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Abstract

We study properties of positive functions satisfying (E) $-\Delta u + u^p - M|\nabla u|^q = 0$ is a domain Ω or in $\mathbb{R}^N_+$ when $p > 1$ and $1 < q < \min\{p, 2\}$. We concentrate our research on the solutions of (E) vanishing on the boundary except at one point. This analysis depends on the existence of separable solutions in $\mathbb{R}^N_+$. We consruct various types of positive solutions with an isolated singularity on the boundary. We also study conditions for the removability of compact boundary sets and the Dirichlet problem associated to (E) with a measure for boundary data.

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Contents

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1 Introduction

The aim of this article is to study some properties of solutions of the following equation

$$\mathcal{L}_{q,M}u := -\Delta u + |u|^{p-1}u - M|\nabla u|^q = 0 \tag{1.1}$$

in a bounded domain Ω of $\mathbb{R}^N$ or in the half-space $\mathbb{R}^N_+$, where $M > 0$ and $p > q > 1$. We are particularly interested in the analysis of boundary singularities of such solutions. If $M = 0$ the boundary singularities problem has been investigated since thirty years, starting with the work of Gmira and Véron [13] who obtained an almost complete description of the solutions with isolated boundary singularities. When $M > 0$ there is a balance between the absorption term $|u|^{p-1}u$ and the source term $M|\nabla u|^q$, a confrontation which can create very new effects. Furthermore, the scale of the two opposed reaction terms depends upon the position of q with respect to $\frac{2p}{p+1}$. This is due to the fact that (1.1) is equivariant with respect to the scaling transformation T_ℓ defined for $\ell > 0$ by $T_\ell[u](x) = \ell^{\frac{2}{p-1}}u(\ell x)$.
If $q < \frac{2p}{p+1}$, the absorption term is dominant and the behaviour of the singular solutions is modelled by the equation studied in [13]

$$-\Delta u + |u|^{p-1}u = 0. \tag{1.2}$$

If $q > \frac{2p}{p+1}$, the source term is dominant and the behaviour of the singular solutions is modelled by positive separable solutions of

$$u^p - M|\nabla u|^q = 0. \tag{1.3}$$

Another associated equation which plays an important role in the construction of singular solutions since its positive solutions are supersolution of (1.1) is

$$-\Delta u - M|\nabla u|^q = 0. \tag{1.4}$$

Note that in (1.3) and (1.4), M can be fixed to be 1 by homothety.
If $q = \frac{2p}{p+1}$, the coefficient $M > 0$ plays a fundamental role in the properties of the

set of solutions, in particular for the existence of singular solutions and removable singularities. This situation is similar in some sense to what happens for equation

$$-\Delta u = |u|^{p-1}u + M|\nabla u|^q \tag{1.5}$$

which is studied thoroughfly in [8], [9].

In the present paper we will consider the case where $1 < q < \min\{2, p\}$, with a special emphasis on the case $q = \frac{2p}{p+1}$ which allows to put into light the role of the value of M. We first analyze the following problem: given a smooth bounded domain $\Omega \subset \mathbb{R}^N$ such that $0 \in \partial\Omega$, under what conditions involving p, q and M is the point 0 a removable singularity for a solution of (1.1) continuous in $\overline{\Omega} \setminus \{0\}$ and vanishing on $\partial\Omega \setminus \{0\}$? Or first result is the following:

Theorem 1.1 *Assume $p \geq \frac{N+1}{N-1}$, $M > 0$ and*
(i) either $p = \frac{N+1}{N-1}$ and $1 < q < 1 + \frac{1}{N}$.
(ii) or $p > \frac{N+1}{N-1}$ and $1 < q \leq \frac{2p}{p+1}$,
Then any nonnegative weak solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{0\})$ of

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^q &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{in } \partial\Omega \setminus \{0\}. \end{aligned} \tag{1.6}$$

verifies $\nabla u \in L^q(\Omega; d)$, $u \in L^p(\Omega; d)$ and is a weak solution of

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^q &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{in } \partial\Omega. \end{aligned} \tag{1.7}$$

Furthermore, if we assume either (i) or
(iii) or $p > \frac{N+1}{N-1}$ and $1 < q < \frac{2p}{p+1}$.
(iv) or $p > \frac{N+1}{N-1}$, $q = \frac{2p}{p+1}$ and

$$M < m^{**} := (p+1)\left(\frac{(N-1)p - (N+1)}{2p}\right)^{\frac{p}{p+1}}, \tag{1.8}$$

then $u = 0$.

Combining the method used in proving Theorem 1.1 with the result of [16] we prove the removability of compact boundary sets on $\partial\Omega$, provided they satisfy some some zero Bessel capacity property.

Theorem 1.2 *Let $p > r > \frac{N+1}{N-1}$ and*

$$m_r^{**} = (p+1)\left(\frac{p-r}{p(r-1)}\right)^{\frac{p}{p+1}}. \tag{1.9}$$

If one of the following conditions is satisfied:

1- either $q = \frac{2p}{p+1}$ *and* $M < m_r^{**}$,
2- or $q < \frac{2p}{p+1}$, $r \leq 3$ *and* M *is arbitrary,*
then for any compact set $K \subset \partial\Omega$ *such that* $cap^{\partial\Omega}_{\frac{2}{r},r'}(K) = 0$, *any solution* u *of*

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^q = 0 \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega \setminus K, \end{aligned} \tag{1.10}$$

is identically 0.

The capacitary framework allows to consider the Dirichlet problem for (1.1) with Radon measure as boundary data. When the two exponents are super-critical with respect to the equations (1.2) and (1.4), the admissibility condition on measure for (1.1) necessitates the introduction of two different capacities.

Theorem 1.3 *Let* $p > 1$, $\frac{N+1}{N} \leq q \leq \frac{2p}{p+1}$ *and* μ *be a nonnegative Radon measure on* $\partial\Omega$ *such that for some constant* $c_1 > 0$, *there holds for any Borel set* $E \subset \partial\Omega$,

$$\begin{aligned} &(i) \quad \mu(E) \leq c_{15} cap^{\partial\Omega}_{\frac{2}{p},p'}(E) && \text{if } \tfrac{N+1}{N} < q \leq \tfrac{2p}{p+1}, \\ &(ii) \quad \mu(E) \leq c_{16}\left(cap^{\partial\Omega}_{\frac{2}{p},p'}(E)\right)^{\frac{2N}{N-1}} && \text{if } p = \tfrac{N+1}{N-1} \text{ and } q = \tfrac{N+1}{N}. \end{aligned} \tag{1.11}$$

There there exists $c_{17} > 0$ *such that for any* $0 < c \leq c_{17}$ *there exists a nonnegative fonction* $u \in L^1(\Omega) \cap L^p_\rho(\Omega)$ *such that* $|\nabla u| \in L^q_\rho(\Omega)$ *solution of*

$$\begin{aligned} -\Delta u + u^p = M|\nabla u|^q \qquad & \text{in } \Omega \\ u = c\mu \qquad & \text{in } \partial\Omega. \end{aligned} \tag{1.12}$$

Furthermore the boundary trace of u *is the measure* $c\mu$.

Since the exponents p and q can be separately super or sub-critical, or even both sub-critical, we have the following result in this configuration of exponents.

Theorem 1.4 *(i) If* $1 < p < \frac{N+1}{N-1}$ *and* $1 < q < \frac{N+1}{N}$, *then there exists* $c_0 > 0$ *such that for* $\mu \in \mathfrak{M}_+(\partial\Omega)$ *with* $\|\mu\|_{\mathfrak{M}} \leq c_0$, *there exists a very weak solution to*

$$\begin{aligned} -\Delta u + u^p = M|\nabla u|^q \qquad & \text{in } \Omega \\ u = \mu \qquad & \text{in } \partial\Omega. \end{aligned} \tag{1.13}$$

(ii) If $1 < p < \frac{N+1}{N-1}$ *and* $\frac{N+1}{N} \leq q < 2$, *there exists* $c_0 > 0$ *such that for* $\mu \in \mathfrak{M}_+(\partial\Omega)$ *satisfying* (2.60), *problem* (1.12) *admits a very weak solution if* $0 < c \leq c_0$.
(iii) If $p \geq \frac{N+1}{N-1}$ *and* $1 < q < \frac{N+1}{N}$ *there exists* $c_0 > 0$ *such that for* $\mu \in \mathfrak{M}_+(\partial\Omega)$ *absolutely continuous with respect to* $cap^{\partial\Omega}_{\frac{2}{p},p'}$ *and with* $\|\mu\|_{\mathfrak{M}} \leq c_0$, *there exists a very weak solution to* (1.13).

In the sub-critical case (i) and when μ is a Dirac mass at 0 on the boundary we have no restriction on its weight.

Theorem 1.5 *Assume $1 < p < \frac{N+1}{N-1}$ and $1 < q < \frac{N+1}{N}$. Then for any $k \geq 0$ there exists a minimal positive solution u_k of*

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^q = 0 \qquad & \text{in } \mathbb{R}^N_+ \\ u = 0 \qquad & \text{in } \partial\mathbb{R}^N_+ \setminus \{0\}, \end{aligned} \tag{1.14}$$

satisfying

$$\lim_{x\to 0} \frac{u_k(x)}{P_N(x)} = k \tag{1.15}$$

where $P_N(x) = c_N x_{_N} |x|^{-N}$ is the Poisson kernel in $\mathbb{R}^N_+$. This function satisfies $u_k \in L^1_{loc}(\mathbb{R}^N_+) \cap L^p_{loc}(\mathbb{R}^N_+; x_{_N} dx)$, $\nabla u_k \in L^q_{loc}(\mathbb{R}^N_+; x_{_N} dx)$ and

$$\int_{\mathbb{R}^N_+} \left(-u_k \Delta\zeta + (u_k^p - M|\nabla u_k|^q)\zeta\right) dx = k \frac{\partial \zeta}{\partial x_{_N}} \quad \text{for all } \zeta \in C^{1,1}_c(\overline{\mathbb{R}^N_+}). \tag{1.16}$$

The proof is completely different from the ones of Theorem 1.3 and Theorem 1.4 and is based upon a delicate construction of supersolutions and subsolutions. A similar result holds if $\mathbb{R}^N_+$ is replaced by a bounded smooth domain $\Omega \subset \mathbb{R}^N_+$ such that $0 \in \partial\Omega$.

Theorem 1.6 *Assume $1 < p < \frac{N+1}{N-1}$ and $0 < q < \frac{N+1}{N}$. Then for any $M > 0$ and $k > 0$ there exists a minimal solution $u_k \in C^1(\overline{\Omega} \setminus \{0\})$ of (1.1) satisfying*

$$\lim_{x\to 0} \frac{u_k(x)}{P_\Omega(x)} = k, \tag{1.17}$$

where P_Ω is the Poisson kernel in Ω. Furthermore $u_k \in L^1(\Omega) \cap L^p(\Omega; \rho dx)$, $\nabla u_k \in L^q(\Omega; \rho dx)$ where where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$, and

$$\int_\Omega \left(-u_k \Delta\zeta + (u_k^p - M|\nabla u_k|^q)\zeta\right) dx = -k \frac{\partial \zeta}{\partial \mathbf{n}} \quad \text{for all } \zeta \in C^{1,1}(\Omega). \tag{1.18}$$

In order to study the behaviour of these solutions u_k when $k \to \infty$ we have to introduce separable solutions of (1.1) in the model case $\mathbb{R}^N_+$. They are solutions of

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^{\frac{2p}{p+1}} = 0 \qquad & \text{in } \mathbb{R}^N_+ \\ u = 0 \qquad & \text{in } \partial\mathbb{R}^N_+ \setminus \{0\}, \end{aligned} \tag{1.19}$$

which have the following expression in spherical coordinates

$$u(r,\sigma) = r^{-\frac{2}{p-1}} \omega(\sigma) \qquad \text{for all } (r,\sigma) \in (0,\infty) \times S^{N-1}_+.$$

Put

$$\alpha = \frac{2}{p-1},$$

and denote by Δ' and ∇' the Laplace-Beltrami operator and the spherical gradient, then ω satisfies

$$\begin{aligned} -\Delta'\omega + \alpha(N-2-\alpha)\omega + |\omega|^{p-1}\omega - M\left(\alpha^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{p+1}} = 0 \qquad & \text{in } S^{N-1}_+ \\ \omega = 0 \qquad & \text{in } \partial S^{N-1}_+. \end{aligned} \tag{1.20}$$

Theorem 1.7 *There exists a positive solution ω to problem* (1.20) *if one of the following conditions is satisfied:*
(i) either $1 < p < \frac{N+1}{N-1}$ *and* $M \geq 0$,
(ii) or $p = \frac{N+1}{N-1}$ *and* $M > 0$,
(iii) or $p > \frac{N+1}{N-1}$ *and* $M \geq M_{N,p}$ *for some explicit value* $M_{N,p} > 0$.

The positive solutions of (1.20) allow to characterize the limit u_∞ of the solutions u_k constructed in Theorem 1.5.

Theorem 1.8 *Let* $1 < p < \frac{N+1}{N-1}$, $1 < q < \frac{N+1}{N}$ *and* $M > 0$, *then*

$$\lim_{x\to 0} \frac{u_\infty(x)}{P_N(x)} = \infty. \tag{1.21}$$

Furthermore
(i) If $1 < q < \frac{2p}{p+1}$

$$\lim_{r\to 0} r^\alpha u_\infty(r,.) = \psi \quad \text{uniformly on } S^{N-1}_+, \tag{1.22}$$

where ψ is the unique positive solution of

$$\begin{aligned} -\Delta'\psi + \alpha(N-2-\alpha)\psi + |\psi|^{p-1}\psi = 0 \qquad & \text{in } S^{N-1}_+ \\ \psi = 0 \qquad & \text{in } \partial S^{N-1}_+. \end{aligned} \tag{1.23}$$

(ii) If $q = \frac{2p}{p+1}$

$$\lim_{r\to 0} r^\alpha u_\infty(r,.) = \omega \quad \text{uniformly on } S^{N-1}_+, \tag{1.24}$$

where ω is the minimal positive solution of (1.20).

A similar result holds if $\mathbb{R}^N_+$ is replaced by a bounded smooth domain $\Omega \subset \mathbb{R}^N_+$, which boundary contains 0 provided some flatness condition near 0 is satisfied. When $\frac{2p}{p+1} < q < \min\{2,p\}$, the situation is completely changed and the solutions with strong boundary blow-up are modelized by equation (1.3). We prove the following result.

Theorem 1.9 *Assume* $M > 0$ *and* $\frac{2p}{p+1} < q < \min\{2,p\}$. *Then there exists a positive solution u of* (1.1) *in* $\mathbb{R}^N_+$, *which vanishes on* $\partial\mathbb{R}^N_+ \setminus \{0\}$ *such that*

$$m\phi_1(\sigma) r^{-\gamma} \leq u(r,\sigma) \leq c_5 \max\left\{ r^{-\alpha}, M^{\frac{1}{p-q}} r^{-\gamma} \right\} \quad \text{for all } (r,\sigma) \in (0,r^*) \times \overline{S^{N-1}_+}. \tag{1.25}$$

for some $m > 0$, $r^* \in (0,\infty]$ *and where* $c_5 = c_5(N,p,q) > 0$. *If* $Nq \geq (N-1)p$, $r^* = \infty$.

Note that our construction which is made by mean of supersolutions and subsolutions does not imply that in the case $\frac{2p}{p+1} < q < \frac{N+1}{N}$, the solution u_∞ obtained in *Theorem* 1.8 satisfies (1.25). Again a similar result holds if $\mathbb{R}^N_+$ is replaced by a bounded smooth domain $\Omega \subset \mathbb{R}^N_+$, such that $0 \in \partial\Omega$ under a flatness flatness condition near 0.

Aknowledgements

2 Singular boundary value problems

2.1 A priori estimates

We give two series of estimates for solutions of (1.1) with a boundary singularity according the sign of M.

