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LOCAL TRANSPARENT BOUNDARY CONDITIONS FOR WAVE PROPAGATION IN FRACTAL TREES (II). ERROR AND COMPLEXITY ANALYSIS

PATRICK JOLY * AND MARYNA KACHANOVSKA *

Abstract. This work is dedicated to a refined error analysis of the high-order transparent boundary conditions introduced in the companion work [8] for the weighted wave equation on a fractal tree. The construction of such boundary conditions relies on truncating the meromorphic series that represents the symbol of the Dirichlet-to-Neumann operator. The error induced by the truncation depends on the behaviour of the eigenvalues and the eigenfunctions of the weighted Laplacian on a self-similar metric tree. In this work we quantify this error by computing asymptotics of the eigenvalues and bounds for Neumann traces of the eigenfunctions. We prove the sharpness of the obtained bounds for a class of self-similar trees.

Key words. Wave equation, Dirchlet-to-Neumann operator, fractal, metric graph, counting function, Weyl asymptotics

AMS subject classifications. 35P20, 34B45, 35L05, 34L20

1. Introduction. Sound propagation in a human respiratory tract can be used to detect various lung diseases, see e.g. the Audible Human Project [1, 17]. Because of the complexity of the underlying physical process, in the respective numerical simulations one often uses simplified models. One of such models, which we study in this work, describes wave propagation inside a bronchiolar tree. It is based on multiple simplifying assumptions, in particular, the absence of the interactions between the bronchioles and the lung tissue, as well as the geometrical self-similarity of the bronchiolar tree [15, 5, 16]. A rigorous asymptotic analysis of this model [10, 18], when the thickness of the bronchioles tends to zero, leads to the weighted 1D wave equation on a fractal tree with infinitely many edges.

To perform numerical simulations on such a tree, it is necessary to be able to truncate the computational domain. To our knowledge, there exist three methods of doing so, all of them based on approximating the Dirichlet-to-Neumann (DtN) operator: local low-order approximate boundary conditions (BCs) [9, 18], convolution quadrature based discrete transparent BCs [6], and local high-order approximate BCs [8]. A refined error analysis of the latter method constitutes the subject of this article.

In [9], it was proven that the symbol (the Fourier-Laplace transform of the convolution kernel) of the DtN operator is a meromorphic function, which can be represented as a partial fraction expansion with infinitely many terms. Truncating this expansion at finitely many terms results in a symbol of a convolution operator that is local in the time domain, see [8]. While this process provides a numerically tractable and stable realization of the DtN, it introduces an approximation error. This error is controlled by a remainder of a convergent series, which, in turn, is related to the poles and residues of the original partial fraction expansion [8]. These arguments prove convergence of the method as the number of terms in the truncated series tends to infinity, but do not provide information on the convergence rate.

The goal of this paper is to prove explicit bounds on the convergence rate of the method of [8]. The principal idea is to exploit the connection between the poles (and residues) of the partial fraction expansion of the DtN symbol and the eigenvalues

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(and Neumann traces of the eigenfunctions) of a weighted Laplacian on a self-similar tree. This allows to relate the convergence rate to the asymptotic estimates on the eigenvalues and bounds on the Neumann traces of the eigenfunctions.

This article is organized as follows. Section 2 introduces the notations and the problem under consideration. In Section 3 we recall the method of approximated transparent boundary conditions from [8], and present the result about the convergence of the approximated BCs from [8]. The quantity controlling the error and depending on the remainder of the convergent series is referred to as the error indicator. Section 3.3 states the goals of the present work based on the results of the preceding sections. In Section 4 we present an upper bound for the error indicator, and in Section 5 a lower bound is studied. Section 6 is dedicated to conclusions and open questions.

2. Problem setting.

This section is not new and follows the works [9, 6, 8].

2.1. Notation.

We consider an infinite (in terms of the number of edges) p-adic tree $\mathcal{T}$ [9, Definiton 2.1]. Let us define it by construction. First of all, let, with $n \in \mathbb{N}$, $\mathcal{G}^n$ be a set of edges, defined as follows: $\mathcal{G}^0$ contains a single edge ('root' edge), $\mathcal{G}^1$ contains p edges (children of the root edge in $\mathcal{G}^0$, i.e. edges which all share the same vertex with the root edge), and $\mathcal{G}^{n+1}$ consists of all p^{n+1} children edges of all the edges $\Sigma \in \mathcal{G}^n$. The edges of $\mathcal{T}$ are given by the collection $\bigcup_{n\in\mathbb{N}} \mathcal{G}^n$. The p^n edges belonging to $\mathcal{G}^n$ will be denoted by

$$\Sigma_{n,k},\ k = 0, \dots, p^n - 1.$$

Each edge $\Sigma_{n,k}$ has p children

$$\Sigma_{n+1,pk+j} \in \mathcal{G}^{n+1}, \qquad j = 0, \dots, p-1. \tag{2.1}$$

The edges $\Sigma_{n+1,pk+j}$ share the vertex $M_{n,k}$ with the parent edge $\Sigma_{n,k}$ (see Figure 1). The root vertex is a vertex incident only to the root edge $\Sigma_{0,0}$, and is denoted by M^*.

We will study metric trees. This means that any edge $\Sigma_{n,k}$ can be identified with a segment of $\mathbb{R}$ of length $\ell_{n,k}$; additionally, we assign to it a constant weight $\mu_{n,k} > 0$. All over the article, we assume that $\mu_{0,0} = 1$. In what follows, we will consider self-

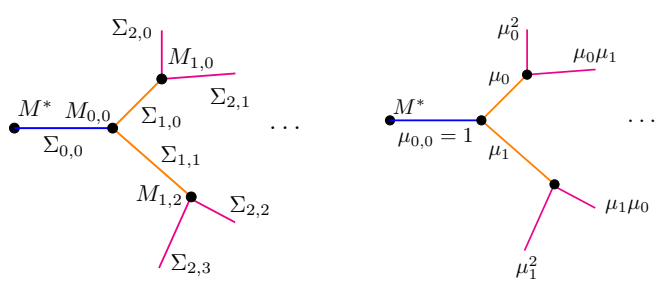

FIGURE 1. *Left: A self-similar p-adic ($p = 2$) infinite tree. In blue we mark the edges that belong to $\mathcal{G}^0$, in orange the edges of $\mathcal{G}^1$, in magenta the edges of $\mathcal{G}^2$. Right: Distribution of weights on the edges of a binary infinite self-similar tree.*

similar (fractal) trees, see [9, Definition 2.3]. Let us explain this in more detail. With

such trees we associate two vectors

$$\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{p-1}), \quad \boldsymbol{\mu} = (\mu_0, \dots, \mu_{p-1}), \quad \boldsymbol{\alpha}, \boldsymbol{\mu} \in \mathbb{R}_*^+. \tag{2.2}$$

The self-similarity condition means that the length/weight of the edge $\Sigma_{n+1,pk+j}$ is related to the length/weight of its parent edge $\Sigma_{n,k}$ according to the following rule:

$$\ell_{n+1,pk+j} = \alpha_j \ell_{n,k}, \qquad \mu_{n+1,pk+j} = \mu_j \mu_{n,k}, \qquad j = 0, \dots, p-1.$$

We will assume that the tree is 'bounded', i.e.

$$|\boldsymbol{\alpha}|_\infty := \sup_{j=0,\dots,p-1} \alpha_j < 1. \tag{2.3}$$

Given $m \in \mathbb{N}_*$, by $\mathcal{T}^m$, we denote the subtree of $\mathcal{T}$ made of the first m generations, i.e., with an obvious abuse of notation:

$$\mathcal{T} := \bigcup_{\ell=0}^{+\infty} \mathcal{G}^\ell, \qquad \mathcal{T}^m := \bigcup_{\ell=0}^{m} \mathcal{G}^\ell. \tag{2.4}$$

Let us call a tree a 'reference tree' if the length of the root edge is $\ell_{0,0} = 1$. Unless stated otherwise, we will always assume that $\mathcal{T}$ is a **reference tree.**

2.2. The weighted wave equation on fractal trees. To write differential equations on the tree $\mathcal{T}$, we need to introduce along each edge $\Sigma_{n,j} \in \mathcal{G}^n$ an abscissa $s \in [0, \ell_{n,j}]$ in such a way that $\Sigma_{n,j}$ is oriented towards the generation $\mathcal{G}^{n+1}$. This allows us, with an obvious abuse of notation, to write a function defined ot $\mathcal{T}$ as a function of s, seen as the space variable.

We then define (formally) the weight function $s \mapsto \mu(s)$ on $\mathcal{T}$ by $\mu(s) = \mu_{n,j}$, $s \in \Sigma_{n,j}$. Denoting by t the time variable, and given a source term $f : \mathcal{T} \times \mathbb{R}^+ \to \mathbb{R}$, we will look for $u : \mathcal{T} \times \mathbb{R}^+ \to \mathbb{R}$, defined as follows. With the notation $u_{n,j} = u|_{\Sigma_{n,j}}$, u satisfies on each edge $\Sigma_{n,j}$ the 1D wave equation:

$$\partial_t^2 u_{n,j} - \partial_s^2 u_{n,j} = f_{n,j} \quad \text{on } \Sigma_{n,j}, \qquad j = 0, \dots p^n - 1, \quad n \geq 0, \tag{2.5}$$

and the following continuity and Kirchoff conditions:

$$\begin{aligned} &u_{n,j}(M_{n,j}, t) = u_{n+1,pj+k}(M_{n,j}, t), \quad k = 0, \dots, p-1, \\ &\partial_s u_{n,j}(M_{n,j}, t) = \sum_{k=0}^{p-1} \mu_k \, \partial_s u_{n+1,pj+k}(M_{n,j}, t), \quad j = 0, \dots p^n - 1, \, n \geq 0. \end{aligned} \tag{2.6}$$

Equations (2.5, 2.6) are completed by a homogeneous Dirichlet condition at the root vertex M^* and vanishing initial conditions:

$$u(M^*, t) = 0, \quad t > 0, \qquad u(., 0) = \partial_t u(., 0) = 0, \quad \text{on } \mathcal{T}. \tag{2.7}$$

Additionally, we need to define the BCs at the 'infinite', fractal boundary of the tree. This is the most delicate point of the model, which we will formalize in Section 2.3.

