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► **To cite this version:**

Thi Thuy Bui, Ion Grama, Quansheng Liu. ASYMPTOTIC EXPANSIONS IN CENTRAL AND LOCAL LIMIT THEOREMS FOR PRODUCTS OF RANDOM MATRICES. 2020. hal-02910314

HAL Id: hal-02910314

<https://hal.science/hal-02910314>

Preprint submitted on 1 Aug 2020

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ASYMPTOTIC EXPANSIONS IN CENTRAL AND LOCAL LIMIT THEOREMS FOR PRODUCTS OF RANDOM MATRICES

THI THUY BUI, ION GRAMA, AND QUANSHENG LIU

ABSTRACT. Let $(A_n)_{n \geq 1}$ be a sequence of independent and identically distributed random $d \times d$ real matrices. Set $G_n = A_n \dots A_1$, $X_n^x = \frac{G_n x}{|G_n x|}$ and $S_n^x := \log |G_n x|$. We consider asymptotic properties of the Markov chain (X_n^x, S_n^x) . For invertible matrices, Le Page [13] established a central limit theorem and a local limit theorem on (X_n^x, S_n^x) with x a starting point on the unit sphere in \mathbb{R}^d . In this paper, motivated by some applications in branching random walks, we improve and extend his theorems in the sense that: 1) we prove that the central limit theorem holds uniformly in x , and give an asymptotic expansion in the local limit theorem with a continuous function f acting on X_n^x and a directly Riemann integrable function h acting on S_n^x ; 2) we extend the results to the case of nonnegative matrices. Our approach is mainly based on the spectral gap theory recently developed for products of random matrices, and smoothing techniques for the approximation of functions.

1. INTRODUCTION

Let μ be a probability measure on the set of $d \times d$ matrices $M(d, \mathbb{R})$ ($d \geq 1$), and let $(A_n)_{n \geq 1}$ be a sequence of independent and identically distributed random matrices with law μ , defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We are interested in the asymptotic behavior of the random walk $G_n x$, where

$$G_n = A_n \dots A_1$$

is the product of the random matrices A_i , x is a starting point on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$, with $|\cdot|$ an arbitrary norm on \mathbb{R}^d . Notice that $G_n x$ is completely determined by its log norm and its projection on the unit sphere, denoted respectively by

$$S_n^x := \log |G_n x| \quad \text{and} \quad X_n^x = G_n \cdot x := \frac{G_n x}{|G_n x|}.$$

Date: July 31, 2020.

2000 Mathematics Subject Classification. 60B12, 60J10.

Key words and phrases. Products of random matrices, Markov walk, central limit theorem, local limit theorem, asymptotic expansion.

We will use the convention that $G_0x = x$, and introduce conditions such that $G_nx \neq 0$. Many authors have contributed to the study of asymptotic properties of S_n^x . For example, central limit theorems have been established by Benoist and Quint [1] for invertible matrices, and by Hennion [11] for nonnegative matrices.

While studying branching random walks in \mathbb{R}^d governed by products of random matrices, we need some asymptotic properties as those given in a central limit theorem and a local limit theorem on the couple (X_n^x, S_n^x) , but we find that the known results on this topic are not sharp enough for our purposes. We thus focus our study on the Markov chain (X_n^x, S_n^x) for establishing finer results. The applications in branching random walks will be considered in a forth coming paper [4].

For invertible matrices, Le Page [13] established a central limit theorem for (X_n^x, S_n^x) with x a given point in \mathbb{S}^{d-1} , and a local limit theorem for (X_n^x, S_n^x) with target functions f and h acting on X_n^x and S_n^x respectively, which are supposed to be continuous and of compact support. Such kind of limit theorems have also been established by Hennion and Hervé [12] in a more general setting by considering (X_n, S_n) instead of (X_n^x, S_n^x) , where (X_n) is a general Markov chain, $S_n = \sum_{i=1}^n \xi(X_i)$ with ξ a measurable and real valued function. Very recently, in parallel to the present work, a Berry-Essen type theorem on the rate of convergence in the central limit theorem has been established for (X_n^x, S_n^x) in [17] for both invertible and nonnegative matrices.

In this paper, our first objective is to improve the central limit theorem of Le Page [13] for invertible matrices with a uniform convergence in x (see Theorem 2.1), and deepen his local limit theorem by giving an asymptotic expansion under the weaker condition that the target functions f and h are respectively continuous and directly Riemann integrable (see Theorem 2.2). Our second objective is to prove that the results also hold for nonnegative matrices.

Our approach is mainly based on the spectral gap theory recently developed for the norm cocycle by Guivarc'h and Le Page [9] for invertible matrices, and by Buraczewski, Damek, Guivarc'h and Mentemeier (see [5, 6]) for nonnegative matrices. Smoothing techniques are also used for the approximation of functions: in the proof of Theorem 2.1, we use a smooth approximation of the indicator function of a Borel set (see Lemma 4.1), while in the proof of Theorem 2.2, we use a suitable approximation of a directly Riemann integrable function with the techniques developed in [16].

The paper is organized as follows. In Section 2, we fix some notation, introduce our assumptions on the branching products of random matrices and state the main results. In Section 3, we recall some results on spectral

theory for products of random matrices which will be used in proofs of main results. These proofs are given in Sections 4 and 5.

We end this section by fixing some notation. We denote by c, C absolute constants whose values may change from line to line. For a set B , $\mathbf{1}_B$ denotes the indicator function, \bar{B}, B° and $\partial B = \bar{B} \setminus B^\circ$ denote respectively the closure, the interior and the boundary of B . For $t \in \mathbb{R}$, we write $\phi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$, $\Phi(t) = \int_{-\infty}^t \phi(u)du$, and $\phi_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}}e^{-t^2/(2\sigma^2)}$. For a measure ν and a measurable function f we denote $\nu(f) = \int f d\nu$. For two functions f and g , we write $f(t) = o(g(t))$ or $f(t) = O(g(t))$ ($t \rightarrow 0$) when $\lim_{t \rightarrow 0} f(t)/g(t) = 0$ or $f(t)/g(t)$ is bounded for $|t|$ small enough, respectively. Denote by L^1 the class of complex valued measurable and Lebesgue integrable functions defined on \mathbb{R} ; for $f \in L^1$, denote its L^1 norm by $\|f\|_{L^1} = \int_{\mathbb{R}} |f(x)|dx$.

2. MAIN RESULTS

2.1. Notation and preliminaries. Let $M(d, \mathbb{R})$ be equipped with the operator norm $\|\mathbf{a}\| = \sup_{x \in \mathbb{S}^{d-1}} |\mathbf{a}x|$ for $\mathbf{a} \in M(d, \mathbb{R})$. Denote by $\Gamma_\mu := [\text{supp } \mu]$ the smallest closed semigroup of $M(d, \mathbb{R})$ generated by the support of μ . Let us recall some definitions in matrix theory. A matrix \mathbf{a} is said to be *proximal* if it has an algebraic simple dominant eigenvalue. Denote by \mathcal{M}_+ the set of matrices with nonnegative entries. A matrix $\mathbf{a} \in \mathcal{M}_+$ is said to be *allowable* if every row and every column has a positive entry.

For invertible matrices, we will use the strong irreducibility and proximality conditions.

C1. (i) (*Strong irreducibility*) There is no finite union $\mathcal{W} = \bigcup_{i=1}^n W_i$ of subspaces $0 \neq W_i \subsetneq \mathbb{R}^d$ which is Γ_μ -invariant (in the sense that $\Gamma_\mu \mathcal{W} = \mathcal{W}$).

(ii) (*Proximality*) Γ_μ contains at least one proximal matrix.

Notice that when $d = 1$, the strong irreducibility and proximality conditions are always satisfied.

For nonnegative matrices, we will need the allowability, positivity and non-arithmeticity conditions.

C2. (i) (*Allowability*) Every $\mathbf{a} \in \Gamma_\mu$ is allowable.

(ii) (*Positivity*) Γ_μ contains at least one matrix belonging to \mathcal{M}_+° .