Theorem 2.1 *Let Ω be a domain such that $0 \in \partial\Omega$, $M \in \mathbb{R}$ and $1 < q < \min\{p, 2\}$. If $u \in C^1(\overline{\Omega} \setminus \{0\})$ is a solution of (1.1) vanishing on $\partial\Omega \setminus \{0\}$, there holds*
1- If $M > 0$, there exists $c_5 = c_5(N, p, q) > 0$ such that

$$u_+(x) \le c_5 \max\left\{ M^{\frac{1}{p-q}} |x|^{-\frac{q}{p-q}}, |x|^{-\frac{2}{p-1}} \right\} \qquad \text{for all } x \in \Omega. \tag{2.1}$$

2- If $M \le 0$, there exist $c_6 = c_6(N, q) > 0$ and $c_7 = c_7(N, p) > 0$ such that

$$u_+(x) \le \min\left\{ c_6 |M|^{-\frac{1}{q-1}} |x|^{-\frac{2-q}{q-1}}, c_7 |x|^{-\frac{2}{p-1}} \right\} \qquad \text{for all } x \in \Omega. \tag{2.2}$$

Proof. We first assume that $\overline{\Omega} \subset B_{R_0}$ for some $R_0 > 0$. Let $\epsilon > 0$, we set

$$j_\epsilon(r) = \begin{cases} 0 & \text{if } r \le 0 \\ \frac{r^2}{2\epsilon} & \text{if } 0 \le r \le \epsilon \\ r - \frac{\epsilon}{2} & \text{if } r \ge \epsilon. \end{cases}$$

If we extend u by 0 in $\overline{\Omega}^c \cap B_{2R_0}$ and set $v_\epsilon = j_\epsilon(u)$ we have

$$\begin{aligned} -\Delta v_\epsilon + v_\epsilon^p - M|\nabla v_\epsilon|^q &= -j'_\epsilon(u)\Delta u - j''_\epsilon(u)|\nabla u|^2 + (j_\epsilon(u))^p - M(j'_\epsilon(u))^q |\nabla u|^q \\ &\le M j'_\epsilon(u) \left(1 - (j'_\epsilon(u))^{q-1}\right) |\nabla u|^q + (j_\epsilon(u))^p - j'_\epsilon(u) u_+^p \\ &\le M \frac{u}{\epsilon} \left(1 - \frac{u^{q-1}}{\epsilon^{q-1}}\right) |\nabla v_\epsilon|^q \chi_{\{0<u<\epsilon\}}. \end{aligned}$$

Letting $\epsilon \to 0$, we deduce from the dominated convergence theorem that $v_0 = \lim_{\epsilon \to 0} v_\epsilon$ is nonnegative and satisfies

$$Lv_0 := -\Delta v_0 + v_0^p - M|\nabla v_0|^q \le 0 \quad \text{in } \mathcal{D}'(B_{2R_0} \setminus \{0\}). \tag{2.3}$$

The case $M > 0$. Following the method of Keller [14] and Osserman [20], we fix $a \in B_{R_0} \setminus \{0\}$, and introduce $U(x) = \lambda(|a|^2 - |x-a|^2)^{-\beta}$ for some $\beta > 0$. Then putting $\rho = |x - a|$ and $\tilde{U}(\rho) = U(x)$, we have

$$\begin{aligned} L\tilde{U} &= -\tilde{U}'' - \frac{N-1}{\rho}\tilde{U}' - M|\tilde{U}'|^q + \tilde{U}^p \\ &= \lambda(|a|^2 - \rho^2)^{-2-\beta} \left[\lambda^{p-1}(|a|^2 - \rho^2)^{2-\beta(p-1)} + 2\beta(N - 2(\beta+1))\rho^2 - 2N\beta|a|^2 \right. \\ &\qquad \left. - M 2^q \beta^q \lambda^{q-1} \rho^q (|a|^2 - \rho^2)^{2+\beta-q(\beta+1)} \right]. \end{aligned}$$

If $M > 0$, the two necessary conditions on $\beta > 0$ to be fulfilled is order $\tilde{U}$ be a supersolution in $B_{|a|}(a)$ are

$$(i) \qquad 2 - \beta(p-1) \le 0 \Longleftrightarrow \beta(p-1) \ge 2,$$

$$(ii) \qquad 2 + \beta - q(\beta+1) \ge 2 - \beta(p-1) \Longleftrightarrow \beta(p-q) \ge q.$$

The above inequalities are satisfied if

$$\beta = \max\left\{\frac{2}{p-1}, \frac{q}{p-q}\right\}. \tag{2.4}$$

If $q > \frac{2p}{p+1}$ then $\beta = \frac{q}{p-q}$ and

$$L\tilde{U} \ge \lambda\left(|a|^2 - \rho^2\right)^{-\frac{2p-q}{p-q}}\left[\lambda^{q-1}\left(\lambda^{p-q} - M2^q\beta^q\rho^q\right)\left(|a|^2-\rho^2\right)^{\frac{2p-q(p+1)}{p-q}} - (3\beta+1)N|a|^2\right].$$

There exists $c_1 > 0$ dependings on N, p and q such that if we choose

$$\lambda = c_1 \max\left\{M^{\frac{1}{p-q}}|a|^{\frac{q}{p-q}}, |a|^{\frac{2p(q-1)}{(p-1)(p-q)}}\right\},$$

there holds

$$L\tilde{U} \ge 0. \tag{2.5}$$

Since $\tilde{U}(x) \to \infty$ when $|x| \to |a|$, we derive by the maximum principle that $v_0 \le \tilde{U}$ in $B_{|a|}(a)$. In particular

$$u_+(x) = v_0(a) \le \tilde{U}(a) = \lambda|a|^{-\frac{2q}{p-q}} = c_1 \max\left\{M^{\frac{1}{p-q}}|a|^{-\frac{q}{p-q}}, |a|^{-\frac{2}{p-1}}\right\}. \tag{2.6}$$

If $q \le \frac{2p}{p+1}$ then $\beta = \frac{2}{p-1}$ and

$$\begin{aligned} L\tilde{U} &\ge \lambda\left(|a|^2-\rho^2\right)^{-\frac{2p}{p-1}}\left[\lambda^{p-1} + \frac{2}{p-1}\left(N - \frac{2(p+1)}{p-1}\right)\rho^2 - \frac{2N}{p-1}|a|^2\right. \\ &\qquad \left. - M2^q\left(\frac{2}{p-1}\right)^q\lambda^{q-1}\rho^q\left(|a|^2-\rho^2\right)^{\frac{2p-q(p+1)}{p-1}}\right] \\ &\ge \lambda\left(|a|^2-\rho^2\right)^{-\frac{2p}{p-1}}\left[\lambda^{p-1} - c_2|a|^2 - c_3\lambda^{q-1}M|a|^{\frac{4p-q(p+3)}{p-1}}\right]. \end{aligned}$$

Hence, if $q = \frac{2p}{p+1}$, (2.5) holds if for some $c_4 > 0$ depending on N, p, q,

$$\lambda = c_4 \max\left\{M^{\frac{p+1}{p(p-1)}}, 1\right\}|a|^{\frac{2}{p-1}},$$

which yields

$$u_+(x) = v_0(a) \le \tilde{U}(a) = \lambda|a|^{-\frac{4}{p-1}} = c_4 \max\left\{M^{\frac{p+1}{p(p-1)}}, 1\right\}|a|^{-\frac{2}{p-1}}. \tag{2.7}$$

While if $q < \frac{2p}{p+1}$, we choose

$$\lambda = c_5 \max\left\{ M^{\frac{1}{p-q}} |a|^{\frac{4p-q(p+3)}{(p-1)(p-q)}}, |a|^{\frac{2}{p-1}} \right\},$$

where $c_5 > 0 = c_5(N,p,q)$, which yields

$$u_+(a) = v_0(a) \le \tilde{U}(a) = \lambda |a|^{-\frac{4}{p-1}} = c_5 \max\left\{ M^{\frac{1}{p-q}} |a|^{-\frac{q}{p-q}}, |a|^{-\frac{2}{p-1}} \right\}. \tag{2.8}$$

The case $M \le 0$. We first assume that $M < 0$. By [19, Lemma 3.3] v_0 satisfies

$$-\Delta v_0 + |M||\nabla v_0|^q \le 0 \qquad \text{in } \mathcal{D}'(B_{2R_0} \setminus \{0\}). \tag{2.9}$$

Therefore

$$u_+(a) = v_0(a) \le c_6 |M|^{-\frac{1}{q-1}} |a|^{-\frac{2-q}{q-1}}. \tag{2.10}$$

If $M \le 0$ there also holds

$$-\Delta v_0 + v_0^p \le 0 \qquad \text{in } \mathcal{D}'(B_{2R_0} \setminus \{0\}). \tag{2.11}$$

Therefore

$$u_+(a) = v_0(a) \le c_7 |a|^{-\frac{2}{p-1}}. \tag{2.12}$$

In the above inequalities $c_6 = c_6(q,N) > 0$ and $c_7 = c_7(p,N) > 0$. Combining these estimates we derive

$$u_+(a) \le \min\left\{ c_7 |a|^{-\frac{2}{p-1}}, c_6 |M|^{-\frac{1}{q-1}} |a|^{-\frac{2-q}{q-1}} \right\}. \tag{2.13}$$

Since the estimate is independent of R_0, the assumption that $\overline{\Omega} \subset B_{R_0}$ is easily ruled out. This ends the proof. □

The equation is not invariant by $u \mapsto -u$ hence the lower and upper estimates are not symmetric.

Corollary 2.2 *Under the assumptions of Theorem 2.1, there holds*
1- If $M > 0$

$$\begin{aligned} &\left\{ -c_6 |M|^{-\frac{1}{q-1}} |x|^{-\frac{2-q}{q-1}}, -c_7 |x|^{-\frac{2}{p-1}} \right\} \le -u_-(x) \le 0 \\ &\qquad \le u_+(x) \le c_5 \max\left\{ M^{\frac{1}{p-q}} |x|^{-\frac{q}{p-q}}, |x|^{-\frac{2}{p-1}} \right\} \qquad \textit{for all } x \in \Omega. \end{aligned} \tag{2.14}$$

2- If $M \le 0$, *there exist* $c_6 = c_6(N,q) > 0$ *and* $c_7 = c_7(N,p) > 0$ *such that*

$$\begin{aligned} &-c_5 \max\left\{ M^{\frac{1}{p-q}} |x|^{-\frac{q}{p-q}}, |x|^{-\frac{2}{p-1}} \right\} \le -u_-(x) \le 0 \\ &\qquad \le u_+(x) \le \min\left\{ c_6 |M|^{-\frac{1}{q-1}} |x|^{-\frac{2-q}{q-1}}, c_7 |x|^{-\frac{2}{p-1}} \right\} \qquad \textit{for all } x \in \Omega. \end{aligned} \tag{2.15}$$

We infer from Theorem 2.1 and estimate of the gradient of u near 0.

Theorem 2.3 *Let Ω be a smooth bounded domain such that $0 \in \partial\Omega$, $M > 0$, $p > 1$ and $1 < q < \min\{2, p\}$. If $u \in C^1(\overline{\Omega}\backslash\{0\})$ is a nonnegative solution of (1.1) vanishing on $\partial\Omega \setminus \{0\}$, for any $r_0 > 0$ there holds there exists $c_8 = c_5(N, p, q, \Omega, r_0) > 0$ such that*

$$|\nabla u(x)| \le c_8 \max\left\{M^{\frac{1}{p-q}}|x|^{-\frac{p}{p-q}}, |x|^{-\frac{p+1}{p-1}}\right\} \qquad \text{for all } x \in \Omega \cap B_{r_0}. \tag{2.16}$$

The restriction that $|x| \le 1$ is not needed if $q = \frac{2p}{p+1}$.

Proof. We assume first that $B_2^+ \subset \Omega$.
Case 1: $1 < q \le \frac{2p}{p+1}$. For $0 < r < 1$ we set

$$u(x) = r^{-\frac{2}{p-1}} u_r(\tfrac{x}{r}) = r^{-\frac{2}{p-1}} u_r(y) \quad \text{with } y = \tfrac{x}{r}.$$

If $\frac{r}{2} < |x| < 2r$, then $\frac{1}{2} < |y| < 2$ and $u_r > 0$ satisfies

$$-\Delta u_r + u_r^p - M r^{\frac{2p-q(p+1)}{p-1}} |\nabla u_r|^q = 0 \qquad \text{in } B_2^+ \setminus B_{\frac{1}{2}}^+,$$

and vanishes on $\partial(B_2^+ \setminus B_{\frac{1}{2}}^+)$. Since $0 < M r^{\frac{2p-q(p+1)}{p-1}} \le M$ as $2p - q(p+1) \ge 0$, it follows that

$$\max\left\{|\nabla u_r(z)| : \tfrac{2}{3} < |z| < \tfrac{3}{2}\right\} \le c_9 \max\left\{|u_r(z)| : \tfrac{1}{2} < |z| < 2\right\}, \tag{2.17}$$

where c_9 depends on N, p, q and M. Now it follows that

$$\max\left\{|u_r(z)| : \tfrac{1}{2} < |z| < 2\right\} \le c_5' \max\left\{M^{\frac{1}{p-q}} r^{\frac{2p-q(p+1)}{(p-1)(p-q)}}, 1\right\}$$

by (2.1) where $c_5' = 2^{\frac{2}{p-1}} c_5$. Therefore

$$\begin{aligned} \max\left\{|\nabla u(y)| : \tfrac{r}{2} < |z| < 2r\right\} &\le c_9 c_5' r^{-\frac{p+1}{p-1}} \max\left\{M^{\frac{1}{p-q}} r^{\frac{2p-q(p+1)}{(p-1)(p-q)}}, 1\right\} \\ &\le c_8 \max\left\{M^{\frac{1}{p-q}}|x|^{-\frac{p}{p-q}}, |x|^{-\frac{p+1}{p-1}}\right\}, \end{aligned} \tag{2.18}$$

which is (2.16).
Case 2: $\frac{2p}{p+1} < q < 2$. For $0 < r < 1$ we set

$$u(x) = r^{-\frac{2-q}{q-1}} u_r(\tfrac{x}{r}) = r^{-\frac{2-q}{q-1}} u_r(y) \quad \text{with } y = \tfrac{x}{r}.$$

If $\frac{r}{2} < |x| < 2r$, then $\frac{1}{2} < |y| < 2$ and $u_r > 0$ satisfies

$$-\Delta u_r + r^{\frac{q(p+1)-2p}{q-1}} u_r^p - M|\nabla u_r|^q = 0 \qquad \text{in } B_2^+ \setminus B_{\frac{1}{2}}^+,$$

We notice that $q(p+1)-2p>0$. Then inequality (2.17) holds. Now

$$\max\left\{|u_r(z)|:\tfrac{1}{2}<|z|<2\right\}\le c_{91}r^{\frac{2-q}{q-1}}\max\left\{r^{-\frac{2}{p-1}},r^{-\frac{q}{p-q}}\right\},$$

thus

$$\max\left\{|\nabla u_r(z)|:\tfrac{2}{3}<|z|<\tfrac{3}{2}\right\}\le c_{92}r^{\frac{2-q}{q-1}-1}\max\left\{r^{-\frac{2}{p-1}},r^{-\frac{q}{p-q}}\right\}, \tag{2.19}$$

which implies

$$\max\left\{|\nabla u(x)|:\tfrac{2r}{3}<|x|<\tfrac{3r}{2}\right\}\le c_{92}\max\left\{r^{-\frac{p+1}{p-1}},r^{-\frac{p}{p-q}}\right\}. \tag{2.20}$$

The general case; If $\partial\Omega$ is not flat near 0 we proceed as in the proof of [19, Lemma 3.4], using the same scaling as in the flat case which transform the domain $B_2^+\setminus B_1^+$) into $(B_2\setminus B_1)\cap\frac{1}{r}\Omega$, the curvature of which is bounded when $0<r<1$. The same estimates holds, up to the value of the constant c_8 and we derive (2.16). □

As a consequence we have the following.

Corollary 2.4 *Under the assumptions of Theorem 2.3 the function u satisfies*

$$u(x)\le c_8 d(x)\max\left\{M^{\frac{1}{p-q}}|x|^{-\frac{p}{p-q}},|x|^{-\frac{p+1}{p-1}}\right\}\qquad \text{for all } x\in\Omega\in\Omega\cap B_1, \tag{2.21}$$

where $d(x)=\operatorname{dist}(x,\partial\Omega)$. The restriction that $|x|\le 1$ is not needed if $q=\frac{2p}{p+1}$.

2.2 Removable singularities

Proof of Theorem 1.1. If $M\le 0$, u is a nonnegative subsolution of $-\Delta u+v^p=0$ which vanishes on $\partial\Omega\setminus\{0\}$, hence it is identically zero by [13].

Step 1. We assume $M>0$ and we prove first that under condition (i) or (ii), $|\nabla u|^q\in L^1(\Omega;d)$ and $u^p\in L^1(\Omega;d)$, and that there holds

$$\int_\Omega(-u\Delta\zeta+u^p\zeta-M|\nabla u|^q\zeta)\,dx=0\qquad\forall\zeta\in W^{2,\infty}(\Omega)\cap C_c^1(\overline{\Omega}). \tag{2.22}$$

For any $\epsilon>0$ we denote by w_ϵ the solution of

$$\begin{array}{rl} -\Delta w+w^p=M|\nabla u|^q & \text{in } \Omega_\epsilon:=\Omega\cap\overline{B}_\epsilon^c\\ w=0 & \text{in } \partial\Omega\cap\overline{B}_\epsilon^c\\ \lim\limits_{|x|\to\epsilon}w(x)=\infty & \text{on } \partial B_\epsilon\cap\Omega, \end{array} \tag{2.23}$$

which exists since $|\nabla u|^q\in L^1(\Omega;d)$ by [17]. Then $u\le w_\epsilon$ in Ω_ϵ. Let z_ϵ be the solution of

$$\begin{array}{rl} -\Delta z+z^p=0 & \text{in } \Omega_\epsilon\\ z=0 & \text{in } \partial\Omega\cap\overline{B}_\epsilon^c\\ \lim\limits_{|x|\to\epsilon}z(x)=\infty & \text{on } \partial B_\epsilon\cap\Omega. \end{array} \tag{2.24}$$

Since $z_\epsilon + M\mathbb{G}_\Omega[|\nabla u|^q]\lfloor_{\Omega_\epsilon}$ is a supersolution of (2.23) in Ω_ϵ we deduce

$$u \le z_\epsilon + M\mathbb{G}_\Omega[|\nabla u|^q]\lfloor_{\Omega_\epsilon} \qquad \text{in } \Omega_\epsilon. \tag{2.25}$$

When $\epsilon \to 0$, w_ϵ decreases to w_0 which satisfies

$$\begin{aligned} -\Delta w + w^p &= 0 \qquad && \text{in } \Omega \\ w &= 0 && \text{in } \partial\Omega \setminus \{0\}. \end{aligned} \tag{2.26}$$

Since $p \ge \frac{N+1}{N-1}$ it is proved in[13] that any solution of (2.26) extends as a continuous solution in Ω with boundary value 0, hence $w_0 = 0$ by the maximum principle. Therefore $u \le M\mathbb{G}_\Omega[|\nabla u|^q]$ in Ω and the boundary trace $Tr_{\partial\Omega}[u]$ of u is zero. By [18] the fact that $|\nabla u|^q \in L^1(\Omega; d)$ jointly with $Tr_{\partial\Omega}[u] = 0$ implies in turn that $u^p \in L^1(\Omega; d)$ and u is a weak solution of

$$\begin{aligned} -\Delta u + u^p &= M|\nabla u|^q \qquad && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.27}$$

in the sense that (2.22) holds.

