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BERRY-ESSEEN BOUNDS AND MODERATE DEVIATIONS FOR THE RANDOM WALK ON $GL_d(\mathbb{R})$

HUI XIAO, ION GRAMA, AND QUANSHENG LIU ¹

ABSTRACT. Let $(g_n)_{n \geq 1}$ be a sequence of independent and identically distributed random real $d \times d$ matrices with law μ . Consider the random walk $G_n := g_n \dots g_1$ on the general linear group $GL_d(\mathbb{R})$ and the Markov chain $X_n^x := G_n x / |G_n x|$, where x is a starting point on the unit sphere \mathbb{S}^{d-1} . Denote respectively by $\|G_n\|$ and $\rho(G_n)$ the operator norm and the spectral radius of G_n . For $\log \|G_n\|$ and $\log \rho(G_n)$, we prove moderate deviation principles under exponential moment and strong irreducibility conditions on μ ; we also establish moderate deviation expansions in the normal range $[0, o(n^{1/6})]$ and Berry-Esseen bounds under the additional proximality condition on μ . Similar results are found for the couples $(X_n^x, \log \|G_n\|)$ and $(X_n^x, \log \rho(G_n))$ with target functions.

1. INTRODUCTION

1.1. Background and previous results. For any integer $d \geq 2$, denote by $\mathbb{G} = GL_d(\mathbb{R})$ the general linear group of real invertible $d \times d$ matrices. We equip \mathbb{R}^d with the canonical Euclidean norm $|\cdot|$. Let $\mathbb{P}^{d-1} = \{x \in \mathbb{R}^d, |x| = 1\} / \pm$ be the projective space in \mathbb{R}^d , obtained from the unit sphere \mathbb{S}^{d-1} by identifying $-x$ with x . For any $g \in \mathbb{G}$, denote by $\|g\| = \sup_{x \in \mathbb{P}^{d-1}} |gx|$ its operator norm, and by $\rho(g)$ its spectral radius. Let $(g_n)_{n \geq 1}$ be a sequence of independent and identically distributed (i.i.d.) random matrices with law μ on the group \mathbb{G} . Consider the random walk $G_n = g_n \dots g_1$ on \mathbb{G} , and the Markov chain $X_n^x := G_n x / |G_n x|$ on \mathbb{P}^{d-1} with any starting point $x \in \mathbb{P}^{d-1}$. The goal of this paper is to investigate Berry-Esseen type bounds and moderate deviation asymptotics for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$, and more generally, for the couples $(X_n^x, \log \|G_n\|)$ and $(X_n^x, \log \rho(G_n))$ with target functions on X_n^x .

¹Corresponding author, quansheng.liu@univ-ubs.fr

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Let Γ_μ be the smallest closed subsemigroup of \mathbb{G} generated by the support of μ . Denote $N(g) = \max\{\|g\|, \|g^{-1}\|\}$ for any $g \in \mathbb{G}$. Consider the following conditions.

A1 (Exponential moments). *There is a constant $\delta > 0$ such that $\mathbb{E}[N(g_1)^\delta] < \infty$.*

A2 (Strong irreducibility). *No finite union of proper subspaces of \mathbb{R}^d is Γ_μ -invariant, that is, there do not exist a finite number of proper subspaces V_1, \dots, V_m of \mathbb{R}^d such that $g(V_1 \cup \dots \cup V_m) = V_1 \cup \dots \cup V_m$ for all $g \in \Gamma_\mu$.*

A3 (Proximality). *Γ_μ contains at least one matrix with a unique eigenvalue of maximal modulus.*

Notice that in condition **A2**, Γ_μ can be replaced by G_μ , the smallest closed subgroup of \mathbb{G} generated by the support of μ . In fact, the set $S = \{g \in \mathbb{G} : g(V_1 \cup \dots \cup V_m) = V_1 \cup \dots \cup V_m\}$ is a subgroup of \mathbb{G} , so that $\Gamma_\mu \subset S$ if and only if $G_\mu \subset S$, which means that $V_1 \cup \dots \cup V_m$ is Γ_μ -invariant if and only if $V_1 \cup \dots \cup V_m$ is G_μ -invariant.

The topic of products of random matrices has a very rich history and has been studied by many authors. The main distinct feature compared with the case of a sum of i.i.d. real-valued random variables lies in the fact that the matrix product is no longer commutative. Furstenberg and Kesten [12] first established the strong law of large numbers for the operator norm $\|G_n\|$: if $\mathbb{E}[\max\{0, \log \|g_1\|\}] < \infty$, then as $n \rightarrow \infty$, $\frac{1}{n} \log \|G_n\| \rightarrow \lambda$ almost surely, where λ is a constant called top Lyapunov exponent of the random walk (G_n) . This result can be considered as an immediate consequence of Kingman's subadditive ergodic theorem [19]. The central limit theorem for $\|G_n\|$ is due to Le Page [20] (see also Bougerol and Lacroix [7]): if conditions **A1**, **A2** and **A3** hold, then for any $y \in \mathbb{R}$ and any continuous function φ on \mathbb{P}^{d-1} , uniformly in $x \in \mathbb{P}^{d-1}$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{\sigma\sqrt{n}} \leq y \right\}} \right] = \nu(\varphi) \Phi(y), \quad (1.1)$$

where $\sigma^2 > 0$ is the asymptotic variance of the random walk $(G_n)_{n \geq 1}$, Φ is the standard normal distribution function, and ν is the unique stationary probability measure of the Markov chain $(X_n^x)_{n \geq 0}$. Recently, using Gordin's martingale approximation method, Benoist and Quint [4] have relaxed the exponential moment condition **A1** to the optimal second moment condition that $\mathbb{E}[\log^2 N(g_1)] < \infty$.

Similar law of large numbers and central limit theorem have been known for the spectral radius $\rho(G_n)$. Using the Hölder regularity of the stationary measure ν (see [15, 13]), Guivarc'h [13] has established the strong law of large numbers for $\rho(G_n)$: under conditions **A1**, **A2** and **A3**, $\frac{1}{n} \log \rho(G_n) \rightarrow \lambda$

almost surely. Recently, under the same conditions, Benoist and Quint [5] established the central limit theorem for $\rho(G_n)$: for any $y \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{\log \rho(G_n) - n\lambda}{\sigma\sqrt{n}} \leq y\right) = \Phi(y).$$

Further improvements have been done very recently: Aoun and Sert [2] proved the strong law of large numbers for $\rho(G_n)$ assuming only the second moment condition $\mathbb{E}[\log^2 N(g_1)] < \infty$, while Aoun [1] proved the central limit theorem for $\rho(G_n)$ under the second moment condition, the strong irreducibility condition **A2** and the unboundedness assumption of the semi-group Γ_μ .

Very little has been known about the Berry-Esseen bounds and moderate and large deviations, for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$. For Berry-Esseen type bounds, Cuny, Dedecker and Jan [10] (see also Cuny, Dedecker and Merlevède [11] in a more general setting) have recently established the following result about the rate of convergence in the central limit theorem for $\|G_n\|$: assuming $\mathbb{E}[\log^3 N(g_1)] < \infty$, **A2** and **A3**, we have

$$\sup_{y \in \mathbb{R}} \left| \mathbb{P}\left(\frac{\log \|G_n\| - n\lambda}{\sigma\sqrt{n}} \leq y\right) - \Phi(y) \right| \leq \frac{C\sqrt{\log n}}{n^{1/4}}. \quad (1.2)$$

However, the rate of convergence in the central limit theorem for the couple $(X_n^x, \log \|G_n\|)$ (cf. (1.1)) has not been known in the literature. For the spectral radius $\rho(G_n)$ and the couple $(X_n^x, \log \rho(G_n))$, such type of result has not yet been considered.

Moderate deviations have not yet been studied neither for $\|G_n\|$ nor for $\rho(G_n)$, to the best of our knowledge. For large deviations, the upper tail large deviation principle for $\|G_n\|$ has been established in [21] and [22] under different conditions; it is conjectured in [21] that the usual large deviation principle would hold for $\rho(G_n)$.

1.2. Objectives. In this paper, we shall establish Berry-Esseen type bounds and moderate deviation results for both the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$. Such kinds of results are important in applications because they give the rate of convergence in the central limit theorem and in the law of large numbers.

Our first objective is to establish the following Berry-Esseen type bound concerning the rate of convergence in the central limit theorem (1.1): under conditions **A1**, **A2** and **A3**, for any continuous function φ on \mathbb{P}^{d-1} , we have,

$$\sup_{x \in \mathbb{P}^{d-1}} \sup_{y \in \mathbb{R}} \left| \mathbb{E}\left[\varphi(X_n^x) \mathbb{1}_{\left\{\frac{\log \|G_n\| - n\lambda}{\sigma\sqrt{n}} \leq y\right\}}\right] - \nu(\varphi)\Phi(y) \right| \leq \frac{C \log n}{\sqrt{n}}. \quad (1.3)$$

In particular, with $\varphi = \mathbf{1}$, the bound (1.3) clearly improves (1.2). See Theorem 2.1 where a slightly stronger conclusion is given.

Our second objective is to prove the moderate deviation principle for the couple $(X_n^x, \log \|G_n\|)$ (see Theorem 2.3): under conditions **A1**, **A2** and **A3**, for any non-negative Hölder continuous function φ on \mathbb{P}^{d-1} satisfying $\nu(\varphi) > 0$, for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n \geq 1}$ satisfying $\frac{b_n}{\sqrt{n}} \rightarrow \infty$ and $\frac{b_n}{n} \rightarrow 0$, we have, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\begin{aligned} - \inf_{y \in B^\circ} \frac{y^2}{2\sigma^2} &\leq \liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \in B \right\}} \right] \\ &\leq \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \in B \right\}} \right] \leq - \inf_{y \in \bar{B}} \frac{y^2}{2\sigma^2}, \end{aligned} \quad (1.4)$$

where B° and \bar{B} are respectively the interior and the closure of B . Note that the moderate deviation principle (1.4) is proved under the proximality condition **A3**. This condition ensures that the Markov chain (X_n^x) has a unique stationary measure ν on the projective space \mathbb{P}^{d-1} . When condition **A3** fails, we are still able to prove the following moderate deviation principle for $\|G_n\|$ (see Theorem 2.4) whenever $\sigma > 0$: there exists a constant $\sigma_0 > 0$ such that for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n \geq 1}$ satisfying $\frac{b_n}{\sqrt{n}} \rightarrow \infty$ and $\frac{b_n}{n} \rightarrow 0$, we have,

$$\begin{aligned} - \inf_{y \in B^\circ} \frac{y^2}{2\sigma_0^2} &\leq \liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left(\frac{\log \|G_n\| - n\lambda}{b_n} \in B \right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left(\frac{\log \|G_n\| - n\lambda}{b_n} \in B \right) \leq - \inf_{y \in \bar{B}} \frac{y^2}{2\sigma_0^2}. \end{aligned} \quad (1.5)$$

This is rather interesting compared with the large deviation case, since when the proximality condition **A3** fails, the rate function in the large deviation principle is not known (in fact we do not even know whether the large deviation principle holds).

Our third objective is to establish the moderate deviation expansion in the normal range $[0, o(n^{1/6})]$ for the couples $(X_n^x, \log \|G_n\|)$ with a target function (see Theorem 2.6 for a slightly stronger version): under conditions **A1**, **A2** and **A3**, for any Hölder continuous function φ on \mathbb{P}^{d-1} , we have, uniformly in $x \in \mathbb{P}^{d-1}$ and $y \in [0, o(n^{1/6})]$,

$$\frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \geq \sqrt{n}\sigma y\}}]}{1 - \Phi(y)} = \nu(\varphi) + o(1). \quad (1.6)$$

The expansion (1.6) has not been known in the literature even when $\varphi = \mathbf{1}$. The result is interesting because it gives a precise comparison between the moderate deviation probability $\mathbb{P}(\log \|G_n\| - n\lambda \geq \sqrt{n}\sigma y)$ with the normal tail $1 - \Phi(y)$. The range $y \in [0, o(n^{1/6})]$ is the best for such a comparison.

All the above mentioned results (1.3), (1.4), (1.5) and (1.6) are concerned with the operator norm $\|G_n\|$, but we shall also establish the analog of these results for the spectral radius $\rho(G_n)$.

