

Flatness: an overview

Part II: new trends

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- 1 Checking flatness: a difficult problem
- 2 Motion planning and tracking with singularities
- 3 Flatness and PDEs

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The initial hope: distributions and Lie brackets

Affine system

- Distribution: module of vector fields
- Lie bracket: $[f, g](x) := Dg(x) \cdot f(x) - Df(x) \cdot g(x)$
- involutive distribution: $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$

Prototype result [JR1980, HSM1983]

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i, \quad x \in \mathbb{R}^n$$

is linearizable by invertible static feedback and coordinate change iff:

- $\mathcal{D}_0 := \text{span}\{g_1, \dots, g_m\}$ is involutive
- $\mathcal{D}_k := \text{span}\{D_{k-1}, [f, D_{k-1}]\}$ is involutive, $i = 1, \dots, n - 2$
- $\text{rank } D_{k-1} = n$ (controllability)

Distributions and Lie brackets, the right tools?

- Not all flat systems are linearizable by static feedback!

A collection of results (for affine systems $\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i$)

- 1 input: flat iff linearizable by static feedback [CLM1989]
- $n - 1$ inputs: flat iff controllable [CLM1989]
- driftless, 2 inputs, $\dot{x} = g_1(x)u_1 + g_2(x)u_2$ [MR1994]:
 - $\mathcal{E}_0 := \text{span}\{g_1, g_2\}$; $\mathcal{E}_k := \text{span}\{\mathcal{E}_{k-1}, [\mathcal{E}_{k-1}, \mathcal{E}_{k-1}]\}$, $k \geq 0$
 - flat iff $\text{rank } \mathcal{E}_k = k + 2$, $i = 0, \dots, n - 2$
- driftless, general, $\sum_{i=1}^m g_i(x)u_i$ [MR1994]:
 - $\mathcal{E}_0 := \text{span}\{g_1, \dots, g_m\}$; $\mathcal{E}_k := \text{span}\{\mathcal{E}_{k-1}, [\mathcal{E}_{k-1}, \mathcal{E}_{k-1}]\}$, $k \geq 0$
 - flat **if** $\text{rank } \mathcal{E}_k = k + m$, $i = 0, \dots, n - m$
- driftless, $n - 2$ inputs, $\sum_{i=1}^{n-2} g_i(x)u_i$: flat iff controllable [MR1995]
- NSC for systems linearizable by one-fold prolongation [NR2017]:
 - 1-fold prolongation: the simplest endogenous feedback
 - (not so simple) condition in terms of distributions and Lie brackets

Beyond distributions and Lie brackets

Sufficient conditions

- Sufficient condition for 1-flatness, $n - 2$ inputs [M1993]
- NSC for 1-flatness, 2 inputs and 4 states [P2002]
- partial extension to r -flatness, 2 inputs and 4 states [AP2007]
- try to find flat output by reduction and elimination [KSS]
- NSC for configuration flatness of Lagrangian systems underactuated by one control (ie, $n - 2$ inputs) [MR1998]
- Several ad hoc methods on many examples

Necessary conditions

- the ruled manifold criterion [S1993, R1994]
- generalization of the ruled manifold criterion [P2009]

For flatness, affine system codimension 2 \Leftrightarrow general system codim 1

Back to criterion for static feedback linearizability

For simplicity, $\dot{x} = f(x) + g(x)u$ with 1 input and 3 states

$$y = h(x) \quad (\text{tentative flat output})$$

$$\begin{aligned} \dot{y} &= Dh(x) \cdot (f(x) + g(x)u) \\ &= L_f h(x) + \cancel{L_g h(x)u} \quad \Rightarrow L_g h = 0 \end{aligned}$$

$$\begin{aligned} \ddot{y} &= DL_f h(x) \cdot (f(x) + g(x)u) \\ &= L_f^2 h(x) + \cancel{L_g L_f h(x)u} \quad \Rightarrow L_g L_f h = 0 \end{aligned}$$

$$\begin{aligned} \dddot{y} &= D^2 L_f h(x) \cdot (f(x) + g(x)u) \\ &= L_f^3 h(x) + L_g L_f^2 h(x)u \quad \Rightarrow L_g L_f^2 h \neq 0 \end{aligned}$$

Recover criterion by “closing” $L_g h = 0 = L_g L_f h = 0, L_g L_f^2 h \neq 0$

- New (higher order) equations by Lie differentiation:

$$L_f L_g = 0, L_f L_g L_f = 0, L_g^2 L_f = 0, \dots, L_f L_g L_f = 0, L_g^2 L_f = 0, \dots$$