We say that the measure μ is *arithmetic* if there are $t > 0$, $\theta \in [0, 2\pi)$ and a function $\vartheta : \mathbb{S}_+^{d-1} \rightarrow \mathbb{R}$ such that for all $\mathbf{a} \in \Gamma_\mu$ and all $x \in V(\Gamma_\mu)$,

$$\exp\{it \log |\mathbf{a}x| - i\theta + i(\vartheta(\mathbf{a}x) - \vartheta(x))\} = 1,$$

where $\mathbb{S}_+^{d-1} = \{x \geq 0 : |x| = 1\}$ is the intersection of the unit sphere with the positive quadrant. Notice when $d = 1$, we have $\mathbb{S}_+^{d-1} = \{1\}$, and the above arithmetic condition reduces to the following more usual form: $\log a$

is almost surely concentrated on an arithmetic progression $a_0 + a_2\mathbb{N}$ for some $a_0, a_2 \geq 0$.

C3. (*Non-arithmeticity*) *The measure μ is non-arithmetic.*

It is known that when $d \geq 2$, condition **C1** implies **C3** (see [10, Proposition 4.6]).

For both invertible matrices and nonnegative matrices, we will need a moment condition. For $\mathbf{a} \in M(d, \mathbb{R})$, set

$$\iota(\mathbf{a}) := \inf_{x \in \mathcal{S}} |\mathbf{a}x|, \quad \text{and} \quad \mathbf{a}.x := \frac{\mathbf{a}x}{|\mathbf{a}x|}$$

for the projective action of a matrix \mathbf{a} on $x \in \mathbb{S}^{d-1}$ when $\mathbf{a}x \neq 0$. Then $\iota(\mathbf{a}) > 0$ for both invertible matrices and allowable nonnegative matrices.

C4. *There is $\alpha \in (0, 1)$ such that*

$$\mathbb{E}N(A_1)^\alpha < \infty.$$

For invertible matrices, this condition is equivalent to the following two-sided exponential moment condition which is usually used in the literature: there is $\alpha_1 > 0$ such that $\mathbb{E}N(A_1)^{\alpha_1} < \infty$, where $N(A_1) = \max\{\|A_1\|, \|A_1^{-1}\|\}$.

We will consider the action of invertible matrices on the projective space \mathbb{P}^{d-1} which is obtained from \mathbb{S}^{d-1} by identifying x and $-x$, and the action of nonnegative matrices on \mathbb{S}_+^{d-1} . When convenient we identify $x \in \mathbb{P}^{d-1}$ with one of its representants in \mathbb{S}^{d-1} . To unify the exposition, we use the symbol \mathcal{S} to denote \mathbb{P}^{d-1} for invertible matrices, and \mathbb{S}_+^{d-1} for nonnegative matrices. The space \mathcal{S} will be equipped with the metric \mathbf{d} , which is the angular distance (see [2]) for invertible matrices, and the Hilbert cross-ratio metric (see [11]) for nonnegative matrices. Moreover, \mathcal{S} is a separable metric space with Borel- σ algebra. For any starting point $x \in \mathcal{S}$, as mentioned in the introduction, $G_n x$ is completely described by (X_n^x, S_n^x) . With the above conditions, X_n^x is well defined and the sequence (X_n^x, S_n^x) is a Markov chain because $X_n^x = A_n \cdot X_{n-1}^x$ and $S_n^x = \log |A_n X_{n-1}^x| + S_{n-1}^x$.

For invertible matrices, it was proved in [9, Theorem 2.6] that if condition **C1** holds, then the Markov chain X_n^x has a unique μ -stationary measure, which is supported on

$$V(\Gamma_\mu) := \overline{\{v_{\mathbf{a}} \in \mathbb{P}^{d-1} : \mathbf{a} \in \Gamma_\mu, \mathbf{a} \text{ is proximal}\}},$$

where $v_{\mathbf{a}}$ denotes the eigenvector with norm $|v_{\mathbf{a}}| = 1$ associated to the dominant eigenvalue of the proximal matrix \mathbf{a} .

For nonnegative matrices, it was shown in [5, Lemma 4.3] that condition **C2** ensures the existence and uniqueness of the invariant measure for the Markov chain (X_n^x) supported on

$$V(\Gamma_\mu) := \overline{\{v_{\mathbf{a}} \in \mathbb{S}_+^{d-1} : \mathbf{a} \in \Gamma_\mu, \mathbf{a} \in \mathcal{M}_+^o\}}.$$

In both cases, we write ν for the unique invariant measure of (X_n^x) .

Central limit theorems have been established in the literature. For invertible matrices, under condition **C1** and the two-sided exponential moment condition, Le Page [13] proved that

$$\frac{1}{\sqrt{n}}(S_n^x - n\gamma) \rightarrow N(0, \sigma^2) \quad \text{in law,} \quad (2.1)$$

where $\gamma = \inf_{n \geq 1} \frac{1}{n} \mathbb{E} \log \|A_n \cdots A_1\|$ is the Lyapunov exponent, and $\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(S_n^x - n\gamma)^2$ is the asymptotic variance which is positive and independent of x . For nonnegative matrices, under condition **C2** and a second moment condition, Hennion [11] proved that (2.1) holds for some $\sigma \geq 0$; he also gave a condition of tightness of the sequence $(G_n)_{n \geq 0}$ to ensure that $\sigma > 0$. As a by-product of our approach, we will show that $\sigma > 0$ under the non-arithmeticity condition **C3** (see Proposition 3.3).

2.2. Main results. We state first a central limit theorem for the couple (X_n^x, S_n^x) with uniform convergence in $x \in \mathcal{S}$.

Theorem 2.1. *For invertible matrices, assume **C1** if $d > 1$, and **C3** if $d = 1$. For nonnegative matrices, assume **C2** and **C3**. For both cases, assume additionally **C4**.*

(1) *For any continuous function f on \mathcal{S} , we have*

$$\lim_{n \rightarrow \infty} \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[f(X_n^x) \mathbf{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(f) \Phi(t) \right| = 0. \quad (2.2)$$

(2) *For any measurable set $B \subset \mathcal{S}$ with $\nu(\partial B) = 0$, we have*

$$\lim_{n \rightarrow \infty} \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{P} \left(X_n^x \in B, \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right) - \nu(B) \Phi(t) \right| = 0. \quad (2.3)$$

For invertible matrices, a point-wise version (by considering a fixed $x \in \mathcal{S}$ instead of $\sup_{x \in \mathcal{S}}$) has been established by Le Page in [13, Theorem 4]. For nonnegative matrices, the asymptotic for the Markov chain (X_n^x, S_n^x) is new even for a fixed x . The uniformity in $x \in \mathcal{S}$ is new for both invertible matrices and nonnegative matrices. Theorem 2.1 will be deduced from a result on the convergence rate in (2.2) which has been established in [17] for the case when f is Hölder continuous.

The following theorem gives the asymptotic expansion in the local limit theorem for products of random matrices.

Theorem 2.2. *Assume the conditions of Theorem 2.1.*

- (1) For any continuous function f on \mathcal{S} and any directly Riemann integrable function h on \mathbb{R} , we have as $n \rightarrow \infty$,

$$\sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |\sigma\sqrt{n}\mathbb{E}[f(X_n^x)h(y + S_n^x - n\gamma)] - \nu(f) \int_{\mathbb{R}} h(z)\phi\left(\frac{y-z}{\sigma\sqrt{n}}\right)H_x\left(\frac{y-z}{\sqrt{n}}\right)dz| \rightarrow 0, \quad (2.4)$$

where

$$H_x(u) = 1 - \frac{b(x)}{\sigma^2\sqrt{n}}u + \frac{m_3}{6\sigma^6\sqrt{n}}(3\sigma^2u - u^3),$$

with m_3 and $b(x)$ defined in Proposition 3.3.

- (2) For any measurable set $B \subset \mathcal{S}$ with $\nu(\partial B) = 0$ and any directly Riemann integrable function h on \mathbb{R} , we have as $n \rightarrow \infty$

$$\sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |\sigma\sqrt{n}\mathbb{E}[\mathbf{1}_B(X_n^x)h(y + S_n^x - n\gamma)] - \nu(B) \int_{\mathbb{R}} h(z)\phi\left(\frac{y-z}{\sigma\sqrt{n}}\right)H_x\left(\frac{y-z}{\sqrt{n}}\right)dz| \rightarrow 0. \quad (2.5)$$

When $y = 0$, $f = 1$ and $h = \mathbf{1}_{[a,b]}$, the integral $\mathbb{E}[f(X_n^x)h(y + S_n^x - n\gamma)]$ reduces to the local probability $\mathbb{P}(S_n^x \in n\gamma + [a, b])$, which is the usual object studied in local limit theorems.

