# Boundary singular solutions of a class of equations with mixed absorption-reaction

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# Boundary singular solutions of a class of equations with mixed absorption-reaction

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**Abstract**

We study properties of positive functions satisfying (E) $-\Delta u + u^p - M|\nabla u|^q = 0$ is a domain $\Omega$ or in $\mathbb{R}^N_+$ when $p > 1$ and $1 < q < \min\{p, 2\}$. We concentrate our research on the solutions of (E) vanishing on the boundary except at one point. This analysis depends on the existence of separable solutions in $\mathbb{R}^N_+$. We construct various types of positive solutions with an isolated singularity on the boundary. We also study conditions for the removability of compact boundary sets and the Dirichlet problem associated to (E) with a measure as boundary data.

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## Contents



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# 1 Introduction

The aim of this article is to study some properties of solutions of the following equation

$$\mathcal{L}_{q,M}u := -\Delta u + |u|^{p-1}u - M|\nabla u|^q = 0 \tag{1.1}$$

in a bounded domain $\Omega$ of $\mathbb{R}^N$ or in the half-space $\mathbb{R}^N_+$, where $M > 0$ and $p > q > 1$. We are particularly interested in the analysis of boundary singularities of such solutions. If $M = 0$ the boundary singularities problem has been investigated since thirty years, starting with the work of Gmira and Véron [15] who obtained an almost complete description of the solutions with isolated boundary singularities. When $M > 0$ there is a balance between the absorption term $|u|^{p-1}u$ and the source term $M|\nabla u|^q$, a confrontation which can create very new effects. Furthermore, the scale of the two opposed reaction terms depends upon the position of $q$ with respect to $\frac{2p}{p+1}$. This is due to the fact that (1.1) is equivariant with respect to the scaling transformation $T_\ell$ defined for $\ell > 0$ by $T_\ell[u](x) = \ell^{\frac{2}{p-1}}u(\ell x)$.

If $q < \frac{2p}{p+1}$, the absorption term is dominant and the behaviour of the singular solutions is modelled by the equation studied in [15]

$$-\Delta u + |u|^{p-1}u = 0. \tag{1.2}$$

If $q > \frac{2p}{p+1}$ , the source term is dominant and the behaviour of the singular solutions is modelled by positive separable solutions of the equation without diffusion

$$u^p - M|\nabla u|^q = 0. \tag{1.3}$$

Another associated equation which plays an important role in the construction of singular solutions since its positive solutions are supersolutions of (1.1) is

$$-\Delta u - M|\nabla u|^q = 0. \tag{1.4}$$

Note that in (1.3) and (1.4), $M$ can be fixed to be 1 by replacing $u$ by $\ell u$.
If $q = \frac{2p}{p+1}$, the coefficient $M > 0$ plays a fundamental role in the properties of the set of solutions, in particular for the existence of singular solutions and removable singularities. This situation is similar in some sense to what happens for equation

$$-\Delta u = |u|^{p-1}u + M|\nabla u|^q \tag{1.5}$$

which is studied thoroughfly in [5], [6] in the case $M > 0$ and in [26] in the case $M < 0$. In this last article the opposition of a forcing term $|u|^{p-1}u$ and an absorption term $M|\nabla u|^q$ creates a very rich configuration of unexpected phenomena and new effects.

In the present paper we will consider the case where $1 < q < 2$, with a special emphasis on the case $q = \frac{2p}{p+1}$ which allows to put into light the role of the value of $M$. We first analyze the following problem: given a smooth bounded domain $\Omega \subset \mathbb{R}^N$ such that $0 \in \partial\Omega$, under what conditions involving $p$, $q$ and $M$ is the point 0 a removable singularity for a solution of (1.1) continuous in $\overline{\Omega} \setminus \{0\}$ and vanishing on $\partial\Omega \setminus \{0\}$ ? In the sequel we denote $\rho(x) = \mathrm{dist}\,(x, \partial\Omega)$ and for $1 \leq s < \infty$, $L^s_\rho(\Omega) := L^s(\Omega; \rho dx)$ and the space of test functions in $\Omega$ is defined by

$$\mathbb{X}(\Omega) = \left\{\zeta \in C^1(\overline{\Omega}) : \zeta = 0 \text{ on } \partial\Omega,\, \Delta\zeta \in L^\infty(\Omega)\right\}. \tag{1.6}$$

If $\Omega$ is replaced by $\mathbb{R}^N_+$, then

$$\mathbb{X}(\mathbb{R}^N_+) = \left\{\zeta \in C^1(\overline{\mathbb{R}}^N_+) \text{ with compact support in } \overline{\mathbb{R}}^N_+,\, \Delta\zeta \in L^\infty(\mathbb{R}^N_+)\right\}. \tag{1.7}$$

Our first result is the following:

**Theorem 1.1** *Assume $p \geq \frac{N+1}{N-1}$, $M > 0$ and*
*(i) either $p = \frac{N+1}{N-1}$ and $1 < q < 1 + \frac{1}{N}$.*
*(ii) or $p > \frac{N+1}{N-1}$ and $1 < q \leq \frac{2p}{p+1}$.*
*Then any nonnegative solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{0\})$ of*

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^q = 0 \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{in } \partial\Omega \setminus \{0\} \end{aligned} \tag{1.8}$$

*verifies $\nabla u \in L^q_\rho(\Omega)$, $u \in L^p_\rho\Omega)$ and is a weak solution of*

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^q = 0 \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{in } \partial\Omega, \end{aligned} \tag{1.9}$$

*in the sense that*

$$\int_\Omega \left(-u\Delta\zeta + (|u|^{p-1}u - M|\nabla u|^q)\zeta\right) dx = 0 \quad \text{for all } \zeta \in \mathbb{X}(\Omega). \tag{1.10}$$

*Furthermore, if we assume either (i), or*

*(iii)* $p > \frac{N+1}{N-1}$ *and* $1 < q < \frac{2p}{p+1}$ *or*
*(iv)* $p > \frac{N+1}{N-1}$, $q = \frac{2p}{p+1}$ *and*

$$M < m^{**} := (p+1)\left(\frac{(N-1)p-(N+1)}{2p}\right)^{\frac{p}{p+1}}, \tag{1.11}$$

*then* $u = 0$.

This result is optimal in the case $p = \frac{N+1}{N-1}$, $q = \frac{2p}{p+1}$ as we will see in Section 4. Combining the method used in proving Theorem 1.1 with the result of [18] we prove the removability of compact boundary sets on $\partial\Omega$, provided they satisfy some zero Bessel capacity property.

**Theorem 1.2** *Assume* $p > \frac{N+1}{N-1}$ *and* $\frac{N+1}{N-1} < r < p$. *If one of the following conditions is satisfied:*
*(i)- either* $q = \frac{2p}{p+1}$ *and*

$$M < m_r^{**} := (p+1)\left(\frac{p-r}{p(r-1)}\right)^{\frac{p}{p+1}} \tag{1.12}$$

*(ii)- or* $1 < q < \frac{2p}{p+1}$, $r \leq 3$ *and* $M$ *is arbitrary.*
*Then if* $K \subset \partial\Omega$ *is a compact set such that* $cap_{\frac{2}{r},r'}^{\partial\Omega}(K) = 0$, *any solution* $u$ *of*

$$\begin{array}{rl} -\Delta u + |u|^{p-1}u - M|\nabla u|^q = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \setminus K, \end{array} \tag{1.13}$$

*is identically* 0.

Note that $m_{\frac{N+1}{N-1}}^{**} = m^{**}$. The capacitary framework allows to consider the Dirichlet problem for (1.1)

$$\begin{array}{rl} -\Delta u + |u|^{p-1}u - M|\nabla u|^q = 0 & \text{in } \Omega \\ u = \mu & \text{in } \partial\Omega, \end{array} \tag{1.14}$$

where $\mu$ is a Radon measure on $\partial\Omega$. By a weak solution of (1.14) we understand a function $u \in L^1(\Omega) \cap L^p_\rho(\Omega)$ such that $|\nabla u| \in L^q_\rho(\Omega)$, which satisfies

$$\int_\Omega \left(-u\Delta\zeta + (|u|^{p-1}u - M|\nabla u|^q)\zeta\right) dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial \mathbf{n}} d\mu \quad \text{for all } \zeta \in \mathbb{X}(\Omega). \tag{1.15}$$

When the two exponents are super-critical with respect to the equations (1.2) and (1.4), the admissibility condition on the measure for (1.1) requires the introduction of two different Bessel capacities defined on Borel subsets of $\partial\Omega$.

**Theorem 1.3** *Let $p > 1$, $1 < q < 2$ and $\mu$ be a nonnegative Radon measure on $\partial\Omega$ which satisfies*

$$\mu(E) \leq C \inf \left\{ cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E), cap^{\partial\Omega}_{\frac{2}{p},p'}(E) \right\} \quad \text{for all Borel set } E \subset \partial\Omega, \tag{1.16}$$

*for some $C > 0$. Then there exists $c_0 > 0$ such that for any $0 < c \leq c_0$ there exists a nonnegative weak solution of* (1.14) *with boundary data $c\mu$. Furthermore the boundary trace of $u$ is the measure $c\mu$.*

The proof is based upon a non-standard application of the sub and supersolutions technique since it relies of the dynamical (and more natural) aspect of the boundary trace as it is exposed in [20]. Another surprising fact is the use of the equation

$$\begin{aligned} -\Delta u &= u^p && \text{in } \Omega \\ u &= \mu && \text{in } \partial\Omega, \end{aligned}$$

which yields key estimates for our construction. The theorem admits several variants the proof of which is based either on imbedding theorems or on properties of Bessel capacities.

**Corollary 1.4** *Assume $1 < p \leq 2\frac{N+1}{N-1}$ and $\frac{(N+1)p}{N+1+p} \leq q < 2$ with $q \leq p$. If $\mu$ is a nonnegative Radon measure on $\partial\Omega$ which satisfies, for some $C > 0$,*

$$\mu(E) \leq C cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E) \quad \text{for all Borel set } E \subset \partial\Omega, \tag{1.17}$$

*there the conclusions of Theorem 1.3 hold.*

The condition on the measure is also fulfilled under the following conditions.

**Corollary 1.5** *Let $p > \frac{N+1}{N-1}$ and $\frac{N+1}{N} < q < \frac{2p}{p+1}$. If $\mu$ is a nonnegative Radon measure on $\partial\Omega$ such that for some constant $C > 0$, there holds for any Borel set $E \subset \partial\Omega$,*

$$\mu(E) \leq C cap^{\partial\Omega}_{\frac{2}{p},p'}(E), \tag{1.18}$$

*then the conclusions of Theorem 1.3 hold.*

Since the exponents $p$ and $q$ can be separately super or sub-critical, or even both sub-critical, we have the following result in different configurations of exponents.

**Corollary 1.6** *Let $p > 1$, $1 < q < 2$ and $\mu \in \mathfrak{M}_+(\partial\Omega)$. There exists a function $u \in L^1(\Omega) \cap L^p_\rho(\Omega)$ such that $\nabla u \in L^q_\rho(\Omega)$ which is a weak solution to* (1.14) *in the following cases:*
*(i) When $p < \frac{N+1}{N-1}$, $q < \frac{N+1}{N}$ and there exists some $c_1 > 0$ such that $\|\mu\|_{\mathfrak{M}} \leq c_2$.*
*(ii) When $p < \frac{N+1}{N-1}$, $\frac{N+1}{N} \leq q < 2$ and $\mu$ satisfies* (1.17)*; in that case $\mu$ has to be replaced by $c\mu$ with $0 < c \leq c_2$, for some $c_2 > 0$, in problem* (1.14).
*(iii) When $p \geq \frac{N+1}{N-1}$, $q < \frac{N+1}{N}$, and $\mu$ satisfies $\|\mu\|_{\mathfrak{M}} \leq c_3$ for some $c_3 > 0$ and*

$$\mu(E) = 0 \quad \text{for all Borel set } E \subset \partial\Omega \text{ such that } cap^{\partial\Omega}_{\frac{2}{p},p'}(E) = 0. \tag{1.19}$$

In [7] the same authors study the problem

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^q &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega, \end{aligned} \tag{1.20}$$

where $\mu$ is a bounded Borel measure in $\Omega$. There too sufficient conditions for solving the problem involves Bessel capacities, but since the boundary trace argument is no longer valid, an intensive utilization of potential theory with various kernels has to be used.

In the sub-critical case (i) and when $\mu$ is a Dirac mass at 0 on the boundary we have no restriction on its weight.

**Theorem 1.7** *Assume $1 < p < \frac{N+1}{N-1}$ and $1 < q < \frac{N+1}{N}$. Then for any $k \geq 0$ there exists a minimal positive solution $u_k$ of*

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^q &= 0 && \textit{in } \mathbb{R}^N_+ \\ u &= 0 && \textit{in } \partial\mathbb{R}^N_+ \setminus \{0\}, \end{aligned} \tag{1.21}$$

*satisfying*

$$\lim_{x\to 0} \frac{u_k(x)}{P_{_N}(x)} = k \tag{1.22}$$

*where $P_{_N}(x) = c_{_N} x_{_N} |x|^{-N}$ is the Poisson kernel in $\mathbb{R}^N_+$. Furthermore this solution is unique among the positve solutions of (1.21)-(1.22) if $q \leq \frac{2p}{p+1}$. This function satisfies $u_k \in L^1_{loc}(\overline{\mathbb{R}^N_+}) \cap L^p_{loc}(\overline{\mathbb{R}^N_+}; x_{_N} dx)$, $\nabla u_k \in L^q_{loc}(\overline{\mathbb{R}^N_+}; x_{_N} dx)$ and*

$$\int_{\mathbb{R}^N_+} \left(-u_k \Delta\zeta + (u_k^p - M|\nabla u_k|^q)\zeta\right) dx = k\frac{\partial\zeta}{\partial x_{_N}}(0) \quad \textit{for all } \zeta \in \mathbb{X}(\mathbb{R}^N_+). \tag{1.23}$$

The proof is completely different from the ones of Theorem 1.3 and Corollary 1.6 and is based upon a delicate construction of supersolutions and subsolutions. A similar result holds if $\mathbb{R}^N_+$ is replaced by a bounded smooth domain $\Omega \subset \mathbb{R}^N_+$ such that $0 \in \partial\Omega$.

**Theorem 1.8** *Assume $1 < p < \frac{N+1}{N-1}$ and $0 < q < \frac{N+1}{N}$. Then for any $M > 0$ and $k > 0$ there exists a minimal solution $u_k \in C^1(\overline{\Omega} \setminus \{0\})$ of (1.8) satisfying*

$$\lim_{x\to 0} \frac{u_k(x)}{P_\Omega(x)} = k, \tag{1.24}$$

*where $P_\Omega$ is the Poisson kernel in $\Omega$. Furthermore $u_k \in L^1(\Omega) \cap L^p_\rho(\Omega)$, $\nabla u_k \in L^q_\rho(\Omega)$, and*

$$\int_\Omega \left(-u_k \Delta\zeta + (u_k^p - M|\nabla u_k|^q)\zeta\right) dx = -k\frac{\partial\zeta}{\partial \mathbf{n}}(0) \quad \textit{for all } \zeta \in \mathbb{X}(\Omega). \tag{1.25}$$

In order to study the behaviour of these solutions $u_k$ when $k \to \infty$ we have to introduce separable solutions of (1.1) in the model case $\mathbb{R}^N_+$. They are solutions of

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^{\frac{2p}{p+1}} = 0 \qquad & \text{in } \mathbb{R}^N_+ \\ u = 0 \qquad & \text{on } \partial\mathbb{R}^N_+ \setminus \{0\}, \end{aligned} \tag{1.26}$$

which have the following expression in spherical coordinates

$$u(r,\sigma) = r^{-\frac{2}{p-1}}\omega(\sigma) \qquad \text{for all } (r,\sigma) \in (0,\infty) \times S^{N-1}_+.$$

Put

$$\alpha = \frac{2}{p-1}, \tag{1.27}$$

and denote by $\Delta'$ and $\nabla'$ the Laplace-Beltrami operator and the spherical gradient, then $\omega$ satisfies

$$\begin{aligned} -\Delta'\omega + \alpha(N-2-\alpha)\omega + |\omega|^{p-1}\omega - M\left(\alpha^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{p+1}} = 0 \qquad & \text{in } S^{N-1}_+ \\ \omega = 0 \qquad & \text{in } \partial S^{N-1}_+. \end{aligned} \tag{1.28}$$

**Theorem 1.9** *There exists a positive solution $\omega$ to problem* (1.28) *if one of the following conditions is satisfied:*
*(i) either $1 < p < \frac{N+1}{N-1}$ and $M \geq 0$,*
*(ii) or $p = \frac{N+1}{N-1}$ and $M > 0$,*
*(iii) or $1 < p < 3$ or $p > \frac{N+1}{N-1}$, and $M \geq M_{N,p}$ for some explicit value $M_{N,p} > 0$.*

The positive solutions of (1.28) allow to characterize the limit $u_\infty$ of the solutions $u_k$ constructed in Theorem 1.7.

**Theorem 1.10** *Let $1 < p < \frac{N+1}{N-1}$, $1 < q < \frac{N+1}{N}$ and $M > 0$, then*

$$\lim_{x\to 0} \frac{u_\infty(x)}{P_{_N}(x)} = \infty. \tag{1.29}$$

*Furthermore*
*(i) If $1 < q < \frac{2p}{p+1}$*

$$\lim_{r\to 0} r^\alpha u_\infty(r,.) = \psi \quad \textit{uniformly on } S^{N-1}_+, \tag{1.30}$$

*where $\psi$ is the unique positive solution of*

$$\begin{aligned} -\Delta'\psi + \alpha(N-2-\alpha)\psi + |\psi|^{p-1}\psi = 0 \qquad & \textit{in } S^{N-1}_+ \\ \psi = 0 \qquad & \textit{in } \partial S^{N-1}_+, \end{aligned} \tag{1.31}$$

*and $u_\infty$ is the unique positive function solution of* (1.21) *and satisfying* (1.29).
*(ii) If $q = \frac{2p}{p+1}$*

$$\lim_{r\to 0} r^\alpha u_\infty(r,.) = \omega \quad \textit{uniformly on } S^{N-1}_+, \tag{1.32}$$

*where $\omega$ is the minimal positive solution of* (1.28).

A similar result holds if $\mathbb{R}^N_+$ is replaced by a bounded smooth domain $\Omega \subset \mathbb{R}^N_+$, which boundary contains 0. In that case we assume that $T_{\partial\Omega}(0) = \partial\mathbb{R}^N_+$ (i.e. $\partial\mathbb{R}^N_+$ is the tangent hyperplane to $\partial\Omega$ at 0 in order to use the spherical coordinates $(r, \sigma)$ as above. Finally, if $(p, q) = \left(\frac{N+1}{N-1}, \frac{N+1}{N}\right)$ and $\partial\Omega$ is "very flat" near 0 in the sense that $\text{dist}\,(x, T_{\partial\Omega}(0)) \leq c|x|^N$ for all $x \in \partial\Omega$ close to 0, we prove that the function $u_\infty$ defined in the previous theorem still satisfies (1.32). Note that the above flatness condition is always satisfied if $N = 2$ since $\partial\Omega$ is locally the graph of a $C^k$ real valued function ($k \geq 2$) defined on $T_{\partial\Omega}(0) \cap B_\delta$ and degenerate at 0.

When $\frac{2p}{p+1} < q < \min\{2, p\}$, the situation is completely changed and the solutions with strong boundary blow-up are modelized by equation (1.3). If $1 < q < 2$ we set

$$\beta = \frac{2-q}{q-1}, \tag{1.33}$$

and if $1 < q < p$

$$\gamma = \frac{q}{p-q}. \tag{1.34}$$

We prove the following result in the statement of which $\phi_1$ denotes the first eigenfunction of $-\Delta'$ in $W^{1,2}_0(S^{N-1}_+)$.

**Theorem 1.11** *Assume $M > 0$ and $\frac{2p}{p+1} < q < \min\{2, p\}$. Then there exists a positive solution $u$ of (1.1) in $\mathbb{R}^N_+$, which vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$ such that*

$$m\phi_1(\sigma)r^{-\gamma} \leq u(r, \sigma) \leq c_4 \max\left\{r^{-\alpha}, M^{\frac{1}{p-q}} r^{-\gamma}\right\} \quad \text{for all } (r, \sigma) \in (0, r^*) \times \overline{S^{N-1}_+}. \tag{1.35}$$

*for some $m > 0$, $r^* \in (0, \infty]$ and where $c_4 = c_4(N, p, q) > 0$. If $Nq \geq (N-1)p$, $r^* = \infty$.*

Note that our construction which is made by mean of supersolutions and subsolutions does not imply that in the case $\frac{2p}{p+1} < q < \frac{N+1}{N}$, the solution $u_\infty$ obtained in Theorem 1.10 satisfies (1.35). A similar result holds if $\mathbb{R}^N_+$ is replaced by a bounded smooth domain $\Omega \subset \mathbb{R}^N_+$, such that $0 \in \partial\Omega$, under the flatness condition $\text{dist}\,(x, T_{\partial\Omega}(0)) \leq c|x|^{\gamma+1}$ for $x \in \partial\Omega$ near 0.

In the sequel $C > 0$ denotes a constant the value of which can change from one occurence to another and $c_j$ ($j = 0, 1, 2, ...$) a more specific positive constant the value of which depends of more precise elements such as $p, q, N$ or other previous constants $c_i$.

In a forthcomming article [8] we study the isolated singularities of positive solution in a domains. Due to the number of parameters even the radial solutions present an amazing rich complexity.

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## 2 Singular boundary value problems

### 2.1 A priori estimates

We give two series of estimates for solutions of (1.1) with a boundary singularity according to the sign of $M$.