- Recombination at order 1 using $L_f L_g h - L_g L_f h = L_{[f,g]} h$, etc:

$$L_g h = 0, L_{[f,g]} = 0, L_{[g,[f,g]]} = 0, L_{[f,[f,g]]} \neq 0!$$

Formal integrability [G1967]

Formal integrability:

- a system of PDE's is *formally integrable* if it has a solution given by a formal power series (no solution when “too many” equations)
- “mild” conditions on the system for a formal solution to be a true (analytic) solution

Main underlying problem:

- as new independent equations may arise by differentiation and recombination, difficult to “close” the system
- a very simple instance: conditions for linearizability by static feedback (the closure is a system of first-order PDE's)

Algebraic sufficient condition for formal integrability:

- 2-acyclicity of some cohomology sequence
- computable in theory, far from easy in practice!

Checking r -flatness: a question of formal integrability

$y = h(x, u, \dots, u^{(r)})$ is an r -flat output iff

$$\text{span}\{dx_1, \dots, dx_n, du_1, \dots, du_n, dh_1^{(\mu)}, \dots, dh_1^{(\mu)}\}_{0 \leq \nu \leq \mu}$$

has dimension $m(\nu + 1)$, with $\nu := n + rm$

This corresponds to the higher order system of PDE's

$$\begin{aligned} dx_i \wedge dh \wedge \dot{dh} \cdots \wedge dh^{(\nu)} &= 0, & i = 1, \dots, n \\ du^j \wedge dh \wedge \dot{dh} \cdots \wedge dh^{(\nu)} &= 0, & i = j, \dots, m \\ dh \wedge \dot{dh} \cdots \wedge dh^{(\nu)} &\neq 0, \end{aligned}$$

where $dh^{(\mu)} := dh_1^{(\mu)} \wedge \cdots \wedge dh_m^{(\mu)}$

- Is this (big!) system formally integrable?
- Does its very special structure yield simplifications?

Conclusion on checking flatness: much to do!

Open problems:

- 1 Tractable computable condition for r -flatness?
- 2 Bound on r for r -flatness, depending on m and n ? Bound possibly growing at least linearly with state dimension:

$$x_1^{(p)} = u_1$$

$$x_2^{(p)} = u_2$$

$$\dot{x}_3 = u_1 u_2$$

$y := (x_2, x_3 + \sum_{i=1}^p (-1)^i x_1^{(p-i)} u_2^{(i-1)})$ is a $(p-1)$ -flat output!

- 3 Flat = linearizable by (non-endogenous) dynamic feedback?

- Even in the absence of a general criterion, a flat output may often be found by “physical reasoning”
- No criterion \nrightarrow not interesting (cf Lyapunov functions)

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There are always singularities for nonlinear systems!

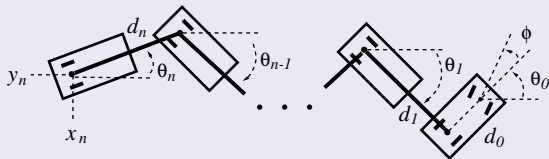
- Locally: controllability may degenerate
- Globally: flatness relations do not hold globally in general

Ingredients to overcome singularities:

- reparametrization of time
- time-varying (dynamic) linearizing feedback
- overparametrization of state
- “chart-by-chart” flatness

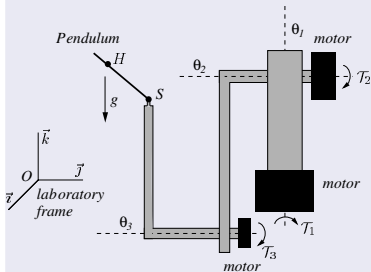
Well-known example: trailer systems

Singularity (of controllability) when system does not move



$2k\pi$, the juggling robot [LMR1998]

- Robot:
3 actuated dofs
- Pendulum:
2 UNactuated dofs



$$m\dot{\vec{V}}_C = m\vec{g} + \vec{F} \quad (\text{translation})$$

$$I\vec{u} \times \ddot{\vec{u}} = l_C \vec{u} \times \vec{F} \quad (\text{rotation})$$

$$\dot{\vec{V}}_S = \vec{A}(\vec{O}\vec{S}, \vec{V}_S, \vec{F}, \mathcal{T}) \quad (\text{robot})$$

- $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ torque inputs
- \vec{u} pendulum direction (unit vector)
- m, I, l_C positive constants
- \vec{F} coupling force