Step 2. Let us assume that $p > \frac{N+1}{N-1}$. If u is nonnegative and not identically zero, then by the maximum principle it is positive in Ω. We set $u = v^b$ with $0 < b \le 1$. Then

$$-\Delta v - (b-1)\frac{|\nabla v|^2}{v} + \frac{1}{b}v^{(p-1)b+1} - Mb^{q-1}v^{(b-1)(q-1)}|\nabla v|^q = 0. \tag{2.28}$$

For $\epsilon > 0$,

$$v^{(b-1)(q-1)}|\nabla v|^q \le \frac{q\epsilon^{\frac{2}{q}}}{2}\frac{|\nabla v|^2}{v} + \frac{2-q}{2\epsilon^{\frac{2}{2-q}}}v^{\frac{(2b-1)q-2(b-1)}{2-q}}.$$

Therefore

$$-\Delta v + \left(1 - b - M\frac{qb^{q-1}\epsilon^{\frac{2}{q}}}{2}\right)\frac{|\nabla v|^2}{v} + \frac{1}{b}v^{(p-1)b+1} - Mb^{q-1}\frac{2-q}{2\epsilon^{\frac{2}{2-q}}}v^{\frac{(2b-1)q-2(b-1)}{2-q}} = 0. \tag{2.29}$$

We notice that the following relation is independent of b

$$\frac{(2b-1)q - 2(b-1)}{2-q} \le (p-1)b + 1 \Longleftrightarrow q \le \frac{2p}{p+1},$$

with simultaneous equality. We take

$$(p-1)b + 1 = \frac{N+1}{N-1} \Longleftrightarrow b = \frac{2}{(N-1)(p-1)}, \tag{2.30}$$

hence $p > \frac{N+1}{N-1}$ if and only if $0 < b < 1$. Next we impose

$$1 - b - M\frac{qb^{q-1}\epsilon^{\frac{2}{q}}}{2} = 0 \Longleftrightarrow \epsilon = \left(\frac{2(1-b)}{Mqb^{q-1}}\right)^{\frac{q}{2}} = \left(\frac{2((N-1)p - N - 1)}{Mqb^{q-1}(N-1)(p-1)}\right)^{\frac{q}{2}}. \tag{2.31}$$

This transforms (2.29) into

$$-\Delta v+\frac{(N-1)(p-1)}{2}v^{\frac{N+1}{N-1}}-\frac{(2-q)b^{q-1}}{2}\left(\frac{q}{2(1-b)}\right)^{\frac{q}{2-q}}M^{\frac{2}{2-q}}v^{\frac{(2b-1)q-2(b-1)}{2-q}}\leq 0. \tag{2.32}$$

We first assume that $0<q<\frac{2p}{p+1}$. Then

$$\frac{(2b-1)q-2(b-1)}{2-q}<\frac{N+1}{N-1}.$$

There exists $A>0$, depending on M, such that

$$-\Delta v+\frac{(N-1)(p-1)}{4}v^{\frac{N+1}{N-1}}\leq A. \tag{2.33}$$

Since v vanishes on $\partial\Omega\setminus\{0\}$, $\tilde{v}=(v-c_1A^{\frac{N-1}{N+1}})_+^{\frac{N+1}{N-1}}$ with $c_1=\left(\frac{4}{(N-1)(p-1)}\right)^{\frac{N+1}{N-1}}$ satisfies

$$-\Delta\tilde{v}+\frac{(N-1)(p-1)}{4}\tilde{v}^{\frac{N+1}{N-1}}\leq 0. \tag{2.34}$$

By [13], $\tilde{v}=0$ which implies $v\leq c_1A^{\frac{N-1}{N+1}}$ and therefore $u(x)\leq c_2A^{\frac{2}{(N+1)(p-1)}}$ in Ω. Since u vanishes on $\partial\Omega\setminus\{0\}$ we extend in a neighborhood of 0 by odd reflexion true $\partial\Omega$ and denote by $\tilde{u}$ the new function defined in B_α where it satisfies

$$-divA(x,\nabla\tilde{u})+\tilde{u}^p+B(x,\nabla\tilde{u})=0\qquad\text{in } B_\alpha\{0\}. \tag{2.35}$$

In this expression the operator $A:(x;\xi)\in B_\alpha\times\mathbb{R}^N\mapsto A(x,\xi)\in\mathbb{R}^N$ is smooth in x and linear in ξ, it and satisfies for all $(x;\xi)\in B_\alpha\times\mathbb{R}^N$,

$$A(x,\xi).\xi\geq 2|\xi|^2\ \text{ and }\ |A(x,\xi)|\leq 4|\xi|\quad\text{for all }\ (x;\xi)\in B_\alpha\times\mathbb{R}^N.$$

Since we can write $|B(.,\nabla\tilde{u})|\leq 2|\nabla\tilde{u}|^q=2|\nabla\tilde{u}|^{q-1}|\nabla\tilde{u}|=C(x)|\nabla\tilde{u}|$ in B_α, then $B:(x;\xi)\in B_\alpha\times\mathbb{R}^N\mapsto B(x,\xi)\in\mathbb{R}$ verifies

$$|B(x,(\xi)|\leq C(x)|\xi|,$$

and $C(x)\leq 2c_8|x|^{-\frac{(p+1)(q-1)}{p-1}}$ by Theorem 2.3. Since $q<\frac{2p}{p+1}$, $\frac{(p+1)(q-1)}{p-1}<1$. Hence $C\in L^{N+\tau}$ for some $\tau>0$. By Serrin's theorem [21, Theorem 10] the singularity at 0 is removable and $\tilde{u}$ can be extended as a regular solution of (2.35) in B_α. Hence $\tilde{u}\in C^1(B_{\frac{\alpha}{2}})$, and as a consequence $u\in C^1(\overline{\Omega})$. Since If u is not zero, it is positive in Ω and achieves its maximum at some $x_0\in\Omega$ where $\Delta u(x_0)\leq 0$ and $\nabla u(x_0)=0$. Contradiction.

Then we assume that $q=\frac{2p}{p+1}$. By the choice of b in (2.30), inequality (2.29) becomes

$$-\Delta v+\left(1-b-\frac{Mpb^{\frac{p-1}{p+1}}\epsilon^{\frac{p+1}{p}}}{p+1}\right)\frac{|\nabla v|^2}{v}+\left(\frac{1}{b}-\frac{Mb^{\frac{p-1}{p+1}}}{(p+1)\epsilon^{p+1}}\right)v^{(p-1)b+1}\leq 0. \tag{2.36}$$

Notice that

$$\frac{1}{b}-\frac{Mb^{\frac{p-1}{p+1}}}{(p+1)\epsilon^{p+1}}=0\Longleftrightarrow\epsilon=\left(\frac{M}{p+1}\right)^{\frac{1}{p+1}}b^{\frac{2p}{(p+1)^2}}, \tag{2.37}$$

and therefore

$$1-b-\frac{Mpb^{\frac{p-1}{p+1}}\epsilon^{\frac{p+1}{p}}}{p+1}=1-b-pb\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}}. \tag{2.38}$$

This coefficient vanishes if

$$p\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}}=\frac{p(N-1)-(N+1)}{2}.$$

Therefore, if M satisfies

$$p\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}}=\frac{p(N-1)-(N+1)}{2}, \tag{2.39}$$

we can choose $\epsilon>0$ so that the coefficient of $v^{(p-1)b+1}$ in (2.39) is equal to some $\tau>0$. Therefore v satisfies

$$\begin{aligned}-\Delta v+\tau v^{\frac{N+1}{N-1}}&\le 0 \qquad \text{in } \Omega\\ v&=0 \qquad \text{on } \partial\Omega\setminus\{0\}.\end{aligned} \tag{2.40}$$

By [13], $v=0$ and the same holds for u. This ends the case $p>\frac{N+1}{N-1}$.

Step 3. Finally we assume $p=\frac{N+1}{N-1}$ and $1<q<\frac{2p}{p+1}=\frac{N+1}{N}$, then

$$M|\nabla u(x)|^q\le c_{10}|x|^{-q\frac{p+1}{p-1}}=c_{10}|x|^{-qN}:=c_{10}Q(x).$$

Hence $u\le u_1=c_{10}\mathbb{G}_\Omega[Q]$. At this point we need the following intermediate result:

Claim Assume $w_\alpha=\mathbb{G}_\Omega[Q_\alpha]$ where $Q_\alpha(x)=|x|^{-\alpha}$ with $\alpha<N+1$, then

$$w_\alpha(x)\le c_\alpha|x|^{2-\alpha} \qquad \text{for all } \ x\in\Omega. \tag{2.41}$$

If this holds true, then $u(x)\le c_{10}c_{qN}|x|^{2-qN}$. By the scaling method of Theorem 2.3, it implies in turn

$$|\nabla u(x)|\le c_8c_{10}c_{qN}|x|^{1-qN}\Longrightarrow|\nabla u(x)|^q\le c_{11}|x|^{q(1-qN)}:=c_{11}Q_{q(Nq-1)}(x), \tag{2.42}$$

and thus

$$w_{q(Nq-1)}(x)=c_{11}\mathbb{G}_\Omega[Q_{q(Nq-1)}](x)\le c_{11}c_{q(Nq-1)}|x|^{2-q(Nq-1)} \qquad \text{for all } \ x\in\Omega. \tag{2.43}$$

Since $q<1+\frac{1}{N}$, $q(Nq-1)-2<Nq-2$. Iterating this process, we finally obtain that u is bounded and we end the proof as in Step 2. □

Remark. It is noticeable that the equation exhibits a phenomenon which is characteristic of Emden-Folwer type equations

$$\Delta u = u^p \quad \text{in } B_1 \setminus \{0\}. \tag{2.44}$$

If u is nonnegative then there exists $\alpha \geq 0$ such that

$$\Delta u = u^p + \alpha\delta_0 \quad \text{in } \mathcal{D}'(B_1). \tag{2.45}$$

If $1 < p < \frac{N}{N-2}$ then α can be positive, but if $p \geq \frac{N}{N-2}$, then $\alpha = 0$. This means that the singularity cannont be seen in the sense of distributions, however there truly exist singular solutions, e.g. if $p > \frac{N}{N-2}$,

$$u_s(x) = c_{N,p}|x|^{-\frac{2}{p-1}}. \tag{2.46}$$

A similar phenomenon exists for solutions of

$$\begin{aligned} \Delta u &= u^p \quad \text{in } B_1^+ \\ u &= 0 \quad \text{in } \partial B_1^+ \setminus \{0\}. \end{aligned} \tag{2.47}$$

In such a case the critical value is $\frac{N+1}{N-1}$ since for $p \geq \frac{N+1}{N-1}$ the boundary value is achieved in $\mathcal{D}'(\partial B_1^+)$.

2.3 Proof of Theorem 1.2

As in the proof of Theorem 1.1, we distinguish according $0 < q < \frac{2p}{p+1}$ or $q = \frac{2p}{p+1}$, and we first assume that $u > 0$. We perform the same change of unknown as in the previous theorem putting $u = v^b$, but now we choose b as follows

$$(p-1)b + 1 = r \Longleftrightarrow b = \frac{r-1}{p-1}, \tag{2.48}$$

and we first assume that

$$1 - b - M\frac{qb^{q-1}\epsilon^{\frac{2}{q}}}{2} = 0 \Longleftrightarrow \epsilon = \left(\frac{2(1-b)}{Mqb^{q-1}}\right)^{\frac{q}{2}} = \left(\frac{2(p-r)}{Mq(p-1)b^{q-1}}\right)^{\frac{q}{2}}. \tag{2.49}$$

Hence (2.32) becomes

$$-\Delta v + \frac{p-1}{r-1}v^r - \frac{(2-q)b^{q-1}}{2}\left(\frac{q}{2(1-b)}\right)^{\frac{q}{2-q}} M^{\frac{2}{2-q}} v^{\frac{(2r-p-1)q+2(p-r)}{(p-1)(2-q)}} \leq 0. \tag{2.50}$$

The condition $r \geq \frac{(2r-p-1)q+2(p-r)}{(p-1)(2-q)}$ is equivalent to $2p - q(p+1) \leq r(2p - q(p+1))$, and $0 < r < 1$ since $1 < r < p$.
Assuming first that $q < \frac{2p}{p+1}$, we obtain from (2.50)

$$-\Delta v + \frac{p-1}{2(r-1)}v^r \leq A. \tag{2.51}$$

for some constant $A \geq 0$. Since $cap^{\partial\Omega}_{\frac{2}{r},r'}(K) = 0$ and v vanishes on $\partial\Omega \setminus K$, it follows from [16] that $v \leq cA^{\frac{1}{r}}$ for some $c > 0$, hence u is also uniformly upper bounded in Ω by some constant a. Next we have to show that $\nabla u \in L^2_\rho(\Omega)$ where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. We also denote by Φ_1 the first eigenfunction of $-\Delta$ in $W^{1,2}_0(\Omega)$ normalized by $\sup \Phi_1 = 1$ and by λ_1 the corresponding eigenvalue. Since $\frac{N+1}{N-1} < r \leq 3$ we infer from [1, Theorem 5.5.1], that

$$\left(cap^{\partial\Omega}_{\frac{1}{2},2}(K)\right)^{\frac{1}{N-2}} \leq B\left(cap^{\partial\Omega}_{\frac{2}{r},r'}(K)\right)^{\frac{1}{N-1-\frac{2}{r-1}}}.$$

Therefore $cap^{\partial\Omega}_{\frac{2}{r},r'}(K) = 0$ implies $cap^{\partial\Omega}_{\frac{1}{2},2}(K) = 0$ and there exists a decreasing sequence $\{\zeta_n\} \subset C^2_0(\partial\Omega)$ such that $\zeta_n = 1$ in a neighborhood of K, $0 \leq \zeta_n \leq 1$ and $\|\zeta_n\|_{W^{1,2}} \to 0$ when $n \to \infty$, furthermore $\zeta_n \to 0$ quasi everywhere. Let $\mathbb{P}_\Omega : C^2(\partial\Omega) \mapsto C^2(\overline{\Omega})$ be the Poisson operator. It is an admissible lifting in the sense of [16, Section 1] in the sense that

$$\mathbb{P}_\Omega[\eta]\lfloor_{\partial\Omega} = \eta \quad \text{and} \quad \eta \geq 0 \Longrightarrow \mathbb{P}_\Omega[\eta] \geq 0.$$

Put $\eta_n = 1 - \zeta_n$. Then, multiplying equation (1.10) by $u(\mathbb{P}_\Omega[\eta_n])^2$ and integrating, we obtain

$$\int_\Omega |\nabla u|^2(\mathbb{P}_\Omega[\eta_n])^2 dx + 2\int_\Omega u\mathbb{P}_\Omega[\eta_n]\nabla u.\nabla\mathbb{P}_\Omega[\eta_n]dx + \int_\Omega u^{p+1}(\mathbb{P}_\Omega[\eta_n])^2 dx - M\int_\Omega |\nabla u|^q u(\mathbb{P}_\Omega[\eta_n])^2 dx = 0,$$

which implies

$$\int_\Omega |\nabla u|^2(\mathbb{P}_\Omega[\eta_n])^2 dx - 2\left(\int_\Omega |\nabla u|^2(\mathbb{P}_\Omega[\eta_n])^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla\mathbb{P}_\Omega[\eta_n]|^2 u^2 dx\right)^{\frac{1}{2}} + \int_\Omega u^{p+1}(\mathbb{P}_\Omega[\eta_n])^2 dx - M\int_\Omega |\nabla u|^q u(\mathbb{P}_\Omega[\eta_n])^2 dx \leq 0.$$

It is standard that

$$\int_\Omega |\nabla\mathbb{P}_\Omega[\eta_n]|^2 dx \leq c_{12}\|\eta_n\|^2_{H^{\frac{1}{2}}(\partial\Omega)} = A_n.$$

Set $X_n = \||\nabla u|\mathbb{P}_\Omega[\eta_n]\|_{L^2}$, then

$$X_n^2 - 2A_nX_n - Ma|\Omega|^{\frac{2-q}{2}}X_n^q \leq 0.$$

Hence there exist two positive real numbers a_1 and a_2 depending only on q, $|\Omega|$ and $a = \|u\|_{L^\infty}$ such that

$$X_n \leq a_1A_n + a_2M^{\frac{1}{2-q}}. \tag{2.52}$$

Now $A_n \to 0$ and X_n to $\|\nabla u\|_{L^2}^2$, therefore by Fatou

$$|\Omega|^{1-\frac{2}{q}} \|\nabla u\|_{L^q}^2 \leq \|\nabla u\|_{L^2}^2 \leq a_2 M^{\frac{1}{2-q}} < \infty.$$

Let $\zeta \in C_0^1(\overline{\Omega})$ and η_n as above. Since η_n vanishes in a neighborhood of K and ζ vanishes on $\partial\Omega$,

$$\int_\Omega \mathbb{P}_\Omega[\eta_n]\nabla u.\nabla\zeta dx + \int_\Omega \zeta\nabla u.\nabla\mathbb{P}_\Omega[\eta_n]dx + \int_\Omega u^p\zeta\mathbb{P}_\Omega[\eta_n]dx = \int_\Omega |\nabla u|^q\zeta\mathbb{P}_\Omega[\eta_n]dx.$$

Letting n to infty and using the fact that $\nabla u \in L^2(\Omega)$ and $\nabla\mathbb{P}_\Omega[\eta_n] \to 0$ in $L^2(\Omega)$, we derive

$$\int_\Omega \nabla u.\nabla\zeta dx + \int_\Omega u^p\zeta dx = \int_\Omega |\nabla u|^q\zeta dx.$$

Hence u is a nonnegative weak solution of

$$\begin{aligned} -\Delta u + |u|^{p-1}u - M|\nabla u|^q = 0 \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega. \end{aligned} \tag{2.53}$$

It is therefore C^2. Again, by the maximum principle we see that u cannot achieve a positive maximum in Ω, contradiction.
Next we assume $q = \frac{2p}{p+1}$. We choose with $b = \frac{r-1}{p-1}$ and (2.36) becomes

$$-\Delta v + \left(1 - b - \frac{Mpb^{\frac{p-1}{p+1}}\epsilon^{\frac{p+1}{p}}}{p+1}\right)\frac{|\nabla v|^2}{v} + \left(\frac{1}{b} - \frac{Mb^{\frac{p-1}{p+1}}}{(p+1)\epsilon^{p+1}}\right)v^r \leq 0. \tag{2.54}$$

If (2.37) holds with this choice of b, (2.38) becomes

$$\begin{aligned} 1 - b - \frac{Mpb^{\frac{p-1}{p+1}}\epsilon^{\frac{p+1}{p}}}{p+1} &= 1 - b - pb\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}} \\ &= \frac{1}{p-1}\left(p - r - p(r-1)\left(\frac{M}{p+1}\right)^{\frac{p+1}{p}}\right). \end{aligned} \tag{2.55}$$