The expansions (2.4) and (2.5) are new for both invertible matrices and nonnegative matrices. The first expansion implies the local limit theorem established in [13, Theorem 6] for invertible matrices, which states that (2.4) holds when the polynomial $H_x(\cdot)$ is replaced by 1 and when f, h are continuous functions with compact supports.

The case $d = 1$ is worth some comments. In this case, Theorem 2.1 follows from Theorem VII.2.7 of Petrov [14], while expansion (2.4) in Theorem 2.2 was proved by Feller (see [7, Theorem XVI.4.1]) under the same non-arithmetic condition on μ and when $h = \mathbf{1}_{[a,b]}$ is the indicator function of an interval. Breuillard (see [3, Theorem 3.2]) proved an expansion like (2.4) but for any finite order, when μ is strongly non-arithmetic (in the sense that its characteristic function $\hat{\mu}(t) = \int e^{itx}\mu(dx)$ satisfies Cramér's condition $\limsup_{|t| \rightarrow \infty} |\hat{\mu}(t)| < 1$) with finite moments of order high enough and when h is integrable and regular enough (he assumed in particular that h has continuous and integrable derivatives $h^{(k)}$ for $0 \leq k \leq K$ with $K \geq 2$ large enough). Compared with the result of Breuillard, the novelty in Theorem 2.2 is that we assume the non-arithmetic condition instead of the strongly arithmetic condition, and we use the direct Riemann integrability of h instead of the smoothness condition on h .

3. SPECTRAL GAP PROPERTY

In this section we recall some spectral gap properties studied in [13, 6, 17] which will be used for the proofs of main results.

For $z \in \mathbb{C}$, define the operator P_z on the set $\mathcal{C}(\mathcal{S})$ of continuous functions on \mathcal{S} by

$$P_z f(x) = \int_{\mathcal{M}} |\mathbf{a}x|^z f(\mathbf{a} \cdot x) \mu(d\mathbf{a}), \quad \text{for all } x \in \mathcal{S}.$$

For $f \in \mathcal{C}(\mathcal{S})$ and $\beta > 0$, we introduce the norms

$$\|f\|_\infty := \sup_{x \in \mathcal{S}} |f(x)|; \quad |f|_\beta := \sup_{x, y \in \mathcal{S}} \frac{|f(x) - f(y)|}{\mathbf{d}^\beta(x, y)}; \quad \|f\|_\beta := \|f\|_\infty + |f|_\beta.$$

Consider the Banach space $\mathcal{B}_\beta := \{f \in \mathcal{C}(\mathcal{S}) : \|f\|_\beta < +\infty\}$. Denote by $\mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$ the set of all bounded linear operators from \mathcal{B}_β to \mathcal{B}_β equipped with the operator norm

$$\|P\|_{\mathcal{B}_\beta \rightarrow \mathcal{B}_\beta} := \sup_{f \neq 0} \frac{\|Pf\|_\infty}{\|f\|_\infty}, \quad \forall P \in \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta).$$

The following result describes the quasi-compactness of P_0 . It can be found in [13, Proposition 4] for invertible matrices and in [6, Proposition 4.2] for nonnegative matrices. For $P \in \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$ and $n \in \mathbb{N}$, denote by P^n the n -fold iteration of P ; by convention P^0 is the identity operator.

Proposition 3.1. *Assume the conditions of Theorem 2.1. Let $\beta > 0$ be small enough. Then $P_0 \in \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$, and there is an operator $L \in \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$ whose spectral radius is strictly less than 1, such that for all $n \in \mathbb{N}$,*

$$P_0^n = \Pi_0 + L^n,$$

where Π_0 is a rank-one projection satisfying $\Pi_0 \mathcal{B}_\beta = \{f \in \mathcal{B}_\beta : P_0 f = f\}$ and $\Pi_0 f(x) = \nu(f)$ for all $f \in \mathcal{B}_\beta$ and $x \in \mathcal{S}$.

For simplicity, in our proofs we will use a slightly different family of operators $(R_t)_{t \in \mathbb{R}}$ defined by

$$R_t f(x) := e^{-it\gamma} P_{it} f(x) = \mathbb{E}[e^{it(S_1^x - \gamma)} f(X_1^x)], \quad \text{for } f \in \mathcal{C}(\mathcal{S}), x \in \mathcal{S}.$$

By the cocycle property $\log |\mathbf{a}_2 \mathbf{a}_1 x| = \log |\mathbf{a}_2(\mathbf{a}_1 x)| + \log |\mathbf{a}_1 x|$ and an induction, we have

$$R_t^n f(x) = \mathbb{E}[e^{it(S_n^x - n\gamma)} f(X_n^x)], \quad n \geq 1.$$

We collect in the following two propositions some results from [17] that we will use. Although these results are stated in [17] only for $d \geq 2$, they remain valid for $d = 1$. The first proposition concerns the perturbation theory.

Proposition 3.2. *Assume the conditions of Theorem 2.1.*

(1) *There exists a real number $\delta > 0$ such that for $t \in [-\delta, \delta]$ we have:*

(a) For all $f \in \mathcal{B}_\beta$ and $n \geq 1$

$$R_t^n f = \lambda^n(t) \Pi_t f + N_t^n f,$$

with $\Pi_t N_t = N_t \Pi_t = 0$.

(b) The mappings

$$\lambda : [-\delta, \delta] \rightarrow \mathbb{R}, \quad \Pi : [-\delta, \delta] \rightarrow \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta), \quad N : [-\delta, \delta] \rightarrow \mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$$

are C^∞ .

(c) For $k_0 \in \mathbb{N}$, there exist $\rho = \rho(k_0) \in (0, 1)$ and $c = c(k_0) > 0$ such that for all $n \geq 1$,

$$\max_{0 \leq k \leq k_0} \sup_{|t| < \delta} \left\| \frac{d^k}{dt^k} N_t^n \right\|_{\mathcal{B}_\beta \rightarrow \mathcal{B}_\beta} \leq c \rho^n.$$

(2) Let $K \subset \mathbb{R} \setminus \{0\}$ be compact. Then for each $f \in \mathcal{B}_\beta$, there is $\rho_1 \in (0, 1)$ such that for all $n \geq 1$,

$$\sup_{t \in K} \|R_t^n f\|_\infty \leq \rho_1^n \|f\|_\infty.$$

The second proposition concerns the Taylor expansion of λ and the positivity of the asymptotic variance.

Proposition 3.3. *Assume the conditions of Theorem 2.1.*

(1) The Taylor expansion of λ at 0 of order 3 is given by

$$\lambda(t) = 1 - \frac{\sigma^2}{2} t^2 - i \frac{m_3}{6} t^3 + o(t^3),$$

where $m_3 = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{V(\Gamma_\mu)} \mathbb{E}(S_n^x - n\gamma)^3 d\nu(x)$.

(2) For each $x \in \mathcal{S}$, the limit $b(x) = \lim_{n \rightarrow \infty} \mathbb{E}(S_n^x - n\gamma)$ exists in \mathbb{R} , the function $x \mapsto b(x)$ is in \mathcal{B}_β , and the derivative Π'_0 of Π_t at 0, satisfies

$$\Pi'_0 f(x) = i \nu(f) b(x), \quad \text{for } f \in \mathcal{B}_\beta, x \in \mathcal{S}.$$

(3) If μ is non-arithmetic, then $\sigma > 0$.