2.3. Neumann and Dirichlet BVPs. To provide a rigorous mathematical formulation of the problem (2.5, 2.6, 2.7), we shall equip it with Neumann or Dirichlet boundary conditions at the 'infinite' boundary of the tree. This is done in the weak sense through the variational formulation on the problem, which requires the introduction of appropriate weighted Sobolev spaces on $\mathcal{T}$.

2.3.1. Sobolev Spaces. For a function $v : \mathcal{T} \to \mathbb{C}$, we adopt the notation

$$\int_{\mathcal{T}} \mu\, v := \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \mu_{n,k} \int_{\Sigma_{n,k}} v(s)\, ds \tag{2.8}$$

Let $C(\mathcal{T})$ be a space of continuous functions on $\mathcal{T}$, and

$$C_0(\mathcal{T}) := \{v \in C(\mathcal{T}) : \quad v = 0 \text{ on } \mathcal{T} \setminus \mathcal{T}^m, \quad \text{for some } m \in \mathbb{N}\}.$$

Let us introduce the following three spaces. First of all,

$$\mathrm{L}^2_\mu(\mathcal{T}) = \{v : \ v|_\Sigma \in \mathrm{L}^2(\Sigma), \text{ for all } \Sigma \in \mathcal{T};\ \|v\| < \infty\},\ \|v\|^2 = \|v\|^2_{\mathrm{L}^2_\mu(\mathcal{T})} := \int_{\mathcal{T}} \mu |v|^2.$$

We denote by $(.,.)$ the corresponding complex scalar product in $\mathrm{L}^2_\mu(\mathcal{T})$. The weighted Sobolev space H^1_μ is defined as follows:

$$\mathrm{H}^1_\mu(\mathcal{T}) := \{v \in C(\mathcal{T}) \cap \mathrm{L}^2_\mu(\mathcal{T}) : \|\partial_s v\| < \infty\}, \quad \|v\|^2_{\mathrm{H}^1_\mu(\mathcal{T})} = \|v\|^2 + \|\partial_s v\|^2.$$

Finally, the last space is an analogue of the classical H^1_0-space:

$$\mathrm{H}^1_{\mu,0}(\mathcal{T}) := \overline{C_0(\mathcal{T}) \cap \mathrm{H}^1_\mu(\mathcal{T})}^{\|\cdot\|_{\mathrm{H}^1_\mu(\mathcal{T})}}.$$

Similarly, we define the corresponding spaces on a truncated tree $\mathcal{T}^m$. The associated L^2_μ-scalar product will be denoted by $(.,.)_{\mathcal{T}^m}$.

Finally, we will use the following result from [9].

THEOREM 2.1. *For $|\boldsymbol{\alpha}|_\infty < 1$, the embedding of $\mathrm{H}^1_\mu(\mathcal{T})$ in $\mathrm{L}^2_\mu(\mathcal{T})$ is compact.*

REMARK 2.2. *In what follows we will not make an explicit distinction between complex and real function spaces, keeping in mind that time-dependent quantities are real-valued; in the rest of the paper the complex function spaces are used.*

2.3.2. The BVP problems. To define the Neumann (resp. Dirichlet) problem, we introduce the space $V_{\mathfrak{n}}$ (resp., $V_{\mathfrak{d}}$) that differs from $\mathrm{H}^1_\mu(\mathcal{T})$ (resp. $\mathrm{H}^1_{\mu,0}(\mathcal{T})$) only by the condition at the root vertex:

$$V_{\mathfrak{n}}(\mathcal{T}) := \{v \in \mathrm{H}^1_\mu(\mathcal{T}) : \ v(M^*) = 0\}, \quad V_{\mathfrak{d}}(\mathcal{T}) := \{v \in \mathrm{H}^1_{\mu,0}(\mathcal{T}) : \ v(M^*) = 0\}.$$

In the sequel, we shall use the index $\mathfrak{a}$ to distinguish between the Neumann ($\mathfrak{a} = \mathfrak{n}$) and Dirichlet problems ($\mathfrak{a} = \mathfrak{d}$), whose weak formulation is essentially the same, the only difference lying in the choice of the trial and test space, i.e. $V_{\mathfrak{n}}(\mathcal{T})$ or $V_{\mathfrak{d}}(\mathcal{T})$.

DEFINITION 2.3 (Neumann and Dirichlet problems).
Find $u_{\mathfrak{a}} \in C^0(\mathbb{R}^+; V_{\mathfrak{a}}(\mathcal{T})) \cap C^1(\mathbb{R}^+; \mathrm{L}^2_\mu(\mathcal{T}))$, s.t. $u_{\mathfrak{a}}(.,0) = \partial_t u_{\mathfrak{a}}(.,0) = 0$, and

$$(\partial_t^2 u_{\mathfrak{a}}, v) + (\partial_s u_{\mathfrak{a}}, \partial_s v) = (f, v), \qquad \textit{for all } v \in V_{\mathfrak{a}}(\mathcal{T}). \tag{2.9}$$

The above problems are well-posed whenever $f \in L^1_{loc}(\mathbb{R}^+; \mathrm{L}^2_\mu(\mathcal{T}))$, [6, Theorem 2.1]. Surprisingly, in some cases the solutions $u_{\mathfrak{n}}$ and $u_{\mathfrak{d}}$ may coincide. To explain this result in more detail, let us introduce

$$\langle \boldsymbol{\mu\alpha} \rangle := \sum_{i=0}^{p-1} \mu_i \alpha_i, \quad \langle \boldsymbol{\mu}/\boldsymbol{\alpha} \rangle := \sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i}, \quad (\text{since } |\boldsymbol{\alpha}|_\infty < 1, \langle \boldsymbol{\mu\alpha} \rangle < \langle \boldsymbol{\mu}/\boldsymbol{\alpha} \rangle). \tag{2.10}$$

THEOREM 2.4 ([9]). *If $\langle \boldsymbol{\mu\alpha} \rangle \geq 1$ or $\langle \boldsymbol{\mu}/\boldsymbol{\alpha} \rangle \leq 1$, the spaces $\mathrm{H}^1_{\mu,0}(\mathcal{T})$ and $\mathrm{H}^1_\mu(\mathcal{T})$ coincide, and thus $u_{\mathfrak{n}} = u_{\mathfrak{d}}$. Otherwise, $\mathrm{H}^1_{\mu,0}(\mathcal{T}) \subsetneq \mathrm{H}^1_\mu(\mathcal{T})$, and $u_{\mathfrak{n}} \neq u_{\mathfrak{d}}$.*

2.4. Reduction to a finite tree. In what follows, we assume that the source term is supported on a finite number of generations:

$$\exists\, m_0 > 1 \text{ such that } \forall\, t > 0, \quad \operatorname{supp} f(.,t) \subseteq \mathcal{T}^{m_0}. \tag{2.11}$$

To perform numerical simulations, we will truncate the tree $\mathcal{T}$ to m generations with $m > m_0$, in order to compute the restriction of the solution $u_{\mathfrak{a}}$ of (2.9) to $\mathcal{T}^m$. This allows to consider (2.5, 2.6) only for $n \leq m$. The problem is then to find boundary conditions at the boundary of $\mathcal{T}^m$ (except M^*), i.e. at points $\{M_{m,j}, 0 \leq j \leq p^m - 1\}$, that should characterize the restriction of $u_{\mathfrak{a}}$ to $\mathcal{T}^m$. Such conditions are called transparent boundary conditions.

Remark 2.5. *All over the article we consider the parameter m to be fixed.*

For the problems (2.9), the transparent BCs were constructed and characterized in [9]. Before presenting them, let us introduce auxiliary notations.

2.4.1. Notations. We will denote by $\mathcal{K}(\partial_t)$ the following convolution operator:

$$\big(\mathcal{K}(\partial_t)v\big)(t) = \int_0^t k(t-\tau)v(\tau)d\tau, \quad \text{where } v : t \in \mathbb{R}^+ \mapsto v(t) \in \mathbb{R}.$$

The integral in the above has to be understood in the sense of a convolution of causal tempered distributions. The notation $\mathcal{K}(\partial_t)$ refers to the symbol $\boldsymbol{\mathcal{K}}(\omega)$ of the convolution operator (i.e. the Fourier-Laplace transform of the convolution kernel k):

$$\boldsymbol{\mathcal{K}}(\omega) := (\mathcal{F}k)(\omega) = \int_0^{+\infty} \mathrm{e}^{i\omega t} k(t)dt, \quad \omega \in \mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Following the above convention, provided a scaling parameter $\delta > 0$, we will denote by $\mathcal{K}(\delta\,\partial_t)$ the convolution operator with the symbol $\boldsymbol{\mathcal{K}}(\delta\omega)$. In what follows we will work with the space $H^1_{0,loc}(\mathbb{R}^+) := \{v \in H^1_{loc}(\mathbb{R}^+) : v(0) = 0\}$.

2.4.2. Transparent boundary conditions. With the above, the transparent boundary condition at the node $M_{m,j}$ takes the form (see Remark 2.6)

$$-\mu_{m,j}\,\partial_s u_{\mathfrak{a}}(M_{m,j},t) = \mathcal{B}^{\mathfrak{a}}_{m,j}(\partial_t)u_{\mathfrak{a}}(M_{m,j},t), \tag{2.12}$$

where the operators $\mathcal{B}^{\mathfrak{a}}_{m,j} \in \mathcal{L}\big(H^1_{0,loc}(\mathbb{R}^+),\, L^2_{loc}(\mathbb{R}^+)\big)$, cf. [7], are related to a single convolution operator $\Lambda_{\mathfrak{a}}(\partial_t)$ (defined further) via

$$\mathcal{B}^{\mathfrak{a}}_{m,j}(\partial_t) = \mu_{m,j}\,\alpha^{-1}_{m,j} \sum_{k=0}^{p-1} \frac{\mu_k}{\alpha_k}\,\Lambda_{\mathfrak{a}}\big(\alpha_k\alpha_{m,j}\,\partial_t\big). \tag{2.13}$$

The operator $\Lambda_{\mathfrak{a}}(\partial_t) \in \mathcal{L}\big(H^1_{0,loc}(\mathbb{R}^+),\, L^2_{loc}(\mathbb{R}^+)\big)$ is the *reference DtN operator*.