If $M < m_r^{**}$ defined by (1.9) we can choose ϵ such that

$$1 - b - \frac{Mpb^{\frac{p-1}{p+1}}\epsilon^{\frac{p+1}{p}}}{p+1} = 0,$$

and

$$\frac{1}{b} - \frac{Mb^{\frac{p-1}{p+1}}}{(p+1)\epsilon^{p+1}} = \tau := \tau(\epsilon) > 0.$$

Then v satisfies

$$\begin{aligned} -\Delta v + \tau v^r \leq 0 \qquad & \text{in } \Omega \\ v = 0 \qquad & \text{in } \partial\Omega \setminus K. \end{aligned}$$

Since $cap_{\frac{2}{r},r'}^{\partial\Omega}(K) = 0$ it follows from [16] that $v = 0$. Hence $u = 0$, which ends the proof. □

2.4 Measure boundary data

Let μ be a nonnegative Radon measure on $\partial\Omega$. The results concerning the following two types of equations

$$\begin{array}{rl} -\Delta v + v^p = 0 & \text{in } \Omega \\ v = \mu & \text{in } \partial\Omega, \end{array} \tag{2.56}$$

and

$$\begin{array}{rl} -\Delta w = M|\nabla w|^q & \text{in } \Omega \\ w = c\mu & \text{in } \partial\Omega, \end{array} \tag{2.57}$$

allows us to consider the measure boundary data for equation (1.1). We recall the results concerning (2.56) and (2.57).
1- Assume $p > 1$. If μ satisfies

$$\text{For all } E \subset \partial\Omega, \text{ E Borel, } cap^{\partial\Omega}_{\frac{2}{p},p'}(E) = 0 \Longrightarrow \mu(E) = 0, \tag{2.58}$$

then problem (2.56) admits a necessarily unique very weak solution $v := v_\mu$, see [16], i.e. $v_\mu \in L^1(\Omega) \cap L^p_\rho(\Omega)$ and for any function $\zeta \in \mathbb{X}(\Omega) := \left\{\eta \in C^1_0(\overline{\Omega}) \text{ s.t. } \Delta\eta \in L^\infty(\Omega)\right\}$, there holds

$$\int_\Omega (-v\Delta\zeta + v^p\zeta)\, dx = -\int_\Omega \frac{\partial\zeta}{\partial \mathbf{n}} d\mu. \tag{2.59}$$

Notice that there is no condition on μ if $1 < p < \frac{N+1}{N-1}$.
2- Assume $1 < q < 2$. If there exists $C_1 > 0$ such that μ satisfies

$$\text{For all } E \subset \partial\Omega, \text{ E Borel, } \mu(E) \le C_1 cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E), \tag{2.60}$$

then problem (2.57) admits at least a positive solution w for $c > 0$ small enough, see [4, Theorem 1.3], in the sense that $w \in L^1(\Omega)$, $\nabla w \in L^q_\rho(\Omega)$ and for any $\zeta \in \mathbb{X}(\Omega)$, there holds

$$\int_\Omega (-w\Delta\zeta - M|\nabla w|^q\zeta)\, dx = -\int_\Omega \frac{\partial\zeta}{\partial \mathbf{n}} d\mu. \tag{2.61}$$

Notice that if $1 < q < \frac{N+1}{N}$ there is no condition on μ.

We use also the following result

Lemma 2.5 *Let $p > \frac{N+1}{N-1}$ and $\mu \in \mathfrak{M}_+(\partial\Omega)$. Then $\mu \in W^{-\frac{2}{p},p}(\partial\Omega)$ if and only if there exists $c > 0$ such that*

$$\mu(E) \le c_{13}\, cap^{\partial\Omega}_{\frac{2}{p},p'}(E) \quad \textit{for all Borel set } E \subset \partial\Omega. \tag{2.62}$$

Proof. Assume $\mu \in W^{-\frac{2}{p},p}(\partial\Omega) \cap \mathfrak{M}_+(\partial\Omega)$. If E is a compact subset of $\partial\Omega$ and let $\zeta \in C^2(\partial\Omega)$ with $0 \le \zeta \le 1$, with $\zeta = 1$ on E, then

$$\mu(E) \le \int_{\partial\Omega} \zeta d\mu = \langle \mu, \zeta\rangle \le \|\mu\|_{W^{-\frac{2}{p},p}} \|\zeta\|_{W^{\frac{2}{p},p'}}.$$

Therefore, by the definition of the capacity,

$$\mu(E) \leq \|\mu\|_{W^{-\frac{2}{p},p}} cap^{\partial\Omega}_{\frac{2}{p},p'}(E).$$

Conversely, we consider the problem

$$\begin{array}{rll} -\Delta z = z^p & \text{in} & \Omega \\ z = c_{14}\mu & \text{in} & \partial\Omega. \end{array} \tag{2.63}$$

It is proved in [4, Theorem 1.2] that if μ satisfies

$$\text{For all } E \subset \partial\Omega, \text{ E Borel, } \mu(E) \leq C cap^{\partial\Omega}_{\frac{2}{p},p'}(E), \tag{2.64}$$

for some constant $C > 0$, then there exists $\tilde{c} > 0$ such that for $0 \leq c < \tilde{c}$ there exists a very weak solution $z \in L^1(\Omega) \cap L^p_\rho(\Omega)$ to (2.63) in the sense that for any $\zeta \in \mathbb{X}(\Omega)$, there holds

$$\int_\Omega (-w\Delta\zeta - u^p\zeta)\, dx = -c_{14}\int_\Omega \frac{\partial\zeta}{\partial\mathbf{n}} d\mu. \tag{2.65}$$

Furthermore $0 \leq c\mathbb{P}_\Omega[\mu] \leq z$. Since $z \in L^p_\rho(\Omega)$, $\mathbb{P}_\Omega[\mu] \in L^p_\rho(\Omega)$. By [16], this implies that $\mu \in W^{-\frac{2}{p},p}(\partial\Omega)$. □

Those very weak solutions are characterized by their boundary trace. Let $\Sigma_\epsilon = \{x \in \Omega : \rho(x) = \epsilon > 0\}$ and $\Sigma_0 = \partial\Omega$. For $0 < \epsilon \leq \epsilon_0$ the hypersurfaces Σ_δ defines a foliation of the set $\Omega_{\epsilon_0} = \{x \in \Omega :\ 0 < \rho(x) \leq \epsilon_0\}$. Let $\pi(x)$ be the orthogonal projection of $x \in \Omega_{\epsilon_0}$. Then $|x - \pi(x)| = \rho(x)$ and $\mathbf{n}_x = (\rho(x))^{-1}(\pi(x) - x)$. The mapping

$$x \mapsto \Pi(x) = (\rho(x), \pi(x)),$$

from Ω_{ϵ_0} onto $(0, \epsilon_0] \times \Sigma_0$ is a C^2 diffeomorphism and the restriction Π_ϵ of Π to Σ_ϵ is a C^2 diffeomorphism from Σ_ϵ onto Σ_0. Let dS_ϵ be the surface measure on Σ_ϵ, then a continuous function u defined in Ω has boundary trace the Radon measure μ on $\partial\Omega$ if

$$\lim_{\epsilon\to 0}\int_{\Sigma_\epsilon} uZ dS_\epsilon = \int_\Sigma Z d\mu \qquad \text{for all } \ Z \in C(\overline{\Omega}). \tag{2.66}$$

Equivalently, if $\zeta \in C(\partial\Omega)$ and $\zeta_\epsilon = \zeta \circ \Pi_\epsilon^{-1} \in C(\Sigma_\epsilon)$, then

$$\lim_{\epsilon\to 0}\int_{\Sigma_\epsilon} u\zeta_\epsilon dS_\epsilon = \int_\Sigma \zeta d\mu \qquad \text{for all } \ \zeta \in C(\partial\Omega). \tag{2.67}$$

The functions v_μ solution of (2.56) and w solution of (2.57) admit for respective boundary trace μ and $c\mu$. Furthermore, for the equations in (2.56) and (2.57), the existence of a boundary trace of a positive solution is equivalent to the fact that $v_\mu \in L^1(\Omega) \cap L^p_\rho(\Omega)$ and $w \in L^1(\Omega)$ with $\nabla w \in L^q_\rho(\Omega)$.

Proof of Theorem 1.3. We recall the formulation of [1, Theorem 5.5.1] in our framework. There exists a constant $A > 0$ such that if $E \subset \Omega$ is a Borel set, then

$$\begin{aligned}
&(i) \quad \left(cap^{\partial\Omega}_{\frac{2}{p},p'}(E)\right)^{\frac{p-1}{(N-1)(p-1)-2}} \le A\left(cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E)\right)^{\frac{q-1}{(N-1)(q-1)+q-2}} \quad \text{if } \tfrac{N+1}{N} < q \le \tfrac{2p}{p+1},\\
&(ii) \quad \left(cap^{\partial\Omega}_{\frac{2}{p},p'}(E)\right)^{\frac{2}{N-1}} \le A\left(cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E)\right)^{\frac{1}{N}} \qquad \text{if } p = \tfrac{N+1}{N-1} \text{ and } q = \tfrac{N+1}{N}.
\end{aligned} \tag{2.68}$$

The condition $\frac{2}{p-1} < \frac{2-q}{q-1}$ (resp. $\frac{2}{p-1} = \frac{2-q}{q-1}$) is equivalent to $q < \frac{2p}{p+1}$ (resp. $q = \frac{2p}{p+1}$). As for $\frac{2-q}{q-1} < N-1$ (resp. $\frac{2-q}{q-1} = N-1$), it is equivalent to $\frac{N+1}{N} < q$ (resp. $\frac{N+1}{N} = q$). Furthermore, if $q = \frac{2p}{p+1}$ and $p > \frac{N+1}{N-1}$, the exponent in condition (1.11)-(i) is equal to 1.

If we assume that (1.11) holds, the measure μ is absoluteley continuous with respect to $cap^{\partial\Omega}_{\frac{2}{p},p'}$ and Lipschitz continuous with respect to $cap^{\partial\Omega}_{\frac{2-q}{q},q'}$. Therefore, for any$c > 0$ there exists a unique solution to (2.56). Furthermore, if $\frac{N+1}{N} < q \le \frac{2p}{p+1}$, there holds

$$\mu(E) \le c_{15} cap^{\partial\Omega}_{\frac{2}{p},p'}(E) \le c_{15}A'\left(cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E)\right)^{\frac{(q-1)((N-1)(p-1)-2)}{(p-1)((N-1)(q-1)+q-2)}},$$

which holds since $q \le \frac{2p}{p+1}$, therefore there exists c_{18} such that

$$\mu(E) \le c_{18}\, cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E).$$

By [4, Theorem 1.3] there exists $c'_{17} > 0$ such that for any $0 < c <\le c'_{17}$ there exists a very weak solution $w = w_{c\mu}$ to (2.57) and there holds

$$w_{c\mu} \le c_{19}\mathbb{P}_\Omega(\mu). \tag{2.69}$$

Furthermore any very weak solution of (2.57) is a supersolution of (2.56) with $c\mu$. Therefore $0 \le v_{c\mu} \le w$. Hence there exists a nonnegative solution $u = u_{c\mu}$ of (1.1) and there holds

$$0 \le v_{c\mu} \le u_{c\mu} \le w_{c\mu}. \tag{2.70}$$

Because $v_{c\mu}$ and $w_{c\mu}$ have boundary trace $c\mu$ in the sense of (2.66) and (2.67), the function $u_{c\mu}$ has the same property. By Lemma 2.5 and (2.56)-(i), $\mu \in W^{-\frac{2}{p},p'}(\partial\Omega)$, hence $\mathbb{P}_\Omega(\mu) \in L^p_\rho(\Omega)$ by [16]. It follows from (2.69) and (2.70) that $u_{c\mu} \in L^p_\rho(\Omega)$. Let $\phi = \mathbb{G}_\Omega[u^p_{c\mu}]$, then $\phi \ge 0$ and

$$-\Delta(u_{c\mu} + \phi) = |\nabla u_{c\mu}|^q.$$

The function $u_{c\mu} + \phi$ is a nonnegative superharmonic function in Ω. By Doob's theorem [6, Chapter II], $-\Delta(u_{c\mu} + \phi) \in L^1_\rho(\Omega)$. Hence $|\nabla u_{c\mu}| \in L^q_\rho(\Omega)$. This implies that $u_{c\mu}$ is a very weak solution of (1.12).

Next we assume that $p = \frac{N+1}{N-1}$ and $q = \frac{N+1}{N}$. By (2.68)-(ii) μ satisfies

$$\mu(E) \leq A'' \left(cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E) \right)^2 \leq c_5\, cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E).$$

Hence there exists a very weak solution $w_{c\mu}$ to problem (2.57) for $c \in (0, c_0]$ and $w_{c\mu}$ satisfies (2.69). Since μ satisfies (1.11)-(ii), it is Lipschitz continuous with respect to $cap^{\partial\Omega}_{\frac{2}{p},p'}$. Hence there exists a unique solution $v_{c\mu}$ to (2.56), thus a unique solution $u_{c\mu}$ of (1.1), and there holds

$$v_{c\mu} \leq u_{c\mu} \leq w_{c\mu} \leq c_4 \mathbb{P}_\Omega(\mu) \in L^p_\rho(\Omega),$$

by Lemma 2.5 and the results of [16] and [4, Theorem 1.3]. We end the proof as in the first case. □

Remark. If $\frac{N+1}{N} = q < \frac{2p}{p+1}$ then [1, Theorem 5.5.1] yields

$$\left(\ln \frac{A}{cap^{\partial\Omega}_{\frac{2}{p},p'}(E)} \right)^{-1} \leq A \left(cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E) \right)^{\frac{1}{N}}. \tag{2.71}$$

Therefore, if we assume that

$$\mu(E) \leq c_{17} \left(\ln \frac{A}{cap^{\partial\Omega}_{\frac{2}{p},p'}(E)} \right)^{-N}, \tag{2.72}$$

then μ is absolutely continuous with respect to $cap^{\partial\Omega}_{\frac{2}{p},p'}$ and Lipschitz continuous with respect to $cap^{\partial\Omega}_{\frac{2-q}{q},q'}$. Consequently there exist $v_{c\mu}$ and $w_{c\mu}$ very weak solutions of (2.56) and (2.57) respectively, and they satisfy $0 \leq v_{c\mu} \leq w_{c\mu}$. Consequently there exists $u_{c\mu}$ which satisfies (1.1) such that $v_{c\mu} \leq u_{c\mu} \leq w_{c\mu}$. Therefore $u_{c\mu}$ has the same boundary trace $c\mu$. However we do not know if $u_{c\mu}$ belongs to $L^p_\rho(\Omega)$. Therefore it is not clear wether $u_{c\mu}$ is a very weak solution of (1.12).

The proof in the partially sub-critical case is simpler.

Proof of Theorem 1.4. If $1 < p < \frac{N+1}{N-1}$ for any $\mu \in \mathfrak{M}_+(\partial\Omega)$ problem (2.56) admits a unique solution v_μ (see [13]). If $1 < q < \frac{N+1}{N}$, then there exists $a_0 > 0$ such that for any non-empty Borel set $E \subset \partial\Omega$, $cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E) \geq a_0$. Therefore

$$\mu(E) \leq \|\mu\|_{\mathfrak{M}} \leq \frac{\|\mu\|_{\mathfrak{M}}}{a_0} cap^{\partial\Omega}_{\frac{2-q}{q},q'}(E).$$

It follows from [4, Theorem 1.3] that problem (2.57) admits a solution w_μ whenever $\|\mu\|_{\mathfrak{M}}$ is small enough. By [5] problem (2.63) admits a solution z_μ with $c\mu$ replaced by μ provided $\|\mu\|_{\mathfrak{M}}$ is small enough. Furthermore

$$w_\mu \leq \mathbb{P}_\Omega[\mu] \leq z_\mu. \tag{2.73}$$

Since $z_\mu \in L^p_\rho(\Omega)$, $w_\mu \in L^p_\rho(\Omega)$. Hence by the same arument as in Theorem 1.3, there exists a solution u_μ of (1.1) which satisfies $v_\mu \le u_\mu \le w_\mu$. Hence $u_\mu \in L^p_\rho(\Omega)$ and by the previous argument $\nabla u_\mu \in L^q_\rho(\Omega)$. This implies again that u_μ is a very weak solution of (1.13).

If $1 < p < \frac{N+1}{N-1}$ and $\frac{N+1}{N} \le q < 2$, then problem (2.56) is uniquely solvable for any $\mu \in \mathfrak{M}_+(\partial\Omega)$, while problem (2.63) admits a solution z_μ with $c\mu$ replaced by μ provided $\|\mu\|_{\mathfrak{M}}$ is small enough and since (2.60) holds, problem (2.57) admits a very weak solution provided $0 < c \le c_0$. Since (2.73) holds with $z_\mu \in L^p_\rho(\Omega)$, the result follows as above.