**Theorem 2.1** *Let $\Omega$ be a domain such that $0 \in \partial\Omega$, $M \in \mathbb{R}$ and $1 < q < \min\{p, 2\}$. If $u \in C^1(\overline{\Omega} \setminus \{0\})$ is a solution of (1.1) vanishing on $\partial\Omega \setminus \{0\}$, there holds*
*1- If $M > 0$, there exists $= c_5(N, p, q) > 0$ such that*

$$u_+(x) \le c_5 \max\left\{ M^{\frac{1}{p-q}} |x|^{-\frac{q}{p-q}}, |x|^{-\frac{2}{p-1}} \right\} \qquad \text{for all } x \in \Omega. \tag{2.1}$$

*2- If $M \le 0$, there exist $c_6 = c_6(N, q) > 0$ and $c_7 = c_7(N, p) > 0$ such that*

$$u_+(x) \le \min\left\{ c_6 |M|^{-\frac{1}{q-1}} |x|^{-\frac{2-q}{q-1}}, c_7 |x|^{-\frac{2}{p-1}} \right\} \qquad \text{for all } x \in \Omega. \tag{2.2}$$

*Proof.* We first assume that $\overline{\Omega} \subset B_{R_0}$ for some $R_0 > 0$. Let $\epsilon > 0$, we set

$$j_\epsilon(r) = \begin{cases} 0 & \text{if } r \le 0 \\ \frac{r^2}{2\epsilon} & \text{if } 0 \le r \le \epsilon \\ r - \frac{\epsilon}{2} & \text{if } r \ge \epsilon. \end{cases}$$

If we extend $u$ by 0 in $\overline{\Omega}^c \cap B_{2R_0}$ and set $v_\epsilon = j_\epsilon(u)$ we have

$$\begin{aligned} -\Delta v_\epsilon + v_\epsilon^p - M|\nabla v_\epsilon|^q &= -j'_\epsilon(u)\Delta u - j''_\epsilon(u)|\nabla u|^2 + (j_\epsilon(u))^p - M(j'_\epsilon(u))^q |\nabla u|^q \\ &\le M j'_\epsilon(u) \left(1 - (j'_\epsilon(u))^{q-1}\right) |\nabla u|^q + (j_\epsilon(u))^p - j'_\epsilon(u) u_+^p \\ &\le M \frac{u}{\epsilon} \left(1 - \frac{u^{q-1}}{\epsilon^{q-1}}\right) |\nabla v_\epsilon|^q \chi_{\{0<u<\epsilon\}}. \end{aligned}$$

Letting $\epsilon \to 0$, we deduce from the dominated convergence theorem that $v_0 = \lim_{\epsilon \to 0} v_\epsilon$ is nonnegative (actually it is the extension of $u^+$ by 0 outside $\overline{\Omega} \setminus \{0\}$) and satisfies

$$Lv_0 := -\Delta v_0 + v_0^p - M|\nabla v_0|^q \le 0 \quad \text{in } \mathcal{D}'(B_{2R_0} \setminus \{0\}). \tag{2.3}$$

*The case $M > 0$.* Following the method of Keller [16] and Osserman [23], we fix $a \in B_{R_0} \setminus \{0\}$, and introduce $U(x - a) = \lambda(|a|^2 - |x - a|^2)^{-b}$ for some $b > 0$. Then putting $r = |x - a|$ and $\tilde{U}(r) = U(x - a)$, we have

$$\begin{aligned} L\tilde{U} &= -\tilde{U}'' - \frac{N-1}{r}\tilde{U}' - M|\tilde{U}'|^q + \tilde{U}^p \\ &= \lambda(|a|^2 - r^2)^{-2-b} \left[ \lambda^{p-1}(|a|^2 - r^2)^{2-b(p-1)} + 2b(N - 2(b+1))r^2 - 2Nb|a|^2 \right. \\ &\qquad \left. - M 2^q b^q \lambda^{q-1} r^q (|a|^2 - r^2)^{2+b-q(b+1)} \right]. \end{aligned}$$

If $M > 0$, the two necessary conditions on $b$ to be fulfilled in order $\tilde{U}$ be a supersolution in $B_{|a|}(a)$ are

$$(i) \qquad 2 - b(p-1) \le 0 \Longleftrightarrow b(p-1) \ge 2,$$

$$(ii) \qquad 2 + b - q(b+1) \ge 2 - b(p-1) \Longleftrightarrow b(p-q) \ge q.$$

The above inequalities are satisfied if

$$b = \max\left\{\frac{2}{p-1}, \frac{q}{p-q}\right\} = \max\{\alpha, \gamma\}. \tag{2.4}$$

If $q > \frac{2p}{p+1}$ then $b = \frac{q}{p-q}$ and

$$L\tilde{U} \ge \lambda\left(|a|^2 - r^2\right)^{-\frac{2p-q}{p-q}} \left[\lambda^{q-1}\left(\lambda^{p-q} - M2^q b^q r^q\right)\left(|a|^2 - r^2\right)^{\frac{2p-q(p+1)}{p-q}} - (3b+1)N|a|^2\right].$$

There exists $c_5 > 0$ depending on $N$, $p$ and $q$ such that if we choose

$$\lambda = c_5 \max\left\{M^{\frac{1}{p-q}}|a|^{\frac{q}{p-q}}, |a|^{\frac{2p(q-1)}{(p-1)(p-q)}}\right\},$$

there holds

$$L\tilde{U} \ge 0. \tag{2.5}$$

Since $\tilde{U}(x) \to \infty$ when $|x| \to |a|$, we obtain by the maximum principle (see [24] for a ) that $v_0 \le \tilde{U}$ in $B_{|a|}(a)$. In particular

$$u_+(a) = v_0(a) \le \tilde{U}(a) = \lambda|a|^{-\frac{2q}{p-q}} = c_5 \max\left\{M^{\frac{1}{p-q}}|a|^{-\frac{q}{p-q}}, |a|^{-\frac{2}{p-1}}\right\}. \tag{2.6}$$

If $q \le \frac{2p}{p+1}$ then $b = \frac{2}{p-1}$ and

$$\begin{aligned} L\tilde{U} &\ge \lambda\left(|a|^2 - r^2\right)^{-\frac{2p}{p-1}} \left[\lambda^{p-1} + \frac{2}{p-1}\left(N - \frac{2(p+1)}{p-1}\right)r^2 - \frac{2N}{p-1}|a|^2 \right. \\ &\qquad \left. - M2^q\left(\frac{2}{p-1}\right)^q \lambda^{q-1} r^q \left(|a|^2 - r^2\right)^{\frac{2p-q(p+1)}{p-1}}\right] \\ &\ge \lambda\left(|a|^2 - r^2\right)^{-\frac{2p}{p-1}} \left[\lambda^{p-1} - C|a|^2 - C'\lambda^{q-1}M|a|^{\frac{4p-q(p+3)}{p-1}}\right]. \end{aligned}$$

Hence, if $q = \frac{2p}{p+1}$, (2.5) holds if for some $c_5 = c_5(N,p,q) > 0$,

$$\lambda = c_5 \max\left\{M^{\frac{p+1}{p(p-1)}}, 1\right\}|a|^{\frac{2}{p-1}},$$

which yields

$$u_+(a) = v_0(a) \le \tilde{U}(a) = \lambda|a|^{-\frac{4}{p-1}} = c_5 \max\left\{M^{\frac{p+1}{p(p-1)}}, 1\right\}|a|^{-\frac{2}{p-1}}. \tag{2.7}$$

While if $q < \frac{2p}{p+1}$, we choose

$$\lambda = c_5 \max\left\{ M^{\frac{1}{p-q}} |a|^{\frac{4p-q(p+3)}{(p-1)(p-q)}}, |a|^{\frac{2}{p-1}} \right\},$$

where $c_5 > 0 = c_5(N,p,q)$, which yields

$$u_+(a) = v_0(a) \le \tilde{U}(a) = \lambda |a|^{-\frac{4}{p-1}} = c_5 \max\left\{ M^{\frac{1}{p-q}} |a|^{-\frac{q}{p-q}}, |a|^{-\frac{2}{p-1}} \right\}. \tag{2.8}$$

*The case* $M \le 0$. We first assume that $M < 0$. By [22, Lemma 3.3] $v_0$ satisfies

$$-\Delta v_0 + |M||\nabla v_0|^q \le 0 \qquad \text{in } \mathcal{D}'(B_{2R_0} \setminus \{0\}). \tag{2.9}$$

Therefore, since $1 < q < 2$,

$$u_+(a) = v_0(a) \le c_6 |M|^{-\frac{1}{q-1}} |a|^{-\frac{2-q}{q-1}}. \tag{2.10}$$

If $M \le 0$ there also holds

$$-\Delta v_0 + v_0^p \le 0 \qquad \text{in } \mathcal{D}'(B_{2R_0} \setminus \{0\}). \tag{2.11}$$

Hence

$$u_+(a) = v_0(a) \le c_7 |a|^{-\frac{2}{p-1}}. \tag{2.12}$$

In the above inequalities $c_6 = c_6(q,N) > 0$ and $c_7 = c_7(p,N) > 0$. Combining these estimates we derive

$$u_+(a) \le \min\left\{ c_7 |a|^{-\frac{2}{p-1}}, c_6 |M|^{-\frac{1}{q-1}} |a|^{-\frac{2-q}{q-1}} \right\}. \tag{2.13}$$

Since the estimate is independent of $R_0$, the assumption that $\overline{\Omega} \subset B_{R_0}$ is easily ruled out. This ends the proof. □

*Remark.* If $M = 0$, estimate (2.1) is just

$$u_+(x) \le c_7 |x|^{-\frac{2}{p-1}}. \tag{2.14}$$

If $M < 0$, (2.14) is valid what ever is the value of $q$. Furthermore there also holds

$$u_+(x) \le c_6 |M| |x|^{-\frac{2-q}{q-1}}, \tag{2.15}$$

whatever is the value of $p$, provided $1 < q < 2$.

The equation is not invariant by $u \mapsto -u$, hence the lower and upper estimates are not symmetric.

**Corollary 2.2** *Under the assumptions of Theorem 2.1, there holds*
*1- If* $M > 0$

$$\begin{aligned} -\min\left\{c_6|M|^{-\frac{1}{q-1}}|x|^{-\frac{2-q}{q-1}}, c_7|x|^{-\frac{2}{p-1}}\right\} &\le u(x) \\ &\le c_5 \max\left\{M^{\frac{1}{p-q}}|x|^{-\frac{q}{p-q}}, |x|^{-\frac{2}{p-1}}\right\} \quad \text{for all } x \in \Omega. \end{aligned} \tag{2.16}$$

*2- If* $M \le 0$, *there exist* $c_6 = c_6(N,q) > 0$ *and* $c_7 = c_7(N,p) > 0$ *such that*

$$\begin{aligned} -c_5 \max\left\{M^{\frac{1}{p-q}}|x|^{-\frac{q}{p-q}}, |x|^{-\frac{2}{p-1}}\right\} &\le u(x) \\ &\le \min\left\{c_6|M|^{-\frac{1}{q-1}}|x|^{-\frac{2-q}{q-1}}, c_7|x|^{-\frac{2}{p-1}}\right\} \quad \text{for all } x \in \Omega. \end{aligned} \tag{2.17}$$

We infer from Theorem 2.1 an estimate of the gradient of $u$ near 0.

**Theorem 2.3** *Let* $\Omega$ *be a smooth bounded domain such that* $0 \in \partial\Omega$ *and* $T_{\partial\Omega}(0) = \partial\mathbb{R}^N_+$, $M > 0$, $p > 1$ *and* $1 < q < \min\{2, p\}$. *If* $u \in C^1(\overline{\Omega} \setminus \{0\})$ *is a nonnegative solution of* (1.1) *vanishing on* $\partial\Omega \setminus \{0\}$, *for any* $r_0 > 0$ *there holds there exists* $c_8 = c_8(N, p, q, \Omega, r_0, M) > 0$ *such that*

$$|\nabla u(x)| \le c_8 \max\left\{|x|^{-\frac{p}{p-q}}, |x|^{-\frac{p+1}{p-1}}\right\} \qquad \text{for all } x \in \Omega \cap B_{r_0}. \tag{2.18}$$

*The restriction that* $|x| \le 1$ *is not needed if* $q = \frac{2p}{p+1}$.

*Proof.* We assume first that $B_2^+ \subset \Omega$.
*Case 1:* $1 < q \le \frac{2p}{p+1}$. For $0 < r < 1$ we set

$$u(x) = r^{-\frac{2}{p-1}} u_r(\tfrac{x}{r}) = r^{-\frac{2}{p-1}} u_r(y) \quad \text{with } y = \tfrac{x}{r}.$$

If $\frac{r}{2} < |x| < 2r$, then $\frac{1}{2} < |y| < 2$ and $u_r > 0$ satisfies

$$-\Delta u_r + u_r^p - M r^{\frac{2p-q(p+1)}{p-1}} |\nabla u_r|^q = 0 \qquad \text{in } B_2^+ \setminus B^+_{\frac{1}{2}},$$

and vanishes on $\partial(B_2^+ \setminus B^+_{\frac{1}{2}})$. Since $0 < M r^{\frac{2p-q(p+1)}{p-1}} \le M$ as $2p - q(p+1) \ge 0$, by the standard regularity theory we have the estimate

$$\max\left\{|\nabla u_r(z)| : \tfrac{2}{3} < |z| < \tfrac{3}{2}\right\} \le c_9 \max\left\{|u_r(z)| : \tfrac{1}{2} < |z| < 2\right\}, \tag{2.19}$$

where $c_9$ depends on $N, p, q$ and $M$. Now it follows that

$$\max\left\{|u_r(z)| : \tfrac{1}{2} < |z| < 2\right\} \le 2^{\frac{2}{p-1}} c_5 \max\left\{M^{\frac{1}{p-q}} r^{\frac{2p-q(p+1)}{(p-1)(p-q)}}, 1\right\},$$

by (2.1). Therefore

$$\begin{aligned} \max\left\{|\nabla u(y)| : \tfrac{r}{2} < |y| < 2r\right\} &\le 2^{\frac{2}{p-1}} c_5 c_9 r^{-\frac{p+1}{p-1}} \max\left\{M^{\frac{1}{p-q}} r^{\frac{2p-q(p+1)}{(p-1)(p-q)}}, 1\right\} \\ &\le c_8 \max\left\{|x|^{-\frac{p}{p-q}}, |x|^{-\frac{p+1}{p-1}}\right\}, \end{aligned} \tag{2.20}$$

which is (2.18).

*Case 2:* $\frac{2p}{p+1} < q < 2$. For $0 < r < 1$ we set

$$u(x) = r^{-\frac{2-q}{q-1}} u_r(\tfrac{x}{r}) = r^{-\frac{2-q}{q-1}} u_r(y) \quad \text{with } y = \tfrac{x}{r}.$$

If $\frac{r}{2} < |x| < 2r$, then $\frac{1}{2} < |y| < 2$ and $u_r > 0$ satisfies

$$-\Delta u_r + r^{\frac{q(p+1)-2p}{q-1}} u_r^p - M|\nabla u_r|^q = 0 \qquad \text{in } B_2^+ \setminus B_{\frac{1}{2}}^+,$$

We notice that $q(p+1) - 2p > 0$. Then inequality (2.19) holds. Now

$$\max\left\{|u_r(z)| : \tfrac{1}{2} < |z| < 2\right\} \le c_9' r^{\frac{2-q}{q-1}} \max\left\{r^{-\frac{2}{p-1}}, r^{-\frac{q}{p-q}}\right\},$$

thus

$$\max\left\{|\nabla u_r(z)| : \tfrac{2}{3} < |z| < \tfrac{3}{2}\right\} \le c_9'' r^{\frac{2-q}{q-1}-1} \max\left\{r^{-\frac{2}{p-1}}, r^{-\frac{q}{p-q}}\right\}, \tag{2.21}$$

which implies

$$\max\left\{|\nabla u(x)| : \tfrac{2r}{3} < |x| < \tfrac{3r}{2}\right\} \le c_8 \max\left\{r^{-\frac{p+1}{p-1}}, r^{-\frac{p}{p-q}}\right\}. \tag{2.22}$$

*The general case;* If $\partial\Omega$ is not flat near 0 we proceed as in the proof of [22, Lemma 3.4], using the same scaling as in the flat case which transform the domain $B_2^+ \setminus B_1^+$) into $(B_2 \setminus B_1) \cap \frac{1}{r}\Omega$, the curvature of which is bounded when $0 < r < 1$. The same estimates holds, up to the value of the constant $c_8$ and we derive (2.18). □

As a consequence we have the following.

**Corollary 2.4** *Under the assumptions of Theorem 2.3 the function $u$ satisfies*

$$u(x) \le c_8 \rho(x) \max\left\{M^{\frac{1}{p-q}}|x|^{-\frac{p}{p-q}}, |x|^{-\frac{p+1}{p-1}}\right\} \qquad \textit{for all } x \in \Omega \cap B_1. \tag{2.23}$$

*The restriction that $|x| \le 1$ is not needed if $q = \frac{2p}{p+1}$.*

## 2.2 Removable singularities

*Proof of Theorem 1.1.* If $M \le 0$, $u$ is a nonnegative subsolution of $-\Delta u + v^p = 0$ which vanishes on $\partial\Omega \setminus \{0\}$, hence it is identically zero by [15].

*Step 1.* We assume $M > 0$. It is straightforward to verify from estimates (2.18) that under conditions (i) or (ii), $|\nabla u(x)| \le c_8|x|^{-a}$ with $a \le N$. Since these conditions imply $q < \frac{N+1}{N}$, it follows that $|\nabla u|^q \in L^1_\rho(\Omega)$.

For any $\epsilon > 0$ we denote by $w_\epsilon$ the solution of

$$\begin{array}{rl} -\Delta w + w^p = M|\nabla u|^q & \text{in } \Omega_\epsilon := \Omega \cap \overline{B}_\epsilon^c \\ w = 0 & \text{in } \partial\Omega \cap \overline{B}_\epsilon^c \\ \lim\limits_{|x|\to\epsilon} w(x) = \infty & \text{on } \partial B_\epsilon \cap \Omega, \end{array} \tag{2.24}$$

which exists since $|\nabla u|^q \in L^1_\rho(\Omega)$, see [19]. Then $u \le w_\epsilon$ in $\Omega_\epsilon$. Let $z_\epsilon$ be the solution of

$$\begin{array}{rl} -\Delta z + z^p = 0 & \text{in } \Omega_\epsilon \\ z = 0 & \text{in } \partial\Omega \cap \overline{B}_\epsilon^c \\ \lim\limits_{|x|\to\epsilon} z(x) = \infty & \text{on } \partial B_\epsilon \cap \Omega. \end{array} \tag{2.25}$$

Denote by $\mathbb{G}_\Omega[.]$ the Green operator in $\Omega$. Since $z_\epsilon + M\mathbb{G}_\Omega[|\nabla u|^q]\lfloor_{\Omega_\epsilon}$ is a supersolution of (2.24) in $\Omega_\epsilon$ we deduce

$$u \le z_\epsilon + M\mathbb{G}_\Omega[|\nabla u|^q]\lfloor_{\Omega_\epsilon} \qquad \text{in } \Omega_\epsilon. \tag{2.26}$$

When $\epsilon \to 0$, $z_\epsilon$ decreases to $z_0$ which satisfies

$$\begin{array}{rl} -\Delta w + w^p = 0 & \text{in } \Omega \\ w = 0 & \text{in } \partial\Omega \setminus \{0\}. \end{array} \tag{2.27}$$

Since $p \ge \frac{N+1}{N-1}$ it is proved in[15] that any solution of (2.27) extends as a continuous solution in $\Omega$ with boundary value 0, hence $z_0 = 0$ by the maximum principle. Therefore $u \le M\mathbb{G}_\Omega[|\nabla u|^q]$ in $\Omega$ and the boundary trace $Tr_{\partial\Omega}[u]$ of $u$ is zero. By [20] the fact that $|\nabla u|^q \in L^1_\rho(\Omega)$ jointly with $Tr_{\partial\Omega}[u] = 0$ implies in turn that $u^p \in L^1_\rho(\Omega)$ and $u$ is a weak solution of

$$\begin{array}{rl} -\Delta u + u^p = M|\nabla u|^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{2.28}$$

in the sense that there holds

$$\int_\Omega (-u\Delta\zeta + u^p\zeta - M|\nabla u|^q\zeta)\, dx = 0 \qquad \forall \zeta \in W^{2,\infty}(\Omega) \cap C^1_c(\overline{\Omega}). \tag{2.29}$$

*Step 2.* Let us assume that $p > \frac{N+1}{N-1}$. If $u$ is nonnegative and not identically zero, then by the maximum principle it is positive in $\Omega$. We set $u = v^b$ with $0 < b \le 1$. Then

$$-\Delta v - (b-1)\frac{|\nabla v|^2}{v} + \frac{1}{b}v^{(p-1)b+1} - Mb^{q-1}v^{(b-1)(q-1)}|\nabla v|^q = 0. \tag{2.30}$$

For $\epsilon > 0$,

$$v^{(b-1)(q-1)}|\nabla v|^q \le \frac{q\epsilon^{\frac{2}{q}}}{2}\frac{|\nabla v|^2}{v} + \frac{2-q}{2\epsilon^{\frac{2}{2-q}}}v^{\frac{(2b-1)q-2(b-1)}{2-q}}.$$

Therefore

$$-\Delta v+\left(1-b-M\frac{qb^{q-1}\epsilon^{\frac{2}{q}}}{2}\right)\frac{|\nabla v|^2}{v}+\frac{1}{b}v^{(p-1)b+1}-Mb^{q-1}\frac{2-q}{2\epsilon^{\frac{2}{2-q}}}v^{\frac{(2b-1)q-2(b-1)}{2-q}}=0. \tag{2.31}$$