2.4.3. Reference DtN operator. Recall that $\mathcal{T}$ is the reference tree. The reference DtN is the DtN associated to the root vertex of this tree, namely:

$$\big(\Lambda_{\mathfrak{a}}(\partial_t)g\big)(t) := -\partial_s u^{\mathfrak{a}}_g(M^*,t), \quad g \in H^1_{0,loc}(\mathbb{R}^+), \text{ where} \tag{2.14}$$

- if $\mathfrak{a} = \mathfrak{n}$, $u^{\mathfrak{a}}_g \in C^1(\mathbb{R}^+; L^2_\mu(\mathcal{T})) \cap C^0(\mathbb{R}^+; \mathrm{H}^1_\mu(\mathcal{T}))$,

- if $\mathfrak{a} = \mathfrak{d}$, $u_g^{\mathfrak{a}} \in C^1(\mathbb{R}^+; L^2_\mu(\mathcal{T})) \cap C^0(\mathbb{R}^+; \mathrm{H}^1_{\mu,0}(\mathcal{T}))$,

and $u_g^{\mathfrak{a}}$ solves

$$\begin{aligned} &(\partial_t^2 u_g^{\mathfrak{a}}, v) + (\partial_s u_g^{\mathfrak{a}}, \partial_s v) = 0, \quad \text{for all } v \in V_{\mathfrak{a}}(\mathcal{T}), \quad u_g^{\mathfrak{a}}(M^*, t) = g(t), \\ &u_g^{\mathfrak{a}}(.,0) = \partial_t u_g^{\mathfrak{a}}(.,0) = 0. \end{aligned} \tag{2.15}$$

Remark 2.6. *Since understanding the construction of the transparent BCs is not required for reading this paper, we refer the interested reader to [9] for more details.*

2.4.4. The truncated problem. We consider the problem consisting in solving (2.5, 2.6) for $n \le m$ completed with the transparent conditions (2.12). To formulate it, let us introduce the space $\boldsymbol{V}_\mu(\mathcal{T}^m)$ of restrictions to $\mathcal{T}^m$ of functions in $V_{\mathfrak{a}}(\mathcal{T})$:

$$\boldsymbol{V}_\mu(\mathcal{T}^m) := \{v \in \mathrm{H}^1_\mu(\mathcal{T}^m) : v(M^*) = 0\},$$

and the vectorial trace operators on $\Gamma_m := \{M_{m,j}, 0 \le j \le p^{m-1}\}$:

$$\boldsymbol{\gamma}_m : \quad \boldsymbol{V}_\mu(\mathcal{T}^m) \to \mathbb{R}^{p^m}, \qquad \boldsymbol{\gamma}_m v = (v(M_{m,0}), \dots, v(M_{m,p^m-1})). \tag{2.16}$$

Accordingly, we introduce the matrix operator

$$\mathcal{B}_m^{\mathfrak{a}}(\partial_t) = \mathrm{diag}\left(\mathcal{B}_{m,j}^{\mathfrak{a}}(\partial_t)\right)_{0 \le j \le p^m-1} \in \mathcal{L}\big(H^1_{0,loc}(\mathbb{R}^+, \mathbb{R}^{p^m}), L^2_{loc}(\mathbb{R}^+, \mathbb{R}^{p^m})\big). \tag{2.17}$$

The weak formulation of such a problem reduced to the finite tree $\mathcal{T}^m$ reads:
Find $u_m^{\mathfrak{a}} \in C(\mathbb{R}^+; \boldsymbol{V}_\mu(\mathcal{T}^m)) \cap C^1(\mathbb{R}^+; \mathrm{L}^2_\mu(\mathcal{T}^m))$, *s.t.* $u_m^{\mathfrak{a}}(.,0) = \partial_t u_m^{\mathfrak{a}}(.,0) = 0$, *and*

$$(\partial_t^2 u_m^{\mathfrak{a}}, v)_{\mathcal{T}^m} + (\partial_s u_m^{\mathfrak{a}}, \partial_s v)_{\mathcal{T}^m} + \int_{\Gamma_m} \mathcal{B}_m^{\mathfrak{a}}(\partial_t) \boldsymbol{\gamma}_m u_m^{\mathfrak{a}} \, \boldsymbol{\gamma}_m v = (f, v)_{\mathcal{T}^m}, \tag{2.18}$$

for any $v \in \boldsymbol{V}_\mu(\mathcal{T}^m)$. *Here, for* $\varphi \in C^0(\mathbb{R}^+; \boldsymbol{V}_\mu(\mathcal{T}^m))$ *and* f_1, $f_2 : \mathbb{R} \to \mathbb{R}$, *we use the notation:*

$$\int_{\Gamma_m} f_1(\boldsymbol{\mu}) f_2(\boldsymbol{\alpha}) \boldsymbol{\gamma}_m \varphi(t) := \sum_{j=0}^{p^m-1} f_1(\mu_{m,j}) f_2(\alpha_{m,j}) \varphi(M_{m,j}, t). \tag{2.19}$$

The problems (2.9) and (2.18) are equivalent in the following sense.

Theorem 2.7 (Theorem 2.6 in [6]). *For all* $f \in L^1_{loc}(\mathbb{R}^+; \mathrm{L}^2_\mu(\mathcal{T}))$ *satisfying* (2.11), *the problem* (2.18) *has a unique solution* $u_m^{\mathfrak{a}}$, *and* $u_m^{\mathfrak{a}} = u_{\mathfrak{a}}|_{\mathcal{T}^m}$, *where* $u_{\mathfrak{a}}$ *solves* (2.9).

Unlike (2.9), the problem (2.18) is suitable for numerical simulations (provided that the boundary term is computable), as it is posed on a tree with finitely many edges.

2.5. Characterization and properties of the reference DtN operator. The resolution of (2.18) relies on approximating the operator $\mathcal{B}_m^{\mathfrak{a}}(\partial_t)$, in other words, through (2.13), on approximating the reference DtN operator $\Lambda_{\mathfrak{a}}(\partial_t)$. The goal of this section is to describe an important property of the reference DtN, which will be used in the construction and analysis of the transparent boundary conditions. For this, let us define the following Hermitian non-negative sesquilinear form on $V_{\mathfrak{a}} := V_{\mathfrak{a}}(\mathcal{T})$:

$$\forall\, (u, v) \in V_{\mathfrak{a}}, \quad a(u, v) = (\partial_s u, \partial_s v) \equiv \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \int_{\Sigma_{n,k}} \mu_{n,k} \, \partial_s u \, \partial_s \overline{v} \, ds. \tag{2.20}$$

It allows to define two unbounded self-adjoint operators $\mathcal{A}_\mathfrak{a}$, $\mathfrak{a}=\mathfrak{n}$ or $\mathfrak{d}$, in $\mathrm{L}^2_\mu(\mathcal{T})$:

$$\mathcal{A}_\mathfrak{a}: \qquad D(\mathcal{A}_\mathfrak{a}) \to \mathrm{L}^2_\mu(\mathcal{T}), \qquad (\mathcal{A}_\mathfrak{a} u, v) = a(u,v), \tag{2.21}$$

$$D(\mathcal{A}_\mathfrak{a}) = \{v \in V_\mathfrak{a} : \text{ with } C(v)>0,\ |a(v,g)| < C(v)\|g\|_{\mathrm{L}^2_\mu(\mathcal{T})}, \text{ for all } g \in V_\mathfrak{a}\}. \tag{2.22}$$

Remark 2.8. *In a strong form, the operator $\mathcal{A}_\mathfrak{a}$ is a weighted Laplace operator $\mu^{-1}(s)\partial_s(\mu(s)\partial_s.)$ on $\mathrm{L}^2_\mu(\mathcal{T})$, with the Dirichlet condition at the root of the tree and the Neumann ($\mathfrak{a}=\mathfrak{n}$) or Dirichlet ($\mathfrak{a}=\mathfrak{d}$) conditions at the fractal boundary of $\mathcal{T}$.*

From Theorem 2.1 it follows that the spectrum of these operators is a pure point spectrum. We define the eigenvalues and normalized eigenfunctions as

$$\mathcal{A}_\mathfrak{a}\varphi_{\mathfrak{a},n} = \omega^2_{\mathfrak{a},n}\,\varphi_{\mathfrak{a},n}, \qquad \|\varphi_{\mathfrak{a},n}\|_{\mathrm{L}^2_\mu(\mathcal{T})} = 1, \qquad 0 < \omega^2_{\mathfrak{a},1} \le \omega^2_{\mathfrak{a},2} \le \ldots \to +\infty. \tag{2.23}$$

Remark that the eigenvalues do not vanish, as shown in [9, Remark 1.20]. A spectral representation of the operator $\mathcal{A}_\mathfrak{a}$ yields the following result.

Theorem 2.9 (Proposition 1.23, discussion after (144) in [9]). *The symbol of the reference DtN operator $\mathbf{\Lambda}_\mathfrak{a}$, $\mathfrak{a}\in\{\mathfrak{n},\mathfrak{d}\}$, satisfies*

$$\mathbf{\Lambda}_\mathfrak{a}(\omega) = \mathbf{\Lambda}_\mathfrak{a}(0) - \omega^2 \sum_{n=1}^{+\infty} \frac{a_{\mathfrak{a},n}}{(\omega_{\mathfrak{a},n})^2 - \omega^2}, \quad a_{\mathfrak{a},n} = \omega^{-2}_{\mathfrak{a},n}\,\partial_s\varphi_{\mathfrak{a},n}(M^*)^2. \tag{2.24}$$

The above series converges uniformly on compact subsets of $\mathbb{C}\setminus\{\pm\omega_{\mathfrak{a},n}, n\ge 1\}$.

