Interacting non-linear reinforced stochastic processes: Synchronization and no-synchronization

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HAL Id: hal-02910341
https://hal.science/hal-02910341

Submitted on 1 Aug 2020

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ABSTRACT. *Rich get richer rule* comforts previously often chosen actions. What is happening to the evolution of individual inclinations to choose an action when agents do interact ? Interaction tends to homogenize while each individual dynamics tends to reinforce its own position. Interacting stochastic systems of reinforced processes were recently considered in many papers, where the asymptotic behavior was proven to exhibit a.s. synchronization. We consider in this paper models where, even if interaction among agents is present, absence of synchronization may happen due to the choice of an individual non-linear reinforcement. We show how these systems can naturally be considered as models for coordination games [67], technological or opinion dynamics.

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Keywords. Interacting agents; Interacting stochastic processes; Reinforced stochastic process; Urn model; Non-linear Pólya urn; Generalized Pólya urn; Reinforcement learning; Stochastic approximation; Evolutionary-game theory; Coordination games; Technological dynamics; CODA models.
JEL Classification.

C6 Mathematical Methods; Programming Models; Mathematical and Simulation Modeling

MSC2010 Classification. 91A12; 91D30 ; 62L20; 60K35; 60F15; 62P25; 62P20

1. INTRODUCTION

The stochastic evolution of systems composed by agents which interact among each other has always been of great interest in several scientific fields. For example, economic and social sciences deal with agents that make decisions under the influence of other agents or some external media. Moreover, preferences and beliefs are partly transmitted by means of various forms of social interaction and opinions are driven by *social influence i.e.* the tendency of individuals to become more similar when they interact (*e.g.* [10, 11, 21, 60]).

A natural description of such systems is provided by Agent Based Modeling [16, 73], where they are modeled as a collection of decision-making agents with a set of rules (defined at a microscopic level) that includes several issues, for instance learning and adaptation, environmental constraints and so on [16, 22, 40]. The character of interactions and influence among agents (or among groups of agents) are crucial in these models and can give rise to emergent phenomena observed in the systems [12, 13]. Agent-based models abound in a variety of disciplines, including economics, game theory, sociology and political science (*e.g.* [23, 25, 39, 44, 46, 48–50, 56, 68, 74]). However, although they are often effective in describing real situations, these models are mainly computational. Due to the many variables involved, it is indeed usually hard to prove analytic results in a rigorous way. On the other hand mathematical literature can be a source of inspiration to improve these models,

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since theoretical results may shed light on aspects that are difficult to be captured with only a numerical approach, and help to guess emergent behaviors that are unexpected in a computational perspective. For example, many mathematical results in the context of urn models have been used to design and study agent-based models both analytically and computationally.

From a mathematical point of view, there exists a growing interest in systems of *interacting reinforced* stochastic processes of different kinds (*e.g.* [4–7, 18, 27, 29, 34–36, 52, 55, 61–63, 69, 72]). Our work is placed in the stream of this scientific literature. Generally speaking, by reinforcement in a stochastic dynamics we mean any mechanism for which the probability that a given event occurs increases with the number of times the same event occurred in the past. This *"reinforcement mechanism"*, also known as *"Rich get richer rule"* or *"Matthew effect"*, is a key feature governing the dynamics of many biological, economic and social systems (see, *e.g.* [71]). The best known example of reinforced stochastic process is the standard Eggenberger-Pólya urn (see [43, 64, 75]), which has been widely studied and generalized (some recent variants can be found in [3,8,9,19,28,32,33,54,58]).

Precisely, in this work we consider a system of $N \geq 2$ interacting stochastic processes $\{I^h = (I_{n,h})_{n\geq 1} : 1 \leq h \leq N\}$ such that each one of them takes values in $\{0,1\}$ and their evolution is modelled as follows: for any $n \geq 0$, the random variables $\{I_{n+1,h} : 1 \leq h \leq N\}$ are conditionally independent given the past information $\mathcal{F}_n$ with

$$P_{n,h} = P(I_{n+1,h} = 1 \,|\, \mathcal{F}_n) = \alpha Z_n + \beta q + (1 - \alpha - \beta) f(Z_{n,h}), \tag{1}$$

where $\alpha,\ \beta \in [0,1),\ \alpha + \beta \in (0,1),\ q \in (0,1]$,

$$Z_{n+1,h} = (1 - r_n)\, Z_{n,h} + r_n I_{n+1,h} \qquad \text{with } r_n \sim \frac{1}{n}, \tag{2}$$

$Z_n = \sum_{i=1}^{N} Z_{n,i}/N$, the function f is a strictly increasing $[0,1]$ valued function belonging to $\mathcal{C}^1([0,1])$ and $r_n \sim 1/n$ means $\lim_{n\to\infty} n r_n = 1$. The starting point for the dynamics (2) is a random variable $Z_{0,h}$ with values in $[0,1]$ and the past information $\mathcal{F}_n$ formally corresponds to the σ-field $\sigma(Z_{0,h} : 1 \leq h \leq N) \vee \sigma(I_{k,h} : 1 \leq k \leq n, 1 \leq h \leq N) = \sigma(Z_{k,h} : 0 \leq k \leq n, 1 \leq h \leq N)$.

The system represents a population of N interacting units whose state at time n is synthesized by the variable $Z_{n,h}$ and whose individual evolution has a reinforcement mechanism driven by a function f. Depending on the choice of the function f, we can give different meanings to such individual evolution.

As a first possible interpretation, let us assume that we are modeling a system of N agents, who at each time-step n have to have to choose an action $s \in \{0,1\}$. Suppose that $s = 1$ represents the "right" choice, that is the one that gives the greater pay-off, and 0 represents the "wrong" one. The processes I^h describes the sequence of actions along the time-steps, that is $I_{n+1,h}$ is the indicator function of the event "agent h makes the right choice at time n". The processes $\{Z^h = (Z_{n,h})_{n\geq 0} : 1 \leq h \leq N\}$ can be interpreted as the "personal inclination" of the agent h in adopting the right choice along time, that is $Z_{n,h}$ is the inclination at time n of agent h toward the right choice. Therefore the above model includes three issues:

- Conditional independence of the agents given the past: Given the past information until time n, the agent h makes a choice at time $n+1$ independently of the other agents' choices at time $n+1$.
- At each time $n+1$, the probability $P_{n,h}$ that agent h makes the right choice is a convex combination of the average value Z_n of all the current agents' inclinations, an external "forcing input" q and a function of her own current inclination $Z_{n,h}$. In the sequel, we will refer to this last factor as the "personal inclination component" of $P_{n,h}$. The term Z_n provides a *mean-field interaction* among the agents. Note that, when $\alpha = 0$, we have not

a proper interaction: indeed, the agents are subject to the same forcing input q, but they evolve idependently of each other. We exclude the case $\alpha = \beta = 0$ because it corresponds to N independent agents who evolve only according to the personal inclination component. We also exclude the case $\alpha + \beta = 1$ because it means that there is not the personal inclination component.

- Since f is strictly increasing, there is a reinforcement mechanism on the personal inclination component: if $I_{n,h} = 1$, then $Z_{n,h} > Z_{n-1,h}$ (provided $Z_{n-1,h} < 1$) and so $f(Z_{n,h}) > f(Z_{n-1,h})$. In other words, the fact that agent h makes the right choice at time n implies a positive increment of her/his inclination toward the adoption of the right choice in the future. As a consequence, the larger the number of times in which an agent has made the right choice until time n, the higher her/his personal inclination component in the probability of a future right choice at time $n + 1$. The justification of this mechanism is twofold: first, higher pay-offs can be related to better physiological conditions and so individuals that are better fed and healthier are less likely to make mistakes in the future choices; second, if the choice is always related to the same action, agents that earn higher pay-offs are not inclined to change their action (see [20] and references therein).
- The forcing input q models the presence of an external force (*e.g.* a political constraint, or an advertising campaign) that leads agents toward the right choice with probability q.

The considered model also fits well in a different context, where there is not a "right" choice, but agents have to choose between two brands $s \in \{0, 1\}$, that are related to a loyalty program: the more they select the same brand, the more loyalty points they gain. This fact motivates the reinforcement mechanism on the personal inclination component and, similarly as before, the forcing input can be interpreted as the possible presence of an external force that leads agents toward the brand 1 with probability q.

Other interpretations can be given in the context of coordination games or opinion dynamics and they will be described in more detail in sections 1.1 and 1.2 below, where we will focus on specific choices of the function f.

The main object of our study is to check if the system has long-run equilibrium configurations as $n \to +\infty$, that is, if, for $h = 1, \dots, N$, the stochastic process $Z^h = (Z_{n,h})_n$ converge almost surely, as n tends to $+\infty$, to some random variable $Z_{\infty,h}$. Second, we want to analyse the limit configurations $[Z_{\infty,1}, \dots, Z_{\infty,N}]$, characterizing the support of their probability distribution. In particular, we are interested in the phenomenon of *synchronization* of the stochastic processes Z^h, which occurs when all the stochastic processes Z^h converge almost surely toward *the same* random variable. Regarding this question, we point out that the above model with f equal to the identity function is essentially included in the models considered in [4, 34] and, in this case, the almost sure asymptotic synchronization always take place (precisely, almost sure synchronization toward a random variable when $\alpha > 0$ and $\beta = 0$ and toward the constant q when $\beta > 0$).

Synchronization phenomena are ubiquitous in Nature and have been observed in a wide variety of models based on randomly interacting units (see the literature cited above and the references therein). Synchronization comes as a result of the interaction and can be enhanced if a reinforcement mechanism is present in the dynamics: for example, in [34] it has been shown that, if reinforcement is sufficiently strong, agents coordinate each other and synchronize in a time scale smaller than the one needed to reach their common (random) limit, giving rise to *synchronized fluctuations*.

Note that the emergence of collective self-organized behaviors in social communities has been frequently described in models based on a Statistical Physics approach (see, *e.g.*, [1, 2, 26, 30, 31]) as a result of a large scale limit. However, we emphasize that synchronization is not a large scale phenomenon in the models studied in this paper. Indeed, for suitable values of the parameters, it

occurs *for any* value of N. In particular, we will prove that for the models under consideration a *phase transition* occurs, depending on the parameter α that tunes the strength of interaction. When α is close enough to 1, synchronization occurs for any N (even in absence of the external input). While, if α is below a threshold, "fragmentation" appears in the population and several limit configurations, where agents are divided into two separated groups with two different inclinations, are possible. In this last scenario, the strength of interaction, even if too weak to produce synchronization, still continues to affect the dynamics, through the number of possible limit configurations and the localization of the limit values for the inclinations.

Remark that some of the systems of interacting reinforced processes previously studied [4, 7, 34–36, 63, 72] can be obtained from the model introduced in this paper taking f equal to the identity function and substituting Z_n with a weighted average of the agents' inclinations. For such models, cases of no-synchronization may occur only in absence of interaction, that is, when agents are divided into two or more groups and at least two of such groups behave independently, *i.e.*, when $\alpha = 0$ or when the matrix describing the strengths of interaction between the various agents is not irreducible. Instead, in the present work, cases of no-synchronization may occur also when $\alpha > 0$, *i.e.*, when all the agents interact with each other. As we will see, the synchronization or no-synchronization of the system is related to the properties of the function f. In particular, in order to have a strictly positive probability of no-synchronization, a necessary condition is f *not linear*.

Finally, it is worthwhile to note that the asymptotic behaviour of the stochastic process Z^h is strictly related to the one of the stochastic process $\{\bar{I}_n^h = \sum_{k=1}^n I_{k,h}/n\}$ (see also [5, 6]), that is, according to the previous interpretation, the average of times in which agent h adopts the right choice. Therefore, the synchronization or no-synchronization of the inclinations of the agents corresponds to the synchronization or no-synchronization of the average of times in which the agents make the right choice.

1.1. **Interacting systems of coordination games.** In this section we illustrate a possible interpretation of our model in the context of Game Theory. Following the approach of [47], each interacting unit h represents a time evolving "economy" *i.e.* a community of agents that grows in time and play a cooperative game. The whole system describes a population of N communities subject to a *mean-field* interaction and to the influence of an external input. The individual evolution of a given community is defined as follows. At time $n = 0$, there are $N_0 > 0$ agents in the community. Each agent is fully described by a binary pure strategy $s \in S = \{-1, +1\}$. Thus, at any n, the state of a given community can be characterized by the current share $X_n \in [0, 1]$ of agents playing strategy $+1$. The system evolves as follows. Given some initial share X_O, at any $n > 0$ a new agent enters the community, observes current strategy share and irreversibly chooses a strategy on the basis of expected pay-offs. More precisely, call $\pi_n(s)$ the expected pay-off associated to strategy $s \in S$ at time n and set $\pi_n = \{\pi_n(s) : s \in S\}$. We assume that the probability, say P_n, that the agent n chooses $s = +1$ is a function of π_n. Moreover, we assume that the expected pay-offs $\pi_n(s)$ are related to a symmetric 2×2 coordination game, that is we assume that the agent entering at time n plays a symmetric 2×2 coordination game against all the present agents according to a standard stage-game pay-off matrix as in Table 1.

TABLE 1. pay-offs Matrix. Left: Original pay-offs. Right: Standardized pay-offs.

	+1	-1
+1	$a_{+1,+1}$	$a_{+1,-1}$
-1	$a_{-1,+1}$	$a_{-1,-1}$

	+1	-1
+1	1	0
-1	A	B

We assume $a_{+1,+1} > a_{-1,+1}$ and $a_{+1,-1} < a_{-1,-1}$ because the game is a coordination one. We also assume $a_{+1,+1} \geq a_{-1,-1}$ and $a_{+1,-1} \leq a_{-1,+1}$. In what follows, we shall focus on the standardized version of the pay-off matrix, obtained from the former (without loosing in generality) by letting $A = (a_{-1,+1} - a_{+1,-1})/(a_{+1,+1} - a_{+1,-1}) \in [0,1)$ and $B = (a_{-1,-1} - a_{+1,-1})/(a_{+1,+1} - a_{+1,-1}) \in (0,1]$.

Expected pay-offs for the agent entering at time $n+1$ associated to any given choice $s \in S$ are given by

$$\pi_n(s) = \begin{cases} X_n & \text{if } s = +1 \\ AX_n + B(1 - X_n) & \text{if } s = -1. \end{cases}$$

Therefore, since P_n is a function of π_n, we get that P_n is a function of X_n, *i.e.* $P_n = f(X_n)$. The dynamics of $\mathcal{X} = (X_n)_n$ is easily given by

$$X_{n+1} = \left(1 - \frac{1}{N_0 + n + 1}\right) X_n + \frac{1}{N_0 + n + 1} I_{n+1}, \tag{3}$$

where I_{n+1} is the indicator function of the event "agent entering the community at time $n+1$ chooses strategy $+1$" and so $P(I_{n+1} = 1 \mid \mathcal{Z}_k, k \leq n) = P_n = f(Z_n)$. Different individual decision rules give different functions f. Two examples are the following:

$$\text{Linear Probability (LP):} \quad P_n = \frac{\pi_n(+1)}{\pi_n(+1) + \pi_n(-1)}$$

which gives

$$f(x) = \begin{cases} x & \text{if } A = 0 \text{ and } B = 1 \\ \frac{x}{\theta(x + x^*)} & \text{if } \theta = (1 + A - B), \end{cases} \tag{4}$$

with $x^* = B/\theta = B/(1 + A - B)$ and so $\theta x^* \in (0,1]$ and $\theta x^* \geq 1 - \theta$.

$$\text{Logit Probability (LogP):} \quad P_n = \frac{\exp(K\pi_n(+1))}{\exp(K\pi_n(+1)) + \exp(K\pi_n(-1))},$$

with $K > 0$, which gives

$$f(x) = \frac{1}{1 + \exp(-\theta(x - x^*))}, \tag{5}$$

with $\theta = K(1 - A + B) > 0$ and $x^* = KB/\theta = B/(1 - A + B) \in (0,1)$.

Under the above individual decision rules, long-run equilibria for one community have been studied in [47]:

- With LP rule, and if the game is not a pure-coordination one (that is $A = 0$ and $B = 1$), the long-run behavior of the system becomes predictable (see definition in Section 2 below): the share of agents playing $+1$ in the limit converges a.s. to the constant $z_\infty = (1 - B)/(1 + A - B)$. Note that, when $B = 1$, we have $z_\infty = 0$ and, when $A = 0$, we have $z_\infty = 1$. In all the other cases (that is $B < 1$ or $A > 0$), coexistence of strategies characterizes equilibrium configuration and we have $z_\infty > 1/2$, or $= 1/2$, or $< 1/2$ if and only if $A + B < 1$, or $A + B = 1$, or $A + B > 1$, respectively.
  With LP rule, if the game is a pure-coordination one, then X_n follows the dynamics of the standard Pólya urn model and so it converges a.s. to a random variable $\mathcal{Z}_\infty$ with beta distribution.
- With LogP, it has been proven that the long-run behavior of the community with $x^* = 1/2$ is predictable if $KB = \theta x^* = \theta/2 \leq 2$: the share of agents playing $+1$ in the limit converges a.s. to $1/2$, that means coexistence of the two strategies in the proportion $1:1$. Moreover, When $x^* \neq 1/2$, some numerical analysis have been performed pointing out the coexistence

of strategies in the limit configuration and the fact that the dynamics is again predictable when $KB = \theta x^*$ is small.

We are interested in analyzing the long-run behavior of a system of $N \geq 2$ interacting games of the above type. More precisely, for each $h \in \{1, \ldots, N\}$, let $Z_{n,h}$ be the share of agents playing strategy $+1$ in the community h. We assume that the dynamics for each $Z_{n,h}$ is of the form

$$Z_{n+1,h} = \left(1 - \frac{1}{N_{0,h} + n + 1}\right) Z_{n,h} + \frac{1}{N_{0,h} + n + 1} I_{n+1,h}, \tag{6}$$

where $I_{n+1,h}$ is the indicator function of the event "agent entering community h at time $n + 1$ chooses strategy $+1$", and we assume that $\{I_{n+1,h} : h = 1, \ldots, N\}$ are conditionally independent given the past information $\mathcal{F}_n$ with

$$P_{n,h} = P(I_{n+1,h} = 1 \,|\, \mathcal{F}_n) = \alpha Z_n + \beta q + (1 - \alpha - \beta) f(Z_{n,h}), \tag{7}$$

where $Z_n = \sum_{i=1}^{N} Z_{n,i}/N$, $q \in (0, 1]$ and $\alpha, \beta \in [0, 1)$ and $\alpha + \beta \in (0, 1)$.

This corresponds to assume that the agent entering a given community at the future time $n + 1$ will choose (independently of the choices of the agents entering the other communities at time $n+1$) the strategy $+1$ with a probability $P_{n,h}$, which is a convex combination of three factors: the present share Z_n of players playing $+1$ in the entire system, an external forcing input q and the expected pay-offs related to the present share $Z_{n,h}$ of players playing $+1$ in the specific game where the agent enters.

It is worthwhile to notice that, although our focus is on the case $N \geq 2$, we are also going to completely describe the asymptotic behaviour of the system when $N = 1$. We point out that, in [47], the case $x^* \neq 1/2$ is studied only by means of simulations, while here we provide analytic results.

1.2. **Technological and Opinion dynamics.** By technological dynamics we mean models which describe the diffusion of some technological assets in a given community. Such diffusion may depend on several factors, such as communication between agents, the influence of an external media and a form of self-reinforcement due to agents' loyalty. On the other hand, opinion dynamics deals with the study of formation and evolution of opinions in a population, which is governed by similar factors; in particular, self-reinforcement can be interpreted in this context as a mechanism for which the agents' personal inclination, after being verbalized through the choice of one out of two (or more) possible actions, is subject to reinforcement in the direction of the expressed choice. Therefore in what follows we will refer to the first context with the implicit assumption that everything can be translated in the language of the second. In the above setting, an interacting unit h of our model may be interpreted either as a single agent, to whom is associated an inclination or opinion $Z_{n,h}$ to adopt one of two different assets (or actions), or as a whole community of agents which has an internal evolution, driven by the function f, and interacts with other similar communities, eventually under the influence of an external media. Below, in order to motivate specific choices of the function f, we describe in details a model based on this last interpretation, where each unit h is modelled as a generalized Pólya urn . In the context of opinion dynamics, our models belong to the class of recently studied CODA models (Continuous Opinions, Discrete Actions) [65,66].

The generalized Pólya urn model [14, 15] has been used in order to model the competitive process among new technologies, which is a fundamental phenomenon in Economics [38]. This urn model is included in the class of reinforced stochastic processes defined by (1) and (2) when the urn function belongs to $\mathcal{C}^1$ and it is strictly increasing. Taking f strictly increasing means that the

considered technologies show increasing returns to adoption: the more they are adopted, the more is learned about them and, consequently the more they are improved, and the more attractive they become [15]. The dynamics for a single "market" of potential adopters is as follows: at each time-step n an agent enters the system and decides to adopt one of two possible technologies $s \in \{0, 1\}$ according to the dynamics (3) with a given urn function f. The present work is related to the study of the long-run behavior of a system of $N \geq 2$ interacting markets of potential adopters of this kind and so described by (6) and (7).

