

# Flatness: an overview

## Part I: basic concepts

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# Motivation for equivalence, flatness, etc

## Historical motivation

- Linearization by (full-state) dynamic feedback
- Input-Output linearization by dynamic feedback

## “Intellectual” motivation

- A central concept in mathematics: equivalence under a “group” of transformations
- Equivalent objects are “identical”
- Main (difficult) problem: classify objects and give “normal” forms
- Important (easier?) subproblem: check if an object may be transformed into the “simplest” normal form

Reference: Martin, Murray, Rouchon (1997). *Flat systems*

Mini-course ECC97, updated as Caltech Technical Report CDS 2003-008 *Flat systems, equivalence and trajectory generation*

- In control theory, the objects are control systems

$$\dot{x} = f(x, u)$$

- “Simplest” systems = chains of integrators

For a sensible equivalence notion, transformations should:

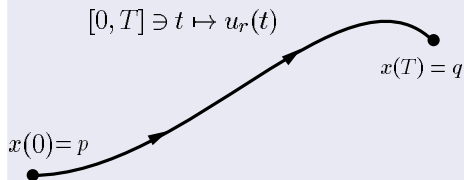
- preserve fundamental system properties (controllability, feedback linearizability, etc)
- be in some sense “realizable”

⇒ importance of transformations consisting of feedbacks and changes of coordinates

What are the interesting notions of equivalence in control theory?

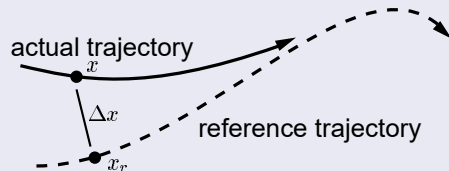
# Two central problems in control theory

1. Plan a reference trajectory  
= *open-loop problem*



Find  $u_r(t)$  such that the solution satisfies  $x(0) = p$  and  $x(T) = q$

2. Track a reference trajectory  
= *closed-loop problem*



Compute  $u = u_r + \Delta u$  such that the error  $\Delta x := x - x_r \rightarrow 0$

- Two very difficult problems in general!
- Very easy for chains of integrators!

$$\dot{x} = Ax + Bu$$

- The transformations

$$u = Fx + Gv \quad (\text{invertible linear static feedback})$$

$$x = Ty \quad (\text{linear change of coordinates})$$

form a group, hence an equivalence relation

- Normal form = Brunovsky form (chains of integrators)
- NSC for being “simple” = controllability

# A first generalization to nonlinear systems

$$\dot{x} = f(x, u)$$

- The transformations

$$u = \alpha(x, v) \quad (\text{invertible nonlinear static feedback})$$

$$x = \phi(y) \quad (\text{nonlinear change of coordinates})$$

form a group, hence an equivalence relation

- Normal (local, “generic”) form = Brunovsky form
- NSC for being “simple” = geometric conditions involving distributions of vector fields and Lie brackets

## Next step:

use dynamic feedback

$$u = \alpha(x, z, v) \quad (\text{invertible nonlinear dynamic feedback})$$

$$\dot{z} = a(x, z, v)$$

$$(x, z) = \phi(y) \quad (\text{"extended" nonlinear change of coordinates})$$

## Motivation:

some systems are linearizable by dynamic feedback, but not by static feedback

No interesting equivalence relation associated with invertible dynamic feedbacks: fundamental properties may be lost (controllability, feedback linearizability, . . .)

## Not possible to do less than work up to:

(i) integrators  $\Rightarrow$  extend  $f$  by  $\nu \in \mathbb{N}^m$  integrators,

$$f^\nu \begin{cases} \dot{x} & = f(x, u) \\ u^{(\nu)} & = v \end{cases}$$

(ii) changes of coordinates  $\Rightarrow \dot{x} = f(x, u)$  and  $\dot{y} = g(y, u)$  are *related* when there is an invertible mapping  $x = \varphi(y)$  such that

$$f(\varphi(y), u) = \partial_y \varphi(y) \cdot g(y, u)$$

## Definition

The feedback  $B$  is *endogenous* if there is a feedback  $B'$  such that  $(f_B)_{B'}$  and  $f^\nu$  are related

Properties preserved by pure integrators and changes of coordinates are preserved by endogenous feedbacks



# Characterization of endogenous feedbacks

## Theorem

*The feedback  $B$  is endogenous iff there is a map  $\kappa$  such that*

$$(z, v) = \kappa(x, \bar{\alpha}(x, z, \bar{v}))$$

*More simply,  $(z, v) = \kappa(x, \bar{u})$ : the feedback is “developing from within”*

## Sketch of the proof:

decouple the IO system  $(f_B, u)$ , which is invertible by assumption, by dynamic feedback

## Typical endogenous feedback:

cascade of integrators and invertible static feedbacks

# From endogenous feedback to equivalence

Endogenous feedbacks define an equivalence relation :

- the composition of endogenous feedbacks is endogenous
- the “inverse” of an endogenous feedback is endogenous

## Definition

$f$  and  $g$  are *equivalent* if there is an endogenous feedback  $B$  such that  $f_B$  and  $g^\nu$  are related

## Interesting equivalence relation:

preserves all the properties preserved by pure integrators and changes of coordinates

Endogenous: “minimal” property for interesting notion of equivalence by feedback

# An important special case: flatness

## Definition (Flat system)

$f$  is *flat* if it is equivalent to a trivial system  
(=  $f$  is linearizable by endogenous feedback)

linearizable by invertible static feedback

⇓ ✗

flat = linearizable by endogenous feedback

⇓ ?