We notice that the following relation is independent of $b$

$$\frac{(2b-1)q-2(b-1)}{2-q}\leq (p-1)b+1\Longleftrightarrow q\leq\frac{2p}{p+1},$$

with simultaneous equality. We take

$$(p-1)b+1=\frac{N+1}{N-1}\Longleftrightarrow b=\frac{2}{(N-1)(p-1)}, \tag{2.32}$$

hence $p>\frac{N+1}{N-1}$ if and only if $0<b<1$.
We first assume that $0<q<\frac{2p}{p+1}$ and choose $\epsilon>0$ such that

$$1-b-M\frac{qb^{q-1}\epsilon^{\frac{2}{q}}}{2}=0\Longleftrightarrow\epsilon=\left(\frac{2(1-b)}{Mqb^{q-1}}\right)^{\frac{q}{2}}=\left(\frac{2((N-1)p-N-1)}{Mqb^{q-1}(N-1)(p-1)}\right)^{\frac{q}{2}}. \tag{2.33}$$

This transforms (2.31) into

$$-\Delta v+\frac{(N-1)(p-1)}{2}v^{\frac{N+1}{N-1}}-\frac{(2-q)b^{\frac{2(q-1)}{2-q}}}{2}\left(\frac{q}{2(1-b)}\right)^{\frac{q}{2-q}}M^{\frac{2}{2-q}}v^{\frac{(2b-1)q-2(b-1)}{2-q}}\leq 0. \tag{2.34}$$

Then, as

$$\frac{(2b-1)q-2(b-1)}{2-q}<\frac{N+1}{N-1},$$

there exists $A>0$, depending on $M$ and $b$, such that

$$-\Delta v+\frac{(N-1)(p-1)}{4}v^{\frac{N+1}{N-1}}\leq A. \tag{2.35}$$

Since $v$ vanishes on $\partial\Omega\setminus\{0\}$, $\tilde{v}=(v-c_{10}A^{\frac{N-1}{N+1}})_+^{\frac{N+1}{N-1}}$ with $c_{10}=\left(\frac{4}{(N-1)(p-1)}\right)^{\frac{N+1}{N-1}}$ satisfies

$$-\Delta\tilde{v}+\frac{(N-1)(p-1)}{4}\tilde{v}^{\frac{N+1}{N-1}}\leq 0. \tag{2.36}$$

By [15], $\tilde{v}=0$ which implies $v\leq c_{10}A^{\frac{N-1}{N+1}}$ and therefore $u(x)\leq c_{11}A^{\frac{2}{(N+1)(p-1)}}$ in $\Omega$. Since $u$ vanishes on $\partial\Omega\setminus\{0\}$ we extend it in a neighborhood of 0 by odd reflection trough $\partial\Omega$ and denote by $\tilde{u}$ the new function defined in $B_\alpha$ where it satisfies

$$-divA(x,\nabla\tilde{u})+\tilde{u}^p+B(x,\nabla\tilde{u})=0\qquad\text{in } B_\alpha\setminus\{0\}. \tag{2.37}$$

In this expression the operator $A : (x;\xi) \in B_\alpha \times \mathbb{R}^N \mapsto A(x,\xi) \in \mathbb{R}^N$ is smooth in $x$ in $B_\alpha$ and linear in $\xi$ (see e.g. [4, Lemma 2.5] in a more general setting), it and satisfies for all $(x;\xi) \in B_\alpha \times \mathbb{R}^N$,

$$A(x,\xi).\xi \geq 2|\xi|^2 \text{ and } |A(x,\xi)| \leq 4|\xi| \quad \text{for all } (x;\xi) \in B_\alpha \times \mathbb{R}^N.$$

Since we can write $|B(.,\nabla\tilde{u})| \leq 2|\nabla\tilde{u}|^q = 2|\nabla\tilde{u}|^{q-1}|\nabla\tilde{u}| = C(x)|\nabla\tilde{u}|$ in $B_\alpha$, then $B : (x,\xi) \in B_\alpha \times \mathbb{R}^N \mapsto B(x,\xi) \in \mathbb{R}$ verifies

$$|B(x,\xi)| \leq C(x)|\xi|,$$

and $C(x) \leq 2c_8|x|^{-\frac{(p+1)(q-1)}{p-1}}$ by Theorem 2.3. Since $q < \frac{2p}{p+1}$, $\frac{(p+1)(q-1)}{p-1} < 1$. Hence $C \in L^{N+\tau}$ for some $\tau > 0$. By Serrin's theorem [25, Theorem 10] the singularity at 0 is removable and $\tilde{u}$ can be extended as a regular solution of (2.37) in $B_\alpha$. Hence $\tilde{u} \in C^1(B_{\frac{\alpha}{2}})$, and as a consequence $u \in C^1(\overline{\Omega})$. If $u$ is not zero, it is positive in $\Omega$ and achieves its maximum at some $x_0 \in \Omega$ where $\Delta u(x_0) \leq 0$ and $\nabla u(x_0) = 0$. Contradiction.

Next we assume that $q = \frac{2p}{p+1}$. By the choice of $b$ in (2.32), inequality (2.31) becomes

$$-\Delta v + \left(1 - b - \frac{Mpb^{\frac{p-1}{p+1}}\epsilon^{\frac{p+1}{p}}}{p+1}\right)\frac{|\nabla v|^2}{v} + \left(\frac{1}{b} - \frac{Mb^{\frac{p-1}{p+1}}}{(p+1)\epsilon^{p+1}}\right)v^{(p-1)b+1} \leq 0. \quad (2.38)$$

We need to make both coefficients positive so that we obtain

$$\begin{aligned} -\Delta v + \tau v^{\frac{N+1}{N-1}} &\leq 0 \qquad \text{in } \Omega \\ v &= 0 \qquad \text{on } \partial\Omega \setminus \{0\}. \end{aligned} \quad (2.39)$$

We first choose

$$\epsilon^{\frac{p+1}{p}} > \left(\frac{M}{p+1}\right)^{\frac{1}{p}} b^{\frac{2}{p+1}},$$

say

$$\epsilon^{\frac{p+1}{p}} = \left(\frac{M}{p+1}\right)^{\frac{1}{p}} b^{\frac{2}{p+1}} + \tilde{\epsilon}, \quad (2.40)$$

with $\tilde{\epsilon} > 0$ so that the coefficient of $v^{\frac{N+1}{N-1}}$ is positive, and we can choose $\tilde{\epsilon}$ thanks to the assumption $m^{**} > M$: we have

$$\begin{aligned} 1 - b - \frac{Mpb^{\frac{p-1}{p+1}}}{p+1}\left(\left(\frac{M}{p+1}\right)^{\frac{1}{p}} b^{\frac{2}{p+1}} + \tilde{\epsilon}\right) &= 1 - b - \left(\frac{M}{p+1}\right)^{\frac{p+1}{p}} pb - \frac{Mpb^{\frac{p+1}{p-1}}}{p+1}\tilde{\epsilon} \\ &= b\left(\frac{1-b}{b} - \left(\frac{M}{p+1}\right)^{\frac{p+1}{p}} p\right) - \frac{Mpb^{\frac{p+1}{p-1}}}{p+1}\tilde{\epsilon} \\ &= pb\left(\frac{(N-1)p-(N+1)}{2p} - \left(\frac{M}{p+1}\right)^{\frac{p+1}{p}}\right) - \frac{Mpb^{\frac{p+1}{p-1}}}{p+1}\tilde{\epsilon} \\ &= pb\left(\left(\frac{m^{**}}{p+1}\right)^{\frac{p+1}{p}} - \left(\frac{M}{p+1}\right)^{\frac{p+1}{p}}\right) - \frac{Mpb^{\frac{p+1}{p-1}}}{p+1}\tilde{\epsilon} \end{aligned} \quad (2.41)$$

and the right-hand side is positive if $\tilde{\epsilon}$ small enough. Hence we obtain (2.39). By [15], $v = 0$ and the same holds for $u$. This ends the case $p > \frac{N+1}{N-1}$.

*Step 3.* Finally we assume $p = \frac{N+1}{N-1}$ and $1 < q < \frac{2p}{p+1} = \frac{N+1}{N}$, then

$$M|\nabla u(x)|^q \le c_{12}|x|^{-q\frac{p+1}{p-1}} = c_{12}|x|^{-qN} := c_{13}Q(x).$$

Hence $u \le u_1 := c_{13}\mathbb{G}_\Omega[Q]$. At this point we need the following intermediate result:

*Claim.* Assume $w_\alpha = \mathbb{G}_\Omega[Q_\alpha]$ where $Q_\alpha(x) = |x|^{-\alpha}$ with $\alpha < N + 1$, then

$$w_\alpha(x) \le c_\alpha|x|^{2-\alpha} \qquad \text{for all } \ x \in \Omega. \tag{2.42}$$

If this holds true, then $u(x) \le c_{13}c_{qN}|x|^{2-qN}$. By the scaling method of Theorem 2.3, we obtain

$$|\nabla u(x)| \le c_8c_{13}c_{qN}|x|^{1-qN} \Longrightarrow |\nabla u(x)|^q \le c_{14}|x|^{q(1-qN)} := c_{14}Q_{q(Nq-1)}(x), \tag{2.43}$$

and thus

$$w_{q(Nq-1)}(x) = c_{14}\mathbb{G}_\Omega[Q_{q(Nq-1)}](x) \le c_{14}c_{q(Nq-1)}|x|^{2-q(Nq-1)} \qquad \text{for all } \ x \in \Omega. \tag{2.44}$$

Since $q < 1 + \frac{1}{N}$, $q(Nq-1) - 2 < Nq - 2$. Iterating this process, we finally obtain that $u$ is bounded and we end the proof as in Step 2. □

*Remark.* It is noticeable that the equation exhibits a phenomenon which is characteristic of Emden-Folwer type equations

$$\Delta u = u^p \quad \text{in } \ B_1 \setminus \{0\}. \tag{2.45}$$

If $u$ is nonnegative then there exists $a \ge 0$ such that

$$\Delta u = u^p + a\delta_0 \quad \text{in } \ \mathcal{D}'(B_1). \tag{2.46}$$

If $1 < p < \frac{N}{N-2}$ then $a$ can be positive, but if $p \ge \frac{N}{N-2}$, then $a = 0$. This means that the singularity cannont be seen in the sense of distributions, however there truly exist singular solutions, e.g. if $p > \frac{N}{N-2}$,

$$u_s(x) = c_{N,p}|x|^{-\frac{2}{p-1}}. \tag{2.47}$$

A similar phenomenon exists for solutions of

$$\begin{aligned} -\Delta u &= u^p \quad \text{in } \ B_1^+ \\ u &= 0 \quad \text{in } \ \partial B_1^+ \setminus \{0\}. \end{aligned} \tag{2.48}$$

In such a case the critical value is $\frac{N+1}{N-1}$ since for $p \ge \frac{N+1}{N-1}$ the boundary value is achieved in the sense of distributions in $\partial B_1^+$.

## 2.3 Proof of Theorem 1.2

As in Theorem 1.1, the proof differs according to whether $0 < q < \frac{2p}{p+1}$ or $q = \frac{2p}{p+1}$, and we first assume that $u > 0$. We perform the same change of unknown as in the previous theorem putting $u = v^b$, but now we choose $b$ as follows

$$(p-1)b + 1 = r \Longleftrightarrow b = \frac{r-1}{p-1}, \tag{2.49}$$

and we first assume that

$$1 - b - M\frac{qb^{q-1}\epsilon^{\frac{2}{q}}}{2} = 0 \Longleftrightarrow \epsilon = \left(\frac{2(1-b)}{Mqb^{q-1}}\right)^{\frac{q}{2}} = \left(\frac{2(p-r)}{Mq(p-1)b^{q-1}}\right)^{\frac{q}{2}}. \tag{2.50}$$

Hence (2.34) becomes

$$-\Delta v + \frac{p-1}{r-1}v^r - \frac{(2-q)b^{q-1}}{2}\left(\frac{q}{2(1-b)}\right)^{\frac{q}{2-q}} M^{\frac{2}{2-q}} v^{\frac{(2r-p-1)q+2(p-r)}{(p-1)(2-q)}} \leq 0. \tag{2.51}$$

The condition $r \geq \frac{(2r-p-1)q+2(p-r)}{(p-1)(2-q)}$ is equivalent to $2p - q(p+1) \leq r(2p - q(p+1))$ since $1 < r < p$.
Assuming first that $q < \frac{2p}{p+1}$, we obtain from (2.51)

$$-\Delta v + \frac{p-1}{2(r-1)}v^r \leq A. \tag{2.52}$$

for some constant $A \geq 0$. Since $cap^{\partial\Omega}_{\frac{2}{r},r'}(K) = 0$ and $v$ vanishes on $\partial\Omega \setminus K$, it follows from [18] that $v \leq cA^{\frac{1}{r}}$ for some $c > 0$, hence $u$ is also uniformly bounded above in $\Omega$ by some constant $a$. Next we have to show that $\nabla u \in L^2(\Omega)$. We also denote by $\Phi_1$ the first eigenfunction of $-\Delta$ in $W^{1,2}_0(\Omega)$ normalized by $\sup \Phi_1 = 1$ and by $\lambda_1$ the corresponding eigenvalue. Since $\frac{N+1}{N-1} < r \leq 3$ we infer from [1, Theorem 5.5.1], that

$$\left(cap^{\partial\Omega}_{\frac{1}{2},2}(K)\right)^{\frac{1}{N-2}} \leq B\left(cap^{\partial\Omega}_{\frac{2}{r},r'}(K)\right)^{\frac{1}{N-1-\frac{2}{r-1}}}.$$

Therefore $cap^{\partial\Omega}_{\frac{2}{r},r'}(K) = 0$ implies $cap^{\partial\Omega}_{\frac{1}{2},2}(K) = 0$ and there exists a decreasing sequence $\{\zeta_n\} \subset C^2_0(\partial\Omega)$ such that $\zeta_n = 1$ in a neighborhood of $K$, $0 \leq \zeta_n \leq 1$ and $\|\zeta_n\|_{W^{1,2}(\partial\Omega)} \to 0$ when $n \to \infty$, furthermore $\zeta_n \to 0$ quasi everywhere. Let $\mathbb{P}_\Omega : C^2(\partial\Omega) \mapsto C^2(\overline{\Omega})$ be the Poisson operator. It is an admissible lifting in the sense of [18, Section 1] in the sense that

$$\mathbb{P}_\Omega[\eta]\lfloor_{\partial\Omega} = \eta \quad \text{and} \quad \eta \geq 0 \Longrightarrow \mathbb{P}_\Omega[\eta] \geq 0.$$

Put $\eta_n = 1 - \zeta_n$. Then, multiplying equation (1.13) by $u(\mathbb{P}_\Omega[\eta_n])^2$ and integrating, we obtain

$$\int_\Omega |\nabla u|^2(\mathbb{P}_\Omega[\eta_n])^2 dx + 2\int_\Omega u\mathbb{P}_\Omega[\eta_n]\nabla u.\nabla\mathbb{P}_\Omega[\eta_n]dx$$
$$+ \int_\Omega u^{p+1}(\mathbb{P}_\Omega[\eta_n])^2 dx - M\int_\Omega |\nabla u|^q u(\mathbb{P}_\Omega[\eta_n])^2 dx = 0,$$

which implies

$$\int_\Omega |\nabla u|^2(\mathbb{P}_\Omega[\eta_n])^2dx - 2\left(\int_\Omega |\nabla u|^2(\mathbb{P}_\Omega[\eta_n])^2dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla \mathbb{P}_\Omega[\eta_n]|^2u^2dx\right)^{\frac{1}{2}} + \int_\Omega u^{p+1}(\mathbb{P}_\Omega[\eta_n])^2dx - M\int_\Omega |\nabla u|^q u(\mathbb{P}_\Omega[\eta_n])^2dx \leq 0.$$

It is standard that

$$\int_\Omega |\nabla \mathbb{P}_\Omega[\eta_n]|^2dx \leq c_{12}\left\|\eta_n\right\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} = A_n.$$

Set $X_n = \|\mathbb{P}_\Omega[\eta_n]|\nabla u|\|_{L^2}$, then

$$X_n^2 - 2A_nX_n - Ma|\Omega|^{\frac{2-q}{2}}X_n^q \leq 0.$$

Hence there exist two positive real numbers $a_1$ and $a_2$ depending only on $q$, $|\Omega|$ and $a = \|u\|_{L^\infty}$ such that

$$X_n \leq a_1A_n^{\frac{1}{q-1}} + a_2M^{\frac{1}{2-q}}. \tag{2.53}$$

Now $A_n \to 0$ and $X_n \to \|\nabla u\|^2_{L^2}$, therefore by Fatou's Lemma

$$|\Omega|^{1-\frac{2}{q}}\|\nabla u\|^2_{L^q} \leq \|\nabla u\|^2_{L^2} \leq a_2M^{\frac{1}{2-q}} < \infty.$$

Let $\zeta \in C^1_0(\overline{\Omega})$ and $\eta_n$ as above. Since $\eta_n$ vanishes in a neighborhood of $K$ and $\zeta$ vanishes on $\partial\Omega$,

$$\int_\Omega \mathbb{P}_\Omega[\eta_n]\nabla u.\nabla\zeta dx + \int_\Omega \zeta\nabla u.\nabla\mathbb{P}_\Omega[\eta_n]dx + \int_\Omega u^p\zeta\mathbb{P}_\Omega[\eta_n]dx = M\int_\Omega |\nabla u|^q\zeta\mathbb{P}_\Omega[\eta_n]dx.$$

Letting $n$ to infty and using the fact that $\nabla u \in L^2(\Omega)$ and $\nabla\mathbb{P}_\Omega[\eta_n] \to 0$ in $L^2(\Omega)$, we derive

$$\int_\Omega \nabla u.\nabla\zeta dx + \int_\Omega u^p\zeta dx = M\int_\Omega |\nabla u|^q\zeta dx.$$

Hence $u$ is a nonnegative bounded weak solution of

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^q &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.54}$$

It is therefore $C^2$. Again, by the maximum principle we see that $u$ cannot achieve a positive maximum in $\Omega$, this yields a contradiction.
Next we assume $q = \frac{2p}{p+1}$. We choose $b = \frac{r-1}{p-1}$ and (2.38) becomes

$$-\Delta v + \left(1 - b - \frac{Mpb^{\frac{p-1}{p+1}}\epsilon^{\frac{p+1}{p}}}{p+1}\right)\frac{|\nabla v|^2}{v} + \left(\frac{1}{b} - \frac{Mb^{\frac{p-1}{p+1}}}{(p+1)\epsilon^{p+1}}\right)v^r \leq 0. \tag{2.55}$$

From there the argument is similar to the one of Step 2-Case $q = \frac{2p}{p+1}$ in the proof of Theorem 1.1: we claim that for some suitable choices the function $v$ satisfies

$$\begin{aligned} -\Delta v + \tau v^r &\le 0 \qquad \text{in } \Omega \\ v &= 0 \qquad \text{in } \partial\Omega \setminus K. \end{aligned}$$

We first choose $\epsilon > 0$ so that (2.40) holds, hence the coefficient of $v$, say $\tau$ is positive. Then the expression

$$1 - b - \frac{Mpb^{\frac{p-1}{p+1}}\epsilon^{\frac{p+1}{p}}}{p+1} = \frac{p(r-1)}{p-1}\left(\left(\frac{m_r^{**}}{p+1}\right)^{\frac{p+1}{p}} - \left(\frac{M}{p+1}\right)^{\frac{p+1}{p}}\right) - \frac{Mpb^{\frac{p+1}{p-1}}}{p+1}\tilde{\epsilon} \tag{2.56}$$

is positive provided $\tilde{\epsilon} > 0$ is small enough. Since $cap^{\partial\Omega}_{\frac{2}{r},r'}(K) = 0$ it follows from [18] that $v = 0$. Hence $u = 0$, which ends the proof. □

## 2.4 Measure boundary data

Let $\mu$ be a nonnegative Radon measure on $\partial\Omega$. The results concerning the following two types of equations

$$\begin{aligned} -\Delta v + v^p &= 0 \qquad \text{in } \Omega \\ v &= \mu \qquad \text{in } \partial\Omega, \end{aligned} \tag{2.57}$$

and

$$\begin{aligned} -\Delta w &= M|\nabla w|^q \qquad \text{in } \Omega \\ w &= c\mu \qquad \text{in } \partial\Omega, \end{aligned} \tag{2.58}$$

allow us to consider the measure boundary data for equation (1.1). We recall the results concerning (2.57) and (2.58).
1- Assume $p > 1$. If $\mu$ satisfies

$$\text{For all Borel set } E \subset \partial\Omega,\ cap^{\partial\Omega}_{\frac{2}{p},p'}(E) = 0 \Longrightarrow \mu(E) = 0, \tag{2.59}$$

then problem (2.57) admits a necessarily unique weak solution $v := v_\mu$, see [18], i.e. $v_\mu \in L^1(\Omega) \cap L^p_\rho(\Omega)$ and for any function $\zeta \in \mathbb{X}(\Omega) := \left\{\eta \in C^1_0(\overline{\Omega}) \text{ s.t. } \Delta\eta \in L^\infty(\Omega)\right\}$, there holds

$$\int_\Omega (-v\Delta\zeta + v^p\zeta)\, dx = -\int_\Omega \frac{\partial\zeta}{\partial\mathbf{n}} d\mu. \tag{2.60}$$

Notice that there is no condition on $\mu$ if $1 < p < \frac{N+1}{N-1}$.
2- Assume $1 < q < 2$. If there exists $C > 0$ such that $\mu$ satisfies

$$\text{For all Borel set } E \subset \partial\Omega,\ \mu(E) \le C cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E), \tag{2.61}$$

then problem (2.58) admits at least a positive solution $w$ for $c > 0$ small enough, see [9, Theorem 1.3], in the sense that $w \in L^1(\Omega)$, $\nabla w \in L^q_\rho(\Omega)$ and for any $\zeta \in \mathbb{X}(\Omega)$, there holds

$$\int_\Omega (-w\Delta\zeta - M|\nabla w|^q\zeta)\, dx = -\int_\Omega \frac{\partial\zeta}{\partial\mathbf{n}} d\mu. \tag{2.62}$$

Notice that if $1 < q < \frac{N+1}{N}$ there is no capacitary condition on $\mu$.