An example of function f used in this framework is

$$f(x) = (1 - \theta) + (2\theta - 1)(3x^2 - 2x^3) \qquad \text{with } \theta \in [0, 1], \tag{8}$$

which belongs to $\mathcal{C}^1$ and is strictly increasing when $\theta \in (1/2, 1]$. The applicative justification behind this function is as follows. See [14,38]. Suppose we have two competing technologies, say $s \in \{0, 1\}$, and represent the population of adopters who are already using one of the two technologies as an urn containing balls of two different colors, say red for technology 1 and black for the other. The composition of the urn evolves along time according to the following agents' decision making rule: at each time-step, the agent extracts with replacement a random sample of $r = 3$ balls from the urn (this means that the agent asks to 3 previous agents which technology they are using) and then the agent selects with probability θ the technology used by the majority of the extracted sample (and an additional ball of the corresponding color is put into the urn) and with probability $(1 - \theta)$ the technology used by the minority of them (and an additional ball of the corresponding color is put into the urn). Notice that, rephrasing the above description in the language of opinion dynamics, we get a variant of the celebrated Galams majority-rule model [53], with the introduction of a reinforcement mechanism in the dynamics.

According to this dynamics, denoting by T and by T_1 the total number of balls and the total number of red balls, respectively, into the urn at time-step n, we have

$$P_n = P(I_{n+1} = 1 | \mathcal{F}_n) = \theta p(T, T_1) + (1 - \theta)(1 - p(T, T_1)) = (1 - \theta) + (2\theta - 1)p(T, T_1) \qquad \text{with}$$

$$p(T, T_1) = \sum_{k=2}^{3} \frac{\binom{T_1}{k}\binom{T-T_1}{3-k}}{\binom{T}{3}} \sim \sum_{k=2}^{3} \binom{3}{k}\left(\frac{T_1}{T}\right)^k \left(1 - \frac{T_1}{T}\right)^{3-k} \qquad \text{for } n \to +\infty.$$

(The above approximation follows from the property of the Gamma function: $\Gamma(n + 1) = n!$ and $\Gamma(n + a) \sim n^a \Gamma(n)$ for $n \to +\infty$.) In other terms, setting $X_n = T_1/T$, that is the proportion of red balls into the urn at time-step n, we have

$$P_n \sim (1 - \theta) + (2\theta - 1) \sum_{k=2}^{3} \binom{3}{k} X_n^k (1 - X_n)^{3-k} = f(X_n) \quad \text{for } n \to +\infty,$$

where f is the function given in (8).

For a single market, the authors [38] find a threshold $1/2$ below which the limit market is shared by the two technologies in the proportion $1 : 1$ that is, if $\theta \leq 1/2$, Z_n converges almost surely to $1/2$, otherwise, two limit market configurations are possible. Although our focus is on the case $N \geq 2$, we are also going to completely describe the asymptotic behaviour of the system when $N = 1$. In particular, we will correct the above mentioned threshold. Indeed, we will prove that, for $N = 1$, when $1/2 < \theta \leq 5/6$, the system has $1/2$ as the unique limit configuration, while, when $5/6 < \theta < 1$, two limit configurations are possible and in both of them the two technologies coexist, with the proportion $z : (1 - z)$ or $(1 - z) : z$ with $z \in (0, 1/2)$. Therefore the threshold is, not at $1/2$, but at $5/6$.

The rest of the paper is organized as follows. In Section 2 we provide some general results regarding the asymptotic behavior of the considered systems. More precisely, we give sufficient conditions for the almost sure convergence of the stochastic processes $Z^h = (Z_{n,h})$ to some random variable $Z_{\infty,h}$ and for the almost sure asymptotic synchronization of the system. Moreover, we give some results concerning the possible values that the limit random vector $[Z_{\infty,1}, \ldots, Z_{\infty,N}]$ can take. In Section 3 we analyze the systems associated to the functions introduced in Subsec. 1.1 and 1.2. Specifically, we show sufficient conditions on the parameters in order to have the almost sure asymptotic synchronization of the system. Moreover, in the case when the interaction parameter and the forcing input are not so strong in order to assure the almost sure asymptotic synchronization, we characterize the possible limit configurations of the system. Furthermore, we point out when the system is predictable (that is, when there exists a unique possible limit configuration) and when the system may almost surely asymptotically synchronize toward the value $1/2$, meaning that in the limit, according to the different applicative frameworks, the two inclinations, or the two strategies in all the games, or the two technologies in all the markets, coexist in the proportion $1:1$. We discuss also if we may have $P(Z_{\infty,h} = 0) > 0$ or $P(Z_{\infty,h} = 1) > 0$, that is if we may have the case when one of the two inclinations (or strategies, or technologies) is asymptotically predominant with respect to the other and it only survives in the limit. Finally, as said before, we also take into account the case $N = 1$. The paper is also enriched by simulations and figures, all collected in Section 4, and by an appendix, containing some recalls about Stochastic Approximation theory and some technical linear algebra results.

2. General results

By means of (1) and (2), the recursive equation for $Z_{n,h}$ can be rewritten as

$$Z_{n+1,h} = Z_{n,h} + r_n \left[\alpha Z_n + \beta q + (1 - \alpha - \beta)f(Z_{n,h}) - Z_{n,h}\right] + r_n \Delta M_{n+1,h}, \tag{9}$$

where $\Delta M_{n+1,h} = I_{n+1,h} - P_{n,h}$ is a martingale difference with respect to $\mathcal{F} = (\mathcal{F}_n)_n$. Moreover, summing over h, we get the equation for Z_n

$$Z_{n+1} = Z_n + r_n \left[\alpha Z_n + \beta q + (1 - \alpha - \beta)\frac{1}{N}\sum_{h=1}^{N} f(Z_{n,h}) - Z_n\right] + r_n \left(\frac{1}{N}\sum_{h=1}^{N} \Delta M_{n+1,h}\right). \tag{10}$$

Let us set $\mathbf{Z}_n = (Z_{n,1}, \ldots, Z_{n,N})^\top$, $\Delta \mathbf{M}_{n+1} = (\Delta M_{n+1,1}, \ldots, \Delta M_{n+1,N})^\top$ and

$$\mathbf{F}(\mathbf{z}) = (F_1(\mathbf{z}), \ldots, F_N(\mathbf{z}))^\top \quad \text{with } F_h(\mathbf{z}) = \alpha \frac{1}{N}\sum_{i=1}^{N} z_i + \beta q + (1 - \alpha - \beta)f(z_h) - z_h \; \forall \mathbf{z} \in [0,1]^N. \tag{11}$$

Using the above notation, we can write (9) in the vectorial form

$$\mathbf{Z}_{n+1} = \mathbf{Z}_n + r_n \mathbf{F}(\mathbf{Z}_n) + r_n \Delta \mathbf{M}_{n+1}. \tag{12}$$

We are interested in proving that

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty, \tag{13}$$

where $\mathbf{Z}_\infty$ is a suitable random variable with values in $[0,1]$, and in the characterization of the support of its distribution.

Throughout this paper we use the symbols $\mathbf{0}$ and $\mathbf{1}$ to denote the vector with all the components equal to zero and the vector with all the components equal to one. When $P(\mathbf{Z}_\infty = \mathbf{z}_\infty) > 0$ with $\mathbf{z}_\infty$ of the form $\mathbf{z}_\infty = z_\infty \mathbf{1}$, with $z_\infty \in [0,1]$, we call $\mathbf{z}_\infty$ a *synchronization point* for the system. If all the possible values for $\mathbf{Z}_\infty$ are synchronization points, that is $\mathbf{Z}_\infty$ is of the form $Z_\infty \mathbf{1}$ with Z_∞ a suitable random variable taking values in $[0,1]$, we say that the system *almost surely asymptotically synchronize*. Moreover, we say that the system is *predictable* when there exists a point $\mathbf{z}_\infty$ such that $\mathbf{Z}_\infty = \mathbf{z}_\infty$ almost surely.

Finally, it is worthwhile to note that the almost sure convergence of $\mathbf{Z}_n$ toward a random variable $\mathbf{Z}_\infty$ implies the almost sure convergence of the empirical means $\bar{\mathbf{I}}_n = \frac{1}{n}\sum_{k=1}^n \mathbf{I}_k$ (where $\mathbf{I}_k$ is the random vector with components $I_{k,h}$, for $h = 1,\dots,N$) toward the same limit.

Let $\mathbf{F}$ and $(\mathbf{Z}_n)_{n\geq 0}$ be defined as in (11) and (12) and let $\mathcal{Z}(\mathbf{F}) = \{\mathbf{z} \in [0,1]^N : \mathbf{F}(\mathbf{z}) = \mathbf{0}\}$ be the zero-set of the function $\mathbf{F}$. Using the Stochastic Approximation methodology (see Appendix A), we obtain the following results. The first one concerns the almost sure convergence of the process $(\mathbf{Z}_n)$.

Theorem 2.1. *(Almost sure convergence)*
The set $\mathcal{Z}(\mathbf{F})$ contains at least one synchronization point and, if $\mathcal{Z}(\mathbf{F})$ is finite and f admits a primitive function, we have

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty,$$

where $\mathbf{Z}_\infty$ is a suitable random variable with values in $\mathcal{Z}(\mathbf{F})$. Moreover, we also have

$$\bar{\mathbf{I}}_n = \frac{1}{n}\sum_{k=1}^n \mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty.$$

Proof. We firstly show that $\mathcal{Z}(\mathbf{F})$ is non empty, since it contains at least one synchronization point. Indeed, points in $\mathcal{Z}(\mathbf{F})$ are the solutions in $[0,1]^N$ of the system of equalities

$$\alpha\frac{1}{N}\sum_{i=1}^N z_i + \beta q + (1 - \alpha - \beta)f(z_h) - z_h = 0 \qquad \forall h = 1,\dots,N. \tag{14}$$

In particular, for the synchronization zero points, that is for the zero points of the form $\mathbf{z} = z\mathbf{1}$, the above system of equalities reduces to the equation

$$\varphi(z) = f(z) - \frac{(1-\alpha)}{(1-\alpha-\beta)}z + \frac{\beta q}{(1-\alpha-\beta)} = 0. \tag{15}$$

See Fig. 2 and Fig. 5 for examples and illustrations. Therefore, since f takes values in $[0,1]$, we have $\varphi(0) \geq 0$ and $\varphi(1) \leq 0$. This fact implies that φ always has at least one zero point in $[0,1]$. Hence, under the above assumptions, the almost sure convergence of $(\mathbf{Z}_n)$ immediately follows from Theorem A.4, because we have

$$\mathbf{F} = -\nabla \mathbf{V} \qquad \text{with}$$

$$\mathbf{V}(\mathbf{z}) = -\frac{\alpha}{2N}\left(\sum_{h=1}^N z_h\right)^2 - \beta q\sum_{h=1}^N z_h - (1 - \alpha - \beta)\sum_{h=1}^N \phi(z_h) + \frac{1}{2}\sum_{h=1}^N z_h^2,$$

where ϕ is a primitive function of f.

Finally, since $E[\mathbf{I}_n|\mathcal{F}_{n-1}] = \mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty$, applying Lemma B.1 in [5] (with $c_k = k$, $v_{n,k} = k/n$ and $\eta = 1$), we get that $\frac{1}{n}\sum_{k=1}^n \mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty$. ∎

The following theorem provides a sufficient condition for the almost sure synchronization of the system.

Theorem 2.2. *(Almost sure asymptotic synchronization)*
If $\mathcal{Z}(\mathbf{F})$ contains a finite number of synchronization points, f admits a primitive function and, for each fixed constant $c \in \left(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\right)$, the function

$$\widetilde{\varphi}(z) = f(z) - \frac{1}{1-\alpha-\beta}z - c \tag{16}$$

has at most one zero point in $[0, 1]$*, then we have the almost sure asymptotic synchronization of the system and the limit random variable* $\mathbf{Z}_\infty$ *is of the form* $Z_\infty \mathbf{1}$*, where* Z_∞ *satisfies Equation* (15).

Remark 2.3. (*Linear case*)
Note that, since c belongs to $\left(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\right)$, we have $\widetilde{\varphi}(0) > 0$ and $\widetilde{\varphi}(1) < 0$ and so the equation $\widetilde{\varphi} = 0$ always has a solution. The above result requires that this solution is unique. A particular case in which this condition is satisfied is when f is *linear*. Indeed, if $f : [0, 1] \to [0, 1]$ is linear and strictly increasing, then $f' = \delta \in (0, 1]$ and hence $\delta \neq 1/(1 - \alpha - \beta)$. It is worthwhile to observe that, when f is linear, Equation (15) has infinite solutions (and so Theorem 2.2 does not apply) only when f is the identity function and $\beta = 0$. However, this case is included in [4, 34], where the almost sure asymptotic synchronization is proven also in this case.

Proof. We first prove that the assumptions of Theorem 2.2 imply that $\mathcal{Z}(\mathbf{F})$ does not contain "no-synchronization" points, that is points that are not synchronization points. To this purpose, we recall that the set $\mathcal{Z}(\mathbf{F})$ is described by the system of equalities (14). In particular, if $\mathbf{z}^*$ is a solution of the system (14) with $z_h^* \neq z_j^*$ for at least a pair of indexes, equation (14) implies

$$(1 - \alpha - \beta)f(z_h^*) < z_h^* \qquad \forall h = 1, \ldots, N \tag{17}$$

and

$$(1 - \alpha - \beta)f(z_h^*) > z_h^* - \alpha - \beta \qquad \forall h = 1, \ldots, N. \tag{18}$$

Moreover, (14) (written for h and j) also implies

$$(z_h^* - z_j^*) = (1 - \alpha - \beta)(f(z_h^*) - f(z_j^*)) \qquad \forall h, j = 1, \ldots, N. \tag{19}$$

Therefore, fixed h, z_j^* is a solution (different from z_h^*) of the equation

$$f(z) = \frac{1}{1 - \alpha - \beta}z + c$$

where $c = f(z_h^*) - \frac{z_h^*}{1-\alpha-\beta} \in \left(-\frac{\alpha+\beta}{1-\alpha-\beta}, 0\right)$ (by (17) and (18)). In other terms, a necessary condition for the existence of no-synchronization zero points is that there exists $c \in Im(f - (1 - \alpha - \beta)^{-1}id) \cap (-\frac{\alpha+\beta}{1-\alpha-\beta}, 0)$ such that the function (16) has more than one zero points in $[0, 1]$. Hence, we can conclude that the assumptions of Theorem 2.2 imply that $\mathcal{Z}(\mathbf{F})$ contains only synchronization points. Therefore, this set is not empty (see Theorem 2.1) and, by assumption, it is finite. Applying Theorem 2.1, we obtain the almost sure convergence of $\mathbf{Z}_n$ toward a random variable $\mathbf{Z}_\infty$ taking values in the set $\mathcal{Z}(\mathbf{F})$, and so of the form $\mathbf{Z}_\infty = Z_\infty \mathbf{1}$, where Z_∞ satisfies Equation (15). $\blacksquare$

Remark 2.4. (*Existence and characterization of the no-synchronization zero points*)
It is worthwhile to underline that, from the above proof, we obtain that a necessary condition for the existence of no-synchronization zero points of $\mathbf{F}$ is that there exists $c \in Im(f - (1 - \alpha - \beta)^{-1}id) \cap (-\frac{\alpha+\beta}{1-\alpha-\beta}, 0)$ such that the corresponding function (16) has more than one zero point in $[0, 1]$. Moreover, if $\mathbf{z}^*$ is a no-synchronization zero point, then, for any fixed component z_h^*, each other component is a solution of $\widetilde{\varphi} = 0$, with $c = f(z_h^*) - z_h^*/(1-\alpha-\beta) \in Im(f - (1 - \alpha - \beta)^{-1}id) \cap (-\frac{\alpha+\beta}{1-\alpha-\beta}, 0)$. Conversely, when $\mathbf{z}^*$ is a point with the above property, it is a zero point of $\mathbf{F}$ if and only if (since (14)) we have

$$\alpha\frac{1}{N}\sum_{i=1}^{N} z_i^* + \beta q + (1 - \alpha - \beta)c = 0. \tag{20}$$

We conclude this section providing a very simple condition that allows us to exclude the linearly unstable zero points (see Appendix A) from the set of possible limit points for the process $(\mathbf{Z}_n)$.

Theorem 2.5. *(No-convergence toward linearly unstable zero points)*
If $f(0) > 0$ and $f(1) < 1$, then, for each $\mathbf{z} \in \mathcal{Z}(\mathbf{F})$ which is linearly unstable, we have

$$P(\mathbf{Z}_n \to \mathbf{z}) = 0.$$

Proof. We can apply Theorem A.5 in Appendix A. Fixed $v \in \mathbb{R}^N$ with $|v| = \sum_{h=1}^N v_h = 1$ and $n \in \mathbb{N}$, consider the random variable

$$X_{n+1} = \sum_{h=1}^N v_h \Delta M_{n+1,h} = \sum_{h=1}^N v_h(I_{n+1,h} - P_{n,h}),$$

where $P_{n,h} = \alpha Z_n + \beta q + (1 - \alpha - \beta)f(Z_{n,h})$. We note that a partition of the sample space is given by the events of the form

$$E_{n+1,H} = \{I_{n+1,h} = 1 \; \forall h \in H, \; I_{n+1,h} = 0 \; \forall h \in H^c\},$$

where H is a subset of $\{1, \ldots, N\}$ (the empty set included). Therefore, we can write

$$X_{n+1} = \sum_H \left(\sum_{h \in H} v_h(1 - P_{n,h}) - \sum_{h \in H^c} v_h P_{n,h} \right) I_{E_{n+1,H}} = \sum_H A_{n,h} I_{E_{n+1,H}},$$

where the first sum is over all the possible subsets of $\{1, \ldots, N\}$ (the empty set included). It follows that

$$X_{n+1}^+ = \sum_H A_{n,H}^+ I_{E_{n+1,H}}$$

and so

$$E[X_{n+1}^+ \,|\, \mathcal{F}_n] = \sum_H A_{n,H}^+ E[I_{E_{n+1,H}} \,|\, \mathcal{F}_n] = \sum_H A_{n,H}^+ \prod_{h \in H} P_{n,h} \prod_{h \in H^c} (1 - P_{n,h})$$

(where we use the convention $\prod = 1$ if H or H^c is empty). Now, by assumption, f has on $[0,1]$ a minimum value $m = f(0) > 0$ and a maximum value $M = f(1) < 1$. Hence, we have

$$0 < (1 - \alpha - \beta)m \le P_{n,h} \le \alpha + \beta + (1 - \alpha - \beta)M < 1$$

and this fact implies $\prod_{h \in H} P_{n,h} \prod_{h \in H^c}(1 - P_{n,h}) \ge p > 0$ for a suitable constant $p > 0$. Moreover, among the possible H, there is $H_* = \{h \in \{1, \ldots, N\} : v_h \ge 0\}$ (possibly equal to the empty set) and, correspondingly, we have

$$A_{n,H_*}^+ = A_{n,H_*} \ge (1 - \alpha - \beta) \min\{m, 1 - M\} \sum_{h \in H_*} v_h + \sum_{h \in H_*^c} (-v_h)$$

$$= (1 - \alpha - \beta) \min\{m, 1 - M\} \sum_{h=1}^N |v_h|$$

$$= (1 - \alpha - \beta) \min\{m, 1 - M\} > 0.$$

Thus, condition (47) of Theorem A.5 is satisfied with $C = (1 - \alpha - \beta) \min\{m, 1 - M\}p > 0$ and so $P(\mathbf{Z}_\infty = \mathbf{z}) = 0$ for all the zero points $\mathbf{z}$ of $\mathbf{F}$ that are linearly unstable. ∎

3. SPECIFIC MODELS

In this section, by means of the above general results, we analyze the asymptotic behaviour of the systems related to the functions f_{LP}, f_{LogP} and f_{Tech}, introduced in Section 1 (Subsec. 1.1 and 1.2). In section 4 some associated numerical illustrations will be presented.

3.1. **Case** $f = f_{LP}$. In this subsection we consider the function

$$f(x) = f_{LP}(x) = \frac{x}{\theta(x + x^*)} \qquad \text{with } \theta > 0, \ \theta x^* \in (0, 1], \ \theta x^* \geq 1 - \theta. \tag{21}$$

Note that we exclude the case f_{LP} defined in (4) with $\theta = 0$, because it coincides with the case of a system of interacting Pólya urns with mean-field interaction and with or without a "forcing input" q and this model has been already analyzed in [4–6, 34–36].

The following result states that, provided that $\mathbf{Z}_0 \neq \mathbf{0}$ (note that, in applications, we generally have $P(\mathbf{Z}_0 \neq \mathbf{0}) = 1$), we always have the almost sure asymptotic synchronization of the system and, moreover, it is predictable.