1.3. Proof outline. In [10, 11], the proof of (1.2) consists of establishing the central limit theorem with rate of convergence in Wasserstein's distance utilizing the martingale approximation method developed in [4]. With this approach, even if we obtain the best rate of convergence $\frac{1}{\sqrt{n}}$ in Wasserstein's distance, while passing to the Kolmogorov distance we can only get the rate $\frac{1}{n^{1/4}}$ in the Berry-Esseen bound. To get a better bound, a new approach is needed. Our proof of (1.3) is based on the Berry-Esseen bound for the couple $(X_n^x, \log |G_n x|)$ recently established in [23] and on the following precise comparison between $\|G_n\|$ and $|G_n x|$ established in [5]: for any $a > 0$, there exist $c > 0$ and $k_0 \in \mathbb{N}$ such that for all $n \geq k \geq k_0$ and $x \in \mathbb{P}^{d-1}$,

$$\mathbb{P} \left(\left| \log \frac{\|G_n\|}{\|G_k\|} - \log \frac{|G_n x|}{|G_k x|} \right| \leq e^{-ak} \right) > 1 - e^{-ck}. \quad (1.7)$$

The basic idea to utilize this powerful inequality consists in carefully choosing certain integer k , taking the conditional expectation with respect to the filtration $\sigma(g_1, \dots, g_k)$ and using the large deviation bounds for $\log \|G_k\|$. This technique, in conjugation with limit theorems for the norm cocycle $\log |G_{n-k} x|$, makes it possible to prove corresponding results for $\log \|G_n\|$; see [5] where a local limit theorem for $\log \|G_n\|$ has been established by taking $k = \lfloor \log^2 n \rfloor$, where $\lfloor a \rfloor$ denotes the integral part of a . In this paper, the proof of (1.3) is carried out by choosing $k = \lfloor C_1 \log n \rfloor$ with a sufficiently large constant $C_1 > 0$ and by using the Berry-Esseen bound for the couple $(X_n^x, \log |G_n x|)$ with a target function φ on X_n^x . In the same spirit, the moderate deviation principle (1.4) for the couple $(X_n^x, \log \|G_n\|)$ is established using the moderate deviation principle for the couple $(X_n^x, \log |G_n x|)$ proved in [23], together with the inequality (1.7) with $k = \lfloor C_1 \frac{b_n^2}{n} \rfloor$, where the constant C_1 is sufficiently large and the sequence $(b_n)_{n \geq 1}$ is given in (1.4).

As to the moderate deviation principle (1.5) for $\log \|G_n\|$ without assuming the proximality condition A3, its proof is more technical and delicate than that of (1.4). Indeed, when condition A3 fails, the transfer operator of the Markov chain $(X_n^x)_{n \geq 0}$ has no spectral gap in general and it may happen that $(X_n^x)_{n \geq 0}$ possesses several stationary measures on the projective space \mathbb{P}^{d-1} . In this case, it becomes hopeless to prove a general form of (1.5) when a target function φ on X_n^x is taken into account. Nevertheless, the proof of (1.5) can be carried out by following the approach of Bougerol and Lacroix [7] (first announced in [6]), where central limit theorems and exponential large deviation bounds for $\|G_n\|$ and $|G_n x|$ were established without giving the rate function. Specifically, employing this approach consists in finding

the proximal dimension p of the semigroup Γ_μ generated by the matrix law μ and then applying Chevalley's algebraic irreducible representation [9] of the exterior powers $\wedge^p \mathbb{R}^d$, to show that the action of the semigroup Γ_μ is strongly irreducible and proximal on $\wedge^p \mathbb{R}^d$. Using this strategy together with (1.4) for $\varphi = \mathbf{1}$, we are able to establish (1.5).

For the proof of the Cramér type moderate deviation expansion (1.6) in the normal range $[0, o(n^{1/6})]$, when $y \in [0, \frac{1}{2}\sqrt{\log n}]$, we deduce the desired result from the Berry-Esseen type bound (1.3); when $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$, we make use of the moderate deviation expansion for the couple $(X_n^x, \log |G_n x|)$ recently established in [23] and the aforementioned inequality (1.7) with $k = \lfloor C_1 y^2 \rfloor$, where $C_1 > 0$ is a sufficiently large constant.

All of the aforementioned results (1.3), (1.4), (1.5) and (1.6) for the operator norm $\|G_n\|$ turn out to be essential to establish analogous Berry-Esseen type bounds and moderate deviation results for the spectral radius $\rho(G_n)$. Another important ingredient in our proof is the precise comparison between $\rho(G_n)$ and $\|G_n\|$ established in [5]; see Lemma 3.3 below.

2. MAIN RESULTS

To formulate our main results, we introduce some notation below. Let $\mathcal{C}(\mathbb{P}^{d-1})$ be the space of continuous complex-valued functions on \mathbb{P}^{d-1} and $\mathbf{1}$ be the constant function with value 1. All over the paper we assume that $\gamma > 0$ is a fixed small enough constant. We equip the projective space \mathbb{P}^{d-1} with the angular distance \mathbf{d} defined by $\mathbf{d}(x, y) = |x \wedge y|$ for $x, y \in \mathbb{P}^{d-1}$, where $x \wedge y$ denotes the exterior product of x and y . Consider the Banach space $\mathcal{B}_\gamma := \{\varphi \in \mathcal{C}(\mathbb{P}^{d-1}) : \|\varphi\|_\gamma < +\infty\}$, where

$$\|\varphi\|_\gamma := \|\varphi\|_\infty + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\mathbf{d}^\gamma(x, y)} \quad \text{with} \quad \|\varphi\|_\infty := \sup_{x \in \mathbb{P}^{d-1}} |\varphi(x)|.$$

For any $g \in \mathbb{G}$ and $x \in \mathbb{P}^{d-1}$, we write $g \cdot x := \frac{gx}{|gx|}$ for the projective action of g on the projective space \mathbb{P}^{d-1} . Consider the Markov chain

$$X_0^x = x, \quad X_n^x = G_n \cdot x, \quad n \geq 1.$$

Under conditions **A1**, **A2** and **A3**, the chain $(X_n^x)_{n \geq 0}$ possesses a unique stationary measure ν on \mathbb{P}^{d-1} such that $\mu * \nu = \nu$ (see [14]), where $\mu * \nu$ denotes the convolution of μ and ν . It is worth mentioning that if the proximality condition **A3** fails, then the stationary measure ν may not be unique (see [7, 5]). It was shown in [5, Proposition 14.17] that the asymptotic variance σ^2 of the random walk $(G_n)_{n \geq 1}$ can be given by

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[(\log \|G_n\| - n\lambda)^2].$$

Throughout the paper, we denote by Φ the standard normal distribution function on \mathbb{R} . We write c, C for positive constants whose values may change from line to line.

2.1. Berry-Esseen type bounds. In this subsection, we present Berry-Esseen type bounds for the couples $(X_n^x, \log \|G_n\|)$ and $(X_n^x, \log \rho(G_n))$ with target functions on the Markov chain $(X_n^x)_{n \geq 0}$. Recall that by Gelfand's formula, it holds that $\rho(g) = \lim_{k \rightarrow \infty} \|g^k\|^{1/k}$ for any $g \in \mathbb{G}$.

Theorem 2.1. *Assume conditions [A1](#), [A2](#) and [A3](#). Then there exists a constant $C > 0$ such that for all $n \geq 1$, $x \in \mathbb{P}^{d-1}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\left| \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{\sigma\sqrt{n}} \leq y \right\}} \right] - \nu(\varphi) \Phi(y) \right| \leq \frac{C \log n}{\sqrt{n}} \|\varphi\|_\gamma \quad (2.1)$$

and

$$\left| \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{\sigma\sqrt{n}} \leq y \right\}} \right] - \nu(\varphi) \Phi(y) \right| \leq \frac{C \log n}{\sqrt{n}} \|\varphi\|_\gamma. \quad (2.2)$$

Using the fact that all matrix norms are equivalent, one can verify that in (2.1), the operator norm $\|\cdot\|$ can be replaced by any matrix norm.

In particular, taking $\varphi = \mathbf{1}$ in (2.1) and (2.2), we have: under conditions [A1](#), [A2](#) and [A3](#), for all $n \geq 1$ and $y \in \mathbb{R}$,

$$\left| \mathbb{P} \left(\frac{\log \|G_n\| - n\lambda}{\sigma\sqrt{n}} \leq y \right) - \Phi(y) \right| \leq \frac{C \log n}{\sqrt{n}}, \quad (2.3)$$

$$\left| \mathbb{P} \left(\frac{\log \rho(G_n) - n\lambda}{\sigma\sqrt{n}} \leq y \right) - \Phi(y) \right| \leq \frac{C \log n}{\sqrt{n}}. \quad (2.4)$$

As mentioned before, the Berry-Esseen type bound (2.3) improves (1.2) established recently by Cuny, Dedecker and Jan [10].

The result with a general target function φ is worth some comments. On the one hand, it concerns the joint distribution of the couples $(X_n^x, \log \|G_n\|)$ and $(X_n^x, \log \rho(G_n))$, which give more information and can lead to interesting applications. On the other hand, the extension from the case $\varphi = \mathbf{1}$ to general φ is not trivial, for which a significant difficulty appears. The difficulty will be overcome by using the Berry Esseen bound for the couple $(X_n^x, \log |G_n x|)$.

It is natural to make the conjecture that the optimal rate of convergence on the right hand sides of (2.1), (2.2), (2.3) and (2.4) should be $\frac{C}{\sqrt{n}}$ instead of $\frac{C \log n}{\sqrt{n}}$. For positive matrices, these optimal bounds have been proved in [24]. However, the proofs of the conjecture for invertible matrices seem to be rather delicate, for which new ideas and techniques are required. Nevertheless, we can prove the optimal bound $\frac{C}{\sqrt{n}}$ for large values of $|y|$, as indicated in the following remark which will be seen in the proof of Theorem 2.1.

Remark 2.2. Under the same conditions as in Theorem 2.1, if we consider $|y| > \sqrt{2 \log \log n}$ instead of $y \in \mathbb{R}$, then the bound $\frac{C \log n}{\sqrt{n}}$ in (2.1), (2.2), (2.3) and (2.4) can be improved to be $\frac{C}{\sqrt{n}}$.

2.2. Moderate deviation principles. We first state moderate deviation principles for the couples $(X_n^x, \log \|G_n\|)$ and $(X_n^x, \log \rho(G_n))$ with target functions on the Markov chain $(X_n^x)_{n \geq 0}$.

Theorem 2.3. Assume conditions A1, A2 and A3. Then, for any non-negative function $\varphi \in \mathcal{B}_\gamma$ satisfying $\nu(\varphi) > 0$, for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n \geq 1}$ satisfying $\frac{b_n}{\sqrt{n}} \rightarrow \infty$ and $\frac{b_n}{n} \rightarrow 0$, we have, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\begin{aligned} - \inf_{y \in B^\circ} \frac{y^2}{2\sigma^2} &\leq \liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \in B \right\}} \right] \\ &\leq \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \in B \right\}} \right] \leq - \inf_{y \in \bar{B}} \frac{y^2}{2\sigma^2}, \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} - \inf_{y \in B^\circ} \frac{y^2}{2\sigma^2} &\leq \liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \in B \right\}} \right] \\ &\leq \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \in B \right\}} \right] \leq - \inf_{y \in \bar{B}} \frac{y^2}{2\sigma^2}, \end{aligned} \quad (2.6)$$

where B° and \bar{B} are respectively the interior and the closure of B .

Note that the target function φ in (2.5) and (2.6) is not necessarily strictly positive, and it can vanish somewhere on the projective space \mathbb{P}^{d-1} . The moderate deviation principles (2.5) and (2.6) are all new, even for $\varphi = \mathbf{1}$.

If we only consider the operator norm $\|G_n\|$ or the spectral radius $\rho(G_n)$, instead of the couples $(X_n^x, \log \|G_n\|)$ and $(X_n^x, \log \rho(G_n))$, we are still able to establish moderate deviation principles without assuming the proximality condition A3:

Theorem 2.4. Assume conditions A1, A2 and $\sigma^2 > 0$. Then, there exists a constant $\sigma_0 > 0$ such that for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n \geq 1}$ satisfying $\frac{b_n}{\sqrt{n}} \rightarrow \infty$ and $\frac{b_n}{n} \rightarrow 0$, we have,

$$\begin{aligned} - \inf_{y \in B^\circ} \frac{y^2}{2\sigma_0^2} &\leq \liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left(\frac{\log \|G_n\| - n\lambda}{b_n} \in B \right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left(\frac{\log \|G_n\| - n\lambda}{b_n} \in B \right) \leq - \inf_{y \in \bar{B}} \frac{y^2}{2\sigma_0^2}, \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} - \inf_{y \in B^\circ} \frac{y^2}{2\sigma_0^2} &\leq \liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left(\frac{\log \rho(G_n) - n\lambda}{b_n} \in B \right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left(\frac{\log \rho(G_n) - n\lambda}{b_n} \in B \right) \leq - \inf_{y \in \bar{B}} \frac{y^2}{2\sigma_0^2}, \end{aligned} \quad (2.8)$$

where B° and \bar{B} are respectively the interior and the closure of B .