4. PROOF OF THEOREM 2.1

Proof of Theorem 2.1, part (1). From [17, Theorem 2.1], for $f \in \mathcal{B}_\beta$, we have

$$\lim_{n \rightarrow \infty} \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[f(X_n^x) \mathbf{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(f) \Phi(t) \right| = 0. \quad (4.1)$$

Combining this with the fact that the Banach space \mathcal{B}_β is dense in $\mathcal{C}(\mathcal{S})$ with respect to the norm $\|\cdot\|_\infty$ gives the conclusion of part (1). \square

To prove part (2), we first introduce a smooth approximation of an indicator function:

Lemma 4.1. (Smooth approximation of an indicator function)

- (1) Let $A, B \subset \mathcal{S}$ be non-empty closed sets with $A \cap B = \emptyset$. Then there is a continuous function $\varphi : \mathcal{S} \rightarrow [0, 1]$ such that $\varphi(x) = 1$ for all $x \in A$ and $\varphi(x) = 0$ for all $x \in B$.
- (2) Let $A \subset \mathcal{S}$ be a non-empty measurable set with $\nu(\partial A) = 0$, and let $\varepsilon > 0$. Then there exist two continuous functions $\varphi^-, \varphi^+ : \mathcal{S} \rightarrow [0, 1]$ such that

$$\varphi^- \leq \mathbf{1}_A \leq \varphi^+ \quad \text{and} \quad \nu\{x \in \mathcal{S} : \varphi^+ \neq \varphi^-\} < \varepsilon.$$

Proof. For a non-empty set $D \subset \mathcal{S}$ and $x \in \mathcal{S}$, define

$$\text{dist}(x, D) = \inf\{\mathbf{d}(x, z) : z \in D\}.$$

(1) Since A, B are closed and disjoint we have $\text{dist}(x, B) + \text{dist}(x, A) \neq 0$ for all $x \in \mathcal{S}$. The function defined below satisfies the desired properties:

$$\varphi(x) := \frac{\text{dist}(x, B)}{\text{dist}(x, B) + \text{dist}(x, A)}, \quad x \in \mathcal{S}.$$

(2) Since ν is a Borel measure on \mathcal{S} , we have

$$\nu(\overline{A}) = \inf\{\nu(U) : \overline{A} \subseteq U, U \text{ open}\},$$

$$\nu(A^\circ) = \sup\{\nu(K) : K \subseteq A^\circ, K \text{ compact}\}.$$

Hence for each $\varepsilon > 0$, there exists a compact $K \subset A^\circ$ such that $\nu(K) > \nu(A^\circ) - \frac{\varepsilon}{2}$. Since K and $(A^\circ)^c$ are disjoint closed sets, by part (1), there exists a continuous function $\varphi^- : \mathcal{S} \rightarrow [0, 1]$ such that $\varphi^-(x) = 1$ for $x \in K$ and $\varphi^-(x) = 0$ for $x \in (A^\circ)^c$.

Similarly, there exists an open set $U \supset \overline{A}$ such that $\nu(U) < \nu(\overline{A}) + \frac{\varepsilon}{2}$. Again by part (1) applied to the disjoint closed sets \overline{A} and U^c , we see that there is a continuous function $\varphi^+ : \mathcal{S} \rightarrow [0, 1]$ such that $\varphi^+(x) = 1$ for $x \in \overline{A}$ and $\varphi^+(x) = 0$ for $x \in U^c$. Therefore,

$$K \cup U^c \subset \{x \in \mathcal{S} : \varphi^+(x) = \varphi^-(x)\}.$$

Consequently,

$$\{x \in \mathcal{S} : \varphi^+(x) \neq \varphi^-(x)\} \subset K^c \cap U = U \setminus K.$$

Since $U \setminus K = (U \setminus \overline{A}) \cup (\overline{A} \setminus A^\circ) \cup (A^\circ \setminus K)$, it follows that

$$\nu\{x \in \mathcal{S} : \varphi^+(x) \neq \varphi^-(x)\} \leq \nu(U \setminus \overline{A}) + \nu(\overline{A} \setminus A^\circ) + \nu(A^\circ \setminus K) < \varepsilon,$$

where we have used the hypothesis that $\nu(\overline{A} \setminus A^\circ) = \nu(\partial A) = 0$. From the construction of φ^- and φ^+ , it is obvious that $\varphi^- \leq \mathbf{1}_B \leq \varphi^+$. \square

Proof of Theorem 2.1, part (2). Let $\varepsilon > 0$ be given. By Lemma 4.1, there exists two continuous functions φ^+ and $\varphi^-: \mathcal{S} \rightarrow [0, 1]$ such that

$$\varphi^- \leq \mathbb{1}_B \leq \varphi^+ \quad \text{and} \quad \nu\{x \in \mathcal{S} : \varphi^+ \neq \varphi^-\} < \varepsilon. \quad (4.2)$$

By the triangular inequality in \mathbb{R} , we have

$$\begin{aligned} & \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[\mathbb{1}_B(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(B)\Phi(t) \right| \\ & \leq \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[(\mathbb{1}_B - \varphi^+)(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] \right| \\ & \quad + \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[\varphi^+(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(\varphi^+)\Phi(t) \right| \\ & \quad + \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \nu(\varphi^+ - \mathbb{1}_B)\Phi(t) \right|. \end{aligned} \quad (4.3)$$

By part (1), the second term in (4.3) is less than ε for n enough large. The third term is also less than ε by the property (4.2) since

$$\nu(\varphi^+ - \mathbb{1}_B) \leq \nu(\varphi^+ - \varphi^-) \leq \nu\{x \in \mathcal{S} : \varphi^+ \neq \varphi^-\} < \varepsilon. \quad (4.4)$$

The first term can be estimated as follows:

$$\begin{aligned} & \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[(\mathbb{1}_B - \varphi^+)(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] \right| \\ & \leq \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[(\varphi^+ - \varphi^-)(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] \right| \\ & \leq \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[\varphi^+(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(\varphi^+)\Phi(t) \right| \\ & \quad + \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \mathbb{E} \left[\varphi^-(X_n^x) \mathbb{1}_{\left\{ \frac{S_n^x - n\gamma}{\sigma\sqrt{n}} \leq t \right\}} \right] - \nu(\varphi^-)\Phi(t) \right| \\ & \quad + \sup_{(x,t) \in \mathcal{S} \times \mathbb{R}} \left| \nu(\varphi^+ - \varphi^-)\Phi(t) \right|. \end{aligned}$$

In the last display, the first two terms are less than ε for n large enough, again by part (1); the third one is also less than ε by (4.4). \square

5. PROOF OF THEOREM 2.2

Proof of Theorem 2.2, part (1). We assume that both f and h are nonnegative; we can do this by considering the positive and negative parts. We will proceed the proof in 4 steps.

Step 1. We first establish (2.4) for $f \in \mathcal{B}_\beta$ and $h \in L^1$ whose Fourier transform

$$\widehat{h}(u) = \int_{\mathbb{R}} e^{-itu} h(t) dt, \quad \forall u \in \mathbb{R},$$

has a compact support $\text{supp}(\widehat{h}) \subset [-k, k]$. By the inversion formula of Fourier transform and Fubini's theorem, for any $x \in \mathcal{S}$ and $y \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E} [f(X_n^x) h(y + S_n^x - n\gamma)] &= \mathbb{E} \left[\frac{f(X_n^x)}{2\pi} \int_{\mathbb{R}} e^{iu(y+S_n^x-n\gamma)} \widehat{h}(u) du \right] \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{iuy} \widehat{h}(u) \mathbb{E} \left[e^{iu(S_n^x-n\gamma)} f(X_n^x) \right] du \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{iuy} \widehat{h}(u) R_u^n f(x) du. \end{aligned}$$

By the change of variables $u = \frac{t}{\sqrt{n}}$ and using Proposition 3.2, we have:

$$\begin{aligned} I &:= \sqrt{n} \mathbb{E} [f(X_n^x) h(y + S_n^x - n\gamma)] \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) R_{\frac{t}{\sqrt{n}}}^n f(x) dt \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{|t| \leq \delta_1 \sqrt{n}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) \lambda^n\left(\frac{t}{\sqrt{n}}\right) \Pi_{\frac{t}{\sqrt{n}}} f(x) dt, \\ I_2 &= \frac{1}{2\pi} \int_{|t| \leq \delta_1 \sqrt{n}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) N_{\frac{t}{\sqrt{n}}}^n f(x) dt, \\ I_3 &= \frac{1}{2\pi} \int_{|t| > \delta_1 \sqrt{n}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) R_{\frac{t}{\sqrt{n}}}^n f(x) dt, \end{aligned}$$

with $\delta_1 \in (0, \delta]$ a parameter which will be fixed later. We will prove that I_1 gives the main term of the desired expansion, while I_2 and I_3 tend to 0.