In (2.23), the eigenvalues are repeated with their multiplicities, and, as a consequence, the series (2.24) may have repeated poles. Moreover, the unique continuation principle for eigenfunctions no longer holds, and it may happen that $\varphi_{\mathfrak{a},n} \not\equiv 0$, but $\partial_s\varphi_{\mathfrak{a},n}(M^*) = 0$ (hence $a_{\mathfrak{a},n}=0$). For this reason, we introduce the sets

$$\begin{cases} \mathcal{S}^{app}_\mathfrak{a} := \{\omega^2 \,/\, \exists\, n\ge 1 \text{ s. t. } \omega^2 = \omega^2_{\mathfrak{a},n}\}, & (\Leftrightarrow \text{ apparent poles}) \\ \mathcal{S}_\mathfrak{a} := \{\omega^2 \in \mathcal{S}^{app}_\mathfrak{a} \,/\, \displaystyle\sum_{\omega^2_{\mathfrak{a},n}=\omega^2} \partial_s\varphi_{\mathfrak{a},n}(M^*)^2 > 0\,\}, & (\Leftrightarrow \text{ poles}) \end{cases} \tag{2.25}$$

and recast the set $\mathcal{S}_\mathfrak{a}$ as

$$\mathcal{S}_\mathfrak{a} := \big\{\Omega^2_{\mathfrak{a},k},\ k\ge 1, \text{ with } 0 < \Omega_{\mathfrak{a},1} < \Omega_{\mathfrak{a},2} < \ldots \to +\infty\big\}, \tag{2.26}$$

so that $\mathcal{P}_\mathfrak{a} = \big\{\pm\,\Omega_{\mathfrak{a},k}, k\ge 1\big\}$ is the set of poles of $\mathbf{\Lambda}_\mathfrak{a}(\omega)$. All the above allows to rewrite (2.24) as follows:

$$\mathbf{\Lambda}_\mathfrak{a}(\omega) = \mathbf{\Lambda}_\mathfrak{a}(0) - \omega^2 \sum_{k=1}^{+\infty} \frac{A_{\mathfrak{a},k}}{\Omega^2_{\mathfrak{a},k} - \omega^2}, \quad \text{with } A_{\mathfrak{a},k} > 0. \tag{2.27}$$

Remark 2.10. *Using Theorem 2.9, it is possible to show that the series (2.27) converges uniformly on the compact subsets of $\mathbb{C}\setminus\mathcal{P}_\mathfrak{a}$.*

3. Approximate transparent boundary conditions: formulation and error analysis. As pointed out in the end of Section 2.4.4, the numerical resolution of (2.18) requires approximating the operators $\mathcal{B}^\mathfrak{a}_m$, or, with (2.13), the operators $\Lambda_\mathfrak{a}(\partial_t)$. This is the subject of this section. All over this section we fix $m\ge 1$.

3.1. A formulation of the method.

3.1.1. Approximating the reference DtN operator. The idea is to realize a local approximation of the operator $\Lambda_{\mathfrak{a}}(\partial_t)$ through a rational approximation of its symbol. Of course, the most natural idea is to truncate the series (2.27) at N terms (with N becoming an approximation parameter):

$$\mathbf{\Lambda}_{\mathfrak{a}}^N(\omega) = \mathbf{\Lambda}_{\mathfrak{a}}(0) - \omega^2 \sum_{k=1}^{N} \frac{A_{\mathfrak{a},k}}{\Omega_{\mathfrak{a},k}^2 - \omega^2}, \tag{3.1}$$

and to define $\Lambda_{\mathfrak{a}}^N(\partial_t)$ as the operator whose symbol is $\mathbf{\Lambda}_{\mathfrak{a}}^N(\omega)$. This operator can be realized through the introduction of N auxiliary unknowns which are coupled to the argument of $\Lambda_{\mathfrak{a}}^N(\partial_t)$ via ODEs. More precisely

$$\begin{aligned} &\big(\Lambda_{\mathfrak{a}}^N(\partial_t)g\big)(t) = \mathbf{\Lambda}_{\mathfrak{a}}(0)\, g(t) + \sum_{k=1}^{N} A_{\mathfrak{a},k} \frac{d\lambda_k}{dt}, \text{ where} \\ &\frac{d^2\lambda_k}{dt^2} + \Omega_{\mathfrak{a},k}^2 \lambda_k = \frac{dg}{dt}, \qquad \lambda_k(0) = \frac{d\lambda_k}{dt}(0) = 0. \end{aligned} \tag{3.2}$$

Remark 3.1. *The coefficients $A_{\mathfrak{a},k}$, $\Omega_{\mathfrak{a},k}$ can be computed with the method of [8].*

3.1.2. Approximating the transparent boundary conditions. According to the previous section, we introduce the truncated DtN operator $\mathcal{B}_m^{\mathfrak{a},N}(\partial_t)$ by replacing in (2.13) $\Lambda_{\mathfrak{a}}$ by $\Lambda_{\mathfrak{a}}^N$, i.e.,

$$\begin{aligned} &\mathcal{B}_m^{\mathfrak{a},N}(\partial_t) = \operatorname{diag}\big(\mathcal{B}_{m,j}^{\mathfrak{a},N}(\partial_t)\big)_{0\le j\le p^m-1}, \\ &\mathcal{B}_{m,j}^{\mathfrak{a},N}(\partial_t) = \mu_{m,j}\, \alpha_{m,j}^{-1} \sum_{k=0}^{p-1} \frac{\mu_k}{\alpha_k} \Lambda_{\mathfrak{a}}^N(\alpha_k \alpha_{m,j} \partial_t). \end{aligned} \tag{3.3}$$

3.1.3. A truncated system. Using the approximation (3.3) in the truncated system (2.18) yields the following problem:
Find $u_m^{\mathfrak{a},N} \in C^1(\mathbb{R}^+; \mathrm{L}^2_\mu(\mathcal{T}^m)) \cap C^0(\mathbb{R}^+; \boldsymbol{V}_\mu(\mathcal{T}^m))$ s.t. $u_m^{\mathfrak{a},N}(0) = \partial_t u_m^{\mathfrak{a},N}(0) = 0$, and s.t. for any $v \in \boldsymbol{V}_\mu(\mathcal{T}^m)$,

$$(\partial_t^2 u_m^{\mathfrak{a},N}, v)_{\mathcal{T}^m} + (\partial_s u_m^{\mathfrak{a},N}, \partial_s v)_{\mathcal{T}^m} + \int_{\Gamma_m} \mathcal{B}_m^{\mathfrak{a},N}(\partial_t) \boldsymbol{\gamma}_m u_m^{\mathfrak{a},N}\, \boldsymbol{\gamma}_m v = (f, v)_{\mathcal{T}^m}. \tag{3.4}$$

The stability of (3.4) (independently of N) was proven in [8]: it relies on the non-negativity of $\mathbf{\Lambda}_{\mathfrak{a}}(0)$ and the coefficients $A_{\mathfrak{a},k}$, $k \in \mathbb{N}_*$, in (3.1).

3.2. Error analysis. The convergence of $u_m^{\mathfrak{a},N}$ towards $u_m^{\mathfrak{a}}$ was established in [8] through an (abstract) error estimate. To state this result, let us introduce the following notation for the error:

$$\varepsilon_m^{\mathfrak{a},N} := u_m^{\mathfrak{a},N} - u_m^{\mathfrak{a}} \quad \text{(recall that } u_m^{\mathfrak{a}} = u_{\mathfrak{a}}|_{\mathcal{T}^m}\text{, cf. Theorem 2.7).} \tag{3.5}$$

We will need an additional regularity assumption on the source term f, namely

$$f \in W^{4,1}(\mathbb{R}^+; \mathrm{L}^2_\mu(\mathcal{T}^m)) \text{ and } f(0) = \ldots = f^{(3)}(0) = 0. \tag{3.6}$$

Additionally, let us introduce the following parameter that will control the error

$$r_{\mathfrak{a},N} := \sum_{k=N+1}^{\infty} A_{\mathfrak{a},k}\,\Omega_{\mathfrak{a},k}^{-2}, \quad \text{with } S_{\mathfrak{a}} := \sum_{k=1}^{\infty} A_{\mathfrak{a},k}\,\Omega_{\mathfrak{a},k}^{-2} < +\infty. \tag{3.7}$$

The convergence of the series $S_{\mathfrak{a}}$ was proven in [8]. Let us introduce the energy norm of $v \in C^1(\mathbb{R}^+; \mathrm{L}^2_\mu(\mathcal{T}^m)) \cap C^0(\mathbb{R}^+; \boldsymbol{V}_\mu(\mathcal{T}^m))$, with $v(0) = 0$:

$$|\!|\!| v |\!|\!|_{[0,T];\mathcal{T}^m} := \sup_{t \le T} \left(\|\partial_t v(\cdot,t)\|_{\mathrm{L}^2_\mu(\mathcal{T}^m)} + \|\partial_s v(\cdot,t)\|_{\mathrm{L}^2_\mu(\mathcal{T}^m)} \right). \tag{3.8}$$

We then have the following abstract error estimate.

THEOREM 3.2 ([8]). *Let $m, N \ge 1$. Let f satisfy (2.11) and (3.6). Then, with $r_{\mathfrak{a},N}$ defined in (3.7), the error (3.5) satisfies, with some $C_m > 0$,*

$$|\!|\!| \varepsilon_m^{\mathfrak{a},N} |\!|\!|_{[0,T];\mathcal{T}^m} \le C_m\, T\, r_{\mathfrak{a},N}\, \|\partial_t^4 \partial_s u_{\mathfrak{a}}\|_{L^1(0,T;\mathrm{L}^2_\mu(\mathcal{T}))}, \text{ for all } T > 0. \tag{3.9}$$

Therefore, for fixed T, m, for all $0 < t < T$, $\|\varepsilon_m^{\mathfrak{a},N}(t)\|_{\mathrm{L}^2_\mu(\mathcal{T}^m)} \to 0$, as $N \to +\infty$.

The error estimate (3.9) shows that the error is governed by the quantity $r_{\mathfrak{a},N}$ (which we will refer to as an *error indicator* in the rest of the paper). It however remains *abstract* because we do not provide any explicit bound for $r_{\mathfrak{a},N}$.