Remark 2.5. Assume conditions **A1** and **A2**. Let

$$\Gamma_{\mu,1} = \{ |\det(g)|^{-1/d} g : g \in \Gamma_\mu \}$$

be the set of elements of Γ_μ normalized to have determinant 1.

- (1) If $\Gamma_{\mu,1}$ is not contained in a compact subgroup of \mathbb{G} , then $\sigma > 0$, as will be seen in the proof of Theorem 2.4.
- (2) If $\Gamma_{\mu,1}$ is contained in a compact subgroup of \mathbb{G} , then $c_1 = \inf \{ \|g\| : g \in \Gamma_{\mu,1} \} > 0$ and $c_2 = \{ \|g\| : g \in \Gamma_{\mu,1} \} < \infty$, so that

$$c_1^d |\det(g)| \leq \|g\|^d \leq c_2^d |\det(g)| \quad \forall g \in \Gamma_\mu. \quad (2.9)$$

Since $\log |\det(G_n)| = \sum_{i=1}^n \log |\det(g_i)|$ is a sum of i.i.d. real-valued random variables, from (2.9) (applied to $g = G_n$) it follows directly that the moderate deviation principle (2.7) holds with $\lambda = \frac{1}{d} \mathbb{E} \log |\det(g_1)|$ and $\sigma_0^2 = \mathbb{E}[(\frac{1}{d} \log |\det(g_1)| - \lambda)^2]$ (which coincide with their original definitions), provided that $|\det(g_1)|$ is not a.s. a constant (which is equivalent to $\sigma_0^2 > 0$).

In fact, in the second case of the remark above, $\log \|G_n\|$ can be expressed exactly as a sum of i.i.d. real-valued random variables when the norm $\|\cdot\|$ is suitably chosen, as observed by Bougerol and Lacroix [7]. See also Lemma 4.5 in subsection 2.3.

2.3. Moderate deviation expansions. In this subsection we formulate the Cramér type moderate deviation expansions in the normal range for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$. Our first result concerns the operator norm $\|G_n\|$.

Theorem 2.6. Assume conditions **A1**, **A2** and **A3**. Then, we have, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [0, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$, as $n \rightarrow \infty$,

$$\frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \geq \sqrt{n}\sigma y\}}]}{1 - \Phi(y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1), \quad (2.10)$$

$$\frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \leq -\sqrt{n}\sigma y\}}]}{\Phi(-y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1). \quad (2.11)$$

In particular, taking $\varphi = \mathbf{1}$, we obtain the following moderate deviation expansion for the operator norm $\|G_n\|$ in the normal range: uniformly in $y \in [0, o(n^{1/6})]$, as $n \rightarrow \infty$,

$$\frac{\mathbb{P}(\log \|G_n\| - n\lambda \geq \sqrt{n}\sigma y)}{1 - \Phi(y)} = 1 + o(1), \quad (2.12)$$

$$\frac{\mathbb{P}(\log \|G_n\| - n\lambda \leq -\sqrt{n}\sigma y)}{\Phi(-y)} = 1 + o(1). \quad (2.13)$$

The expansions (2.12) and (2.13) are also new.

The proof of Theorem 2.6 is based on the Cramér type moderate deviation expansion for the couple $(X_n^x, \log |G_n x|)$ proved recently in [23], and on a fine comparison between the operator norm $\|G_n\|$ and the vector norm $|G_n x|$ established in [5] (see Lemma 3.2 below). Note that Theorem 2.6 covers the special case where $\nu(\varphi) = 0$; in this case the exact comparison with the normal tail remains open.

Our second result concerns the moderate deviation expansions for the spectral radius $\rho(G_n)$, also in the normal range.

Theorem 2.7. *Assume conditions A1, A2 and A3. Then, we have, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [0, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$, as $n \rightarrow \infty$,*

$$\frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log \rho(G_n) - n\lambda \geq \sqrt{n}\sigma y\}}]}{1 - \Phi(y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1), \quad (2.14)$$

$$\frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log \rho(G_n) - n\lambda \leq -\sqrt{n}\sigma y\}}]}{\Phi(-y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1). \quad (2.15)$$

Taking $\varphi = \mathbf{1}$, we obtain the following moderate deviation expansions for the spectral radius $\rho(G_n)$ which are also new: uniformly in $y \in [0, o(n^{1/6})]$, as $n \rightarrow \infty$,

$$\frac{\mathbb{P}(\log \rho(G_n) - n\lambda \geq \sqrt{n}\sigma y)}{1 - \Phi(y)} = 1 + o(1), \quad (2.16)$$

$$\frac{\mathbb{P}(\log \rho(G_n) - n\lambda \leq -\sqrt{n}\sigma y)}{\Phi(-y)} = 1 + o(1). \quad (2.17)$$

The proof of Theorem 2.7 relies on Theorem 2.6 and on an estimate of the difference between spectral radius $\rho(G_n)$ and the operator norm $\|G_n\|$ established in [5] (see Lemma 3.3). Like in Theorem 2.6, Theorem 2.7 also covers the case where $\nu(\varphi) = 0$, for which the exact comparison with the normal tail is not known.

3. BERRY-ESSEEN TYPE BOUNDS

The goal of this section is to prove Theorem 2.1 about Berry-Esseen type bounds for the operator norm $\|G_n\|$ and for the spectral radius $\rho(G_n)$. We

will need the following Berry-Esseen bound for the norm cocycle $\log |G_n x|$ established in [23].

Lemma 3.1. *Assume conditions A1, A2 and A3. Then, we have, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,*

$$\left| \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\lambda}{\sigma\sqrt{n}} \leq y \right\}} \right] - \nu(\varphi) \Phi(y) \right| \leq \frac{C}{\sqrt{n}} \|\varphi\|_\gamma.$$

The following result is an interesting comparison theorem for $\log \|G_n\|$ and $\log |G_n x|$. It shows that the difference between $\log \frac{\|G_n\|}{\|G_k\|}$ and $\log \frac{|G_n x|}{|G_k x|}$ is smaller than any exponential rate e^{-ak} for any $a > 0$ and $k \leq n$, with large probability.

Lemma 3.2. *Assume conditions A1, A2 and A3. Then, for any $a > 0$, there exist $c > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geq k \geq k_0$ and $x \in \mathbb{P}^{d-1}$,*

$$\mathbb{P} \left(\left| \log \frac{\|G_n\|}{\|G_k\|} - \log \frac{|G_n x|}{|G_k x|} \right| \leq e^{-ak} \right) > 1 - e^{-ck}.$$

Lemma 3.2 was established in [5, Lemma 17.8] and has been used to prove the local limit theorem for the operator norm $\|G_n\|$: see [5, Theorem 17.9].

Proof of (2.1) of Theorem 2.1. Without loss of generality, we assume that the target function φ is non-negative. On the one hand, using Lemma 3.1 and the fact that $\log \|G_n\| \geq \log |G_n x|$, we easily get the following upper bound: there exists a constant $C > 0$ such that for all $x \in \mathbb{P}^{d-1}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,

$$I_n := \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{\sigma\sqrt{n}} \leq y \right\}} \right] \leq \nu(\varphi) \Phi(y) + \frac{C}{\sqrt{n}} \|\varphi\|_\gamma. \quad (3.1)$$

On the other hand, applying Lemma 3.2, we deduce that for any $a > 0$, there exist $c > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geq k \geq k_0$, it holds uniformly in $x \in \mathbb{P}^{d-1}$ and $\varphi \in \mathcal{B}_\gamma$ that

$$\begin{aligned} I_n &\geq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{\sigma\sqrt{n}} \leq y \right\}} \mathbb{1}_{\left\{ \left| \log \|G_n\| - \log \frac{|G_n x|}{|G_k x|} - \log \|G_k\| \right| \leq e^{-ak} \right\}} \right] \\ &\geq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - \log |G_k x| + \log \|G_k\| - n\lambda + e^{-ak}}{\sigma\sqrt{n}} \leq y \right\}} \right] - e^{-ck} \|\varphi\|_\infty. \end{aligned} \quad (3.2)$$

For simplicity, for any $n > k \geq 1$, we write $G_n = G_{n,k} G_k$ with

$$G_{n,k} = g_n \cdots g_{k+1}, \quad G_k = g_k \cdots g_1.$$

From the large deviation bounds for $\log \|G_k\|$ (see [5] or [22]), we have that for any $q > \lambda$, there exists a constant $c > 0$ such that for sufficiently large $k \geq 1$,

$$\mathbb{P}(\log \|G_k\| > kq) \leq e^{-ck}. \quad (3.3)$$

Denote the σ -algebra $\mathcal{F}_k = \sigma(g_1, \dots, g_k)$. From (3.2), taking the conditional expectation with respect to the filtration \mathcal{F}_k , we derive that for any $q > \lambda$,

$$\begin{aligned} I_n &\geq \mathbb{E} \left\{ \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - \log |G_k x| + \log \|G_k\| - n\lambda + e^{-ak}}{\sigma\sqrt{n}} \leq y \right\}} \middle| \mathcal{F}_k \right] \right\} - e^{-ck} \|\varphi\|_\infty \\ &\geq \mathbb{E} \left\{ \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - \log |G_k x| + \log \|G_k\| - n\lambda + e^{-ak}}{\sigma\sqrt{n}} \leq y \right\}} \mathbb{1}_{\{\log \|G_k\| \leq kq\}} \middle| \mathcal{F}_k \right] \right\} \\ &\quad - e^{-ck} \|\varphi\|_\infty \\ &\geq \mathbb{E} \left\{ \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - \log |G_k x| + kq - n\lambda + e^{-ak}}{\sigma\sqrt{n}} \leq y \right\}} \middle| \mathcal{F}_k \right] \right\} - 2e^{-c_1 k} \|\varphi\|_\infty, \end{aligned}$$

where in the last step we use the large deviation bound (3.3) and the constant $c_1 > 0$ is taken to be small enough. Note that $X_n^x = G_n \cdot x = G_{n,k} \cdot X_k^x$ and $\log |G_n x| - \log |G_k x| = \log |G_{n,k} X_k^x|$. It follows that

$$I_n \geq \mathbb{E} \left\{ \mathbb{E} \left[\varphi(G_{n,k} \cdot X_k^x) \mathbb{1}_{\left\{ \frac{\log |G_{n,k} X_k^x| + kq - n\lambda + e^{-ak}}{\sigma\sqrt{n}} \leq y \right\}} \middle| \mathcal{F}_k \right] \right\} - 2e^{-c_1 k} \|\varphi\|_\infty.$$

Since it is shown in Lemma 3.1 that the Berry-Esseen bound for the norm cocycle $\log |G_n x|$ holds uniformly in $x \in \mathbb{P}^{d-1}$, we obtain

$$I_n \geq \nu(\varphi) \Phi(y_1) - \frac{C}{\sqrt{n-k}} \|\varphi\|_\gamma - 2e^{-c_1 k} \|\varphi\|_\infty,$$

where

$$y_1 = \frac{\sqrt{n}}{\sqrt{n-k}} y - \frac{k(q-\lambda) + e^{-ak}}{\sigma\sqrt{n-k}}.$$

Taking $k = [C_1 \log n]$ with $C_1 = \frac{1}{2c_1}$, we get that there exists a constant $C > 0$ such that $\frac{1}{\sqrt{n-k}} \leq \frac{C}{\sqrt{n}}$ and $e^{-c_1 k} \leq \frac{C}{\sqrt{n}}$. Since Φ is the standard normal distribution function, we write

$$\Phi(y_1) = \Phi(y) - \frac{1}{\sqrt{2\pi}} \int_{y_1}^y e^{-\frac{t^2}{2}} dt.$$

To estimate the above integral, by elementary calculations, there exists a constant $C_2 > 0$ such that

$$|y - y_1| \leq C_2 \left(\frac{\log n}{n} |y| + \frac{\log n}{\sqrt{n}} \right),$$

and for $n > k_0$ large enough,

$$\begin{aligned} e^{-\frac{y_1^2}{2}} &\leq \exp \left\{ -\frac{1}{2} \frac{n}{n-k} y^2 + \sqrt{n} \frac{k(q-\lambda) + e^{-ak}}{n-k} y \right\} \\ &\leq \exp \left\{ -\frac{1}{2} y^2 + C_2 \frac{\log n}{\sqrt{n}} |y| \right\}. \end{aligned}$$

Thus, it follows that there exists a constant $C > 0$ such that

$$\begin{aligned} \left| \int_{y_1}^y e^{-\frac{t^2}{2}} dt \right| &\leq |y - y_1| \max \left(e^{-\frac{y^2}{2}}, e^{-\frac{y_1^2}{2}} \right) \\ &\leq C_2 \left(\frac{\log n}{n} |y| + \frac{\log n}{\sqrt{n}} \right) \exp \left\{ -\frac{1}{2} y^2 + C_2 \frac{\log n}{\sqrt{n}} |y| \right\} \\ &\leq \begin{cases} C \frac{\log n}{\sqrt{n}} & \forall y \in \mathbb{R}, \\ C \frac{1}{\sqrt{n}} & \text{if } |y| > \sqrt{2 \log \log n}. \end{cases} \end{aligned}$$

Consequently, we get the following lower bound for I_n : there exists a constant $C > 0$ such that for all $x \in \mathbb{P}^{d-1}$ and $\varphi \in \mathcal{B}_\gamma$,

$$I_n \geq \begin{cases} \nu(\varphi)\Phi(y) - \frac{C \log n}{\sqrt{n}} \|\varphi\|_\gamma & \forall y \in \mathbb{R}, \\ \nu(\varphi)\Phi(y) - \frac{C}{\sqrt{n}} \|\varphi\|_\gamma & \text{if } |y| > \sqrt{2 \log \log n}. \end{cases}$$

Together with the upper bound (3.1), this concludes the proof of (2.1) of Theorem 2.1 and the corresponding results in Remark 2.2. \square

We now proceed to prove the Berry-Esseen type bound (2.2) of Theorem 2.1 for the spectral radius $\rho(G_n)$. The proof relies on the following comparison lemma between the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$, established in [5, Lemma 14.13] using the Hölder regularity properties of the stationary measure ν . It shows that $\rho(G_n)$ behaves like $\|G_n\|$ up to a factor lying between 1 and $e^{-\varepsilon k}$ for any $\varepsilon > 0$ and $k \leq n$, with high probability. Notice that $\rho(G_n) \leq \|G_n\|$ by Gelfand's formula and the submultiplicity of the operator norm.