Theorem 3.1. *Let* $f = f_{LP}$. *Set*

$$\widehat{P} = \begin{cases} P & \text{when } \beta > 0, \\ P & \text{when } \beta = 0 \text{ and } \theta x^* = 1, \\ P(\cdot | \mathbf{Z}_0 \neq \mathbf{0}) & \text{when } \beta = 0 \text{ and } \theta x^* < 1. \end{cases}$$

and

$$z_\infty = \begin{cases} \hat{z} & \text{when } \beta > 0, \\ \frac{1 - \theta x^*}{\theta} & \text{when } \beta = 0. \end{cases} \tag{22}$$

where $\hat{z} \in (0, 1)$ *depends on the model parameters and it is defined as in* (24). *Then, under* $\widehat{P}$, *the system almost surely asymptotically synchronizes and it is predictable: indeed, we have*

$$\mathbf{Z}_n \xrightarrow{a.s.} z_\infty \mathbf{1}$$

and

$$\bar{\mathbf{I}}_n = \frac{1}{n} \sum_{k=1}^{n} \mathbf{I}_k \xrightarrow{a.s.} z_\infty \mathbf{1}.$$

Observe that, when $\beta > 0$, the limit point $z_\infty \in (0, 1)$. In the first interpretation, this means that in the limit configuration the N agents keep a positive inclination for both actions; while in the interpretation regarding the games (see Subsec. 1.1), this means that in the limit configuration, both strategies coexist in all the N games. When $\beta = 0$, the limit point z_∞ belongs to the entire interval $[0, 1]$, including the extremes: precisely, it is equal to 0 when $\theta x^* = 1$ and equal to 1 when $\theta x^* = 1 - \theta$. Therefore, there is the possibility that, in the limit configuration, only one inclination (or strategy) survives. Furthermore, we note that the limit value depends only on θ and x^*, but not on the parameter α, that rules the interaction.

Proof. Let us look for the solutions of the equation $\mathbf{F}(\mathbf{z}) = \mathbf{0}$ in $[0, 1]^N$, that is of the system (14). *Synchronization zero points.* We start looking for the solutions of (14) of the type $\mathbf{z} = z\mathbf{1}$, that is for the solution of (15). Taking into account that $f = f_{LP}$, we obtain the second-order equation

$$\widehat{\varphi}(z) = (1 - \alpha)\theta z^2 + [(1 - \alpha)\theta x^* - \beta\theta q - (1 - \alpha - \beta)]z - \beta q\theta x^* = 0. \tag{23}$$

We recall that, since The discriminant associated to this equation is

$$\Delta = [(1 - \alpha)\theta x^* - \beta\theta q - (1 - \alpha - \beta)]^2 + 4(1 - \alpha)\beta\theta^2 q x^*.$$

When $\beta = 0$ and $\theta x^* < 1$, we have two distinct solutions in $[0, 1]$, that is 0 and $\frac{1 - \theta x^*}{\theta}$, while, if $\beta = 0$ and $\theta x^* = 1$, we have only one solution $z^* = 0$. When $\beta > 0$, we have $\Delta > 0$ and so there are two distinct solutions of (23). However, we are interested only in solutions belonging to $[0, 1]$. Since $\varphi(0) > 0$ and $\varphi(1) < 0$, there is at least one solution in $(0, 1)$. Moreover, since in Δ we have

the term $4(1-\alpha)\beta\theta^2 qx^* > 0$, one of the solutions is obviously strictly negative. Therefore, there is a unique solution in $(0,1)$ given by

$$\hat{z} = \frac{-[(1-\alpha)\theta x^* - \beta\theta q - (1-\alpha-\beta)] + \sqrt{\Delta}}{2(1-\alpha)\theta}. \tag{24}$$

Summing up, synchronization zero points are of the form $\mathbf{z}^* = z^*\mathbf{1}$ with

$$z^* \begin{cases} \in \{0, \frac{1-\theta x^*}{\theta}\} & \text{if } \beta = 0, \ \theta x^* < 1. \\ = 0 & \text{if } \beta = 0, \ \theta x^* = 1. \\ = \hat{z} & \text{if } \beta > 0, \end{cases} \tag{25}$$

No-syncronization zero points. Such zero points do not exist: indeed, writing equation (16) of Theorem 2.2 for $f = f_{LP}$ we obtain

$$\theta z^2 + [c(1-\alpha-\beta)\theta + \theta x^* - (1-\alpha-\beta)]z + c(1-\alpha-\beta)\theta x^* = 0,$$

which, since $c < 0$, admits at most one solution in $[0,1]$.

Almost sure asymptotic synchronization. We have proven above that the set $\mathcal{Z}(\mathbf{F})$ contains only a finite number of points. Moreover, f admits the primitive function

$$\phi(x) = \frac{1}{\theta}\left[x - x^* \ln(x + x^*)\right] + const.$$

Then, by Theorem 2.1 and Theorem 2.2, we can conclude that the system almost surely asymptotically synchronizes:

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty = Z_\infty \mathbf{1}$$

and

$$\bar{\mathbf{I}}_n = \frac{1}{n}\sum_{k=1}^n \mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty = Z_\infty \mathbf{1},$$

where Z_∞ can take the values z^* specified above. In particular, when we are in the case $\beta > 0$ or in the case $\beta = 0$ and $\theta x^* = 1$, we have a unique possible value for z^* and so the system is predictable. It remains to prove that, under $\widehat{P} = P(\cdot|\mathbf{Z}_0 \neq \mathbf{0})$, the system is predictable with the unique limit point $\frac{1-\theta x^*}{\theta}\mathbf{1}$. The following step provides the proof of this fact.

Case $\beta = 0$ and $\theta x^* < 1$: predicibility under $\widehat{P}$. Let us consider the case $\beta = 0$ and $\theta x^* < 1$, for which we have $\mathcal{Z}(\mathbf{F}) = \{\mathbf{0}, (\frac{1-\theta x^*}{\theta})\mathbf{1}\}$. For $\mathbf{z}^* = z^*\mathbf{1}$, Corollary B.3 provides the eigenvalues of $J(\mathbf{F})(\mathbf{z}^*)$, that is

$$(1-\alpha-\beta)f'(z^*) - 1 \qquad \text{and} \qquad (1-\alpha-\beta)f'(z^*) - 1 + \alpha.$$

Now, the eigenvalues for $\mathbf{z}^* = (\frac{1-\theta x^*}{\theta})\mathbf{1}$ are $(1-\alpha)\theta x^* - 1 < 0$ and $-(1-\alpha)(1-\theta x^*) < 0$, and so $\mathbf{z}^*$ is strictly stable; while the eigenvalues for $\mathbf{0}$ are $(1-\alpha)(\theta x^*)^{-1} - 1$, that can be positive or negative, and $-(1-\alpha)(1-1/\theta x^*) > 0$, so that $\mathbf{0}$ is linearly unstable. However, we cannot exclude convergence toward $\mathbf{0}$ by means of Theorem 2.5, because $f(0) = 0$. Anyway, we observe that, if $\mathbf{Z}_0 \neq \mathbf{0}$, then $\mathbf{Z}_n \neq \mathbf{0}$ for all n. Hence, if we prove for $\mathbf{z}^* = \frac{1-\theta x^*}{\theta}\mathbf{1}$, that

$$\langle \mathbf{F}(\mathbf{z}), \mathbf{z} - \mathbf{z}^* \rangle = \langle \mathbf{F}(\mathbf{z}) - \mathbf{F}(\mathbf{z}^*), \mathbf{z} - \mathbf{z}^* \rangle < 0 \tag{26}$$

for all $\mathbf{z} = (z_1, \ldots, z_N)^T \in [0,1]^N \setminus \mathcal{Z}(\mathbf{F})$, then we can conclude by Theorem A.3 that, under $\widehat{P} = P(\cdot|\mathbf{Z}_0 \neq \mathbf{0})$, the system is predictable. In order to prove (26), we observe that f' is positive and strictly decreasing on $[0,1]$ and $f'(z^*) = \theta x^* < 1$ by hypothesis. Then, recalling that $f(z) - f(z^*) <$

0 for $z < z^*$ and, using the mean value theorem for $z > z^*$, we obtain that $f(z) - f(z^*) < |z - z^*|$ for all $z \in [0,1], z \neq z^*$. Then, since $z_h \neq z^*$ for at least one $h \in \{1, \ldots, N\}$, we have

$$\langle \mathbf{F}(\mathbf{z}) - \mathbf{F}(\mathbf{z}^*), \mathbf{z} - \mathbf{z}^* \rangle = \frac{\alpha}{N} \left[\sum_{h=1}^{N} (z_h - z^*) \right]^2 - \sum_{h=1}^{N} (z_h - z^*)^2 + (1 - \alpha) \sum_{h=1}^{N} (f(z_h) - f(z^*))(z_h - z^*)$$

$$< -(1 - \alpha) \sum_{h=1}^{N} (z_h - z^*)^2 + (1 - \alpha) \sum_{h=1}^{N} (z_h - z^*)^2 = 0. \tag{27}$$

Finally, regarding the almost sure convergence of the empirical means under $\widehat{P}$, we observe that the proof given for Theorem 2.1 also works with $\widehat{P} = P(\cdot | \mathbf{Z}_0 \neq 0)$, because $\{\mathbf{Z}_0 \neq \mathbf{0}\} \in \mathcal{F}_0$. ∎

Remark 3.2. (*Possible asymptotic synchronization toward* $1/2$)
We recall that the almost sure asymptotic synchronization of the system toward the value $1/2$ means that in the limit the two inclinations (in the first interpretation) or the two strategies in all the games (in the second interpretation) coexist in the proportion $1:1$. To this regard, we observe that $1/2\mathbf{1}$ is a synchronization zero point for the case $f = f_{LP}$ if and only if we have

$$f_{LP}(1/2) - \frac{1 - \alpha}{2(1 - \alpha - \beta)} + \frac{\beta q}{1 - \alpha - \beta} = 0, \qquad \text{that is}$$

$$\frac{(\theta + 2\theta x^*)}{2} (1 - \alpha - 2\beta q) = 1 - \alpha - \beta, \tag{28}$$

that, in particular, implies $(1 - \alpha) > 2\beta[q \vee (1 - q)]$ (because $f_{LP}(1/2) \in (0, 1)$). Therefore, only when condition (28) is satisfied, the system almost surely asymptotically synchronizes toward $1/2$. Note that, in the special case when $\beta = 0$ (which includes the case $N = 1$, $\alpha = \beta = 0$ that corresponds to the one studied in [47]), condition (28) simply becomes $\theta x^* = 1 - \theta/2$.

Applying Theorem A.6, we can provide also the rate of convergence of $(\mathbf{Z}_n)$. More precisely, we have the following result:

Remark 3.3. (*Rate of convergence*)
With the same assumptions and notation as in Theorem 3.1, we have

$$\Delta M_{n+1,h} \Delta M_{n+1,j} = (I_{n+1,h} - P_{n,h})(I_{n+1,j} - P_{n,j}),$$

where $P_{n,h}$ is defined in (1), and so, for $h \neq j$, by conditional independence, we get $E[\Delta M_{n+1,h} \Delta M_{n+1,j} | \mathcal{F}_n] = 0$, and, for $h = j$, taking into account that $\mathbf{F}(\mathbf{z}_\infty) = \mathbf{0}$ for $\mathbf{z}_\infty = z_\infty \mathbf{1}$,

$$E[(\Delta M_{n+1,h})^2 | \mathcal{F}_n] = P_{n,h} - P_{n,h}^2 \xrightarrow{a.s.} z_\infty - z_\infty^2 \qquad \text{w.r.t. } \widehat{P}.$$

Moreover, by Corollary B.3, the smallest eigenvalue of $-J(\mathbf{F})(z_\infty \mathbf{1})$ is $\lambda = (1 - \alpha) - (1 - \alpha - \beta)f'(z_\infty)$. Therefore, applying Theorem A.6 when $x_\infty \in (0, 1)$, we obtain, under $\widehat{P}$:

- if $\lambda > 1/2$, then

$$\sqrt{n}(\mathbf{Z}_n - z_\infty \mathbf{1}) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma),$$

  where Σ is a suitable matrix of the form $z_\infty(1 - z_\infty)\widehat{\Sigma}$;
- if $\lambda = 1/2$, then

$$\sqrt{\frac{n}{\ln(n)}}(\mathbf{Z}_n - z_\infty \mathbf{1}) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma),$$

  where Σ is a suitable matrix of the form $z_\infty(1 - z_\infty)\widehat{\Sigma}$;

• if $0 < \lambda < 1/2$, then

$$n^\lambda(\mathbf{Z}_n - z_\infty \mathbf{1}) \xrightarrow{a.s.} V,$$

where V is a suitable finite random variable.

In particular, when $\lambda > 1/2$, using Remark A.7, we obtain $\Sigma = z_\infty(1 - z_\infty)(-2J(\mathbf{F})(\mathbf{z}_\infty) - Id)^{-1}$.

3.2. Case $f = f_{LogP}$. In this subsection, we consider the function

$$f(x) = f_{LogP}(x) = \frac{1}{1 + \exp(-\theta(x - x^*))} \qquad \text{with } x^* \in (0,1), \ \theta > 0. \tag{29}$$

It is a sigmoid function, *i.e.* its first derivative is a strictly positive function, which is strictly increasing on $[0, x^*)$ and strictly decreasing on $(x^*, 1]$ with a maximum given by $f'(x^*) = \theta/4$. Furthermore, we have $f'(x) = f'(2x^* - x)$ for all $x \in [0,1]$.

The following lemma provides a description of the subset of $\mathcal{Z}(\mathbf{F})$ containing all the zero points of $\mathbf{F}$ that are synchronization points (more briefly, "synchronization zero points").

Lemma 3.4. *(Synchronization zero points)*
Let $f = f_{LogP}$. Then, accordingly to the values of the parameters, $\mathcal{Z}(\mathbf{F})$ contains at least three synchronization zero points. Moreover, at most two of them are stable. In particular, if one of the following conditions is satisfied, $\mathbf{F}$ has a unique stable synchronization zero point:

- U1) $\theta/4 \leq (1-\alpha)/(1-\alpha-\beta)$ *or*
- U2) $f'(0) \vee f'(1) \geq (1-\alpha)/(1-\alpha-\beta)$ *or*
- U3) $f'(0) \vee f'(1) < (1-\alpha)/(1-\alpha-\beta) < \theta/4$ *and either $f(\widehat{x}_1) > (1-\alpha)\widehat{x}_1/(1-\alpha-\beta) - \beta q/(1-\alpha-\beta)$ or $f(\widehat{x}_2) < (1-\alpha)\widehat{x}_2/(1-\alpha-\beta) - \beta q/(1-\alpha-\beta)$, where $\widehat{x}_1 \in (0, x^*)$ and $\widehat{x}_2 = 2x^* - \widehat{x}_1 \in (x^*, 1)$ are the solutions of $f' = (1-\alpha)/(1-\alpha-\beta)$.*

Otherwise, $\mathbf{F}$ has two stable synchronization zero points belonging to $(0, \widehat{x}_1] \cup [\widehat{x}_2, 1)$ (more precisely, one in each of these two intervals).

Proof. We recall (see Theorem 2.1) that there exists at least one synchronization zero point of $\mathbf{F}$ and points of this type are of the form $\mathbf{z} = z\mathbf{1}$ with $\varphi(z) = 0$, where

$$\varphi(z) = f(z) - (1-\alpha)/(1-\alpha-\beta)z + \beta q/(1-\alpha-\beta).$$

Note that $\varphi(z)$ is of the form $f(z) - \delta z + cost$ with $\delta = (1-\alpha)/(1-\alpha-\beta)$ and a suitable constant $cost$ such that $\varphi(0) > 0$ and $\varphi(1) < 0$ (note that $f(0) > 0$ and $f(1) < 1$). Hence, we have $\varphi' = f' - \delta$ and $\varphi'' = f''$. Therefore, recalling that f is a sigmoid function with $\max_{[0,1]} f' = f'(x^*) = \theta/4$ and the symmetry of f', we get that equation $\varphi'(x) = 0$, *i.e.* $f'(x) = \delta$, has at most two solutions on $[0,1]$ and we can have the following cases:

1) $\theta/4 \leq \delta$ or
2) $f'(0) \vee f'(1) \geq \delta$ (and so $\theta/4 > \delta$) or
3) $f'(0) \vee f'(1) < \delta < \theta/4$ and, letting $\widehat{x}_1 \in (0, x^*)$ and $\widehat{x}_2 = 2x^* - \widehat{x}_1 \in (x^*, 1)$ be the solutions of $\varphi' = 0$, we have either $\varphi(\widehat{x}_1) > 0$ or $\varphi(\widehat{x}_2) < 0$, or
4) $f'(0) \vee f'(1) < \delta < \theta/4$ and, letting $\widehat{x}_1, \widehat{x}_2$ as in c), we have either $\varphi(\widehat{x}_1) = 0$ or $\varphi(\widehat{x}_2) = 0$,
5) $f'(0) \vee f'(1) < \delta < \theta/4$ and, letting $\widehat{x}_1, \widehat{x}_2$ as in c), we have $\varphi(\widehat{x}_1) < 0$ and $\varphi(\widehat{x}_2) > 0$.

(Note that 1), 2) and 3) coincide, respectively, with conditions U1), U2) and U3) in the statement.) In case 1), φ' is strictly negative on $[0,1] \setminus \{x^*\}$, that is φ is strictly decreasing on $[0,1]$, and, since $\varphi(0) > 0$ and $\varphi(1) < 0$, this fact implies that $\varphi = 0$ has a unique solution in $(0,1)$. Now, assume to be in case 2). Observe that, since $\varphi(0) > 0$ and $\varphi(1) < 0$, it holds $\varphi'(0) \wedge \varphi'(1) < 0$ (otherwise φ would be increasing on $[0,1]$, yielding a contradiction). Now, when $\varphi'(0) \geq 0$, then $\varphi'(x) > 0$ for all $x \in (0, x^*]$, *i.e.*, φ is strictly increasing on $(0, x^*]$; this fact implies that $\varphi(x) > 0$

for all $x \in [0, x^*]$. Consequently, φ has at most one zero point $z^* \in (x^*, 1)$, because $\varphi(1) < 0$ and φ' is strictly decreasing on $(x^*, 1]$. Analogously, if $\varphi'(1) \geq 0$, then φ has at most one zero point $z^* \in (0, x^*)$. In case 3), $\widehat{x}_1 < x^*$ and $\widehat{x}_2 = 2x^* - \widehat{x}_1 > x^*$ are respectively points of local minimum and local maximum of φ in $(0, 1)$. Now, if $\varphi(\widehat{x}_1) > 0$, then φ has a unique zero $z^* \in (\widehat{x}_2, 1)$. Analogously, if $\varphi(\widehat{x}_2) < 0$, then φ has a unique zero $z^* \in (0, \widehat{x}_1)$. In case 4), the function φ has two zero points: more precisely, if $\widehat{x}_1$ is a zero point of φ, then $\varphi(\widehat{x}_2) > 0$ and the other zero point of φ belongs to $(\widehat{x}_2, 1)$; if $\widehat{x}_2$ is a zero point of φ, then $\varphi(\widehat{x}_1) < 0$ and the other zero point of φ belongs to $(0, \widehat{x}_1)$. Finally, in case 5), then φ has three zero points: one in $(0, \widehat{x}_1)$, one in $(\widehat{x}_1, \widehat{x}_2)$ and the last in $(\widehat{x}_2, 1)$.

Regarding the stability of the synchronization zero points of $\mathbf{F}$, we observe that, when $\mathbf{z} = z\mathbf{1}$, where z is a zero point of φ, by Corollary B.3, the eigenvalues of $J(\mathbf{F})(\mathbf{z})$ are given by

$$(1 - \alpha - \beta)f'(z) - 1 \qquad \text{and} \qquad (1 - \alpha - \beta)f'(z) - 1 + \alpha \,,$$

that is

$$(1 - \alpha - \beta)\varphi'(z) - \alpha \qquad \text{and} \qquad (1 - \alpha - \beta)\varphi'(z).$$

Therefore, in cases 1), 2) and 3), the unique synchronization zero point is stable, because the corresponding eigenvalues are both negative. In case 4), recalling that φ' is strictly negative on $(0, \widehat{x}_1)$ and on $(\widehat{x}_1, 1)$ and strictly positive on $(\widehat{x}_1, \widehat{x}_2)$, both synchronization zero points are stable. For the same reason, in case 5), the synchronization zero point strictly smaller than $\widehat{x}_1$ and the one strictly bigger than $\widehat{x}_2$ are stable, while the one in $(\widehat{x}_1, \widehat{x}_2)$ is linearly unstable. ∎

Remark 3.5. Note that if $x^*\mathbf{1}$ is a synchronization zero point of $\mathbf{F}$, that is $(1-\alpha)(1-2x^*) - \beta(1 - 2q) = 0$, then U2) is not possible, because, as shown in the above proof, in that case the unique zero point of φ is necessarily different from x^*. Moreover, cases U3) and 4) are also not possible. Indeed, $f - (1-\alpha)id/(1-\alpha-\beta)$ is strictly increasing on $(\widehat{x}_1, \widehat{x}_2)$ and so we have $f(\widehat{x}_1) - (1-\alpha)\widehat{x}_1/(1-\alpha-\beta) < f(x^*) - (1-\alpha)x^*/(1-\alpha-\beta) = -\beta q/(1-\alpha-\beta) < f(\widehat{x}_2) - (1-\alpha)\widehat{x}_2/(1-\alpha-\beta)$. Therefore, when $x^*\mathbf{1}$ is a synchronization zero point of $\mathbf{F}$, it is stable if and only if U1) is satisfied. Otherwise, there are three synchronization zero points: x^* (linearly unstable) and two stable, say $\mathbf{z}_1^* = z_1^*\mathbf{1}$ and $\mathbf{z}_2^* = z_2^*\mathbf{1}$, with $0 < z_1^* < \widehat{x}_1 < x^* < \widehat{x}_2 < z_2^* = 2x^* - z_1^* < 1$.