Implicit system (\vec{F} must be eliminated)

Flat output given by the *center of oscillation* H

- $\vec{OH} = \vec{OC} + \frac{I}{ml_C} \vec{u}$
- H is the only point such that $\dot{\vec{V}}_H - \vec{g}$ is colinear to the pendulum

$$\vec{OH}(t) \text{ known} \Rightarrow \vec{V}_H(t), \dot{\vec{V}}_H(t) \text{ known}$$

$$\Rightarrow \vec{u}(t) // \dot{\vec{V}}_H - \vec{g}, \dot{\vec{u}}(t), \ddot{\vec{u}}(t) \text{ known}$$

$$\Rightarrow \vec{OS}(t), \vec{OC}(t), \vec{V}_S(t), \vec{V}_C(t) \text{ known}$$

$$\Rightarrow \vec{F}, \mathcal{T} \text{ known}$$

Problem around *intrinsic* singularity $\dot{\vec{V}}_H = \vec{g}$:

$$\vec{u} = \pm \frac{\dot{\vec{V}}_H - \vec{g}}{\|\dot{\vec{V}}_H - \vec{g}\|} \quad (+ \text{ rest down, } - \text{ rest up})$$

Motion planning: crossing the singularity

To go from rest down to rest up, vertical semi-circle for H :

$$\vec{OH} = \chi_x \vec{i} + \chi_y \vec{j} + \chi_z \vec{k}$$

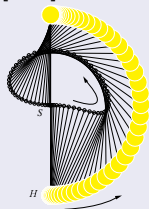
$$\chi_x(t) := r \cos(\alpha(t/T))$$

$$\chi_y(t) := 0$$

$$\chi_z(t) := r \sin(\alpha(t/T))$$

- T duration of motion
- Reparametrization of time:

$$s \in [0, 1] \mapsto t = \alpha(s)$$



Adjust T to have $t_c < T$ such that

$$\ddot{\chi}_x(t_c) = \ddot{\chi}_z(t_c) + g = 0,$$

with well-defined tangent around t_c

$$\ddot{\chi}_x(\tau + t_c) = a\tau + \mathcal{O}(\tau^2)$$

$$\ddot{\chi}_z(\tau + t_c) + g = b\tau + \mathcal{O}(\tau^2)$$

$$a^2 + b^2 \neq 0$$

On this trajectory, the orientation is always well-defined by

$$\vec{u}(t) = \text{sign}(t - t_c) \frac{\dot{\vec{V}}_H(t) - \vec{g}}{\|\dot{\vec{V}}_H(t) - \vec{g}\|}$$

The tracking feedback law

Two-level design using singular perturbations

- 1 Low-level: high-gain loops with compensation of gyroscopic effects transform the robot into a “fast” velocity actuator

$$\varepsilon \dot{\vec{V}}_S = \vec{V} - \vec{V}_S + \mathcal{O}(\varepsilon) \quad (\varepsilon \text{ is “small”})$$

- 2 High-level: a 2-step tracking controller takes care of the “slow” pendulum with input \vec{V} :
 - 2a a *global* dynamic feedback transforms the system into a “non-holonomic 3D-car” rolling without sliding on \mathbb{R}^3 (*over-parametrization* of state)
 - 2b a *time-varying* dynamic “linearizing feedback” allows the crossing of the singularity

This is not only theory!

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Flatness of PDEs with boundary control?

- Generalization of parametrization to infinite dimension?
 - Issues with convergence
 - Motion planning with initial conditions in what space?
 - Several notions of controllability (exact controllability may be impossible, except between “prepared” states)
-
- Fairly general theory for 1D linear parabolic PDEs
 - Approach somehow generalizes to PDEs with order in time $<$ order in space
 - Some generalizations to nonlinear PDEs
 - Some generalizations to 2D (linear) PDEs

A typical example

1-D piecewise constant heat equation (e.g. composite material)

$$\theta_t(x, t) = \begin{cases} \frac{a_0}{\rho_0} \theta_{xx}(x, t), & 0 < x < X \\ \frac{a_1}{\rho_1} \theta_{xx}(x, t), & X < x < 1 \end{cases}$$

$$\alpha_0 \theta(0, t) + \beta_0 a_0 \theta_x(0, t) = 0$$

$$\alpha_1 \theta(1, t) + \beta_1 a_1 \theta_x(1, t) = u(t)$$

$$\theta(X^-, t) = \theta(X^+, t)$$

$$a_0 \theta_x(X^-, t) = a_1 \theta_x(X^+, t)$$

- X location of interface; $a_0, a_1, \rho_0, \rho_1, \alpha_0, \alpha_1, \beta_0, \beta_1$ constants > 0