If $p \ge \frac{N+1}{N-1}$, $1 < q < \frac{N+1}{N}$ and $\mu \in \mathfrak{M}_+(\partial\Omega)$ absolutely continuous with respect to $cap^{\partial\Omega}_{\frac{2}{p},p'}$, there exists u_μ solution of (2.56) and w_μ solution of (2.57) provided $c\,\|\mu\|_{\mathfrak{M}}$ is small enough. Since $|\nabla w_\mu|^q \in L^1_\rho(\Omega)$ the function w_μ belongs to the Marcinkiewicz space $M^{\frac{N+1}{N-1}}_\rho(\Omega)$ (see eg. [25]). Since $M^{\frac{N+1}{N-1}}_\rho(\Omega) \subset L^p_\rho(\Omega)$ as $1 < p < \frac{N+1}{N-1}$, it implies that w_μ and therefore u_μ, belongs to $L^p_\rho(\Omega)$. The end of the proof is as above. □

3 Separable solutions

Separable solutions of (1.1) in $\mathbb{R}^N \setminus \{0\}$ are solutions which have the form

$$u(x) = u(r,\sigma) = r^{-\kappa}\omega(\sigma) \qquad \text{for } (r,\sigma) \in \mathbb{R}_+ \times S^{N-1}.$$

This forces q to be equal to $\frac{2p}{p+1}$, κ to $\frac{2}{p-1}$ (recall that this defines α) and ω satisfies

$$-\Delta'\omega + \alpha(N-2-\alpha)\omega + |\omega|^{p-1}\omega - M\left(\alpha^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{p+1}} = 0 \qquad \text{in } S^{N-1}. \tag{3.1}$$

Constant positive solutions are solutions of

$$X^{p-1} - M\alpha^{\frac{2p}{p+1}}X^{\frac{p-1}{p+1}} + \alpha(N-2-\alpha) = 0. \tag{3.2}$$

This existence of solutions to (3.2) and their stability properties will be detailled in a forthcoming article [10]. The understanding of boundary singularities of solutions of (1.1) is conditioned by the knowledge of separable solutions in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N \setminus \{0\}$. Then ω is a solution of

$$\begin{aligned} -\Delta'\omega + \alpha(N-2-\alpha)\omega + |\omega|^{p-1}\omega - M\left(\alpha^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{p+1}} = 0 \qquad &\text{in } S^{N-1}_+ \\ \omega = 0 \qquad &\text{in } \partial S^{N-1}_+. \end{aligned} \tag{3.3}$$

3.1 Existence of singular solutions

We recall the following result due to Boccardo-Murat-Puel dealing with the quasi-linear equation in a domain $G \subset \mathbb{R}^N$

$$\mathcal{Q}(u) := -\Delta u + B(.,u,\nabla u) = 0 \qquad \text{in } \mathcal{D}'(G), \tag{3.4}$$

where $B \in C(G \times \mathbb{R} \times \mathbb{R}^N)$ satisfies, for some continuous increasing function Γ from $\mathbb{R}^+$ to $\mathbb{R}^+$,

$$|B(x,r,\xi)| \leq \Gamma(|r|)(1+|\xi|^2) \quad \text{for all } (x,r,\xi) \in G \times \mathbb{R} \times \mathbb{R}^N. \tag{3.5}$$

Theorem 3.1 *Let G be a bounded domain in $\mathbb{R}^N$. If there exist a supersolution ϕ and a subsolution ψ of the equation $\mathcal{Q}v = 0$ belonging to $W^{1,\infty}(G)$ and such that $\psi \leq \phi$, then for any $\chi \in W^{1,\infty}(G)$ satisfying $\psi \leq \chi \leq \phi$ there exists a function $u \in W^{1,2}(G)$ solution of $\mathcal{Q}u = 0$ such that $\psi \leq u \leq \phi$ and $u - \chi \in W_0^{1,2}(G)$.*

Remark. Mutatis mutandi, the same result holds if $\mathbb{R}^N$ is replaced by a Riemannian manifold.

Their result is actually more general since the Laplacian can be replaced by a quasilinear p-laplacian-type operator and B by a perturbation with the *natural* p-growth. This theorem has direct applications in the construction of solution on S_+^{N-1}, but aslo for the construction of singular solutions in several configurations

Proposition 3.2 *Let Ω be a bounded smooth domain containing 0, $p > 1$, $1 \leq q \leq 2$ and $M \in \mathbb{R}$. Assume that equation*

$$-\Delta u + u^p - M|\nabla u|^q = 0, \tag{3.6}$$

admits a radial positive and decreasing solution v in $\mathbb{R}^N \setminus \{0\}$ satisfying

$$\lim_{|x| \to 0} v(x) = \infty. \tag{3.7}$$

Then there exists a positive function u satisfying (3.6) in $\Omega \setminus \{0\}$, vanishing on $\partial\Omega$ and such that

$$(v(x) - \max\{v(z) : |z| = \delta_0\})_+ \leq u(x) \leq v(x) \quad \text{for all } x \in \Omega \setminus \{0\}. \tag{3.8}$$

where $\delta_0 = \operatorname{dist}(0, \partial\Omega)$.

Proof. Put $m = \max\{v(z) : |z| = \delta_0\}$. The function $v_m = (v - m)_+$ is a radial subsolution of (3.6) in Ω, positive in $B_{\delta_0} \setminus \{0\}$ and vanishing in $\Omega \setminus B_{\delta_0}$. For $\epsilon > 0$ set $\Omega_\epsilon = \Omega \setminus \overline{B}_\epsilon$. The function v_m is a subsolution and it is dominated by the supersolution v. Hence there exists a solution u_ϵ of (3.6) in Ω_ϵ such that $v_m \leq u_\epsilon \leq v$ and $u_\epsilon - v_m \in H_0^1(\Omega_\epsilon)$. By standard regularity estimates, u_ϵ is C^2, hence it solves

$$\begin{aligned} -\Delta u_\epsilon + u_\epsilon^p - M|\nabla u_\epsilon|^q = 0 \qquad & \text{in } \Omega_\epsilon \\ u_\epsilon = v_m \qquad & \text{on } \partial B_\epsilon \\ u_\epsilon = 0 \qquad & \text{on } \partial\Omega. \end{aligned} \tag{3.9}$$

Notice that u_ϵ is unique by the comparison principle. If $0 < \epsilon' < \epsilon$ the function $u_{\epsilon'}$ solution of (3.9) in $\Omega_{\epsilon'}$ with the corresponding boundary data is larger than v_m and in particular $u_{\epsilon'} \lfloor_{\partial B_\epsilon} \geq v_m \lfloor_{\partial B_\epsilon} = u_\epsilon \lfloor_{\partial B_\epsilon}$. Hence $u_{\epsilon'} \geq u_\epsilon$ in Ω_ϵ. When $\epsilon \downarrow 0$,

u_ϵ increase and converges in the $C^{1,\theta}_{loc}(\overline{\Omega}\setminus\{0\})$-topology to some function u which satisfies (3.6) in $\Omega\setminus\{0\}$, is larger that v_m and smaller than v, vanishes on $\partial\Omega$ and such that (3.9) holds. □

The previous result admits can be adapted to the study of solutions with a boundary singularity in bounded domains which are flat enough near the singular point or in $\mathbb{R}^N_+$.

Proposition 3.3 *Let $p>1$, $1\le q\le 2$ and $M\in\mathbb{R}$. Assume that the equation* (3.6) *admits a positive solution w in $\mathbb{R}^N_+$ belonging to $C(\overline{\mathbb{R}^N_+}\setminus\{0\})$, radially decreasing in $\mathbb{R}^N_+$ and satisfying*

$$\lim_{t\to 0} w(t\sigma)=\infty \quad \textit{uniformly on compact sets } K\subset S^{N-1}_+. \tag{3.10}$$

Assume also

(i) either $w\lfloor_{\partial\mathbb{R}^N_+\setminus\{0\}}$ is bounded,

(ii) or $\Omega\subset\mathbb{R}^N_+$ is a bounded smooth domain such that $0\in\partial\Omega$ starshapped with respect to 0 and such that $w\lfloor_{\partial\Omega\setminus\{0\}}$ is bounded.

Then there exists a positive function u satisfying (3.6) *in $\mathbb{R}^N_+$ in case (i), or Ω in case (ii) , vanishing on $\partial\mathbb{R}^N_+\setminus\{0\}$ in case (i), or $\partial\Omega\setminus\{0\}$ in case (ii), and such that*

$$\left(w(x)-\sup\left\{w(z): z\in\partial\mathbb{R}^N_+\setminus\{0\}\right\}\right)_+\le u(x)\le w(x) \quad \textit{for all } x\in\mathbb{R}^N_+, \tag{3.11}$$

where $K=\sup\left\{\limsup_{|z|\to\infty} w(z), \sup\left\{w(z): z\in\partial\mathbb{R}^N_+\setminus\{0\}\right\}\right\}$ *in case (i) or*

$$\left(w(x)-\sup\left\{w(z): z\in\partial\Omega\setminus\{0\}\right\}\right)_+\le u(x)\le w(x) \quad \textit{for all } x\in\Omega. \tag{3.12}$$

in case (ii).

Proof. The proof is a variant of the preceding one, only the geometry of the domains is changed.

In case (ii) set $m=\sup\{w(z): z\in\partial\Omega\setminus\{0\}\}$. Then the function $z\mapsto w_m:=(w(z)-m)_+$ is a subsolution of (3.6) in Ω. It vanishes on $\partial\Omega\setminus\{0\}$ and is dominated by w. For $\epsilon<\delta_0$, let Ω_ϵ denote $\Omega\cap\overline{B_\epsilon}^c$. We consider the problem of finding u_ϵ solution of

$$\begin{aligned} -\Delta u_\epsilon+u_\epsilon^p-M|\nabla u_\epsilon|^q&=0 && \text{in } \Omega_\epsilon\\ u_\epsilon&=w_m && \text{on } \partial B_\epsilon\cap\Omega\\ u_\epsilon&=0 && \text{on } B^c_\epsilon\cap\partial\Omega. \end{aligned} \tag{3.13}$$

Again since $u_\epsilon-w_m\in H^1_0(\Omega_\epsilon)$ and since w_m is smaller than $w\lfloor_{\Omega_\epsilon}$, the solution u_ϵ exists and it satisfies $w_m\le u_\epsilon\le w$ in Ω_ϵ. If $0<\epsilon'<\epsilon$, $u_{\epsilon'}\lfloor_{\partial\Omega_\epsilon}\ge u_\epsilon\lfloor_{\partial\Omega_\epsilon}=v_m$. Hence $u_{\epsilon'}\ge\epsilon$ in Ω_ϵ. As in the proof of Proposition 3.2 the sequence $\{u_\epsilon\}$ is relatively compact in the $C^{1,\theta}_{loc}(\overline{\Omega}\setminus\{0\})$-topology, which ends the proof.

In case (i), for $n > 0$ set $K_n = \sup\{w(z) : z \in \partial B_n^+ \setminus \{0\}\}$ where, we recall it, $B_n^+ = B_n \cap \mathbb{R}^N_+$. The function $w_{K_n} = (w - K_n)_+$ is a subsolution of (3.6) in B_n^+ which vanishes on $\partial B_n^+ \setminus \{0\}$ and is smaller than w. For $0 < \epsilon < n$ we denote by $u_{\epsilon,n}$ the unique function satisfying

$$\begin{array}{rll} -\Delta u_{\epsilon,n} + u_{\epsilon,n}^p - M|\nabla u_{\epsilon,n} e|^q = 0 & \text{in } \Gamma_{\epsilon,n} := B_n^+ \setminus \overline{B}_\epsilon^+ \\ u_{\epsilon,n} = w_m & \text{on } \partial B_\epsilon \cap \mathbb{R}^N_+ \\ u_{\epsilon,n} = 0 & \text{on } (\partial B_n^+ \cap \mathbb{R}^N_+) \cup (\overline{\Gamma}_{\epsilon,n} \cap \partial \mathbb{R}^N_+). \end{array} \tag{3.14}$$

For $\epsilon' \le \epsilon < n \le n'$ there holds $w_{K_n} \le u_{\epsilon,n} \le u_{\epsilon',n'} \le w$ in $\Gamma_{\epsilon,n}$. Letting $n \to \infty$ and $\epsilon \to 0$ there exists a subsequence still denoted by $\{u_{\epsilon,n}\}$ which converges to a solution of u of (3.6) in $\mathbb{R}^N_+$ vanishing on $\partial \mathbb{R}^N_+ \setminus \{0\}$ and satisfying (3.11). □

Remark. The assumption that $w\lfloor_{\partial\Omega\setminus\{0\}}$ is bounded is restricctive. For example if $w(t\sigma) = t^{-a}\omega(\sigma)$ the flatness assumption means that $\text{dist}\,(x, \mathbb{R}^N_+) = O(|x|^{\alpha+1})$ for all $x \in \partial\Omega$ near 0. It can be avoided in case of the existence of a subsolution.

Proposition 3.4 *Let $p > 1$, $1 \le q \le 2$ and $M \in \mathbb{R}$. Assume that the equation* (3.6) *admits a positive supersolution w in $\mathbb{R}^N_+$ belonging to $C(\overline{\mathbb{R}^N_+} \setminus \{0\})$ satisfying* (3.10). *Assume also*

(i) either there exists a positive subsolution $Z \in C(\overline{\mathbb{R}^N_+}\setminus\{0\})$ vanishing on $\partial\mathbb{R}^N_+\setminus\{0\}$, smaller than w and satisfying (3.10),

(ii) or $\Omega \subset \mathbb{R}^N_+$ is a bounded smooth domain such that $0 \in \partial\Omega$ and there exists a positive subsolution $Z \in C(\overline{\Omega} \setminus \{0\})$, vanishing on $\partial\Omega \setminus \{0\}$ such that $Z \le w\lfloor_\Omega$ and satisfying (3.10).

Then there exists a positive function u satisfying (3.6) *in $\mathbb{R}^N_+$ (resp. Ω), vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ (resp. $\partial\Omega \setminus \{0\}$) and such that*

$$Z(x) \le u(x) \le w(x) \quad \text{for all } x \in \mathbb{R}^N + \ \ (\text{resp. } x \in \Omega). \tag{3.15}$$

Example. If $1 < p < \frac{N+1}{N-1}$ it is proved in [13] that if $\Omega \subset \mathbb{R}^N_+$ is a smooth bounded domain such that $0 \in \partial\Omega$, there exists a nonnegative function $Z_\infty \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ satisfying the equation

$$\begin{array}{rll} -\Delta Z + Z^p = 0 & \text{in } \Omega \\ Z = 0 & \text{on } \partial\Omega \setminus \{0\}, \end{array} \tag{3.16}$$

and such that $t^{\frac{2}{p-1}} Z_\infty(t\sigma) \to \psi(\sigma)$ uniformly on compact sets $K \subset S^{N-1}_+$ as $t \to 0$ where ψ is the unique a positive solution of

$$\begin{array}{rll} -\Delta'\psi - \dfrac{2}{p-1}\left(\dfrac{2}{p-1} + 2 - N\right)\psi + \psi^p = 0 & \text{in } S^{N-1}_+ \\ \psi = 0 & \text{on } \partial S^{N-1}_+. \end{array} \tag{3.17}$$

Furthermore, for any $k > 0$ there exists a nonnegative function $Z_k \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ satisfying (3.16) and such that $t^{N-1} Z_k(t\sigma) \to k\phi_1(\sigma)$ uniformly on compact

subsets of S^{N-1}_+ where ϕ_1 is the first eigenfunction of $-\Delta'$ in $W^{1,2}_0(S^{N-1}_+)$. Furthermore $Z_k \uparrow Z_\infty$ when $k \to \infty$. If the equation (3.6) admits a positive supersolution w in $\mathbb{R}^N_+$ belonging to $C(\overline{\mathbb{R}^N_+} \setminus \{0\})$ and such that $Z_k \le w$ in Ω for some $0 < k \le \infty$, then there exists a positive function u satisfying (3.6) in Ω, vanishing on $\partial\Omega \setminus \{0\}$ and such that

$$Z_k(x) \le u(x) \le w(x) \quad \text{for all } \ x \in \Omega. \tag{3.18}$$

The same result holds if Ω is replaced by $\mathbb{R}^N_+$.

3.2 Existence or non-existence of separable solutions

Since any large enough constant is a supersolution of (3.1), it follows by Theorem 3.1 that if there exists a nonnegative subsolution $z \in W^{1,\infty}_0(S^{N-1}_+)$, there exists a solution in between.

3.2.1 Proof of Theorem 1.7

We recall that ϕ_1 is the first eigenfunction of $-\Delta'$ in $W^{1,2}_0(S^{N-1}_+)$ with corresponding eigenvalue $\lambda_1 = N-1$. Put

$$H(\omega) = -\Delta'\omega + \alpha(N-2-\alpha)\omega + |\phi|^{p-1}\omega - M\left(\alpha^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{p+1}},$$

then

$$H(\phi_1) = (N-1+\alpha(N-2-\alpha))\,\phi_1 + \phi_1^p - M\left(\alpha^2\phi_1^2 + |\nabla'\phi_1|^2\right)^{\frac{p}{p+1}}.$$

If ϕ_1 is small enough, there holds $\phi_1^p - M\left(\alpha^2\phi_1^2 + |\nabla'\phi_1|^2\right)^{\frac{p}{p+1}} < 0$, hence ϕ_1 is a subsolution. However the condition $N-1+\alpha(N-2-\alpha) \le 0$ is too stringent. We can use the fact that, up to a good choice of coordinates, $\phi_1 = \phi_1(\sigma) = \cos\sigma$ with $\sigma \in [0, \frac{\pi}{2}]$. Furthermore the statement "ϕ_1 is small enough" can be replaced by $\phi_1 = \delta\cos\sigma$ with $\delta > 0$ small enough. Then

$$\begin{aligned}\delta^{-1}H(\delta^{\frac{p+1}{p-1}}\cos\sigma)&\\ &= (N-1+\alpha(N-2-\alpha))\cos\sigma + \delta^{p+1}\cos^p\sigma - M\delta(\alpha^2\cos^2\sigma + \sin^2\sigma)^{\frac{p}{p+1}}.\end{aligned}$$

The problem is to find $\delta > 0$ such that for all $\sigma \in [0, \frac{\pi}{2}]$ we have $H(\delta^{\frac{p+1}{p-1}}\cos\sigma) \le 0$. Put $Z = \cos\sigma$ and $\delta^{-1}H(\delta^{\frac{p+1}{p-1}}\cos\sigma) = \delta^{-1}H(\delta^{\frac{p+1}{p-1}}Z) = K(Z)$, then

$$K_\delta(Z) = (N-1+\alpha(N-2-\alpha))\,Z + \delta^{p+1}Z^p - M\delta((\alpha^2-1)Z^2+1)^{\frac{p}{p+1}},$$

where $0 \le Z \le 1$. We use the fact that

$$\alpha^2\cos^2\sigma + \sin^2\sigma \ge \min\{\alpha^2, 1\}(\cos^2\sigma + \sin^2\sigma) := \kappa^2 > 0,$$

hence

$$\delta^{-1}H(\delta^{\frac{p+1}{p-1}}\cos\sigma) \le (N-1+\alpha(N-2-\alpha))\cos\sigma + \delta^{p+1}\cos^p\sigma - M\delta\kappa^{\frac{2p}{p+1}}.$$

Then

$$K_\delta(Z) \le \tilde{K}_\delta(Z) := (N-1+\alpha(N-2-\alpha))\, Z + \delta^{p+1} Z^p - M\delta\kappa^{\frac{2p}{p+1}}, \tag{3.19}$$

and

$$\tilde{K}'_\delta(Z) = N-1+\alpha(N-2-\alpha) + p\delta^{p+1} Z^{p-1}. \tag{3.20}$$