As an immediate consequence of Lemma 3.4, we get that, if the system almost surely asymptotically synchronizes and one of the conditions U1), U2) and U3) holds true, then it is predictable. In the next results (see Theorems 3.6 and 3.7) we give sufficient conditions for the almost sure asymptotic synchronization of the system. Moreover, we provide a characterization of the possible limit points, that are not synchronization points (see Theorem 3.7).

Theorem 3.6. *Let $f = f_{LogP}$. Assume that one of the following conditions hold:*
   S1) $\theta/4 \leq 1/(1 - \alpha - \beta)$ *or*
   S2) $f'(0) \vee f'(1) \geq 1/(1 - \alpha - \beta)$ *(and so $\theta/4 > 1/(1 - \alpha - \beta)$) or*
   S3) $f'(0) \vee f'(1) < 1/(1 - \alpha - \beta) < \theta/4$ *and, letting $x_1^* \in (0, x^*)$ and $x_2^* = 2x^* - x_1^* \in (x^*, 1)$ be the solutions of $f' = 1/(1 - \alpha - \beta)$, we have either $f(x_1^*) \geq x_1^*/(1 - \alpha - \beta)$ or $f(x_2^*) \leq x_2^*/(1 - \alpha - \beta) - (\alpha + \beta)/(1 - \alpha - \beta)$.*

Then, we have the almost sure asymptotic synchronization of the system, i.e.

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty$$

and

$$\bar{\mathbf{I}}_n = \frac{1}{n} \sum_{k=1}^{n} \mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty,$$

where $\mathbf{Z}_\infty$ is a random variable of the form $\mathbf{Z}_\infty = Z_\infty \mathbf{1}$. Moreover, the random variable $\mathbf{Z}_\infty$ takes values in the set of the stable zero points of $\mathbf{F}$, which is contained in $(0,1)^N$ and consists of at most two different points.

Proof. We want to apply Theorem 2.2 and Theorem 2.5. Observe first that $f = f_{LogP}$ admits the primitive function

$$\phi(x) = x + \frac{1}{\theta} \ln\left(1 + e^{-\theta(x-x^*)}\right) + const$$

and, by Lemma 3.4, the set of the synchronization zero points of $\mathbf{F}$ is finite. Now, consider the function $\widetilde{\varphi}$ defined in Theorem 2.2. Observe that this function has the same form of φ: indeed, we have $\widetilde{\varphi}(z) = f(z) - \delta z + cost$, with $\delta = 1/(1 - \alpha - \beta)$ and $cost = -c \in (0, \frac{\alpha+\beta}{1-\alpha-\beta})$ and so such that $\widetilde{\varphi}(0) > 0$ and $\widetilde{\varphi}(1) < 0$. Therefore, arguing exactly as in the proof of Lemma 3.4, with $\widetilde{\varphi}$ in place of φ and $\delta = 1/(1-\alpha-\beta)$, we obtain that each of the above conditions S1), S2) and S3) implies that, for all $c \in (-\frac{\alpha+\beta}{1-\alpha-\beta}, 0)$, the function $\widetilde{\varphi}$ has exactly one zero point in $[0,1]$. Indeed, S1) and S2) correspond to condition 1) and 2) in the proof of Lemma 3.4, while condition S3) implies, for all $c \in (-\frac{\alpha+\beta}{1-\alpha-\beta}, 0)$, that $\widetilde{\varphi}$ satisfies condition 3) in the proof of Lemma 3.4. Applying Theorem 2.2 we obtain the almost sure asymptotic synchronization of the system, that is $\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty = Z_\infty \mathbf{1}$, where Z_∞ takes values in the set of the zero points of φ. Moreover, recalling that $f(0) > 0$ and $f(1) < 1$, we can also apply Theorem 2.5 and conclude that the support of the limit random variable $\mathbf{Z}_\infty$ only consists of the zero points of φ that give rise to a stable synchronization zero point of $\mathbf{F}$. By Lemma 3.4, such points belong to $(0,1)$ and they are at most two. ∎

Next theorem deal with the case not covered by Theorem 3.6. In particular, analyzing the stability of eventual "no-synchronization zero point" of $\mathbf{F}$, we provide another condition under which we have the almost sure asymptotic synchronization of the system (see condition S4) below). Moreover, we characterize the possible "no-synchronization limit configurations" for the system.

Theorem 3.7. *Let $f = f_{LogP}$ and suppose*

$$f'(0) \vee f'(1) < \frac{1}{(1-\alpha-\beta)} < \theta/4, \quad f(x_1^*) < \frac{x_1^*}{(1-\alpha-\beta)}, \quad f(x_2^*) > \frac{x_2^* - (\alpha+\beta)}{(1-\alpha-\beta)}, \qquad (30)$$

where $x_1^* \in (0, x^*)$ and $x_2^* = 2x^* - x_1^* \in (x^*, 1)$ are the solutions of $f' = 1/(1-\alpha-\beta)$. Moreover, assume that $\mathcal{Z}(\mathbf{F})$ is finite. Then

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty$$

and

$$\bar{\mathbf{I}}_n = \frac{1}{n}\sum_{k=1}^n \mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty,$$

where $\mathbf{Z}_\infty$ takes values in the set $\mathcal{SZ}(\mathbf{F})$ of the stable zero points of $\mathbf{F}$, which is contained in $(0,1)^N$. Such set always contains at most two synchronization zero points. Moreover, if $(1-\alpha-\beta)[f'(0) + f'(1)] < 1 + (1-\alpha)$, any $\mathbf{z}_\infty \in \mathcal{SZ}(\mathbf{F})$ which is not a synchronization point, has the form, up to permutations, $\mathbf{z}_\infty = (z_{\infty,1}, \ldots, z_{\infty,N})^T$ with

$$z_{\infty,h} = \begin{cases} \tilde{z}_{\infty,1} \in (0, x_1^*) & \text{for } h = 1, \ldots, N_1 \\ \tilde{z}_{\infty,2} \in (x_2^*, 1) & \text{for } h = N_1 + 1, \ldots, N, \end{cases} \qquad (31)$$

and $N_1 \in \{1, \ldots, N-1\}$. On the other hand, if

$$(1-\alpha-\beta)[f'(0) + f'(1)] \geq 1 + (1-\alpha), \qquad (32)$$

$\mathcal{SZ}(\mathbf{F})$ contains only synchronization points (and so we have the almost sure asymptotic synchronization of the system).

Summing up, taking $f = f_{LogP}$, if we are in the case S1) or S2) or S3) or S4) $\mathcal{Z}(\mathbf{F})$ finite and (30) and (32) are satisfied, then, we have the almost sure asymptotic synchronization of the system. Otherwise, we may have a non-zero probability that the system does not synchronize as time goes to $+\infty$. More precisely, provided that $\mathcal{Z}(\mathbf{F})$ is finite, the system almost surely converges, and we have a non-zero probability of asymptotic synchronization, but we may also have a non-zero probability of observing the system splitting into two groups of components that converge towards two different values. We also point out that the above results state that the random limit $\mathbf{Z}_\infty$ always belongs to $(0,1)^N$. In the first interpretation, this fact means that in the limit configuration the N agents always keep a strictly positive inclination for both actions; while in the interpretation regarding the games, this fact means that in the limit configuration, both strategies coexist in all the N games. Moreover, regarding the possible "no-synchronization limit configurations" we have that, independently from the value of N, we always have at most two groups of agents (or games) that approach in the limit two different values. We never have a more complicated asymptotic fragmentation of the whole system. Furthermore, we are able to localize the two limit values: one is strictly smaller than $x_1^* < x^*$ and the other strictly bigger than $x_2^* > x^*$, where the points x_i^* only depend on x^*, θ and on $(1 - \alpha - \beta)$, which is the "weight" of the personal inclination component of $P_{n,h}$ in (1). In addition, Remark 3.8 (after the proof) may provide some information on the sizes of the two groups. In section 4 some numerical illustrations are presented. In Fig. 4 the chosen set of parameters is such that there is only one (stable) synchronization zero point. In Fig. 6 ($\beta \neq 0$) and Fig. 9 ($\beta = 0$) there are two stable synchronization zero points and there are stable no-synchronization zero points.

Proof. Almost sure convergence follows from the fact that f admits a primitive function (see the proof of Theorem 3.6 above) and $\mathcal{Z}(\mathbf{F})$ is finite, so that we can apply Theorem 2.1. Moreover, since $f(0) > 0$ and $f(1) < 1$, by Theorem 2.5 we get that the random variable $\mathbf{Z}_\infty$ takes values in the set of the stable points of $\mathcal{Z}(\mathbf{F})$. By Lemma 3.4, this set always contains one or two synchronization points. Let us now investigate about the existence of stable no-synchronization zero points of $\mathbf{F}$. According to Remark 2.4, a necessary condition for the existence of a solution $\mathbf{z}^* = (z_1^*, \ldots, z_N^*)^T$ of (14) with $z_h^* \neq z_j^*$ for at least one pair of indexes h, j, is that there exists $c \in Im(f - (1 - \alpha - \beta)^{-1} id) \cap \left(-\frac{\alpha + \beta}{1 - \alpha - \beta}, 0\right)$ such that the corresponding function $\widetilde{\varphi}$ defined in (16) has more than one zero point in $[0, 1]$. To this regard, we observe that the assumptions (30) and the fact that $f - (1 - \alpha - \beta)^{-1} id$ is continuous and strictly increasing on (x_1^*, x_2^*) imply that $Im(f - (1 - \alpha - \beta)^{-1} id) \cap \left(-\frac{\alpha + \beta}{1 - \alpha - \beta}, 0\right)$ coincides with

$$I = \left(f(x_1^*) - \frac{x_1^*}{(1 - \alpha - \beta)} \vee -\frac{(\alpha + \beta)}{1 - \alpha - \beta}, \, f(x_2^*) - \frac{x_2^*}{(1 - \alpha - \beta)} \wedge 0\right)$$

and it is not empty. Moreover, for each c belonging to this set, the corresponding function $\widetilde{\varphi}$ has the same form of φ: indeed, we have $\widetilde{\varphi}(z) = f(z) - \delta z + cost$, with $\delta = 1/(1 - \alpha - \beta)$ and $cost = -c$ such that $\widetilde{\varphi}(0) > 0$ and $\widetilde{\varphi}(1) < 0$. Therefore, arguing exactly as in the proof of Lemma 3.4, with $\widetilde{\varphi}$ in place of φ and $\delta = 1/(1 - \alpha - \beta)$, we obtain that the assumptions (30) imply that equation $\widetilde{\varphi} = 0$ has two or three distinct solutions in $(0, 1)$ (see cases 4) and 5) in the proof of Lemma 3.4). More precisely, the equation $\widetilde{\varphi}' = 0$, that is $f' = \delta$, has exactly two solutions $x_1^*, x_2^* \in (0, 1)$ (with $x_1^* < x^* < x_2^* = 2x^* - x_1^*$), which are respectively points of local minimum and local maximum of $\widetilde{\varphi}$ in $(0, 1)$; moreover, since $c \in I$, we have $\widetilde{\varphi}(x_1^*) \leq 0$ and $\widetilde{\varphi}(x_2^*) \geq 0$. Therefore, $\widetilde{\varphi}$ has two zero points (case 4)), one in $\{x_1^*, x_2^*\}$ and the other in $(0, x_1^*) \cup (x_2^*, 1)$, or it has three zero points, one in (x_1^*, x_2^*) and the other two in $(0, x_1^*) \cup (x_2^*, 1)$. Hence, recalling again Remark 2.4, if $\mathbf{z}^*$ is a no-synchronization zero point of $\mathbf{F}$, then, fixed a component z_h^*, each other component is a solution of $\widetilde{\varphi} = 0$ with $c = f(z_h^*) - (1 - \alpha - \beta)^{-1} z_h^* \in I$, and so its components belong to $(0, 1)$ and are, up

to permutations, of the following form

$$
z_h^* = \begin{cases} \tilde{z}_1 = \zeta_1(\tilde{z}_2) & \text{for } h = 1, \ldots, N_1 \\ \tilde{z}_2 & \text{for } h = N_1 + 1, \ldots, N_2 \\ \tilde{z}_3 = \zeta_3(\tilde{z}_2) & \text{for } h = N_2 + 1, \ldots, N_3, \end{cases} \tag{33}
$$

where $N_i \in \{0, \ldots, N-1\}$, $\tilde{z}_1 \leq \tilde{z}_2 \leq \tilde{z}_3$, $\tilde{z}_2 \in [x_1^*, x_2^*]$,

$$
\zeta_1(\tilde{z}_2) \begin{cases} = \tilde{z}_2 = x_1^* & \text{if } \tilde{z}_2 = x_1^* \\ < x_1^* & \text{if } \tilde{z}_2 \in (x_1^*, x_2^*] \end{cases} \qquad \zeta_3(\tilde{z}_2) \begin{cases} = \tilde{z}_2 = x_2^* & \text{if } \tilde{z}_2 = x_2^* \\ > x_2^* = 2x^* - x_1^* & \text{if } \tilde{z}_2 \in [x_1^*, x_2^*), \end{cases}
$$

and (see (20) in Remark 2.4)

$$
\alpha \frac{1}{N} \left(N_1 \zeta_1(\tilde{z}_2) + N_2 \tilde{z}_2 + N_3 \zeta_3(\tilde{z}_2) \right) + \beta q + (1 - \alpha - \beta)c = 0. \tag{34}
$$

Finally, let us study the stability of such a point. Note that, since $\tilde{z}_2 \in [x_1^*, x_2^*]$, we have $\tilde{\varphi}'(\tilde{z}_2) = f'(\tilde{z}_2) - 1/(1 - \alpha - \beta) \geq 0$. Moreover, for $\tilde{z}_2 \in (x_1^*, x_2^*)$, we have $\tilde{\varphi}'(\tilde{z}_i) = f'(\tilde{z}_i) - 1/(1 - \alpha - \beta) < 0$, for $i = 1, 3$, while if $\tilde{z}_2 = x_1^* = \tilde{z}_1$ (respectively, $\tilde{z}_2 = x_2^* = \tilde{z}_3$), we have necessarily $\tilde{\varphi}'(\tilde{z}_3) = f'(\tilde{z}_3) - 1/(1 - \alpha - \beta) < 0$ (respectively, $\tilde{\varphi}'(\tilde{z}_1) = f'(\tilde{z}_1) - 1/(1 - \alpha - \beta) < 0$). Therefore, if $N_2 \neq 0$, we have $f'(z_h^*) > (1 - \frac{\alpha}{N})/(1 - \alpha - \beta)$ for all $h \in \{N_1 + 1, \ldots, N_2\}$. Now, for $\mathbf{w} = (w_1, \ldots, w_N)^T \in [0, 1]^N$, consider

$$
\langle \mathbf{F}(\mathbf{w}), \mathbf{w} - \mathbf{z}^* \rangle = \langle \mathbf{F}(\mathbf{w}) - \mathbf{F}(\mathbf{z}^*), \mathbf{w} - \mathbf{z}^* \rangle
$$

$$
= \frac{\alpha}{N} \left[\sum_{h=1}^N (w_h - z_h^*) \right]^2 + (1 - \alpha - \beta) \sum_{h=1}^N \left(f(w_h) - f(z_h^*) \right) (w_h - z_h^*) - \sum_{h=1}^N (w_h - z_h^*)^2. \tag{35}
$$

If we choose an index k such that $z_k^* = \tilde{z}_2$ and we take $w_h = z_h^*$ for all $h \neq k$ and $w_k = \tilde{z}_2 + \epsilon$, with $\epsilon \neq 0$, the above scalar product (35) can be written as

$$
- \left(1 - \frac{\alpha}{N} \right) \epsilon^2 + (1 - \alpha - \beta) f'(\xi) \epsilon^2,
$$

with ξ a suitable point in the interval with extremes $\tilde{z}_2$ and $\tilde{z}_2 + \epsilon$. Hence, if we take $\epsilon \neq 0$ sufficiently small so that $f'(\xi) > (1 - \frac{\alpha}{N})/(1 - \alpha - \beta)$, the above quantity is strictly positive. This fact implies that $\mathbf{z}^*$ is linearly unstable (see Appendix A). A similar argument shows that if $N_2 = 0, \tilde{z}_1 = x_1^*$ or $N_2 = 0, \tilde{z}_2 = x_2^*$, the point $\mathbf{z}^*$ is linearly unstable.

Now, let us consider a zero point $\mathbf{z}^*$ of the form (33) with $N_2 = 0$. If $f'(\tilde{z}_1) = f'(\tilde{z}_3)$ we can apply Corollary B.3 and conclude that $\mathbf{z}^*$ is stable if and only if $(1 - \alpha - \beta)f'(\tilde{z}_1) \leq 1 - \alpha$. Since $2(1 - \alpha) \leq 1 + (1 - \alpha)$, this last condition implies

$$
(1 - \alpha - \beta)[f'(\tilde{z}_1) + f'(\tilde{z}_3)] < 1 + (1 - \alpha). \tag{36}
$$

If $f'(\tilde{z}_1) \neq f'(\tilde{z}_3)$, Corollary B.4 provides conditions for the stability of $\mathbf{z}^*$ and, by Remark B.5, a necessary condition for the stability of $\mathbf{z}^*$ is given by (53), that is

$$
\alpha \frac{N_i}{N} < 1 - (1 - \alpha - \beta)f'(\tilde{z}_i) \qquad \forall i = 1, 3.
$$

Since $N_3 = N - N_1$, we find

$$
-(1 - \alpha) + (1 - \alpha - \beta)f'(\tilde{z}_3) < \alpha \frac{N_1}{N} < 1 - (1 - \alpha - \beta)f'(\tilde{z}_1). \tag{37}
$$

Note that the above inequalities implies condition (36) again. Moreover, since f' is strictly increasing on $[0, x^*)$ and strictly decreasing on $(x^*, 1]$ and $\tilde{z}_1 < x^* < \tilde{z}_3$, condition (36) necessarily implies

$$(1 - \alpha - \beta)\left[f'(0) + f'(1)\right] < 1 + (1 - \alpha). \tag{38}$$

Summing up, under the assumptions of the considered theorem, if condition (38) is not satisfied (that is (32) is satisfied), then we have the almost sure asymptotic synchronization of the system. Otherwise, if (38) is satisfied, then we always have a strictly positive probability that $\mathbf{Z}_\infty$ is equal to a synchronization zero point of $\mathbf{F}$, but we may also have a strictly positive probability that it is equal to a no-synchronization zero point of $\mathbf{F}$ of the form (31) (note that $\tilde{z}_{\infty,1} = \tilde{z}_1$ and $\tilde{z}_{\infty,2} = \tilde{z}_3$). ∎

Remark 3.8. (*Restrictions on the possible values for N_1*)
Suppose to be under the same assumptions of Theorem 3.7. It could be useful to observe that, as seen in the above proof, when $f'(\tilde{z}_1) \vee f'(\tilde{z}_3) \geq (1 - \alpha)/(1 - \alpha - \beta)$ (and so $f'(\tilde{z}_1) \neq f'(\tilde{z}_3)$), relation (37) may provide a restriction on the possible values for N_1. Note that the two bounds depend on the values $\tilde{z}_i$, $i = 1, 3$. However, when $f'(0) \vee f'(1) \geq (1 - \alpha)/(1 - \alpha - \beta)$, recalling that f' is strictly increasing on $[0, x^*)$ and strictly decreasing on $(x^*, 1]$ and $\tilde{z}_1 < x^* < \tilde{z}_3$, we obtain

$$-(1 - \alpha) + (1 - \alpha - \beta)f'(1) < \alpha\frac{N_1}{N} < 1 - (1 - \alpha - \beta)f'(0),$$

that may provide two bounds not depending on the values of the component of the limit point.

In the following remark we discuss the possible asymptotic synchronization of the system toward the values $1/2$.

