathbb{Q}_s^x\text{-a.s.}$$

This lemma can be proved in the same way as in the proof of Theorem 5 of [40], using Berry-Esseen's bound for S_n^x under the changed measure \mathbb{Q}_s^x established in [55, Theorem 2.1] for $s \in (-\eta_1, 0]$ and in [17, Theorem 8.1] for $s \in (0, s_\infty)$. Since the proof is very similar, we omit the details.

Case 3. Assume that $\mathbb{E} \left[\min_{x \in \mathcal{S}} W_{s,1}^x \log_+ \min_{x \in \mathcal{S}} W_{s,1}^x \right] = \infty$ and $\Lambda^*(q_s) - \log m < 0$. We shall prove that (4.4) holds for all $y > 0$. By the definition of \mathbb{Q}_s^x , we have

$$\mathbb{E} \left[W_{s,1}^{X_{1n}^x} \mathbf{1}_{\{H(n, G_{1n} x) W_{s,1}^{X_{1n}^x} > y\}} \mid \mathcal{F}_n \right] = \mathbb{E}_{\mathbb{Q}_s^x} \left[\mathbf{1}_{\{H(n, G_{1n} x) W_{s,1}^{X_{1n}^x} > y\}} \mid \mathcal{F}_n \right].$$

By the extended Borel- Cantelli lemma, we get, for $y > 0$,

$$\begin{aligned} & \left\{ \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[\mathbb{1}_{\{H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} > y\}} \mid \mathcal{F}_n \right] = \infty \right\} \\ & \stackrel{\mathbb{Q}_s^x\text{-a.s.}}{=} \limsup_{n \rightarrow \infty} \left\{ H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} > y \right\}. \end{aligned}$$

Therefore, for $y > 0$, (4.4) holds if

$$\mathbb{Q}_s^x \left(\limsup_{n \rightarrow \infty} \left\{ H(n, G_{1_n} x) W_{s,1}^{X_{1_n}^x} > y \right\} \right) = 1. \quad (4.11)$$

By (4.8), we see that (4.11) achieves if

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log_+ W_{s,1}^{X_{1_n}^x} = +\infty, \quad \mathbb{Q}_s^x\text{-a.s.},$$

which is equivalent to

$$\mathbb{Q}_s^x \left(\limsup_{n \rightarrow \infty} \underbrace{\left\{ \log_+ W_{s,1}^{X_{1_n}^x} > Mn \right\}}_{=: B_{n+1}} \right) = 1 \quad \forall M > 0. \quad (4.12)$$

We see that (4.12) follows from

$$\mathbb{Q}_s^x \left(\sum_{n=0}^{\infty} \mathbb{1}_{B_{n+1}} = \infty \right) = 1. \quad (4.13)$$

To prove (4.13), notice that by the extended Borel-Cantelli lemma we have

$$\left\{ \sum_{n=0}^{\infty} \mathbb{1}_{B_{n+1}} < \infty \right\} \stackrel{\mathbb{Q}_s^x\text{-a.s.}}{=} \left\{ \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} [\mathbb{1}_{B_{n+1}} \mid \mathcal{F}_n] < \infty \right\}. \quad (4.14)$$

By the definition of \mathbb{Q}_s^x and Fubini's theorem, we have, \mathbb{Q}_s^x -a.s.,

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} [\mathbb{1}_{B_{n+1}} \mid \mathcal{F}_n] &= \sum_{n=0}^{\infty} \mathbb{E} \left[W_{s,1}^{X_{1_n}^x} \mathbb{1}_{\{\log_+ W_{s,1}^{X_{1_n}^x} > Mn\}} \mid \mathcal{F}_n \right] \\ &\geq \sum_{n=0}^{\infty} \mathbb{E} \left[\min_{x \in S} W_{s,1}^x \mathbb{1}_{\{\log_+ \min_{x \in S} W_{s,1}^x > Mn\}} \mid \mathcal{F}_n \right] \\ &= \mathbb{E} \left[\sum_{n=0}^{\infty} \min_{x \in S} W_{s,1}^x \mathbb{1}_{\{\log_+ \min_{x \in S} W_{s,1}^x > Mn\}} \right] \\ &\geq \frac{1}{M} \mathbb{E} [\min_{x \in S} W_{s,1}^x \log_+ \min_{x \in S} W_{s,1}^x - 1] \\ &= +\infty, \end{aligned}$$

where the second equality holds since $\min_{x \in S} W_{s,1}^x$ is independent of \mathcal{F}_n , and the last equality holds by hypothesis. Hence (4.13) follows from (4.14).

4.2. Proof of Corollary 2.3. We will need the following result which was established in [45] in a slightly weaker form. We use the convention that the empty sum is taken to be 0.

Lemma 4.3. *Let N, X_1, X_2, \dots be independent random variables with $N \in \mathbb{N}$, $X_i \in \mathbb{R}_+$, $\mathbb{P}(N = 0) < 1$ and $\mathbb{P}(X = 0) < 1$. Assume that all the X_i have the same law. Then*

$$\mathbb{E} \left[\left(\sum_{i=1}^N X_i \right) \log_+ \left(\sum_{i=1}^N X_i \right) \right] < \infty$$

if and only if

$$\mathbb{E}(X_1 \log_+ X_1) < \infty \quad \text{and} \quad \mathbb{E}(N \log_+ N) < \infty.$$

Proof. The "if" part has been proved in [45]. The "only if" part is slightly stronger than that in [45]. Let us give a short proof which is different to that used in [45]. Since the function $f(x) = x \log_+ x$ (with $f(0) = 0$) is increasing, we have

$$\mathbb{E} \left[f \left(\sum_{i=1}^N X_i \right) \right] \geq \mathbb{E} \left(X_1 \log_+ X_1 \mathbf{1}_{\{N \geq 1\}} \right) = \mathbb{E}(X_1 \log_+ X_1) \mathbb{P}(N \geq 1).$$

Therefore $\mathbb{E}(X_1 \log_+ X_1) < \infty$. Together with $\mathbb{P}(X_1 = 0) < 1$, this implies that $c := \mathbb{E}X_1 \in (0, \infty)$. Since f is convex on \mathbb{R}_+ , by Jensen's inequality, we have

$$\mathbb{E} \left[f \left(\sum_{i=1}^N X_i \right) \middle| N \right] \geq f \left(\mathbb{E} \left[\sum_{i=1}^N X_i \middle| N \right] \right) = f(cN) = (cN) \log_+(cN).$$

Taking expectation, we get $\mathbb{E} \left[\sum_{i=1}^N X_i \log_+ \left(\sum_{i=1}^N X_i \right) \right] \geq \mathbb{E}[(cN) \log_+(cN)]$. Hence $\mathbb{E}[(cN) \log_+(cN)] < \infty$, which is equivalent to $\mathbb{E}N \log_+ N < \infty$. \square

Then the three conditions (2.10), (2.13) and (2.14) are equivalent, and (2.11) holds for all $x \in \mathcal{S}$ if and only if (2.9) and (2.14) hold. Moreover, if (2.11) holds for some $x \in \mathcal{S}$, then it holds for all $x \in \mathcal{S}$.

Proof of Corollary 2.3. (1) Note that for $s \in (-\eta_1, s_\infty)$, the function $r_s(\cdot)$ is strictly positive and continuous on the compact set \mathcal{S} . It is therefore bounded from above and from below by two positive constants. From the definition of $W_{s,1}^x$ and $\|\cdot\|$, we observe that

$$\mathbb{E} \left[\max_{x \in \mathcal{S}} W_{s,1}^x \log_+ \max_{x \in \mathcal{S}} W_{s,1}^x \right] \leq \mathbb{E} \left[\sum_{i=1}^N c \|A_i\|^s \log_+ \left(\sum_{i=1}^N c \|A_i\|^s \right) \right].$$

Therefore by Lemma 4.3, (2.14) implies (2.10). This ends the proof of the first part of Corollary 2.3.

(2) By Theorem 2.2, to prove the second part of Corollary 2.3, it is enough to show that (2.10), (2.13) and (2.14) are all equivalent.

Now we prove the equivalence according to the scheme: (2.10) \Rightarrow (2.13) \Rightarrow (2.14) \Rightarrow (2.10). The implication (2.10) \Rightarrow (2.13) is obvious; the implication (2.14) \Rightarrow (2.10) is just proved above in part (1). So we only need to show that (2.13) \Rightarrow (2.14). Set for $k \geq 1$,

$$\bar{A}_k = \max_{1 \leq i, j \leq d} |A_k(i, j)| \quad \text{and} \quad \underline{A}_k = \min_{1 \leq i, j \leq d} |A_k(i, j)|.$$

Since all norms on \mathbb{R}^d are equivalent, we can take the norm $|x| = |x_1| + |x_2| + \dots + |x_d|$. Then for $k \geq 1$ and $x \in \mathcal{S}$,

$$d\underline{A}_k \leq |A_k x| = \sum_{1 \leq i, j \leq d} |A_k(i, j)x_j| \leq d\bar{A}_k.$$

Since the function r_s on \mathcal{S} is bounded from above and from below by two positive constants, this implies that for some constant $c_1 > 0$,

$$\mathbb{E} \left[\min_{x \in \mathcal{S}} W_{s,1}^x \log_+ \min_{x \in \mathcal{S}} W_{s,1}^x \right] \geq \mathbb{E} \left[\sum_{k=1}^N c_1 \underline{A}_k^s \log_+ \left(\sum_{k=1}^N c_1 \underline{A}_k^s \right) \right].$$

Remark that under the Furstenberg-Kesten condition (2.15), $\mathbb{E} [\underline{A}_1^s \log_+ \underline{A}_1^s] < \infty$ if and only if $\mathbb{E} [\|A_1\|^s \log_+ \|A_1\|^s] < \infty$. Therefore, by Lemma 4.3 and the above inequality, (2.13) implies (2.14). \square

5. PROOF OF THEOREM 2.4

In this section we will prove Theorem 2.4, the precise large deviation asymptotic of Bahadur-Rao type on the counting measure Z_n^x , using a uniform local limit theorem for products of random matrices that we recently established in [14].