We use also the following result.

**Lemma 2.5** *Let $p > \frac{N+1}{N-1}$ and $\mu \in \mathfrak{M}_+(\partial\Omega)$. If $\mu \in W^{-\frac{2}{p},p}(\partial\Omega)$, then there exists $C > 0$ such that*

$$\mu(E) \leq C \left( cap^{\partial\Omega}_{\frac{2}{p},p'}(E) \right)^{\frac{1}{p'}} \quad \text{for all Borel set } E \subset \partial\Omega. \tag{2.63}$$

*Conversely, if $\mu$ satisfies*

$$\mu(E) \leq C cap^{\partial\Omega}_{\frac{2}{p},p'}(E) \quad \text{for all Borel set } E \subset \partial\Omega, \tag{2.64}$$

*for some $C > 0$, then $\mu \in W^{-\frac{2}{p},p}(\partial\Omega)$.*

*Proof.* Assume $\mu \in W^{-\frac{2}{p},p}(\partial\Omega) \cap \mathfrak{M}_+(\partial\Omega)$. If $E$ is a compact subset of $\partial\Omega$ and $\zeta \in C^2(\partial\Omega)$ with $0 \leq \zeta \leq 1$, with $\zeta = 1$ on $E$, then

$$\mu(E) \leq \int_{\partial\Omega} \zeta d\mu = \langle \mu, \zeta \rangle \leq \|\mu\|_{W^{-\frac{2}{p},p}} \|\zeta\|_{W^{\frac{2}{p},p'}}.$$

Therefore, by the definition of the capacity,

$$\mu(E) \leq \|\mu\|_{W^{-\frac{2}{p},p}} \left( cap^{\partial\Omega}_{\frac{2}{p},p'}(E) \right)^{\frac{1}{p'}}.$$

Conversely, if (2.64) holds, then there exists $c_{16}$ such that for any $0 < c \leq c_{16}$ there exists a $z_{c\mu}$ to

$$\begin{aligned} -\Delta z &= z^p \qquad \text{in } \Omega \\ z &= c\mu \qquad \text{in } \partial\Omega, \end{aligned} \tag{2.65}$$

(see [9, Theorem 1.5]) in the sense that $z_{c\mu} \in L^1(\Omega) \cap L^p_\rho(\Omega)$ and $c\mathbb{P}_\Omega[\mu] \leq z_{c\mu}$. Hence $\mathbb{P}_\Omega[\mu] \in L^p_\rho(\Omega)$, which implies $\mu \in W^{-\frac{2}{p},p}(\Omega)$ by [18]. □

Those weak solutions are characterized by their boundary trace. Let $\Sigma_\epsilon = \{x \in \Omega : \rho(x) = \epsilon > 0\}$ and $\Sigma_0 = \partial\Omega$. For $0 < \epsilon \leq \epsilon_0$ the hypersurfaces $\Sigma_\delta$ defines a foliation of the set $\Omega_{\epsilon_0} = \{x \in \Omega : 0 < \rho(x) \leq \epsilon_0\}$. Let $\pi(x)$ be the orthogonal projection of $x \in \Omega_{\epsilon_0}$ on $\partial\Omega$. Then $|x - \pi(x)| = \rho(x)$ and $\mathbf{n}_x = (\rho(x))^{-1}(\pi(x) - x)$. The mapping

$$x \mapsto \Pi(x) = (\rho(x), \pi(x)),$$

from $\Omega_{\epsilon_0}$ onto $(0, \epsilon_0] \times \Sigma_0$ is a $C^2$ diffeomorphism and the restriction $\Pi_\epsilon$ of $\Pi$ to $\Sigma_\epsilon$ is a $C^2$ diffeomorphism from $\Sigma_\epsilon$ onto $\Sigma_0$. Let $dS_\epsilon$ be the surface measure on $\Sigma_\epsilon$, then a continuous function $u$ defined in $\Omega$ has boundary trace the Radon measure $\mu$ on $\partial\Omega$ if

$$\lim_{\epsilon \to 0} \int_{\Sigma_\epsilon} u Z dS_\epsilon = \int_\Sigma Z d\mu \qquad \text{for all } Z \in C(\overline{\Omega}). \tag{2.66}$$

Equivalently, if $\zeta \in C(\partial\Omega)$ and $\zeta_\epsilon = \zeta \circ \Pi_\epsilon^{-1} \in C(\Sigma_\epsilon)$, then

$$\lim_{\epsilon\to 0} \int_{\Sigma_\epsilon} u\zeta_\epsilon dS_\epsilon = \int_\Sigma \zeta d\mu \qquad \text{for all } \ \zeta \in C(\partial\Omega). \tag{2.67}$$

The functions $v_\mu$ solution of (2.57) and $w$ solution of (2.58) admit for respective boundary trace $\mu$ and $c\mu$. Furthermore, for the equations in (2.57) and (2.58), the existence of a boundary trace of a positive solution is equivalent to the fact that $v_\mu \in L^1(\Omega) \cap L^p_\rho(\Omega)$ and $w \in L^1(\Omega)$ with $\nabla w \in L^q_\rho(\Omega)$ respectively.

*Proof of Theorem 1.3.* If we assume that (1.16) holds, the measure $\mu$ is Lipschitz continuous with respect to $cap^{\partial\Omega}_{\frac{2}{p},p'}$ and $cap^{\partial\Omega}_{\frac{2-q}{q},q'}$. By [9, Theorem 1.3] there exists $c_{17} > 0$ such that for any $0 < c \le c_{17}$ there exists a weak solution $w = w_{c\mu}$ to (2.58) and there holds for some positive constant $c_{18}$ depending on $q$ and $\Omega$

$$w_{c\mu} \le c_{18} c\mathbb{P}_\Omega[\mu]. \tag{2.68}$$

By [18] there exists a unique solution $v_{c\mu}$ to (2.57) with $\mu$ replaced by $c\mu$. The functions $w_{c\mu}$ and $v_{c\mu}$ are respectively supersolution and subsolution of (2.57) with boundary data $c\mu$ and there holds,

$$v_{c\mu} \le c\mathbb{P}_\Omega[\mu] \le w_{c\mu} \tag{2.69}$$

Hence there exists a nonnegative function $u$ satisfying (1.1) and such that

$$0 \le v_{c\mu} \le u \le w_{c\mu} \le c_{18} c\mathbb{P}_\Omega[\mu]. \tag{2.70}$$

Moreover $v_{c\mu} \in L^p_\rho(\Omega)$ and $\nabla w_{c\mu} \in L^q_\rho(\Omega)$. Because $v_{c\mu}$ and $w_{c\mu}$ have boundary trace $c\mu$ in the sense of (2.66) and (2.67), the function $u$ has the same property and we denote it by $u_{c\mu}$. Assuming that $c \le \min\{c_{16}, c_{17}\}$, there exists also $z_{c\mu}$ solution of (2.65) which satisfies $z_{c\mu} \in L^p_\rho(\Omega)$ and $c\mathbb{P}_\Omega[\mu] \le z_{c\mu}$ by the maximum principle. Therefore $w_{c\mu} \in L^p_\rho(\Omega)$ and finally $u_{c\mu} \in L^p_\rho(\Omega)$.

Let $\phi = \mathbb{G}_\Omega[u^p_{c\mu}]$, then $\phi \ge 0$ and

$$-\Delta(u_{c\mu} + \phi) = |\nabla u_{c\mu}|^q.$$

The function $u_{c\mu} + \phi$ is a nonnegative superharmonic function in $\Omega$. By Doob's theorem [12, Chapter II], $-\Delta(u_{c\mu} + \phi) \in L^1_\rho(\Omega)$. Hence $|\nabla u_{c\mu}| \in L^q_\rho(\Omega)$. This implies that $u_{c\mu}$ is a weak solution of (1.15). □

*Proof of Corollary 1.4.* We give two proofs, both interesting, the first one based on the theory of Bessel capacities and the second one on harmonic analysis.

*First proof.* Since inequality $\frac{p(N+1)}{N+1+p} > \frac{2p}{p+2}$ always holds as $N \ge 2$, and $\frac{2}{p} > \frac{2-q}{q}$ is equivalent $q > \frac{2p}{p+2}$, we note that $q \ge \frac{p(N+1)}{N+1+p}$ implies $\frac{2}{p} > \frac{2-q}{q}$. Therefore the Bessel space $\mathcal{L}_{\frac{2}{p},p'}(\partial\Omega)$ (constructed by using local charts on $\partial\Omega$ as it is indicated in [9, p.

3]) is continuously imbedded into the Bessel space $\mathcal{L}_{\frac{2-q}{q},q'}(\partial\Omega)$ (see e.g. [1], [27]), since

$$\frac{1}{q'} \geq \frac{1}{p'} - \frac{1}{N-1}\left(\frac{2}{p} - \frac{2-q}{q}\right) \Longleftrightarrow q \geq \frac{p(N+1)}{N+1+p} \tag{2.71}$$

$$\|\zeta\|_{\mathcal{L}_{\frac{2-q}{q},q'}} \leq c_{20} \|\zeta\|_{\mathcal{L}_{\frac{2}{p},p'}} \quad \text{for all } \zeta \in C_0^2(\partial\Omega). \tag{2.72}$$

Let $K \subsetneq \partial\Omega$ be a compact set and $\zeta_n$ a sequence of nonnegative functions in $C_0^2(\partial\Omega)$ such that $0 \leq \zeta_n \leq 1$, $\zeta_n \geq 1$ on $K$ and such that

$$\|\zeta_n\|^{p'}_{\mathcal{L}_{\frac{2}{p},p'}} \downarrow cap^{\partial\Omega}_{\frac{2}{p},p'}(K) \quad \text{as } n \to \infty.$$

Then

$$\left(cap^{\partial\Omega}_{\frac{2-q}{q},q'}(K)\right)^{\frac{p'}{q'}} \leq \liminf_{n\to\infty} \|\zeta_n\|^{p'}_{\mathcal{L}_{\frac{2-q}{q},q'}} \leq c_{20}^{p'} cap^{\partial\Omega}_{\frac{2}{p},p'}(K).$$

Since $q \leq p$, we deduce

$$\left(cap^{\partial\Omega}_{\frac{2-q}{q},q'}(\partial\Omega)\right)^{\frac{p'}{q'}-1} cap^{\partial\Omega}_{\frac{2-q}{q},q'}(K) \leq \left(cap^{\partial\Omega}_{\frac{2-q}{q},q'}(K)\right)^{\frac{p'}{q'}} \leq c_{20}^{p'} cap^{\partial\Omega}_{\frac{2}{p},p'}(K).$$

Hence

$$cap^{\partial\Omega}_{\frac{2-q}{q},q'}(K) \leq c_{21} cap^{\partial\Omega}_{\frac{2}{p},p'}(K). \tag{2.73}$$

Combining (2.73) and (1.17) we infer

$$\begin{aligned}\mu(E) \leq C cap^{\partial\Omega}_{\frac{2-q}{q},q'}(K) &= C \min\left\{cap^{\partial\Omega}_{\frac{2-q}{q},q'}(K), c_{21} cap^{\partial\Omega}_{\frac{2}{p},p'}(K)\right\} \\ &\leq C \max\{1, c_{21}\} \min\left\{cap^{\partial\Omega}_{\frac{2-q}{q},q'}(K), cap^{\partial\Omega}_{\frac{2}{p},p'}(K)\right\}\end{aligned}$$

then (1.16) holds. The proof follows by Theorem 1.3.

*Second proof.* Under the assumption (1.17) there exists a weak solution $w_{c\mu}$ of (2.58) which satisfies $\mathbb{P}_\Omega[\mu] \leq w_{c\mu}$ and $\nabla w_{c\mu} \in L^q_\rho(\Omega)$. By classical imbedding theorems between weighted Sobolev spaces (see e.g. [27])

$$\left(\int_\Omega |w_{c\mu}|^p \rho dx\right)^{\frac{1}{p}} \leq C\left(\int_\Omega |\nabla w_{c\mu}|^q \rho dx\right)^{\frac{1}{q}} + \int_\Omega |w_{c\mu}| \rho dx, \tag{2.74}$$

for all $r > 1$ such that

$$\frac{1}{p} \geq \frac{1}{q} - \frac{1}{N+1} > 0 \Longleftrightarrow q > \frac{p(N+1)}{N+1+p}.$$

Furthermore the condition $q < 2$ yields $p < \frac{2(N+1)}{N-1}$. Under this condition, $\mathbb{P}_\Omega[\mu] \in L^p_\rho(\Omega)$. By [18] $\mu \in W^{-\frac{2}{p},p}(\partial\Omega)$. Again by [18] there exists a solution $v_{c\mu}$ to problem

(2.57) and inequality (2.69) holds. The end of the proof is similar to the one of Theorem 1.3. □

*Proof of Corollary 1.5.* We recall the formulation of [1, Theorem 5.5.1] in our framework. There exists a constant $A > 0$ such that if $E \subset \partial\Omega$ is a Borel set, then

$$\left(cap^{\partial\Omega}_{\frac{2}{p},p'}(E)\right)^{\frac{p-1}{(N-1)(p-1)-2}} \leq A\left(cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E)\right)^{\frac{q-1}{(N-1)(q-1)+q-2}} \text{ if } \frac{2p'}{p} \leq \frac{(2-q)q'}{q} < N-1. \tag{2.75}$$

Now

$$\frac{2p'}{p} \leq \frac{(2-q)q'}{q} < N-1 \Longleftrightarrow \frac{N+1}{N} < q \leq \frac{2p}{p+1} \text{ and } p > \frac{N+1}{N-1}.$$

Since $\frac{(q-1)((N-1)(p-1)-2)}{(p-1)((N-1)(q-1)+q-2)} \geq 1$ as $q \leq \frac{2p}{p+1}$, we infer

$$cap^{\partial\Omega}_{\frac{2}{p},p'}(E) \leq A'\left(cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E)\right)^{\frac{(q-1)((N-1)(p-1)-2)}{(p-1)((N-1)(q-1)+q-2)}} \leq c_{22}cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E). \tag{2.76}$$

Then

$$\begin{aligned}\mu(E) \leq Ccap^{\partial\Omega}_{\frac{2}{p},p'}(K) &= C\min\left\{cap^{\partial\Omega}_{\frac{2}{p},p'}(K), c_{22}cap^{\partial\Omega}_{\frac{2-q}{q},q'}(K)\right\} \\ &\leq C\max\{1,c_{22}\}\min\left\{cap^{\partial\Omega}_{\frac{2-q}{q},q'}(K), cap^{\partial\Omega}_{\frac{2}{p},p'}(K)\right\}.\end{aligned}$$

This implies the claim. □

*Remark.* If $\frac{N+1}{N} = q < \frac{2p}{p+1}$ then [1, Theorem 5.5.1] yields

$$\left(\ln\frac{A}{cap^{\partial\Omega}_{\frac{2}{p},p'}(E)}\right)^{-1} \leq A\left(cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E)\right)^{\frac{1}{N}}. \tag{2.77}$$

Therefore, if we assume that

$$\mu(E) \leq c_{17}\left(\ln\frac{A}{cap^{\partial\Omega}_{\frac{2}{p},p'}(E)}\right)^{-N}, \tag{2.78}$$

then $\mu$ is absolutely continuous with respect to $cap^{\partial\Omega}_{\frac{2}{p},p'}$ and Lipschitz continuous with respect to $cap^{\partial\Omega}_{\frac{2-q}{q},q'}$. Consequently there exist $v_{c\mu}$ and $w_{c\mu}$ weak solutions of (2.57) and (2.58) respectively, and they satisfy $0 \leq v_{c\mu} \leq w_{c\mu}$. Consequently there exists $u_{c\mu}$ which satisfies (1.1) such that $v_{c\mu} \leq u_{c\mu} \leq w_{c\mu}$. Therefore $u_{c\mu}$ has the same boundary trace $c\mu$. However we do not know if $u_{c\mu}$ belongs to $L^p_\rho(\Omega)$. Therefore it is not clear wether $u_{c\mu}$ is a weak solution of (1.15).

The proof in the partially sub-critical case is simpler.

*Proof of Corollary 1.6.* If $1 < p < \frac{N+1}{N-1}$ for any $\mu \in \mathfrak{M}_+(\partial\Omega)$ problem (2.57) admits a unique solution $v_\mu$ (see [15]). If $1 < q < \frac{N+1}{N}$, then there exists $a_0 > 0$ such that for any non-empty Borel set $E \subset \partial\Omega$, $cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E) \geq a_0$. Therefore

$$\mu(E) \leq \|\mu\|_{\mathfrak{M}} \leq \frac{\|\mu\|_{\mathfrak{M}}}{a_0} cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E).$$

It follows from [9, Theorem 1.3] that problem (2.58) admits a solution $w_\mu$ whenever $\|\mu\|_{\mathfrak{M}}$ is small enough. By [10] problem (2.65) admits a solution $z_\mu$ with $c\mu$ replaced by $\mu$ provided $\|\mu\|_{\mathfrak{M}}$ is small enough. Furthermore

$$w_\mu \leq \mathbb{P}_\Omega[\mu] \leq z_\mu. \tag{2.79}$$

Since $z_\mu \in L^p_\rho(\Omega)$, $w_\mu \in L^p_\rho(\Omega)$. Hence by the same arument as in Theorem 1.3, there exists a solution $u_\mu$ of (1.1) which satisfies $v_\mu \leq u_\mu \leq w_\mu$. Hence $u_\mu \in L^p_\rho(\Omega)$ and by the previous argument $\nabla u_\mu \in L^q_\rho(\Omega)$. This implies again that $u_\mu$ is a weak solution of (1.15).

If $1 < p < \frac{N+1}{N-1}$ and $\frac{N+1}{N} \leq q < 2$, then problem (2.57) is uniquely solvable for any $\mu \in \mathfrak{M}_+(\partial\Omega)$, while problem (2.65) admits a solution $z_\mu$ with $c\mu$ replaced by $\mu$ provided $\|\mu\|_{\mathfrak{M}}$ is small enough and since (2.61) holds, problem (2.58) admits a weak solution provided $0 < c \leq c_0$. Since (2.79) holds with $z_\mu \in L^p_\rho(\Omega)$, the result follows as above.

If $p \geq \frac{N+1}{N-1}$, $1 < q < \frac{N+1}{N}$ and $\mu \in \mathfrak{M}_+(\partial\Omega)$ absolutely continuous with respect to $cap^{\partial\Omega}_{\frac{2}{p},p'}$, there exists $u_\mu$ solution of (2.57) and $w_\mu$ solution of (2.58) provided $c\,\|\mu\|_{\mathfrak{M}}$ is small enough. Since $|\nabla w_\mu|^q \in L^1_\rho(\Omega)$ the function $w_\mu$ belongs to the Marcinkiewicz space $M^{\frac{N+1}{N-1}}_\rho(\Omega)$ (see eg. [31]). Since $M^{\frac{N+1}{N-1}}_\rho(\Omega) \subset L^p_\rho(\Omega)$ as $1 < p < \frac{N+1}{N-1}$, it implies that $w_\mu$ and therefore $u_\mu$, belongs to $L^p_\rho(\Omega)$. The end of the proof is as before. □

# 3 Separable solutions

Separable solutions of (1.1) in $\mathbb{R}^N \setminus \{0\}$ are solutions which have the form

$$u(x) = u(r,\sigma) = r^{-\kappa}\omega(\sigma) \qquad \text{for } (r,\sigma) \in \mathbb{R}_+ \times S^{N-1}.$$

This forces $q$ to be equal to $\frac{2p}{p+1}$, $\kappa$ to $\frac{2}{p-1}$ (recall that this defines $\alpha$) and $\omega$ satisfies

$$-\Delta'\omega + \alpha(N-2-\alpha)\omega + |\omega|^{p-1}\omega - M\left(\alpha^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{p+1}} = 0 \qquad \text{in } S^{N-1}. \tag{3.1}$$

Constant positive solutions are solutions of

$$X^{p-1} - M\alpha^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} + \alpha(N-2-\alpha) = 0. \tag{3.2}$$

This existence of solutions to (3.2) and their stability properties will be detailled in a forthcoming article [8]. The understanding of boundary singularities of solutions of (1.1) is conditioned by the knowledge of separable solutions in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N \setminus \{0\}$. Then $\omega$ is a solution of

$$\begin{aligned} -\Delta'\omega + \alpha(N-2-\alpha)\omega + |\omega|^{p-1}\omega - M\left(\alpha^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{p+1}} = 0 \qquad &\text{in } S^{N-1}_+ \\ \omega = 0 \qquad &\text{in } \partial S^{N-1}_+. \end{aligned} \tag{3.3}$$

### 3.1 Existence of singular solutions

We recall the following result proved in [32, Corollary 1.4.5] is a variant of Boccardo-Murat-Puel's result [11, Theorem 2.1] dealing with the quasilinear equation in a domain $G \subset \mathbb{R}^N$.

$$\mathcal{Q}(u) := -\Delta u + B(., u, \nabla u) = 0 \qquad \text{in } \mathcal{D}'(G), \tag{3.4}$$

where $B \in C(G \times \mathbb{R} \times \mathbb{R}^N)$ satisfies, for some continuous increasing function $\Gamma$ from $\mathbb{R}^+$ to $\mathbb{R}^+$,

$$|B(x, r, \xi)| \le \Gamma(|r|)(1 + |\xi|^2) \quad \text{for all } (x, r, \xi) \in G \times \mathbb{R} \times \mathbb{R}^N. \tag{3.5}$$

**Theorem 3.1** *Let $G$ be a bounded domain in $\mathbb{R}^N$. If there exist a supersolution $\phi$ and a subsolution $\psi$ of the equation $\mathcal{Q}v = 0$ belonging to $W^{1,\infty}(G)$ and such that $\psi \le \phi$, then for any $\chi \in W^{1,\infty}(G)$ satisfying $\psi \le \chi \le \phi$ there exists a function $u \in W^{1,2}(G)$ solution of $\mathcal{Q}u = 0$ such that $\psi \le u \le \phi$ and $u - \chi \in W^{1,2}_0(G)$.*

*Remark. Mutatis mutandi*, the same result holds if $\mathbb{R}^N$ is replaced by a Riemannian manifold.