Remark 3.9. (*Possible asymptotic synchronization toward $1/2$*)
As already said, the almost sure asymptotic synchronization toward the value $1/2$ means that in the limit the two inclinations (in the first interpretation) or the two strategies in all the games (in the second interpretation) coexist in the proportion $1 : 1$. With $f = f_{LogP}$, the point $1/2\mathbf{1}$ is a synchronization zero point if and only if we have

$$f(1/2) + \frac{2\beta q - (1 - \alpha)}{2(1 - \alpha - \beta)} = 0, \tag{39}$$

that, since f takes values in $(0, 1)$, implies $(1 - \alpha) > 2\beta[q \vee (1 - q)]$. Moreover, by Lemma 3.4, if $(1/2)\mathbf{1}$ is a zero point of $\mathbf{F}$, then it is stable (and so a possible limit point for the system) if and only if one of the conditions U1) or U2) is satisfied or when $1/2$ belongs to $(0, \hat{x}_1] \cup [\hat{x}_2, 1)$ (note that U3) is included in this last condition).

In the next two remarks, we discuss some of the conditions introduced in the above results, providing simple conditions on x^*, α and β sufficient to guarantee or to exclude them.

Remark 3.10. (*Regarding conditions S2), S4) and U2)*)
In this remark we show that if α, β and x^* satisfy a particular condition (see (41) below), the above cases S2) and S4) are not possible. Indeed, taking $f = f_{LogP}$, we have $|x - x^*|f'(x) = g(\theta|x - x^*|)$, where

$$g(x) := \frac{x\exp(x)}{(1 + \exp(x))^2} = \frac{x\exp(-x)}{(1 + \exp(-x))^2}. \tag{40}$$

Therefore, observing that $\max_{[0,+\infty[} g < 1/4$ (see Fig. 1), we get that the condition

$$\min\{x^*, (1 - x^*)\} \geq \frac{(1 - \alpha - \beta)}{4(1 - \alpha)} \tag{41}$$

implies $f'(0) \vee f'(1) < (1-\alpha)/(1-\alpha-\beta)$. Hence, if (41) holds true, then S2) and (32) (and so S4)) are not possible. Furthermore, under (41), case U2) of Lemma (3.4) is also not possible.

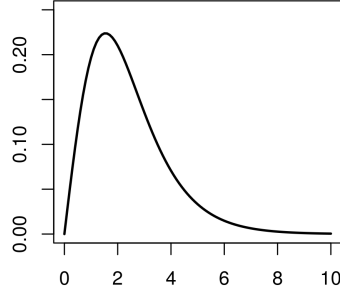Note that the above condition (41) is verified when $x^* = 1/2$.

FIGURE 1. Graph of the function g.

Remark 3.11. (*Regarding conditions S3) and* (30))
Take $f = f_{LogP}$ and suppose to be in the case $f'(0) \vee f'(1) < \frac{1}{(1-\alpha-\beta)} < \theta/4$ and let $x_1^* < x^* < x_2^*$ such that $f'(x_i^*) = 1/(1-\alpha-\beta)$. If x^* belongs to the interval

$$\left(\frac{1}{2} - \frac{(\alpha+\beta)}{2}, \frac{1}{2} + \frac{(\alpha+\beta)}{2} \right)$$

(for instance, this is the case when $x^* = 1/2$), then we necessarily have $f(x_1^*) < x_1^*/(1-\alpha-\beta)$ and $f(x_2^*) > x_2^*/(1-\alpha-\beta) - (\alpha+\beta)/(1-\alpha-\beta)$. Indeed, when x^* belongs to the above interval, then $f(x^*) - x^*/(1-\alpha-\beta) = 1/2 - x^*/(1-\alpha-\beta)$ belongs to $(-(\alpha+\beta)/(1-\alpha-\beta), 0)$ and so, since the function $f - (1-\alpha-\beta)^{-1}id$ is strictly increasing on (x_1^*, x_2^*), we get the two desired inequalities for $f(x_i^*) - x_i^*/(1-\alpha-\beta)$, $i = 1, 2$. As a consequence, case S3) is not possible.

As a consequence of the above results and remarks, we obtain the following corollary, that deals with the special case $x^* = 1/2$ and either $\beta = 0$ or $q = 1/2$. See Fig. 9.

Corollary 3.12. (*Special case:* $x^* = 1/2$ *and either* $\beta = 0$ *or* $q = 1/2$)
Take $f = f_{LogP}$ with $x^* = 1/2$ and suppose that one of the conditions $\beta = 0$ or $q = 1/2$ is satisfied. Assume $\mathcal{Z}(\mathbf{F})$ is finite. Then, using the same notation as in Lemma (3.4) and Theorem 3.7, only the following cases are possible:

a) *$\theta/4 \le (1-\alpha)/(1-\alpha-\beta)$ and, if this is the case, the system almost surely asymptotically synchronizes and it is predictable, and the unique limit point is $x^* = 1/2$;*
b) *$(1-\alpha)/(1-\alpha-\beta) < \theta/4 \le 1/(1-\alpha-\beta)$ and, if this is the case, the system almost surely synchronizes, but there are two possible limit points, $\mathbf{z}_i^* = z_i^*\mathbf{1}$, $i = 1, 2$, with $0 < z_1^* < \widehat{x}_1 < 1/2 < \widehat{x}_2 < z_2^* = 1 - z_1^* < 1$;*
c) *$\theta/4 > 1/(1-\alpha-\beta)$ and, if this the case, the system almost surely converges to a random variable $\mathbf{Z}_\infty$, taking values in the set of the stable zero points of $\mathbf{F}$, which is contained in $(0,1)^N$. Such set always contains two stable synchronization zero points, $\mathbf{z}_i^* = z_i^*\mathbf{1}$, $i = 1, 2$,*

with $0 < z_1^* < \widehat{x}_1 < 1/2 < \widehat{x}_2 < z_2^* = 1 - z_1^* < 1$, and it may contain also no-syncronization zero points of the form (31). In particular, when

$$x_1^* \leq x^* - \frac{1-\alpha-\beta}{4(1-\alpha)} = \frac{1}{2} - \frac{1-\alpha-\beta}{4(1-\alpha)}, \tag{42}$$

the points of the form (31) with $0 < \tilde{z}_{\infty,1} < x_1^* < 1/2$, $1/2 < x_2^* < \tilde{z}_{\infty,2} = 1 - \tilde{z}_{\infty,1} < 1$, $N_1 = N/2$ and $(1 - \alpha - \beta)f(\tilde{z}_{\infty,1}) - \tilde{z}_{\infty,1} = -(\alpha + \beta)/2$, are stable no-synchronization zero points of $\mathbf{F}$.

(Note that, for $\beta = 0$, the above condition (42) simply becomes $x_1^* \leq 1/4$.)

Proof. By Remark 3.5, when $x^* = 1/2$ and one of the conditions $\beta = 0$ or $q = 1/2$ is satisfied, then $(1/2)\mathbf{1}$ is a synchronization zero point of $\mathbf{F}$. Moreover, it is stable if and only if U1) is satisfied and, if this is the case, then the system almost surely asymptotically synchronizes and it is predictable. If U1) is not satisfied, we have two stable synchronization zero points, whose components are symmetric with respect to $x^* = 1/2$, that is $\mathbf{z}_i^* = z_i^*\mathbf{1}$, $i = 1, 2$, with $0 < z_1^* < \widehat{x}_1 < 1/2 < \widehat{x}_2 < z_2^* = 1 - z_1^* < 1$. Moreover, by Remarks 3.10 and 3.11, cases S2), S3) and S4) are not possible (because $x^* = 1/2$ implies $f'(0) \vee f'(1) < (1-\alpha)/(1-\alpha-\beta)$, $f(x_1^*) < x_1^*/(1-\alpha-\beta)$ and $f(x_2^*) > x_2^*/(1-\alpha-\beta) - (\alpha+\beta)/(1-\alpha-\beta)$). Summing up, we can have only: case S1), in which we have the almost sure asymptotic synchronization of the system and, when U1) is satisfied, also its predictability (see cases a) and b) in the statement); or the case when (30) is satisfied, but not (32), and so the convergence toward no-synchronization zero points of the form (31) may be possible (see case c) in the statement). Moreover, we observe that, when we are in this last case, taking $c = f(x^*) - x^*/(1-\alpha-\beta) = -(\alpha+\beta)/[2(1-\alpha-\beta)]$, the zero points of the corresponding function $\widetilde{\varphi}$ are symmetric with respect to $x^* = 1/2$ (that is $\widetilde{\varphi}(z) = 0 \Leftrightarrow \widetilde{\varphi}(1-z)$). It follows that points of the form (33) with $0 < \tilde{z}_1 < x_1^* < 1/2$, $\tilde{z}_2 = x^* = 1/2$, $1/2 < x_2^* < \tilde{z}_3 = 1 - \tilde{z}_1 < 1$ are zero points of $\mathbf{F}$ if and only if $f(\tilde{z}_1) - z_1/(1-\alpha-\beta) = c$ and (34) is satisfied, that is

$$\alpha \frac{1}{N}\left(N_1\tilde{z}_1 + N_2/2 + N_3\tilde{z}_3\right) + \beta q + (1 - \alpha - \beta)c = 0. \tag{43}$$

In particular, if we take $N_1 = N_3$ and we use the symmetry between $\tilde{z}_1$ and $\tilde{z}_3$, we obtain that (43) is satisfied if and only if $\beta = 0$ or $q = 1/2$. Therefore, when $x^* = 1/2$ and one of the conditions $\beta = 0$ or $q = 1/2$ is satisfied, $\mathbf{F}$ has no-synchronization zero points of the form (33) with $0 < \tilde{z}_1 < x_1^* < 1/2$, $\tilde{z}_2 = x^* = 1/2$, $1/2 < x_2^* < \tilde{z}_3 = 1 - \tilde{z}_1$, $N_1 = N_3$ and $f(\tilde{z}_1) - z_1/(1-\alpha-\beta) = c$. Now, such points are stable if and only if $N_2 = 0$ (and so $N_1 = N_3 = N/2$) and $(1 - \alpha - \beta)f'(\tilde{z}_1) \leq 1 - \alpha$ (note that we are in the case $f'(\tilde{z}_1) = f'(\tilde{z}_3)$). Since f' is strictly increasing on $[0, x^*)$, this last condition is satisfied when $(1 - \alpha - \beta)f'(x_1^*) \leq 1 - \alpha$. Finally, using the fact that, for $f = f_{LogP}$, we have $f'(x) = g(\theta|x-x^*|)/|x-x^*|$, where g is the function defined in (40) and such that $\max_{[0,+\infty[} g < 1/4$, in order to guarantee the stability of the considered no-synchronization zero points, it is enough to require (42). ∎

We conclude this section with two remarks: one regarding the case $N = 1$ and the other the rate of convergence.

Remark 3.13. (*Case $N = 1$*)
This remark is devoted to the case $N = 1$ and the relationship with the results obtained in [47].

The above proofs (with the due simplifications) also work in the case $N = 1$ and $\alpha = \beta = 0$, that corresponds to the case studied in [47]. Indeed, in this case, we have to consider only Theorem 2.1, Lemma 3.4 and Remark 3.5 (with $N = 1$ and $\alpha = \beta = 0$). As a consequence, when $x^* = 1/2$, if $\theta \leq 4$, then the system is predictable and the unique limit configuration is given by

$z_\infty = 1/2$; otherwise it almost surely converges, but it is not predictable, and the two possible limit configurations belong to $(0, \widehat{x}_1] \cup [\widehat{x}_2, 1) \subset (0,1) \setminus \{1/2\}$ and they are symmetric with respect to $1/2$. This is the same result obtained in [47]. For the case $x^* \neq 1/2$, in [47] there are only some numerical analyses; while here we have proven a precise result: when one of the conditions U1), U2) or U3) is satisfied, then the system is predictable and the limit configuration belongs to $(0,1) \setminus \{x^*, 1/2\}$ (more precisely, it is strictly smaller than $1/2$ when $x^* > 1/2$ and strictly greater than $1/2$ when $x^* < 1/2$, because $\varphi(1/2) = f(1/2) - 1/2 = f(1/2) - f(x^*)$ and f is strictly increasing); otherwise it almost surely converges, but the system is not predictable, and the two possible limit configurations belong to $(0,1) \setminus \{x^*, 1/2\}$ (more precisely, one belongs to $(0, \widehat{x}_1] \setminus \{1/2\} \subset (0, x^*) \setminus \{1/2\}$ and one in $[\widehat{x}_2, 1) \setminus \{1/2\} \subset (x^*, 1) \setminus \{1/2\}$ and so as before, taking into account that $\varphi(1/2) = f(1/2) - f(x^*)$, one is strictly smaller than $1/2$ and the other strictly greater than $1/2$).

Remark 3.14. (*Rate of convergence*)
When the system is predictable with $\mathbf{z}_\infty = z_\infty \mathbf{1}$ as the unique possible limit value for $\mathbf{Z}_\infty$, applying the same arguments used in Remark 3.3, we can obtain a central limit theorem where the rate of convergence is driven by $\lambda = (1 - \alpha) - (1 - \alpha - \beta) f'(z_\infty)$ (see Theorem A.6 and Remark A.7).

When the system almost surely converges to $\mathbf{Z}_\infty$, but it is not predictable, applying Remark A.8), we get $1/\sqrt{n}$ as the rate of convergence, for any $\mathbf{z}_\infty$ with $P(\mathbf{Z}_n \to \mathbf{z}_\infty) > 0$ and $\lambda(\mathbf{z}_\infty) > 0$. In particular, with some computations similar to those done in Remark 3.3, we have that the matrix $\Gamma = \Gamma(\mathbf{z}_\infty)$ is a diagonal matrix with diagonal elements equal to $z_{\infty,h}(1 - z_{\infty,h})$ with $z_{\infty,h} = z_\infty$ (synchronization point) or $z_{\infty,h} \in \{\tilde{z}_{\infty,1}, \tilde{z}_{\infty,2}\}$ (no-synchronization point). Finally, note that, when $\mathbf{z}_\infty$ is a possible no-synchronization limit point with $\tilde{z}_{\infty,2} = 2x^* - \tilde{z}_{\infty,1}$, we have $f'(\tilde{z}_{\infty,1}) = f'(\tilde{z}_{\infty,2})$ and so again $\lambda(\mathbf{z}_\infty) = (1 - \alpha) - (1 - \alpha - \beta) f'(\tilde{z}_{\infty,1})$.

3.3. Case $f = f_{\mathbf{Tech}}$. In this subsection we consider the function f_{Tech} defined in (8) with $\theta \in (1/2, 1)$, that is

$$f(x) = f_{\text{Tech}}(x) = (1 - \theta) + (2\theta - 1)(3x^2 - 2x^3) \qquad \text{with } \theta \in (1/2, 1). \tag{44}$$

Similarly to f_{LogP}, the function $f = f_{\text{Tech}}$ is a sigmoid function, *i.e.* its first derivative is a strictly positive function, which is strictly increasing on $[0, 1/2]$ and strictly decreasing on $(1/2, 1]$ with a maximum given by $f'(1/2) = 3\theta - 3/2$. Furthermore, we have $f'(x) = f'(1 - x)$ for all $x \in [0, 1]$. Differently from f_{LogP}, we have $f'(0) = f'(1) = 0$. Therefore, arguing exactly as in the proof of Lemma 3.4 and Remark 3.5, but using the fact that $x^* = 1/2$ and $f'(0) = f'(1)$, we obtain the following lemma and remark.

Lemma 3.15. (*Synchronization zero points*)
Let $f = f_{Tech}$. Then, accordingly to the values of the parameters, $\mathcal{Z}(\mathbf{F})$ contains at least three synchronization zero points. Moreover, at most two of them are stable. In particular, if one of the following conditions is satisfied, $\mathbf{F}$ has a unique stable synchronization zero point:

U1) $3\theta - 3/2 \leq (1 - \alpha)/(1 - \alpha - \beta)$ *or*

U2) $(1 - \alpha)/(1 - \alpha - \beta) < 3\theta - 3/2$ *and either* $f(\widehat{x}_1) > (1 - \alpha)/(1 - \alpha - \beta)\widehat{x}_1 - \beta q/(1 - \alpha - \beta)$ *or* $f(\widehat{x}_2) < (1 - \alpha)/(1 - \alpha - \beta)\widehat{x}_2 - \beta q/(1 - \alpha - \beta)$, *where $\widehat{x}_1 \in (0, 1/2)$ and $\widehat{x}_2 = 1 - \widehat{x}_1 \in (1/2, 1)$ are the solutions of $f' = (1 - \alpha)/(1 - \alpha - \beta)$.*

Otherwise, $\mathbf{F}$ has two stable synchronization zero points belonging to $(0, \widehat{x}_1] \cup [\widehat{x}_2, 1)$ (more precisely, one in each of these two intervals).

Remark 3.16. Note that if $1/21$ is a synchronization zero point of $\mathbf{F}$, that is $\beta = 0$ or $q = 1/2$, then U2) is not possible. Indeed, $f - (1 - \alpha)id/(1 - \alpha - \beta)$ is strictly increasing on $(\widehat{x}_1, \widehat{x}_2)$ and so we have $f(\widehat{x}_1) - (1 - \alpha)\widehat{x}_1/(1 - \alpha - \beta) < f(1/2) - (1 - \alpha)/2(1 - \alpha - \beta) = -\beta q/(1 - \alpha - \beta) < f(\widehat{x}_2) - (1 -$

$\alpha) \widehat{x}_2/(1-\alpha-\beta)$. Moreover, when $1/2\mathbf{1}$ is a synchronization zero point of $\mathbf{F}$, it is stable if and only if U1) is satisfied. Otherwise, there are three synchronization zero points: $1/2$ (linearly unstable) and two stable, say $\mathbf{z}_1^* = z_1^*\mathbf{1}$ and $\mathbf{z}_2^* = z_2^*\mathbf{1}$, with $0 < z_1^* < \widehat{x}_1 < 1/2 < \widehat{x}_2 < z_2^* = 1 - z_1^* < 1$.

As an immediate consequence of Lemma 3.15, we get that, if the system almost surely asymptotically synchronizes and one of the conditions U1) or U2) holds true, then it is predictable.

Now, we observe that $f = f_{\text{Tech}}$ admits the primitive function

$$\phi(x) = (1 - \theta)x + (2\theta - 1)\left(1 - \frac{x}{2}\right)x^3 + const.$$

Moreover, for $\theta \in (1/2, 1)$, we have $f(0) = 1 - \theta > 0$ and $f(1) = \theta < 1$. Therefore, arguing exactly as done in the proof of Theorem 3.6, but with some simplifications due to the fact that $x^* = 1/2$ (see Remark 3.11) and $f'(0) = f'(1) = 0$, we obtain the following result.

See in section 4, simulations and illustrations associated in Fig. 10, Fig. 11, Fig. 12.

Theorem 3.17. *Let $f = f_{\text{Tech}}$. If $3\theta - 3/2 \leq 1/(1-\alpha-\beta)$, then we have the almost sure asymptotic synchronization of the system, i.e.*

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty$$

and

$$\bar{\mathbf{I}}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty,$$

where $\mathbf{Z}_\infty$ is a random variable of the form $\mathbf{Z}_\infty = Z_\infty\mathbf{1}$. Moreover, the random variable $\mathbf{Z}_\infty$ takes values in the set of the stable zero points of $\mathbf{F}$, which is contained in $(0,1)^N$ and consists of at most two different points.

Next theorem deal with the case not covered by Theorem 3.17. In particular, we characterize the possible "no-synchronization limit configurations" for the system. The proof is exactly the same given for Theorem 3.7, but taking into account that $x^* = 1/2$ and $f'(0) = f'(1) = 0$ and, above all, that $\mathcal{Z}(\mathbf{F})$ is finite (see Lemma C.1 in Appendix).

Theorem 3.18. *Let $f = f_{\text{Tech}}$. If $1/(1 - \alpha - \beta) < 3\theta - 3/2$, then*

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty$$

and

$$\bar{\mathbf{I}}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{I}_k \xrightarrow{a.s.} \mathbf{Z}_\infty,$$

where $\mathbf{Z}_\infty$ takes values in the set $\mathcal{SZ}(\mathbf{F})$ of the stable zero points of $\mathbf{F}$, which is contained in $(0,1)^N$. Such set always contains at most two synchronization zero points. Moreover, any $\mathbf{z}_\infty \in \mathcal{SZ}(\mathbf{F})$ which is not a synchronization point, has the form, up to permutations, $\mathbf{z}_\infty = (z_{\infty,1},\ldots,z_{\infty,N})^T$ with

$$z_{\infty,h} = \begin{cases} \tilde{z}_{\infty,1} \in (0, x_1^*) & \text{for } h = 1,\ldots,N_1 \\ \tilde{z}_{\infty,2} \in (x_2^*, 1) & \text{for } h = N_1 + 1,\ldots,N, \end{cases} \tag{45}$$

where $x_1^* \in (0, 1/2)$ and $x_2^* = 1 - x_1^* \in (1/2, 1)$ are the solutions of $f' = 1/(1 - \alpha - \beta)$, and $N_1 \in \{1,\ldots,N-1\}$.