5.1. Auxiliary results. In the proof of Theorem 2.4 we make use of the following three assertions. The first one is a local limit theorem for products of random matrices under the changed measure \mathbb{Q}_s^x (see Proposition 5.1). The second is an exponential bound of the large deviation probability of the products of random matrices under \mathbb{Q}_s^x (see Proposition 5.2). The third gives a relationship between moment conditions on $W_{s,1}^x$ and on $W_{s,*}^x := \sup_n W_{s,n}^x$ (see Proposition 5.3).

We start with a uniform local limit theorem for products of random matrices under the changed measure \mathbb{Q}_s^x . Under the initial measure (when $s = 0$), it has been established in [14].

Proposition 5.1. *Under the conditions of Theorem 2.4, we have, for any continuous function f on \mathcal{S} and any directly Riemann integrable function h*

on \mathbb{R} ,

$$\lim_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| \sigma_s \sqrt{n} \mathbb{E}_{\mathbb{Q}_s^x} [f(X_n^x) h(y + S_n^x - n\Lambda'(s))] - \pi_s(f) \int_{\mathbb{R}} h(z) \phi\left(\frac{z-y}{\sigma_s \sqrt{n}}\right) dz \right| = 0, \quad (5.1)$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the density function of the standard normal law.

Proof. For $\lambda > 0$ sufficiently small, we introduce the Banach space $\mathcal{B}_\lambda = \{f \in \mathcal{C}(\mathcal{S}) : \|f\|_\lambda < +\infty\}$, where

$$\|f\|_\lambda := \|f\|_\infty + |f|_\lambda,$$

with

$$\|f\|_\infty := \sup_{x \in \mathcal{S}} |f(x)|, \quad |f|_\lambda := \sup_{x,y \in \mathcal{S}, x \neq y} \frac{|f(x) - f(y)|}{\mathbf{d}^\lambda(x,y)}.$$

For $s \in (-\eta_1, s_\infty)$ and $t \in \mathbb{R}$, define the perturbed operator $R_{s,it}$ on \mathcal{B}_λ as follows: for any $\varphi \in \mathcal{B}_\lambda$,

$$R_{s,it} \varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{it[S_1^x - \Lambda'(s)]} \varphi(X_1^x) \right], \quad x \in \mathcal{S}.$$

By induction, it follows that for any $n \geq 1$,

$$R_{s,it}^n \varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{it[S_n^x - n\Lambda'(s)]} \varphi(X_n^x) \right], \quad x \in \mathcal{S}.$$

For properties of this operator, we refer the reader to [54] for $s \in (0, s_\infty)$ and [55] for $s \in (-\eta_1, 0]$. Since the proof of Theorem 2.4 is quite similar to that of Theorem 2.2(1) in [14], we will not give the details here. The only difference is that, instead of the properties of the operator $R_{0,it}$ used in the proof in [14], here we use the properties of the operator $R_{s,it}$ proved in [54, 55]. □

We next present an exponential bound of the large deviation probability of the products of random matrices under \mathbb{Q}_s^x . For $s = 0$, it has been established in [54].

Proposition 5.2. *Assume the conditions of Theorem 2.4. Let $\varepsilon > 0$. There are $C > 0$ and $0 < c < 1$ such that for all $n \geq 1$,*

$$\sup_{x \in \mathcal{S}} \mathbb{Q}_s^x \left(\frac{|S_n^x - n\Lambda'(s)|}{n} > \varepsilon \right) \leq Cc^n. \quad (5.2)$$

Proof. By the monotonicity in ε of the large deviation probability, it is clear that it suffices to prove the inequality for $\varepsilon > 0$ small enough. By the formula

of the changed measure (4.1), for any nonnegative and Borel function φ and any point $t \in (-\eta_1, s_\infty)$, we have

$$\mathbb{E}_{\mathbb{Q}_s^x} \varphi(S_n^x) = \frac{\kappa(t)^n r_t(x)}{\kappa(s)^n r_s(x)} \mathbb{E}_{\mathbb{Q}_t^x} \left[e^{-(t-s)S_n^x} (r_t^{-1} r_s)(X_n^x) \varphi(S_n^x) \right].$$

Take $\varphi(x) = \mathbf{1}_{(n\Lambda'(s)+n\varepsilon, +\infty)}(x)$. Because $\Lambda''(s) > 0$ for all $s \in (-\eta_1, s_\infty)$ and $\Lambda'(s)$ is continuous in s , for $\varepsilon > 0$ small enough, there is $t \in (-\eta_1, s_\infty)$ with $t > s$ such that $\Lambda'(t) = \Lambda'(s) + \varepsilon$. Hence

$$\begin{aligned} & \mathbb{Q}_s^x \left(S_n^x - n\Lambda'(s) > n\varepsilon \right) \\ &= \frac{\kappa(t)^n r_t(x)}{\kappa(s)^n r_s(x)} \mathbb{E}_{\mathbb{Q}_t^x} \left[\frac{e^{-(t-s)S_n^x} r_s(X_n^x)}{r_t(X_n^x)} \mathbf{1}_{\{S_n^x - n\Lambda'(s) > n\varepsilon\}} \right] \\ &= e^{n[-(t-s)\Lambda'(t) + \Lambda(t) - \Lambda(s)]} \frac{r_t(x)}{r_s(x)} \mathbb{E}_{\mathbb{Q}_t^x} [f(X_n^x) h(S_n^x - n\Lambda'(t))], \end{aligned} \quad (5.3)$$

where $f(x) = \frac{r_s(x)}{r_t(x)}$ and $h(x) = e^{-(t-s)x} \mathbf{1}_{\{x > 0\}}$. Notice that $h(x) \leq 1$ and that $f(x)$ is bounded from below and above by two positive constants because r_t and r_s are continuous and strictly positive on the compact set \mathcal{S} . Therefore from (5.3), we see that there exists a constant $C_1 > 0$ such that for all $n \geq 1$,

$$\mathbb{Q}_s^x \left(S_n^x - n\Lambda'(s) > n\varepsilon \right) \leq C_1 e^{n[-(t-s)\Lambda'(t) + \Lambda(t) - \Lambda(s)]}. \quad (5.4)$$

We now prove that

$$-(t-s)\Lambda'(t) + \Lambda(t) - \Lambda(s) < 0. \quad (5.5)$$

To do this, we consider the function

$$\psi(y) = -y\Lambda'(s+y) + \Lambda(s+y) - \Lambda(s), \quad y \in [0, s_\infty - s],$$

which is continuous on the interval $[0, s_\infty - s]$. For $y \in (0, s_\infty - s)$, $\psi'(y) = -y\Lambda''(s+y) < 0$, so that $\psi(y) < \psi(0) = 0$. With $y = t - s$, this implies (5.5). From (5.4) and (5.5), we see that for all $n \geq 1$,

$$\sup_{x \in \mathcal{S}} \mathbb{Q}_s^x \left(S_n^x - n\Lambda'(s) > n\varepsilon \right) \leq C_1 c_1^n, \quad (5.6)$$

where $c_1 = \exp \{ -(t-s)\Lambda'(t) + \Lambda(t) - \Lambda(s) \} < 1$. In the same way, if we take $\varphi(x) = \mathbf{1}_{(-\infty, n\Lambda'(s) - n\varepsilon)}(x)$ and $t \in (-\eta_1, s)$ such that $\Lambda'(t) = \Lambda'(s) - \varepsilon$, then there are constants $C_2 > 0$ and $0 < c_2 < 1$ such that

$$\sup_{x \in \mathcal{S}} \mathbb{Q}_s^x \left(S_n^x - n\Lambda'(s) < -n\varepsilon \right) \leq C_2 c_2^n. \quad (5.7)$$

The conclusion of the proposition follows from (5.6) and (5.7). \square

We finally establish a relationship between moment conditions on $W_{s,1}^x$ and on $W_{s,*}^x$.