Estimation of I_1 . By Proposition 3.3 and an elementary calculation, we obtain, as $\frac{t}{\sqrt{n}} \rightarrow 0$,

$$\lambda^n\left(\frac{t}{\sqrt{n}}\right) = e^{-\frac{\sigma^2 t^2}{2}} \exp\left(-\frac{im_3 t^3}{6\sqrt{n}} + o\left(\frac{t^3}{\sqrt{n}}\right)\right). \quad (5.1)$$

By Taylor's expansion of the operator Π_t on a neighborhood of 0, we have

$$\Pi_{\frac{t}{\sqrt{n}}} f(x) = \Pi_0 f(x) + \frac{t}{\sqrt{n}} \Pi'_0 f(x) + O\left(\frac{t^2}{n}\right), \quad (5.2)$$

where Π_0 and Π'_0 are bounded operators on \mathcal{B}_β defined in Propositions 3.1 and 3.3(2). Notice that $\Pi_0 f(x) = \nu(f)$ and $\Pi'_0 f(x) = i\nu(f)b(x)$. With (5.1)

and (5.2) in mind, we do the composition

$$\begin{aligned}
\lambda^n\left(\frac{t}{\sqrt{n}}\right)\Pi_{\frac{t}{\sqrt{n}}}f(x) &= \lambda^n\left(\frac{t}{\sqrt{n}}\right)\left[\Pi_{\frac{t}{\sqrt{n}}}f(x) - \nu(f) - \frac{it\nu(f)b(x)}{\sqrt{n}}\right] \\
&\quad + \left[\lambda^n\left(\frac{t}{\sqrt{n}}\right) - e^{-\frac{\sigma^2 t^2}{2}}\left(1 - \frac{im_3 t^3}{6\sqrt{n}}\right)\right]\nu(f) \\
&\quad + \left[\lambda^n\left(\frac{t}{\sqrt{n}}\right) - e^{-\frac{\sigma^2 t^2}{2}}\right]\frac{it\nu(f)b(x)}{\sqrt{n}} \\
&\quad + \nu(f)e^{-\frac{\sigma^2 t^2}{2}}\left(1 + \frac{itb(x)}{\sqrt{n}} - \frac{im_3 t^3}{6\sqrt{n}}\right). \tag{5.3}
\end{aligned}$$

Choosing $\delta_1 \leq \delta$ small enough such that when $|t|/\sqrt{n} \leq \delta_1$, we have

$$\left| -\frac{im_3 t^3}{6\sigma^3 \sqrt{n}} + o\left(\frac{t^3}{\sqrt{n}}\right) \right| \leq \frac{\sigma^2 t^2}{4} \text{ and } \lambda^n\left(\frac{t}{\sqrt{n}}\right) \leq e^{-\frac{\sigma^2 t^2}{4}}. \tag{5.4}$$

In the definition of I_1 , we substitute $\lambda^n\left(\frac{t}{\sqrt{n}}\right)\Pi_{\frac{t}{\sqrt{n}}}f(x)$ by the decomposition (5.3); this leads to a decomposition of I_1 which we write accordingly as

$$I_1 = J_1 + J_2 + J_3 + J_4.$$

We first estimate the integral J_1 . Using (5.2), (5.4) and the fact that \widehat{h} is bounded, we obtain

$$\begin{aligned}
|J_1| &\leq \frac{1}{2\pi} \sup_{t \in \mathbb{R}} |\widehat{h}(t)| \int_{|t| \leq \delta_1 \sqrt{n}} e^{-\frac{\sigma^2 t^2}{4}} O\left(\frac{t^2}{n}\right) dt \\
&\leq \int_{\mathbb{R}} e^{-\frac{\sigma^2 t^2}{4}} |t| dt \cdot O\left(\frac{1}{\sqrt{n}}\right) \leq \frac{C}{\sqrt{n}}.
\end{aligned}$$

To estimate J_2 , we use the following inequality (see inequality XVI(2.8) in [7]): for all $u, v \in \mathbb{C}$,

$$|e^u - 1 - v| \leq \left(|u - v| + \frac{1}{2}|v|^2\right) e^{\max(|u|, |v|)}. \tag{5.5}$$

Using (5.5) with $u = -\frac{im_3 t^3}{6\sqrt{n}} + o\left(\frac{t^3}{\sqrt{n}}\right)$ and $v = -\frac{im_3 t^3}{6\sqrt{n}}$, we have

$$\begin{aligned}
|J_2| &\leq \frac{\nu(f)}{2\pi} \sup_{t \in \mathbb{R}} |\widehat{h}(t)| \int_{|t| \leq \delta_1 \sqrt{n}} e^{-\frac{\sigma^2 t^2}{2}} \left[|t|^3 o\left(\frac{1}{\sqrt{n}}\right) + t^6 O\left(\frac{1}{n}\right)\right] e^{\frac{\sigma^2 t^2}{4}} dt \\
&\leq \int_{\mathbb{R}} (|t|^3 + |t^5|) e^{-\frac{\sigma^2 t^2}{4}} dt \cdot O\left(\frac{1}{\sqrt{n}}\right) \leq \frac{C}{\sqrt{n}}.
\end{aligned}$$

For J_3 , we use again inequality (5.5) and the fact that the mapping b belongs to \mathcal{B}_β , to conclude that

$$\begin{aligned} |J_3| &\leq \int_{|t| \leq \delta_1 \sqrt{n}} \frac{|t\nu(f)b(x)|}{\sqrt{n}} e^{-\frac{\sigma^2 t^2}{2}} \left| 1 - \lambda_{\frac{it}{\sqrt{n}}}^n \right| \\ &\leq \nu(f) \|b\|_\infty \delta_1 \int_{\mathbb{R}} e^{-\frac{\sigma^2 t^2}{4}} \left| -\frac{im_3 t^3}{6\sqrt{n}} + o\left(\frac{t^3}{\sqrt{n}}\right) \right| dt \\ &\leq \int_{\mathbb{R}} e^{-\frac{\sigma^2 t^2}{4}} |t|^3 dt \cdot O\left(\frac{1}{\sqrt{n}}\right) \leq \frac{C}{\sqrt{n}}. \end{aligned}$$

Hence, we obtain the following estimate of I_1 :

$$|I_1 - J_4| \leq \frac{C}{\sqrt{n}}. \quad (5.6)$$

Estimation of I_2 . From Proposition 3.2, we know that for $|t| \leq \delta_1 \sqrt{n}$ there exists a constant $c > 0$ and $\rho \in (0, 1)$ such that $\left\| N_{\frac{it}{\sqrt{n}}}^n f \right\|_\beta \leq c\rho^n$. Hence,

$$|I_2| \leq C\rho^n \int_{\mathbb{R}} |\widehat{h}(t)| dt. \quad (5.7)$$

Estimation of I_3 . From Proposition 3.2(2), we have

$$\sup_{\delta_1 \sqrt{n} \leq |t| \leq c\sqrt{n}} \left\| R_{\frac{it}{\sqrt{n}}}^n f \right\|_\infty < \rho_1^n \|f\|_\infty.$$

Using this together with the condition that $\text{supp}(\widehat{h}) \subset [-k, k]$, we get

$$|I_3| \leq C\rho^n \int_{\mathbb{R}} |\widehat{h}(t)| dt. \quad (5.8)$$

Collecting the bounds (5.6), (5.7) and (5.8), we have

$$|I - J_4| \leq C \left(\frac{1}{\sqrt{n}} + \rho_1^n + \rho^n \right). \quad (5.9)$$

Set

$$Q(t) = 1 + \frac{it}{\sqrt{n}} b(x) - \frac{im_3 t^3}{6\sqrt{n}}.$$

It is not difficult to see that

$$\begin{aligned} &\left| \frac{\nu(f)}{2\pi} \int_{|t| > \delta_1 \sqrt{n}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) e^{-\frac{\sigma^2 t^2}{2}} Q(t) dt \right| \\ &\leq C \left(e^{-\frac{\sigma^2 \delta_1^2 n}{2}} \int_{\mathbb{R}} |\widehat{h}(t)| dt + \frac{1}{\sqrt{n}} \sup_{t \in \mathbb{R}} |\widehat{h}(t)| \right). \end{aligned}$$

Hence we replace the integral on $|t| \leq \delta_1 \sqrt{n}$ of J_4 in (5.9) by an integral on \mathbb{R} . We get

$$\sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| I - \frac{\nu(f)}{2\pi} \int_{\mathbb{R}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) e^{-\frac{\sigma^2 t^2}{2}} Q(t) dt \right| \xrightarrow{n \rightarrow \infty} 0.$$

Note that $\widehat{\phi}(\sigma t) = e^{-\frac{\sigma^2 t^2}{2}}$ is the Fourier transform of $\phi_\sigma(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$.