Their result is actually more general since the Laplacian can be replaced by a quasilinear $p$-Laplacian-type operator and $B$ by a perturbation with the *natural* $p$-growth. This theorem has direct applications in the construction of solutions on $S^{N-1}_+$, but also for the construction of singular solutions in several configurations.

**Proposition 3.2** *Let $\Omega$ be a bounded smooth domain containing 0, $p > 1$, $1 \le q \le 2$ and $M \in \mathbb{R}$. Assume that equation*

$$-\Delta u + u^p - M|\nabla u|^q = 0, \tag{3.6}$$

*admits a radial positive and decreasing solution $v$ in $\mathbb{R}^N \setminus \{0\}$ satisfying*

$$\lim_{|x|\to 0} v(x) = \infty. \tag{3.7}$$

*Then there exists a positive function $u$ satisfying (3.6) in $\Omega \setminus \{0\}$, vanishing on $\partial\Omega$ and such that*

$$\left(v(x) - \max\{v(z) : |z| = \delta_0\}\right)_+ \le u(x) \le v(x) \quad \text{for all } x \in \Omega \setminus \{0\}. \tag{3.8}$$

*where $\delta_0 = \operatorname{dist}(0, \partial\Omega)$.*

*Proof.* Put $m = \max\{v(z) : |z| = \delta_0\} = v(\delta_0)$. The function $v_m = (v - m)_+$ is a radial subsolution of (3.6) in $\Omega$, positive in $B_{\delta_0} \setminus \{0\}$ and vanishing in $\Omega \setminus B_{\delta_0}$. For $\epsilon > 0$ set $\Omega_\epsilon = \Omega \setminus \overline{B}_\epsilon$. Since $v_m$ is dominated by the supersolution $v$, there exists a solution $u_\epsilon$ of (3.6) in $\Omega_\epsilon$ such that $v_m \le u_\epsilon \le v$ and $u_\epsilon - v_m \in H^1_0(\Omega_\epsilon)$. By standard regularity estimates, $u_\epsilon$ is $C^2$, hence it solves

$$\begin{array}{rll} -\Delta u_\epsilon + u_\epsilon^p - M|\nabla u_\epsilon|^q = 0 & \text{in } \Omega_\epsilon \\ u_\epsilon = v_m & \text{on } \partial B_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{array} \tag{3.9}$$

Notice that $u_\epsilon$ is unique by the comparison principle. If $0 < \epsilon' < \epsilon$ the function $u_{\epsilon'}$ solution of (3.9 ) in $\Omega_{\epsilon'}$ with the corresponding boundary data is larger than $v_m$ and in particular $u_{\epsilon'}\lfloor_{\partial B_\epsilon} \ge v_m\lfloor_{\partial B_\epsilon} = u_\epsilon\lfloor_{\partial B_\epsilon}$. Hence $u_{\epsilon'} \ge u_\epsilon$ in $\Omega_\epsilon$. When $\epsilon \downarrow 0$, $u_\epsilon$ increase and converges in the $C^{1,\theta}_{loc}(\overline{\Omega} \setminus \{0\})$-topology toward some function $u$ which satisfies (3.6) in $\Omega \setminus \{0\}$, is larger that $v_m$ and smaller than $v$, vanishes on $\partial\Omega$ and such that (3.9) holds. □

The previous result can be adapted to the study of solutions with a boundary singularity in bounded domains which are flat enough near the singular point or in $\mathbb{R}^N_+$.

**Proposition 3.3** *Let $p > 1$, $1 \le q \le 2$ and $M \in \mathbb{R}$. Assume that the equation* (3.6) *admits a positive solution $w$ in $\mathbb{R}^N_+$ belonging to $C(\overline{\mathbb{R}^N_+} \setminus \{0\})$, radially decreasing in $\mathbb{R}^N_+$ and satisfying*

$$\lim_{t\to 0} w(t\sigma) = \infty \quad \textit{uniformly on compact sets } K \subset S^{N-1}_+. \tag{3.10}$$

*Assume also*

*(i) either $w\lfloor_{\partial\mathbb{R}^N_+\setminus\{0\}}$ is bounded,*

*(ii) or $\Omega \subset \mathbb{R}^N_+$ is a bounded smooth domain such that $0 \in \partial\Omega$ starshapped with respect to 0 and such that $w\lfloor_{\partial\Omega\setminus\{0\}}$ is bounded.*

*Then there exists a positive function $u$ satisfying* (3.6) *in $\mathbb{R}^N_+$ in case (i), or $\Omega$ in case (ii) , vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ in case (i), or $\partial\Omega \setminus \{0\}$ in case (ii), and such that*

$$\left(w(x) - \sup\left\{w(z) : z \in \partial\mathbb{R}^N_+ \setminus \{0\}\right\}\right)_+ \le u(x) \le w(x) \quad \textit{for all } x \in \mathbb{R}^N_+, \tag{3.11}$$

*where* $K = \sup\left\{\limsup_{|z|\to\infty} w(z), \sup\left\{w(z) : z \in \partial\mathbb{R}^N_+ \setminus \{0\}\right\}\right\}$ *in case (i) or*

$$\left(w(x) - \sup\left\{w(z) : z \in \partial\Omega \setminus \{0\}\right\}\right)_+ \le u(x) \le w(x) \quad \textit{for all } x \in \Omega. \tag{3.12}$$

*in case (ii).*

*Proof.* The proof is a variant of the preceding one, only the geometry of the domains is changed.

In case (ii) set $m = \sup\{w(z) : z \in \partial\Omega \setminus \{0\}\}$. Then the function $z \mapsto w_m := (w(z) - m)_+$ is a subsolution of (3.6) in $\Omega$. It vanishes on $\partial\Omega \setminus \{0\}$ and is dominated by $w$. For $\epsilon < \delta_0$, let $\Omega_\epsilon$ denote $\Omega \cap \overline{B_\epsilon}^c$. We consider the problem of finding $u_\epsilon$ solution of

$$\begin{array}{ll} -\Delta u_\epsilon + u_\epsilon^p - M|\nabla u_\epsilon|^q = 0 & \text{in } \Omega_\epsilon \\ u_\epsilon = w_m & \text{on } \partial B_\epsilon \cap \Omega \\ u_\epsilon = 0 & \text{on } B_\epsilon^c \cap \partial\Omega. \end{array} \tag{3.13}$$

Again since $u_\epsilon - w_m \in H_0^1(\Omega_\epsilon)$ and since $w_m$ is smaller than $w\lfloor_{\Omega_\epsilon}$, the solution $u_\epsilon$ exists and it satisfies $w_m \le u_\epsilon \le w$ in $\Omega_\epsilon$. If $0 < \epsilon' < \epsilon$, $u_{\epsilon'}\lfloor_{\partial\Omega_\epsilon} \ge u_\epsilon\lfloor_{\partial\Omega_\epsilon} = v_m$. Hence $u_{\epsilon'} \ge \epsilon$ in $\Omega_\epsilon$. As in the proof of Proposition 3.2 the sequence $\{u_\epsilon\}$ is relatively compact in the $C^{1,\theta}_{loc}(\overline{\Omega} \setminus \{0\})$-topology, which ends the proof.

In case (i), for $n > 0$ set $K_n = \sup\{w(z) : z \in \partial B_n^+ \setminus \{0\}\}$ where, we recall it, $B_n^+ = B_n \cap \mathbb{R}^N_+$. The function $w_{K_n} = (w - K_n)_+$ is a subsolution of (3.6) in $B_n^+$ which vanishes on $\partial B_n^+ \setminus \{0\}$ and is smaller than $w$. For $0 < \epsilon < n$ we denote by $u_{\epsilon,n}$ the unique function satisfying

$$\begin{array}{ll} -\Delta u_{\epsilon,n} + u_{\epsilon,n}^p - M|\nabla u_{\epsilon,n} e|^q = 0 & \text{in } \Gamma_{\epsilon,n} := B_n^+ \setminus \overline{B}_\epsilon^+ \\ u_{\epsilon,n} = w_m & \text{on } \partial B_\epsilon \cap \mathbb{R}^N_+ \\ u_{\epsilon,n} = 0 & \text{on } (\partial B_n^+ \cap \mathbb{R}^N_+) \cup (\overline{\Gamma}_{\epsilon,n} \cap \partial\mathbb{R}^N_+). \end{array} \tag{3.14}$$

For $\epsilon' \le \epsilon < n \le n'$ there holds $w_{K_n} \le u_{\epsilon,n} \le u_{\epsilon',n'} \le w$ in $\Gamma_{\epsilon,n}$. Letting $n \to \infty$ and $\epsilon \to 0$ there exists a subsequence still denoted by $\{u_{\epsilon,n}\}$ which converges to a solution of $u$ of (3.6) in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and satisfying (3.11). □

*Remark.* The assumption that $w\lfloor_{\partial\Omega\setminus\{0\}}$ is bounded is restrictive. For example if $w(t\sigma) = t^{-a}\omega(\sigma)$ the flatness assumption means that $\text{dist}\,(x, \mathbb{R}^N_+) = O(|x|^{\alpha+1})$ for all $x \in \partial\Omega$ near 0 (remember that $T_{\partial\Omega}(0) = \partial\mathbb{R}^N_+$). This assumption is always satisfied if $p \ge 3$ since $\alpha \le 1$, and it can be avoided if there exists a subsolution.

**Proposition 3.4** *Let $p > 1$, $1 \le q \le 2$ and $M \in \mathbb{R}$. Assume that the equation* (3.6) *admits a positive supersolution $w$ in $\mathbb{R}^N_+$ belonging to $C(\overline{\mathbb{R}^N_+} \setminus \{0\})$ satisfying* (3.10). *Assume also*

*(i) either there exists a positive subsolution $Z \in C(\overline{\mathbb{R}^N_+} \setminus \{0\})$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$, smaller than $w$ and satisfying* (3.10),

*(ii) or $\Omega \subset \mathbb{R}^N_+$ is a bounded smooth domain such that $0 \in \partial\Omega$ and there exists a positive subsolution $Z \in C(\overline{\Omega} \setminus \{0\})$, vanishing on $\partial\Omega \setminus \{0\}$ such that $Z \le w\lfloor_\Omega$ and satisfying* (3.10).

*Then there exists a positive function $u$ satisfying* (3.6) *in $\mathbb{R}^N_+$ (resp. $\Omega$), vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ (resp. $\partial\Omega \setminus \{0\}$) and such that*

$$Z(x) \le u(x) \le w(x) \quad \text{for all } x \in \mathbb{R}^N + \ \ (\text{resp. } x \in \Omega). \tag{3.15}$$

*Example.* If $1 < p < \frac{N+1}{N-1}$ it is proved in [15] that if $\Omega \subset \mathbb{R}^N_+$ is a smooth bounded domain such that $0 \in \partial\Omega$, there exists a nonnegative function $Z_\infty \in C(\overline{\Omega} \setminus \{0\}) \cap$

$C^2(\Omega)$ satisfying the equation

$$\begin{aligned} -\Delta Z + Z^p = 0 \qquad & \text{in } \Omega \\ Z = 0 \qquad & \text{on } \partial\Omega \setminus \{0\}, \end{aligned} \tag{3.16}$$

and such that $t^{\frac{2}{p-1}} Z_\infty(t\sigma) \to \psi(\sigma)$ uniformly on compact sets $K \subset S_+^{N-1}$ as $t \to 0$ where $\psi$ is the unique a positive solution of

$$\begin{aligned} -\Delta'\psi + \alpha\,(N-2-\alpha)\,\psi + \psi^p = 0 \qquad & \text{in } S_+^{N-1} \\ \psi = 0 \qquad & \text{on } \partial S_+^{N-1}. \end{aligned} \tag{3.17}$$

Furthermore, for any $k > 0$ there exists a nonnegative function $Z_k \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ satisfying (3.16) and such that $t^{N-1} Z_k(t\sigma) \to k\phi_1(\sigma)$ where $\phi_1$ has been introduced in Theorem 1.11, uniformly on compact subsets of $S_+^{N-1}$. Furthermore $Z_k \uparrow Z_\infty$ when $k \to \infty$. If the equation (3.6) admits a positive supersolution $w$ in $\mathbb{R}^N_+$ belonging to $C(\overline{\mathbb{R}^N_+} \setminus \{0\})$ and such that $Z_k \le w$ in $\Omega$ for some $0 < k \le \infty$, then there exists a positive function $u$ satisfying (3.6) in $\Omega$, vanishing on $\partial\Omega \setminus \{0\}$ and such that

$$Z_k(x) \le u(x) \le w(x) \quad \text{for all } x \in \Omega. \tag{3.18}$$

The same result holds if $\Omega$ is replaced by $\mathbb{R}^N_+$.

## 3.2 Existence or non-existence of separable solutions

Since any large enough constant is a supersolution of (3.1), it follows by Theorem 3.1 that if there exists a nonnegative subsolution $z \in W_0^{1,\infty}(S_+^{N-1})$, there exists a solution in between.

### 3.2.1 Proof of Theorem 1.9

We recall that $\phi_1$ is the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$ with corresponding eigenvalue $\lambda_1 = N-1$. Put

$$H(\omega) = -\Delta'\omega + \alpha(N-2-\alpha)\omega + |\phi|^{p-1}\omega - M\left(\alpha^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{p+1}},$$

then

$$H(\phi_1) = (N-1+\alpha(N-2-\alpha))\,\phi_1 + \phi_1^p - M\left(\alpha^2\phi_1^2 + |\nabla'\phi_1|^2\right)^{\frac{p}{p+1}}.$$

If $\phi_1$ is small enough, there holds $\phi_1^p - M\left(\alpha^2\phi_1^2 + |\nabla'\phi_1|^2\right)^{\frac{p}{p+1}} < 0$, hence $\phi_1$ is a subsolution. However the condition $N-1+\alpha(N-2-\alpha) \le 0$ is too stringent. We can use the fact that, up to a good choice of coordinates, $\phi_1 = \phi_1(\sigma) = \cos\sigma$ with $\sigma \in [0, \frac{\pi}{2}]$. Furthermore the statement "$\phi_1$ is small enough" can be achieved by $\phi_1 = \delta\cos\sigma$ with $\delta > 0$ small enough. Then

$$\begin{aligned} &\delta^{-1} H(\delta^{\frac{p+1}{p-1}}\cos\sigma) \\ &\qquad = (N-1+\alpha(N-2-\alpha))\cos\sigma + \delta^{p+1}\cos^p\sigma - M\delta(\alpha^2\cos^2\sigma + \sin^2\sigma)^{\frac{p}{p+1}}. \end{aligned}$$

The problem is to find $\delta > 0$ such that for all $\sigma \in [0, \frac{\pi}{2}]$ we have $H(\delta^{\frac{p+1}{p-1}} \cos\sigma) \leq 0$. Put $Z = \cos\sigma$ and $\delta^{-1} H(\delta^{\frac{p+1}{p-1}} \cos\sigma) = \delta^{-1} H(\delta^{\frac{p+1}{p-1}} Z) = K_\delta(Z)$, then

$$K_\delta(Z) = (N - 1 + \alpha(N - 2 - \alpha))\, Z + \delta^{p+1} Z^p - M\delta((\alpha^2 - 1)Z^2 + 1)^{\frac{p}{p+1}},$$

where $0 \leq Z \leq 1$. We use the fact that

$$\alpha^2 \cos^2\sigma + \sin^2\sigma \geq \min\{\alpha^2, 1\}(\cos^2\sigma + \sin^2\sigma) := \kappa^2 > 0,$$

hence

$$\delta^{-1} H(\delta^{\frac{p+1}{p-1}} \cos\sigma) \leq (N - 1 + \alpha(N - 2 - \alpha)) \cos\sigma + \delta^{p+1} \cos^p\sigma - M\delta\kappa^{\frac{2p}{p+1}}.$$

Then

$$K_\delta(Z) \leq \tilde{K}_\delta(Z) := (N - 1 + \alpha(N - 2 - \alpha))\, Z + \delta^{p+1} Z^p - M\delta\kappa^{\frac{2p}{p+1}}, \tag{3.19}$$

and

$$\tilde{K}'_\delta(Z) = N - 1 + \alpha(N - 2 - \alpha) + p\delta^{p+1} Z^{p-1}. \tag{3.20}$$

If $N - 1 + \alpha(N - 2 - \alpha) \geq 0$, equivalently $p \geq \frac{N+1}{N-1}$, then $\tilde{K}'_\delta \geq 0$ on $[0, 1]$, hence

$$\tilde{K}_\delta(Z) \leq \tilde{K}_\delta(1) = N - 1 + \alpha(N - 2 - \alpha) + \delta^{p+1} - M\delta\kappa^{\frac{2p}{p+1}}.$$

The function $\delta \mapsto \tilde{K}_\delta(1)$ achieves its minimum for $\delta = \delta_0 := \beta^{\frac{2}{p+1}} \left(\frac{M}{p+1}\right)^{\frac{1}{p}}$ and

$$\tilde{K}_{\delta_0}(1) = N - 1 + \alpha(N - 2 - \alpha) - p\kappa^2 \left(\frac{M}{p+1}\right)^{\frac{p+1}{p}}.$$

Therefore, when $p \geq \frac{N+1}{N-1}$, $K_{\delta_0} \leq 0$ on $[0, 1]$ if

$$\begin{aligned} \left(\tfrac{M}{p+1}\right)^{\frac{p+1}{p}} \geq \left(\tfrac{M_{N,p}}{p+1}\right)^{\frac{p+1}{p}} &:= \frac{N - 1 + \alpha(N - 2 - \alpha)}{p \min\{1, \alpha^2\}} \\ &= \frac{(p+1)\,(p(N-1) - (N+1))}{p \min\{(p-1)^2, 4\}}. \end{aligned} \tag{3.21}$$

*If* $N - 1 + \alpha(N - 2 - \alpha) \leq 0$, equivalently $p \leq \frac{N+1}{N-1}$, it is clear from (3.19 ) that $\tilde{K}_\delta(Z) \leq 0$ for any $Z \in [0, 1]$ as soon as $\delta \leq \kappa^{\frac{1}{p+1}} M^{\frac{1}{p}}$.
*Improvement in the case* $\alpha > 1$*, equivalently* $1 < p < 3$ . We set

$$F(Z) = \frac{(\alpha^2 - 1)Z^2 + 1}{Z^{\frac{p+1}{p}}}.$$

Then

$$\frac{F'(Z)}{F(Z)} = \frac{(p-1)(\alpha^2 - 1)Z^2 - (p+1)}{p((\alpha^2 - 1)Z^2 + 1)Z}.$$

Since

$$K_\delta(Z) \le 0 \Longleftrightarrow (N-1+\alpha(N-2-\alpha)) + \delta^{p+1} Z^{p-1} \le M\delta F^{\frac{p}{p+1}}(Z) \tag{3.22}$$

for all $Z \in (0,1]$, it is sufficient to prove

$$(N-1+\alpha(N-2-\alpha)) + \delta^{p+1} \le M\delta \min_{Z\in(0,1]} F^{\frac{p}{p+1}}(Z) \tag{3.23}$$

The function $F$ is minimal on $(0,1]$ at $Z = Z_0 = \sqrt{\alpha^2-1}$ (remember that $\alpha = \frac{2}{p-1}$) and $F(Z_0) = (\alpha+2)(\alpha-1)^{\frac{\alpha+1}{\alpha+2}}$.
If $Z_0 \le 1$, equivalently $\alpha \ge 2$, inequality (3.23) is satisfied if one find $\delta$ such that

$$(N-1+\alpha(N-2-\alpha)) + \delta^{p+1} \le M\delta F^{\frac{p}{p+1}}(Z_0),$$

and a sufficient condition is

$$p\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}} \ge p\left(\frac{M_{N,p}}{p+1}\right)^{\frac{p+1}{p}} := \frac{N-1+\alpha(N-2-\alpha)}{F(Z_0)} \tag{3.24}$$

If $Z_0 > 1$, equivalently $1 < \alpha < 2$, the minimum of $F$ on $(0,1]$ is achieved at $Z = 1$ with value $F(1) = \alpha^2$, hence a sufficient condition is

$$(N-1+\alpha(N-2-\alpha)) + \delta^{p+1} \le M\delta\alpha^{\frac{2p}{p+1}}),$$

and we obtain the desired inequality as soon as

$$p\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}} \ge p\left(\frac{M_{N,p}}{p+1}\right)^{\frac{p+1}{p}} := \frac{N-1+\alpha(N-2-\alpha)}{\alpha^2}. \tag{3.25}$$

This ends the proof. □

*Remark.* Introducing $m^{**}$ defined in (1.11), inequality (3.21) takes the form

$$M \ge \left(\frac{2(p+1)}{\min\{(p-1)^2, 4\}}\right)^{\frac{p}{p+1}} m^{**}, \tag{3.26}$$

in the general case and a more complicated expression in the case $\alpha > 1$.