Proposition 5.3. *Assume the conditions of Theorem 2.4. Then*

$$\sup_{x \in \mathcal{S}} \mathbb{E} \left[(W_{s,*}^x + 1) \log^\delta (W_{s,*}^x + 1) \right] < \infty.$$

For the proof, we will adapt the approach of Biggins [10] on the classical branching random walk. The following recursive relations on $W_{s,n}^x$ and W_s^x will be used. First, it can be easily seen that for $1 \leq k \leq n$,

$$W_{s,n}^x = \sum_{u \in \mathbb{T}_k} H(k, G_u x) W_{s,n-k}^{X_u^x}, \quad \text{where } H(k, G_u x) = \frac{e^{s S_u^x} r_s(X_u^x)}{[m \kappa(s)]^k r_s(x)}. \quad (5.8)$$

From this recursive relation on $W_{s,n}^x$, taking $n \rightarrow \infty$ we obtain the following recursive relation on W_s^x : for $k \geq 1$,

$$W_s^x = \sum_{u \in \mathbb{T}_k} H(k, G_u x) W_s^{X_u^x}, \quad (5.9)$$

by our notation. The proof of Proposition 5.3 will be done with the help of three lemmas.

Lemma 5.4. *Assume the conditions of Theorem 2.4. Then*

$$\sup_{x \in \mathcal{S}} \mathbb{E} \left[(W_s^x + 1) \log^\delta (W_s^x + 1) \right] < \infty. \quad (5.10)$$

Proof of Lemma 5.4. Let

$$h(u) := \begin{cases} c_0 u & \text{for } 0 < u \leq x_0, \\ c_1 + c_2 \log^\delta u & \text{for } u \geq x_0 > 1, \end{cases}$$

where x_0, c_0, c_1 and c_2 are constants with $x_0 > 1, c_0, c_2 > 0$, which make h concave (and hence subadditive) and increasing. Then

$$h(uv) \leq C(1 + \log_+^\delta u + \log_+^\delta v), \quad \forall u, v > 0, \quad (5.11)$$

when $C > 0$ is a large enough constant. Notice that to prove (5.10) we only need show that

$$\sup_{x \in \mathcal{S}} \mathbb{E}[W_s^x h(W_s^x)] < \infty. \quad (5.12)$$

Using (5.8) and the subadditivity of h , we have

$$\begin{aligned}
& \mathbb{E}[W_{s,n+1}^x h(W_{s,n+1}^x) | \mathcal{F}_n] \\
&= \mathbb{E}\left[\left(\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x}\right) h\left(\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x}\right) \middle| \mathcal{F}_n\right] \\
&\leq \mathbb{E}\left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h\left(\sum_{t \in \mathbb{T}_n, t \neq u} H(n, G_t x) W_{s,1}^{X_t^x}\right) \middle| \mathcal{F}_n\right] \\
&\quad + \mathbb{E}\left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h\left(H(n, G_u x) W_{s,1}^{X_u^x}\right) \middle| \mathcal{F}_n\right]. \tag{5.13}
\end{aligned}$$

For the first term, we see that $H(n, G_t x)$ is \mathcal{F}_n -measurable, $W_{s,1}^{X_u^x}$ and $\sum_{t \in \mathbb{T}_n, t \neq u} H(n, G_t x) W_{s,1}^{X_t^x}$ are conditionally independent given \mathcal{F}_n . Hence

$$\begin{aligned}
& \mathbb{E}\left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h\left(\sum_{t \in \mathbb{T}_n, t \neq u} H(n, G_t x) W_{s,1}^{X_t^x}\right) \middle| \mathcal{F}_n\right] \\
&= \sum_{u \in \mathbb{T}_n} H(n, G_u x) \mathbb{E}\left[W_{s,1}^{X_u^x} \middle| \mathcal{F}_n\right] \mathbb{E}\left[h\left(\sum_{t \in \mathbb{T}_n, t \neq u} H(n, G_t x) W_{s,1}^{X_t^x}\right) \middle| \mathcal{F}_n\right] \\
&\leq \sum_{u \in \mathbb{T}_n} H(n, G_u x) h\left(\sum_{t \in \mathbb{T}_n, t \neq u} H(n, G_t x) \mathbb{E}\left[W_{s,1}^{X_t^x} \middle| \mathcal{F}_n\right]\right) \\
&\leq W_{s,n}^x h(W_{s,n}^x),
\end{aligned}$$

where the last two inequalities hold by Jensen's inequality and the fact that $\mathbb{E}\left[W_{s,1}^{X_u^x} \middle| \mathcal{F}_n\right] = 1$ and h is a concave and increasing function. Therefore, from (5.13),

$$\begin{aligned}
\mathbb{E}[W_{s,n+1}^x h(W_{s,n+1}^x)] &\leq \mathbb{E}[W_{s,n}^x h(W_{s,n}^x)] \\
&\quad + \mathbb{E}\left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h\left(H(n, G_u x) W_{s,1}^{X_u^x}\right)\right].
\end{aligned}$$

So by recurrence on n and Fatou's lemma, we obtain

$$\begin{aligned}
\mathbb{E}[W_s^x h(W_s^x)] &\leq \liminf_{n \rightarrow \infty} \mathbb{E}[W_n^x(s) h(W_n^x(s))] \\
&\leq \mathbb{E}[W_{s,1}^x h(W_{s,1}^x)] + \sum_{n=0}^{\infty} \mathbb{E}\left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h\left(H(n, G_u x) W_{s,1}^{X_u^x}\right)\right].
\end{aligned}$$

Hence to prove (5.12), it is enough to prove that

$$\sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} \mathbb{E}\left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h\left(H(n, G_u x) W_{s,1}^{X_u^x}\right)\right] < \infty. \tag{5.14}$$

Note that the hypothesis $\Lambda^*(q_s) - \log m < 0$ implies that there exists $b > 0$ such that

$$\Lambda^*(q_s) - \log m < -b < 0. \quad (5.15)$$

We write

$$\mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(H(n, G_u x) W_{s,1}^{X_u^x} \right) \right] = I_{n,1}(x) + I_{n,2}(x),$$

where

$$I_{n,1}(x) = \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(H(n, G_u x) W_{s,1}^{X_u^x} \right) \mathbf{1}_{\{H(n, G_u x) \leq \frac{e^{-bn}}{r_s(x)}\}} \right],$$

$$I_{n,2}(x) = \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(H(n, G_u x) W_{s,1}^{X_u^x} \right) \mathbf{1}_{\{H(n, G_u x) > \frac{e^{-bn}}{r_s(x)}\}} \right].$$

Control of $I_{n,1}(x)$. Because h is an increase function, $\frac{1}{r_s}$ is bounded from above by a positive constant c and $\mathbb{E}[\sum_{u \in \mathbb{T}_n} H(n, G_u x)] = \mathbb{E}[W_{s,n}^x] = 1$, we have

$$\begin{aligned} I_{n,1}(x) &\leq \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} h \left(\frac{e^{-bn}}{r_s(x)} W_{s,1}^{X_u^x} \right) \right] \\ &\leq \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) \mathbb{E} \left[W_{s,1}^{X_u^x} h \left(\frac{e^{-bn}}{r_s(x)} W_{s,1}^{X_u^x} \right) \middle| \mathcal{F}_n \right] \right] \\ &\leq \mathbb{E} \left[\check{W}_{s,1} h \left(c e^{-bn} \check{W}_{s,1} \right) \right], \end{aligned}$$

where $\check{W}_{s,1} = \max_{x \in \mathcal{S}} W_{s,1}^x$. Set $U = \{c e^{-bn} \check{W}_{s,1} \geq x_0\}$ and its complement $U^c = \{c e^{-bn} \check{W}_{s,1} < x_0\}$. From the definition and the property (5.11) of h , we have

$$\begin{aligned} I_{n,1}(x) &\leq C \mathbb{E} \left[\check{W}_{s,1} \left(1 + \log_+^\delta (c e^{-bn}) + \log_+^\delta \check{W}_{s,1} \right) \mathbf{1}_U \right] \\ &\quad + c_0 \mathbb{E} \left[\check{W}_{s,1} (c e^{-bn} \check{W}_{s,1}) \mathbf{1}_{U^c} \right] \\ &\leq C \mathbb{E} \left[\check{W}_{s,1} \left(1 + \log_+^\delta \check{W}_{s,1} \right) \mathbf{1}_U \right] + c c_0 e^{-bn} \mathbb{E} \left[(\check{W}_{s,1})^2 \mathbf{1}_{U^c} \right] \\ &= I_{n,1,1}(x) + I_{n,1,2}(x). \end{aligned} \quad (5.16)$$

We observe that $ce^{-bn}\check{W}_{s,1} \geq x_0$ leads to $n \leq \frac{1}{b} \log \frac{c\check{W}_{s,1}}{x_0}$. Let $J = \left\lceil \frac{1}{b} \log_+ \frac{c\check{W}_{s,1}}{x_0} \right\rceil$. By Fubini's theorem and hypothesis (2.17), we have

$$\begin{aligned} \sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} I_{n,1,1}(x) &\leq C\mathbb{E} \left[\sum_{n=1}^{\infty} \check{W}_{s,1} \left(1 + \log_+^{\delta} \check{W}_{s,1}\right) \mathbf{1}_{\{n \leq J\}} \right] \\ &\leq C_1 \mathbb{E} \left[\check{W}_{s,1} \left(1 + \log_+^{\delta} \check{W}_{s,1}\right) \log_+ \check{W}_{s,1} \right] < \infty, \end{aligned}$$

and

$$\begin{aligned} \sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} I_{n,1,2}(x) &\leq cc_0 \mathbb{E} \left[(\check{W}_{s,1})^2 \sum_{n=1}^{\infty} e^{-bn} \mathbf{1}_{\{n \geq J+1\}} \right] \\ &\leq C_2 \mathbb{E} \left[(\check{W}_{s,1})^2 e^{-b(J+1)} \right] \\ &\leq C_3 \mathbb{E} \left[\check{W}_{s,1} \right] < \infty. \end{aligned}$$