Then

$$\lim_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| I - \frac{\nu(f)}{2\pi} \int_{\mathbb{R}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) \widehat{\phi}(t\sigma) Q(t) dt \right| = 0.$$

But one has for all $p \geq 0$, $(\sigma t)^p \widehat{\phi}(\sigma t) = (-i)^p \widehat{\phi_\sigma^{(p)}}(t)$ where the notation $f^{(p)}$ is the derivative of order p of f , it implies

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{\frac{ity}{\sqrt{n}}} \widehat{h}\left(\frac{t}{\sqrt{n}}\right) \widehat{\phi}(t\sigma) Q(t) dt = \sqrt{n} \int_{\mathbb{R}} h(y + u\sqrt{n}) \phi_\sigma(u) H_x(u) du,$$

where $H_x(u)$ is a polynomial such that $Q\left(-i \frac{d}{dx}\right) \phi_\sigma(u) = H_x(u) \phi_\sigma(u)$. With an elementary calculation, we get

$$H_x(u) = 1 - \frac{b(x)}{\sigma^2 \sqrt{n}} u + \frac{m_3}{6\sigma^6 \sqrt{n}} (3\sigma^2 u - u^3).$$

Using the change of variables $z = y + u\sqrt{n}$, we get

$$\lim_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| I - \nu(f) \int_{\mathbb{R}} h(z) \phi_\sigma\left(\frac{y-z}{\sqrt{n}}\right) H_x\left(\frac{y-z}{\sqrt{n}}\right) dz \right| = 0,$$

or, equivalently,

$$\lim_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} \left| \sigma I - \nu(f) \int_{\mathbb{R}} h(z) \phi\left(\frac{y-z}{\sigma\sqrt{n}}\right) H_x\left(\frac{y-z}{\sqrt{n}}\right) dz \right| = 0.$$

So we have established (2.4) for $f \in \mathcal{B}_\beta$ and Lebesgue integrable function h whose Fourier transform \widehat{h} has a compact support.

Step 2. We establish (2.4) for $f \in \mathcal{B}_\beta$ and $h \in L^1$ satisfying $h \in \mathcal{H}_\varepsilon$ for all $\varepsilon \in (0, \frac{1}{4})$ and

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \overline{h}_\varepsilon(u) du = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \underline{h}_\varepsilon(u) du = \int_{\mathbb{R}} h(u) du, \quad (5.10)$$

where \mathcal{H}_ε , \overline{h}_ε and $\underline{h}_\varepsilon$ are defined below. For any nonnegative Lebesgue integrable function h defined on \mathbb{R} , and for any $\varepsilon > 0$ and $u \in \mathbb{R}$, set $\mathbb{B}_\varepsilon(u) = \{u' \in \mathbb{R} : |u' - u| \leq \varepsilon\}$,

$$\overline{h}_\varepsilon(u) = \sup_{v \in \mathbb{B}_\varepsilon(u)} h(v) \quad \text{and} \quad \underline{h}_\varepsilon(u) = \inf_{v \in \mathbb{B}_\varepsilon(u)} h(v).$$

For any $\varepsilon > 0$, denote by \mathcal{H}_ε the set of nonnegative Lebesgue integrable functions h such that \bar{h}_ε and $\underline{h}_\varepsilon$ are measurable and Lebesgue integrable:

$$\mathcal{H}_\varepsilon = \{h \in L^1 : h \geq 0, \bar{h}_\varepsilon \text{ and } \underline{h}_\varepsilon \text{ are measurable and integrable}\}.$$

We shall use the following result proved in [8, Lemma 5.2].

Lemma 5.1. *Let $h \in L^1$ be such that $h \in \mathcal{H}_\varepsilon$ for all $\varepsilon \in (0, 1/4)$. Then we have, for all $\varepsilon \in (0, 1/4)$ and $u \in \mathbb{R}$,*

$$\underline{h}_\varepsilon * \kappa_{\varepsilon^2}(u) - \int_{|v| \geq \varepsilon} \underline{h}_\varepsilon(u-v) \kappa_{\varepsilon^2}(v) dv \leq h(u) \leq (1 + 4\varepsilon) \bar{h}_\varepsilon * \kappa_{\varepsilon^2}(u),$$

where κ is defined on \mathbb{R} by

$$\kappa(u) = \frac{1}{2\pi} \left(\frac{\sin\left(\frac{u}{2}\right)}{\frac{u}{2}} \right)^2 \text{ for } u \in \mathbb{R}^*, \text{ and } \kappa(0) = \frac{1}{2\pi}.$$

Moreover, we need some properties of the kernel κ that we state in the following. The function κ is integrable and its Fourier transform is given by

$$\widehat{\kappa}(t) = 1 - |t| \text{ for all } t \in [-1, 1], \text{ and } \widehat{\kappa}(t) = 0 \text{ otherwise.}$$

Note that

$$\int_{\mathbb{R}} \kappa(u) du = \widehat{\kappa}(0) = 1 = \int_{\mathbb{R}} \widehat{\kappa}(t) dt.$$

For any $\varepsilon > 0$, we define the function κ_ε on \mathbb{R} by

$$\kappa_\varepsilon(u) = \frac{1}{\varepsilon} \kappa\left(\frac{u}{\varepsilon}\right), \quad u \in \mathbb{R}. \quad (5.11)$$

Its Fourier transform is given by $\widehat{\kappa}_\varepsilon(t) = \widehat{\kappa}(\varepsilon t)$. Note also that, for any $\varepsilon > 0$, we have

$$\int_{|u| \geq \frac{1}{\varepsilon}} \kappa(u) du \leq \frac{1}{\pi} \int_{\frac{1}{\varepsilon}}^{+\infty} \frac{4}{u^2} du = \frac{4\varepsilon}{\pi}. \quad (5.12)$$

For simplicity, we denote, for any $f \in \mathcal{C}(\mathcal{S})$ and $h \in L^1$,

$$I(n, f, h) = \sigma \sqrt{n} \mathbb{E} [f(X_n^x) h(y + S_n^x - n\gamma)], \quad (5.13)$$

$$K(n, f, h) = \nu(f) \int_{\mathbb{R}} h(z) \phi\left(\frac{y-z}{\sigma\sqrt{n}}\right) H_x\left(\frac{y-z}{\sqrt{n}}\right) dz. \quad (5.14)$$

Notice that $\sup_{x \in \mathcal{S}, u \in \mathbb{R}} |\phi\left(\frac{u}{\sigma}\right) H_x(u)| < \infty$. This implies the following uniform bound in $x \in \mathcal{S}, y, z \in \mathbb{R}, n \geq 1$:

$$\phi\left(\frac{y-z}{\sigma\sqrt{n}}\right) H_x\left(\frac{y-z}{\sqrt{n}}\right) \leq C. \quad (5.15)$$

From this we see that for $f \in \mathcal{C}(\mathcal{S})$ and $h \in L^1$,

$$K(n, f, h) \leq C \nu(f) \|h\|_{L^1}. \quad (5.16)$$

Remark that, for f and h fixed as in the beginning of Step 2, with the notation (5.13) and (5.14), to prove the desired conclusion (2.4), it suffices to establish the following result: for all $\varepsilon > 0$ small enough, when n is large enough,

$$\sup_{(x,y) \in (\mathcal{S}, \mathbb{R})} |I(n, f, h) - K(n, f, h)| \leq \nu(f)\delta_h(\varepsilon) + \varepsilon, \quad (5.17)$$

where

$$\begin{aligned} \delta_h(\varepsilon) &= C \left(\|h - \underline{h}_{2\varepsilon}\|_{L^1} + \|\bar{h}_{2\varepsilon} - h\|_{L^1} \right) + C \|\bar{h}_{2\varepsilon}\|_{L^1} (\varepsilon + \varepsilon^2) \\ &\rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ (due to (5.10)).} \end{aligned}$$

Below we will prove (5.17) by giving upper and lower bounds of $I(n, f, h) - K(n, f, h)$.