Main result: null controllability

Given final time $T > 0$ and initial state $\theta(\cdot, 0) := \theta_0 \in L^2(0, 1)$ find (explicit and smooth) control $t \in [0, T] \mapsto u(t)$ such that $\theta(\cdot, T) = 0$

Interlude: Gevrey functions

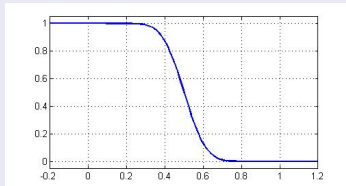
Definition (Gevrey function of one variable)

$y \in C^\infty(I, \mathbb{R})$ is in $G^s(I, \mathbb{R})$ if there exists $M, R > 0$ such that

$$|y^{(i)}(t)| \leq M \frac{i!^s}{R^i} \quad \text{for all } t \in I \text{ and } i \geq 0$$

“Gevrey step function” $\phi_s(t)$, $s = 1 + \frac{1}{k} > 1$:

$$\begin{cases} 1 & \text{if } t \leq 0 \\ 0 & \text{if } t \geq 1 \\ \frac{e^{-(1-t)^{-k}}}{e^{-(1-t)^{-k}} + e^{-t^{-k}}} & \text{if } t \in (0, 1) \end{cases}$$



Special case of general null controllability result

1-D parabolic system with fairly irregular coefficients

$$\rho\theta_t - (a\theta_x)_x - b\theta_x - c\theta = 0, \quad (x, t) \in (0, 1) \times (0, T)$$

$$\alpha_0\theta(0, t) + \beta_0(a\theta_x)(0, t) = 0$$

$$\alpha_1\theta(1, t) + \beta_1(a\theta_x)(1, t) = u(t)$$

with a, b, c, ρ functions on $(0, 1)$ such that:

- $a(x) > 0$ and $\rho(x) > 0$ for almost all $x \in (0, 1)$
- $\exists K \geq 0$ such that $c(x) \leq K\rho(x)$ for almost all $x \in (0, 1)$
- $\frac{1}{a}, \frac{b}{a}, c, \rho \in L^1(0, 1)$; $a^{1-\frac{1}{p}}\rho \in L^p(0, 1)$ for some $p \in (1, \infty]$

Theorem

Let $\theta_0 \in L^1_\rho(0, 1)$, $T > 0$ and $s \in (1, 2 - \frac{1}{p})$. Then:

- *there exists a control $u \in G^s([0, T], \mathbb{R})$ steering the system from the initial state θ_0 to the final state $\theta(\cdot, T) = 0$*
- *$\theta, a\theta_x \in G^s([\varepsilon, T], W^{1,1}(0, 1))$ for all $\varepsilon \in (0, T)$*

The system is “flat”

Parametrization by “flat output” $y(t) := \beta_0\theta(0, t) - \alpha_0(a\theta_x)(0, t)$

$$\theta(t, x) = \sum_{i \geq 0} g_i(x)y^{(i)}(t)$$

$$u(t) = \sum_{i \geq 0} (\alpha_1 g_i(1) + \beta_1 (ag'_i)(1))y^{(i)}(t)$$

with the generating functions g_i solutions of

$$\begin{aligned} (ag'_0)' &= 0, & (ag'_i)' &= \rho g_{i-1} \\ \alpha_0 g_0(0) + \beta_0 (ag'_0)(0) &= 0, & \alpha_0 g_i(0) + \beta_0 (ag'_i)(0) &= 0 \\ \beta_0 g_0(0) - \alpha_0 (ag'_0)(0) &= 1, & \beta_0 g_i(0) - \alpha_0 (ag'_i)(0) &= 0 \end{aligned}$$

Proposition (regularity of parametrization)

Assume $y \in G^s(I, \mathbb{R})$, $s \in (0, 2)$. Then:

- $u \in G^s(I, \mathbb{R})$, $s \in (0, 2)$
- $\theta, a\theta_x \in G^s(I, W^{1,1}(0, 1))$

Construct control with two-step procedure

First step (specific of PDE, not based on flatness)