3.3. The goal of this article. Because the convergence of the approximate transparent boundary conditions is defined by the remainder $r_{\mathfrak{a},N}$ of the convergent series $S_{\mathfrak{a}}$, the main goal of the present article is to provide an explicit upper bound on $r_{\mathfrak{a},N}$ in terms of N. While Theorem 3.2 provides only an upper bound for the error induced by the approximation (3.3), in practical computations (see the numerical experiments in [8]) this bound often appears to be tight (i.e. the error $\varepsilon_m^{\mathfrak{a},N}$ converges with N like $r_{\mathfrak{a},N}$). Therefore, we will also obtain a lower bound on $r_{\mathfrak{a},N}$ explicit in N.

REMARK 3.3. *The number of terms in the approximation (3.1) plays a role in the complexity estimates. As seen from (3.2), in the time domain we introduce N auxiliary unknowns coupled to the original unknown $u_m^{\mathfrak{a},N}$ through N ODEs. The numerical resolution of (3.4) discretized as suggested in [8] is then of $O(N)$ complexity.*

4. Explicit upper bounds for the error indicator.

4.1. Main results. Here we summarize the principal results of Section 4, while the proofs will be provided in the sections that follow.

REMARK 4.1. ***In what follows we will use the notation $\sum$ instead of $\sum_{j=0}^{p-1}$.***

To formulate these results, let us introduce the Minkowski dimension of $\mathcal{T}$, cf. [14]:

$$d_s > 0 \text{ is a unique number s.t. } \sum \alpha_j^{d_s} = 1. \tag{4.1}$$

To prove the existence and uniqueness of $d_s > 0$, we remark that the function $d \mapsto \sum \alpha_j^d$ is continuous strictly monotonically decaying on $\mathbb{R}^+$, and takes values from p to 0. In what follows we will also use the following notation:

$$\langle \boldsymbol{\alpha} \rangle := \sum \alpha_j.$$

It is clear that $d_s < 1$ if and only if $\langle \boldsymbol{\alpha} \rangle < 1$. This case corresponds to the fact that the total length of all the branches of the tree $\mathcal{T}$ is finite. We then have

THEOREM 4.2 (Convergence rate). *There exists $c_{\mathfrak{a}}^+ > 0$, depending on $\boldsymbol{\mu}$, $\boldsymbol{\alpha}$, s.t., for all $N \geq 2$, $r_{\mathfrak{a},N}$ defined in (3.7) satisfies:*

- *if $\langle \boldsymbol{\alpha} \rangle < 1$ ($d_s < 1$), $r_{\mathfrak{a},N} \leq c_{\mathfrak{a}}^+ \, N^{-1}$.*
- *if $\langle \boldsymbol{\alpha} \rangle = 1$ ($d_s = 1$), $r_{\mathfrak{a},N} \leq c_{\mathfrak{a}}^+ \, N^{-1} \log N$.*
- *if $\langle \boldsymbol{\alpha} \rangle > 1$ ($d_s > 1$), $r_{\mathfrak{a},N} \leq c_{\mathfrak{a}}^+ \, N^{-\frac{1}{d_s}}$.*

Alternatively, we can reformulate the above statement as a result for

$$N_{\mathfrak{a},\varepsilon} := \inf\{N \in \mathbb{N}_* : r_{\mathfrak{a},N} < \varepsilon\}. \tag{4.2}$$

This quantity shows how many terms should be taken in the approximation (3.3) in order to ensure that $\|\!|\varepsilon_m^{\mathfrak{a},N}|\!\|_{[0,T];\mathcal{T}^m} < C\varepsilon$, with some $C > 0$, and is important for complexity estimates, cf. Remark 3.3.

THEOREM 4.3. *There exists $C_{\mathfrak{a}}^+ > 0$, depending on $\boldsymbol{\mu}$, $\boldsymbol{\alpha}$, s.t., for all $0 < \varepsilon < 1/2$, $N_{\mathfrak{a},\varepsilon}$ defined in (4.2) satisfies:*

- *if $\langle \boldsymbol{\alpha} \rangle < 1$ ($d_s < 1$), $N_{\mathfrak{a},\varepsilon} \leq C_{\mathfrak{a}}^+ \, \varepsilon^{-1}$.*
- *if $\langle \boldsymbol{\alpha} \rangle = 1$ ($d_s = 1$), $N_{\mathfrak{a},\varepsilon} \leq C_{\mathfrak{a}}^+ \, \varepsilon^{-1} \log \varepsilon^{-1}$.*
- *if $\langle \boldsymbol{\alpha} \rangle > 1$ ($d_s > 1$), $N_{\mathfrak{a},\varepsilon} \leq C_{\mathfrak{a}}^+ \, \varepsilon^{-d_s}$.*

4.2. Motivation and plan of the section. Proving Theorem 4.2 requires finding an explicit bound on $r_{\mathfrak{a},N}$ in terms of N. For this we remark that comparing the two series representations (2.24) and (2.27) of $\boldsymbol{\Lambda}_{\mathfrak{a}}(\omega)$, it is possible to re-express $r_{\mathfrak{a},N}$ using the eigenvalues $\omega_{\mathfrak{a},n}^2$ of $\mathcal{A}_{\mathfrak{a}}$ and the coefficients $a_{\mathfrak{a},n}$ (as defined by (2.23)). This yields two definitions of $r_{\mathfrak{a},N}$ (the first definition below is the definition (3.7)):

$$r_{\mathfrak{a},N} = \sum_{k=N+1}^{\infty} \frac{A_{\mathfrak{a},k}}{\Omega_{\mathfrak{a},k}^2}, \qquad r_{\mathfrak{a},N} \equiv \sum_{\omega_{\mathfrak{a},n} > \Omega_{\mathfrak{a},N}} \frac{a_{\mathfrak{a},n}}{\omega_{\mathfrak{a},n}^2}, \quad a_{\mathfrak{a},n} = \frac{\partial_s \varphi_{\mathfrak{a},n}(M^*)^2}{\omega_{\mathfrak{a},n}^2}. \tag{4.3}$$

From the above it is clear that estimating $r_{\mathfrak{a},N}$ requires

(i) an estimate on the asymptotic (as $n \to +\infty$) growth of the eigenvalues $\omega_{\mathfrak{a},n}^2$;
(ii) an estimate on the coefficients $a_{\mathfrak{a},n}$, which, in turn, requires an estimate on $\partial_s \varphi_{\mathfrak{a},n}(M^*)$.

To obtain (i), it is sufficient to find the asymptotic behaviour of the counting function

$$\rho_{\mathfrak{a}} : \mathbb{R}^+ \to \mathbb{N}, \quad \rho_{\mathfrak{a}}(\lambda) = \#\{\ell : \omega_{\mathfrak{a},\ell}^2 < \lambda\}, \tag{4.4}$$

of the operator $\mathcal{A}_{\mathfrak{a}}$, as $\lambda \to +\infty$. For addressing (ii), the most natural approach would be to estimate individually each of the coefficients $a_{\mathfrak{a},n}$. It is not difficult to prove that there exists $C_a > 0$, s.t. $a_{\mathfrak{a},n} \leq C_a$ for all $n \in \mathbb{N}_*$. Combining this upper bound with the asymptotic estimate provided by (i), i.e. $\omega_{\mathfrak{a},n}^2 = c(n) + o(c(n))$ as $n \to +\infty$, yields an upper bound on $r_{\mathfrak{a},N}$:

$$r_{\mathfrak{a},N} \leq C C_a \sum_{\omega_{\mathfrak{a},n} > \Omega_{\mathfrak{a},N}} c(n)^{-1}, \qquad \text{with some } C > 0.$$

However, depending on the geometry of the tree $\mathcal{T}$, the series in the right-hand side may not converge. Because numerical experiments indicate that the uniform bound $a_{\mathfrak{a},n} \leq C_a$ is optimal (we conjecture that there exists $c_a > 0$, s.t. for all $n \in \mathbb{N}$ there exists $n^* > n$ s.t. $a_{\mathfrak{a},n^*} > c_a$), a different strategy for estimating $r_{\mathfrak{a},N}$ is needed.

An alternative strategy. To obtain an estimate on $r_{\mathfrak{a},N}$, we replace an individual estimate of $a_{\mathfrak{a},n}$ by an estimate on a sum of $a_{\mathfrak{a},n}$ lying in a certain frequency window. This idea was inspired by the work of Barnett and Hassell [3], where the authors prove a certain quasi-orthogonality property of the Neumann traces of the eigenfunctions of the Dirichlet Laplacian $-\Delta$ in a open set in $\mathbb{R}^d$.

As we will see, for this it is more advantageous to work with the second definition of $r_{\mathfrak{a},N}$ in (4.3). The price to pay when using this strategy is that the obtained bound is of the type $r_{\mathfrak{a},N} \le C\Omega_{\mathfrak{a},N}^{-1}$, which is not explicit in N. Nonetheless, it is possible to make this bound explicit in N by exploiting estimates on the counting function (4.4).

Plan of the section. The rest of this section is organized as follows. In Section 4.3, we prove the bound $r_{\mathfrak{a},N} \le C\Omega_{\mathfrak{a},N}^{-1}$. Section 4.4 is dedicated to the study of the asymptotics of the counting function (4.4). In Section 4.5 we combine these two results to prove Theorem 4.3. We could have started by the proof of Theorem 4.2, but this seemed somewhat less natural to us. In Section 4.6, we prove Theorem 4.2.

4.3. An implicit bound on $r_{\mathfrak{a},N}$. Based on the second expression in (4.3), let us introduce the following quantity

$$R_{\mathfrak{a},\omega} := \sum_{\omega_{\mathfrak{a},n}>\omega} \frac{a_{\mathfrak{a},n}}{\omega_{\mathfrak{a},n}^2}, \text{ so that } r_{\mathfrak{a},N} \equiv R_{\mathfrak{a},\Omega_{\mathfrak{a},N}}. \tag{4.5}$$

We then have the first result (cf. the discussion in the end of Section 4.2).

PROPOSITION 4.4. *There exists $\tilde{c}_{\mathfrak{a}}^+ > 0$, that depends on $\boldsymbol{\alpha}, \boldsymbol{\mu}$, s.t.*

$$R_{\mathfrak{a},\omega} < \tilde{c}_{\mathfrak{a}}^+ \, \omega^{-1}, \text{ for all } \omega > 0. \tag{4.6}$$

As discussed in the end of Section 4.2, the proof of Proposition 4.4 relies on an estimate on the Neumann traces of 'packages' of eigenfunctions. We need a corresponding auxiliary result, which is summarized and proven in the section that follows.