### 3.2.2 Non-existence

**Theorem 3.5** *Let $p > \frac{N+1}{N-1}$ and $M \le m^{**}$, defined by* (1.11)*. Then equation* (3.1) *admits no positive solution.*

*Proof.* If $\omega$ is a positive solution of (3.1) the function $\eta$ defined by $\omega = \eta^b$ for some $b > 0$ satisfies

$$-\Delta'\eta + (1-b)\frac{|\nabla'\eta|^2}{\eta} + \frac{\alpha(N-2-\alpha)}{b}\eta + \frac{1}{b}\eta^{1+(p-1)b} - \frac{M\eta^{\frac{(b-1)(p-1)}{p+1}}}{b}\left(\alpha^2\eta^2 + b^2|\nabla'\eta|^2\right)^{\frac{p}{p+1}} = 0.$$

Since for any $\epsilon > 0$ we have by Hölder's inequality,

$$\int_{S^{N-1}_+} \eta^{1+\frac{(b-1)(p-1)}{p+1}} \left(\alpha^2\eta^2 + b^2|\nabla'\eta|^2\right)^{\frac{p}{p+1}} dS \\ \leq \frac{\epsilon^{\frac{p+1}{p}} p}{p+1} \int_{S^{N-1}_+} (\alpha^2\eta^2 + b^2|\nabla'\eta|^2) + \frac{1}{(p+1)\epsilon^{p+1}} \int_{S^{N-1}_+} \eta^{2+(p-1)b} dS,$$

it follows that

$$\left(2 - b - M\frac{\epsilon^{\frac{p+1}{p}} pb}{p+1}\right) \int_{S^{N-1}_+} |\nabla'\eta|^2 dS + \frac{\alpha}{b}\left(N - 2 - \alpha - M\frac{\epsilon^{\frac{p+1}{p}} \alpha p}{p+1}\right) \int_{S^{N-1}_+} \eta^2 dS \\ + \frac{1}{b}\left(1 - \frac{M}{(p+1)\epsilon^{p+1}}\right) \int_{S^{N-1}_+} \eta^{2+(p-1)b} dS \leq 0. \tag{3.27}$$

If $b \in (0,2)$, $\epsilon > 0$ and $M > 0$ are linked by the relation

$$2 - b - M\frac{\epsilon^{\frac{p+1}{p}} pb}{p+1} \geq 0 \Longleftrightarrow M\epsilon^{\frac{p+1}{p}} \leq \frac{(2-b)(p+1)}{bp}, \tag{3.28}$$

inequality (3.27) turns into

$$\left((2-b)(N-1) + \frac{\alpha(N-2-\alpha)}{b} - \frac{M\epsilon^{\frac{p+1}{p}} p}{p+1}\left((N-1)b + \frac{\alpha^2}{b}\right)\right) \int_{S^{N-1}_+} \eta^2 dS \\ + \frac{1}{b}\left(1 - \frac{M}{(p+1)\epsilon^{p+1}}\right) \int_{S^{N-1}_+} \eta^{2+(p-1)b} dS \leq 0. \tag{3.29}$$

Next we choose

$$\epsilon^{p+1} = \frac{M}{p+1}, \tag{3.30}$$

and we define the function $b \mapsto L(b)$ by

$$L(b) := (2-b)(N-1) + \frac{\alpha(N-2-\alpha)}{b} - p\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}} \left((N-1)b + \frac{\alpha^2}{b}\right). \tag{3.31}$$

Because $N-1$ is the first eigenvalue of $-\Delta'$ in $W^{1,2}_0(S^{N-1})$, (3.29) combined with (3.30) yields

$$L(b) \int_{S^{N-1}_+} \eta^2 dS \leq 0. \tag{3.32}$$

Furthermore, if inequality (3.28) is strict, and since $\eta$ is not a first eigenfunction, inequality (3.32) is also strict. Then $L(b) \geq 0$ if

$$p\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}} \leq f(b) := \frac{b(2-b)(N-1) + \alpha(N-2-\alpha)}{(N-1)b^2 + \alpha^2}. \tag{3.33}$$

Now

$$f'(b) = \frac{-2(N-1)^2}{\left((N-1)b^2+\alpha^2\right)^2}(b+\alpha)\left(b-\frac{\alpha}{N-1}\right).$$

Notice that

$$\frac{\alpha}{N-1} \leq 1 \Longleftrightarrow p \geq \frac{N+1}{N-1}.$$

If $1 < p \leq \frac{N+1}{N-1}$, then $f' \geq 0$ and in such a case the maximum of $f$ over $(0,1]$ is achieved at $b = 1$ and for such a value $L(b) \leq 0$.

If $p > \frac{N+1}{N-1}$, then $f$ is increasing on $[0, \frac{\alpha}{N-1})$ and decreasing on $(\frac{\alpha}{N-1}, 1]$, hence the maximum is achieved at $b = \frac{\alpha}{N-1}$, which gives

$$f\left(\frac{\alpha}{N-1}\right) = \frac{N-1-\alpha}{\alpha} = \frac{(N-1)p-(N+1)}{2}. \tag{3.34}$$

Therefore there exists no solution if $p \geq \dfrac{N+1}{N-1}$ and

$$\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}} \leq \left(\frac{m^{**}}{p+1}\right)^{\frac{p+1}{p}} := \frac{(N-1)p-(N+1)}{2p}. \tag{3.35}$$

□

*Remark.* Using Theorem 1.1 we can prove the previous result in the case $M < m^{**}$ . Indeed, if $\omega$ is a positive solution of (3.1), $u_\omega(r,.) = r^{-\frac{2}{p-1}}\omega(.)$ is a positive solution of (1.1) in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$. Let $\Omega \subset \mathbb{R}^N_+$ be any smooth domain such that $0 \in \partial\Omega$ and $\partial\Omega$ is flat near 0. Then $u_\omega \leq K$ on $\partial\Omega$ for some $K > 0$. Put $v = (u_\omega - K)_+$, then it is a nonnegative subsolution of (3.1). For any $\epsilon > 0$ small enough there exists a solution $u_\epsilon$ of

$$\begin{aligned} -\Delta u + u^p - M|\nabla u|^{\frac{2p}{p+1}} = 0 \qquad & \text{in } \ \Omega_\epsilon := \Omega \cap \overline{B}^c_\epsilon \\ u = v \qquad & \text{on } \ \partial B_\epsilon \cap \Omega \\ u = 0 \qquad & \text{on } \ B^c_\epsilon \cap \partial\Omega. \end{aligned} \tag{3.36}$$

Then $v \leq u_\epsilon \leq u_\omega$. Furthermore, for $0 < \epsilon' < \epsilon$, $u_\epsilon \leq u_{\epsilon'}$ in $\Omega_\epsilon$. Hence $\{u_\epsilon\}$ converges, when $\epsilon \to 0$ to a solution $u_0$ of (1.9), which satisfies $v \leq u_0 \leq u_\omega$ and therefore vanishes on $\partial\Omega \setminus \{0\}$, which is a contradiction.

# 4 Solutions with an isolated boundary singularity

## 4.1 Uniqueness of singular solutions

In this section we use scaling transormations to prove uniqueness of singular solutions.

**Theorem 4.1** *Assume $N \geq 2$, $p > 1$, $1 < q < 2$ and $M > 0$. Let $a$ such that*

$$\begin{array}{lll} (i) & 0 \leq a < \beta & \text{if } q \leq \frac{2p}{p+1} \\ (ii) & \beta < a & \text{if } q > \frac{2p}{p+1}. \end{array} \tag{4.37}$$

*Let $\theta \in C^1(\overline{S_+^{N-1}})$ be a nonnegative function, positive in $S_+^{N-1}$, vanishing on $\partial S_+^{N-1}$, and $\tilde{a}$ be a real number. Then there exists at most one nonnegative solution of* (1.21) *satisfying*

$$u(r,\sigma) = r^{-a}|\ln r|^{\tilde{a}}\theta(\sigma)(1+o(1)) \qquad \text{as } r \to 0. \tag{4.38}$$

*Proof.* The proof is an adaptation to the configuration where the singularity lies on the boundary of [8, Theorem 1.15]. If $u$ is a solution of (1.1) in $\mathbb{R}^N_+$, $\ell > 0$ and $b > 0$, define $u_{\ell,b}$ by

$$u_{\ell,b}(x) = \ell^b u(\ell).$$

Then

$$-\Delta u_{\ell,b} + \ell^{2-b(p-1)}u_{\ell,b}^p - M\ell^{2-q-b(q-1)}|\nabla u_{\ell,b}|^q = 0 \quad \text{in } \mathbb{R}^N_+.$$

If $\ell > 1$, $u_{\ell,b}$ is a supersolution of (1.1) in $\mathbb{R}^N_+$ if and only if

$$\alpha \leq b \leq \beta.$$

These conditions are compatible if and only if $1 < q \leq \frac{2p}{p+1}$. Then we take $b \in (a,\infty) \cap [\alpha,\beta]$, then

$$u_{\ell,b}(r,\sigma) = \ell^{b-a}r^{-a}|\ln r|^{\tilde{a}}\theta(\sigma)(1+o(1)) \quad \text{as } r \to 0.$$

By (2.1) all $u(x)$ tends to 0 when $|x| \to \infty$. Hence, for any $\epsilon > 0$ the super solution $u_{\ell,b} + \epsilon$ which is larger than $u$ for $|x|$ small enough and large enough is larger than another solution $\tilde{u}$ in $\mathbb{R}^N_+$. Letting $\ell \downarrow 1$ and $\epsilon \to 0$ yields $u \geq \tilde{u}$. In the same way $\tilde{u} \geq u$.
If $\ell < 1$, $u_{\ell,b}$ is a supersolution of (1.1) in $\frac{1}{\ell}G$ if and only if

$$\beta \leq b \leq \alpha,$$

and these conditions are compatible if and only if $\frac{2p}{p+1} \leq q < 2$. If $\alpha > \beta$ we choose $b \in (0,a) \cap [\beta,\alpha]$ and we conclude as in the first case. □

*Remark.* In the case $a = \beta$ a more precise expansion of the singular solution $u$ at $x = 0$ yields uniqueness as it is proved in [8] in the case of an internal singularity. Since the proof of the next result is based upon a easy adaptation of the ideas in [8, Theorem 4.4], we omit it.

**Theorem 4.2** *Assume $N \geq 2$, $p > 1$, $1 < q \leq \frac{2p}{p+1}$, $M > 0$ and $a \in [0,\beta]$. Assume $\theta$ and $\tilde{\theta}$ are $C^1(\overline{S_+^{N-1}})$ functions positive in $S_+^{N-1}$ and vanishing on $\partial S_+^{N-1}$ and $\tilde{a}$ is a real smaller than $a$. Then there exists at most one nonnegative solution of* (1.21) *satisfying*

$$u(r,\sigma) = r^{-a}\theta(\sigma) + r^{-\tilde{a}}\tilde{\theta}(\sigma)(1+o(1)) \qquad \text{as } r \to 0. \tag{4.39}$$

When problem (1.21) is replaced by (1.8) the scaling method becomes much more delicate to apply. However we give below an easy extension when $\partial\Omega$ is flat near $x = 0$.

**Theorem 4.3** *Assume $N \geq 2$, $p > 1$, $1 < q < 2$, $M > 0$ and $\Omega$ is a bounded smooth domain such that $0 \in \partial\Omega$ and there exists $\delta > 0$ such that $\partial\Omega \cap B_\delta = T_{\partial\Omega}(0) \cap B_\delta$. Let $a$ such that*

$$\begin{array}{lll} (i) & 0 \leq a < \beta & \text{if } q \leq \frac{2p}{p+1} \\ (ii) & \beta < a & \text{if } q > \frac{2p}{p+1}. \end{array} \tag{4.40}$$

*Let $\theta \in C^1(\overline{S_+^{N-1}})$ be a nonnegative function, positive in $S_+^{N-1}$, vanishing on $\partial S_+^{N-1}$, and $\tilde{a}$ be a real number. If $\Omega$ is starshaped with respect to 0, then there exists at most one nonnegative solution of (1.21) satisfying*

$$u(r,\sigma) = r^{-a}|\ln r|^{\tilde{a}}\theta(\sigma)(1 + o(1)) \qquad \text{as } r \to 0. \tag{4.41}$$

*Proof.* We use the same change of scale as in Theorem 4.1. In case (i) with $\ell > 1$ and $b \in (a, \infty) \cap [\alpha, \beta]$ , $u_{\ell,b}$ is a supersolution in $\Omega_\ell = \frac{1}{\ell}\Omega \subset \Omega$ and $\partial\Omega_\ell \cap B_{\delta/\ell} = T_{\partial\Omega}(0) \cap B_{\delta/\ell}$. If $\tilde{u}$ is another solution, $\tilde{u}\lfloor_{\partial\Omega_\ell} = h_\ell$ and $h_\ell \to 0$ uniformly as $\ell \downarrow 1$ since $u \in C^1(\overline{\Omega} \cap B_\epsilon^c)$ for any $\epsilon > 0$. The function $u_{\ell,b} + \max_{\partial\Omega_\ell} h_\ell$ is a supersolution of 1.1 in $\Omega_\ell$ larger than $\tilde{u}$ on $\partial\Omega_\ell \setminus \{0\}$ and near $x = 0$, hence it is larger than $\tilde{u}$ in $\Omega_\ell$. Letting $\ell \downarrow 1$ yields $u \geq \tilde{u}$.
In case (ii), for $\ell < 1$, $\Omega \subset \Omega_\ell$ and with $b \in (0, a) \cap [\beta, \alpha]$, $u_{\ell,b}$ is a supersolution in $\Omega$ larger that $\tilde{u}$ on $\partial\Omega \setminus \{0\}$ and near $x = 0$ it is larger than $\tilde{u}$ in $\Omega$. We conclude as in case (i). □

## 4.2 Construction of fundamental solutions

Let $\Omega$ be either $\mathbb{R}_+^N$ or a bounded domain with $0 \in \partial\Omega$. A function $u$ satisfying (1.8) is a *fundamental solution* if it has a singularity of potential type, that is

$$\lim_{x\to 0} \frac{|x|^N u(x)}{\rho(x)} = c_{_N} k, \tag{4.42}$$

for some $k > 0$. The function $u$ can also be looked for as a solution of

$$\begin{array}{rl} -\Delta u + u^p - M|\nabla u|^q = 0 & \text{in } \Omega \\ u = k\delta_0 & \text{in } \partial\Omega, \end{array} \tag{4.43}$$

in the sense that $u \in L^p_\rho(\Omega \cap B_r)$, $\nabla u \in L^q_{\rho,loc}(\Omega \cap B_r, )$ for any $r > 0$, and for any $\zeta \in C_c^1(\overline{\Omega}) \cap W^{2,\infty}(\Omega)$ there holds

$$\int_\Omega (-u\Delta\zeta + u^p\zeta - M|\nabla u|^q\zeta)\, dx = -k\frac{\partial\zeta}{\partial\mathbf{n}}(0). \tag{4.44}$$

We first consider the problem in $\mathbb{R}_+^N$.

*Proof of Theorem 1.7.* The scheme of the proof is surprising since we first show that, in the case $q = \frac{2p}{p+1}$, there exists $M_1 > 0$ such that for any $k > 0$ and any $0 < M < M_1$ there exists a solution. Using this result we prove that if $1 < q < \frac{2p}{p+1}$, then for any $M > 0$ and $k > 0$ there exists a solution. Then we return to the case $q = \frac{2p}{p+1}$ and using the result in the previous case, we prove that when $q = \frac{2p}{p+1}$ we can get rid of the restriction on $M > 0$ and $k > 0$ for the existence of solutions.

*I- The case $q = \frac{2p}{p+1}$ and $M$ upper bounded.*

For $\ell > 0$ the transformation $T_\ell$ defined by

$$T_\ell[u(x)] = \ell^{\frac{2}{p-1}} u(\ell x), \tag{4.45}$$

leaves the operator $\mathcal{L}_{\frac{2p}{p+1},M}$ invariant. We can therefore *write*

$$T_\ell[u_k] = u_{k\ell^{\frac{2}{p-1}+1-N}},$$

in the sense that if $u_k$ satisfies (4.42) then $T_\ell[u_k]$ satisfies the same limit with $k$ replaced by $k\ell^{\frac{2}{p-1}+1-N}$. However this identity to hold needs some uniqueness for the solutions under consideration satisfying (4.42). This is achieved if $u_k$ is the minimal solution satisfying (4.42) in which case $T_\ell[u_k]$ is the minimal solution satisfying (4.42) with $k$ replaced by $k\ell^{\frac{2}{p-1}+1-N}$. Therefore if there exists a solution to (1.8) in $\mathbb{R}^N_+$, vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ satisfying (4.47) for some $k > 0$, then there exists such a solution for any $k > 0$.

*Step 1- Construction of a subsolution.* For $k > 0$ we denote by $v_k$ the solution of

$$\begin{aligned} -\Delta v + v^p &= 0 \qquad && \text{in } \mathbb{R}^N_+ \\ v &= k\delta_0 \qquad && \text{on } \partial\mathbb{R}^N_+ \setminus \{0\}. \end{aligned} \tag{4.46}$$

Such a solution exists thanks to [15] if $\mathbb{R}^N_+$ is replaced by a bounded domain $\Omega$. If case of a half-space the problem is first solved in $B^+_n$ and by letting $n \to \infty$, we obtain the solution in $\mathbb{R}^N_+$. Clearly $v_k$ is a subsolution of problem (1.8), and it satisfies

$$\lim_{x\to 0} \frac{u(x)}{P_{_N}(x)} = k, \tag{4.47}$$

for some $c'_{_N} > 0$, where $P_{_N}(x) = c_{_N} \frac{x_N}{|x|^N}$ is the Poisson kernel in $\mathbb{R}^N_+$.

*Step 2- Construction of a supersolution.* It is known that

$$|\nabla P_{_N}(x)|^2 = |x|^{-2N} c^2(x), \tag{4.48}$$

where $c(.)$ is smooth and verifies

$$0 < \tilde{c}_1 \le c(x) \le \tilde{c}_2 \quad \text{for some } \tilde{c}_1, \tilde{c}_2 > 0.$$

We construct $w_k$ in $\mathbb{R}^N_+$ of the form

$$w_k = kP_{_N} + w, \tag{4.49}$$

where $w$ satisfies

$$\begin{array}{ll} -\Delta w + w^p = a\gamma_2 |x|^{-\frac{2Np}{p+1}} & \text{in } \mathbb{R}^N_+ \\ w = 0 & \text{on } \partial\mathbb{R}^N_+, \end{array} \tag{4.50}$$

for some $a > 0$ to be chosen later on. Then

$$\begin{aligned} \mathcal{L}_{\frac{2p}{p+1},M} w_k &= -\Delta w + (kP_{_N} + w)^p - M\left(|k\nabla P_{_N} + \nabla w|^2\right)^{\frac{p}{p+1}} \\ &= (kP_{_N} + w)^p - w^p + a\gamma_2 |x|^{-\frac{2Np}{p+1}} - M\left(|k\nabla P_{_N} + \nabla w|^2\right)^{\frac{p}{p+1}} \\ &\geq pkP_{_N} w^{p-1} + a\gamma_2 |x|^{-\frac{2Np}{p+1}} - 2M\left(k^{\frac{2p}{p+1}}\gamma_2^{\frac{2p}{p+1}} |x|^{-\frac{2Np}{p+1}} + |\nabla w|^{\frac{2p}{p+1}}\right). \end{aligned}$$

Now it is easy to check using Osserman's type construction as in [29, Lemma 2.1] and scaling techniques that

$$w(x) \leq \gamma_3 \min\left\{a^{\frac{1}{p}} |x|^{-\frac{2N}{p+1}}, a|x|^{2(1-\frac{Np}{p+1})}\right\},$$

and

$$\begin{gathered} |\nabla w(x)| \leq \gamma_4 \min\left\{a^{\frac{1}{p}} |x|^{-\frac{2N}{p+1}-1}, a|x|^{1-\frac{2Np}{p+1}}\right\} \\ \Longrightarrow \\ |\nabla w(x)|^{\frac{2p}{p+1}} \leq \gamma_5 \min\left\{a^{\frac{2}{p+1}} |x|^{-\frac{2p(2N+p+1))}{(p+1)^2}}, a^{\frac{2p}{p+1}} |x|^{\frac{2p(p+1-2Np)}{(p+1)^2}}\right\}. \end{gathered}$$

Therefore, if we put

$$\tau = \frac{p^2 - 1}{2p(N + 1 - p(N-1))},$$

then $\tau > 0$ since $N + 1 > p(N-1)$ and

$$\begin{aligned} |x|^{\frac{2Np}{p+1}} \mathcal{L}_{\frac{2p}{p+1},M} w_k &\geq \gamma_2 \left(a - 2Mk^{\frac{2p}{p+1}}\gamma_2^{\frac{p-1}{p+1}}\right) - 2M\gamma_5 a^{\frac{2p}{p+1}} |x|^{\frac{2p(N+1-p(N-1))}{(p+1)^2}} \\ &\geq \gamma_2 \left(a - 2Mk^{\frac{2p}{p+1}}\gamma_2^{\frac{p-1}{p+1}}\right) - 2M\gamma_5 a \quad \text{in } B^+_{a^\tau}, \end{aligned} \tag{4.51}$$

and similarly,

$$|x|^{\frac{2Np}{p+1}} \mathcal{L}_{\frac{2p}{p+1},M} w_k \geq \gamma_2 \left(a - 2Mk^{\frac{2p}{p+1}}\gamma_2^{\frac{p-1}{p+1}}\right) - 2M\gamma_5 a \text{ in } (B^+_{a^\tau})^c. \tag{4.52}$$

Replacing $\tau$ by its value, we obtain a very simple expression from (4.51) and (4.52), valid both in $B^+_{a^\tau}$ and $(B^+_{a^\tau})^c$, namely

$$|x|^{\frac{2Np}{p+1}} \mathcal{L}_{\frac{2p}{p+1},M} w_k \geq \gamma_2 \left(a - 2Mk^{\frac{2p}{p+1}}\gamma_2^{\frac{p-1}{p+1}}\right) - 2M\gamma_5 a \quad \text{in } \mathbb{R}^N_+. \tag{4.53}$$

When

$$M < M_1 := \frac{\gamma_2}{2\gamma_5}, \tag{4.54}$$

then for fixed $k$, if we take

$$a > \frac{2M_1\gamma_2^{\frac{2p}{p+1}}k^{\frac{2p}{p+1}}}{\gamma_2 - 2M\gamma_5},$$

we infer that the right-hand side of (4.53) is nonnegative, hence $w_k$ is a supersolution.