Find a “regular” trajectory $t \in (0, T] \mapsto \theta^-(\cdot, t)$ starting from the initial condition θ_0 . Choose control on $[0, \tau]$, $\tau < T$ as

$$u(t) := \alpha_1 \theta^-(1, t) + \beta_1 (a \theta_x^-)(1, t)$$

Second step: use flatness to connect $\theta^-(\cdot, \tau)$ to 0

$$u(t) := \underbrace{\sum_{i \geq 0} (\alpha_1 g_i(1) + \beta_1 (a g_i')(1)) Y^{(i)}(t)}_{\text{flat parametrization of input}}, \quad \tau \leq t \leq T$$

$$Y(t) := \phi_s \left(\frac{t - \tau}{T - \tau} \right) \underbrace{\left(\beta_0 \theta^-(0, t) - \alpha_0 (a \theta_x^-)(0, t) \right)}_{\text{trace of } \theta^- \text{ on flat output}}$$

- ϕ_s “Gevrey step”, $s \in (1, 2)$
- ϕ_s , hence y and u are in $G^s([\tau, T], \mathbb{R})$

First step: null control on $[0, \tau]$ (“smoothing effect”)

Sturm-Liouville problem corresponding to system with $u(t) := 0$

\exists orthonormal basis e_n of $L^2(0, 1)$ and sequence $\lambda_n > 0$ solution of

$$-(ae'_n)' = \lambda_n \rho e_n$$

$$\alpha_0 e_n(0) + \beta_0 (ae'_n)(0) = 0$$

$$\alpha_1 e_n(1) + \beta_1 (ae'_n)(1) = 0$$

\Rightarrow Fourier expansion of initial state θ_0

$$\left\| \theta_0 - \sum_{n \geq 0} c_n e_n \right\|_{L^2(0,1)} = 0$$

Explicit expression for the e_n , numerics for the λ_n, c_n

The solution of the system starting from θ_0 with $u(t) := 0$ is then

$$\theta^-(x, t) := \sum_{n \geq 0} c_n e^{-\lambda_n t} e_n(x)$$

Summary: null controllability (+ regularity)

Theorem

Let $\theta_0 \in L^2(0, 1)$, $T > \tau > 0$, $s \in (1, 2)$. There is a function $Y \in G^s([0, T], \mathbb{R})$ such that the control

$$u(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq \tau \\ \sum_{i \geq 0} (\alpha_1 g_i(1) + \beta_1 (a g_i')(1)) Y^{(i)}(t) & \text{if } \tau < t \leq T \end{cases}$$

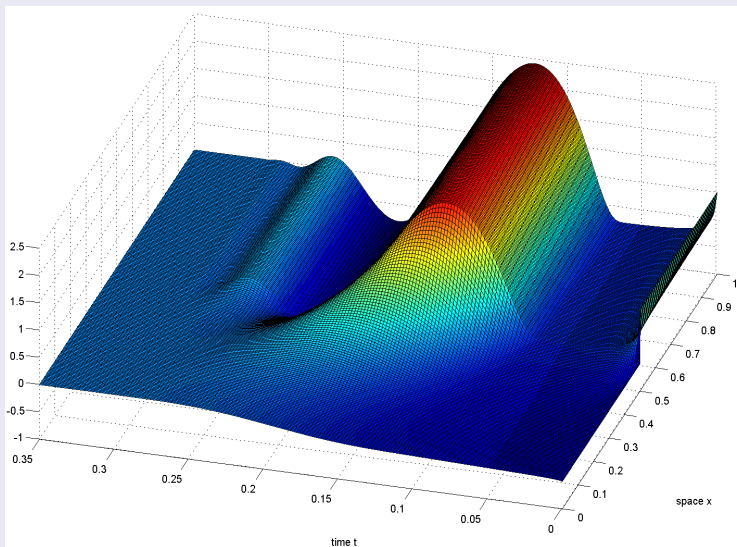
steers the system from θ_0 at time 0 to 0 at time T . Moreover

- $u \in G^s([0, T], \mathbb{R})$
- $\theta, a\theta_x \in G^s((0, T], W^{1,1}(0, 1))$

The construction of Y hence of u is explicit

Simulation: $\theta_0 = \text{step}$, $\tau = 0.05$, $T = 0.35$, $s = 1.65$

Truncated expansions: $\bar{n} = 60$, $\bar{i} = 130$



Conclusion on flatness for PDEs

Null controllability in 1-D using flatness:

- explicit formula for the control
- “straightforward” implementation
- very regular control

Still much to do!

- Fairly general theory for 1D linear parabolic PDEs
- Approach somehow generalizes to PDEs with order in time $<$ order in space
- Some generalizations to nonlinear PDEs
- Some generalizations to 2D (linear) PDEs
- There are results, with completely different methods, for some PDEs with order in time = order in space