4.3.1. An upper bound for the Neumann traces of eigenfunctions.

LEMMA 4.5. *For any $\eta > 0$, there exists $C_\eta^+ > 0$ (that depends on $\boldsymbol{\mu}, \boldsymbol{\alpha}, \mathfrak{a}$) s.t.*

$$\sum_{j:\,|\omega_{\mathfrak{a},j}-\omega|<\eta} \left(\partial_s \varphi_{\mathfrak{a},j}(M^*)\right)^2 \omega_{\mathfrak{a},j}^{-2} \equiv \sum_{j:\,|\omega_{\mathfrak{a},j}-\omega|<\eta} a_{\mathfrak{a},j} \le C_\eta^+, \quad \text{for all } \omega > 0. \tag{4.7}$$

The above lemma shows that the sum of the coefficients $a_{\mathfrak{a},j}$ corresponding to the eigenfrequencies lying in the frequency window $(\omega-\eta, \omega+\eta)$ is bounded independently of ω. This result is stronger than the simple uniform estimate $|a_{\mathfrak{a},j}| \le C_a$ because it is independent on the number of the eigenfrequencies on the interval $(\omega-\eta, \omega+\eta)$.

The proof of this result is adapted from the proof of a similar estimate for the Laplacian in a bounded domain in $\mathbb{R}^d$ by Barnett, Hassell [3]. The only modification compared to [3] is required in the proof of Lemma 2.1 [3], where we choose a smooth multiplier $\chi(s)$ (in the notation of [3], $a(s)$) supported on $\Sigma_{0,0}$, s.t. $\chi(M^*) = -1$. We will nonetheless present its proof here, for the sake of self-consistency of the paper. To prove Lemma 4.5, we start with the following auxiliary lemma.

LEMMA 4.6. *Let $\mathcal{J}_{\omega,\eta}^{\mathfrak{a}} := \{j : |\omega_{\mathfrak{a},j} - \omega| < \eta\}$, where $\omega \ge 1$, $\eta > 0$. Let*

$$\forall\, \boldsymbol{c} = (c_j)_{j\in\mathcal{J}_{\omega,\eta}^{\mathfrak{a}}} \in \mathbb{R}^{K_{\mathfrak{a}}} \quad (\text{where } K_{\mathfrak{a}} = \#\mathcal{J}_{\omega,\eta}^{\mathfrak{a}}), \quad \varphi_{\boldsymbol{c}} := \sum_{j\in\mathcal{J}_{\omega,\eta}^{\mathfrak{a}}} c_j \varphi_{\mathfrak{a},j}.$$

Then, with some $C_\eta > 0$,

$$|\varphi_{\boldsymbol{c}}'(M^*)| \le C_\eta \omega \|\boldsymbol{c}\|_{\mathbb{R}^{K_{\mathfrak{a}}}}. \tag{4.8}$$

Proof. Let $\chi : \mathcal{T} \to \mathbb{R}$ be a C^1-function supported on the root branch $\Sigma_{0,0}$, with $\chi(M^*) = 1$ and $\chi(M_{0,0}) = 0$. Identifying $\Sigma_{0,0}$ with $[0,1]$ (and M^* with 0), we have

$$\begin{aligned} &\varphi_{\boldsymbol{c}}'(0)^2 = -\int_0^1 \frac{d}{ds}\left(\chi(s)\varphi_{\boldsymbol{c}}'(s)\right)^2 = I_1 + I_2, \\ &I_1 = -\int_0^1 \chi'(s)(\varphi_{\boldsymbol{c}}'(s))^2 ds, \quad I_2 = -2\int_0^1 \chi(s)\varphi_{\boldsymbol{c}}'(s)\varphi_{\boldsymbol{c}}''(s)ds. \end{aligned} \tag{4.9}$$

Our goal is to bound I_1, I_2. First of all, from the definition (2.23) it follows that

$$\|\varphi_{\mathfrak{a},j}\| = 1, \;\; \|\partial_s\varphi_{\mathfrak{a},j}\|^2 = \omega_{\mathfrak{a},j}^2, \;\; (\varphi_{\mathfrak{a},j}, \varphi_{\mathfrak{a},k}) = (\partial_s\varphi_{\mathfrak{a},j}, \partial_s\varphi_{\mathfrak{a},k}) = 0, \;\; \text{for } j \neq k. \tag{4.10}$$

The above with the definition of $\mathcal{J}_{\omega,\eta}^{\mathfrak{a}}$ yields the bound:

$$\|\varphi_{\boldsymbol{c}}\|_{\mathrm{L}^2_\mu} = \|\boldsymbol{c}\|, \qquad \|\varphi_{\boldsymbol{c}}'\|^2 = \sum_{j\in\mathcal{J}_{\omega,\eta}^{\mathfrak{a}}} \omega_{\mathfrak{a},j}^2 c_j^2 \leq (\omega+\eta)^2\|\boldsymbol{c}\|^2. \tag{4.11}$$

The first term in (4.9) thus satisfies

$$|I_1| \leq |\chi'|_\infty(\omega+\eta)^2\|\boldsymbol{c}\|^2. \tag{4.12}$$

By Remark 2.8, because μ is piecewise-constant on $\mathcal{T}$, on $\Sigma_{0,0}$ it holds that $-\varphi_{\mathfrak{a},j}'' = \omega_{\mathfrak{a},j}^2\varphi_{\mathfrak{a},j}$. We use this expression to handle $\varphi_{\boldsymbol{c}}''$ in I_2, cf. (4.9):

$$I_2 = 2\int_0^1 \chi(s)\varphi_{\boldsymbol{c}}'(s) \sum_{j\in\mathcal{J}_{\omega,\eta}^{\mathfrak{a}}} \omega_{\mathfrak{a},j}^2 c_j\varphi_{\mathfrak{a},j}(s)ds. \tag{4.13}$$

Bounding the above term with the Cauchy-Schwarz inequality would result in

$$|I_2| \leq C\|\varphi_{\boldsymbol{c}}'\| \left\| \sum_{j\in\mathcal{J}_{\omega,\eta}^{\mathfrak{a}}} \omega_{\mathfrak{a},j}^2 c_j\varphi_{\mathfrak{a},j} \right\| \overset{(4.11)}{\leq} C(\omega+\eta)^3\|\boldsymbol{c}\|^2,$$

which, for large ω, is ω times more than what we would like to have. To overcome this problem, we will use two observations: 1) $\omega_{\mathfrak{a},j}$ in the above sum are 'close' to ω; 2) the orthogonality of eigenfunctions (cf. the first equation of (4.11)). We rewrite

$$\sum_{j\in\mathcal{J}_{\omega,\eta}^{\mathfrak{a}}} \omega_{\mathfrak{a},j}^2 c_j\varphi_{\mathfrak{a},j}(s) = \sum_{j\in\mathcal{J}_{\omega,\eta}^{\mathfrak{a}}} (\omega_{\mathfrak{a},j}^2 - \omega^2)c_j\varphi_{\mathfrak{a},j}(s) + \omega^2\varphi_{\boldsymbol{c}}(s) = \psi_{\boldsymbol{c}}(s) + \omega^2\varphi_{\boldsymbol{c}}(s).$$

Then (4.13) rewrites

$$\begin{aligned} I_2 &= 2\int_0^1 \chi(s)\varphi_{\boldsymbol{c}}'(s)\psi_{\boldsymbol{c}}(s)ds + 2\omega^2\int_0^1 \chi(s)\varphi_{\boldsymbol{c}}'(s)\varphi_{\boldsymbol{c}}(s)ds \\ &= 2\int_0^1 \chi(s)\varphi_{\boldsymbol{c}}'(s)\psi_{\boldsymbol{c}}(s)ds - \omega^2\int_0^1 \varphi_{\boldsymbol{c}}^2(s)\chi'(s)ds, \end{aligned} \tag{4.14}$$

where we integrated by parts using $\chi(1) = 0$ and $\varphi_{\boldsymbol{c}}(0) = 0$. Next we bound $\psi_{\boldsymbol{c}}$ using first the orthogonality of $\varphi_{\mathfrak{a},j}$, cf. (4.11), and next $|\omega_{\mathfrak{a},j}^2 - \omega^2| \leq \eta(2\eta + \omega)$, $j \in \mathcal{J}_{\omega,\eta}^{\mathfrak{a}}$:

$$\|\psi_{\boldsymbol{c}}\|^2 = \sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathfrak{a}}} (\omega_{\mathfrak{a},j}^2 - \omega^2)^2 c_j^2 \leq \eta(2\eta + \omega)\|\boldsymbol{c}\|^2. \tag{4.15}$$

Therefore, by the Cauchy-Schwarz inequality, using the bounds (4.11) and (4.15), applied to (4.14), we have

$$|I_2| \leq 2|\chi|_\infty \sqrt{\eta(2\eta + \omega)}(\omega + \eta)\|\boldsymbol{c}\|^2 + |\chi'|_\infty (\omega + \eta)^2 \|\boldsymbol{c}\|^2.$$

Combining the above bound and (4.12) in (4.9) results in

$$|\varphi_{\boldsymbol{c}}'(0)|^2 \leq C(\eta)\omega^2 \|\boldsymbol{c}\|^2,$$

which is the desired statement (with an abuse of notation $M^* \equiv 0$). ∎

Lemma 4.5 is then almost an immediate corollary of Lemma 4.6.