Hence we conclude that

$$\sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} I_{n,1}(x) < \infty. \quad (5.17)$$

Control of $I_{n,2}(x)$. Using the property (5.11) of the function h , we obtain

$$\begin{aligned} I_{n,2}(x) &\leq C\mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} \left(1 + \log_+^{\delta} W_{s,1}^{X_u^x}\right) \mathbf{1}_{\{H(n, G_u x) > \frac{e^{-bn}}{r_s(x)}\}} \right] \\ &\quad + C\mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) W_{s,1}^{X_u^x} \log_+^{\delta} \left(H(n, G_u x)\right) \mathbf{1}_{\{H(n, G_u x) > \frac{e^{-bn}}{r_s(x)}\}} \right]. \end{aligned}$$

From hypothesis (2.17), we get for each $u \in \mathbb{U}$,

$$\mathbb{E} \left[W_{s,1}^{X_u^x} \left(1 + \log_+^{\delta} W_{s,1}^{X_u^x}\right) \right] < \infty.$$

Taking $C_1 > 0$ sufficiently large, we have

$$\begin{aligned} I_{n,2}(x) &\leq C_1 \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) \left\{1 + \log_+^{\delta} \left(H(n, G_u x)\right)\right\} \mathbf{1}_{\{H(n, G_u x) > \frac{e^{-bn}}{r_s(x)}\}} \right] \\ &\leq C_1 \mathbb{E} \left[\sum_{u \in \mathbb{T}_n} H(n, G_u x) \left\{1 + \log_+^{\delta} \left(H(n, G_u x)\right)\right\} \mathbf{1}_{\{sS_u^x > n(\log[m\kappa(s)] - b)\}} \right]. \end{aligned}$$

By the definition of \mathbb{Q}_s^x , we have

$$I_{n,2}(x) \leq C_1 \mathbb{E}_{\mathbb{Q}_s^x} \left[\left\{1 + \log_+^{\delta} \left(H(n, G_{1_n} x)\right)\right\} \mathbf{1}_{\{sS_u^x > n(\log[m\kappa(s)] - b)\}} \right], \quad (5.18)$$

where $1_n = (1, \dots, 1) \in \mathbb{N}^{*n}$ denotes the sequence of length n whose components are all equal to 1, and $H(n, G_{1_n} x) = \frac{e^{sS_n^x} r_s(X_n)}{[m\kappa(s)]^n r_s(x)}$ by our notation.

It is easy to see that if $s = 0$ then $I_{n,2}(x) = 0$ by the choice of b . Hence we only consider the case where $s \neq 0$. We will prove that

$$\sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[\left(1 + \log_+^\delta H(n, G_{1_n} x) \right) \mathbf{1}_{\{sS_u^x > n(\log[m\kappa(s)] - b)\}} \right] < \infty.$$

We observe that

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_s^x} \left[\left(1 + \log_+^\delta H(n, G_{1_n} x) \right) \mathbf{1}_{\{sS_u^x > n(\log[m\kappa(s)] - b)\}} \right] \\ &= \sum_{k=n}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[\left(1 + \log_+^\delta H(k, G_{1_k} x) \right) \mathbf{1}_{\{k(\log[m\kappa(s)] - b) < sS_k \leq (k+1)(\log[m\kappa(s)] - b)\}} \right] \\ &\leq \left(1 + \log_+ \left(\frac{m\kappa(s)}{r_s(x)} \right) \right) \sum_{k=n}^{\infty} \mathbb{Q}_s^x \left(sS_k > k(\log[m\kappa(s)] - b) \right). \end{aligned}$$

Set $w = \frac{\log[m\kappa(s)] - b}{s}$, which is equal to $\frac{\log m + \Lambda(s) - b}{s}$. By (5.15), we see that $w > \Lambda'(s)$. From Proposition 5.2, we have for some constants $0 < c < 1$, $C > 0$, and all $k \geq 1$,

$$\sup_{x \in \mathcal{S}} \mathbb{Q}_s^x \left(sS_k > ksw \right) \leq \sup_{x \in \mathcal{S}} \mathbb{Q}_s^x \left(\frac{|S_k - k\Lambda'(s)|}{k} > w - \Lambda'(s) \right) \leq Cc^k.$$

This implies that for all $x \in \mathcal{S}$,

$$\sum_{n=1}^{\infty} \mathbb{E}_{\mathbb{Q}_s^x} \left[\left(1 + \log_+^\delta H(n, G_{1_n} x) \right) \mathbf{1}_{\{sS_u^x > n(\log[m\kappa(s)] - b)\}} \right] \leq C_3 \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} c^k < \infty.$$

Therefore, by (5.18),

$$\sup_{x \in \mathcal{S}} \sum_{n=1}^{\infty} I_{n,2}(x) < \infty.$$

Hence Lemma 5.4 is proved. \square

Lemma 5.5. *Assume the conditions of Theorem 2.4. For any $\varepsilon > 0$, there exists a constant $B > 0$ such that for any $x \in \mathcal{S}$ and any $n \geq 0$,*

$$\mathbb{P} \left(\frac{W_s^x}{W_{s,n}^x} - 1 > -\varepsilon \mid \mathcal{F}_n \right) \geq B.$$

Proof of Lemma 5.5. Let $\varepsilon > 0$, $x \in \mathcal{S}$ and $n \geq 0$. Let $T > 0$. For $u \in \mathbb{T}_n$, set $Y_u = W_s^{X_u} - 1$ and

$$Y_u^T := \begin{cases} Y_u & \text{if } Y_u < T \\ T & \text{if } Y_u \geq T. \end{cases}$$

Then $Y_u^T \leq Y_u$, and

$$\begin{aligned} \mathbb{P}\left(\frac{W_s^x}{W_{s,n}^x} - 1 > -\varepsilon \mid \mathcal{F}_n\right) &= \mathbb{P}\left(\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u > -\varepsilon \mid \mathcal{F}_n\right) \\ &\geq \mathbb{P}\left(\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T > -\varepsilon \mid \mathcal{F}_n\right). \end{aligned} \quad (5.19)$$

Using the facts that $Y_u^T \leq T$ and $\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) = 1$, we have

$$\begin{aligned} &\mathbb{E}\left[\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T \mid \mathcal{F}_n\right] \\ &= \mathbb{E}\left[\left(\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T\right) \mathbb{1}_{\left\{\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T \leq -\varepsilon\right\}} \mid \mathcal{F}_n\right] \\ &\quad + \mathbb{E}\left[\left(\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T\right) \mathbb{1}_{\left\{\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T > -\varepsilon\right\}} \mid \mathcal{F}_n\right] \\ &\leq (-\varepsilon) + T \mathbb{P}\left(\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T > -\varepsilon \mid \mathcal{F}_n\right). \end{aligned} \quad (5.20)$$

We now prove that the expectation in the above display is uniformly bounded from below by $-\varepsilon/2$ when T is large enough. By Theorem 2.2, for each $u \in \mathbb{T}_n$, Y_u satisfies

$$\mathbb{E}(Y_u \mid \mathcal{F}_n) = \mathbb{E}\left[W_s^{X_u^x} \mid \mathcal{F}_n\right] - 1 = 0.$$

Using this and the definition of Y_u^T , we have

$$\begin{aligned} \mathbb{E}\left[Y_u^T \mid \mathcal{F}_n\right] &= \mathbb{E}\left[Y_u \mathbb{1}_{\{Y_u < T\}} \mid \mathcal{F}_n\right] + \mathbb{E}\left[T \mathbb{1}_{\{Y_u \geq T\}} \mid \mathcal{F}_n\right] \\ &= \mathbb{E}\left[Y_u (1 - \mathbb{1}_{\{Y_u \geq T\}}) \mid \mathcal{F}_n\right] + \mathbb{E}\left[(T \mathbb{1}_{\{Y_u \geq T\}}) \mid \mathcal{F}_n\right] \\ &= -\mathbb{E}\left[(Y_u - T)_+ \mid \mathcal{F}_n\right], \end{aligned}$$

where $(Y_u - T)_+ = \max(Y_u - T, 0)$. Therefore

$$\mathbb{E}\left[\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T \mid \mathcal{F}_n\right] = -\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) \mathbb{E}\left[(Y_u - T)_+ \mid \mathcal{F}_n\right].$$

Now

$$\begin{aligned} \mathbb{E}\left[(Y_u - T)_+ \mid \mathcal{F}_n\right] &\leq \sup_{y \in S} \mathbb{E}\left[(W_s^y - 1 - T)_+ \mid \mathcal{F}_n\right] = \sup_{y \in S} \mathbb{E}\left[(W_s^y - 1 - T)_+\right] \\ &\leq \sup_{y \in S} \mathbb{E}\left[W_s^y \mathbb{1}_{\{W_s^y > T\}}\right] \xrightarrow{T \rightarrow +\infty} 0, \end{aligned}$$

where the last step holds because by Lemma 5.10, the family of random variables $W_s^y, y \in S$ is uniformly integrable. Let $T > 0$ be sufficiently large such that

$$\sup_{y \in S} \mathbb{E} \left[W_s^y 1_{\{W_s^y > T\}} \right] < \varepsilon/2.$$