linearizable by invertible dynamic feedback

⇓ ✗

controllable

# Equivalence, step 1: getting rid of feedback

## Theorem

$\dot{x} = f(x, u)$  and  $\dot{y} = g(y, v)$  are equivalent iff there are 4 maps

$$\begin{aligned}x &= \varphi(y, \bar{v}) & y &= \psi(x, \bar{u}) \\u &= \alpha(y, \bar{v}) & v &= \beta(x, \bar{u})\end{aligned}$$

which are reciprocal

$$\begin{aligned}x &= \varphi(\psi(x, \bar{u}), \bar{\beta}(x, \bar{u})) & y &= \psi(\varphi(y, \bar{v}), \bar{\alpha}(y, \bar{v})) \\u &= \alpha(\psi(x, \bar{u}), \bar{\beta}(x, \bar{u})) & v &= \beta(\varphi(y, \bar{v}), \bar{\alpha}(y, \bar{v}))\end{aligned}$$

and “exchange” solutions

$$\begin{aligned}f(\varphi(y, \bar{v}), \alpha(y, \bar{v})) &= \partial_y \varphi(y, \bar{v}) \cdot g(y, v) + \partial_{\bar{v}} \varphi(y, \bar{v}) \cdot \dot{\bar{v}} \\g(\psi(x, \bar{u}), \beta(x, \bar{u})) &= \partial_x \psi(x, \bar{u}) \cdot f(x, u) + \partial_{\bar{u}} \psi(x, \bar{u}) \cdot \dot{\bar{u}}\end{aligned}$$

Equivalence = 1 – 1 correspondence between solutions

## Corollary (“usual” definition of flatness)

$\dot{x} = f(x, u)$  is flat iff there is a map  $y = h(x, \bar{u})$  such that

$$\begin{aligned}x &= \varphi(\bar{y}) \\ u &= \alpha(\bar{y})\end{aligned}$$

$y = h(x, \bar{u})$  is called a flat (or linearizing) output

Flatness = 1 – 1 correspondence between solutions  $(x(t), u(t))$  of the system and arbitrary functions  $y(t) = (y_1(t), \dots, y_m(t))$

Flat output = generalization of the “Brunovsky output” of a controllable linear system

## Equivalence, step 2: getting rid of state dimension

Replace  $\dot{x} = f(x, u)$  with the “infinite” vector field

$$\begin{aligned}\dot{x} &= f(x, u) \\ \dot{u} &= u^1 \\ \dot{u}^1 &= u^2 \\ &\vdots\end{aligned}$$

i.e.,  $\xi := (x, u, u^1, \dots) \mapsto F(\xi) := (f(x, u), u^1, u^2, \dots)$

### Definition

The infinite vector fields  $F$  and  $G$  are *equivalent* if there is a “smooth” invertible mapping  $\xi = \Psi(\zeta)$  such that

$$\forall \xi, \quad G(\Psi(\xi)) = \partial_\xi \Psi(\xi) \cdot F(\xi)$$

Equivalence = 1 – 1 correspondence between solutions

## Equivalence, step 3: getting rid of control theory

System: underdetermined set of differential equations (no state/input)

$$F(\bar{\xi}) = 0$$

### Definition

$F(\bar{\xi}) = 0$  and  $G(\bar{\zeta}) = 0$  are *equivalent* if there are maps  $\xi = \Phi(\bar{\zeta})$  and  $\zeta = \Psi(\bar{\xi})$  which are “reciprocal”,

$$\xi = \Phi \circ \bar{\Psi}(\bar{\xi}) \quad \zeta = \Psi \circ \bar{\Phi}(\bar{\zeta}),$$

and “exchange” solutions

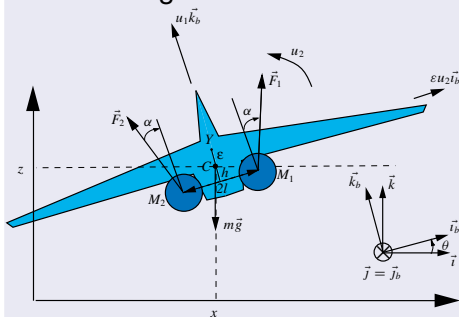
$$\begin{aligned} \forall \xi, F(\bar{\xi}) = 0 &\Rightarrow G(\bar{\Psi}(\bar{\xi})) = 0 \\ \forall \zeta, G(\bar{\zeta}) = 0 &\Rightarrow F(\bar{\Phi}(\bar{\zeta})) = 0 \end{aligned}$$

Not new! Transformations reversible “without integrals” (Hilbert, 1912)

# Example: the planar VTOL aircraft

$$\begin{aligned}\ddot{x} &= -u_1 \sin \theta + \varepsilon u_2 \cos \theta \\ \ddot{z} &= u_1 \cos \theta + \varepsilon u_2 \sin \theta - 1 \\ \ddot{\theta} &= u_2\end{aligned}$$

- $u_1$  thrust,  $u_2$  roll moment
- $(x, z)$  center of mass
- $\theta$  roll angle



Flat output = *center of oscillation*

$$(y_1, y_2) := (x - \varepsilon \sin \theta, z + \varepsilon \cos \theta)$$

Flatness relations:

$$(y_1 - x)^2 + (y_2 - z)^2 = \varepsilon^2$$

$$(y_1 - x)(\ddot{y}_2 + 1) - (y_2 - z)\ddot{y}_1 = 0$$

$$(\ddot{y}_2 + 1) \sin \theta + \ddot{y}_1 \cos \theta = 0$$



## Two easy problems for flat systems:

- motion planning
- trajectory tracking

Many systems in control engineering are flat!