Lemma 3.3. *Assume conditions A1 and A2. Then, for any $\varepsilon > 0$, there exist $c > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geq k \geq k_0$,*

$$\mathbb{P} \left(1 \geq \frac{\rho(G_n)}{\|G_n\|} > e^{-\varepsilon k} \right) \geq 1 - e^{-ck}.$$

Proof of (2.2) of Theorem 2.1. Without loss of generality, we assume that the target function φ is non-negative.

The lower bound is a direct consequence of (2.1) together with Remark 2.2 on it, and the inequality $\log \rho(G_n) \leq \log \|G_n\|$, from which we get that, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} I_n &:= \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{\sigma \sqrt{n}} \leq y \right\}} \right] \\ &\geq \begin{cases} \nu(\varphi)\Phi(y) - \frac{C \log n}{\sqrt{n}} \|\varphi\|_\gamma & \forall y \in \mathbb{R}, \\ \nu(\varphi)\Phi(y) - \frac{C}{\sqrt{n}} \|\varphi\|_\gamma & \text{if } |y| > \sqrt{2 \log \log n}. \end{cases} \end{aligned}$$

The upper bound is a consequence of (2.1) together with Remark 2.2 on it and Lemma 3.3. In fact, applying Lemma 3.3, we deduce that for any $\varepsilon > 0$, there exist $c_1 > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geq k \geq k_0$,

$$\begin{aligned} I_n &\leq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{\sigma\sqrt{n}} \leq y \right\}} \mathbb{1}_{\left\{ \log \rho(G_n) - \log \|G_n\| > -\varepsilon k \right\}} \right] + e^{-c_1 k} \|\varphi\|_\infty \\ &\leq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - \varepsilon k - n\lambda}{\sigma\sqrt{n}} \leq y \right\}} \right] + e^{-c_1 k} \|\varphi\|_\infty. \end{aligned}$$

Taking $k = \lfloor C_1 \log n \rfloor$ with $C_1 = \frac{1}{2c_1}$, we have $e^{-c_1 k} \leq \frac{C}{\sqrt{n}}$ for some constant $C > 0$. Using the bound (2.1) with y replaced by $y_1 := y + \frac{\varepsilon k}{\sigma\sqrt{n}}$, we obtain the following upper bound for I_n : there exists a constant $C > 0$ such that for all $x \in \mathbb{P}^{d-1}$, $y \in \mathbb{R}$, $\varphi \in \mathcal{B}_\gamma$, and $n \geq k_0$ with k_0 large enough,

$$I_n \leq \begin{cases} \nu(\varphi)\Phi(y_1) + \frac{C \log n}{\sqrt{n}} \|\varphi\|_\gamma & \forall y \in \mathbb{R}, \\ \nu(\varphi)\Phi(y_1) + \frac{C}{\sqrt{n}} \|\varphi\|_\gamma & \text{if } |y| > \sqrt{2 \log \log n}. \end{cases}$$

(Notice that $|y| > \sqrt{2 \log \log n}$ implies $|y_1| > \sqrt{2 \log \log n}$ for n large enough.) By an argument similar to that used in the proof of (2.1), it can be seen that

$$\Phi(y_1) \leq \begin{cases} \Phi(y) + \frac{C \log n}{\sqrt{n}} & \forall y \in \mathbb{R}, \\ \Phi(y) + \frac{C}{\sqrt{n}} & \text{if } |y| > \sqrt{2 \log \log n}. \end{cases}$$

This concludes the proof of (2.2) and Remark 2.2 on it. \square

4. MODERATE DEVIATION PRINCIPLES

The goal of this section is to establish Theorems 2.3 and 2.4 about moderate deviation principles for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$. Notice that in the first theorem, we need the proximality condition, while in the second, we do not need it.

4.1. Proof of Theorem 2.3. We shall make use of the following moderate deviation principle for the couple $(X_n^x, \log |G_n x|)$, which is a direct consequence of Cramér type moderate deviation expansion for $(X_n^x, \log |G_n x|)$ recently established in [23].

Lemma 4.1. *Assume conditions A1, A2 and A3. Then, for any non-negative function $\varphi \in \mathcal{B}_\gamma$ satisfying $\nu(\varphi) > 0$, for any Borel set $B \subseteq \mathbb{R}$ and any positive sequence $(b_n)_{n \geq 1}$ satisfying $\frac{b_n}{\sqrt{n}} \rightarrow \infty$ and $\frac{b_n}{n} \rightarrow 0$, we have,*

uniformly in $x \in \mathbb{P}^{d-1}$,

$$\begin{aligned} - \inf_{y \in B^\circ} \frac{y^2}{2\sigma^2} &\leq \liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\lambda}{b_n} \in B \right\}} \right] \\ &\leq \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\lambda}{b_n} \in B \right\}} \right] \leq - \inf_{y \in \bar{B}} \frac{y^2}{2\sigma^2}, \end{aligned}$$

where B° and \bar{B} are respectively the interior and the closure of B .

Proof of (2.5) of Theorem 2.3. Since the rate function $I(y) := \frac{y^2}{2\sigma^2}$, $y \in \mathbb{R}$, is strictly increasing on $[0, \infty)$ and strictly decreasing on $(-\infty, 0]$ with $I(0) = 0$, by Lemma 4.4 of [17], it suffices to prove the following moderate deviation asymptotics: for any $y > 0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \geq y \right\}} \right] = -\frac{y^2}{2\sigma^2}, \quad (4.1)$$

$$\lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \leq -y \right\}} \right] = -\frac{y^2}{2\sigma^2}. \quad (4.2)$$

We first prove (4.1) using the moderate deviation principle for the norm cocycle $\log |G_n x|$ stated in Lemma 4.1.

For the lower bound, by Lemma 4.1 and the fact that $\log |G_n x| \leq \log \|G_n\|$, we easily get: for any $y > 0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \geq y \right\}} \right] \geq -\frac{y^2}{2\sigma^2}.$$

We now prove the upper bound. Denote by $(e_i)_{1 \leq i \leq d}$ the standard orthonormal basis of \mathbb{R}^d . Since all matrix norms in \mathbb{R}^d are equivalent, and both $g \mapsto \|g\|$ and $g \mapsto \max_{1 \leq i \leq d} |ge_i|$ are matrix norms, there exists a positive constant c_1 such that $\log \|G_n\| \leq \max_{1 \leq i \leq d} \log |G_n e_i| + c_1$. From this inequality, we derive that

$$\mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \geq y \right\}} \right] \leq \sum_{i=1}^d \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n e_i| - n\lambda + c_1}{b_n} \geq y \right\}} \right].$$

Since $b_n \rightarrow \infty$ as $n \rightarrow \infty$, we have that for any $\varepsilon > 0$, it holds that $\frac{c_1}{b_n} < \varepsilon$ for large enough n . Thus using Lemma 4.1, we obtain for any $y > 0$,

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n e_i| - n\lambda + c_1}{b_n} \geq y \right\}} \right] \\ &\leq \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n e_i| - n\lambda}{b_n} \geq y - \varepsilon \right\}} \right] = -\frac{(y - \varepsilon)^2}{2\sigma^2}. \end{aligned}$$

Since $\varepsilon > 0$ can be arbitrary small, we get the desired upper bound. This concludes the proof of (4.1).

We next prove (4.2). The upper bound is easy: using Lemma 4.1 and the fact that $\log |G_n x| \leq \log \|G_n\|$, we get that for any $y > 0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \leq -y \right\}} \right] \leq -\frac{y^2}{2\sigma^2}. \quad (4.3)$$

We now come to the proof of the lower bound using Lemma 3.2. For any $n \geq k$, consider the event

$$A_{n,k} = \left\{ \left| \log \|G_n\| - \log \frac{|G_n x|}{|G_k x|} - \log \|G_k\| \right| \leq e^{-ak} \right\},$$

and denote by $A_{n,k}^c$ its complement. From Lemma 3.2, we know that for any $a > 0$, there exist $c_1 > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geq k \geq k_0$ and $x \in \mathbb{P}^{d-1}$,

$$\mathbb{P}(A_{n,k}^c) \leq e^{-c_1 k}. \quad (4.4)$$

Using (4.4), we see that

$$\begin{aligned} I_n &:= \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \leq -y \right\}} \right] \\ &\geq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \leq -y \right\}} \mathbb{1}_{\{A_{n,k}\}} \right] \\ &\geq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - \log |G_k x| + \log \|G_k\| - n\lambda + e^{-ak}}{b_n} \leq -y \right\}} \mathbb{1}_{\{A_{n,k}\}} \right] \\ &\geq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - \log |G_k x| + \log \|G_k\| - n\lambda + e^{-ak}}{b_n} \leq -y \right\}} \right] - e^{-c_1 k} \|\varphi\|_\infty. \end{aligned} \quad (4.5)$$

As in the proof of (2.1), for any $n \geq k \geq k_0$, we write $G_n = G_{n,k} G_k$ with

$$G_{n,k} = g_n \cdots g_{k+1}, \quad G_k = g_k \cdots g_1.$$

Taking the conditional expectation with respect to the filtration $\mathcal{F}_k = \sigma(g_1, \dots, g_k)$, and using the large deviation bound (3.3) for the operator norm $\|G_k\|$, we derive that for any $q > \lambda$, there exists a constant $c_2 > 0$ such that

$$\begin{aligned} I_n &\geq \mathbb{E} \left\{ \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - \log |G_k x| + \log \|G_k\| - n\lambda + e^{-ak}}{b_n} \leq -y \right\}} \middle| \mathcal{F}_k \right] \right\} - e^{-c_1 k} \|\varphi\|_\infty \\ &\geq \mathbb{E} \left\{ \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - \log |G_k x| + \log \|G_k\| - n\lambda + e^{-ak}}{b_n} \leq -y \right\}} \mathbb{1}_{\{\log \|G_k\| \leq kq\}} \middle| \mathcal{F}_k \right] \right\} \\ &\quad - e^{-c_1 k} \|\varphi\|_\infty \\ &\geq \mathbb{E} \left\{ \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - \log |G_k x| + kq - n\lambda + e^{-ak}}{b_n} \leq -y \right\}} \middle| \mathcal{F}_k \right] \right\} - 2e^{-c_2 k} \|\varphi\|_\infty \\ &=: J_n - 2e^{-c_2 k} \|\varphi\|_\infty. \end{aligned} \quad (4.6)$$

According to Lemma 4.1, for any $y > 0$ and $\epsilon > 0$, we have that for sufficiently large n , uniformly in $x \in \mathbb{P}^{d-1}$,

$$e^{-\frac{b_n^2}{n}(\frac{y^2}{2\sigma^2} + \epsilon)} \leq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log |G_n x| - n\lambda}{b_n} \leq -y \right\}} \right] \leq e^{-\frac{b_n^2}{n}(\frac{y^2}{2\sigma^2} - \epsilon)}. \quad (4.7)$$