Then for all $x \in \mathcal{S}$ and $n \geq 0$,

$$\mathbb{E} \left[\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T \middle| \mathcal{F}_n \right] > -\frac{1}{2} \varepsilon. \quad (5.21)$$

Therefore, from (5.20), we obtain

$$\mathbb{P} \left(\frac{1}{W_{s,n}^x} \sum_{u \in \mathbb{T}_n} H(n, G_u x) Y_u^T > -\varepsilon \middle| \mathcal{F}_n \right) \geq \frac{\varepsilon}{2T}.$$

Hence, it follows from (5.19) that the inequality in the lemma holds with $B = \frac{\varepsilon}{2T}$. \square

Lemma 5.6. *Assume the conditions of Theorem 2.4. For any $0 < a < 1$, there exists a constant $B > 0$ such that for any $x \in \mathcal{S}$ and any $t > 0$,*

$$\mathbb{P}(W_s^x \geq at) \geq B \mathbb{P}(W_{s,*}^x \geq t) \geq B \mathbb{P}(W_s^x \geq t).$$

Proof of Lemma 5.6. The second inequality is evident. We now prove the first one. For $t > 0$, let

$$E_n = \{W_{s,n}^x \geq t, W_{s,k}^x < t \text{ for } 0 \leq k < n\}, \quad n \geq 1.$$

As E_n are pairwise disjoint sets, for each $a \in (0, 1)$ and each $t > 0$,

$$\mathbb{P}(W_s^x > at) \geq \sum_{n=1}^{\infty} \mathbb{P}(W_s^x > at \mid E_n) \mathbb{P}(E_n). \quad (5.22)$$

By (5.9), we have for each $a \in (0, 1)$ and each $t > 0$,

$$\begin{aligned} \mathbb{P}(W_s^x > at \mid E_n) &= \mathbb{P} \left(\frac{W_s^x}{W_{s,n}^x} - 1 > \frac{at}{W_{s,n}^x} - 1 \mid E_n \right) \\ &\geq \mathbb{P} \left(\frac{W_s^x}{W_{s,n}^x} - 1 > a - 1 \mid E_n \right), \end{aligned}$$

where the last step holds because $W_{s,n}^x \geq t$ on E_n . By using the fact that $E_n \in \mathcal{F}_n$ and applying Lemma 5.5, we have

$$\mathbb{P} \left(\frac{W_s^x}{W_{s,n}^x} - 1 > a - 1 \mid E_n \right) = \mathbb{E} \left[\mathbb{P} \left(\frac{W_s^x}{W_{s,n}^x} - 1 > a - 1 \mid \mathcal{F}_n \right) \middle| E_n \right] \geq B > 0,$$

where B is a constant independent of n . It follows from (5.22) that

$$\mathbb{P}(W_s^x > at) \geq B \sum_{n=1}^{\infty} \mathbb{P}(E_n) = B \mathbb{P}(W_{s,*}^x \geq t),$$

which proves the first inequality of the lemma.

□

5.2. Proof of Theorem 2.4. By the definition of $\Lambda^*(q_s)$ and $Z_n^x(dy, dz)$, we have

$$\begin{aligned} I &:= \frac{e^{n\Lambda^*(q_s)}}{m^n r_s(x)} \int_{\mathcal{S} \times \mathbb{R}} f(y)g(z - nq_s)Z_n^x(dy, dz) \\ &= \frac{e^{snq_s}}{[m\kappa(s)]^n r_s(x)} \sum_{u \in \mathbb{T}_n} f(X_u^x)g(S_u^x - nq_s). \end{aligned}$$

Set $h(z) = e^{-sz}g(z)$, $z \in \mathbb{R}$. For $n \geq 1$ and $0 \leq k \leq n$, we have the decomposition

$$\begin{aligned} I &= \frac{1}{[m\kappa(s)]^n r_s(x)} \sum_{u \in \mathbb{T}_n} f(X_u^x)h(S_u^x - nq_s)e^{sS_u^x} \\ &= \frac{1}{[m\kappa(s)]^n r_s(x)} \sum_{u \in \mathbb{T}_{k_n}} e^{sS_u^x} \sum_{v \in \mathbb{T}_{n-k_n}(u)} e^{sS_v^{X_u^x}} f(X_v^{X_u^x})h(S_v^{X_u^x} + S_u^x - nq_s) \\ &= \frac{1}{[m\kappa(s)]^k r_s(x)} \sum_{u \in \mathbb{T}_k} e^{sS_u^x} r_s(X_u^x) \int_{\mathcal{S} \times \mathbb{R}} \frac{f(y)h(z + S_u^x - nq_s)}{r_s(y)} W_{s, n-k_n}^{X_u^x}(dy, dz), \end{aligned}$$

where $W_{s, n-k_n}^{X_u^x}(dy, dz)$ is the probability measure defined as follows: for measurable sets $B \subset \mathcal{S}$ and $C \subset \mathbb{R}$,

$$W_{s, n-k}^{X_u^x}(B, C) = \sum_{v \in \mathbb{T}_{n-k}(u)} \frac{e^{sS_v^{X_u^x}} r_s(X_v^{X_u^x})}{[m\kappa(s)]^{n-k} r_s(X_u^x)} \mathbb{1}_{\{X_v^{X_u^x} \in B, S_v^{X_u^x} \in C\}}.$$

Recall that for $u \in \mathbb{T}_{k_n}$, $0 \leq k_n \leq n$, $H(k_n, G_u x) = \frac{e^{sS_u^x} r_s(X_u^x)}{[m\kappa(s)]^{k_n} r_s(x)}$. From the preceding decomposition of I , we obtain

$$\sigma_s \sqrt{2\pi n} I - W_s^x \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} e^{-sz} g(z) dz = A_n + B_n + C_n, \quad (5.23)$$

where

$$\begin{aligned}
A_n &= \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \int_{\mathcal{S} \times \mathbb{R}} \frac{f(y)}{r_s(y)} h(z + S_u^x - nq_s) \\
&\quad \left(W_{s, n-k_n}^{X_u^x}(dy, dz) - \mathbb{E}_{k_n} [W_{s, n-k_n}^{X_u^x}(dy, dz)] \right), \\
B_n &= \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \left\{ \sigma_s \sqrt{2\pi n} \int_{\mathcal{S} \times \mathbb{R}} \frac{f(y)}{r_s(y)} h(z + S_u^x - nq_s) \right. \\
&\quad \left. \mathbb{E}_{k_n} [W_{s, n-k_n}^{X_u^x}(dy, dz)] - \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} e^{-sz} g(z) dz \right\}, \\
C_n &= \left(W_{s, k_n}^x - W_s^x \right) \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} e^{-sz} g(z) dz.
\end{aligned}$$

We choose k_n as follows. Let β be such that $\frac{3}{2\delta} < \beta < 1$ and $\alpha > \frac{2}{\beta-1-1}$. For each n , let $j = j(n) \in \mathbb{N}$ be such that $j^{\alpha/\beta} \leq n < (j+1)^{\alpha/\beta}$; set $k_n = a_j = \lfloor j^\alpha \rfloor$. Then $k_n \sim n^\beta$.

We will prove that with the above choice of (k_n) , $A_n, B_n, C_n \rightarrow 0$ a.s. By the decomposition (5.23), this will imply Theorem 2.4. By the convergence of the martingale $W_{s, n}^x$ to W_s^x , we have clearly $C_n \rightarrow 0$, \mathbb{P} -a.s. It remains to show that $A_n \rightarrow 0$ and $B_n \rightarrow 0$ \mathbb{P} -a.s.

A) We first prove that $A_n \rightarrow 0$ \mathbb{P} -a.s. For $u \in \mathbb{T}_{k_n}$, write

$$\begin{aligned}
Y_u &= \int_{\mathcal{S} \times \mathbb{R}} \frac{f(y)}{r_s(y)} h(z + S_u^x - nq_s) \left(W_{s, n-k_n}^{X_u^x}(dy, dz) - \mathbb{E}_{k_n} [W_{s, n-k_n}^{X_u^x}(dy, dz)] \right), \\
\tilde{Y}_u &= Y_u \mathbb{1}_{\{|Y_u| < \frac{1}{H(k_n, G_u x)}\}}
\end{aligned}$$

Then

$$\begin{aligned}
A_n &= \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) Y_u \\
&= \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \mathbb{E}_{k_n} [\tilde{Y}_u] + \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) (Y_u - \tilde{Y}_u) \\
&\quad + \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) (\tilde{Y}_u - \mathbb{E}_{k_n} [\tilde{Y}_u]) \\
&= A_{n,1} + A_{n,2} + A_{n,3},
\end{aligned}$$

with $A_{n,i}$ denoting the corresponding sum. We will show that each of these three terms tends to zero a.s. as $n \rightarrow \infty$. We divide the proof into 3 steps.