If $N-1+\alpha(N-2-\alpha) \ge 0$, equivalently $p \ge \frac{N+1}{N-1}$, then $\tilde{K}'_\delta \ge 0$ on $[0,1]$, hence

$$\tilde{K}_\delta(Z) \le \tilde{K}_\delta(1) = N-1+\alpha(N-2-\alpha) + \delta^{p+1} - M\delta\kappa^{\frac{2p}{p+1}}.$$

The function $\delta \mapsto \tilde{K}_\delta(1)$ achieves its minimum for $\delta = \delta_0 := \beta^{\frac{2}{p+1}} \left(\frac{M}{p+1}\right)^{\frac{1}{p}}$ and

$$\tilde{K}_{\delta_0}(1) = N-1+\alpha(N-2-\alpha) - p\kappa^2 \left(\frac{M}{p+1}\right)^{\frac{p+1}{p}}.$$

Therefore, when $p \ge \frac{N+1}{N-1}$, $K_{\delta_0} \le 0$ on $[0,1]$ if

$$\begin{aligned} \left(\tfrac{M}{p+1}\right)^{\frac{p+1}{p}} \ge \left(\tfrac{M_{N,p}}{p+1}\right)^{\frac{p+1}{p}} &:= \frac{N-1+\alpha(N-2-\alpha)}{p\min\{1,\alpha^2\}} \\ &= \frac{(p+1)\,(p(N-1)-(N+1))}{p\min\{(p-1)^2,4\}}. \end{aligned} \tag{3.21}$$

If $N-1+\alpha(N-2-\alpha) \le 0$, equivalently $p \le \frac{N+1}{N-1}$, it is clear from (3.19) that $\tilde{K}_\delta(Z) \le 0$ for any $Z \in [0,1]$ as soon as $\delta \le \kappa^{\frac{1}{p+1}} M^{\frac{1}{p}}$. This ends the proof. □

Remark. Introducing m^{**} defined in (1.8), inequality (3.21) endows the form

$$M \ge \left(\frac{2(p+1)}{\min\{(p-1)^2,4\}}\right)^{\frac{p}{p+1}} m^{**}. \tag{3.22}$$

3.2.2 Non-existence

Theorem 3.5 *Let $p > \frac{N+1}{N-1}$ and $M < m^{**}$, defined by* (1.8)*. Then problem* (3.1) *admits no positive solution.*

Proof. If ω is a positive solution of (3.1), $u_\omega(r,.) = r^{-\frac{2}{p-1}}\omega(.)$ is a positive solution of (1.1) in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$. Let $\Omega \subset \mathbb{R}^N_+$ be any smooth domain such that $0 \in \partial\Omega$ and $\partial\Omega$ is flat near 0. Then $u_\omega \le K$ on $\partial\Omega$ for some $K > 0$. Put $v = (u_\omega - K)_+$, then it is a nonnegative subsolution of (3.1). For any $\epsilon > 0$ small enough there exists a solution u_ϵ of

$$\begin{aligned} -\Delta u + u^p - M|\nabla|^{\frac{2p}{p+1}} &= 0 \qquad && \text{in } \ \Omega_\epsilon := \Omega \cap \overline{B}^c_\epsilon \\ u &= v \qquad && \text{on } \ \partial B_\epsilon \cap \Omega \\ u &= 0 \qquad && \text{on } \ B^c_\epsilon \cap \partial\Omega. \end{aligned} \tag{3.23}$$

Then $v \le u_\epsilon \le u_\omega$. Furthermore, for $0 < \epsilon' < \epsilon$, $u_\epsilon \le u_{\epsilon'}$ in Ω_ϵ. Hence $\{u_\epsilon\}$ converges, when $\epsilon \to 0$ to a solution u_0 of (1.7), which satisfies $v \le u_0 \le u_\omega$ and therefore vanishes on $\partial\Omega \setminus \{0\}$, contradiction. □

4 Solutions with an isolated boundary singularity

4.1 Construction of fundamental solutions

Let Ω be either $\mathbb{R}^N_+$ or a bounded domain with $0 \in \partial\Omega$. A function u satisfying (1.6) is a *fundamental solution* if it has a singularity of potential type, that is

$$\lim_{x\to 0} \frac{|x|^N u(x)}{\rho(x)} = k, \tag{4.24}$$

for some $k > 0$. We call . The function u can also be looked for as a solution of

$$\begin{aligned} -\Delta u + u^p - M|\nabla u|^q = 0 \qquad & \text{in } \Omega \\ u = k\delta_0 \qquad & \text{in } \mathcal{D}'(\partial\Omega), \end{aligned} \tag{4.25}$$

in the sense that $u \in L^p(\Omega \cap B_r, \rho dx)$, $\nabla u \in L^q_{loc}(\Omega \cap B_r, \rho dx)$ for any $r > 0$, where $\rho(x) = \text{dist}\,(x, \partial\Omega)$, and for any $\zeta \in C^1_c(\overline{\Omega}) \cap W^{2,\infty}(\Omega)$ there holds

$$\int_\Omega (-u\Delta\zeta + u^p\zeta - M|\nabla u|^q\zeta)\, dx = -k\frac{\partial\zeta}{\partial \mathbf{n}}(0). \tag{4.26}$$

We first consider the problem in $\mathbb{R}^N_+$.

Theorem 4.1 *Assume $1 < p < \frac{N+1}{N-1}$ and $1 < q < \frac{N+1}{N}$. Then for any $M > 0$ and any $k \geq 0$ there exists a minimal solution $u := u_k$ to (1.6) in $\mathbb{R}^N_+$ such that (4.24) holds.*

Proof. The scheme of the proof is surprising since we first show that, in the case $q = \frac{2p}{p+1}$, there exists $M_1 > 0$ such that for any $k > 0$ and any $0 < M < M_1$ there exists a solution. Using this result we prove that if $1 < q < \frac{2p}{p+1}$, then for any $M > 0$ and $k > 0$ there exists a solution. Finally we return to the case $q = \frac{2p}{p+1}$ and using the result in the previous case, we prove that when $q = \frac{2p}{p+1}$ we can get rid of the restriction on $M > 0$ and $k > 0$ for the existence of solutions.
I- The case $q = \frac{2p}{p+1}$ and M upper bounded.
For $\ell > 0$ the transformation T_ℓ defined by

$$T_\ell[u(x)] = \ell^{\frac{2}{p-1}} u(\ell x), \tag{4.27}$$

leaves the operator $\mathcal{L}_{\frac{2p}{p+1},M}$ invariant. We can therefore write

$$T_\ell[u_k] = u_{k\ell^{\frac{2}{p-1}+1-N}},$$

in the sense that if u_k satisfies (4.24) then $T_\ell[u_k]$ satisfies the same limit with k replaced by $k\ell^{\frac{2}{p-1}+1-N}$. This observation will take its complete value as we will prove later on than u_k can be the minimal solution for this k. Therefore if there exists a solution to (1.6) in $\mathbb{R}^N_+$, vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ satisfying (4.29) for some $k > 0$, then there exists such a solution for any $k > 0$.

Step 1- Construction of a subsolution. For $k>0$ we denote by v_k the solution of

$$\begin{array}{rl} -\Delta v + v^p = 0 & \text{in } \mathbb{R}^N_+ \\ v = k\delta_0 & \text{on } \partial\mathbb{R}^N_+ \setminus \{0\}. \end{array} \tag{4.28}$$

Such a solution exists thanks to Gmira-Véron if $\mathbb{R}^N_+$ is replaced by a bounded domain Ω. If case of a half-space the problem is first solved in B_n^+ and by letting $n\to\infty$, we obtain the solution in $\mathbb{R}^N_+$. Clearly v_k is a subsolution of problem (1.6), and it satisfies

$$\lim_{x\to 0}\frac{u(x)}{R(x)} = c_N k, \tag{4.29}$$

for some $c_N>0$, where $R(x)=c'_N\frac{x_N}{|x|^N}$ is the Poisson kernel in $\mathbb{R}^N_+$.

Step 2- Construction of a supersolution. It is known that

$$|\nabla R(x)|^2 = |x|^{-2N}c^2(x), \tag{4.30}$$

where $c(.)$ is smooth and verifies

$$0<\gamma_1\le c(x)\le\gamma_2.$$

We construct $w=w_k$ in $\mathbb{R}^N_+$ under the form

$$w_k = kR + w, \tag{4.31}$$

where w satisfies

$$\begin{array}{rl} -\Delta w + w^p = a\gamma_2|x|^{-\frac{2Np}{p+1}} & \text{in } \mathbb{R}^N_+ \\ w = 0 & \text{on } \partial\mathbb{R}^N_+, \end{array} \tag{4.32}$$

for some $a>0$ to be chosen later on. Then

$$\begin{aligned} \mathcal{L}_{\frac{2p}{p+1},M} w_k &= -\Delta w + (kR+w)^p - M\left(|k\nabla R+\nabla w|^2\right)^{\frac{p}{p+1}} \\ &= (kR+w)^p - w^p + a\gamma_2|x|^{-\frac{2Np}{p+1}} - M\left(|k\nabla R+\nabla w|^2\right)^{\frac{p}{p+1}} \\ &\ge pkRw^{p-1} + a\gamma_2|x|^{-\frac{2Np}{p+1}} - 2M\left(k^{\frac{2p}{p+1}}\gamma_2^{\frac{2p}{p+1}}|x|^{-\frac{2Np}{p+1}} + |\nabla w|^{\frac{2p}{p+1}}\right). \end{aligned}$$

Now it is easy to check using Osserman's type construction as in [23, Lemma 2.1] and scaling techniques that

$$w(x)\le\gamma_3\min\left\{a^{\frac{1}{p}}|x|^{-\frac{2N}{p+1}}, a|x|^{2(1-\frac{Np}{p+1})}\right\},$$

and

$$\begin{gathered} |\nabla w(x)|\le\gamma_4\min\left\{a^{\frac{1}{p}}|x|^{-\frac{2N}{p+1}-1}, a|x|^{1-\frac{2Np}{p+1}}\right\} \\ \Longrightarrow \\ |\nabla w(x)|^{\frac{2p}{p+1}}\le\gamma_5\min\left\{a^{\frac{2}{p+1}}|x|^{-\frac{2p(2N+p+1))}{(p+1)^2}}, a^{\frac{2p}{p+1}}|x|^{\frac{2p(p+1-2Np)}{(p+1)^2}}\right\}. \end{gathered}$$

Therefore, if we put

$$\tau = -\frac{p^2-1}{2p(N+1-p(N-1))},$$

then $\tau > 0$ since $N+1 > p(N-1)$ and

$$\begin{aligned} |x|^{\frac{2Np}{p+1}}\mathcal{L}_{\frac{2p}{p+1},M}w_k &\geq \gamma_2\left(a - 2Mk^{\frac{2p}{p+1}}\gamma_2^{\frac{p-1}{p+1}}\right) - 2M\gamma_5 a^{\frac{2p}{p+1}}|x|^{\frac{2p(N+1-p(N-1))}{(p+1)^2}} \\ &\geq \gamma_2\left(a - 2Mk^{\frac{2p}{p+1}}\gamma_2^{\frac{p-1}{p+1}}\right) - 2M\gamma_5 a \quad \text{in } \ B^+_{a^\tau}. \end{aligned} \tag{4.33}$$

Thus, by the definition of τ,

$$|x|^{\frac{2Np}{p+1}}\mathcal{L}_{\frac{2p}{p+1},M}w_k \geq \gamma_2\left(a - 2Mk^{\frac{2p}{p+1}}\gamma_2^{\frac{p-1}{p+1}}\right) - 2M\gamma_5 a \text{ in } \ (B^+_{a^\tau})^c. \tag{4.34}$$

Replacing τ by its value, we obtain a very simple expression from (4.33) and (4.34), valid both in $B^+_{a^\tau}$ and $(B^+_{a^\tau})^c$, namely

$$|x|^{\frac{2Np}{p+1}}\mathcal{L}_{\frac{2p}{p+1},M}w_k \geq \gamma_2\left(a - 2Mk^{\frac{2p}{p+1}}\gamma_2^{\frac{p-1}{p+1}}\right) - 2M\gamma_5 a \quad \text{in } \mathbb{R}^N_+. \tag{4.35}$$

Since k can be changed by scaling, we conclude that if

$$M < M_1 := \frac{\gamma_2}{2\gamma_5}, \tag{4.36}$$

there exists $k_0 > 0$ such that for any $0 < k \leq k_0$, w_k is a supersolution.

Step 3-Existence. For $0 < k \leq k_0$ w_k is a supersolution which dominates the subsolution v_k. Hence, by [26, Theorem 1-4-6] there exists a solution u_k to (1.6) in $\mathbb{R}^N_+$, vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and such that $v_k \leq u_k \leq w_k$. Since

$$\lim_{x\to 0}\frac{v_k(x)}{R(x)} = \lim_{x\to 0}\frac{w_k(x)}{R(x)} = c_N k,$$

it follows that u_k inherits the same asymptotic behaviour. Since $k < k_0$ can be replaced by any $k > 0$, the existence of a solution follows.

II- The case $1 < q < \frac{2p}{p+1}$. Assume $M < M_1$, $k > 0$ and $\tilde{u}_k$ is the minimal solution of (1.6) in $\mathbb{R}^N_+$ with $q = \frac{2p}{p+1}$, vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and such that (4.29). Since $|\nabla\phi|^{\frac{2p}{2p}} \geq |\nabla\phi|^q - 1$, there holds

$$-\Delta\tilde{u}_k + \tilde{u}_k^p + M - M|\nabla\tilde{u}_k|^q \geq 0.$$

Hence $\tilde{u}^*_k = \tilde{u}_k + M^{\frac{1}{p}}$ is a supersolution (1.6) in $\mathbb{R}^N_+$ and it dominates v_k defined in (4.28). By [26, Theorem 1-4-6] there exists a solution u_k of (1.6), vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and satisfying (4.29) under the following weaker form

$$\lim_{t\to 0}\frac{u_k(tx)}{R(tx)} = c_N k \quad \text{uniformly on compact subsets of } \mathbb{R}^N_+. \tag{4.37}$$

Since $|x|^{N-1}u_k(x)$ is uniformly bounded and vanishes on $\partial\mathbb{R}^N_+\setminus\{0\}$, it is bounded in the $C^1_{loc}(\overline{\mathbb{R}^N_+})$-topology. Hence (4.29) holds. This proves the result when $M<M_1$.

Next let $M>0$ arbitrary and $k>0$. In order to find a solution $u:=u_k$ to (1.6), we set $u(x)=\ell^{-\frac{2}{p-1}}U_\ell(\frac{x}{\ell})$. Then $\mathcal{L}_{q,M}u=0$ is equivalent to

$$\mathcal{L}_{q,M_\ell}U_\ell:=-\Delta U_\ell+U_\ell^p-M\ell^{\frac{2p-q(p+1)}{p-1}}|\nabla U|^q=0\quad\text{with}\ \ M_\ell=M\ell^{\frac{2p-q(p+1)}{p-1}},$$

and (4.29) is equivalent to

$$\lim_{x\to0}\frac{U_\ell(x)}{R(x)}=c_N\ell^{\frac{2}{p-1}+1-N}k.$$

Since $2p-q(p+1)>0$ it is enough to choose $\ell>0$ such that $M\ell^{\frac{2p-q(p+1)}{p-1}}<M_1$, and we end the proof using the result when $M<M_1$.

III- The case $q=\frac{2p}{p+1}$ revisited. Let $p<\tilde p<\frac{N+1}{N-1}$. Then $\frac{2p}{p+1}<\frac{2\tilde p}{\tilde p+1}$. This implies that for any $M>0$ and $k>0$ there exists a positive solution $\tilde u_k$ to

$$-\Delta\tilde u_k+\tilde u_k^{\tilde p}-M|\nabla\tilde u_k|^{\frac{2p}{p+1}}=0\qquad\text{in }\ \mathbb{R}^N_+,$$

vanishing on $\partial\mathbb{R}^N_+\setminus\{0\}$ and such that

$$\lim_{x\to0}\frac{\tilde u_k(x)}{R(x)}=c_Nk.$$

Since $\tilde p>p$ we have $\tilde u_k^{\tilde p}>\tilde u_k^p-1$ and therefore

$$-\Delta\tilde u_k+\tilde u_k^p-M|\nabla\tilde u_k|^{\frac{2p}{p+1}}\ge1>0\qquad\text{in }\ \mathbb{R}^N_+.\tag{4.38}$$

The function $\tilde v_k$ solution of

$$\begin{aligned}-\Delta v+v^p&=0&&\text{in }\mathbb{R}^N_+\\ v&=k\delta_0&&\text{on }\partial\mathbb{R}^N_+\setminus\{0\},\end{aligned}\tag{4.39}$$

is a subsolution of (4.38), hence the exists a solution u_k of such that $\tilde v_k<u_k<\tilde u_k$ of (1.6) in $\mathbb{R}^N_+$, vanishing on $\partial\mathbb{R}^N_+\setminus\{0\}$ and such that (4.24) holds.