Note that $X_n^x = G_{n,k} \cdot X_k^x$ and $\log |G_n x| - \log |G_k x| = \log |G_{n,k} X_k^x|$. In the sequel, we take

$$k = \left\lfloor C_1 \frac{b_n^2}{n} \right\rfloor, \quad (4.8)$$

where $C_1 > 0$ is a constant whose value will be chosen sufficiently large. If we denote

$$b'_n = b_n + \frac{k(q - \lambda) + e^{-ak}}{y},$$

then the term J_n defined in (4.6) can be rewritten as

$$J_n = \mathbb{E} \left\{ \mathbb{E} \left[\varphi(G_{n,k} \cdot X_k^x) \mathbb{1}_{\left\{ \frac{\log |G_{n,k} X_k^x| - (n-k)\lambda}{b'_n} \leq -y \right\}} \middle| \mathcal{F}_k \right] \right\}.$$

Note that the sequence $(b'_n)_{n \geq 1}$ satisfies $\frac{b'_n}{\sqrt{n}} \rightarrow \infty$ and $\frac{b'_n}{n} \rightarrow 0$, as $n \rightarrow \infty$. Applying the moderate deviation bound (4.7) with n replaced by $n - k$, and with b_n replaced by b'_n , we obtain the following upper and lower bounds for J_n : with fixed $y > 0$ and $\epsilon > 0$, for n large enough, uniformly in $x \in \mathbb{P}^{d-1}$,

$$e^{-\frac{(b'_n)^2}{n-k}(\frac{y^2}{2\sigma^2} + \epsilon)} \leq J_n \leq e^{-\frac{(b'_n)^2}{n-k}(\frac{y^2}{2\sigma^2} - \epsilon)}. \quad (4.9)$$

From (4.6) and (4.9), it follows that, there exists a constant $c_3 > 0$ such that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log I_n &\geq \liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \left[e^{-\frac{(b'_n)^2}{n-k}(\frac{y^2}{2\sigma^2} + \epsilon)} - 2e^{-c_2 k} \right] \\ &\geq \liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \left[e^{-\frac{(b'_n)^2}{n-k}(\frac{y^2}{2\sigma^2} + \epsilon)} (1 - 2e^{-c_3 k}) \right], \end{aligned}$$

where the last inequality holds due to the fact that as $n \rightarrow \infty$,

$$\frac{(b'_n)^2}{k(n-k)} \left(\frac{y^2}{2\sigma^2} + \epsilon \right) \rightarrow \frac{1}{C_1} \left(\frac{y^2}{2\sigma^2} + \epsilon \right) < c_2$$

by choosing $C_1 > (\frac{y^2}{2\sigma^2} + \epsilon)/c_2$. Recalling that $k = \lfloor C_1 \frac{b_n^2}{n} \rfloor \rightarrow \infty$, by elementary calculations, we get $\lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log(1 - 2e^{-c_3 k}) = 0$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log e^{-\frac{(b'_n)^2}{n-k}(\frac{y^2}{2\sigma^2} + \epsilon)} &= - \lim_{n \rightarrow \infty} \frac{n}{n-k} \left[1 + \frac{k(q-\lambda) + e^{-ak}}{yb_n} \right]^2 \left(\frac{y^2}{2\sigma^2} + \epsilon \right) \\ &= - \left(\frac{y^2}{2\sigma^2} + \epsilon \right). \end{aligned}$$

Taking $\epsilon \rightarrow 0$, we get the desired lower bound: for any $y > 0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \leq -y \right\}} \right] \geq -\frac{y^2}{2\sigma^2}.$$

Combining this with the upper bound (4.3), we obtain (4.2) and thus conclude the proof of (2.5). \square

Using the moderate deviation principle (2.5) for the couple $(X_n^x, \log \|G_n\|)$ and Lemma 3.3, we are now in a position to establish the moderate deviation principle (2.6) for the couple $(X_n^x, \log \rho(G_n))$.

Proof of (2.6) of Theorem 2.3. As explained in the proof of (2.5), according to Lemma 4.4 of [17], it is sufficient to prove the following moderate deviation asymptotics: for any $y > 0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \geq y \right\}} \right] = -\frac{y^2}{2\sigma^2}, \quad (4.10)$$

$$\lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \leq -y \right\}} \right] = -\frac{y^2}{2\sigma^2}. \quad (4.11)$$

We first prove (4.10). On the one hand, since the function φ is non-negative, using the moderate deviation principle (2.5) for the operator norm $\|G_n\|$ and the fact that $\rho(G_n) \leq \|G_n\|$, we immediately get the upper bound: for any $y > 0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \geq y \right\}} \right] \leq -\frac{y^2}{2\sigma^2}. \quad (4.12)$$

On the other hand, using Lemma 3.3, we obtain that for any $\varepsilon > 0$, there exist $c_1 > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geq k \geq k_0$,

$$\begin{aligned}
I_n &:= \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \geq y \right\}} \right] \\
&\geq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \geq y \right\}} \mathbb{1}_{\{\log \rho(G_n) - \log \|G_n\| \geq -\varepsilon k\}} \right] \\
&\geq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \geq y b_n + \varepsilon k\}} \mathbb{1}_{\{\log \rho(G_n) - \log \|G_n\| \geq -\varepsilon k\}} \right] \\
&\geq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \geq y b_n + \varepsilon k\}} \right] - e^{-c_1 k} \|\varphi\|_\infty.
\end{aligned} \tag{4.13}$$

As in the proof of (2.5) of Theorem 2.3, we take

$$k = \left\lfloor C_1 \frac{b_n^2}{n} \right\rfloor, \tag{4.14}$$

where $C_1 > 0$ is a constant whose value will be chosen sufficiently large. By the moderate deviation principle (2.5) for the couple $(X_n^x, \log \|G_n\|)$, it follows that for any $y > 0$ and $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \geq y \right\}} \right] \geq e^{-\frac{b_n^2}{n} \left(\frac{y^2}{2\sigma^2} + \eta \right)}. \tag{4.15}$$

Set

$$b'_n = b_n + \frac{\varepsilon k}{y}.$$

It is easy to see that the sequence $(b'_n)_{n \geq 1}$ satisfies $\frac{b'_n}{\sqrt{n}} \rightarrow \infty$ and $\frac{b'_n}{n} \rightarrow 0$, as $n \rightarrow \infty$. Using the bound (4.15), we get that uniformly in $x \in \mathbb{P}^{d-1}$,

$$\mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \geq y b_n + \varepsilon k\}} \right] \geq e^{-\frac{(b'_n)^2}{n} \left(\frac{y^2}{2\sigma^2} + \eta \right)}.$$

Substituting this bound into (4.13), we obtain

$$I_n \geq e^{-\frac{(b'_n)^2}{n} \left(\frac{y^2}{2\sigma^2} + \eta \right)} \left[1 - e^{-c_1 k + \frac{(b'_n)^2}{n} \left(\frac{y^2}{2\sigma^2} + \eta \right)} \|\varphi\|_\infty \right].$$

Taking into account of (4.14), by elementary calculations, choosing $C_1 > \frac{1}{c_1} \left(\frac{y^2}{2\sigma^2} + \eta \right)$, we have, as $n \rightarrow \infty$,

$$\frac{(b'_n)^2}{kn} \left(\frac{y^2}{2\sigma^2} + \eta \right) \rightarrow \frac{1}{C_1} \left(\frac{y^2}{2\sigma^2} + \eta \right) < c_1.$$

Hence we get for some constant $c_2 > 0$,

$$I_n \geq e^{-\frac{(b'_n)^2}{n} \left(\frac{y^2}{2\sigma^2} + \eta \right)} \left[1 - e^{-c_2 k} \|\varphi\|_\infty \right].$$

Therefore, using $k = \lfloor C_1 \frac{b_n^2}{n} \rfloor \rightarrow \infty$, we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log I_n &\geq \lim_{n \rightarrow \infty} \frac{n}{b_n^2} \left[-\frac{(b'_n)^2}{n} \left(\frac{y^2}{2\sigma^2} + \eta \right) \right] + \lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log(1 - e^{-c_2 k} \|\varphi\|_\infty) \\ &= \lim_{n \rightarrow \infty} \left[-\left(1 + \frac{\varepsilon k}{y b_n}\right)^2 \left(\frac{y^2}{2\sigma^2} + \eta \right) \right] + 0 \\ &= -\left(\frac{y^2}{2\sigma^2} + \eta \right). \end{aligned}$$

Taking $\eta \rightarrow 0$, we obtain the desired lower bound: for any $y > 0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \geq y \right\}} \right] \geq -\frac{y^2}{2\sigma^2}.$$

Together with the upper bound (4.12), this concludes the proof of (4.10).

We next prove (4.11). Using (4.2) and the fact that $\rho(G_n) \leq \|G_n\|$, we easily get the desired lower bound: for any $y > 0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \leq -y \right\}} \right] \geq -\frac{y^2}{2\sigma^2}. \quad (4.16)$$

For the upper bound, we still choose k as before:

$$k = \left\lfloor C_2 \frac{b_n^2}{n} \right\rfloor, \quad (4.17)$$

where $C_2 > 0$ is a constant whose value will be chosen sufficiently large. By Lemma 3.3, we see that for any $\varepsilon > 0$, there exist $c_3 > 0$ and $k_0 \in \mathbb{N}$, such that for all $n \geq k \geq k_0$,

$$\begin{aligned} J_n &:= \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \leq -y \right\}} \right] \\ &= \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \leq -y \right\}} \mathbb{1}_{\left\{ \log \rho(G_n) - \log \|G_n\| \geq -\varepsilon k \right\}} \right] \\ &\quad + \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \leq -y \right\}} \mathbb{1}_{\left\{ \log \rho(G_n) - \log \|G_n\| < -\varepsilon k \right\}} \right] \\ &\leq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \log \|G_n\| - n\lambda \leq -y b_n + \varepsilon k \right\}} \right] + e^{-c_3 k} \|\varphi\|_\infty. \end{aligned}$$

By the moderate deviation principle (4.2) for $\|G_n\|$, we have that for any $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$,

$$\mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{b_n} \leq -y \right\}} \right] \leq e^{-\frac{b_n^2}{n} \left(\frac{y^2}{2\sigma^2} - \eta \right)}. \quad (4.18)$$

Let $b'_n = b_n - \frac{\varepsilon k}{y}$. In view of (4.17), we see that $\frac{b'_n}{\sqrt{n}} \rightarrow \infty$ and $\frac{b'_n}{n} \rightarrow 0$, as $n \rightarrow \infty$. From (4.18), it follows that uniformly in $x \in \mathbb{P}^{d-1}$,

$$J_n \leq e^{-\frac{(b'_n)^2}{n}(\frac{y^2}{2\sigma^2} - \eta)} + e^{-c_3 k} \|\varphi\|_\infty. \quad (4.19)$$

Since $b'_n = b_n - \frac{\varepsilon k}{y}$ and $k = \lfloor C_1 \frac{b_n^2}{n} \rfloor$, it holds that as $n \rightarrow \infty$, $\frac{b'_n}{b_n} \rightarrow 1$ and, choosing $C_1 > \frac{1}{c_3}(\frac{y^2}{2\sigma^2} - \eta)$, we have,

$$\frac{(b'_n)^2}{kn}(\frac{y^2}{2\sigma^2} - \eta) \rightarrow \frac{1}{C_1}(\frac{y^2}{2\sigma^2} - \eta) < c_3.$$

Thus,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log J_n &\leq \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log e^{-\frac{(b'_n)^2}{n}(\frac{y^2}{2\sigma^2} - \eta)} \\ &= - \lim_{n \rightarrow \infty} \left(\frac{b'_n}{b_n} \right)^2 \left(\frac{y^2}{2\sigma^2} - \eta \right) = - \left(\frac{y^2}{2\sigma^2} - \eta \right). \end{aligned}$$

Since $\eta > 0$ is arbitrary, we get the desired upper bound for J_n :

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log J_n \leq -\frac{y^2}{2\sigma^2}.$$

Combining this with the lower bound (4.16), we conclude the proof of (4.11).

Combining (4.10) and (4.11), we obtain the moderate deviation principle (2.6), as mentioned in the beginning of the proof. \square

4.2. Proof of Theorem 2.4. We now come to the proof of Theorem 2.4 without assuming the proximality condition A3. The proof is based on Theorem 2.3 applied to $\|\wedge^p G_n\|$ which will be introduced below and which satisfies the proximality condition A3, using the p -th exterior power representation approach developed in [7]. In [7, Theorem V. 6.2], this approach is used to establish large deviation bounds for the vector norm $|G_n x|$ and the operator norm $\|G_n\|$; it allows to relax the proximality condition A3, but fails to give the rate function. For moderate deviations, the situation is different: with this approach we are able to get the rate function explicitly.