Step 1. We prove that $A_{n,1} \xrightarrow{n \rightarrow \infty} 0$ a.s. From the fact that $0 = \mathbb{E}_{k_n}[Y_u] = \mathbb{E}_{k_n}[\tilde{Y}_u] + \mathbb{E}_{k_n}[Y_u \mathbb{1}_{\{|Y_u| > \frac{1}{H(k_n, G_u x)}\}}]$, we have

$$|A_{n,1}| \leq \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \mathbb{E}_{k_n} [|Y_u| \mathbb{1}_{\{|Y_u| > \frac{1}{H(k_n, G_u x)}\}}]. \quad (5.24)$$

Notice that for $C > 0$ large enough, $\sup_{(y,z) \in \mathcal{S} \times \mathbb{R}} \left| \frac{f(y)h(z)}{r_s(y)} \right| \leq C$. Using this and the fact that $\mathbb{E}_{k_n}[W_{s,n-k_n}^{X_u^x}(\mathcal{S}, \mathbb{R})] = \mathbb{E}_{k_n}[W_{s,n-k_n}^{X_u^x}] = 1$, we obtain

$$|Y_u| \leq C(W_{s,n-k_n}^{X_u^x} + 1). \quad (5.25)$$

This implies that

$$\begin{aligned} \mathbb{E}_{k_n} [|Y_u| \mathbb{1}_{\{|Y_u| > \frac{1}{H(k_n, G_u x)}\}}] &\leq C \mathbb{E}_{k_n} \left[(W_{s,n-k_n}^{X_u^x} + 1) \mathbb{1}_{\{W_{s,n-k_n}^{X_u^x} + 1 > \frac{1}{CH(k_n, G_u x)}\}} \right] \\ &=: \mathcal{E}(W_{s,n-k_n}^{X_u^x}). \end{aligned}$$

Let

$$U = \left\{ H(k_n, G_u x) > \frac{e^{-bk_n}}{r_s(x)} \right\}, \quad (5.26)$$

where b is chosen as in the proof of Lemma 5.4. Denote its complement by $U^c = \left\{ H(k_n, G_u x) \leq \frac{e^{-bk_n}}{r_s(x)} \right\}$. We have

$$|A_{n,1}| \leq I_{n,1} + I_{n,2}, \quad (5.27)$$

where

$$\begin{aligned} I_{n,1} &= \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \mathcal{E}(W_{s,n-k_n}^{X_u^x}) \mathbb{1}_U, \\ I_{n,2} &= \sigma_s \sqrt{2\pi n} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \mathcal{E}(W_{s,n-k_n}^{X_u^x}) \mathbb{1}_{U^c}. \end{aligned}$$

For the first term $I_{n,1}$, by using the facts that $\mathcal{E}(W_{s,n-k_n}^{X_u^x}) \leq 2C$ and $U \subseteq \{sS_u^x > k_n(\log[m\kappa(s)] - b)\}$, and the definition of \mathbb{Q}_s^x , we have

$$\begin{aligned} \mathbb{E}I_{n,1} &\leq \sigma_s 2C \sqrt{2\pi n} \mathbb{E} \left[\sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \mathbb{1}_{\{sS_u^x > k_n(\log[m\kappa(s)] - b)\}} \right] \\ &= \sigma_s 2C \sqrt{2\pi n} \mathbb{Q}_s^x \left(sS_{1_{k_n}}^x > k_n(\log[m\kappa(s)] - b) \right) \end{aligned} \quad (5.28)$$

(recall that $1_{k_n} = (1, \dots, 1)$ is the sequence of length k_n whose components are all equal to 1). If $s = 0$, then $\mathbb{E}[I_{n,1}] = 0$ by the choice of b . Hence we only need to consider the case where $s \neq 0$, which we assume below. As in the proof of Lemma 5.4, setting $w = \frac{\log[m\kappa(s)] - b}{s}$, we have $w > \Lambda'(s)$ by the

choice of b . From Proposition 5.2, there are constants $0 < c < 1$ and $C_1 > 0$ such that

$$\begin{aligned} \mathbb{Q}_s^x \left(sS_{1_{k_n}}^x > k_n(\log[m\kappa(s)] - b) \right) &\leq \mathbb{Q}_s^x \left(\frac{|S_{1_{k_n}}^x - k_n\Lambda'(s)|}{k_n} > w - \Lambda'(s) \right) \\ &\leq C_1 c^{k_n}. \end{aligned}$$

Hence, by (5.28), we get

$$\sum_{n=1}^{\infty} \mathbb{E}[I_{n,1}] \leq C_2 \sum_{n=1}^{\infty} c^{n\beta} n^{\frac{1}{2}} < \infty. \quad (5.29)$$

For the second term $I_{n,2}$, we see that

$$\begin{aligned} \mathcal{E}(W_{s,n-k_n}^{X_u^x}) \mathbf{1}_{U^c} &\leq C \mathbf{1}_{U^c} \mathbb{E}_{k_n} \left[(W_{s,n-k_n}^{X_u^x} + 1) \mathbf{1}_{\{W_{s,n-k_n}^{X_u^x} + 1 \geq \frac{r_s(x)}{C e^{-bk_n}}\}} \right] \\ &\leq \frac{C \mathbf{1}_{U^c}}{\log_+^\delta \left(\frac{r_s(x)}{C e^{-bk_n}} \right)} \mathbb{E}_{k_n} \left[(W_{s,n-k_n}^{X_u^x} + 1) \log^\delta (W_{s,n-k_n}^{X_u^x} + 1) \right] \\ &\leq \frac{CC_3 \mathbf{1}_{U^c}}{1 + \log_+^\delta \left(\frac{r_s(x)}{C} \right) + (bk_n)^\delta}, \end{aligned}$$

where $C_3 = \sup_{x \in \mathcal{S}} \mathbb{E} \left[(W_{s,*}^x + 1) \log^\delta (W_{s,*}^x + 1) \right] < \infty$ by Proposition 5.3. Therefore, since $k_n^\delta \sim j^{\alpha\delta} \sim n^{\beta\delta}$ and $\beta\delta > 3/2$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{E}[I_{n,2}] &\leq \sum_{n=1}^{\infty} \frac{CC_3 \sigma_s \sqrt{2\pi n}}{1 + \log_+^\delta \left(\frac{r_s(x)}{C} \right) + (bk_n)^\delta} \mathbb{E}[W_{s,k_n}^x] \\ &\leq C_4 \sum_{n=1}^{\infty} \frac{\sqrt{n}}{1 + n^{\beta\delta}} < \infty. \end{aligned} \quad (5.30)$$

Putting together (5.27), (5.29) and (5.30), we get

$$\sum_{n=1}^{\infty} \mathbb{E}[|A_{n,1}|] < \infty.$$

Thus $\sum_{n=0}^{\infty} |A_{n,1}| < \infty$ a.s., which implies that $A_{n,1} \xrightarrow{n \rightarrow \infty} 0$ a.s.

Step 2. We prove that $A_{n,2} \xrightarrow{n \rightarrow \infty} 0$ a.s. By the definition of \tilde{Y}_u and inequality (5.25), for any $\varepsilon > 0$,

$$\begin{aligned}
& \mathbb{P}_{k_n}(|A_{n,2}| > \varepsilon) \leq \mathbb{P}_{k_n} \left(\sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) (Y_u - \tilde{Y}_u) \neq 0 \right) \\
& \leq \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n} (Y_u \neq \tilde{Y}_u) = \sum_{u \in \mathbb{T}_{k_n}} \mathbb{P}_{k_n} \left(|Y_u| \geq \frac{1}{H(k_n, G_u x)} \right) \\
& \leq \sum_{u \in \mathbb{T}_{k_n}} \mathbb{E}_{k_n} \left[\mathbf{1}_{\{W_{s,n-k_n}^{X_{k_n}^x} + 1 > \frac{1}{CH(k_n, G_u x)}\}} \right] \\
& \leq \sum_{u \in \mathbb{T}_{k_n}} \mathbb{E}_{k_n} \left[CH(k_n, G_u x) (W_{s,n-k_n}^{X_{k_n}^x} + 1) \mathbf{1}_{\{W_{s,n-k_n}^{X_{k_n}^x} + 1 > \frac{1}{CH(k_n, G_u x)}\}} \right] \\
& = \frac{C}{\sigma_s \sqrt{2\pi n}} (I_{n,1} + I_{n,2}),
\end{aligned}$$

where $I_{n,1}$ and $I_{n,2}$ are defined in Step 1. Therefore, from (5.29) and (5.30), we get

$$\sum_{n=1}^{\infty} \mathbb{P}(|A_{n,2}| > \varepsilon) \leq \sum_{n=1}^{\infty} \frac{C}{\sigma_s \sqrt{2\pi n}} \mathbb{E} (I_{n,1} + I_{n,2}) < \infty.$$

So by the lemma of Borel-Cantelli, we conclude that $A_{n,2} \rightarrow 0$ \mathbb{P} -a.s.