Proof of Lemma 4.5. Clearly, it suffices to prove the bound for ω large enough. Take in Lemma 4.6 $\boldsymbol{c} := (\partial_s \varphi_{\mathfrak{a},j}(M^*))_{j \in \mathcal{J}_{\omega,\eta}^{\mathfrak{a}}}$. Then (4.8) rewrites

$$\sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathfrak{a}}} \partial_s \varphi_{\mathfrak{a},j}(M^*)^2 \leq C_\eta \omega \Big(\sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathfrak{a}}} \partial_s \varphi_{\mathfrak{a},j}(M^*)^2 \Big)^{\frac{1}{2}}.$$

This yields $\sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathfrak{a}}} \partial_s \varphi_{\mathfrak{a},j}(M^*)^2 \leq C_\eta^2 \omega^2$. For ω large enough, for all $j \in \mathcal{J}_{\omega,\eta}^{\mathfrak{a}}$, one has that $\omega_{\mathfrak{a},j} \geq \omega - \eta > 0$, and therefore,

$$\sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathfrak{a}}} (\partial_s \varphi_{\mathfrak{a},j}(M^*))^2 \omega_{\mathfrak{a},j}^{-2} \leq C_\eta \frac{\omega^2}{|\omega - \eta|^2} \leq \widetilde{C}_\eta,$$

for all ω large enough. This proves Lemma 4.5. ∎

4.3.2. Proof of Proposition 4.4. Because $\omega \mapsto R_{\mathfrak{a},\omega}$ is non-increasing and piecewise-constant, it suffices to prove Proposition 4.4 for $\omega = M \in \mathbb{N}_*$. We start by rewriting (4.5) in the following form

$$R_{\mathfrak{a},M} = \sum_{n:\, \omega_{\mathfrak{a},n} \geq M} \frac{a_{\mathfrak{a},n}}{\omega_{\mathfrak{a},n}^2} = \sum_{n=M}^{\infty} \sum_{n \leq \omega_{\mathfrak{a},j} < n+1} \frac{a_{\mathfrak{a},j}}{\omega_{\mathfrak{a},j}^2},$$

where, in order to apply Lemma 4.5, we split the interval $[m, \infty)$ into subintervals $[k, k+1)$, $k \geq m$, and sum over the eigenvalues belonging to these intervals. Obviously,

$$R_{\mathfrak{a},M} \leq \sum_{n=M}^{\infty} \frac{1}{n^2} \sum_{n \leq \omega_{\mathfrak{a},j} < n+1} a_{\mathfrak{a},j} \leq \sum_{n=M}^{\infty} \frac{1}{n^2} \sum_{|\omega_{\mathfrak{a},j} - n| \leq 1} a_{\mathfrak{a},j} \leq C_1^+ \sum_{n=M}^{\infty} \frac{1}{n^2},$$

where the last bound follows from Lemma 4.5 applied with $\eta = 1$. The above remainder is then $O(M^{-1})$, which proves the desired statement.

4.4. Asymptotics of the counting function. The second result, necessary for the proof of Theorem 4.2, cf. Section 4.2, summarizes estimates on the counting function $\rho_{\mathfrak{a}}$. We shall need the following definition:

Definition 4.7 (Arithmetic set, cf. [14]). *A set $E = \{\gamma_j, 0 \le j \le p-1\} \subset \mathbb{R}_*^+$ is called arithmetic if $\gamma_j/\gamma_k \in \mathbb{Q}$ for all j, k. The span of E is the largest $\gamma \in \mathbb{R}_*^+$, s.t. $\gamma_j/\gamma \in \mathbb{N}$ for all j.*

Theorem 4.8. *The counting function $\rho_{\mathfrak{a}}$, cf. (4.4), satisfies, as $\lambda \to +\infty$,*

1. *if $\langle \boldsymbol{\alpha} \rangle < 1$ ($d_s < 1$), then $\rho_{\mathfrak{a}}(\lambda) = \pi^{-1} \left(1 - \langle \boldsymbol{\alpha} \rangle\right)^{-1} \lambda^{\frac{1}{2}} + O(\lambda^{\frac{d_s}{2}})$.*
2. *if $\langle \boldsymbol{\alpha} \rangle = 1$ ($d_s = 1$), then $\rho_{\mathfrak{a}}(\lambda) = \pi^{-1}\, C_{\boldsymbol{\alpha}}\, \lambda^{\frac{1}{2}} \log \lambda^{\frac{1}{2}} + O(\lambda^{\frac{1}{2}})$, where*
$$C_{\boldsymbol{\alpha}} = \Big(\sum \alpha_j \log \alpha_j^{-1} \Big)^{-1}.$$
3. *if $\langle \boldsymbol{\alpha} \rangle > 1$ ($d_s > 1$), there exists a bounded function $\rho_{\mathfrak{a}}^{\infty} : \mathbb{R}^+ \to \mathbb{R}^+$, s.t.*
$$\rho_{\mathfrak{a}} = \rho_{\mathfrak{a}}^{\infty}(\lambda)\, \lambda^{\frac{d_s}{2}} + o(\lambda^{\frac{d_s}{2}}), \quad \mathfrak{a} \in \{\mathfrak{d}, \mathfrak{n}\}, \tag{4.16}$$
where, with $E := \{\log \alpha_j^{-1}, j = 0, \dots, p-1\}$, the function $\rho_{\mathfrak{a}}^{\infty}$ satisfies:
   - *if the set E is arithmetic with the span γ, then $\rho_{\mathfrak{a}}^{\infty}(\lambda) = \psi_{\mathfrak{a}}^{\infty}(\frac{1}{2}\log\lambda)$ with $\psi_{\mathfrak{a}}^{\infty}$ piecewise-continuous and γ-periodic, s.t. $0 < \psi_- < \psi_{\mathfrak{a}}^{\infty} < \psi_+$.*
   - *otherwise, $\rho_{\mathfrak{a}}^{\infty}(\lambda) = \rho_{\mathfrak{a}}^{\infty}$ (const > 0).*

The goal of this section is to prove Theorem 4.8. Asymptotics of counting functions of the discrete and continuous Laplacian on various types of fractals were obtained in e.g. [13, 14, 2], see as well [11] and references therein. Our geometric setting bears some similarities to the one from the work by Levitin, Vassiliev [14], however, we do not use the same geometrical and boundary condition assumptions.

We will make use of the ideas from the seminal article by Kigami and Lapidus [13], whether the authors study the counting function for the discrete Laplacian on the post-critically finite self-similar fractals (the geometry considered in the present paper does not belong to this class). In particular, like for the classical Weyl's estimates (for the Laplacian in bounded domains of $\mathbb{R}^d$ for instance), the study of $\rho_{\mathfrak{a}}(\lambda)$ relies on the min-max principle and the so-called Dirichlet-Neumann bracketing technique. Using properties of the fractal geometry, it is possible to write a recursive equation for $\rho_{\mathfrak{a}}(\lambda)$, which next is investigated using the renewal theorem from [14].

4.4.1. A recursive equation for $\rho_{\mathfrak{a}}(\lambda)$. This section is dedicated to the proof of the recursive equation for $\rho_{\mathfrak{a}}(\lambda)$. For this, we will extensively use the well-known min-max characterization of the eigenvalues:

$$\omega_{\mathfrak{a},n}^2 = \inf_{Q \in \mathcal{F}^n(V_{\mathfrak{a}})} \sup_{v \in Q} \frac{a(v,v)}{\|v\|^2_{\mathrm{L}^2_\mu(\mathcal{T})}} \quad \text{where } \mathcal{F}^n(V_{\mathfrak{a}}) := \{Q : \ Q \subset V_{\mathfrak{a}}, \dim Q = n\}. \tag{4.17}$$

Lemma 4.9 (A recursive equation). *The functions $\rho_{\mathfrak{a}}$, $\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$, satisfy*

$$\rho_{\mathfrak{a}}(\lambda) = r_{\mathfrak{a}}(\lambda) + \sum \rho_{\mathfrak{a}}(\alpha_j^2 \lambda), \qquad \lambda > 0, \tag{4.18}$$

where $r_{\mathfrak{a}} : \mathbb{R}_*^+ \to \mathbb{R}_*^+$ is a piecewise-continuous function, s.t.

$$\left\lceil \pi^{-1} \lambda^{\frac{1}{2}} \right\rceil - 1 \le r_{\mathfrak{a}}(\lambda) \le \left\lceil \pi^{-1} \lambda^{\frac{1}{2}} \right\rceil + p. \tag{4.19}$$

Moreover, there exists $\lambda^*_{\mathfrak{a}} > 0$*, s.t.* $r_{\mathfrak{a}}(\lambda) = 0$ *and* $\rho_{\mathfrak{a}}(\lambda) = 0$ *for* $\lambda < \lambda^*_{\mathfrak{a}}$.

Proof. Note that showing (4.18, 4.19) amounts to proving the double inequality

$$\sum \rho_{\mathfrak{a}}(\alpha_j^2\lambda) + \left\lceil \pi^{-1}\lambda^{\frac{1}{2}} \right\rceil - 1 \le \rho_{\mathfrak{a}}(\lambda) \le \sum \rho_{\mathfrak{a}}(\alpha_j^2\lambda) + \left\lceil \pi^{-1}\lambda^{\frac{1}{2}} \right\rceil + p, \tag{4.20}$$

so that, setting $r_{\mathfrak{a}}(\lambda) := \rho_{\mathfrak{a}}(\lambda) - \sum \rho_{\mathfrak{a}}(\alpha_j^2\lambda)$, $r_{\mathfrak{a}}$ inherits piecewise-continuity from $\rho_{\mathfrak{a}}$ and satisfies (4.19). We will prove (4.20) for $\mathfrak{a} = \mathfrak{n}$, the case $\mathfrak{a} = \mathfrak{d}$ being almost verbatim the same. The proof is based on two ideas: the classical Dirichlet-Neumann bracketing technique and a rescaling argument.

Step 1. Dirichlet-Neumann bracketing.