In order to prove Theorem 2.4, we need to introduce some additional notation. For any integer $1 \leq p \leq d$, the p -th exterior power $\wedge^p(\mathbb{R}^d)$ is the $\binom{d}{p}$ -dimensional vector space with basis

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}, 1 \leq i_1 < i_2 < \cdots < i_p \leq d\},$$

where $(e_i)_{1 \leq i \leq d}$ is the standard orthonormal basis of \mathbb{R}^d ; it is endowed with the standard norm still denoted by $|\cdot|$ as in the case of \mathbb{R}^d (there should be no confusion in the context). For any $v_1, \dots, v_p \in \mathbb{R}^d$, the vector $v_1 \wedge \cdots \wedge v_p$ is nonzero if and only if v_1, \dots, v_p are linearly independent in \mathbb{R}^d . We write $\wedge^p g$ for the image of $g \in GL_d(\mathbb{R})$ under the representation $\wedge^p(\mathbb{R}^d)$; for any

$v_1, \dots, v_p \in \mathbb{R}^d$, the action of the matrix $\wedge^p g$ on the vector $v_1 \wedge \dots \wedge v_p$ is given by

$$\wedge^p g(v_1 \wedge \dots \wedge v_p) = gv_1 \wedge \dots \wedge gv_p.$$

The associated operator norm of $\wedge^p g$ is defined by

$$\|\wedge^p g\| = \sup\{|\langle \wedge^p g v, v \rangle| : v \in \wedge^p(\mathbb{R}^d), |v| = 1\}.$$

Since $\wedge^p(gg') = (\wedge^p g)(\wedge^p g')$ for any $g, g' \in GL_d(\mathbb{R})$, the submultiplicative property holds: $\|\wedge^p(gg')\| \leq \|\wedge^p g\| \|\wedge^p g'\|$. If the singular values of a matrix $g \in GL_d(\mathbb{R})$ is given by a_{11}, \dots, a_{dd} (arranged in decreasing order), then it holds that

$$\|\wedge^p g\| = a_{11} \dots a_{pp}. \quad (4.20)$$

As a consequence, we have $\|\wedge^p g\| \leq \|g\|^p$ and $\|\wedge^p g\| \|\wedge^{p+2} g\| \leq \|\wedge^{p+1} g\|^2$.

Let V be a subspace of $\wedge^p(\mathbb{R}^d)$. A set $S \subset \wedge^p(\mathbb{G}) := \{\wedge^p g : g \in \mathbb{G}\}$ is said to be *irreducible* on V if there is no proper subspace V_1 of V such that $gV_1 = V_1$ for all $g \in S$. A set $S \subset \wedge^p(\mathbb{G})$ is said to be *strongly irreducible* on V if there are no finite number of subspaces V_1, \dots, V_m of V such that $g(V_1 \cup \dots \cup V_m) = V_1 \cup \dots \cup V_m$ for all $g \in S$. The strong irreducibility condition **A2** means that Γ_μ is strongly irreducible on \mathbb{R}^d , or, equivalently, G_μ is strongly irreducible on \mathbb{R}^d . We refer to [7] for more details.

The following purely algebraic result is due to Chevalley [9]; see also Bougerol and Lacroix [7].

Lemma 4.2. *Let G be an irreducible subgroup of $GL_d(\mathbb{R})$. Then, for any integer $1 \leq p \leq d$, there exists a direct-sum decomposition of the p -th exterior power: $\wedge^p(\mathbb{R}^d) = V_1 \oplus \dots \oplus V_k$ such that $(\wedge^p g)V_j = V_j$ for any $g \in G$ and $1 \leq j \leq k$. Moreover, $\wedge^p(G) := \{\wedge^p g : g \in G\}$ is irreducible on each subspace $V_j, j = 1, \dots, k$.*

We say that an integer $1 \leq p \leq d$ is the *proximal dimension* of the semi-group Γ_μ , if p is the smallest integer with the following property: there exists a sequence of matrices $\{M_n\}_{n \geq 1} \subset \Gamma_\mu$ such that $\frac{M_n}{\|M_n\|}$ converges to a matrix with rank p . By definition, it is easy to verify that the proximality condition **A3** implies that the proximal dimension of Γ_μ is 1. The converse is also true if we assume that Γ_μ is irreducible, see [5] for the proof. Under the first moment condition $\mathbb{E}(\log N(g_1)) < \infty$, according to Kingman's subadditive ergodic theorem [19], the Lyapunov exponents $(\lambda_p)_{1 \leq p \leq d}$ of μ are defined recursively by

$$\lambda_1 + \dots + \lambda_p = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(\log \|\wedge^p G_n\|) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\wedge^p G_n\|, \quad a.s..$$

This formula, together with the fact that $\|\wedge^{p-1} G_n\| \|\wedge^{p+1} G_n\| \leq \|\wedge^p G_n\|^2$, yields that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$. The following fundamental result is due to

Guivarc'h and Raugi [15] and gives a sufficient condition for ensuring that two successive Lyapunov exponents are distinct. It can also be found in [7, Proposition III. 6.2].

Lemma 4.3. *Assume condition A2. If $\mathbb{E} \log N(g_1) < \infty$ and the proximal dimension of the semigroup Γ_μ is p , then we have*

$$\lambda_1 = \lambda_2 = \dots = \lambda_p > \lambda_{p+1}.$$

From (4.20) we have seen that $\|\wedge^p g\| \leq \|g\|^p$ for $1 \leq p \leq d$. The following result (see [7, Lemma III. 1.4]) provides a two-sided comparison between $\|\wedge^p g\|$ and $\|g\|^p$, where p is the proximal dimension of the semigroup Γ_μ .

Lemma 4.4. *Assume condition A2. If $\mathbb{E} \log N(g_1) < \infty$ and the proximal dimension of the semigroup Γ_μ is p , then there exists a constant $c > 0$ such that for all $g \in \Gamma_\mu$,*

$$c\|g\|^p \leq \|\wedge^p g\| \leq \|g\|^p.$$

The following lemma was proved in [7, Proposition III. 1.7 and Remark III. 1.8]. Recall that $\Gamma_{\mu,1} = \{|\det(g)|^{-1/d}g : g \in \Gamma_\mu\}$.

Lemma 4.5. (a) *If the set $\Gamma_{\mu,1}$ is not contained in a compact subgroup of \mathbb{G} , then the proximal dimension p of the semigroup Γ_μ satisfies $1 \leq p \leq d-1$.*

(b) *If the set $\Gamma_{\mu,1}$ is contained in a compact subgroup of \mathbb{G} , then there exists a scalar product on \mathbb{R}^d for which all the matrices in $\Gamma_{\mu,1}$ are orthogonal. In this case, $\log \|G_n\|$ can be written as a sum of i.i.d. real-valued random variables.*

Now we are equipped to prove the moderate deviation principle (2.7) for the operator norm $\|G_n\|$ without assuming the proximality condition A3.

Proof of (2.7) of Theorem 2.4. We assume that $\Gamma_{\mu,1}$ is not contained in a compact subgroup of \mathbb{G} ; the opposite case was already proved in Remark 2.5(2). Note that $\lambda = \lambda_1$. Without loss of generality, we assume that $\lambda_1 = 0$ since otherwise we can replace each matrix $g \in \Gamma_\mu$ by $e^{-\lambda_1}g$. As mentioned before, by Lemma 4.4 of [17], in order to prove (2.7), it is sufficient to establish the following moderate deviation asymptotics: for any $y > 0$,

$$\lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log \|G_n\|}{b_n} \geq y\right) = -\frac{y^2}{2\sigma_0^2}, \quad (4.21)$$

$$\lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log \|G_n\|}{b_n} \leq -y\right) = -\frac{y^2}{2\sigma_0^2}. \quad (4.22)$$

We first give a proof of (4.21). Let p be the proximal dimension of the semigroup Γ_μ . Since the set $\Gamma_{\mu,1}$ is not contained in a compact subgroup of \mathbb{G} , by Lemma 4.5 (a), we have $1 \leq p \leq d-1$. Using Lemma 4.3, under

condition **A2**, this implies that the Lyapunov exponents $(\lambda_p)_{1 \leq p \leq d}$ of μ satisfy

$$\lambda_1 = \dots = \lambda_p = 0 > \lambda_{p+1}.$$

It follows that the two largest Lyapunov exponents of $\wedge^p G_n$ are given by $\lambda_1 + \dots + \lambda_p = 0$ and $\lambda_2 + \dots + \lambda_{p+1} = \lambda_{p+1} < 0$ (see [7, Proposition III. 1.2]). Applying Lemma 4.2 to $G = G_\mu$ (the smallest closed subgroup of \mathbb{G} generated by the support of μ), we get the following direct-sum decomposition of the p -th exterior power $\wedge^p(\mathbb{R}^d)$:

$$\wedge^p(\mathbb{R}^d) = V_1 \oplus V_2 \oplus \dots \oplus V_k,$$

where V_j are subspaces of $\wedge^p(\mathbb{R}^d)$ such that $(\wedge^p g)V_j = V_j$ for any $g \in G_\mu$ and $1 \leq j \leq k$, i.e. each V_j is invariant under $\wedge^p(G_\mu) := \{\wedge^p g : g \in G_\mu\}$. Moreover, $\wedge^p(G_\mu)$ is irreducible on each subspace V_j . Note that the set of all Lyapunov exponents of $\wedge^p G_n$ on the space $\wedge^p(\mathbb{R}^d)$ coincides with the union of all the Lyapunov exponents of $(\wedge^p G_n)$ restricted to each subspace V_j , $1 \leq j \leq k$. Hence we can choose V_1 in such a way that the restrictions of $\wedge^p G_n$ to V_1 and $V_2 \oplus \dots \oplus V_k$, denoted respectively by G'_n and G''_n (as usual we identify the linear transform with the corresponding matrice), satisfy:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|G'_n\| = \lambda_1 + \dots + \lambda_p = 0, \quad \text{a.s.}, \quad (4.23)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|G''_n\| = \lambda_2 + \dots + \lambda_{p+1} = \lambda_{p+1} < 0, \quad \text{a.s.}, \quad (4.24)$$

$$\|\wedge^p G_n\| = \max\{\|G'_n\|, \|G''_n\|\}. \quad (4.25)$$

As in the case of G_n , G'_n and G''_n are products of i.i.d. invertible matrices of the form $G'_n = g'_n \cdots g'_1$ and $G''_n = g''_n \cdots g''_1$. We denote by μ_1 the law of the random matrix g_1 , by d_1 the dimension of the vector space V_1 , and by Γ_{μ_1} the smallest closed subsemigroup of $GL_{d_1}(\mathbb{R})$ generated by the support of μ_1 . Then, following the analogous argument used in the proof of the central limit theorem for $\|G_n\|$ (see [7, Theorem V.5.4]), one can verify, under Condition **A2** on μ , that the semigroup Γ_{μ_1} is strongly irreducible and proximal on \mathbb{R}^{d_1} . Therefore, μ_1 satisfies Conditions **A2** and **A3**, so that we are allowed to apply the moderate deviation principle (2.5) with $\varphi = 1$ and G_n replaced by G'_n , to get the following moderate deviation asymptotics: for any $y > 0$,

$$\lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log \|G'_n\|}{b_n} \geq y\right) = -\frac{y^2}{2\sigma_1^2}, \quad (4.26)$$

$$\lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log \|G'_n\|}{b_n} \leq -y\right) = -\frac{y^2}{2\sigma_1^2}, \quad (4.27)$$

where $\sigma_1^2 > 0$ is the asymptotic variance of the sequence $(G'_n)_{n \geq 1}$ given by

$$\sigma_1^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[(\log \|G'_n\|)^2]. \quad (4.28)$$

From (4.25), it follows that $\log \|\wedge^p G_n\| \geq \log \|G'_n\|$. By (4.26), we get the lower bound for $\|\wedge^p G_n\|$: for any $y > 0$,

$$\liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log \|\wedge^p G_n\|}{b_n} \geq y\right) \geq -\frac{y^2}{2\sigma_1^2}. \quad (4.29)$$

On the other hand, since the upper Lyapunov exponent of the sequence $(G''_n)_{n \geq 1}$ is strictly less than 0 (see (4.24)), we have $\mathbb{E}(\log \|G''_m\|) < 0$ for sufficiently large integer $m \geq 1$. Write $n = km + r$ with $k \geq 1$ and $0 \leq r < m$. By the identity $G''_n = [G''_n(G''_{km})^{-1}] [G''_{km}(G''_{(k-1)m})^{-1}] \cdots [G''_{2m}(G''_m)^{-1}] G''_m$, it follows that

$$\log \|G''_n\| \leq \log \|G''_n(G''_{km})^{-1}\| + \log \|G''_{km}(G''_{(k-1)m})^{-1}\| + \cdots + \log \|G''_m\|. \quad (4.30)$$