*Step 3-Existence.* For $0 < k \leq k_0$ $w_k$ is a supersolution which dominates the subsolution $v_k$. Hence, by [32, Theorem 1-4-6] there exists a solution $u_k$ to (1.8) in $\mathbb{R}^N_+$, vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and such that $v_k \leq u_k \leq w_k$. Since

$$\lim_{x\to 0} \frac{v_k(x)}{P_{_N}(x)} = \lim_{x\to 0} \frac{w_k(x)}{P_{_N}(x)} = k,$$

it follows that $u_k$ inherits the same asymptotic behaviour. Since $k < k_0$ can be replaced by any $k > 0$, the existence of a solution follows.

*II- The case* $1 < q < \frac{2p}{p+1}$. Assume $M < M_1$, $k > 0$ and $\tilde{u}_k$ is the minimal solution of (1.8) in $\mathbb{R}^N_+$ with $q = \frac{2p}{p+1}$, vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and such that (4.47). Since $|\nabla\phi|^{\frac{2p}{2p}2p} \geq |\nabla\phi|^q - 1$, there holds

$$-\Delta\tilde{u}_k + \tilde{u}_k^p + M - M|\nabla\tilde{u}_k|^q \geq 0.$$

Hence $\tilde{u}_k^* = \tilde{u}_k + M^{\frac{1}{p}}$ is a supersolution (1.8) in $\mathbb{R}^N_+$ and it dominates $v_k$ defined in (4.46). By [32, Theorem 1-4-6] there exists a solution $u_k$ of (1.8), vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and satisfying (4.47) under the following weaker form

$$\lim_{t\to 0} \frac{u_k(tx)}{P_{_N}(tx)} = k \quad \text{uniformly on compact subsets of } \mathbb{R}^N_+. \tag{4.55}$$

Since $|x|^{N-1}u_k(x)$ is uniformly bounded and vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$, it is bounded in the $C^1_{loc}(\overline{\mathbb{R}^N_+})$-topology. Hence (4.47) holds. This proves the result when $M < M_1$.

Next let $M > 0$ arbitrary and $k > 0$. In order to find a solution $u := u_k$ to (1.8), we set $u(x) = \ell^{-\frac{2}{p-1}}U_\ell(\frac{x}{\ell})$. Then $\mathcal{L}_{q,M}u = 0$ is equivalent to

$$\mathcal{L}_{q,M_\ell}U_\ell := -\Delta U_\ell + U_\ell^p - M_\ell|\nabla U|^q = 0 \quad \text{with} \quad M_\ell = M\ell^{\frac{2p-q(p+1)}{p-1}},$$

and (4.47) is equivalent to

$$\lim_{x\to 0} \frac{U_\ell(x)}{P_{_N}(x)} = \ell^{\frac{2}{p-1}+1-N}k.$$

Since $2p - q(p+1) > 0$ it is enough to choose $\ell > 0$ such that $M\ell^{\frac{2p-q(p+1)}{p-1}} < M_1$, and we end the proof using the result when $M < M_1$.

*III- The case* $q = \frac{2p}{p+1}$ *revisited.* Let $p < \tilde{p} < \frac{N+1}{N-1}$. Then $\frac{2p}{p+1} < \frac{2\tilde{p}}{\tilde{p}+1}$. This implies that for any $M > 0$ and $k > 0$ there exists a positive solution $\tilde{u}_k$ to

$$-\Delta\tilde{u}_k + \tilde{u}_k^{\tilde{p}} - M|\nabla\tilde{u}_k|^{\frac{2p}{p+1}} = 0 \qquad \text{in } \mathbb{R}^N_+,$$

vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and such that

$$\lim_{x\to 0} \frac{\tilde{u}_k(x)}{P_{_N}(x)} = k.$$

Since $\tilde{p} > p$ we have $\tilde{u}_k^{\tilde{p}} > \tilde{u}_k^p - 1$ and therefore

$$-\Delta\tilde{u}_k + \tilde{u}_k^p - M|\nabla\tilde{u}_k|^{\frac{2p}{p+1}} \geq 1 > 0 \qquad \text{in } \mathbb{R}^N_+. \tag{4.56}$$

The function $\tilde{v}_k$ solution of

$$\begin{array}{rl} -\Delta v + v^p = 0 & \text{in } \mathbb{R}^N_+ \\ v = k\delta_0 & \text{on } \partial\mathbb{R}^N_+ \setminus \{0\}, \end{array} \tag{4.57}$$

is a subsolution of (4.56), hence the exists a solution $u_k$ of such that $\tilde{v}_k < u_k < \tilde{u}_k$ of (1.8) in $\mathbb{R}^N_+$, vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and such that (4.42) holds.

*IV- The case* $\frac{2p}{p+1} < q < \frac{1+N}{N}$. We follow the ideas of Case I. We look for a supersolution $w_k$ of the form (4.49) where $w_k$ satisfies

$$\begin{array}{rl} -\Delta w + w^p = a\gamma_2|x|^{-Nq} & \text{in } \mathbb{R}^N_+ \\ w = 0 & \text{on } \partial\mathbb{R}^N_+, \end{array} \tag{4.58}$$

for some $a > 0$. Then

$$\begin{aligned} \mathcal{L}_{q,M} w_k &= -\Delta w + (kP_{_N} + w)^p - M\left(|k\nabla P_{_N} + \nabla w|^2\right)^{\frac{q}{2}} \\ &= (kP_{_N} + w)^p - w^p + a\gamma_2|x|^{-Nq} - M\left(|k\nabla P_{_N} + \nabla w|^2\right)^{\frac{q}{2}} \\ &\geq pkP_{_N} w^{p-1} + a\gamma_2|x|^{-Nq} - 2M\left(k^q\gamma_2^q|x|^{-Nq} + |\nabla w|^q\right). \end{aligned}$$

As in Case I, by scaling techniques,

$$w(x) \leq \gamma_3 \min\left\{a^{\frac{1}{p}}|x|^{-\frac{Nq}{p}}, a|x|^{2-Nq}\right\}$$

and

$$|\nabla w(x)| \leq \gamma_4 \min\left\{a^{\frac{1}{p}}|x|^{-\frac{Nq}{p}-1}, a|x|^{1-Nq}\right\}.$$

Hence

$$|\nabla w(x)|^q \leq \gamma_5 \min\left\{a^{\frac{q}{p}}|x|^{-\frac{Nq^2}{p}-q}, a^q|x|^{q(1-Nq)}\right\}.$$

We set

$$\tau = -\frac{1}{2p' - Nq} = -\frac{p-1}{2p - Nq(p-1)}.$$

Then, by the definition of $\tau$,

$$\begin{aligned} |x|^{Nq}\mathcal{L}_{q,M} w_k &\geq \gamma_2\left(a - 2Mk^q\gamma_2^{q-1}\right) - 2M\gamma_5 a^q|x|^{q(N+1-Nq)} \\ &\geq \gamma_2\left(a - 2Mk^q\gamma_2^{q-1}\right) - 2M\gamma_5 a^{\frac{1+N-p(N-1)}{\frac{2p}{q}-N(p-1)}} \quad \text{in } B^+_{a^\tau}, \end{aligned} \tag{4.59}$$

and

$$|x|^{\frac{2Np}{p+1}}\mathcal{L}_{\frac{2p}{p+1},M}w_k \geq \gamma_2\left(a-2Mk^q\gamma_2^{q-1}\right)-2M\gamma_5 a^{\frac{1+N-p(N-1)}{\frac{2p}{q}-N(p-1)}} \text{ in } (B^+_{a^\tau})^c. \tag{4.60}$$

We obtain a very simple expression from (4.59) and (4.60), valid both in $B^+_{a^\tau}$ and $(B^+_{a^\tau})^c$, hence

$$|x|^{\frac{2Np}{p+1}}\mathcal{L}_{\frac{2p}{p+1},M}w_k \geq \gamma_2\left(a-2Mk^q\gamma_2^{q-1}\right)-2M\gamma_5 a^{\frac{1+N-p(N-1)}{\frac{2p}{q}-N(p-1)}} \text{ in } \mathbb{R}^N_+. \tag{4.61}$$

Using the scaling transformation $T_\ell$ defined in (4.45), the problem of finding $u_k$ solution of (4.43) is equivalent to looking for a solution of

$$\begin{aligned} -\Delta u+u^p-M\ell^{\frac{2p-q(p+1)}{p-1}}|\nabla u|^q=0 \quad &\text{in } \mathbb{R}^N_+\\ u=k\ell^{\frac{p+1}{p-1}-N}\delta_0 \quad &\text{in } \partial\mathbb{R}^N_+. \end{aligned} \tag{4.62}$$

If we replace $M$ by $M_\ell := M\ell^{\frac{2p-q(p+1)}{p-1}}$ and $k$ by $k_\ell := k\ell^{\frac{p+1}{p-1}-N}$, the inequality (4.53) turns into

$$|x|^{\frac{2Np}{p+1}}\mathcal{L}_{\frac{2p}{p+1},M}w_{k,\ell} \geq \gamma_2\left(a-2M_\ell k_\ell^q\gamma_2^{q-1}\right)-2M_\ell\gamma_5 a^{\frac{1+N-p(N-1)}{\frac{2p}{q}-N(p-1)}} \text{ in } \mathbb{R}^N_+, \tag{4.63}$$

where $w_{k,\ell}=w+k_\ell R$ instead of (4.49). Notice that $M_\ell k_\ell^q = M\ell^{\frac{2p}{p-1}-Nq}k^q$. We choose $\ell>0$ such that $M_\ell k_\ell^q\gamma_2^{q-1}=\frac{a}{4}$, hence

$$|x|^{\frac{2Np}{p+1}}\mathcal{L}_{\frac{2p}{p+1},M}w_{k,\ell} \geq \frac{a\gamma_2}{2}\left(1-\gamma_5\gamma_2^{-q}k^{-q}a^{\frac{1+N-p(N-1)}{\frac{2p}{q}-N(p-1)}}\right) \text{ in } \mathbb{R}^N_+. \tag{4.64}$$

It is now sufficient to choose $a>0$ such that the right-hand side of (4.64) is nonnegative and thus $w_{k,\ell}$ is a supersolution. Since $\tilde{v}_{k,\ell}$ is a subsolution smaller that $w_{k,\ell}$, we end the proof as in Case I.

*V- Uniqueness or existence of a minimal solution.* If $1<q\leq\frac{2p}{p+1}$, uniqueness follows from Theorem 4.1 applied with $a=N-1<\beta=\frac{2-q}{q-1}$. If $\frac{2p}{p+1}<q<\frac{N+1}{N}$ and if $u_{k,1}$ and $u_{k,2}$ are solutions, they are larger than $v_k$ and the function $u_{k,1,2}=\inf\{u_{k,1},u_{k,2}\}$ is a supersolution larger than $v_k$. Hence there exists a solution $\tilde{u}_k$ such that

$$v_k\leq\tilde{u}_k\leq u_{k,1,2}.$$

Let $\mathcal{E}_k$ be the set of nonnegative solutions of (1.8) in $\mathbb{R}^N_+$, vanishing on $\partial\mathbb{R}^N_+\setminus\{0\}$ and such that (4.42) and put

$$u_k=\inf\{v : v\in\mathcal{E}_k\}.$$

Then there exists a decreasing sequence $\{v_j\}$ such that $v_j$ converges to $u_k$ on a countable dense subset of $\mathbb{R}^N_+$. By standard elliptic equation regularity theory, $v_j$

converges to $u_k$ on any compact subset of $\overline{\mathbb{R}}^N_+ \setminus \{0\}$. Hence $u_k$ is a solution of (1.8) in $\mathbb{R}^N_+$, it vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and (4.47) since $u_k \geq v_k$. Hence $u_k$ is the minimal solution. □

Next of we consider the same problem in a bounded domain $\Omega$.

*Proof of Theorem 1.8.* We give first proof when $\Omega \subset \mathbb{R}^N_+$. We adapt the proof of Theorem 1.7. The solution $v_k$ of

$$\begin{array}{rl} -\Delta v + v^p = 0 & \text{in } \Omega \\ v = k\delta_0 & \text{on } \partial\Omega, \end{array} \tag{4.65}$$

is a subsolution for (1.8) in $\Omega$ and satisfies (1.24 ). The solution $u_k$ of (1.8) in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and satisfying (4.42) is larger than $v_k$ in $\Omega$. Hence the result follows by Proposition 3.4.

When $\Omega$ is not included in $\mathbb{R}^N_+$, estimates (4.48 ) is valid with the same type of bounds on $c$. We also consider separately the cases $q = \frac{2p}{p+1}$ and $M$ upper bounded, $q < \frac{2p}{p+1}$ and $M > 0$ arbitrary and $q = \frac{2p}{p+1}$ and $M > 0$ arbitrary and finally $\frac{2p}{p+1} < q < \frac{N+1}{N}$. As supersolution we consider the function $w_k := kP^\Omega + w$ where $w$ satisfies

$$\begin{array}{rl} -\Delta w + w^p = a\gamma_2 |x|^{-\frac{2Np}{p+1}} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{array} \tag{4.66}$$

for some $a > 0$. The estimates on $w$ endow the form

$$w(x) \leq \gamma_3 a^{\frac{1}{p}} |x|^{2(1-\frac{Np}{p+1})},$$

and

$$|\nabla w(x)| \leq \gamma_4 a^{\frac{1}{p}} |x|^{1-\frac{2Np}{p+1}},$$

where $\gamma_3$ and $\gamma_4$ depend on $\Omega$. Hence (4.53) holds in $\Omega$ instead of $\mathbb{R}^N_+$, and we have existence for $M < M_1$, where $M_1$ is defined by (4.54). Then we prove existence for any $M > 0$ and $k > 0$ when $q < \frac{2p}{p+1}$ then for any $M > 0$ when $q = \frac{2p}{p+1}$ and finally when $\frac{2p}{p+1} < q < \frac{N+1}{N}$ as in Theorem 1.7. □

## 4.3 Solutions with a strong singularity

### 4.3.1 The case $1 < q \leq \frac{2p}{p+1}$

If $p = \frac{N+1}{N-1}$ and $1 < q < \frac{N+1}{N}$ and if $p > \frac{N+1}{N-1}$ and either $1 < q < \frac{2p}{p+1}$ and $M > 0$ or $q = \frac{2p}{p+1}$ and $M > m^{**}$ defined in (1.11), the singularity is removable by Theorem 1.1. Thus the ranges of exponents that we consider are the following,

$$\begin{array}{ll} (i) & 1 < q \leq \frac{2p}{p+1} \text{ and } 1 < p < \frac{N+1}{N-1}, \\ (ii) & (p,q) = \left(\frac{N+1}{N-1}, \frac{N+1}{N}\right). \end{array} \tag{4.67}$$

If (4.67)-(i) holds, $q < \frac{N+1}{N}$, and in this range the limit of the fundamental solutions $u_k$ when $k \to \infty$ is a solution with a strong singularity with an explicit blow-up rate. In the case of a bounded domain our construction requires a geometric *flatness* condition of $\partial\Omega$ near 0. We consider first the case $\Omega = \mathbb{R}^N_+$.

**Theorem 4.4** *Assume* (4.67)*-(i) holds, then for any $M \geq 0$ there exists a positive solution $u$ of* (1.1) *in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ such that*

$$\lim_{x\to 0} \frac{u(x)}{P_{_N}(x)} = \infty. \tag{4.68}$$

*Furthermore,*
*(i) If $1 < q < \frac{2p}{p+1}$,*

$$\lim_{r\to 0} r^{\frac{2}{p-1}} u(r,.) = \psi \qquad \textit{uniformly in } S^{N-1}_+, \tag{4.69}$$

*where $\psi$ is the unique positive solution of* (3.17).
*(ii) If $q = \frac{2p}{p+1}$,*

$$\lim_{r\to 0} r^{\frac{2}{p-1}} u(r,.) = \omega \qquad \textit{uniformly in } S^{N-1}_+, \tag{4.70}$$

*where $\omega$ is the minimal positive solution of* (1.28).

*Proof.* If $k > 0$, we denote by $u = u_{k,M}$ the solution of

$$\begin{array}{rl} -\Delta u + u^p = M|\nabla u|^q & \text{in } \mathbb{R}^N_+ \\ u = k\delta_0 & \text{in } \partial\mathbb{R}^N_+. \end{array} \tag{4.71}$$

The mapping $k \mapsto u_k$ is increasing. We set $T_\ell[u] = u_\ell$, where $T_\ell$ is defined in (4.45). Since $1 < q \leq \frac{2p}{p+1}$,

$$T_\ell[u_{k,M}] = u_{k\ell^{\frac{2}{p-1}+1-N}, M\ell^{\frac{2p-q(p+1)}{p-1}}}.$$

It follows from Theorem 2.1 and Theorem 2.3 that the sequences $\{u_{k,M}\}$ and $\{\nabla u_{k,M}\}$ converge locally uniformly in $\mathbb{R}^N_+$, when $k \to \infty$, to a function $u_{\infty,M}$ which satisfies (1.1) in $\mathbb{R}^N_+$. Furthermore

$$T_\ell[u_{\infty,M}] = u_{\infty, M\ell^{\frac{2p-q(p+1)}{p-1}}} \qquad \text{for all } \ell > 0. \tag{4.72}$$

In the case $q = \frac{2p}{p+1}$ the function $u_{\infty,M}$ is self-similar, hence

$$u_{\infty,M}(r,\sigma) = r^{-\alpha}\tilde{\omega}(\sigma),$$

where $\tilde{\omega}$ is a nonnegative solution of (1.28). Inasmuch $u_{k,M} \geq u_{k,0} = v_k$ (already defined by (4.46)), it follows that

$$u_{\infty,M}(r,\sigma) \geq u_{\infty,0}(r,\sigma) = r^{-\alpha}\psi(\sigma) \Longrightarrow \tilde{\omega} \geq \psi \quad \text{in } S^{N-1}_+. \tag{4.73}$$

Since $u_{k,M}$ is dominated by any self-similar solution of (1.1), it implies that $\tilde{\omega}$ is the minimal positive solution of (1.28) that we denote by $\omega$ hereafter. Up to a subsequence, $\{T_{\ell_n}[u_{\infty,M}]\}$ converges locally uniformly in $\overline{\mathbb{R}^N_+}\setminus\{0\}$ to $u_{\infty,M}$. Consequently

$$\lim_{\ell_n\to 0}\ell_n^\alpha u_{\infty,M}(\ell_n,\sigma)=\omega(\sigma)\quad\text{uniformly in }\ S^{N-1}_+.$$

Because of uniqueness, the whole sequence converges, which implies (4.70).
In the case $q<\frac{2p}{p+1}$, using the a priori estimates from Theorem 2.1 and Theorem 2.3, we obtain that $T_{\ell_n}[u_{\infty,M}](1,\sigma)=\ell_n^\alpha u_{\infty,M}(\ell_n,\sigma)$ converges locally uniformly in $\overline{S^{N-1}_+}$ to $u_{\infty,0}(1,\sigma)$. Since $u_{\infty,0}(1,.)\ge\psi$, it follows that

$$\lim_{\ell_n\to 0}\ell_n^\alpha u_{\infty,M}(\ell_n,\sigma)=\psi(\sigma)\quad\text{uniformly in }\ S^{N-1}_+.$$

Hence (4.70) follows by uniqueness of the function $\psi$.
Uniqueness of positive solution of (1.21) satisfying (4.68) follows from Theorem 4.1 applied with $a=\alpha=\frac{2}{p-1}<\frac{2-q}{q-1}=\beta$.

□

As a consequence of Theorem 1.9-(ii) we have

**Theorem 4.5** *Assume* (4.67)*-(ii) holds, then for any $M>0$ there exists a positive separable solution $u$ of* (1.1) *in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+\setminus\{0\}$*

When $\mathbb{R}^N_+$ is replaced by a bounded domain there holds.