Since f_{Tech} is a polynom of degree 3, thanks to Lemma C.1, it holds $\mathcal{Z}(\mathbf{F})$, set of roots, is finite. Summing up, taking $f = f_{\text{Tech}}$, if $3\theta - 3/2 \leq 1/(1 - \alpha - \beta)$, then, we have the almost sure asymptotic synchronization of the system. Otherwise, we may have a non-zero probability that the system does not synchronize as time goes to $+\infty$. More precisely, the system almost

surely converges, and we have a non-zero probability of asymptotic synchronization, but we may also have a non-zero probability of observing the system splitting into two groups of components that converge towards two different values. We also point out that the above results state that the random limit $\mathbf{Z}_\infty$ always belongs to $(0,1)^N$. In the first interpretation, this fact means that in the limit configuration the N agents always keep a strictly positive inclination for both actions; while in the interpretation regarding the technological dynamics, this fact means that in the limit configuration, both technologies coexist in all the N markets. Moreover, regarding the possible "no-synchronization limit configurations" we have that, independently from the value of N, we always have at most two groups of agents (or markets) that approach in the limit two different values. We never have a more complicated asymptotic fragmentation of the whole system. Furthermore, we are able to localize the two limit values: one is strictly smaller than $x_1^* < 1/2$ and the other strictly bigger than $x_2^* > 1/2$, where the points x_i^* only depend on θ and on $(1 - \alpha - \beta)$, which is the "weight" of the personal inclination component of $P_{n,h}$ in (1). Moreover, arguing as in Remark 3.8, the following inequality may provide restrictions on the possible values for N_1:

$$-(1 - \alpha) + (1 - \alpha - \beta)f'(\tilde{z}_{\infty,2}) < \alpha \frac{N_1}{N} < 1 - (1 - \alpha - \beta)f'(\tilde{z}_{\infty,1}).$$

In the following remark we discuss the possible asymptotic synchronization of the system toward the values $1/2$.

Remark 3.19. (*Possible asymptotic synchronization toward* $1/2$)
As already said, the almost sure asymptotic synchronization toward the value $1/2$ means that in the limit the two inclinations (in the first interpretation) or the two technologies in all the markets (in the third interpretation) coexist in the proportion $1:1$. With $f = f_{\text{Tech}}$, the point $1/2\mathbf{1}$ is a synchronization zero point if and only if $\beta = 0$ or $q = 1/2$. Moreover, by Remark 3.16, if $(1/2)\mathbf{1}$ is a zero point of $\mathbf{F}$, then it is stable (and so a possible limit point for the system) if and only if U1) is satisfied.

As a consequence of the above results and remarks, arguing as in the proof of Corollary 3.12, we obtain the following corollary, that deals with the special case $\beta = 0$ or $q = 1/2$.

Corollary 3.20. (Special case: $\beta = 0$ or $q = 1/2$)
Take $f = f_{Tech}$ and suppose that one of the conditions $\beta = 0$ or $q = 1/2$ is satisfied. Then, using the same notation as in Lemma (3.15) and Theorem 3.18, only the following cases are possible:
   a) $3\theta - 3/2 \leq (1-\alpha)/(1-\alpha-\beta)$ *and, if this is the case, the system almost surely asymptotically synchronizes and it is predictable, and the unique limit point is $x^* = 1/2$;*
   b) $(1 - \alpha)/(1 - \alpha - \beta) < 3\theta - 3/2 \leq 1/(1 - \alpha - \beta)$ *and, if this is the case, the system almost surely synchronizes, but there are two possible limit points, $\mathbf{z}_i^* = z_i^*\mathbf{1}$, $i = 1, 2$, with $0 < z_1^* < \hat{x}_1 < 1/2 < \hat{x}_2 < z_2^* = 1 - z_1^* < 1$;*
   c) $3\theta - 3/2 > 1/(1-\alpha-\beta)$ *and, if this the case, the system almost surely converges to a random variable $\mathbf{Z}_\infty$, taking values in the set of the stable zero points of $\mathbf{F}$, which is contained in $(0, 1)^N$. Such set always contains two stable synchronization zero points, $\mathbf{z}_i^* = z_i^*\mathbf{1}$, $i = 1, 2$, with $0 < z_1^* < \hat{x}_1 < 1/2 < \hat{x}_2 < z_2^* = 1 - z_1^* < 1$, and it may contain also no-syncronization zero points of the form (45). In particular, when*
   $$(1 - \alpha - \beta)f'(x_1^*) \leq 1 - \alpha,$$
   the points of the form (45) with $0 < \tilde{z}_{\infty,1} < x_1^* < 1/2$, $1/2 < x_2^* < \tilde{z}_{\infty,2} = 1 - \tilde{z}_{\infty,1} < 1$, $N_1 = N/2$ and $(1 - \alpha - \beta)f(\tilde{z}_{\infty,1}) - \tilde{z}_{\infty,1} = -(\alpha + \beta)/2$, are stable no-synchronization zero points of $\mathbf{F}$

We conclude this section with two remarks: one regarding the case $N = 1$ and the other the rate of convergence.

Remark 3.21. (*Case $N = 1$*)
This remark is devoted to the case $N = 1$ and the relationship with the results obtained in [14,38]. Indeed, we have to consider only Theorem 2.1, Lemma 3.15 and Remark 3.16 (with $N = 1$ and $\alpha = \beta = 0$, that corresponds to the case studied in [14,38]). As a consequence, when $1/2 < \theta \le 5/6$ the system is predictable and the unique limit configuration is $1/2$; otherwise it almost surely converges, but it is not predictable, and the two possible limit configurations z_1^*, z_2^* belong to $(0,1) \setminus \{1/2\}$ and they are such that $0 < z_1^* < \widehat{x}_1 < 1/2$ and $1/2 < \widehat{x}_2 = 1 - \widehat{x}_1 < z_2^* = 1 - z_1^* < 1$.

Remark 3.22. (*Rate of convergence*)
When the system is predictable with $\mathbf{z}_\infty = z_\infty \mathbf{1}$ as the unique possible limit value for $\mathbf{Z}_\infty$, applying the same arguments used in Remark 3.3, we can obtain a central limit theorem where the rate of convergence is driven by $\lambda = (1 - \alpha) - (1 - \alpha - \beta) f'(z_\infty)$ (see Theorem A.6 and Remark A.7).

When the system almost surely converges to $\mathbf{Z}_\infty$, but it is not predictable, applying the same arguments used in Remark 3.14, we get $1/\sqrt{n}$ as the rate of convergence, for any $\mathbf{z}_\infty$ with $P(\mathbf{Z}_n \to \mathbf{z}_\infty) > 0$ and $\lambda(\mathbf{z}_\infty) > 0$ (see Remark A.8). Note that, when $\mathbf{z}_\infty$ is a possible no-synchronization limit point with $\tilde{z}_{\infty,2} = 1 - \tilde{z}_{\infty,1}$, we have $f'(\tilde{z}_{\infty,1}) = f'(\tilde{z}_{\infty,2})$ and so again $\lambda(\mathbf{z}_\infty) = (1 - \alpha) - (1 - \alpha - \beta) f'(\tilde{z}_{\infty,1})$.

4. Simulations and figures

In the following section, we do consider some graphical illustrations and numerical simulations or sampling of the stochastic dynamical systems. This can be easily coded thanks to the iterative equations defining the dynamical evolution.

We have chosen particular parameter sets for each specific f considered previously. The sets were chosen for their own interest or for their interest in comparison with other sets. We used different values for N. We considered either deterministic initial conditions or random ones. When random, we chose independent values, uniformly distributed on [0,1]. Note that, when we assume as initial condition $(Z_{0,h}, \ldots, Z_{N,h})$ exchangeable, the variables $(Z_{n,1}, \ldots, Z_{n,N})$ are exchangeable for all n, and so the set $\mathcal{Z}$ where $\mathbf{Z}_\infty$ takes values is permutation invariant, *i.e.* if $(z_1, \ldots, z_N)^\top$ belongs to $\mathcal{Z}$, then, for any permutation σ of $\{1, \ldots, N\}$, the vector $(z_{\sigma(1)}, \ldots, z_{\sigma(N)})^\top$ also belongs to $\mathcal{Z}$.

As previously noticed, when $\alpha = 0$, there is no interaction *i.e.* stochastic independence between the components holds. Since f is non linear, contrary to the models where f is linear, combinatorics can create multiple limit points. For $f = f_{LP}$ only synchronization limit points are possible. For $f = f_{LogP}$ or $f = f_{\text{Tech}}$, according to the choice of parameters, many limit points are possible which can be of synchronization (on the diagonal) or of no-synchronization type (off the diagonal). When f is linear synchronization was proved to hold as soon as there is interaction ($\alpha > 0$). Here, on the contrary, it needs specific conditions between the parameters to be fulfilled in order to have synchronization almost surely. Finally, as observed in the following samplings, in some region of parameters, limit points may be difficult to observe computationally due to slow dynamical evolution.

4.1. **Case $f = f_{LP}$.** For the set of parameters $\theta = 0.9$, $x^* = 1/3$, $\alpha = 0.1$, $\beta = 0.2$, $q = 0.4$, Fig. 2 shows the graph of the function f_{LP} intersecting the straight line defined through (15). There is a unique zero synchronisation point at ≈ 0.664.

For the same set of parameters Fig. 3 shows some numerical simulations samples. Fig. 3 (A) presents the trajectories of the components values of one sample of the whole system, when $N = 30$. The associated empirical means are represented in Fig. 3 (B). In both cases, a.s. convergence towards the unique (stable) synchronization point is observed in coherence with the previously stated theoretical result. Fig. 3 (C) pictures through an histogram the values observed for a large time ($T = 5000$). Remark these values are components' values of 100 independent samples of the

system. Fig. 3 (D) is a representation when $N = 2$ of the tangent/gradient field of $-V$. Additively values of $-V$ are represented through colors. Blue color is used for low values, showing a unique minimum of $-V$. Red color is used for high values.

4.2. **Case** $f = f_{\mathrm{LogP}}$. In the following Figures some specific set of parameters were chosen.

In Fig. 4 parameters are chosen such that there is a unique stable synchronization zero point. Fig. 4 (A) illustrates, for one sample, the a.s. convergence towards this value for $(Z_n(k))_n$ and Fig. 4 (B) for the associated empirical means $(I_n(k))_n$. Fig. 4 (C) pictures the histogram of the components' values $Z_n(k)$ for n large and $k \in \{1, \ldots, N\}$. There were 100 independent samples of the whole system. Please pay attention the $(N = 15) * 100$ values used for the histogram are not independent. As previously, Fig. 4 (D) represents, when $N = 2$, the "landscape" of $F = -\nabla V$.

In Fig. 5, two parameters sets are considered: the one from Fig. 9 and the one from Fig. 6, Fig. 7 and Fig. 8. For illustration, stable synchronization zero points are found at the intersection of the curve associated to f and the straight line given by Eq.(15). In both cases there are two stable synchronization zeros and some no-synchronization stable zero points. In Fig. 9 the components' values are different, contrary to the set of parameters from Fig. 6, Fig. 7 and Fig. 8, where the component's values are close between synchronization points and no-synchronization points.

The parameters' set of Fig. 9 is related to Corollary 3.12.

In Fig. 6, the initial condition is always $1/2$ and $N = 30$ was chosen. In Fig. 7 initial conditions are independent and uniformly distributed. Case $N = 5$ is considered differently from Fig. 8 where $N = 30$ is chosen.

As deduced from Fig. 7 (B) and Fig. 8 (B), the convergence is not always towards the same zero point. No-synchronization zero points are observed as limit. It is possible that the synchronization zero points are rarely observed since Fig. 8 (B) and (C) show no observation going to the synchronization values. For large values of N it seems that synchronization is never observed (unless starting with very specific starting conditions close to the synchronization points).

In Fig. 9, the "landscape" associated to this parameter set when $N = 2$ is shown in Subfig. (D). Convergence towards no-synchonization points is observed in particular in the sample Fig. 9 (A) with $N = 100$. Subfig. 9 (C) depicts trajectories represented in the potential landscape when $N = 2$.
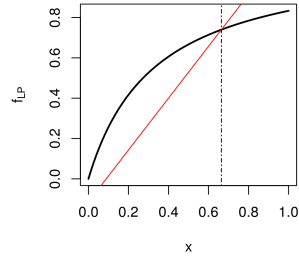
FIGURE 2. Case $f = f_{LP}$. Graph of the function f_{LP} intersecting the straight line $y = ((1 - \alpha)x - \beta q)/(1 - \alpha - \beta)$, giving a unique zero synchronisation point at ≈ 0.664 . Set of parameters $\theta = 0.9$, $x^* = 1/3$, $\alpha = 0.1$, $\beta = 0.2$, $q = 0.4$.
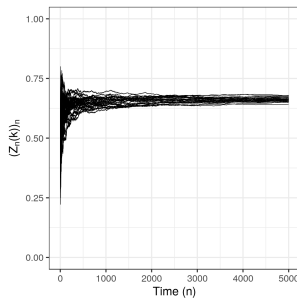
(A) One sample of the trajectories $(Z_n(k))_n$ $(1 \leq k \leq N)$ of a system with $N = 30$. Starting condition is $1/2$ for all components.
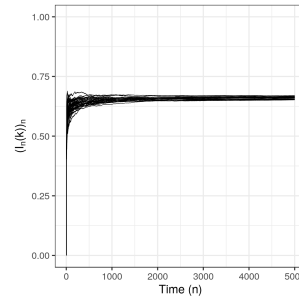
(B) One sample of the trajectories of the associated empirical means $(I_n(k))_n$ $(1 \leq k \leq N)$.
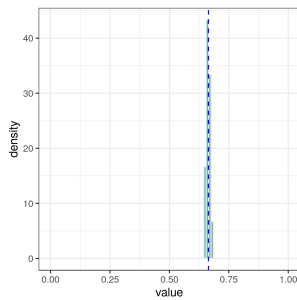
(C) Histogram of system's $(N = 30)$ components values from 100 (independent) samples, at time 5.000. Starting conditions are always $1/2$.

(D) Representation of the field $F = -\nabla V$ when $N = 2$ computed using the software Mathematica 11.3.

FIGURE 3. Case $f = f_{LP}$. Set of parameters is $\theta = 0.9$, $x^* = 1/3$, $\alpha = 0.1$, $\beta = 0.2$, $q = 0.4$. There is a unique (stable) synchronisation point at ≈ 0.664. The system is predictable.
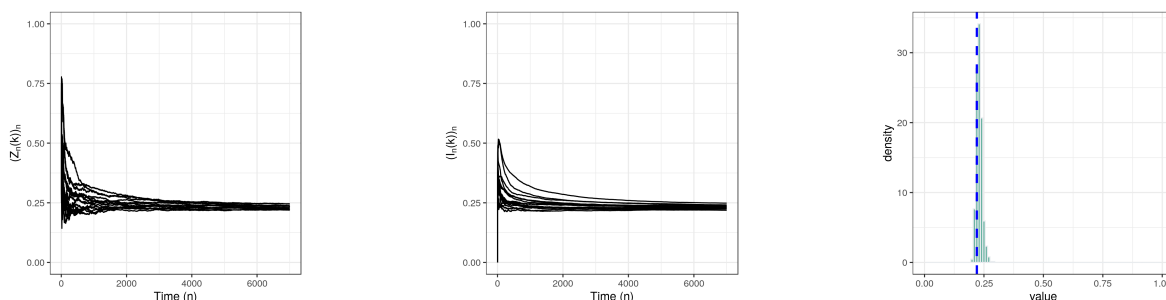
(A) One sample of the trajecto-
ries $(Z_n(k))_n$ $(1 \leq k \leq N)$.

(B) One sample of the trajecto-
ries of the associated empirical
means $(I_n(k))_n$ $(1 \leq k \leq N)$.

(C) Histogram of system's com-
ponents values, for 100 indepen-
dent samples, at time 7.000.

FIGURE 4. Case $f = f_{LogP}$. Parameters are $x^\star = 0.6$, $\theta = 5$, $\alpha = 0.1$, $\beta = 0.3$,
$q = 0.4$. There is a unique zero of F and synchronisation point at ≈ 0.22 (vertical
dashed line in subfig. (C)). System's size is $N = 15$. Starting condition is $1/2$ for
all components.
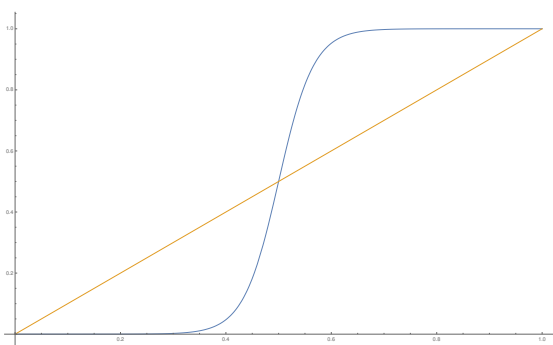
(A) Set of parameters $\theta = 30$, $x^* = 1/2$, $\alpha = 0.4$,
$\beta = 0$, $q = 0$ as in Fig. 9. It gives two stable zero
synchronisation points close to 0 and close to 1
at the intersection of both curves
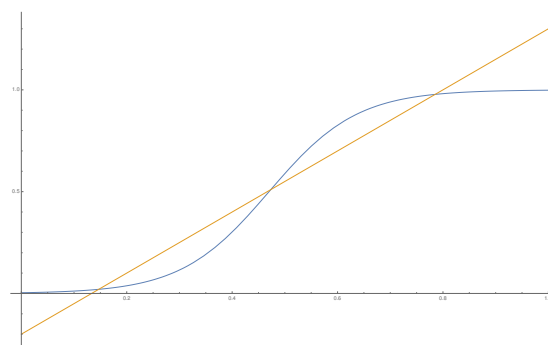
(B) Set of parameters $\theta = 12$, $x^* = 0.47$, $\alpha = 0.1$,
$\beta = 0.3$, $q = 0.4$ as in Fig. 6. It gives two stable
zero synchronisation points at the intersection of
both curves.

FIGURE 5. Case $f = f_{LogP}$. Graph of the function f_{LogP} in blue intersecting the
straight line (orange) defined through $y = ((1 - \alpha)x - \beta q)/(1 - \alpha - \beta)$ Eq.(15)

.

(A) One sample of the trajectories $(Z_n(k))_n$ $(1 \leq k \leq N)$. $N = 30$.
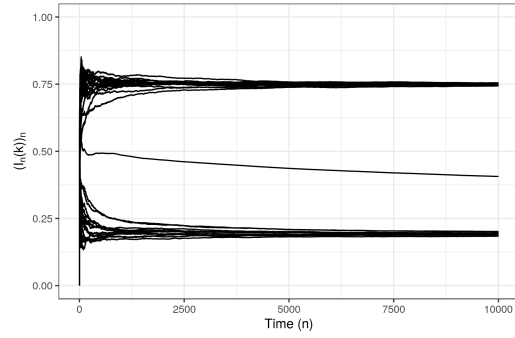
(B) One sample of the trajectories of the associated empirical means $(I_n(k))_n$ $(1 \leq k \leq N)$.
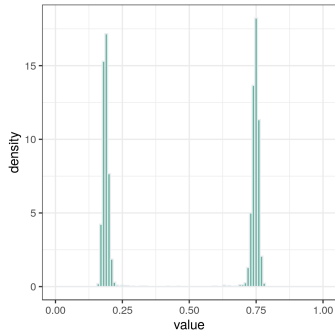
(C) $N = 30$. Histogram of components' values at $T = 10.000$ iterations. Independent system's sample of size 100.
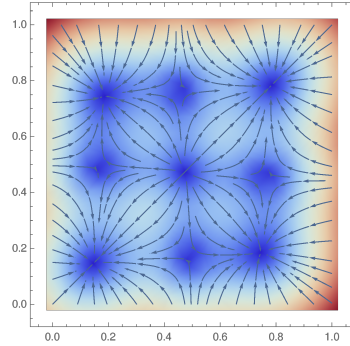
(D) Field $F = -\nabla V$ associated when $N = 2$.

FIGURE 6. Case $f = f_{LogP}$. Parameters are $x^\star = 0.47$, $\theta = 12$, $\alpha = 0.1$, $\beta = 0.3$, $q = 0.4$. There are two stable synchronization zeros $\approx \{0.14, \ 0.78\}$. Components' values of (stable) non-synchronization points are close to 0.2 and 0.8. Starting conditions are $1/2$ for every component and sample.
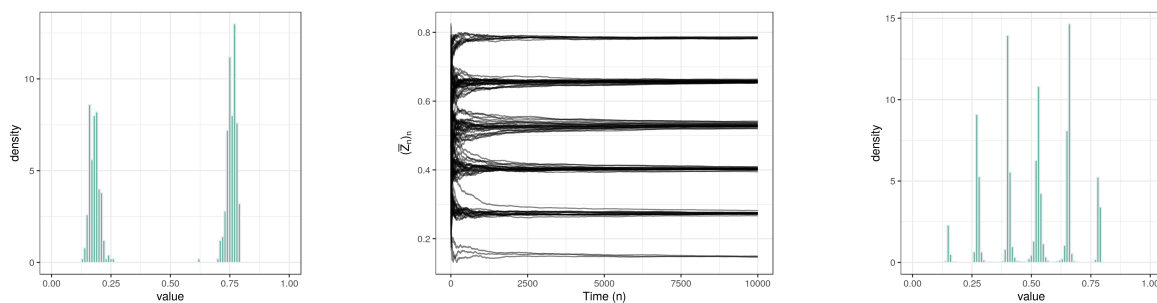
(A) Histogram of component's values at time 10000.