For fixed integer $m \geq 1$, we denote $u_m := -\mathbb{E}(\log \|G''_m\|) > 0$. Notice that

$$\begin{aligned} \mathbb{P}(\log \|G''_n\| \geq 0) &\leq \mathbb{P}\left(\log \|G''_n(G''_{km})^{-1}\| \geq k \frac{u_m}{2}\right) \\ &\quad + \mathbb{P}\left(\log \|G''_{km}(G''_{(k-1)m})^{-1}\| + \cdots + \log \|G''_m\| + k u_m \geq k \frac{u_m}{2}\right). \end{aligned} \quad (4.31)$$

Using (4.25) and the fact that $\|\wedge^p g\| \leq \|g\|^p$ for any $g \in \Gamma_\mu$, we get that for constant $c > 0$ small enough,

$$\mathbb{E}(\|G''_n(G''_{km})^{-1}\|^c) = \mathbb{E}(\|G''_r\|^c) \leq \mathbb{E}(\|\wedge^p G_r\|^c) \leq \mathbb{E}(\|G_r\|^{cp}) \leq [\mathbb{E}(\|g_1\|^{cp})]^r,$$

which is finite by Condition A1. By Markov's inequality and the fact that $u_m > 0$ is a constant, it follows that there exist constants $c, C > 0$ such that

$$\mathbb{P}\left(\log \|G''_n(G''_{km})^{-1}\| \geq k \frac{u_m}{2}\right) \leq \mathbb{E}(\|G''_n(G''_{km})^{-1}\|^c) e^{-ck \frac{u_m}{2}} \leq C e^{-ck}. \quad (4.32)$$

Using the large deviation bounds for sums of i.i.d. real-valued random variables, the second term in the right hand side of (4.31) is dominated by $C e^{-ck}$. Substituting this bound and (4.32) into (4.31), and taking into account $k \geq n/(m+1)$, we obtain

$$\mathbb{P}(\log \|G''_n\| \geq 0) \leq C e^{-cn}. \quad (4.33)$$

From (4.26) we derive that for any $y > 0$ and $\epsilon > 0$, there exists $n_0 \geq 1$ such that for any $n \geq n_0$,

$$\mathbb{P}(\log \|G'_n\| \geq y b_n) \leq \exp\left\{-\frac{b_n^2}{n} \left(\frac{y^2}{2\sigma_1^2} - \epsilon\right)\right\}.$$

This, together with (4.33) and (4.25), yields the upper bound for $\|\wedge^p G_n\|$: for any $y > 0$ and $\epsilon > 0$,

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left(\frac{\log \|\wedge^p G_n\|}{b_n} \geq y \right) \\
& \leq \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \left[\mathbb{P} \left(\frac{\log \|G'_n\|}{b_n} \geq y \right) + \mathbb{P} \left(\frac{\log \|G''_n\|}{b_n} \geq y \right) \right] \\
& \leq \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \left[e^{-\frac{b_n^2}{n} \left(\frac{y^2}{2\sigma_1^2} - \epsilon \right)} + C e^{-cn} \right] \\
& = \limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \left[e^{-\frac{b_n^2}{n} \left(\frac{y^2}{2\sigma_1^2} - \epsilon \right)} \left(1 + C e^{-cn + \frac{b_n^2}{n} \left(\frac{y^2}{2\sigma_1^2} - \epsilon \right)} \right) \right] \\
& = - \left(\frac{y^2}{2\sigma_1^2} - \epsilon \right).
\end{aligned}$$

Since $\epsilon > 0$ can be arbitrary small, it follows that

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left(\frac{\log \|\wedge^p G_n\|}{b_n} \geq y \right) \leq - \frac{y^2}{2\sigma_1^2}. \quad (4.34)$$

The lower and upper bounds (4.29) and (4.34) lead to

$$\lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left(\frac{\log \|\wedge^p G_n\|}{b_n} \geq y \right) = - \frac{y^2}{2\sigma_1^2}. \quad (4.35)$$

Since the proximal dimension of the semigroup Γ_μ is p , by Lemma 4.4, the sequence $\{\log \|\wedge^p G_n\| - p \log \|G_n\|\}_{n \geq 1}$ is bounded by a constant from above and below. Combining this with (4.35), we deduce the following moderate deviation asymptotic for the operator norm $\|G_n\|$: for any $y > 0$,

$$\lim_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left(\frac{\log \|G_n\|}{b_n} \geq y \right) = - \frac{y^2}{2\sigma_0^2}, \quad (4.36)$$

where $\sigma_0^2 = (\sigma_1^2)/p^2 > 0$.

We next give a proof of (4.22). From (4.25) and (4.27), the upper bound easily follows: for any $y > 0$,

$$\limsup_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P} \left(\frac{\log \|\wedge^p G_n\|}{b_n} \leq -y \right) \leq - \frac{y^2}{2\sigma_1^2}. \quad (4.37)$$

To prove the lower bound, observe that from (4.25) we have

$$\begin{aligned}
\mathbb{P}(\log \|\wedge^p G_n\| \leq -yb_n) &= \mathbb{P}(\log \|G'_n\| \leq -yb_n, \log \|G''_n\| \leq -yb_n) \\
&\geq \mathbb{P}(\log \|G'_n\| \leq -yb_n) - \mathbb{P}(\log \|G''_n\| > -yb_n).
\end{aligned}$$

Similarly to (4.33), with fixed integer $m \geq 1$ and $u_m = -\mathbb{E}(\log \|G_m''\|) > 0$, taking into account (4.30), we write

$$\begin{aligned} \mathbb{P}(\log \|G_n''\| > -yb_n) &\leq \mathbb{P}\left(\log \|G_n''(G_{km}'')^{-1}\| > k\frac{u_m}{2} - yb_n\right) \\ &+ \mathbb{P}\left(\log \|G_{km}''(G_{(k-1)m}'')^{-1}\| + \cdots + \log \|G_m''\| + ku_m > k\frac{u_m}{2}\right). \end{aligned}$$

In an analogous way as in the proof of (4.32), by Markov's inequality and the fact that $k = O(n)$ and $b_n = o(n)$, the first term on the right hand side of the above inequality is bounded by Ce^{-ck} , where $c, C > 0$ are constants. It has been shown in the proof of (4.33) that the second term is also bounded by Ce^{-ck} . Therefore, taking into account $k \geq n/(m+1)$, we get that there exist constants $c, C > 0$ such that

$$\mathbb{P}(\log \|G_n''\| > -yb_n) \leq Ce^{-cn}.$$

Combining this bound with (4.27), we obtain

$$\liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log \|\wedge^p G_n\|}{b_n} \leq -y\right) \geq -\frac{y^2}{2\sigma_1^2}.$$

By Lemma 4.4, this implies

$$\liminf_{n \rightarrow \infty} \frac{n}{b_n^2} \log \mathbb{P}\left(\frac{\log \|G_n\|}{b_n} \leq -y\right) \geq -\frac{y^2}{2\sigma_0^2}, \quad (4.38)$$

where $\sigma_0^2 = (\sigma_1^2)/p^2 > 0$. Putting together (4.37) and (4.38), we conclude the proof of (4.22). Combining (4.21) and (4.22), we get the desired moderate deviation principle (2.7) for the operator norm $\|G_n\|$. \square

Proof of (2.8) of Theorem 2.4. Using Lemma 3.3, we can obtain (2.8) from (2.7) just as we obtained (2.6) from (2.5). The details are omitted. \square

5. MODERATE DEVIATION EXPANSIONS

This section is devoted to proving Theorems 2.6 and 2.7 about Cramér type moderate deviation expansions in the normal range, for the operator norm $\|G_n\|$ and the spectral radius $\rho(G_n)$. We will use the following moderate deviation expansion for the norm cocycle $\log |G_n x|$ recently established in [23] (where the usual range $y \in [0, o(n^{1/2})]$ is considered):

Lemma 5.1. *Assume conditions A1, A2 and A3. Then, we have, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [0, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$, as $n \rightarrow \infty$,*

$$\begin{aligned} \frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - n\lambda \geq \sqrt{n}\sigma y\}}]}{1 - \Phi(y)} &= \nu(\varphi) + \|\varphi\|_\gamma O\left(\frac{y+1}{\sqrt{n}}\right), \\ \frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - n\lambda \leq -\sqrt{n}\sigma y\}}]}{\Phi(-y)} &= \nu(\varphi) + \|\varphi\|_\gamma O\left(\frac{y+1}{\sqrt{n}}\right). \end{aligned}$$

Proof of Theorem 2.6. Using the Berry-Esseen type bound (2.1) for the couple $(X_n^x, \log \|G_n\|)$, we get that there exists a constant $C > 0$ such that for all $n \geq 1$, $x \in \mathbb{P}^{d-1}$, $y > 0$ and $\varphi \in \mathcal{B}_\gamma$,

$$\left| \frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \leq -\sqrt{n}\sigma y\}}]}{\Phi(-y)} - \nu(\varphi) \right| \leq C \frac{\log n}{\sqrt{n}\Phi(-y)} \|\varphi\|_\gamma. \quad (5.1)$$

Using the basic inequality

$$\frac{1}{\sqrt{2\pi}} \left(\frac{1}{y} - \frac{1}{y^3} \right) e^{-\frac{y^2}{2}} \leq \Phi(-y) \leq \frac{1}{\sqrt{2\pi}y} e^{-\frac{y^2}{2}} \quad \text{for } y > 0, \quad (5.2)$$

it is easy to see that $\frac{\log n}{\sqrt{n}\Phi(-y)} = O(n^{-3/8}(\log n)^{3/2}) \rightarrow 0$, as $n \rightarrow \infty$, uniformly in $y \in [0, \frac{1}{2}\sqrt{\log n}]$. Therefore, from (5.1) we see that the expansion (2.11) holds uniformly in $y \in [0, \frac{1}{2}\sqrt{\log n}]$. In the same way, using again the Berry-Esseen type bound (2.1) but together with the fact that $|\mathbb{E}\varphi(X_n^x) - \nu(\varphi)| \leq Ce^{-cn}\|\varphi\|_\gamma$, one can also verify that the expansion (2.10) also holds uniformly in $y \in [0, \frac{1}{2}\sqrt{\log n}]$.

It remains to prove that expansions (2.10) and (2.11) hold uniformly in $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$. Without loss of generality we assume that the target function φ is non-negative (otherwise we can consider the positive and negative parts of φ). We only give a proof of (2.11), since (2.10) can be established in a similar way. For simplicity, we denote for any $x \in \mathbb{P}^{d-1}$ and $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$,

$$I_n := \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \leq -\sqrt{n}\sigma y\}} \right].$$

The proof consists of establishing the upper and lower bounds. From Lemma 5.1 on $\log |G_n x|$ and the fact that $|G_n x| \leq \|G_n\|$, the upper bound of I_n immediately follows: there exists a constant $C > 0$ such that for all $n \geq 1$, $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{I_n}{\Phi(-y)} \leq \nu(\varphi) + C \|\varphi\|_\gamma \frac{y+1}{\sqrt{n}}. \quad (5.3)$$

For the lower bound of I_n , we shall use Lemma 3.2. For any $a > 0$ and $n > k \geq 1$, consider the event

$$A_{n,k} = \left\{ \left| \log \|G_n\| - \log \frac{|G_n x|}{|G_k x|} - \log \|G_k\| \right| \leq e^{-ak} \right\},$$

and we write $A_{n,k}^c$ for its complement. Since the function φ is assumed to be non-negative, using Lemma 3.2 we get that for any $a > 0$, there exist

$c_1 > 0$ and $k_0 \in \mathbb{N}$ such that for all $n \geq k \geq k_0$, uniformly in $x \in \mathbb{P}^{d-1}$,

$$\begin{aligned}
I_n &\geq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \leq -\sqrt{n}\sigma y\}} \mathbb{1}_{\{A_{n,k}\}} \right] \\
&\geq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - \log |G_k x| + \log \|G_k\| - n\lambda + e^{-ak} \leq -\sqrt{n}\sigma y\}} \mathbb{1}_{\{A_{n,k}\}} \right] \\
&\geq \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - \log |G_k x| + \log \|G_k\| - n\lambda + e^{-ak} \leq -\sqrt{n}\sigma y\}} \right] - e^{-c_1 k} \|\varphi\|_\infty \\
&=: J_n - e^{-c_1 k} \|\varphi\|_\infty.
\end{aligned} \tag{5.4}$$