**Theorem 4.6** *Assume $\Omega\subset\mathbb{R}^N_+$ is a bounded smooth domain such that $0\in\partial\Omega$ and $T_{\partial\Omega}(0)=\partial\mathbb{R}^N_+$, and $(p,q)$ satisfies* (4.67)*-(i). Then for any $M\ge 0$ there exists a positive solution $u$ of* (1.1) *in $\Omega$ vanishing on $\partial\Omega\setminus\{0\}$ such that*

$$\lim_{x\to 0}\frac{u(x)}{P_\Omega(x)}=\infty,\tag{4.74}$$

*where $P_\Omega$ is the Poisson kernel in $\Omega$. Furthermore*
*(i) If $1<q<\frac{2p}{p+1}$, then*

$$\lim_{r\to 0}r^\alpha u(r,.)=\psi\qquad\text{locally uniformly in }\ S^{N-1}_+,\tag{4.75}$$

*where $\psi$ is the unique positive solution of*

$$\begin{aligned}-\Delta'\psi+\alpha(N-2-\alpha)\psi+\psi^p&=0\qquad&&\text{in } S^{N-1}_+\\ \psi&=0\qquad&&\text{in } \partial S^{N-1}_+.\end{aligned}$$

*(ii) If $q=\frac{2p}{p+1}$, then*

$$\psi\le\liminf_{r\to 0}r^\alpha u(r,.)\le\limsup_{r\to 0}r^\alpha u(r,.)\le\omega\qquad\text{locally uniformly in }\ S^{N-1}_+.\tag{4.76}$$

*Proof.* As in the proof of Theorem 4.4, the sequence $\{u_k\}$ of the solution of (1.8) which satisfy (4.42) is increasing. Since it is bounded from above by the restriction to $\Omega$ of the solutions of the same equation in $\mathbb{R}^N_+$, vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and satisfying (4.68), it admits a limit $u_\infty$ which is a solution of 1.8 which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies (4.74). In order to have an estimate of the blow-up rate, we recall that the solution $v_k$ of (4.65) is a subsolution of (1.1) and $u_k \geq v_k$ Furthermore $\{v_k\}$ converges to $\{v_\infty\}$ which is a positive solution of (1.1) in $\Omega$, vanishing on $\partial\Omega \setminus \{0\}$ and such that

$$\lim_{r\to 0} r^\alpha v_\infty(r,\sigma) = \psi(\sigma) \quad \text{locally uniformly in } S^{N-1}_+. \tag{4.77}$$

Combined with (4.69) and (4.70) it implies (4.75) and (4.76) since the solution $u_k$ in $\Omega$ is bounded from above by the solution in $\mathbb{R}^N_+$.

$$\liminf_{r\to 0} r^\alpha u_\infty(r,\sigma) \geq \psi(\sigma) \quad \text{locally uniformly in } S^{N-1}_+. \tag{4.78}$$

□

**Theorem 4.7** *Assume $\Omega \subset \mathbb{R}^N_+$ is a bounded smooth domain such that $0 \in \partial\Omega$ and $T_{\partial\Omega}(0) = \partial\mathbb{R}^N_+$, $p = \frac{N+1}{N-1}$ and $q = \frac{2p}{p+1} = \frac{N+1}{N}$. If*

$$\text{dist}\,(x, \mathbb{R}^N_+) \leq c_{21}|x|^N \qquad \textit{for all } \ x \in \partial\Omega \cap B_\delta, \tag{4.79}$$

*for some constants $\delta, c_{21} > 0$, then there exists a positive solution $u$ of (1.1) in $\Omega$, vanishing on $\partial\Omega \setminus \{0\}$ such that*

$$\lim_{r\to 0} r^\alpha u(r,\sigma) = \omega(\sigma) \quad \textit{locally uniformly in } S^{N-1}_+. \tag{4.80}$$

*Proof.* The function $u_\omega(r,.) = r^{1-N}\omega$ satisfies (1.1) in $\mathbb{R}^N_+$ and vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$. Since $\nabla\omega$ is bounded, it satisfies

$$u(x) \leq c_{22} \qquad \text{for all } \ x \in \partial\Omega \setminus \{0\},$$

for some constant $c_{22} > 0$. Then the result follows from Proposition 3.3. □

### 4.3.2 The case $\frac{2p}{p+1} < q < p$

If

$$1 < p < \frac{N+1}{N-1} \ \text{ and } \ \frac{2p}{p+1} < q < \frac{N+1}{N}, \tag{4.81}$$

there exists fundamental solutions $u_k$ in $\mathbb{R}^N_+$ by Theorem 1.7, or in $\Omega$ by Theorem 1.8. Since the mapping $k \mapsto u_k$ is increasing and $u_k$ is bounded from above the function $u_\infty = \lim_{k\to\infty} u_k$ is a solution of (1.1) in $\mathbb{R}^N_+$ (resp. $\Omega$) vanishing on $\mathbb{R}^N_+ \setminus \{0\}$ (resp. $\Omega \setminus \{0\}$) which satisfies (4.68) (resp. (4.74)). However the blow-up rate of $u_\infty$ is not easy to obtain from scaling methods since the transformation $T_\ell$ transform (1.1) into (4.62) where $M$ is replaced by $M\ell^{\frac{2p-q(p+1)}{p-1}}$ which is not bounded when $\ell \to 0$.

When $q > \frac{2p}{p+1}$, the natural exponent is $\gamma$ defined by (1.34) The transformation $S_\ell$ defined for $\ell > 0$ by

$$S_\ell[u](x) = \ell^\gamma u(\ell x), \tag{4.82}$$

transforms (1.1) into

$$-\ell^{\frac{q(p+1)-2p}{p-q}}\Delta u + |u|^{p-1}u - M|\nabla u|^q = 0. \tag{4.83}$$

When $\ell \to 0$, the limit equation is an eikonal equation (up to change of unknown),

$$|u|^{p-1}u - M|\nabla u|^q = 0. \tag{4.84}$$

Separable solutions of (1.3) in $\mathbb{R}^N_+$ are under the form $u_\eta(r,.) = r^{-\gamma}\eta$ and $\eta$ satisfies

$$|\eta|^{p-1}\eta - M(\gamma^2\eta^2 + |\nabla'\eta|^2)^{\frac{q}{2}} = 0 \quad \text{in } S^{N-1}_+. \tag{4.85}$$

Clearly this equation admits no $C^1$ solution but for the constant ones. As limit of solutions with vanishing viscosity, the solutions that we obtain are viscosity solutions outside the origin. We will look for solutions having a strong singularity by the method of sub and supersolutions. Note that (1.3) admits an explicit radial singular solution, namely

$$U(x) = \omega_0|x|^{-\gamma} := \gamma^\gamma M^{\frac{1}{p-q}}|x|^{-\gamma}. \tag{4.86}$$

*Proof of Theorem 1.11.* For $n > 0$ set $U_n(r) = nr^{-\gamma}$. As

$$\gamma(p-1) + 2 = -q(p+1) + \gamma + 2 = \frac{2p - q(p+1)}{p-q},$$

we have

$$n^{-1}r^{-2-\gamma}\mathcal{L}_{q,M}U_n = -\gamma(\gamma + 2 - N) + n^{q-1}(n^{p-q} - \gamma^q M)r^{2-(p-1)\gamma}.$$

Since $\gamma + 2 - N > 0$ because $q > \frac{2p}{p+1}$ and $p < \frac{N+1}{N-1}$, for any $n > \omega_0$ there exists $r_n > 0$ such that

$$n^{q-1}(n^{p-q} - \gamma^q M)r_n^{2-(p-1)\gamma} = \gamma(\gamma + 2 - N).$$

It implies that $U_n$ is a super solution of (1.1) in $B_{r_n} \setminus \{0\}$. Furthermore

$$r_n = \left(\frac{n^{p-1}}{\gamma(\gamma + 2 - N)}\right)^{\frac{1}{(p-1)\gamma - 2}}(1 + o(1)) \quad \text{when } \ n \to \infty. \tag{4.87}$$

For a subsolution we set

$$W_m(r,\sigma) = mr^{-\gamma}\phi_1(\sigma), \tag{4.88}$$

where $m > 0$. Then

$$\begin{aligned} r^{p\gamma}\mathcal{L}_{q,M}W_m = -mr^{\frac{q(p+1)-2p}{p-q}} &\left(\gamma^2 - (N-2)\gamma + 1 - N\right)\phi_1 \\ &+ m^q\left(m^{p-q}\phi_1^p - M\left(\gamma^2\phi_1^2 + |\nabla'\phi_1|^2\right)^{\frac{q}{2}}\right), \end{aligned} \tag{4.89}$$

and this expression is negative for $m > 0$ small enough. Set

$$P(X) = X^2 - (N-2)X + 1 - N = (X+1)(X+1-N).$$

Then

$$P(\gamma) = \frac{p\,(Nq - (N-1)p)}{(p-q)^2}.$$

*We first give the proof when* $Nq \geq (N-1)p$. In such case $P(\gamma) \geq 0$. Hence there exists $m_0 > 0$ such that for any $0 < m \leq m_0$, $W_m$ is a subsolution in $\mathbb{R}^N_+$, smaller than $U_n$ and it is bounded on $\partial B^+_{r_n} \setminus \{0\}$. When $m \leq m_0$, the function $W_m$ defined in (4.88) is a subsolution of (1.1) in $\mathbb{R}^N_+$. Since $W_m$ is bounded on $\partial B^+_{r_n} \setminus \{0\}$ there exists a nonnegative solution $u_n$ of (1.1) in $B^+_{r_n}$ which vanishes on $B^+_{r_n} \setminus \{0\}$ and there holds

$$(W_m(x) - mr_n^{-\gamma})_+ \leq u_n(x) \leq U_n(x) \quad \text{for all } \ x \in B^+_{r_n}. \tag{4.90}$$

The fact that $B^+_{r_n}$ is just a Lipschitz domain is easily bypassed by smoothing it in a neighborhood of $\partial B_{r_n} \cap \mathbb{R}^N_+$. Furthermore, by (2.1) and (2.18),

$$u_n(x) \leq c_5 \max\left\{|x|^{-\alpha}, M^{\frac{1}{p-q}}|x|^{-\gamma}\right\}. \tag{4.91}$$

and for any $r_0 > 0$, there exists $c_8 > 0$ depending on $r_0$ such that

$$|\nabla u_n(x)| \leq c_8 \max\left\{|x|^{-\alpha-1}, M^{\frac{1}{p-q}}|x|^{-\gamma-1}\right\}. \tag{4.92}$$

By standard local regularity theory, there exists a subsequence $\{u_{n_j}\}$ which converges in the $C^1(K)$-topology for any compact set $K \subset \overline{\mathbb{R}^N_+} \setminus \{0\}$ to a positive solution $u$ of (1.1) in $\mathbb{R}^N_+$ which vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and satisfies (1.35).
*Next we assume* $Nq < (N-1)p$. Observe that $\gamma^2\phi_1^2 + |\nabla'\phi_1|^2 \geq \delta^2 > 0$, then

$$m^p\phi_1^p - Mm^q\left(\gamma^2\phi_1^2 + |\nabla'\phi_1|^2\right)^{\frac{q}{2}} \leq m^p - Mm^q\delta^q.$$

Thus, from (4.89) we obtain

$$r^{p\gamma}\mathcal{L}_{q,M}W_m \leq -mr^{\frac{q(p+1)-2p}{p-q}}P(\gamma) + m^p - Mm^q\delta^q, \tag{4.93}$$

and $P(\gamma) < 0$. If we choose

$$m = \delta^\gamma\left(\frac{M}{2}\right)^{\frac{1}{p-q}},$$

then

$$m^p - Mm^q\delta^q = -\frac{Mm^q\delta^q}{2}.$$

Therefore $\mathcal{L}_{q,M}W_m \leq 0$ on $B^+_{r_*}$ where

$$r_* = \left(\frac{Mm^{q-1}\delta^q}{-2P(\gamma)}\right)^{\frac{p-q}{q(p+1)-2p}}.$$

If $a = mr_*^{-\gamma}$, then $W_m \le a$ in $\partial B^+_{r_*}$, thus $W_{m,a} = (W_m - a)+$ is nonnegative in $B^+_{r_*}$ and it is a subsolution of (1.1) in $B^+_{r_*}$ which vanishes on $\partial B^+_{r_*} \setminus \{0\}$. If we extend it by 0 in $\mathbb{R}^N_+$, the new function is a a subsolution of (1.1) which belongs to $W^{1,\infty}_{loc}(\overline{\mathbb{R}^N_+} \setminus \{0\})$. We end the proof using Proposition 3.4 as in the previous case. □

If $\mathbb{R}^N_+$ is replaced by a bounded domain we have the following result.

**Theorem 4.8** *Let $M > 0$ and $\frac{2p}{p+1} < q < p$. If $\Omega \subset \mathbb{R}^N_+$ is a bounded smooth domain such that $0 \in \partial\Omega$ and $T_{\partial\Omega}(0) = \partial\mathbb{R}^N_+$. If*

$$\operatorname{dist}(x, \partial\mathbb{R}^N_+) \le c_{23}|x|^{\frac{p}{p-q}} \quad \text{for all } x \in \partial\Omega \text{ near } 0, \tag{4.94}$$

*for some constant $c_{23} > 0$. Then there exists a positive solution $u$ of (1.1) in $\Omega$ vanishing on $\partial\Omega \setminus \{0\}$ satisfying, for some $m > 0$,*

$$m\phi_1(\sigma) \le \liminf_{r\to 0} r^\gamma u(r,\sigma) \le \limsup_{r\to 0} r^\gamma u(r,\sigma) \le \omega_0, \tag{4.95}$$

*uniformly on any compact set $K \subset S^{N-1}_+$.*

*Proof.* We recall that $\phi_1$ is the first eigenfunction of $-\Delta'$ in $W^{1,2}_0(S^{N-1})$. Let $R > 0$ and $B := B_R(a) \subset \Omega$ be an open ball tangent to $\partial\Omega$ at 0. Up to rescaling and since the result does not depend on the value of $M$ we can assume that $R = 1$. We set $w_m(x) = m|x|^{-\theta}P_B(x)$ where $\theta = \gamma + 1 - N$ and $P_B$ is the Poisson kernel in $B$ expressed by

$$P_B(x) = \frac{1 - |x-a|^2}{\sigma_N|x|^N},$$

where $\sigma_N$ is the volume of the unit sphere in $\mathbb{R}^N$. Then

$$\begin{aligned} m^{-1}&\mathcal{L}_{q,M} w_m \\ &= -(\theta^2 + (2-N)\theta)|x|^{-\theta-2}P_B(x) + 2\theta|x|^{-\theta-1}\langle \nabla P_B(x), \tfrac{x}{|x|}\rangle + m^{p-1}|x|^{-p\theta}P_B^p(x) \\ &\quad - Mm^{q-1}\left(\theta^2|x|^{-2(\theta-1)}P_B^2(x) + |x|^{-2\theta}|\nabla P_B(x)|^2 - 2\theta|x|^{-2\theta-1}\langle \nabla P_B(x), \tfrac{x}{|x|}\rangle\right)^{\frac{q}{2}}. \end{aligned} \tag{4.96}$$

Since

$$\nabla P_B(x) = -\frac{1}{\sigma_N}\left(\frac{N(1-|x-a|^2)}{|x|^{N+1}}\frac{x}{|x|} + \frac{2(x-a)}{|x|^N}\right),$$

then

$$\begin{aligned} \langle \nabla P_B(x), \tfrac{x}{|x|}\rangle &= -\frac{1}{\sigma_N|x|^{N+1}}\left((N-1)(1-|x-a|^2) + |x|^2\right) \\ &= -\frac{N-1}{|x|}P_B(x) - \frac{1}{\sigma_N|x|^{N-1}}, \end{aligned}$$

which implies in particular

$$|\nabla P_B(x)| \ge \frac{N-1}{|x|}P_B(x) + \frac{1}{\sigma_N|x|^{N-1}}.$$

If $q \geq \frac{N-1}{N}p$, equivalently $\theta \geq 0$, we have

$$\begin{aligned}|\nabla w_m|^2 &= \theta^2|x|^{-2(\theta+1)}P_B^2(x) + |x|^{-2\theta}|\nabla P_B(x)|^2 - 2\theta|x|^{-2\theta-1}\langle \nabla P_B(x), \tfrac{x}{|x|}\rangle \\ &\geq \theta^2|x|^{-2(\theta+1)}P_B^2(x) + |x|^{-2\theta}\left(\frac{N-1}{|x|}P_B(x) + \frac{1}{\sigma_N|x|^{N-1}}\right)^2 \\ &\qquad + 2\theta|x|^{-2\theta-1}\left(\frac{N-1}{|x|}P_B(x) + \frac{1}{\sigma_N|x|^{N-1}}\right) \\ &\geq (\theta^2 + (N-1)^2)|x|^{-2(\theta+1)}P_B^2(x).\end{aligned} \tag{4.97}$$

Hence

$$\begin{aligned}m^{-1}\mathcal{L}_{q,M}w_m &\leq -(\theta^2 + N\theta)|x|^{-\theta-2}P_B(x) + m^{p-1}|x|^{-p\theta}P_B^p(x) \\ &\qquad - m^{q-1}M(\theta^2 + (N-1)^2)^{\frac{q}{2}}|x|^{-q(\theta+1)}P_B^q(x) \\ &\leq m^{q-1}|x|^{-p\theta}P_B^q(x)\left(m^{p-q}P_B^{p-q}(x) - M(\theta^2 + (N-1)^2)|x|^{(p-q)\theta-q}\right).\end{aligned} \tag{4.98}$$

Now

$$P_B(x) \leq \frac{2}{\sigma_N|x|^{N-1}} \Longrightarrow P_B^{p-q}(x) \leq \left(\frac{2}{\sigma_N}\right)^{p-q}|x|^{(1-N)(p-q)}.$$

Since $(1-N)(p-q) = (p-q)\theta - q$, we obtain finally that,

$$m^{-1}\mathcal{L}_{q,M}w_m \leq m^{q-1}|x|^{-q\theta}P_B^q(x)\left(m^{p-q}\left(\frac{2}{\sigma_N}\right)^{p-q} - M(\theta^2 + (N-1)^2)\right).$$

Choosing $m$ small enough we deduce that $w_m$ is a subsolution in $B$. If we extend it by 0 in $\Omega \setminus B$, the new function denoted by $\tilde{w}$ is a nonnegative subsolution of (1.1) in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies (4.95). The proof follows from Proposition 3.4.

If $q < \frac{N-1}{N}p$, then $\theta < 0$. Since $\langle \nabla P_B(x), \frac{x}{|x|}\rangle \leq 0$, (4.97) is replaced by

$$\begin{aligned}|\nabla w_m|^2 &= \theta^2|x|^{-2(\theta+1)}P_B^2(x) + |x|^{-2\theta}|\nabla P_B(x)|^2 - 2\theta|x|^{-2\theta-1}\langle \nabla P_B(x), \tfrac{x}{|x|}\rangle \\ &\geq \theta^2|x|^{-2(\theta+1)}P_B^2(x) + |x|^{-2\theta}\left(\frac{N-1}{|x|}P_B(x) + \frac{1}{\sigma_N|x|^{N-1}}\right)^2 \\ &\qquad + 2\theta|x|^{-2\theta-1}\left(\frac{N-1}{|x|}P_B(x) + \frac{1}{\sigma_N|x|^{N-1}}\right) \\ &\geq (\theta^2 + (N-1)^2)|x|^{-2(\theta+1)}P_B^2(x) + \left(\frac{1}{\sigma_N^2|x|^{2(N+\theta-1)}} + \frac{2\theta}{\sigma_N|x|^{N+2\theta}}\right) \\ &\qquad + 2(N-1)\left(\frac{1}{\sigma_N|x|^{N+2\theta}} + \frac{\theta}{|x|^{2\theta+2}}\right)P_B(x).\end{aligned} \tag{4.99}$$

Set

$$\tilde{r} = \min\left\{2, \left(\frac{1}{2\sigma_N|\theta|}\right)^{\frac{1}{N-2}}\right\}. \tag{4.100}$$

If $x \in B \cap B_{\tilde{r}}(0)$, the two last terms in (4.99) are nonnegative, hence

$$|\nabla w|^2 \geq (\theta^2 + (N-1)^2)|x|^{-2(\theta+1)} P_B^2(x) \qquad \text{for all } x \in B \cap B_{\tilde{r}}(0). \tag{4.101}$$

Note that $B \cap B_{\tilde{r}}(0) = B$ if $\tilde{r} = 2$. Choosing $m > 0$ small enough we infer that $w_m$ is a subsolution of (1.1) in $B \cap B_{\tilde{r}}(0)$. Denoting by $\hat{m}$ the maximum of $w_m$ on $\partial(B \cap B_{\tilde{r}}(0)) \setminus \{0\}$, then $(w_m - \hat{m})_+$ is a subsolution in $\Omega$. Since the restriction to $\Omega$ of the solution constructed in Theorem 1.11 dominates $(w_m - \hat{m})_+$, the proof follows as in the first case. □

### 4.3.3 Open problems

*Problem 1.* Under what conditions are the positive solutions of problem (1.28) unique ? If instead of separable solutions in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ one looks for separable radial solutions of (1.1) in $\mathbb{R}^N \setminus \{0\}$ (with $q = \frac{2p}{p+1}$) , then they are under the form

$$U(x) = A|x|^{-\alpha} \tag{4.102}$$

and $A$ is a positive root of the polynomial

$$P(X) = X^{p-1} - M\alpha^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} + \alpha(N-2-\alpha). \tag{4.103}$$

A complete study of the radial solutions of (1.1) is provided in [7], however it is straightforward to check that if $1 < p < \frac{N}{N-2}$, there exists a unique positive root, hence a unique positive separable solution, while if $p > \frac{N}{N-2}$, there exists a unique positive root (resp. two positive roots) if

$$M = (p+1)\left(\frac{p(N-2)-N}{2p}\right)^{\frac{p}{p+1}} := m^*, \tag{4.104}$$

(resp. $M > m^*$). Uniqueness of solution plays a fundamental role in the description and classification of all the positive solutions with an isolated singularity at 0.

*Problem 2.* It is proved in [7] that if $\max\{\frac{N}{N-1}, \frac{2p}{p+1}\} < q < \min\{2, p\}$ and $M > 0$, there exist infinitely many local radial solutions of of (1.1) in $\mathbb{R}^N \setminus \{0\}$ which satisfies

$$u(r) = \xi_M r^{-\beta}(1 + o(1)) \quad \text{as } r \to 0 \tag{4.105}$$

where

$$\beta = \frac{2-q}{q-1} \quad \text{and } \xi_M = \frac{1}{\beta}\left(\frac{(N-1)q-N}{M(p-1)}\right)^{\frac{1}{p-1}}. \tag{4.106}$$

These solutions present the property that there blow-up is smaller than the one of the explicit radial separable solution. It would be interesting to construct such solutions of (1.1) in $\mathbb{R}^N_+$ (or more likely $B_R^+$), vanishing on $\partial\mathbb{R}^N \setminus \{0\}$.

*Problem 3.* Is it possible to define a boundary trace for any positive solution of (1.1) in $\mathbb{R}^N_+$, noting the fact such a result holds separately for positive solutions of (1.2) and (1.4) ? A related problem would be to define an initial trace for any positive solution of the parabolic equation

$$\partial_t u - \Delta u + u^p - M|\nabla u|^q = 0, \tag{4.107}$$

in $(0,T) \times \mathbb{R}^N$. Initial trace of semilinear parabolic equations ($M = 0$ in (4.107)) are studied in [17], [14].

*Problem 4.* Are the positive solutions of 1.8 satisfying (1.24) or (4.75) unique without the flatness and the starshapedness assumptions of Theorem 4.2. More generaly, are the weak solutions of the Dirichlet problem with measure boundary data (1.14) unique ?

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