IV- The case $\frac{2p}{p+1}<q<\frac{1+N}{N}$. We follow the ideas of Case I. We look for a supersolution w_k under the form (4.31) where w_k satisfies

$$\begin{aligned}-\Delta w+w^p&=a\gamma_2|x|^{-Nq}&&\text{in }\ \mathbb{R}^N_+\\ w&=0&&\text{on }\ \partial\mathbb{R}^N_+,\end{aligned}\tag{4.40}$$

for some $a>0$. Then

$$\begin{aligned}\mathcal{L}_{q,M}w_k&=-\Delta w+(kR+w)^p-M\left(|k\nabla R+\nabla w|^2\right)^{\frac{q}{2}}\\ &=(kR+w)^p-w^p+a\gamma_2|x|^{-Nq}-M\left(|k\nabla R+\nabla w|^2\right)^{\frac{q}{2}}\\ &\ge pkRw^{p-1}+a\gamma_2|x|^{-Nq}-2M\left(k^q\gamma_2^q|x|^{-Nq}+|\nabla w|^q\right).\end{aligned}$$

As in Case I, by scaling techniques,

$$w(x) \le \gamma_3 \min\left\{a^{\frac{1}{p}}|x|^{-\frac{Nq}{p}}, a|x|^{2-Nq}\right\}$$

and

$$|\nabla w(x)| \le \gamma_4 \min\left\{a^{\frac{1}{p}}|x|^{-\frac{Nq}{p}-1}, a|x|^{1-Nq}\right\}.$$

Hence

$$|\nabla w(x)|^q \le \gamma_5 \min\left\{a^{\frac{q}{p}}|x|^{-\frac{Nq^2}{p}-q}, a^q|x|^{q(1-Nq)}\right\}.$$

We set

$$\tau = -\frac{1}{2p' - Nq} = -\frac{p-1}{2p - Nq(p-1)}.$$

Then, by the definition of τ,

$$\begin{aligned} |x|^{Nq}\mathcal{L}_{q,M} w_k &\ge \gamma_2\left(a - 2Mk^q\gamma_2^{q-1}\right) - 2M\gamma_5 a^q|x|^{q(N+1-Nq)} \\ &\ge \gamma_2\left(a - 2Mk^q\gamma_2^{q-1}\right) - 2M\gamma_5 a^{\frac{1+N-p(N-1)}{\frac{2p}{q}-N(p-1)}} \quad \text{in } B^+_{a^\tau}, \end{aligned} \tag{4.41}$$

and

$$|x|^{\frac{2Np}{p+1}}\mathcal{L}_{\frac{2p}{p+1},M} w_k \ge \gamma_2\left(a - 2Mk^q\gamma_2^{q-1}\right) - 2M\gamma_5 a^{\frac{1+N-p(N-1)}{\frac{2p}{q}-N(p-1)}} \text{ in } (B^+_{a^\tau})^c. \tag{4.42}$$

We obtain a very simple expression from (4.41) and (4.42), valid both in $B^+_{a^\tau}$ and $(B^+_{a^\tau})^c$, hence

$$|x|^{\frac{2Np}{p+1}}\mathcal{L}_{\frac{2p}{p+1},M} w_k \ge \gamma_2\left(a - 2Mk^q\gamma_2^{q-1}\right) - 2M\gamma_5 a^{\frac{1+N-p(N-1)}{\frac{2p}{q}-N(p-1)}} \quad \text{in } \mathbb{R}^N_+ \tag{4.43}$$

Using the scaling transformation T_ℓ defined in (4.27), the problem of finding u_k solution of (4.25) is equivalent to looking for a solution of

$$\begin{aligned} -\Delta u + u^p - M\ell^{\frac{2p-q(p+1)}{p-1}}|\nabla u|^q = 0 \quad &\text{in } \mathbb{R}^N_+ \\ u = k\ell^{\frac{p+1}{p-1}-N}\delta_0 \quad &\text{in } \mathcal{D}'(\partial\mathbb{R}^N_+). \end{aligned} \tag{4.44}$$

If we replace M by $M_\ell := M\ell^{\frac{2p-q(p+1)}{p-1}}$ and k by $k_\ell := k\ell^{\frac{p+1}{p-1}-N}$, the inequality (4.35) turns into

$$|x|^{\frac{2Np}{p+1}}\mathcal{L}_{\frac{2p}{p+1},M} w_{k,\ell} \ge \gamma_2\left(a - 2M_\ell k_\ell^q\gamma_2^{q-1}\right) - 2M_\ell\gamma_5 a^{\frac{1+N-p(N-1)}{\frac{2p}{q}-N(p-1)}} \quad \text{in } \mathbb{R}^N_+, \tag{4.45}$$

where $w_{k,\ell} = w + k_\ell R$ instead of (4.31). Notice that $M_\ell k_\ell^q = M\ell^{\frac{2p}{p-1}-Nq}k^q$. We choose $\ell > 0$ such that $M_\ell k_\ell^q\gamma_2^{q-1} = \frac{a}{4}$, hence

$$|x|^{\frac{2Np}{p+1}}\mathcal{L}_{\frac{2p}{p+1},M} w_{k,\ell} \ge \frac{a\gamma_2}{2}\left(1 - \gamma_5\gamma_2^{-q}k^{-q}a^{\frac{1+N-p(N-1)}{\frac{2p}{q}-N(p-1)}}\right) \quad \text{in } \mathbb{R}^N_+. \tag{4.46}$$

It is now sufficient to choose $a > 0$ such that the right-hand side of (4.46) is nonnegative and thus $w_{k,\ell}$ is a supersolution. Since $\tilde{v}_{k,\ell}$ is a subsolution smaller that $w_{k,\ell}$, we end the proof as in Case I.

V- Existence of a minimal solution. Next, if u_k^1 and u_k^2 are solutions they dominate v_k and the function $u_k^{1,2} = \inf\{u_{k,1}, u_{k,2}\}$ is a supersolution which dominates v_k. Hence there exists a solution $\tilde{u}_k$ such that

$$v_k \leq \tilde{u}_k \leq u_k^{1,2}.$$

Let $\mathcal{E}_k$ be the set of nonnegative solutions of (1.6) in $\mathbb{R}^N_+$, vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and such that (4.24) and put

$$u_k = \inf\{v : v \in \mathcal{E}_k\}.$$

Then there exists a decreasing sequence $\{v_j\}$ such that v_j converges to u_k on a countable dense subset of $\mathbb{R}^N_+$. By standard elliptic equation regularity theory, v_j converges to u_k on any compact subset of $\overline{\mathbb{R}}^N_+ \setminus \{0\}$. Hence u_k is a solution of (1.6) in $\mathbb{R}^N_+$, it vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and (4.29) since $u_k \geq v_k$. Hence u_k is the minimal solution. □

Next of we consider the same problem in a bounded domain Ω.

Theorem 4.2 *Let Ω be a bounded smooth domain such that $0 \in \partial\Omega$. If $1 < p < \frac{N+1}{N-1}$ and $1 < q < \frac{N+1}{N}$, then for any $M > 0$ and any $k \geq 0$ there exists a minimal positive solution $u := u_k$ to (1.6) in Ω vanishing on $\partial\Omega \setminus \{0\}$ and satisfying*

$$\lim_{x\to 0} \frac{v_k(x)}{P^\Omega(x)} = c_N k, \tag{4.47}$$

where P^Ω is the Martin kernel in Ω with a singularity at 0.

Proof. We give first proof when $\Omega \subset \mathbb{R}^N_+$. We adapt the proof of Theorem 4.1. The solution v_k of

$$\begin{aligned} -\Delta v + v^p &= 0 && \text{in } \Omega \\ v &= k\delta_0 && \text{on } \partial\Omega, \end{aligned} \tag{4.48}$$

is a subsolution for (1.6) in Ω and satisfies (4.47). The solution u_k of (1.6) in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and satisfying (4.24) dominates v_k in Ω. Hence the result follows by Proposition 3.4.

When Ω is not included in $\mathbb{R}^N_+$, estimates (4.30) is valid with the same type of bounds on c. We also consider separately the cases $q = \frac{2p}{p+1}$ and M upper bounded, $q < \frac{2p}{p+1}$ and $M > 0$ arbitrary and $q = \frac{2p}{p+1}$ and$M > 0$ arbitrary and finally $\frac{2p}{p+1} < q < \frac{N+1}{N}$. For supersolution we consider the function $w_k := kP^\Omega + w$ where w satisfies

$$\begin{aligned} -\Delta w + w^p &= a\gamma_2 |x|^{-\frac{2Np}{p+1}} && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.49}$$

for some $a > 0$. The estimates on w endow the form

$$w(x) \le \gamma_3 a^{\frac{1}{p}} |x|^{2(1-\frac{Np}{p+1})},$$

and

$$|\nabla w(x)| \le \gamma_4 a^{\frac{1}{p}} |x|^{1-\frac{2Np}{p+1}},$$

where γ_3 and γ_4 depend on Ω. Hence (4.35) holds in Ω instead of $\mathbb{R}^N_+$, and we have existence for $M < M_1$, where M_1 is defined by (4.36). Then we prove existence for any $M > 0$ and $k > 0$ when $q < \frac{2p}{p+1}$ then for any $M > 0$ when $q = \frac{2p}{p+1}$ and finally when $\frac{2p}{p+1} < q < \frac{N+1}{N}$ as in Theorem 4.1. □

4.2 Solutions with a strong singularity

4.2.1 The case $1 < q \le \frac{2p}{p+1}$

If $p = \frac{N+1}{N-1}$ and $1 < q < \frac{N+1}{N}$ and if $p > \frac{N+1}{N-1}$ and either $1 < q < \frac{2p}{p+1}$ and $M > 0$ or $q = \frac{2p}{p+1}$ and $M > m^{**}$ defined in (1.8), the singularity is removable by Theorem 1.1. Thus the ranges of exponents that we consider are the following,

$$\begin{aligned} &(i) \qquad 1 < q \le \tfrac{2p}{p+1} \ \text{ and } \ 1 < p < \tfrac{N+1}{N-1}, \\ &(ii) \qquad (p,q) = \left(\tfrac{N+1}{N-1}, \tfrac{N+1}{N}\right). \end{aligned} \tag{4.50}$$

If (4.50)-(i) holds, $q < \frac{N+1}{N}$, and in this range the limit of the fundamental solutions u_k when $k \to \infty$ is a solution with a strong singularity with an explicit blow-up rate. In the case of a bounded domain oir construction necessitates a geometric *flatness* condition of $\partial\Omega$ near 0. We consider first the case $\Omega = \mathbb{R}^N_+$.

Theorem 4.3 *Assume* (4.50)*-(i) holds, then for any $M \ge 0$ there exists a positive solution u of* (1.1) *in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ such that*

$$\lim_{x\to 0} \frac{u(x)}{R(x)} = \infty. \tag{4.51}$$

Furthermore,
(i) If $1 < q < \frac{2p}{p+1}$,

$$\lim_{r\to 0} r^\alpha u(r,.) = \psi \qquad \textit{uniformly in } \ S^{N-1}_+, \tag{4.52}$$

where ψ is the unique positive solution of (3.17).
(ii) If $q = \frac{2p}{p+1}$,

$$\lim_{r\to 0} r^\alpha u(r,.) = \omega \qquad \textit{uniformly in } \ S^{N-1}_+, \tag{4.53}$$

where ω is the minimal positive solution of (1.20).

Proof. If $k > 0$, we denote by $u = u_{k,M}$ the solution of

$$\begin{array}{rl} -\Delta u + u^p = M|\nabla u|^q & \text{in } \mathbb{R}^N_+ \\ u = k\delta_0 & \text{in } \partial\mathbb{R}^N_+. \end{array} \tag{4.54}$$

The mapping $k \mapsto u_k$ is increasing. We set $T_\ell[u] = u_\ell$, where T_ℓ is defined in (4.27). Since $1 < q \le \frac{2p}{p+1}$,

$$T_\ell[u_{k,M}] = u_{k\ell^{\frac{2}{p-1}+1-N}, M\ell^{\frac{2p-q(p+1)}{p-1}}}.$$

It follows from Theorem 2.1 and Theorem 2.3 that the sequences $\{u_{k,M}\}$ and $\{\nabla u_{k,M}\}$ converge locally uniformly in $\mathbb{R}^N_+$, when $k \to \infty$, to a function $u_{\infty,M}$ which satisfies (1.1) in $\mathbb{R}^N_+$. Furthermore

$$T_\ell[u_{\infty,M}] = u_{\infty, M\ell^{\frac{2p-q(p+1)}{p-1}}} \qquad \text{forall } \ell > 0. \tag{4.55}$$

In the case $q = \frac{2p}{p+1}$ the function $u_{\infty,M}$ is self-similar, hence

$$u_{\infty,M}(r,\sigma) = r^{-\alpha}\tilde{\omega}(\sigma),$$

where $\tilde{\omega}$ is a nonnegative solution of (1.20). Inasmuch $u_{k,M} \ge u_{k,0} = v_k$ (already defined by (4.28)), it follows that

$$u_{\infty,M}(r,\sigma) \ge u_{\infty,0}(r,\sigma) = r^{-\alpha}\psi(\sigma) \Longrightarrow \tilde{\omega} \ge \psi \quad \text{in } S^{N-1}_+. \tag{4.56}$$

Since $u_{k,M}$ is dominated by any self-similar solution of (1.1), it implies that $\tilde{\omega}$ is the minimal positive solution of (1.20) that we denote by ω hereafter. Up to a subsequence, $\{T_{\ell_n}[u_{\infty,M}]\}$ converges locally uniformly in $\overline{\mathbb{R}^N_+} \setminus \{0\}$ to $u_{\infty,M}$. Consequently

$$\lim_{\ell_n \to 0} \ell_n^\alpha u_{\infty,M}(\ell_n,\sigma) = \omega(\sigma) \quad \text{uniformly in } S^{N-1}_+.$$

Because of uniqueness, the whole sequence converges, which implies 4.53.
In the case $q = \frac{2p}{p+1}$, using the a priori estimates from Theorem 2.1 and Theorem 2.3, we obtain that $T_{\ell_n}[u_{\infty,M}](1,\sigma) = \ell_n^\alpha u_{\infty,M}(\ell_n,\sigma)$ converges locally uniformly in $\overline{S^{N-1}_+}$ to $u_{\infty,0}(1,\sigma)$. Since $u_{\infty,0}(1,.) \ge \psi$, it follows that

$$\lim_{\ell_n \to 0} \ell_n^\alpha u_{\infty,M}(\ell_n,\sigma) = \psi(\sigma) \quad \text{uniformly in } S^{N-1}_+.$$

Hence 4.53 follows by uniqueness.

□

As a consequence of Theorem 1.7-(ii) we have

Theorem 4.4 *Assume* (4.50)*-(ii) holds, then for any* $M \ge 0$ *there exists a positive separable solution* u *of* (1.1) *in* $\mathbb{R}^N_+$ *vanishing on* $\partial\mathbb{R}^N_+ \setminus \{0\}$

When $\mathbb{R}^N_+$ is replaced by a bounded domain there holds.

Theorem 4.5 *Assume $\Omega \subset \mathbb{R}^N_+$ is a bounded smooth domain such that $0 \in \partial\Omega$, and (p,q) satisfies (4.50)-(ii). Then for any $M \geq 0$ there exists a positive solution u of (1.1) in Ω vanishing on $\partial\Omega \setminus \{0\}$ such that*

$$\lim_{x\to 0} \frac{u(x)}{P_\Omega(x)} = \infty, \tag{4.57}$$

where P_Ω is the Poisson kernel in Ω. Furthermore
(i) If $1 < q < \frac{2p}{p+1}$, then

$$\lim_{r\to 0} r^\alpha u(r,.) = \psi \qquad \text{locally uniformly in } S^{N-1}_+, \tag{4.58}$$

where ψ is the unique positive solution of (3.17).
(ii) If $q = \frac{2p}{p+1}$, then

$$\psi \leq \liminf_{r\to 0} r^\alpha u(r,.) \leq \limsup_{r\to 0} r^\alpha u(r,.) \leq \omega \qquad \text{locally uniformly in } S^{N-1}_+. \tag{4.59}$$

Proof. As in the proof of Theorem 4.3, the sequence $\{u_k\}$ of the solution of (1.6) which satisfy (4.24) is increasing. Since it is bounded from above by the restriction to Ω of the solutions of the same equation in $\mathbb{R}^N_+$, vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and satisfying (4.51), it admits a limit u_∞ which is a solution of 1.6 which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies (4.57). In order to have an estimate of the blow-up rate, we recall that the solution v_k of (4.48) is a subsolution of (1.1) and $u_k \geq v_k$ Furthermore $\{v_k\}$ converges to $\{v_\infty\}$ which is a positive solution of (1.1) in Ω, vanishing on $\partial\Omega \setminus \{0\}$ and such that

$$\lim_{r\to 0} r^\alpha v_\infty(r,\sigma) = \psi(\sigma) \quad \text{locally uniformly in } S^{N-1}_+. \tag{4.60}$$

Combined with (4.52) and (4.53) it implies (4.58) and (4.59) since the solution u_k in Ω is bounded from above by the solution in $\mathbb{R}^N_+$.

$$\liminf_{r\to 0} r^\alpha u_\infty(r,\sigma) \geq \psi(\sigma) \quad \text{locally uniformly in } S^{N-1}_+. \tag{4.61}$$

□

Theorem 4.6 *Assume $\Omega \subset \mathbb{R}^N_+$ is a bounded smooth domain such that $0 \in \partial\Omega$, $p = \frac{N-1}{N-1}$ and $q = \frac{2p}{p+1} = \frac{N+1}{N}$. If*

$$|x|^{-N}\text{dist}\,(x, \mathbb{R}^N_+) \leq c_{20} \qquad \text{for all } x \in \partial\Omega, \tag{4.62}$$

for some constant $c_{20} > 0$, there exists a positive solution u of (1.1) in Ω, vanishing on $\partial\Omega \setminus \{0\}$ such that

$$\lim_{r\to 0} r^\alpha u(r,\sigma) = \omega(\sigma) \quad \text{locally uniformly in } S^{N-1}_+. \tag{4.63}$$

Proof. The function $u_\omega(r,.) = r^{1-N}\omega$ satisfies (1.1) in $\mathbb{R}^N_+$ and vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$. Since $\nabla\omega$ is bounded, it satisfies

$$u(x) \leq c_{21} \qquad \text{for all } x \in \partial\Omega \setminus \{0\},$$

for some constant $c_{21} > 0$. Then the rsult follows from Proposition 3.3. □

4.2.2 The case $\frac{2p}{p+1} < q < p$

If

$$1 < p < \frac{N+1}{N-1}\,,\ \frac{2p}{p+1} < q < \frac{N+1}{N}, \tag{4.64}$$

there exists fundamental solutions u_k in $\mathbb{R}^N_+$ by Theorem 4.1, or in Ω by Theorem 4.2. Since the mapping $k \mapsto u_k$ is increasing and u_k is bounded from above the function $u_\infty = \lim_{k\to\infty} u_k$ is a solution of (1.1) in $\mathbb{R}^N_+$ (resp. Ω) vanishing on $\mathbb{R}^N_+ \setminus \{0\}$ (resp. $\Omega \setminus \{0\}$) which satisfies (4.51) (resp. (4.57)). However the blow-up rate of u_∞ is not easy to obtain from scaling methods since the transformation T_ℓ transform (1.1) into (4.44) where M is replaced by $M\ell^{2p-q(p+1)}$ which is not bounded when $\ell \to 0$. When $q > \frac{2p}{p+1}$, the natural exponent is γ defined by

$$\gamma = \frac{q}{p-q}. \tag{4.65}$$

The transformation S_ℓ defined for $\ell > 0$ by

$$S_\ell[u](x) = \ell^\gamma u(\ell x), \tag{4.66}$$

transforms (1.1) into

$$-\ell^{\frac{q(p+1)-2p}{p-q}}\Delta u + |u|^{p-1}u - M|\nabla u|^q = 0. \tag{4.67}$$