(B) Trajectories of mean field for each system sampled. Thus these are 100 independent curves.

(C) Histogram of 100 independent mean field's values at time 10.000

FIGURE 7. Case $f = f_{LogP}$. Same set of parameters as Fig. 6: $x^\star = 0.47$, $\theta = 12$, $\alpha = 0.1$, $\beta = 0.3$, $q = 0.4$. Here $N = 5$ and starting conditions are chosen i.i.d. uniformly distributed on [0,1]. Subfigures (A), (B) and (C) are related to the same 100 independent systems' samples.
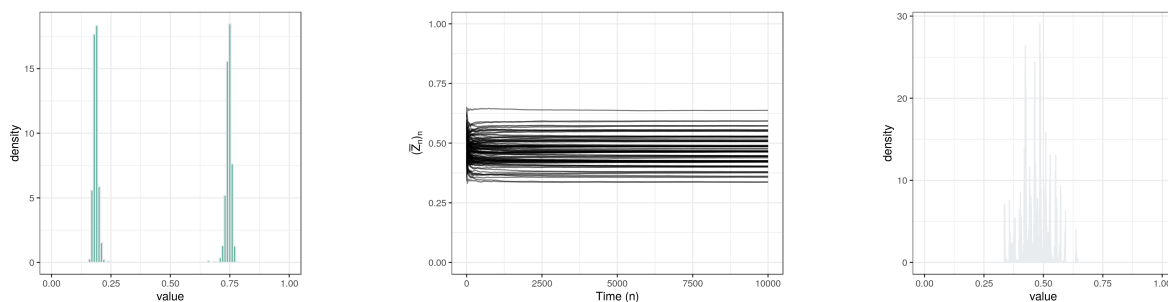
(A) Histogram of components' values at time 10.000

(B) Trajectories of mean field from 100 independent samples.

(C) Histogram of the mean field's values at time 10.000, from a set of 100 independent system's samples.

FIGURE 8. Case $f = f_{LogP}$. Parameters as Fig. 6 and Fig. 7 but with uniformly distributed starting conditions. Here, $N = 30$.
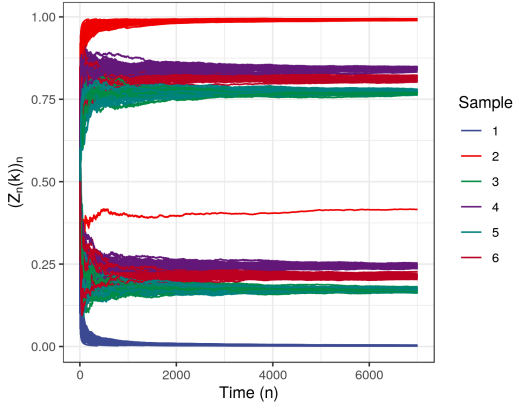
(A) Six samples of the system $(Z_n(k))_n$ $(1 \le k \le N)$ with $N = 100$. Sample 1 starts with 0.2 on the diagonal. Sample 2 starts with 0.7 on the diagonal. Samples 3 to 6 starts with 0.5 on the diagonal. All trajectories of one system's sample share the same color.
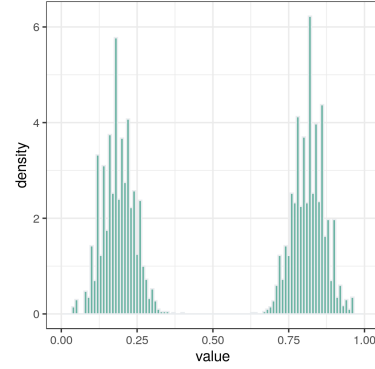
(B) Histogram of components' values at $T = 6000$ for 200 samples of the systems with $N = 20$. For each component of each sample, the starting condition is chosen, independetly, uniformly distributed on [0,1].
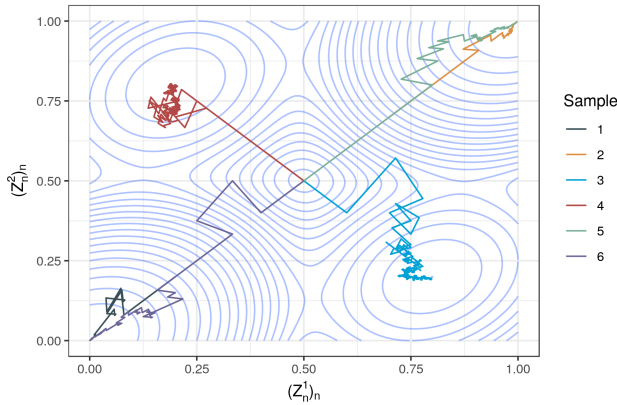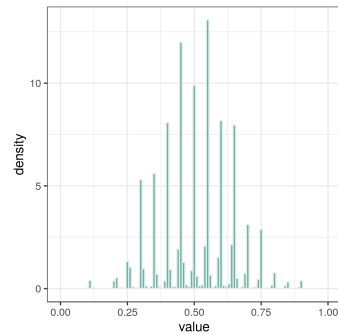

(C) Six samples of the system when $N = 2$. Representation of trajectories $(Z_n(1), Z_n(2))_n$ up to 30.000 iterations. Each color means a different sample. Sample 1 starts with $(0.2, 0.2)$. Sample 2 starts with $(0.7, 0.7)$. Samples 3 to 6 start with 0.5 on the diagonal. In background some level sets of $-V$.
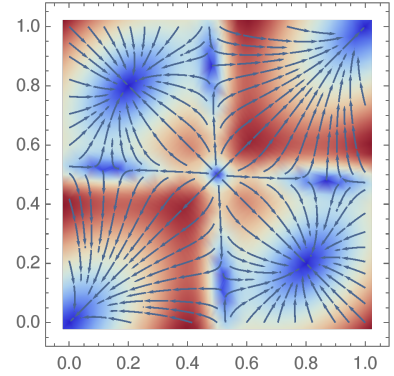
(D) Field $F = -\nabla V$ when $N = 2$.
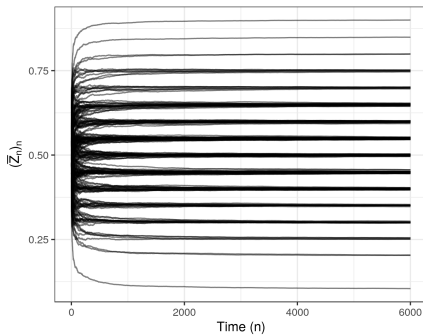
(E) Mean fields trajectories associated to (B).

(F) Histogram of mean field values at $T = 6000$ for 200 independent samples of the system with $N = 20$. Same sample as in (B) and (E)

FIGURE 9. Case $f = f_{LogP}$. Parameters are $x^\star = 0.5$, $\theta = 30$, $\alpha = 0.4$, $\beta = 0$. This is related to Corollary 3.12. Synchronization points are close to 0 and 1 (stable) and $x^* = 0.5$ (unstable). Components' values of no-synchronization (stable) points are close to 0.2 and 0.8.

4.3. **Case** $f = f_{\mathbf{Tech}}$. Parameters related to Fig. 10 are such that there are two stable synchronization zero points and there exist stable no-synchronization zero points. As it can be observed from simulations in (C) and in the landscape (D), the dynamics can be very slow close to these points. Samples from (A) comforts this observation.

Parameters related to Fig. 11 are such that there are two stable synchronization zero points and there are no stable no-synchronization zero points. In Subfig. 11 (C) it can be seen on these samples that the dynamical behavior is slow in the neighborhood of the unstable no-synchronization points. Contrary to what is observed, due to finite number of iterations, thanks to the previously mentioned theoretical results, we know convergence will eventually happen towards stable synchronization points.

In Fig. 12, parameters are set up such that there are only two stable synchronization points. From $N = 2$ cases, unstable no-synchronization points can be guessed to be in regions where the dynamics are slow. For instance, in Subfig. Fig. 12 (B), the sample 6 starts at $(0.6, 0.1)$ and does not succeed to reach the neighborhood of a synchronization point before 150.000 iterations.
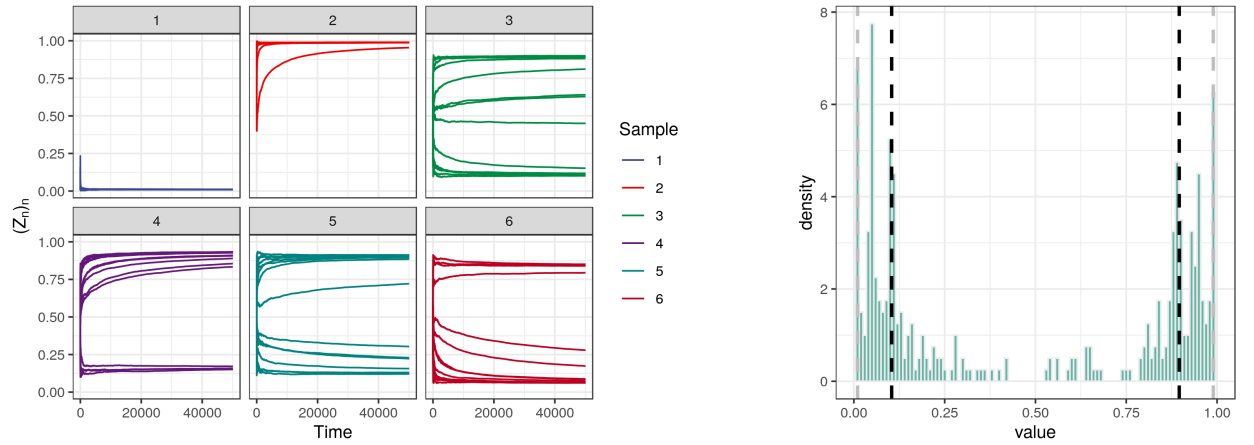
(A) Six samples of the system's trajectories when $N = 15$. Each component starts at 0.5 on the diagonal.

(B) Histogram of components' values at time $T = 50.000$ when $N = 4$. All trajectories start in $1/2$. Vertical grey dashed lines are at stable synchronization points values 0.010 and 0.989. Vertical black dashed lines are at non-synchronization values 0.104, and 0.896.
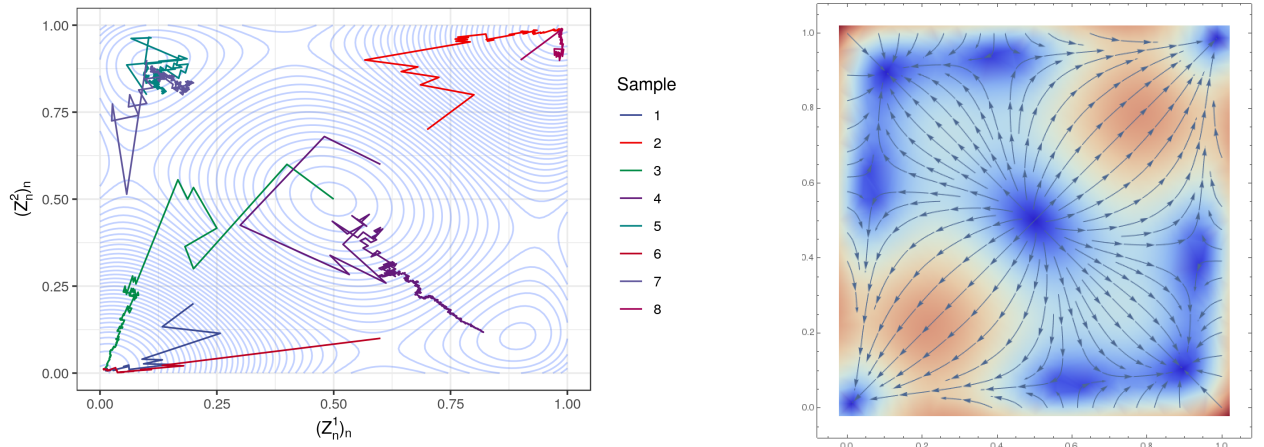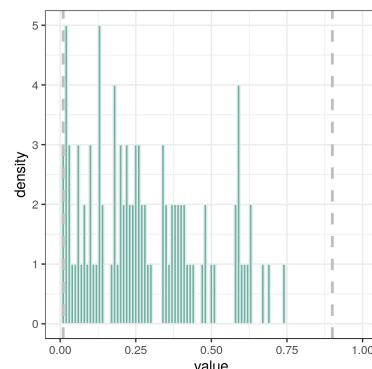
(C) Eight samples of the systems when $N = 2$. Representation of trajectories $(Z_n(1), Z_n(2))_n$. Each color means a different sample. Sample 1 starts with $(0.2, 0.2)$. Sample 2 starts with $(0.7, 0.7)$. Samples 3 and 4 start with $(0.5, 0.5)$. Sample 5 starts with $(0.1, 0.8)$. Sample 6 starts with $(0.6, 0.1)$. Sample 7 starts with $(0.1, 0.9)$. Sample 8 starts with $(0.4, 0.95)$. In background some level sets of the associated $-V$. Time up to 20.000 iterations.

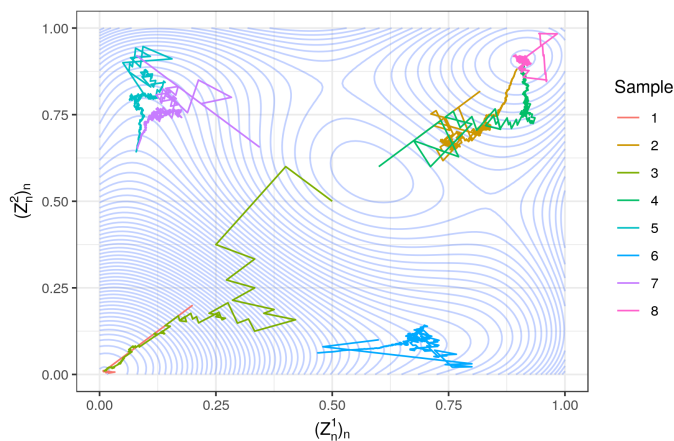(D) Representation of the field $F = -\nabla V$ when $N = 2$.

FIGURE 10. Case $f = f_{\text{Tech}}$. Parameters are $\theta = 0.99$, $\alpha = 0.14$, $\beta = 0$. There are two stable synchronization points $\{\approx 0.0103, \approx 0.989\}$ and there are stable non-synchronization points. Components' values of non-synchronization zero points belong to $\{\approx 0.104, \approx 0.896\}$.
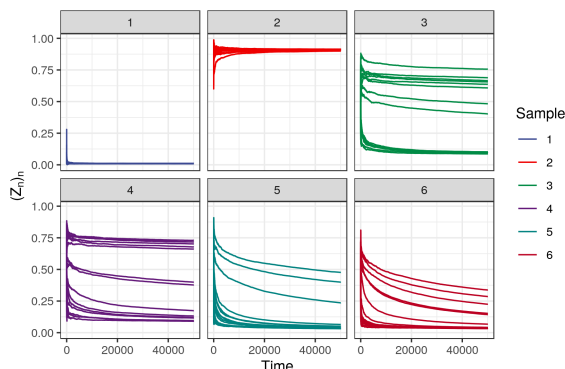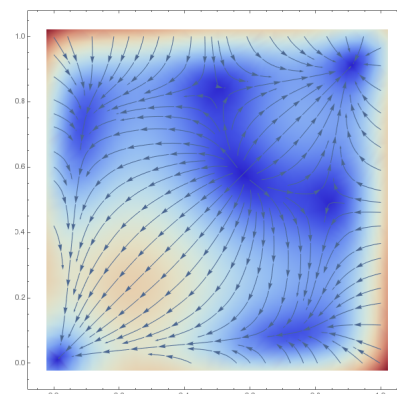
(A) Six samples of the system's trajectories when $N = 15$. Each system starts at $1/2$ on the diagonal.

(B) Histogram of mean field's final values, at $T = 50.000$, when $N = 10$, sample of size 100, uniformly distributed initial condition.
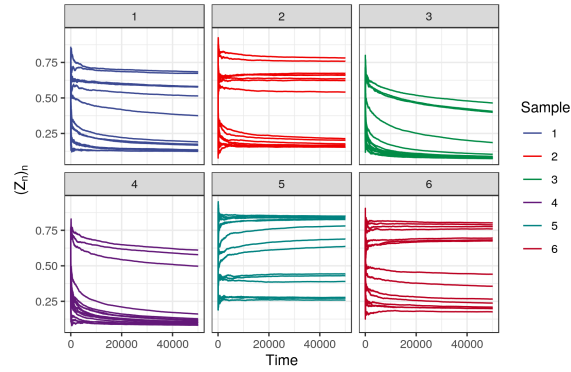

(C) Eight samples of the systems when $N = 2$. Representation of trajectories $(Z_n(1), Z_n(2))_n$. Each color means a different sample. Sample 1 starts with $(0.2, 0.2)$. Sample 2 starts with $(0.7, 0.7)$. Samples 3 starts with $(0.5, 0.5)$. Samples 4 starts with $(0.6, 0.6)$. Sample 5 starts with $(0.1, 0.8)$. Sample 6 starts with $(0.6, 0.1)$. Sample 7 starts with $(0.1, 0.9)$. Sample 8 starts with $(0.9, 0.9)$. In background some level sets of the associated $-V$. Time up to 200.000 iterations.
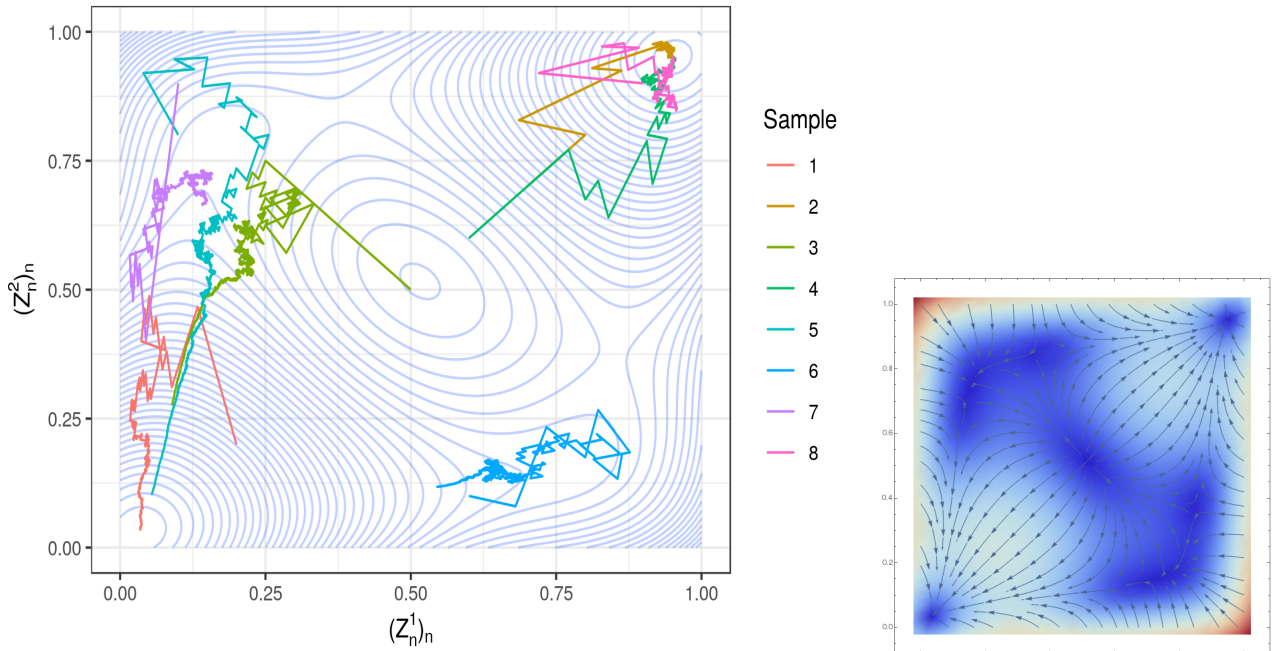
(D) Representation of the field when $N = 2$.

FIGURE 11. Case $f = f_{\text{Tech}}$. Parameters are $\theta = 0.99$, $\alpha = 0.15$, $\beta = 0.05$, $q = 0.005$. There are only two stable synchronization zero points. No-synchronization points exist but are unstable.

(A) Six samples of the system's trajectories when $N = 10$. Each component start at $1/2$.

(B) Six samples of the systems when $N = 2$. Representation of trajectories $(Z_n(1), Z_n(2))_n$. Time up to 150.000 iterations. Each color means a different sample. Sample 1 starts with 0.2 on the diagonal. Sample 2 starts with 0.7 on the diagonal. Samples 3 to 8 start diversely, for instance sample 6 starts at $(0.6, 0.1)$. In background some level sets of the associated $-V$.