Step 3. We prove that $A_{n,3} \xrightarrow{n \rightarrow \infty} 0$ a.s. By Markov's inequality and von Bahr-Esseen's inequality [53, Theorem 2] or Marcinkiewicz-Zygmund's inequality [20, Theorem 1.5],

we have for any $\varepsilon > 0$ and $1 < \theta < 2$

$$\begin{aligned}
\mathbb{P}_{k_n}(|A_{n,3}| > \varepsilon) & \leq \frac{(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \mathbb{E}_{k_n} \left[\left| \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) (\tilde{Y}_u - \mathbb{E}_{k_n} \tilde{Y}_u) \right|^\theta \right] \\
& \leq \frac{2(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbb{E}_{k_n} [|\tilde{Y}_u - \mathbb{E}_{k_n} \tilde{Y}_u|^\theta] \\
& \leq \frac{4(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbb{E}_{k_n} [|\tilde{Y}_u|^\theta] \\
& \leq K_{n,1} + K_{n,2},
\end{aligned} \tag{5.31}$$

where

$$\begin{aligned}
K_{n,1} & = \frac{4(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbf{1}_U \mathbb{E}_{k_n} [|\tilde{Y}_u|^\theta], \\
K_{n,2} & = \frac{4(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbf{1}_{U^c} \mathbb{E}_{k_n} [|\tilde{Y}_u|^\theta].
\end{aligned}$$

For the first term $K_{n,1}$, by the definition of \tilde{Y}_u , it is easy to see that

$$\begin{aligned} K_{n,1} &= \frac{4(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbb{1}_U \mathbb{E}_{k_n} \left[|Y_u|^\theta \mathbb{1}_{\{|Y_u| < \frac{1}{H(k_n, G_u x)}\}} \right] \\ &\leq \frac{4(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \mathbb{1}_U \mathbb{E}_{k_n} \left[|Y_u| \mathbb{1}_{\{|Y_u| < \frac{1}{H(k_n, G_u x)}\}} \right], \end{aligned}$$

where the last inequality holds because on $\{|Y_u| < \frac{1}{H(k_n, G_u x)}\}$, we have $[H(k_n, G_u x)|Y_u|]^{\theta-1} < 1$. Using the facts that $\mathbb{E}_{k_n}[|Y_u|] \leq 2$ and $U \subset \{sS_u^x > k_n(\log[m\kappa(s)] - b)\}$, we get that

$$\mathbb{E}[K_{n,1}] \leq \frac{8(\sigma_s \sqrt{2\pi n})^\theta}{\varepsilon^\theta} \mathbb{Q}_s^x \left(sS_{1_{k_n}}^x > k_n(\log[m\kappa(s)] - b) \right).$$

Similar to (5.29), with the same reason we get, for some constants $0 < c < 1$ and $C > 0$,

$$\sum_{n=1}^{\infty} \mathbb{E}[K_{n,1}] < C \sum_{n=1}^{\infty} c^{n^\beta} n^{\frac{\theta}{2}} < \infty. \quad (5.32)$$

For the second term $K_{n,2}$, using the definition of \tilde{Y}_u , Fubini's theorem and inequality (5.25), we have

$$\begin{aligned} \mathbb{E}_{k_n}[|\tilde{Y}_u|^\theta] &= \int_0^\infty \theta y^{\theta-1} \mathbb{P}_{k_n}(|\tilde{Y}_u| > y) dy \\ &= \theta \int_0^\infty y^{\theta-1} \mathbb{P}_{k_n}(|Y_u| \mathbb{1}_{\{|Y_u| < \frac{1}{H(k_n, G_u x)}\}} > y) dy \\ &= \theta \mathbb{E}_{k_n} \left[\int_0^{\frac{1}{H(k_n, G_u x)}} y^{\theta-1} \mathbb{1}_{\{|Y_u| \mathbb{1}_{\{|Y_u| < \frac{1}{H(k_n, G_u x)}\}} > y\}} dy \right] \\ &\leq \theta \int_0^{\frac{1}{H(k_n, G_u x)}} y^{\theta-1} \mathbb{P}_{k_n}(W_{s,n-k_n}^{X_u^x} + 1 > \frac{y}{C}) dy. \end{aligned}$$

By the change of variables $z = (\frac{y}{C})^{\theta-1}$, we obtain

$$\begin{aligned} \mathbb{E}_{k_n}[|\tilde{Y}_u|^\theta] &\leq \theta C^\theta \int_0^{[CH(k_n, G_u x)]^{1-\theta}} z^{\frac{1}{\theta-1}} \mathbb{P}_{k_n} \left((W_{s,n-k_n}^{X_u^x} + 1)^{\theta-1} > z \right) dz \\ &\leq \theta C^\theta \int_0^{[CH(k_n, G_u x)]^{1-\theta}} \mathbb{E}_{k_n} \left[(W_{s,n-k_n}^{X_u^x} + 1) \mathbb{1}_{\left\{ (W_{s,n-k_n}^{X_u^x} + 1)^{\theta-1} > z \right\}} \right] dz. \end{aligned} \quad (5.33)$$

We split the above integral according to $z \in [0, e]$ and $z \in (e, [CH(k_n, G_u x)]^{1-\theta})$. For the integral over $z \in [0, e]$, we use

$$\mathbb{E}_{k_n} \left[(W_{s,n-k_n}^{X_u^x} + 1) \mathbb{1}_{\left\{ (W_{s,n-k_n}^{X_u^x} + 1)^{\theta-1} > z \right\}} \right] \leq \mathbb{E}_{k_n} (W_{s,n-k_n}^{X_u^x} + 1) = 2.$$

For the integral over $z \in (e, [CH(k_n, G_u x)]^{1-\theta}]$, we use

$$\begin{aligned} & \mathbb{E}_{k_n} \left[(W_{s, n-k_n}^{X_u} + 1) \mathbf{1}_{\{(W_{s, n-k_n}^{X_u} + 1)^{\theta-1} > z\}} \right] \\ & \leq \frac{\theta - 1}{\log^\delta z} \mathbb{E}_{k_n} \left[(W_{s, n-k_n}^{X_{k_n}} + 1) \log^\delta (W_{s, n-k_n}^{X_{k_n}} + 1) \right] \leq \frac{C_3(\theta - 1)}{\log^\delta z}, \end{aligned}$$

where $C_3 = \sup_{x \in \mathcal{S}} \mathbb{E} \left[(W_{s,*}^x + 1) \log^\delta (W_{s,*}^x + 1) \right] < \infty$ by Proposition 5.3. Hence, by (5.33),

$$\begin{aligned} \mathbb{E}_{k_n} [|\tilde{Y}_u|^\theta] & \leq 2\theta C^2 \theta e \\ & \quad + \theta(\theta - 1) C^\theta C_3 \mathbf{1}_{\{[CH(k_n, G_u x)]^{1-\theta} > e\}} \int_e^{[CH(k_n, G_u x)]^{1-\theta}} \frac{1}{\log^\delta z} dz. \end{aligned} \quad (5.34)$$

By the definition of $K_{n,2}$ and inequality (5.34), we get

$$\begin{aligned} K_{n,2} & \leq C_4 n^{\frac{\theta}{2}} \sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbf{1}_{U^c} \\ & \quad \cdot \left(1 + \mathbf{1}_{\{[CH(k_n, G_u x)]^{1-\theta} > e\}} \int_e^{[CH(k_n, G_u x)]^{1-\theta}} \frac{1}{\log^\delta z} dz \right). \end{aligned} \quad (5.35)$$

Now consider the integral in the last expression. Take a constant $1 < d_2 < e^b$, we see that on U^c , we have $\frac{1}{CH(k_n, G_u x)} \geq \frac{1}{C} r_s(x) e^{bk_n} \geq C_5 d_2^{k_n}$. Let $n_0 \in \mathbb{N}^*$ be large enough such that $[C_5 d_2^{k_{n_0}}]^{\theta-1} > e$. Using $\log z \geq 1$ for $z \in [e, [C_5 d_2^{k_n}]^{\theta-1}]$, and $\log z \geq (\theta - 1) \log(C_5 d_2^{k_n})$ for $z > [C_5 d_2^{k_n}]^{\theta-1}$, we see that when $[CH(k_n, G_u x)]^{1-\theta} > e$ and $n > n_0$, we have

$$\begin{aligned} & \int_e^{[CH(k_n, G_u x)]^{1-\theta}} \frac{1}{\log^\delta z} dz \\ & = \int_e^{[C_5 d_2^{k_n}]^{\theta-1}} \frac{1}{\log^\delta z} dz + \int_{[C_5 d_2^{k_n}]^{\theta-1}}^{[CH(k_n, G_u x)]^{1-\theta}} \frac{1}{\log^\delta z} dz \\ & \leq [C_5 d_2^{k_n}]^{\theta-1} + \frac{[CH(k_n, G_u x)]^{1-\theta}}{((\theta - 1) \log(C_5 d_2^{k_n}))^\delta} \\ & \leq C_6 \left(d_2^{(\theta-1)k_n} + \frac{[CH(k_n, G_u x)]^{1-\theta}}{k_n^\delta} \right). \end{aligned} \quad (5.36)$$

When $[CH(k_n, G_u x)]^{1-\theta} > e$ and $n \leq n_0$, the above inequality (5.36) remains valid by choosing C_6 large enough such that $C_6 > k_{n_0}^\delta / C^{\theta-1}$, since

for all $1 \leq n \leq n_0$,

$$\int_e^{[CH(k_n, G_u x)]^{1-\theta}} \frac{1}{\log^\delta z} dz \leq [CH(k_n, G_u x)]^{1-\theta} \leq \frac{C_6 [H(k_n, G_u x)]^{1-\theta}}{k_n^\delta}.$$