Upper bound of $I(n, f, h) - K(n, f, h)$. By Lemma 5.1, we have, for any $x \in \mathcal{S}, n \geq 1, y \in \mathbb{R}$ and $\varepsilon \in (0, 1/4)$,

$$I(n, f, h) \leq (1 + 4\varepsilon)I(n, f, \bar{h}_\varepsilon * \kappa_{\varepsilon^2}).$$

Since \bar{h}_ε and κ_{ε^2} are integrable, the function $\bar{h}_\varepsilon * \kappa_{\varepsilon^2}$ is also integrable; its Fourier transform $\widehat{\bar{h}_\varepsilon * \kappa_{\varepsilon^2}}$ has a compact support included in $[-1/\varepsilon^2, 1/\varepsilon^2]$. Consequently, we can use the result proved in Step 1, applied to f and $\bar{h}_\varepsilon * \kappa_{\varepsilon^2}$, to conclude that for n large enough,

$$I(n, f, h) \leq (1 + 4\varepsilon)K(n, f, \bar{h}_\varepsilon * \kappa_{\varepsilon^2}) + \varepsilon. \quad (5.18)$$

Notice that for $|v| \leq \varepsilon$ and $u \in \mathbb{R}$, we have $[u - v - \varepsilon, u - v + \varepsilon] \subset [u - 2\varepsilon, u + 2\varepsilon]$. Therefore, by definition,

$$\underline{h}_\varepsilon(u - v) \geq \underline{h}_{2\varepsilon}(u) \quad \text{and} \quad \bar{h}_\varepsilon(u - v) \leq \bar{h}_{2\varepsilon}(u). \quad (5.19)$$

Consequently, for any $u \in \mathbb{R}$,

$$\begin{aligned} \bar{h}_\varepsilon * \kappa_{\varepsilon^2}(u) &\leq \bar{h}_{2\varepsilon}(u) \int_{|v| \leq \varepsilon} \kappa_{\varepsilon^2}(v) dv + \int_{|v| \geq \varepsilon} \bar{h}_\varepsilon(u - v) \kappa_{\varepsilon^2}(v) dv \\ &\leq \bar{h}_{2\varepsilon}(u) + \int_{|v| \geq \varepsilon} \bar{h}_\varepsilon(u - v) \kappa_{\varepsilon^2}(v) dv. \end{aligned}$$

From this together with the bound (5.15), inequality (5.18) implies

$$\begin{aligned} I(n, f, h) &\leq (1 + 4\varepsilon)K(n, f, \bar{h}_{2\varepsilon}) \\ &\quad + (1 + 4\varepsilon)C\nu(f) \int_{\mathbb{R}} \int_{|v| \geq \varepsilon} \bar{h}_\varepsilon(z - v) \kappa_{\varepsilon^2}(v) dv dz + \varepsilon. \end{aligned}$$

For a bound of the first term in the right hand side, we use the decomposition $K(n, f, \bar{h}_{2\varepsilon}) = K(n, f, h) + K(n, f, \bar{h}_{2\varepsilon} - h)$ and the inequality (5.16)

for $K(n, f, \bar{h}_{2\varepsilon} - h)$ to get $K(n, f, \bar{h}_{2\varepsilon}) \leq K(n, f, h) + C\nu(f) \|\bar{h}_{2\varepsilon} - h\|_{L^1}$.
Therefore

$$\begin{aligned} I(n, f, h) &\leq (1 + 4\varepsilon) \left[K(n, f, h) + C\nu(f) \|\bar{h}_{2\varepsilon} - h\|_{L^1} \right] \\ &\quad + (1 + 4\varepsilon) C\nu(f) \int_{\mathbb{R}} \int_{|v| \geq \varepsilon} \bar{h}_{\varepsilon}(z - v) \kappa_{\varepsilon^2}(v) dv dz + \varepsilon. \end{aligned}$$

For a bound of the last integral, we use (5.11) and (5.12), to obtain

$$\int_{\mathbb{R}} \int_{|v| \geq \varepsilon} \bar{h}_{\varepsilon}(z - v) \kappa_{\varepsilon^2}(v) dv dz \leq \|\bar{h}_{\varepsilon}\|_{L^1} \frac{4\varepsilon}{\pi}.$$

Using this and the bound of $K(n, f, h)$ in (5.16), from the preceding bound of $I(n, f, h)$ we get for $f \in \mathcal{B}_{\beta}$ and $h \in \mathcal{H}_{\varepsilon}$,

$$\begin{aligned} I(n, f, h) - K(n, f, h) &\leq 4\nu(f)\varepsilon \|h\|_{L^1} + C\nu(f)(1 + 4\varepsilon) \|\bar{h}_{2\varepsilon} - h\|_{L^1} \\ &\quad + \frac{4C\nu(f)(1 + 4\varepsilon)\varepsilon}{\pi} \|\bar{h}_{\varepsilon}\|_{L^1} + \varepsilon \\ &\leq C\nu(f) \|\bar{h}_{2\varepsilon} - h\|_{L^1} + C\nu(f)(\varepsilon + \varepsilon^2) \|\bar{h}_{2\varepsilon}\|_{L^1} + \varepsilon. \end{aligned} \quad (5.20)$$

Lower bound of $I(n, f, h) - K(n, f, h)$. With the notation $g_{v,\varepsilon}(u) = \underline{h}_{\varepsilon}(u - v)$ and by Lemma 5.1, we have:

$$I(n, f, h) \geq I(n, f, \underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}) - \int_{|v| \geq \varepsilon} I(n, f, g_{v,\varepsilon}) \kappa_{\varepsilon^2}(v) dv. \quad (5.21)$$

*Bound of $I(n, f, \underline{h}_{\varepsilon} * \kappa_{\varepsilon^2})$.* The Fourier transform of $\underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}$ has a compact support included in $[-1/\varepsilon^2, 1/\varepsilon^2]$. So by the the result proved in Step 1, for n large enough,

$$I(n, f, \underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}) \geq K(n, f, \underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}) - \varepsilon. \quad (5.22)$$

By (5.19) and (5.12),

$$\underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}(z) \geq \underline{h}_{2\varepsilon}(z) \int_{|v| \leq \varepsilon} \kappa_{\varepsilon^2}(v) dv \geq \left(1 - \frac{4\varepsilon}{\pi}\right) \underline{h}_{2\varepsilon}(z).$$

From this and the bound (5.16) for $K(n, f, \underline{h}_{2\varepsilon})$, the inequality (5.22) implies

$$I(n, f, \underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}) \geq K(n, f, \underline{h}_{2\varepsilon}) - \frac{4\varepsilon}{\pi} C\nu(f) \|\underline{h}_{2\varepsilon}\|_{L^1} - \varepsilon.$$

Using $K(n, f, \underline{h}_{2\varepsilon}) = K(n, f, h) + K(n, f, \underline{h}_{2\varepsilon} - h)$ and the bound (5.16) for $K(n, f, \underline{h}_{2\varepsilon} - h)$, we have

$$I(n, f, \underline{h}_{\varepsilon} * \kappa_{\varepsilon^2}) - K(n, f, h) \geq -C\nu(f) \|h - \underline{h}_{2\varepsilon}\|_{L^1} - \frac{4C\nu(f)\varepsilon}{\pi} \|\underline{h}_{2\varepsilon}\|_{L^1} - \varepsilon. \quad (5.23)$$

Bound of $\int_{|v| \geq \varepsilon} I(n, f, g_{v, \varepsilon}) \kappa_{\varepsilon^2}(v) dv$ in (5.21). Using (5.20) with h replaced by $g_{v, \varepsilon}$ (which lies in \mathcal{H}_ε) and (5.16) for $K(n, f, g_{v, \varepsilon})$, we get

$$\begin{aligned} I(n, f, g_{v, \varepsilon}) &\leq C\nu(f) \|g_{v, \varepsilon}\|_{L^1} + C\nu(f) \|(\overline{g_{v, \varepsilon}})_{2\varepsilon} - g_{v, \varepsilon}\|_{L^1} \\ &\quad + C\nu(f) (\varepsilon + \varepsilon^2) \|(\overline{g_{v, \varepsilon}})_{2\varepsilon}\|_{L^1} + \varepsilon. \end{aligned}$$

This implies that

$$I(n, f, g_{v, \varepsilon}) \leq C\nu(f) (1 + \varepsilon + \varepsilon^2) \|(\overline{g_{v, \varepsilon}})_{2\varepsilon}\|_{L^1} + \varepsilon. \quad (5.24)$$