We now give a lower bound for J_n . As before, for any $n > k \geq 1$, we write $G_n = G_{n,k} G_k$ with $G_{n,k} = g_n \dots g_{k+1}$ and $G_k = g_k \dots g_1$. We take the conditional expectation with respect to the filtration $\mathcal{F}_k = \sigma(g_1, \dots, g_k)$ and use the large deviation bound (3.3) for the operator norm $\|G_k\|$, to obtain that, for any $q > \lambda$, there exists a constant $c_2 > 0$ such that

$$\begin{aligned}
J_n &= \mathbb{E} \left\{ \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - \log |G_k x| + \log \|G_k\| - n\lambda + e^{-ak} \leq -\sqrt{n}\sigma y\}} \middle| \mathcal{F}_k \right] \right\} \\
&\geq \mathbb{E} \left\{ \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - \log |G_k x| + \log \|G_k\| - n\lambda + e^{-ak} \leq -\sqrt{n}\sigma y\}} \mathbb{1}_{\{\log \|G_k\| \leq kq\}} \middle| \mathcal{F}_k \right] \right\} \\
&\geq \mathbb{E} \left\{ \mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log |G_n x| - \log |G_k x| + kq - n\lambda + e^{-ak} \leq -\sqrt{n}\sigma y\}} \middle| \mathcal{F}_k \right] \right\} - e^{-c_2 k} \|\varphi\|_\infty \\
&=: J'_n - e^{-c_2 k} \|\varphi\|_\infty.
\end{aligned} \tag{5.5}$$

For brevity, we set

$$y_1 = y \sqrt{\frac{n}{n-k}} - \frac{k(q-\lambda)}{\sigma \sqrt{n-k}} - \frac{e^{-ak}}{\sigma \sqrt{n-k}}, \tag{5.6}$$

then J'_n can be rewritten as

$$J'_n = \mathbb{E} \left\{ \mathbb{E} \left[\varphi(G_{n,k} \cdot X_k^x) \mathbb{1}_{\{\log |G_{n,k} X_k^x| - (n-k)\lambda \leq -\sqrt{n-k}\sigma y_1\}} \middle| \mathcal{F}_k \right] \right\}.$$

For any $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$, we take

$$k = \lfloor C_1 y^2 \rfloor, \tag{5.7}$$

where $C_1 > 0$ is a constant whose value will be chosen large enough. From (5.6) and (5.7), we see that $y \sim y_1 = o(n^{1/6})$ as $n \rightarrow \infty$. Hence we are allowed to apply Lemma 5.1 to obtain the following moderate deviation expansion for J'_n : as $n \rightarrow \infty$, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{J'_n}{\Phi(-y_1)} = \nu(\varphi) + \|\varphi\|_\gamma O\left(\frac{y_1 + 1}{\sqrt{n}}\right). \tag{5.8}$$

Using the asymptotic expansion $\sqrt{2\pi}\Phi(-y) = \frac{1}{y}e^{-\frac{y^2}{2}}[1 + O(\frac{1}{y^2})]$ as $y \rightarrow \infty$ (cf. (5.2)), we get that as $n \rightarrow \infty$, uniformly in $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$,

$$\frac{\Phi(-y)}{\Phi(-y_1)} = \frac{\frac{1}{y}e^{-\frac{y^2}{2}}[1 + O(\frac{1}{y^2})]}{\frac{1}{y_1}e^{-\frac{y_1^2}{2}}[1 + O(\frac{1}{y_1^2})]} = \frac{y_1}{y}e^{-\frac{y^2}{2} + \frac{y_1^2}{2}}(1 + o(1)).$$

Taking into account (5.6) and (5.7), one can find that, as $n \rightarrow \infty$, uniformly in $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$, we have $\frac{y_1}{y} = 1 + o(1)$ and

$$\begin{aligned} e^{-\frac{y^2}{2} + \frac{y_1^2}{2}} &= \exp \left\{ \frac{ky^2}{2(n-k)} - \frac{y\sqrt{n}(e^{-ak} + k(q-\lambda))}{\sigma(n-k)} + \frac{(e^{-ak} + k(q-\lambda))^2}{2\sigma^2(n-k)} \right\} \\ &= 1 + o(1). \end{aligned}$$

Consequently, substituting the above estimates into (5.8), we get that, as $n \rightarrow \infty$, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{J'_n}{\Phi(-y)} = \nu(\varphi) + \|\varphi\|_\gamma o(1).$$

This, together with (5.5), implies the following lower bound for I_n : there exists a constant $c_3 > 0$ such that uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} \frac{I_n}{\Phi(-y)} &\geq \nu(\varphi) + \|\varphi\|_\gamma o(1) - \|\varphi\|_\infty \frac{2e^{-c_3k}}{\Phi(-y)} \\ &= \nu(\varphi) + \|\varphi\|_\gamma o(1) - \|\varphi\|_\infty \frac{2e^{-c_3\lfloor C_1 y^2 \rfloor}}{\Phi(-y)}. \end{aligned}$$

Using the inequality (5.2) and taking $C_1 > \frac{1}{c_3}$, it follows that, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} \frac{I_n}{\Phi(-y)} &\geq \nu(\varphi) + \|\varphi\|_\gamma o(1) - 4\|\varphi\|_\infty y e^{-c_3\lfloor C_1 y^2 \rfloor + \frac{y^2}{2}} \\ &= \nu(\varphi) + \|\varphi\|_\gamma o(1). \end{aligned} \tag{5.9}$$

Combining this with the upper bound (5.3), we conclude the proof of the expansion (2.11). \square

We proceed to establish Theorem 2.7 based on Theorem 2.6, Lemma 3.3 and the Berry-Esseen type bound (2.2).

Proof of Theorem 2.7. We only prove the first expansion (2.14) since the proof of the second one (2.15) can be carried out in an analogous way.

We first remark that for the range of small values $y \in [0, \frac{1}{2}\sqrt{\log n}]$, the moderate deviation expansion (2.14) is a direct consequence of the Berry-Esseen type bound (2.2). Indeed, from (2.2) and the fact that $|\mathbb{E}\varphi(X_n^x) - \nu(\varphi)| \leq Ce^{-cn}\|\varphi\|_\gamma$, we derive that uniformly in $y > 0$,

$$\left| \frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log \rho(G_n) - n\lambda \geq \sqrt{n}\sigma y\}}]}{1 - \Phi(y)} - \nu(\varphi) \right| \leq C \frac{\log n}{\sqrt{n}(1 - \Phi(y))} \|\varphi\|_\gamma.$$

Using the inequality (5.2), Using the inequality (5.2), one can verify that $\frac{\log n}{\sqrt{n}(1 - \Phi(y))} \rightarrow 0$, as $n \rightarrow \infty$, uniformly in $y \in [0, \frac{1}{2}\sqrt{\log n}]$. Hence the expansion (2.14) holds uniformly in $y \in [0, \frac{1}{2}\sqrt{\log n}]$.

In the sequel we prove that (2.14) holds uniformly in $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$. Without loss of generality, we assume that the target function φ is non-negative. For brevity, we denote for $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$,

$$I_n := \mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log \rho(G_n) - n\lambda \geq \sqrt{n}\sigma y\}}].$$

The proof consists of establishing upper and lower bounds.

For the upper bound, we use the inequality $\rho(G_n) \leq \|G_n\|$, and the moderate deviation expansion (2.10) for the operator norm $\|G_n\|$, to get the desired bound: as $n \rightarrow \infty$, uniformly in $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$,

$$\frac{I_n}{1 - \Phi(y)} \leq \nu(\varphi) + \|\varphi\|_\gamma o(1). \quad (5.10)$$

For the lower bound, we shall apply Lemma 3.3 for a precise comparison between $\rho(G_n)$ and $\|G_n\|$. For any $\varepsilon > 0$ and $n > k \geq 1$, we denote

$$A_{n,k} = \left\{ \log \rho(G_n) - \log \|G_n\| > -\varepsilon k \right\}.$$

From Lemma 3.3 we know that for any $\varepsilon > 0$, there exist $c_1 > 0$ and $k_0 \in \mathbb{N}$ such that for all $n \geq k \geq k_0$, we have $\mathbb{P}(A_{n,k}) > 1 - e^{-c_1 k}$. Thus,

$$\begin{aligned} I_n &\geq \mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log \rho(G_n) - n\lambda \geq \sqrt{n}\sigma y\}} \mathbb{1}_{\{A_{n,k}\}}] \\ &\geq \mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \geq \sqrt{n}\sigma y + \varepsilon k\}} \mathbb{1}_{\{A_{n,k}\}}] \\ &\geq \mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \geq \sqrt{n}\sigma y + \varepsilon k\}}] - e^{-c_1 k} \|\varphi\|_\infty. \end{aligned} \quad (5.11)$$

By Theorem 2.6, we have, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E}[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \geq \sqrt{n}\sigma y + \varepsilon k\}}]}{1 - \Phi(y + \frac{\varepsilon k}{\sqrt{n}\sigma})} = \nu(\varphi) + \|\varphi\|_\gamma o(1). \quad (5.12)$$

For $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$, we take

$$k = \lfloor C_1 y^2 \rfloor, \quad (5.13)$$

where $C_1 > 0$ is a constant whose value will be chosen large enough. From the asymptotic expansion $\int_y^\infty e^{-\frac{t^2}{2}} dt = \frac{1}{y} e^{-\frac{y^2}{2}} [1 + O(\frac{1}{y^2})]$ as $y \rightarrow \infty$ (see (5.2)), we infer that as $n \rightarrow \infty$, uniformly in $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$,

$$\begin{aligned} \frac{1 - \Phi(y + \frac{\varepsilon k}{\sqrt{n}\sigma})}{1 - \Phi(y)} &= \frac{y}{y + \frac{\varepsilon k}{\sqrt{n}\sigma}} \exp \left\{ \frac{1}{2} y^2 - \frac{1}{2} \left(y + \frac{\varepsilon k}{\sqrt{n}\sigma} \right)^2 \right\} (1 + o(1)) \\ &= \frac{y}{y + \frac{\varepsilon k}{\sqrt{n}\sigma}} \exp \left\{ -y \frac{\varepsilon k}{\sqrt{n}\sigma} - \frac{\varepsilon^2 k^2}{2n\sigma^2} \right\} (1 + o(1)). \end{aligned}$$

Since $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$, taking into account (5.13), we get

$$\frac{y}{y + \frac{\varepsilon k}{\sqrt{n}\sigma}} = 1 - \frac{\frac{\varepsilon k}{\sqrt{n}\sigma}}{y + \frac{\varepsilon k}{\sqrt{n}\sigma}} \geq 1 - \frac{\varepsilon k}{y\sqrt{n}\sigma} = 1 - \frac{\varepsilon \lfloor C_1 y^2 \rfloor}{y\sqrt{n}\sigma} = 1 + o(1),$$

and

$$\exp \left\{ -y \frac{\varepsilon k}{\sqrt{n}\sigma} - \frac{\varepsilon^2 k^2}{2n\sigma^2} \right\} = \exp \left\{ -y \frac{\varepsilon \lfloor C_1 y^2 \rfloor}{\sqrt{n}\sigma} - \frac{\varepsilon^2 \lfloor C_1 y^2 \rfloor^2}{2n\sigma^2} \right\} = 1 + o(1).$$

Hence, substituting the above estimates into (5.12), we get, uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\frac{\mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\left\{ \log \|G_n\| - n\lambda \geq \sqrt{n}\sigma y + \varepsilon k \right\}} \right]}{1 - \Phi(y)} \geq \nu(\varphi) + \|\varphi\|_\gamma o(1).$$

This, together with (5.11), implies the lower bound for I_n : uniformly in $x \in \mathbb{P}^{d-1}$, $y \in [\frac{1}{2}\sqrt{\log n}, o(n^{1/6})]$ and $\varphi \in \mathcal{B}_\gamma$,

$$\begin{aligned} \frac{I_n}{1 - \Phi(y)} &\geq \nu(\varphi) + \|\varphi\|_\gamma o(1) - \frac{e^{-c_1 k}}{1 - \Phi(y)} \|\varphi\|_\infty \\ &\geq \nu(\varphi) + \|\varphi\|_\gamma o(1) - 2y \exp \left\{ -c_1 \lfloor C_1 y^2 \rfloor + \frac{y^2}{2} \right\} \|\varphi\|_\infty \\ &\geq \nu(\varphi) + \|\varphi\|_\gamma o(1), \end{aligned} \quad (5.14)$$

where in the second inequality we use (5.2) and in the last inequality we take $C_1 > \frac{1}{2c_1}$.

Combining this lower bound (5.14) with the upper bound (5.10), we conclude the proof of Theorem 2.7. \square

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Current address, Xiao, H., Grama, I., Liu, Q.: Université de Bretagne-Sud, LMBA UMR CNRS 6205, Vannes, France.

Email address: `quansheng.liu@univ-ubs.fr`