From (5.35) and (5.36) we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{E}[K_{n,2}] &\leq C_6 \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} \mathbb{E} \left[\sum_{u \in \mathbb{T}_{k_n}} H^\theta(k_n, G_u x) \mathbf{1}_{U^c} \right. \\ &\quad \cdot \left. \left(1 + d_2^{(\theta-1)k_n} + \frac{[H(k_n, G_u x)]^{1-\theta}}{k_n^\delta} \right) \right] \\ &\leq \frac{C_6}{r_s(x)^{\theta-1}} \sum_{n=1}^{\infty} \mathbb{E}[W_{s,k_n}^x] \frac{n^{\frac{\theta}{2}} (1 + d_2^{(\theta-1)k_n})}{e^{b(\theta-1)k_n}} + C_6 \sum_{n=1}^{\infty} \mathbb{E}[W_{s,k_n}^x] \frac{n^{\frac{\theta}{2}}}{k_n^\delta}, \end{aligned} \quad (5.37)$$

where the last inequality holds because on U^c (see Eq. (5.26)), $H(k_n, G_u x) \leq \frac{1}{r_s(x)e^{bk_n}}$, so that $H^\theta(k_n, G_u x) = H(k_n, G_u x)H^{\theta-1}(k_n, G_u x) \leq \frac{H(k_n, G_u x)}{[r_s(x)e^{bk_n}]^{\theta-1}}$ (for the second term we just use the identity $H^\theta(k_n, G_u x)[H(k_n, G_u x)]^{1-\theta} = H(k_n, G_u x)$). We choose θ sufficiently close to 1. Since $\mathbb{E}[W_{s,k_n}^x] = 1$, $k_n \sim j^\alpha \sim n^\beta$, $1 < d_2 < e^b$ and $\beta\delta > \frac{3}{2}$, the two series $\sum_{n=1}^{\infty} \frac{n^{\frac{\theta}{2}}(1+d^{(\theta-1)k_n})}{e^{b(\theta-1)k_n}}$ and $\sum_{n=1}^{\infty} \frac{n^{\frac{\theta}{2}}}{1+k_n^\delta}$ converge. Therefore from (5.37), we get

$$\sum_{n=1}^{\infty} \mathbb{E}K_{n,2} < \infty. \quad (5.38)$$

Combining (5.31), (5.32) and (5.38), we conclude that for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \mathbb{P}(|A_{n,3}| > \varepsilon) < \infty.$$

By the Lemma of Borel-Cantelli, it follows that $A_{n,3} \xrightarrow{n \rightarrow \infty} 0$ a.s.

B) We then prove that $B_n \rightarrow 0$ \mathbb{P} -a.s. By the definition of $W_{s,n-k_n}^{X_u^x}(dy, dz)$, the branching property, and the definition of \mathbb{Q}_s^x (cf.(4.1)), we obtain successively,

$$\begin{aligned}
& \int_{\mathcal{S} \times \mathbb{R}} \frac{f(y)}{r_s(y)} h(z + S_u^x - nq_s) \mathbb{E}_{k_n} [W_{s,n-k_n}^{X_u^x}(dy, dz)] \\
&= \frac{\mathbb{E}_{k_n} \left[\sum_{v \in \mathbb{T}_{n-k_n}(u)} e^{sS_v^{X_u^x}} r_s(X_v^{X_u^x}) \frac{f(X_v^{X_u^x})}{r_s(X_v^{X_u^x})} h(S_v^{X_u^x} + S_u^x - nq_s) \right]}{[m \kappa(s)]^{n-k_n} r_s(X_u^x)} \\
&= \frac{\mathbb{E}_{k_n} \left[e^{sS_{n-k_n}^{X_u^x}} r_s(X_{n-k_n}^{X_u^x}) \frac{f(X_{n-k_n}^{X_u^x})}{r_s(X_{n-k_n}^{X_u^x})} h(S_{n-k_n}^{X_u^x} + S_u^x - nq_s) \right]}{[\kappa(s)]^{n-k_n} r_s(X_u^x)} \\
&= \mathbb{E}_{\mathbb{Q}_s^x} \left[\frac{f(X_{n-k_n}^{X_u^x})}{r_s(X_{n-k_n}^{X_u^x})} h(S_{n-k_n}^{X_u^x} - (n-k_n)q_s + S_u^x - k_nq_s) \mid \mathcal{F}_{k_n} \right] \quad (5.39) \\
&=: \text{RHS}_{(5.39)}.
\end{aligned}$$

Hence, by the definition of B_n ,

$$\begin{aligned}
B_n &= \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \left[\sigma_s \sqrt{2\pi n} \text{RHS}_{(5.39)} - \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} h(z) dz \right] \\
&= \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \sqrt{\frac{2\pi n}{n-k_n}} \left[\sigma_s \sqrt{n-k_n} \text{RHS}_{(5.39)} \right. \\
&\quad \left. - \sqrt{\frac{n-k_n}{2\pi n}} \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} h(z) dz \right].
\end{aligned}$$

By Proposition 5.1,

$$\begin{aligned}
& \left| \sigma_s \sqrt{n-k_n} \text{RHS}_{(5.39)} - \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} h(z) \phi \left(\frac{z - S_u^x + k_n q_s}{\sigma_s \sqrt{n-k_n}} \right) dz \right| \\
&\leq \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| \sigma_s \sqrt{n-k_n} \mathbb{E}_{\mathbb{Q}_s^x} \left[\frac{f(X_{n-k_n}^x)}{r_s(X_{n-k_n}^x)} h(y + S_{n-k_n}^x - (n-k_n)q_s) \right] \right. \\
&\quad \left. - \pi_s \left(\frac{f}{r_s} \right) \int_{\mathbb{R}} h(z) \phi \left(\frac{z - y}{\sigma_s \sqrt{n-k_n}} \right) dz \right| \\
&\xrightarrow{n \rightarrow \infty} 0.
\end{aligned}$$

Since $\sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \sqrt{\frac{2\pi n}{n-k_n}} \sim W_{s, k_n}^x \sqrt{2\pi} \rightarrow W_s^x \sqrt{2\pi}$ a.s., it follows that $B_n \xrightarrow{n \rightarrow \infty} 0$ a.s. if and only if

$$\left| \int_{\mathbb{R}} h(z) \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \left[\sqrt{\frac{2\pi n}{n-k_n}} \phi\left(\frac{z - S_u^x + k_n q_s}{\sigma_s \sqrt{n-k_n}}\right) - 1 \right] dz \right| \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.}$$

We shall prove this convergence by the dominated convergence theorem. Notice that the function in the above integral is bounded by $Ch(z)W_{s, k_n}^x \leq CW_{s, * }^x h(z)$ which is integrable on \mathbb{R} . So it suffices to prove that for $z \in \mathbb{R}$,

$$D_n(z) := \left| \sqrt{\frac{2\pi n}{n-k_n}} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \phi\left(\frac{z - S_u^x + k_n q_s}{\sigma_s \sqrt{n-k_n}}\right) - W_{s, k_n}^x \right| \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.} \quad (5.40)$$

Using the fact that $|\phi(x) - \phi(y)| \leq C|x - y|$, we see that for all $z \in \mathbb{R}$

$$\begin{aligned} D_n(z) &\leq \sqrt{\frac{2\pi n}{n-k_n}} \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \left| \phi\left(\frac{z - S_u^x + k_n q_s}{\sigma_s \sqrt{n-k_n}}\right) - \phi\left(\frac{z}{\sigma_s \sqrt{n-k_n}}\right) \right| \\ &\quad + W_{s, k_n}^x \left| \sqrt{\frac{2\pi n}{n-k_n}} \phi\left(\frac{z}{\sigma_s \sqrt{n-k_n}}\right) - 1 \right| \\ &\leq C \sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \frac{|S_u^x - k_n q_s|}{\sigma_s \sqrt{n-k_n}} + W_{s, k_n}^x \left| \sqrt{\frac{2\pi n}{n-k_n}} \phi\left(\frac{z}{\sigma_s \sqrt{n-k_n}}\right) - 1 \right|. \end{aligned}$$

It is clear that the second term converges to 0 a.s. as $n \rightarrow \infty$. For the first term, we use the same argument as the proof of (3.19), noting that

$$\mathbb{E} \left[\sum_{u \in \mathbb{T}_{k_n}} H(k_n, G_u x) \frac{|S_u^x - k_n q_s|}{\sigma_s \sqrt{n-k_n}} \right] = \frac{\mathbb{E}_{\mathbb{Q}_s^x} [|S_{k_n}^x - k_n q_s|]}{\sigma_s \sqrt{n-k_n}},$$

and (see [17, Lemma 7.1] for $s > 0$ and [55, Proposition 3.14] for $s \leq 0$)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\mathbb{Q}_s^x} (S_n^x - n q_s)^2 = \sigma_s^2.$$

Therefore, (5.40) holds. This shows that $B_n \xrightarrow{n \rightarrow \infty} 0$ a.s. The proof of Theorem 2.4 is therefore completed.

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