Note that, for any $v \in \mathbb{R}$,

$$\overline{(g_{\varepsilon, v})_{2\varepsilon}}(u) = \sup_{w \in [u-2\varepsilon, u+2\varepsilon]} \underline{h}_\varepsilon(w - v) \leq \sup_{w \in [u-2\varepsilon, u+2\varepsilon]} h(w - v) = \overline{h}_{2\varepsilon}(u - v).$$

So, $\|(\overline{g_{\varepsilon, v})_{2\varepsilon}}\|_{L^1} \leq \|\overline{h}_{2\varepsilon}\|_{L^1}$. This together with (5.24) and (5.12) implies that

$$\begin{aligned} \int_{|v| \geq \varepsilon} I(n, f, g_{v, \varepsilon}) \kappa_{\varepsilon^2}(v) dv &\leq \left(C\nu(f) (1 + \varepsilon + \varepsilon^2) \|\overline{h}_{2\varepsilon}\|_{L^1} + \varepsilon \right) \int_{|v| \geq \varepsilon} \kappa_{\varepsilon^2}(v) dv \\ &\leq \left(C\nu(f) (1 + \varepsilon + \varepsilon^2) \|\overline{h}_{2\varepsilon}\|_{L^1} + \varepsilon \right) \frac{4\varepsilon}{\pi}. \end{aligned} \quad (5.25)$$

Putting together (5.21), (5.23) and (5.25), we obtain

$$I(n, f, h) - K(n, f, h) \geq -C\nu(f) \|h - \underline{h}_{2\varepsilon}\|_{L^1} + C\nu(f) \|\overline{h}_{2\varepsilon}\|_{L^1} (\varepsilon + \varepsilon^2) - \varepsilon. \quad (5.26)$$

Combining the upper bound (5.20), the lower bound (5.26) and the condition (5.10), the desired result (5.17) or (2.4) follows for $f \in \mathcal{B}_\beta$ and $h \in \mathcal{H}_\varepsilon$ for all $\varepsilon \in (0, \frac{1}{4})$ and h satisfies (5.10).

Step 3. We prove (2.4) for $f \in \mathcal{B}_\beta$ and h which is nonnegative and directly Riemann integrable. Since h is directly Riemann integrable, $M := \sup_{y \in \mathbb{R}} h(y) < +\infty$. Let $\eta \in (0, 1)$ and $\varepsilon \in (0, M\eta)$.

By a result of approximation in the proof of Theorem 2.2 in [16], there exist two functions $h_{\eta, \varepsilon}^-$ and $h_{\eta, \varepsilon}^+$ which belong to $\mathcal{H}_{\varepsilon_1}$ for all $\varepsilon_1 \in (0, \min\{1/4, M\eta, \eta/3\})$ small enough, and which satisfy (5.10), together with

$$h_{\eta, \varepsilon}^- \leq h \leq h_{\eta, \varepsilon}^+ \quad \text{and} \quad \int_{\mathbb{R}} [h_{\eta, \varepsilon}^+(t) - h_{\eta, \varepsilon}^-(t)] dt < 3\varepsilon. \quad (5.27)$$

The first inequality in (5.27) gives $K(n, f, h_{\eta, \varepsilon}^+ - h) \leq K(n, f, h_{\eta, \varepsilon}^+ - h_{\eta, \varepsilon}^-)$, so that

$$\begin{aligned} |I(n, f, h) - K(n, f, h)| &\leq |I(n, f, h) - I(n, f, h_{\eta, \varepsilon}^+)| \\ &\quad + |I(n, f, h_{\eta, \varepsilon}^+) - K(n, f, h_{\eta, \varepsilon}^+)| + |K(n, f, h_{\eta, \varepsilon}^+ - h_{\eta, \varepsilon}^-)|. \end{aligned} \quad (5.28)$$

In the right hand side, as $n \rightarrow \infty$, the second term tends to 0 uniformly in $x \in \mathcal{S}$ and $y \in \mathbb{R}$ by the result proved in Step 2. The third one is bounded

by $C\nu(f)3\varepsilon$ from the bound (5.16) for $K(n, f, h_{\eta,\varepsilon}^+ - h_{\eta,\varepsilon}^-)$ and the property (5.27). Therefore, using (5.28) and passing to the limit as $n \rightarrow \infty$, we obtain

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, f, h) - K(n, f, h)| \\ & \leq \limsup_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, f, h) - I(n, f, h_{\eta,\varepsilon}^+)| + C\nu(f)3\varepsilon. \end{aligned} \quad (5.29)$$

Now

$$\begin{aligned} I(n, f, h) - I(n, f, h_{\eta,\varepsilon}^+) & \leq I(n, f, h_{\eta,\varepsilon}^+) - I(n, f, h_{\eta,\varepsilon}^-) \\ & = [I(n, f, h_{\eta,\varepsilon}^+) - K(n, f, h_{\eta,\varepsilon}^+)] + K(n, f, h_{\eta,\varepsilon}^+ - h_{\eta,\varepsilon}^-) \\ & \quad + [K(n, f, h_{\eta,\varepsilon}^-) - I(n, f, h_{\eta,\varepsilon}^-)]. \end{aligned} \quad (5.30)$$

As in the preceding, in the last display, as $n \rightarrow \infty$, the first and third terms tend to 0 uniformly in $x \in \mathcal{S}$ and $y \in \mathbb{R}$ by the result proved in Step 2; the second one is bounded by $C\nu(f)3\varepsilon$ (by (5.16) and (5.27)). Therefore,

$$\limsup_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, f, h) - I(n, f, h_{\eta,\varepsilon}^+)| \leq C\nu(f)3\varepsilon. \quad (5.31)$$

Combining (5.29) and (5.31), we obtain

$$\limsup_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, f, h) - K(n, f, h)| \leq C\nu(f)6\varepsilon. \quad (5.32)$$

Since $\varepsilon > 0$ est arbitrary, this gives (2.4).

Step 4. We establish (2.4) for $f \in \mathcal{C}(\mathcal{S})$ and h which is directly Riemann integrable. Let $\varepsilon > 0$. From the fact that \mathcal{B}_β is dense in $\mathcal{C}(\mathcal{S})$ with respect to the norm $\|\cdot\|_\infty$, there is a function $\tilde{f} \in \mathcal{B}_\beta$ such that $\|\tilde{f} - f\|_\infty < \varepsilon$. Hence we have

$$\begin{aligned} I(n, f, h) - K(n, f, h) & = I(n, f - \tilde{f}, h) + [I(n, \tilde{f}, h) - K(n, \tilde{f}, h)] \\ & \quad - K(n, f - \tilde{f}, h). \end{aligned} \quad (5.33)$$

It follows from the result proved in Step 3 that

$$\lim_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, \tilde{f}, h) - K(n, \tilde{f}, h)| = 0.$$

Consequently, by (5.33)

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, f, h) - K(n, f, h)| \\ & \leq \limsup_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, f - \tilde{f}, h) - K(n, f - \tilde{f}, h)| \\ & \leq \|f - \tilde{f}\|_\infty \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} [I(n, 1, h) + K(n, 1, h)], \end{aligned} \quad (5.34)$$

where the last inequality follows directly from the definition of I and K . By the result proved in Step 3 (applied to $f = 1$) and the bound (5.16) applied

for $K(n, 1, h)$, we see that $I(n, 1, h) - K(n, 1, h)$ and $I(n, 1, h)$ are bounded uniformly in x, y and $n \geq 1$. Hence

$$\sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} [I(n, 1, h) + K(n, 1, h)] < \infty.$$

Since $\|\tilde{f} - f\|_\infty < \varepsilon$ and $\varepsilon > 0$ is arbitrary, this together with (5.34) implies

$$\limsup_{n \rightarrow \infty} \sup_{(x,y) \in \mathcal{S} \times \mathbb{R}} |I(n, f, h) - K(n, f, h)| = 0,$$

which completes the proof of part (1) of Theorem 2.2. □

Proof of Theorem 2.2, part (2). For the proof of part (2), we use the conclusion of part (1) and the approximation of the indicator function by a continuous function (see Lemma 4.1). Because the argument is quite similar to the proof of part (2) of Theorem 2.1, we omit the details. □

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