(C) Representation of the field $F = -\nabla V$ when $N = 2$.

FIGURE 12. Case $f = f_{\text{Tech}}$. Parameters are $\theta = 0.97$, $\alpha = 0.18$, $\beta = 0.001$, $q = 0.01$. There are only two stable synchronization zero points.

Acknowledgments

Funding Sources

Declaration

All the authors developed the theoretical results, performed the numerical simulations, contributed to the final version of the manuscript.

Appendix A. Stochastic approximation

Here, we briefly recall the results of Stochastic Approximation theory used in the present work. We refer the interested reader to more complete monographs (*e.g.* [17, 24, 37, 41, 42, 57, 58, 71]).

Let $\mathbf{Z} = (\mathbf{Z}_n)_{n \geq 0}$ be an N-dimensional stochastic process with values in $[0, 1]^N$, adapted to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n \geq 0}$. Suppose that $\mathbf{Z}$ satisfies

$$\mathbf{Z}_{n+1} = \mathbf{Z}_n + r_n \mathbf{F}(\mathbf{Z}_n) + r_n \Delta \mathbf{M}_{n+1} , \tag{46}$$

where $r_n \sim 1/n$, $\mathbf{F}$ is a bounded $\mathcal{C}^1$ vector-valued function on an open subset $\mathcal{O}$ of $\mathbb{R}^N$, with $[0, 1]^N \subset \mathcal{O}$, and $(\Delta \mathbf{M}_n)_n$ is a bounded martingale difference with respect to $\mathcal{F}$.

We have the following results:

Theorem A.1. *(e.g. [58, 59])*
The limit set of $\mathbf{Z}$*, i.e. the set defined as* $\mathcal{L}(\mathbf{Z}) = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} \mathbf{Z}_m}$*, is almost surely a compact connected set stable by the flow of the differential equation* $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$*.*

Therefore, the asymptotic behaviour of the stochastic process $\mathbf{Z}$ is related to the properties of the zero points of the vector field $\mathbf{F}$. In the next definition, we give a classification of these points.

Definition A.2. A zero point of $\mathbf{F}$ is a point $\mathbf{z}$ such that $\mathbf{F}(\mathbf{z}) = \mathbf{0}$. We denote by $\mathcal{Z}(\mathbf{F})$ the set of all the zero points of $\mathbf{F}$. Moreover, denoting by $J(\mathbf{F})(\mathbf{z})$ the Jacobian matrix of $\mathbf{F}$ computed at the point $\mathbf{z}$, we classify the zero points of $\mathbf{F}$ according to the sign of the real part of the eigenvalues of $J(\mathbf{F})(\mathbf{z})$ as follows:

- $\mathbf{x}$ is said a *stable* zero point if all the eigenvalues of $J(\mathbf{F})(\mathbf{z})$ have negative (we mean "non-strictly positive", that is ≤ 0) real parts;
- $\mathbf{x}$ is said a *strictly stable* zero point if all the eigenvalues of $J(\mathbf{F})(\mathbf{z})$ have strictly negative real parts;
- $\mathbf{x}$ is said a *linearly unstable* (or *unstable*) zero point if $J(\mathbf{F})(\mathbf{z})$ has at least one eigenvalue with strictly positive real part;
- $\mathbf{x}$ is said a *repulsive* zero point if all the eigenvalues of $J(\mathbf{F})(\mathbf{z})$ have strictly positive real parts.

Suppose that, for each $\mathbf{z}$, the Jacobian matrix $J(\mathbf{F})(\mathbf{z})$ is symmetric. Then all its eigenvalues are real and, since the sign of the scalar product $\langle \mathbf{F}(\mathbf{z}') - \mathbf{F}(\mathbf{z}), \mathbf{z}' - \mathbf{z} \rangle$ for $\mathbf{z}'$ in a neighborhood of $\mathbf{z}$ is related to the property of $J(\mathbf{F})(\mathbf{z})$ of being positive/negative (semi)definite, and this last property is related to the sign of the eigenvalues of $J(\mathbf{F})(\mathbf{z})$, we can state:

- $\mathbf{z}$ is a stable zero point if and only if $\langle \mathbf{F}(\mathbf{z}'), \mathbf{z}' - \mathbf{z} \rangle \leq 0$ for all $\mathbf{z}'$ in a neighborhood of $\mathbf{z}$;
- $\mathbf{z}$ is a linearly unstable zero point if and only if, for any neighborhood $\mathcal{B}_{\mathbf{z}}$ of $\mathbf{z}$, there exists $\mathbf{z}' \in \mathcal{B}_{\mathbf{z}}$ such that $\langle \mathbf{F}(\mathbf{z}'), \mathbf{z}' - \mathbf{z} \rangle > 0$.

Theorem A.3. *(e.g. [59])*
If there exists a stable zero point $\mathbf{z}$ *of* $\mathbf{F}$ *such that*

$$\langle \mathbf{F}(\mathbf{Z}_n), \mathbf{Z}_n - \mathbf{z} \rangle < 0 \qquad \forall n \text{ with } \mathbf{Z}_n \neq \mathbf{z},$$

then $\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{z}$*.*

Theorem A.4. *([37, Ch. 2, Th. 2] or [57, Ch. 5, Th. 2.1] or [71, Th. 2.18])*
If $\mathbf{F} = -\nabla \mathbf{V}$ *and* $\mathcal{Z}(\mathbf{F})$ *is not empty and finite, then there exists a random variable* $\mathbf{Z}_\infty$*, which takes values in* $\mathcal{Z}(\mathbf{F})$ *and such that*

$$\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{Z}_\infty.$$

Theorem A.5. ([70, Th. 1])
If there exists a constant $C > 0$, such that we have

$$E\left[\langle\Delta\mathbf{M}_{n+1},\mathbf{v}\rangle)^+|\mathcal{F}_n\right] \geq C \qquad \forall\mathbf{v}\in\mathbb{R}^N \text{ with } |v| = \sum_{h=1}^N v_h = 1, \tag{47}$$

then, for each linearly unstable zero point $\mathbf{z}$ of $\mathbf{F}$, we have $P(\mathbf{Z}_n \to \mathbf{z}) = 0$.

If $\mathbf{F}$ belongs to $\mathcal{C}^2$ and, for each $\mathbf{z}$, the Jacobian matrix $J(\mathbf{F})(\mathbf{z})$ is symmetric (and so all its eigenvalues are real and it is diagonalizable), then from [76] we get the following Central Limit Theorem (CLT):

Theorem A.6. *Suppose $\mathbf{F} \in \mathcal{C}^2$, $J(\mathbf{F})(\mathbf{z})$ symmetric for each $\mathbf{z}$. Let $\mathbf{z}_\infty \in (0,1)^N$ be a strictly stable zero point of $\mathbf{F}$ such that $\mathbf{Z}_n \xrightarrow{a.s.} \mathbf{z}_\infty$. Suppose that*

$$E[\Delta\mathbf{M}_{n+1}(\Delta\mathbf{M}_{n+1})^\top|\mathcal{F}_n] \xrightarrow{a.s.} \Gamma, \tag{48}$$

where $\Gamma = \Gamma(\mathbf{z}_\infty)$ is a deterministic symmetric positive definite matrix. Denote by $\lambda = \lambda(\mathbf{z}_\infty)$ be the smallest eigenvalue of $-J(\mathbf{F})(\mathbf{z}_\infty)$. Then, we have:

- *If $\lambda > 1/2$, then*

$$\sqrt{n}(\mathbf{Z}_n - \mathbf{z}_\infty) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma),$$

  where

$$\Sigma = \Sigma(\mathbf{z}_\infty) = \int_0^{+\infty} e^{(J(\mathbf{F})(\mathbf{z}_\infty)+\frac{Id}{2})u}\Gamma e^{(J(\mathbf{F})(\mathbf{z}_\infty)+\frac{Id}{2})u}\, du.$$

- *If $\lambda = 1/2$, then*

$$\sqrt{\frac{n}{\ln(n)}}(\mathbf{Z}_n - \mathbf{z}_\infty) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma),$$

  where

$$\Sigma = \Sigma(\mathbf{z}_\infty) = \lim_{n\to+\infty}\frac{1}{\ln(n)}\int_0^{\ln(n)} e^{(J(\mathbf{F})(\mathbf{z}_\infty)+\frac{Id}{2})u}\Gamma e^{(J(\mathbf{F})(\mathbf{z}_\infty)+\frac{Id}{2})u}\, du.$$

- *If $0 < \lambda < 1/2$, then*

$$n^\lambda(\mathbf{Z}_n - \mathbf{z}_\infty) \xrightarrow{a.s.} V,$$

  where V is a suitable finite random variable.

Note that Assumption 2.2. in [76] is satisfied since we take $\mathbf{F} \in \mathcal{C}^2$. Equation (2.3) of Assumption 2.3 in [76] holds because we assume $(\Delta\mathbf{M}_n)_n$ bounded. Equation (2.4) of the same assumption is implied by the above condition (48). All the assumptions on the remainder term $\mathbf{r}_n$ in [76] are verified because we have $\mathbf{r}_n = \mathbf{0}$. Finally, since $J(\mathbf{F})(\mathbf{z}_\infty)$ is symmetric, also the matrix $e^{(J(\mathbf{F})(\mathbf{z}_\infty)+\frac{Id}{2})u}$ is symmetric.

Remark A.7. With the same assumptions and notation as in Theorem A.6, the limit covariance matrix Σ in the case $\lambda > 1/2$ can be rewritten using the Lyapounov equation (*e.g.* [51] or [37]):

$$\left(J(\mathbf{F})(\mathbf{z}_\infty) + \frac{1}{2}Id\right)\Sigma + \Sigma\left(J(\mathbf{F})(\mathbf{z}_\infty)^\top + \frac{1}{2}Id\right) = -\Gamma$$

Since $J(\mathbf{F})(\mathbf{z}_\infty)$ is symmetric by assumption and Σ is symmetric by definition, we have

$$\Sigma = (-2J(\mathbf{F})(\mathbf{z}_\infty) - Id)^{-1}\Gamma.$$

Remark A.8. In [51] we have a CLT also when there exist more limit points for $(\mathbf{Z}_n)_n$. Indeed, under the same assumptions on $\mathbf{F}$ as in Theorem A.6, when condition (48) is satisfied and $\mathbf{z}_\infty \in (0,1)^N$ is a strictly stable zero point of $\mathbf{F}$ such that $P(\mathbf{Z}_n \to \mathbf{z}_\infty) > 0$ and $\lambda(\mathbf{z}_\infty) > 1/2$, we have to consider the convergence in distribution under the probability measure $P(\cdot | \mathbf{Z}_n \to \mathbf{z}_\infty)$ and the corresponding limit distribution is the one with characteristic function

$$\mathbf{u} \mapsto E\left[\exp(-\frac{1}{2}\mathbf{u}^\top \Sigma(\mathbf{z}_\infty)\mathbf{u}) \,|\, \mathbf{Z}_n \to \mathbf{z}_\infty\right],$$

where $\Sigma(\mathbf{z}_\infty)$ is defined as in Theorem (A.6) or, equivalently, as in Remark (A.7).

APPENDIX B. EIGENVALUES OF THE JACOBIAN MATRIX

We observe that, letting $d_i := (1 - \alpha - \beta)f'(z_i) - 1$ for $i = 1, \ldots, N$, the Jacobian matrix of $\mathbf{F}$ is given by

$$J\mathbf{F}(\mathbf{z}) = \frac{\alpha}{N}\begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ldots & \vdots \\ 1 & \ldots & 1 \end{pmatrix} + diag\,(d_1, \ldots, d_N)\,, \tag{49}$$

where $diag(d_1, \ldots, d_N)$ denotes the diagonal matrix with diagonal elements $d_1, \ldots, d_N$.

In order to compute its eigenvalues, we use the following results:

Lemma B.1. *Assume that the matrix A has the form*

$$A = c^2\begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ldots & \vdots \\ 1 & \ldots & 1 \end{pmatrix} + diag(d_1, \ldots, d_N),$$

where $c > 0$ and $diag(d_1, \ldots, d_N)$ denotes the diagonal matrix with diagonal elements equal to d_i for $i = 1, \ldots, N$. Then the characteristic polynomial of A can be written as

$$p(\lambda) = \prod_{i=1}^{N}(d_i - \lambda) + c^2 \sum_{i=1}^{N}\prod_{j\neq i}(d_j - \lambda). \tag{50}$$

Proof. Set $\mathbf{v} = (c, \ldots, c)^\top$. By the matrix determinant lemma, we get for all $\lambda \notin \{d_1, \ldots, d_N\}$

$$p(\lambda) = det(A - \lambda I) = det(c^2\mathbf{v}\mathbf{v}^\top + diag(d_1 - \lambda, \ldots, d_N - \lambda))$$

$$= \left(1 + c^2\sum_{i=1}^{N}\frac{1}{d_i - \lambda}\right)\prod_{i=1}^{N}(d_i - \lambda) = \prod_{i=1}^{N}(d_i - \lambda) + c^2\sum_{i=1}^{N}\prod_{j\neq i}(d_j - \lambda).$$

By continuity, we can conclude that $p(\lambda)$ has the above expression for all λ. ∎

Corollary B.2. *With the same assumptions and notation of the above lemma, the number d_k is an eigenvalue of $J\mathbf{F}(\mathbf{z})$ if and only if there exists at least one $j \neq k$ such that $d_j = d_k$.*

Proof. Clearly, if $d_k = d_j$ for at least one pair $k \neq j$ we have $p(d_k) = 0$. On the other hand, if $p(d_k) = 0$ we have necessarily $\prod_{j\neq k}(d_j - d_k) = 0$ which implies $d_j = d_k$ for at least one $j \neq k$. ∎

Corollary B.3. *With the same assumption and notation notation of the above lemma, if $d_i = d$ for $i = 1, \ldots, N$, then the eigenvalues of A are:*

$$\lambda_1 = d \qquad and \qquad \lambda_2 = d + c^2 N$$

Proof. In this case, we have

$$p(\lambda) = (d - \lambda)^{N-1}(d - \lambda + c^2 N)$$

and so d is an eigenvalue with multiplicity $N - 1$ and $d + c^2 N$ is an eigenvalue with multiplicity 1. ■

Corollary B.4. *With the same assumptions and notation of the above lemma, suppose that $d_i \in \{D_1, D_2\}$, with $D_1 \neq D_2$, for all $i = 1, \ldots, N$, and assume that $d_i \neq d_j$ for at least a pair of different indexes. Moreover, denote by $N_1 \in \{1, \ldots, N - 1\}$ the number of indexes such that $d_i = D_1$ and $N_2 = N - N_1$. The eigenvalues of $J\mathbf{F}(\mathbf{z})$ are*

- *$\lambda = D_1$ with multiplicity $N_1 - 1$;*
- *$\lambda = D_2$ with multiplicity $N_2 - 1$;*
- *$\lambda = \lambda_i$ with $i = 1, 2$, where the λ_i's are the solutions of the equation*

$$\lambda^2 - (D_1 + D_2 + c^2 N)\lambda + D_1 D_2 + c^2 N_1 D_2 + c^2 N_2 D_1 = 0. \tag{51}$$

Then, in particular:

(a) *If $D_1 \geq 0$ and $D_2 \geq 0$, then all the eigenvalues are positive.*

(b) *If $D_1 = 0$ and $D_2 < 0$ (or vice-versa), then there exists a strictly positive eigenvalue.*

(c) *When $D_1 < 0$, $D_2 < 0$, all the eigenvalues are negative if and only if we have*

$$D_1 + D_2 + c^2 N < 0 \qquad and \qquad D_1 D_2 + c^2 N_1 D_2 + c^2 N_2 D_1 \geq 0. \tag{52}$$

Proof. We first observe that the matrix A is symmetric, and so all its eigenvalues are real numbers and this, together with formula (50), proves the first assertion. Let us write the ploynomial in (51) as $r(\lambda) = \lambda^2 - B\lambda + C$ where $B = D_1 + D_2 + c^2 N$ and $C = D_1 D_2 + c^2 N_1 D_2 + c^2 N_2 D_1$. For statement (a) observe that, if $D_1, D_2 \geq 0$, then $B > 0$ and $C \geq 0$ and this implies that the zeros of $r(\lambda)$ are both positive. Similarly, in case (b), if $D_1 = 0, D_2 < 0$ (or vicevera), we have $C < 0$ and so one of the zeros of $r(\lambda)$ must be strictly positive. Finally, in case (c), it is enough to observe that the zeros of $r(\lambda)$ are both negative if and only if $B < 0$ and $C \geq 0$. ■

Remark B.5. A necessary condition for (52) is

$$D_i < -c^2 N_i \quad \text{for } i = 1, 2\,; \tag{53}$$

while a sufficient condition is

$$D_i \leq -c^2 N_i (1 + \delta_i) \quad \text{for } i = 1, 2, \tag{54}$$

with δ_1, $\delta_2 > 0$ and $\delta_1 \delta_2 \geq 1$.

Indeed, observe that the second condition in (52) can be written as $(D_1 + c^2 N_1)(D_2 + c^2 N_2) - c^4 N_1 N_2 \geq 0$, which implies $(D_1 + c^2 N_1)(D_2 + c^2 N_2) > 0$ (because c, N_1, $N_2 > 0$) and so we have either $D_i < -c^2 N_i$ for both $i = 1, 2$ or $D_i > -c^2 N_i$ for both $i = 1, 2$ and the first equation in (52) excludes the second case. Hence we necessarily have $D_i < -c^2 N_i$ for $i = 1, 2$. On the other hand, a simple computation shows that condition (54) implies (52): we have $D_1 + D_2 + c^2 N \leq -c^2(N_1 \delta_1 + N_2 \delta_2) < 0$ and $(D_1 + c^2 N_1)(D_2 + c^2 N_2) - c^4 N_1 N_2 = (-D_1 - c^2 N_1)(-D_2 - c^2 N_2) - c^4 N_1 N_2 \geq c^4 N_1 N_2 (\delta_1 \delta_2 - 1) \geq 0$.

Remark B.6. If in (54) we take $N_1(1 + \delta_1) = N_2(1 + \delta_2) = N$ (that is $\delta_i = NN_i - 1 \geq 1$), we obtain

$$D_i \leq -c^2 N \quad \forall i = 1, 2.$$

Appendix C. Roots of polynoms systems

Lemma C.1. *Assume f is a real polynom of degree $d \geq 2$ such that $F = (F_1, \ldots, F_N) : [0,1]^N \mapsto [0,1]^N$ ($N \geq 1$) with*

$$F_i(z) = \alpha \bar{z} + \beta q + (1 - \alpha - \beta) f(z_i) - z_i$$

where $\alpha, \beta \in [0,1]^2$ such that $1 - \alpha - \beta \neq 0$. Then the set $\mathcal{Z}(\mathbf{F})$ of roots of the system $(F_1, \ldots, F_N)$ is finite.

Proof. Let $\mathcal{Z}_{\mathbb{C}}(\mathbf{F})$ be the algebraic set of the solutions $z \in \mathbb{C}$ such that $\forall i, 1 \leq i \leq N, F_i(z) = 0$. Let I be the ideal generated by the polynomials $(F_1, \ldots, F_N)$ in $\mathbb{C}[z_1, \ldots, z_N]$. Let $I(\mathcal{Z}_{\mathbb{C}}(\mathbf{F}))$ be the ideal generated by all polynomials from $\mathbb{C}[z_1, \ldots, z_N]$ vanishing in $\mathcal{Z}_{\mathbb{C}}(\mathbf{F})$. It holds $I \subset I(\mathcal{Z}_{\mathbb{C}}(\mathbf{F}))$. Using Corollary 2.15 in [45], we get $\mathcal{Z}_{\mathbb{C}}(\mathbf{F})$ is finite if and only if the dimension of $\mathbb{C}[z_1, \ldots, z_N] \big/ I(\mathcal{Z}_{\mathbb{C}}(\mathbf{F}))$ as $\mathbb{C}$-vector space is finite. Since $I \subset I(\mathcal{Z}_{\mathbb{C}}(\mathbf{F}))$ there is a surjective morphism from $\mathbb{C}[z_1, \ldots, z_N] \big/ I$ to $\mathbb{C}[z_1, \ldots, z_N] \big/ I(\mathcal{Z}_{\mathbb{C}}(\mathbf{F}))$. Thus, it is enough to state that the dimension of $\mathbb{C}[z_1, \ldots, z_N] \big/ I$ as $\mathbb{C}$ vector space is finite. Since f is a polynom of degree $d \geq 2$, it remains in $\mathbb{C}[z_1, \ldots, z_N] \big/ I$ the monomes $z_1^{a_1} z_2^{a_2} z_N^{a_N}$ (with $\forall i, 1 \leq i \leq N, 1 \leq a_i \leq (d-1)$) whose cardinal is d^N. Thus the cardinal of $\mathcal{Z}_{\mathbb{C}}(\mathbf{F})$ is bounded from above by d^N. Since it holds on the field $\mathbb{C}$, it holds on the field $\mathbb{R}$. ∎

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