When ℓ to 0, the limit equation is of eikonal type (up to change of unknown) (4.68).

$$|u|^{p-1}u - M|\nabla u|^q = 0. \tag{4.68}$$

Separable solution of (1.3) in $\mathbb{R}^N_+$ are under the form $u_\eta(r,.) = r^{-\gamma}\eta$ and η satisfies

$$|\eta|^{p-1}\eta - M(\gamma^2\eta^2 + |\nabla'\eta|^2)^{\frac{q}{2}} = 0 \quad \text{in } S^{N-1}_+. \tag{4.69}$$

Clearly this equation admits no C^1 solution but for the constant ones. For avoiding the study the use of viscosity solutions it is better to look directly for solutions with strong blow-up by the method of supersolutions and subsolutions. Note that (1.3) admits an explicit radial singular solution namely

$$U(x) = \omega_0|x|^{-\gamma} := \gamma^\gamma M^{\frac{1}{p-q}}|x|^{-\gamma}. \tag{4.70}$$

Proof of Theorem 1.9. For $n > 0$ set $U_n = nr^{-\gamma}$

$$n^{-1}r^{-2-\gamma}\mathcal{L}_{q,M}U_n = -\gamma(\gamma+2-N) + n^{q-1}(n^{p-q} - \gamma^q M)r^{2-(p-1)\gamma}.$$

Since $q > \frac{2p}{p+1}$, then $2-(p-1)\gamma < 0$ and for any $n > \omega_0$ there exists $r_n > 0$ such that

$$n^{q-1}(n^{p-q} - \gamma^q M)r_n^{2-(p-1)\gamma} = \gamma(\gamma+2-N) > 0.$$

It implies that U_n is a super solution of (1.1) in $B_{r_n} \setminus \{0\}$. Furthermore

$$r_n = \left(\frac{n^{p-1}}{\gamma(\gamma + 2 - N)} \right)^{\frac{1}{(p-1)\gamma - 2}} (1 + o(1)) \quad \text{when} \ \ n \to \infty. \tag{4.71}$$

For a subsolution we set

$$W_m(r, \sigma) = m r^{-\gamma} \phi_1(\sigma), \tag{4.72}$$

where ϕ_1 is the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$ and $m > 0$. Then

$$\begin{aligned} r^{p\gamma} \mathcal{L}_{q,M} W_m = -m r^{\frac{q(p+1)-2p}{p-q}} & \left(\gamma^2 - (N-2)\gamma + 1 - N \right) \phi_1 \\ & + \left(m^p \phi_1^p - M m^q \left(\gamma^2 \phi_1^2 + |\nabla' \phi_1|^2 \right)^{\frac{q}{2}} \right). \end{aligned} \tag{4.73}$$

Set $P(X) = X^2 - (N-2)X + 1 - N = (X+1)(X+1-N)$. Then

$$P(\gamma) = \frac{p\,(Nq - (N-1)p)}{(p-q)^2}.$$

We first give the proof when $Nq \geq (N-1)p$. In such a case $P(\gamma) \geq 0$. Hence there exists $m_0 > 0$ such that for any $0 < m \leq m_0$, W_m is a subsolution in $\mathbb{R}^N_+$, smaller than U_n and it is bounded on $\partial B^+_{r_n} \setminus \{0\}$. When $m \leq m_0$, the function W_m defined in (4.72) is a subsolution of (1.1) in $\mathbb{R}^N_+$. Since W_m is bounded on $\partial B^+_{r_n} \setminus \{0\}$ there exists a nonnegative solution u_n of (1.1) in $B^+_{r_n}$ which vanishes on $B^+_{r_n} \setminus \{0\}$ and there holds

$$(W_m(x) - m r_n^{-\gamma})_+ \leq u_n(x) \leq U_n(x) \quad \text{for all} \ \ x \in B^+_{r_n}. \tag{4.74}$$

The fact that $B^+_{r_n}$ is just a Lipschitz domain is easily bypassed by smoothing it in a neighborhood of $\partial B_{r_n} \cap \mathbb{R}^N_+$. Furthermore, by (2.1) and (2.16),

$$u_n(x) \leq c_5 \max \left\{ |x|^{-\alpha}, M^{\frac{1}{p-q}} |x|^{-\gamma} \right\}. \tag{4.75}$$

and for any $r_0 > 0$, there exists $c_8 > 0$ depending on r_0 such that

$$|\nabla u_n(x)| \leq c_8 \max \left\{ |x|^{-\alpha-1}, M^{\frac{1}{p-q}} |x|^{-\gamma-1} \right\}. \tag{4.76}$$

By standard local regularity theory, there exists a subsequence $\{u_{n_j}\}$ which converges in the $C^1(K)$-topology for any compact set $K \subset \overline{\mathbb{R}^N_+} \setminus \{0\}$ to a positive solution u of (1.1) in $\mathbb{R}^N_+$ which vanishes on $\partial \mathbb{R}^N_+ \setminus \{0\}$ and satisfies (1.25).
Next we assume $Nq < (N-1)p$. Observe that $\gamma^2 \phi_1^2 + |\nabla' \phi_1|^2 \geq \delta^2 > 0$, then

$$m^p \phi_1^p - M m^q \left(\gamma^2 \phi_1^2 + |\nabla' \phi_1|^2 \right)^{\frac{q}{2}} \leq m^p - M m^q \delta^q.$$

Thus, from (4.73) we obtain

$$r^{p\gamma} \mathcal{L}_{q,M} W_m \leq -m r^{\frac{q(p+1)-2p}{p-q}} P(\gamma) + m^p - M m^q \delta^q, \tag{4.77}$$

and $P(\gamma) < 0$. If we choose

$$m = \delta^\gamma \left(\frac{M}{2}\right)^{\frac{1}{p-q}},$$

then

$$m^p - Mm^q\delta^q \geq -\frac{Mm^q\delta^q}{2}.$$

Therefore $\mathcal{L}_{q,M} W_m \leq 0$ on $B^+_{r^*}$ where

$$r^* = \left(\frac{Mm^{q-1}\delta^q}{-2P(\gamma)}\right)^{\frac{p-q}{q(p+1)-2p}}.$$

If $a = mr^{*\,-\gamma}$, then $W_m \leq a$ in $\partial B^+_{r^*}$, thus $W_{m,a} = (W_m - a)+$ is nonnegative in $B^+_{r^*}$ and it is a subsolution of (1.1) in $B^+_{r^*}$ which vanishes on $\partial B^+_{r^*} \setminus \{0\}$. If we extend it by 0 in $\mathbb{R}^N_+$, the new function is a a subsolution of (1.1) which belongs to $W^{1,\infty}_{loc}(\overline{\mathbb{R}^N_+} \setminus \{0\})$. We end the proof using Proposition 3.4 as in the previous case. □

If $\mathbb{R}^N_+$ is replaced by a bounded domain we have the following result.

Theorem 4.7 *Let $M > 0$ and $\frac{2p}{p+1} < q < p$. If $\Omega \subset \mathbb{R}^N_+$ is a bounded smooth domain such that $0 \in \partial\Omega$. If*

$$\operatorname{dist}(x, \partial\mathbb{R}^N_+) \leq c_{23}|x|^{\frac{p}{p-q}} \qquad \text{for all } x \in \partial\Omega, \tag{4.78}$$

for some constant $c_{23} > 0$. Then there exists a positive solution u of (1.1) in Ω vanishing on $\partial\Omega \setminus \{0\}$ satisfying, for some $m > 0$,

$$m\phi_1(\sigma) \leq \liminf_{r\to 0} r^\gamma u(r,\sigma) \leq \limsup_{r\to 0} r^\gamma u(r,\sigma) \leq \omega_0, \tag{4.79}$$

uniformly on any compact set $K \subset S^{N-1}_+$.

Proof. Let $R > 0$ and $B := B_R(a) \subset \Omega$ be an open ball tangent to $\partial\Omega$ at 0. Up to rescaling and since the result does not depend on the value of M we can assume that $R = 1$. We set $w_m(x) = m|x|^{-\theta}P_B(x)$ where $\theta = \gamma + 1 - N$ and P_B is the Poisson kernel in B expressed by

$$P_B(x) = \frac{1 - |x-a|^2}{\sigma_N |x|^N},$$

where σ_N is the volume of the unit sphere in $\mathbb{R}^N$. Then

$$\begin{aligned} m^{-1}&\mathcal{L}_{q,M} w_m \\ &= -(\theta^2 + (2-N)\theta)|x|^{-\theta-2}P_B(x) + 2\theta|x|^{-\theta-1}\langle \nabla P_B(x), \tfrac{x}{|x|}\rangle + m^{p-1}|x|^{-p\theta}P_B^p(x) \\ &\quad - Mm^{q-1}\left(\theta^2|x|^{-2(\theta-1)}P_B^2(x) + |x|^{-2\theta}|\nabla P_B(x)|^2 - 2\theta|x|^{-2\theta-1}\langle \nabla P_B(x), \tfrac{x}{|x|}\rangle\right)^{\frac{q}{2}}. \end{aligned} \tag{4.80}$$

Since

$$\nabla P_B(x) = -\frac{1}{\sigma_N}\left(\frac{N(1-|x-a|^2)}{|x|^{N+1}}\frac{x}{|x|} + \frac{2(x-a)}{|x|^N}\right),$$

then

$$\begin{aligned}\langle \nabla P_B(x), \tfrac{x}{|x|}\rangle &= -\frac{1}{\sigma_N|x|^{N+1}}\left((N-1)(1-|x-a|^2)+|x|^2\right)\\ &= -\frac{N-1}{|x|}P_B(x) - \frac{1}{\sigma_N|x|^{N-1}},\end{aligned}$$

which implies in particular

$$|\nabla P_B(x)| \geq \frac{N-1}{|x|}P_B(x) + \frac{1}{\sigma_N|x|^{N-1}}.$$

If $q \geq \frac{N-1}{p}p$, equivalently $\theta \geq 0$, we have

$$\begin{aligned}|\nabla w_m|^2 &= \theta^2|x|^{-2(\theta+1)}P_B^2(x) + |x|^{-2\theta}|\nabla P_B(x)|^2 - 2\theta|x|^{-2\theta-1}\langle \nabla P_B(x), \tfrac{x}{|x|}\rangle\\ &\geq \theta^2|x|^{-2(\theta+1)}P_B^2(x) + |x|^{-2\theta}\left(\frac{N-1}{|x|}P_B(x) + \frac{1}{\sigma_N|x|^{N-1}}\right)^2\\ &\qquad + 2\theta|x|^{-2\theta-1}\left(\frac{N-1}{|x|}P_B(x) + \frac{1}{\sigma_N|x|^{N-1}}\right)\\ &\geq (\theta^2+(N-1)^2)|x|^{-2(\theta+1)}P_B^2(x).\end{aligned} \tag{4.81}$$

Hence

$$\begin{aligned}m^{-1}\mathcal{L}_{q,M}w_m &\leq -(\theta^2+N\theta)|x|^{-\theta-2}P_B(x) + m^{p-1}|x|^{-p\theta}P_B^p(x)\\ &\qquad - m^{q-1}M(\theta^2+(N-1)^2)^{\frac{q}{2}}|x|^{-q(\theta+1)}P_B^q(x)\\ &\leq m^{q-1}|x|^{-p\theta}P_B^q(x)\left(m^{p-q}P_B^{p-q}(x) - M(\theta^2+(N-1)^2)|x|^{(p-q)\theta-q}\right).\end{aligned} \tag{4.82}$$

Now

$$P_B(x) \leq \frac{2}{\sigma_N|x|^{N-1}} \Longrightarrow P_B^{p-q}(x) \leq \left(\frac{2}{\sigma_N}\right)^{p-q}|x|^{(1-N)(p-q)}.$$

Since $(1-N)(p-q) = (p-q)\theta - q$, we obtain finally that,

$$m^{-1}\mathcal{L}_{q,M}w_m \leq m^{q-1}|x|^{-q\theta}P_B^q(x)\left(m^{p-q}\left(\frac{2}{\sigma_N}\right)^{p-q} - M(\theta^2+(N-1)^2)\right).$$

Choosing m small enough we deduce that w is a subsolution in B. If we extend it by 0 in $\Omega \setminus B$, the new function denoted by $\tilde{w}$ is a nonnegative subsolution of (1.1) in Ω which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies (4.79). The proof follows from Proposition 3.4.

If $q < \frac{N-1}{p}p$, then $\theta < 0$. Since $\langle \nabla P_B(x), \frac{x}{|x|}\rangle \le 0$, (4.81) is replaced by

$$\begin{aligned}
|\nabla w_m|^2 &= \theta^2|x|^{-2(\theta+1)}P_B^2(x) + |x|^{-2\theta}|\nabla P_B(x)|^2 - 2\theta|x|^{-2\theta-1}\langle \nabla P_B(x), \tfrac{x}{|x|}\rangle \\
&\ge \theta^2|x|^{-2(\theta+1)}P_B^2(x) + |x|^{-2\theta}\left(\frac{N-1}{|x|}P_B(x) + \frac{1}{\sigma_N|x|^{N-1}}\right)^2 \\
&\qquad + 2\theta|x|^{-2\theta-1}\left(\frac{N-1}{|x|}P_B(x) + \frac{1}{\sigma_N|x|^{N-1}}\right) \\
&\ge (\theta^2 + (N-1)^2)|x|^{-2(\theta+1)}P_B^2(x) + \left(\frac{1}{\sigma_N^2|x|^{2(N+\theta-1)}} + \frac{2\theta}{\sigma_N|x|^{N+2\theta}}\right) \\
&\qquad + 2(N-1)\left(\frac{1}{\sigma_N|x|^{N+2\theta}} + \frac{\theta}{|x|^{2\theta+2}}\right)P_B(x).
\end{aligned} \tag{4.83}$$

Set

$$\tilde r = \min\left\{2, \left(\frac{1}{2\sigma_N|\theta|}\right)^{\frac{1}{N-2}}\right\}. \tag{4.84}$$

If $x \in B \cap B_{\tilde r}(0)$, the two last terms in (4.83) are nonnegative, hence

$$|\nabla w|^2 \ge (\theta^2 + (N-1)^2)|x|^{-2(\theta+1)}P_B^2(x) \qquad \text{for all } x \in B \cap B_{\tilde r}(0). \tag{4.85}$$

Note that $B \cap B_{\tilde r}(0) = B$ if $\tilde r = 2$. Choosing $m > 0$ small enough we infer that w_m is a subsolution of (1.1) in $B \cap B_{\tilde r}(0)$. Denoting by $\hat m$ the maximum of w_m on $\partial(B \cap B_{\tilde r}(0)) \setminus \{0\}$, then $(w_m - \hat m)_+$ is a subsolution in Ω. Since the restriction to Ω of the solution constructed in Theorem 1.9 dominates $w_m - \hat m)_+$, the proof follows as in the first case. □

4.2.3 Open problems

Problem 1. Under what conditions are the posiitive solutions of problem (1.20) unique ? If instead of separable solutions in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ one looks for separable radial solutions of (1.1) in $\mathbb{R}^N \setminus \{0\}$ (with $q = \frac{2p}{p+1}$) , then they are under the form

$$U(x) = A|x|^{-\alpha} \tag{4.86}$$

and A is a positive root of the polynomial

$$P(X) = X^{p-1} - M\alpha^{\frac{2p}{p+1}}X^{\frac{p-1}{p+1}} + \alpha(N-2-\alpha). \tag{4.87}$$

A complete study of the radial solutions of (1.1) is provided in [10], however it is straightforward to check that if $1 < p\frac{N}{N-2}$, there exists a unique positive root, hence a unique positive separable solution, while if $p > \frac{N}{N-2}$, there exists a unique positive root (resp. two positive roots) if

$$M = (p+1)\left(\frac{p(N-2)-N}{2p}\right)^{\frac{p}{p+1}} := m^*, \tag{4.88}$$

(resp. $M > m^*$). Uniqueness of solution plays a fundamental role in the description and classification of all the positive solutions with an isolated singularity at 0.

Problem 2. It is proved in [10] that if $\max\{\frac{N}{N-1}, \frac{2p}{p+1}\} < q < \min\{2, p\}$ and $M > 0$, there exist infinitely many local radial solutions of of (1.1) in $\mathbb{R}^N \setminus \{0\}$ which satisfies

$$u(r) = \xi_M r^{-\beta}(1 + o(1)) \quad \text{as} \quad r \to 0 \tag{4.89}$$

where

$$\beta = \frac{2-q}{q-1} \quad \text{and} \quad \xi_M = \frac{1}{\beta}\left(\frac{(N-1)q - N}{M(p-1)}\right)^{\frac{1}{p-1}}. \tag{4.90}$$

These solutions present the property that there blow-up is smaller than the one of the explicit radial separable solution. It would be interesting to construct such solutions of (1.1) in $\mathbb{R}^N_+$ (or more likely B_R^+), vanishing on $\partial\mathbb{R}^N \setminus \{0\}$.

Problem 3. Is it possible to define a boundary trace for any positive solution of (1.1) in $\mathbb{R}^N_+$, noting the fact such a result holds separately for positive solutions of (1.2) and (1.4) ? A related problem would be to define an initial trace for any positive solution of the parabolic equation

$$\partial_t u - \Delta u + u^p - M|\nabla u|^q = 0, \tag{4.91}$$

in $(0, T) \times \mathbb{R}^N$. Initial trace of semilinear parabolic equations ($M = 0$ in (4.91)) are studied in [15], [12].

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