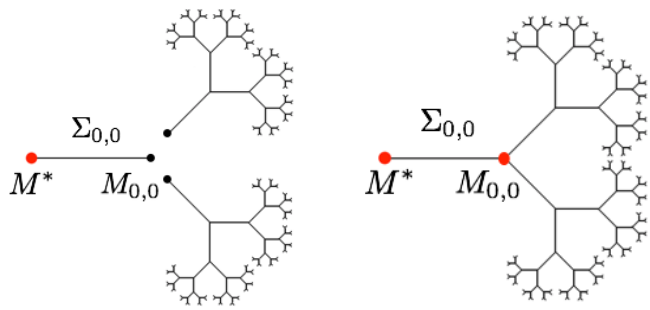

FIGURE 2. *Left: we illustrate the space* $\widetilde{V}_{\mathfrak{n}}(\mathcal{T})$*, which includes functions that may be discontinuous in the point* $M_{0,0}$*. Right: an illustration to the space* $\widetilde{V}_{\mathfrak{n},0}(\mathcal{T})$*, where all functions vanish in the point* $M_{0,0}$*. By red we mark the points where the functions vanish.*

Step 1.1. Let $\mathcal{A}_N$ be the self-adjoint operator in $\mathrm{L}^2_\mu(\mathcal{T})$ associated to the sesquilinear form $a(u,v)$ (2.20), defined like $\mathcal{A}_{\mathfrak{n}}$ in (2.21), but with the following domain (remark that $V_{\mathfrak{n}}$ is replaced by $\mathrm{H}^1_\mu(\mathcal{T})$, i. e. the Dirichlet condition in M^* is removed):

$$\mathcal{D}(\mathcal{A}_N;\mathcal{T}) = \{u \in \mathrm{H}^1_\mu(\mathcal{T}) : \ |a(u,v)| \le C(u)\|v\|_{\mathrm{L}^2_\mu(\mathcal{T})}, \quad \forall v \in \mathrm{H}^1_\mu(\mathcal{T})\}.$$

In a strong form, this operator corresponds to the weighted Laplacian on $\mathcal{T}$, with the Neumann conditions at M^* and at the fractal boundary of the tree $\mathcal{T}$. By Theorem 2.1, it has a compact resolvent. Its spectrum is denoted by $\sigma_N(\mathcal{T})$ and the associated counting function by $\rho_N(\lambda;\mathcal{T})$.

Let $\mathcal{T}_{1,j}$ be the self-similar p-adic subtree of $\mathcal{T}$ whose root edge is $\Sigma_{1,j}$. Let us introduce a broken space that contains $V_{\mathfrak{n}}(\mathcal{T})$ (see Figure 2 for an illustration)

$$\widetilde{V}_{\mathfrak{n}}(\mathcal{T}) := \{u \in \mathrm{L}^2_\mu(\mathcal{T}) : \ u|_{\Sigma_{0,0}} \in H^1(\Sigma_{0,0}),\ u(M^*) = 0,\ u|_{\mathcal{T}_{1,j}} \in \mathrm{H}^1_\mu(\mathcal{T}_{1,j})), \quad \forall j\}.$$

Let us define the operator $\widetilde{\mathcal{A}}_{\mathfrak{n}}$ like in (2.21), with the domain

$$\mathcal{D}(\widetilde{\mathcal{A}}_{\mathfrak{n}}) = \{u \in \widetilde{V}_{\mathfrak{n}}(\mathcal{T}) : \ |a(u,v)| \le C(u)\|v\|_{\mathrm{L}^2_\mu(\mathcal{T})}, \quad \forall v \in \widetilde{V}_{\mathfrak{n}}(\mathcal{T})\}.$$

Again, by Theorem 2.1, it has a compact resolvent. We denote by $\widetilde{\sigma}_{\mathfrak{n}}(\mathcal{T})$ and by $\widetilde{\rho}_{\mathfrak{n}}(\lambda;\mathcal{T})$ the spectrum and counting function of the operator $\widetilde{\mathcal{A}}_{\mathfrak{n}}$.

Finally, let $\widetilde{V}_{\mathfrak{n},0}$ be a subspace of $V_{\mathfrak{n}}$ of functions vanishing in $M_{0,0}$,

$$\widetilde{V}_{\mathfrak{n},0}(\mathcal{T}) = \{v \in V_{\mathfrak{n}}(\mathcal{T}) : \ v(M_{0,0}) = 0\},$$

see Figure 2 for an illustration. The spectrum of the associated operator $\widetilde{\mathcal{A}}_{\mathfrak{n},0}$ (defined like above) will be denoted by $\widetilde{\sigma}_{\mathfrak{n},0}(\mathcal{T})$, and the counting function by $\widetilde{\rho}_{\mathfrak{n},0}(\lambda;\mathcal{T})$.

Let us remark that we will need to use the counting functions for different trees, and hence the notation $\rho(\lambda;\mathcal{T})$, where the tree is made explicit. The notation $\rho(\lambda)$ is reserved for the counting functions for the operators on the original tree $\mathcal{T}$.

Step 1.2. Relations between the counting functions.
Relating $\rho_{\mathfrak{n}}$ and ρ_N. We will need an upper bound for $\rho_N(\lambda;\mathcal{T})$ in terms of $\rho_{\mathfrak{n}}(\lambda) \equiv \rho_{\mathfrak{n}}(\lambda;\mathcal{T})$. It can be obtained from the same arguments as [13, Corollary 4.7] because the co-dimension of $V_{\mathfrak{n}}(\mathcal{T})$, as a subspace of $\mathrm{H}^1_\mu(\mathcal{T})$, is finite. Namely,

$$\rho_N(\lambda;\mathcal{T}) \le \rho_{\mathfrak{n}}(\lambda;\mathcal{T}) + \dim\left(\mathrm{H}^1_\mu(\mathcal{T}) \setminus V_{\mathfrak{n}}(\mathcal{T})\right) = \rho_{\mathfrak{n}}(\lambda;\mathcal{T}) + 1. \tag{4.21}$$

Relating $\rho_{\mathfrak{n}}$ and $\widetilde{\rho}_{\mathfrak{n}}$. The min-max principle (4.17), since $V_{\mathfrak{n}}(\mathcal{T}) \subset \widetilde{V}_{\mathfrak{n}}(\mathcal{T})$, yields

$$\rho_{\mathfrak{n}}(\lambda;\mathcal{T}) \le \widetilde{\rho}_{\mathfrak{n}}(\lambda;\mathcal{T}). \tag{4.22}$$

Relating $\rho_{\mathfrak{n}}$ and $\widetilde{\rho}_{\mathfrak{n},0}$. As $\widetilde{V}_{\mathfrak{n},0}(\mathcal{T}) \subset V_{\mathfrak{n}}(\mathcal{T})$, we obtain, by the min-max principle,

$$\widetilde{\rho}_{\mathfrak{n},0}(\lambda;\mathcal{T}) \le \rho_{\mathfrak{n}}(\lambda;\mathcal{T}). \tag{4.23}$$

Step 2. An equation for $\widetilde{\rho}_{\mathfrak{n}}(\lambda;\mathcal{T})$.
Step 2.1. An immediate identity. Because, with an obvious abuse of notation,

$$\widetilde{V}_{\mathfrak{n}}(\mathcal{T}) = V_{\mathfrak{n}}(\Sigma_{0,0}) \oplus \bigoplus_{j=0}^{p-1} \mathrm{H}^1_\mu(\mathcal{T}_{1,j}), \quad V_{\mathfrak{n}}(\Sigma_{0,0}) := \{u \in H^1(\Sigma_{0,0}) : u(M^*) = 0\},$$

the spectrum $\widetilde{\sigma}_{\mathfrak{n}}$ of $\widetilde{\mathcal{A}}_{\mathfrak{n}}$ consists of the eigenvalues of $-\Delta$ on $V_{\mathfrak{n}}(\Sigma_{0,0})$ (cf. Remark 2.8; the associated counting function is denoted by $\rho_{\mathfrak{n}}(\lambda;\Sigma_{0,0})$ and is known explicitly) and the union $\bigcup_{j=0}^{p-1} \sigma_N(\mathcal{T}_{1,j})$. The counting function $\widetilde{\rho}_{\mathfrak{n}}$ is a sum of the respective counting functions:

$$\widetilde{\rho}_{\mathfrak{n}}(\lambda;\mathcal{T}) = \rho_{\mathfrak{n}}(\lambda;\Sigma_{0,0}) + \sum \rho_N(\lambda;\mathcal{T}_{1,j}), \quad \rho_{\mathfrak{n}}(\lambda;\Sigma_{0,0}) = \left\lceil \pi^{-1}\lambda^{\frac{1}{2}} + 1/2 \right\rceil - 1. \tag{4.24}$$

Step 2.2. Scaling argument. The goal of this step is to show that

$$\rho_N(\lambda;\mathcal{T}_{1,j}) = \rho_N(\alpha_j^2\lambda;\mathcal{T}). \tag{4.25}$$

To see this, we get back to the min-max characterization of eigenvalues (4.17). Since $\mathcal{T}_{1,j}$ is obtained from $\mathcal{T}$ by a similitude transformation γ_j of ratio α_j (cf. [9]), setting formally $\tilde{s} = \gamma_j(s)$, we obtain the following identities, valid for any $v \in \mathrm{H}^1_\mu(\mathcal{T})$,

$$\|v\|^2_{\mathrm{L}^2_\mu(\mathcal{T})} = \int_{\mathcal{T}} |v(s)|^2\, \mu(s)\, ds = \mu_j^{-1} \int_{\mathcal{T}_{1,j}} |v(\gamma_j^{-1}(\tilde{s}))|^2\, \alpha_j^{-1}\, \mu(\tilde{s})\, d\tilde{s},$$
$$a(v,v) = \int_{\mathcal{T}} |\partial_s v(s)|^2\, \mu(s)\, ds = \mu_j^{-1} \int_{\mathcal{T}_{1,j}} |\partial_{\tilde{s}} v(\gamma_j^{-1}(\tilde{s}))|^2\, \alpha_j\, \mu(\tilde{s})\, d\tilde{s},$$

and thus, see the min-max principle (4.17),

$$\omega^2_{N,\ell} \in \sigma_N(\mathcal{T}) \iff \alpha_j^{-2}\omega^2_{N,\ell} \in \sigma_N(\mathcal{T}_{1,j}). \tag{4.26}$$

Therefore, $\rho_N(\lambda;\mathcal{T}_{1,j}) = \#\{\ell : \alpha_j^{-2}\omega_{N,\ell}^2 < \lambda\} = \rho_N(\alpha_j^2\lambda;\mathcal{T})$, i.e. (4.25).

Step 2.3. Summary. Combining (4.25) with (4.24), we obtain

$$\widetilde{\rho}_{\mathfrak{n}}(\lambda) = \left\lceil \pi^{-1}\lambda^{\frac{1}{2}} + 1/2 \right\rceil - 1 + \sum \rho_N(\alpha_j^2\lambda). \tag{4.27}$$

Step 3. An equation for $\widetilde{\rho}_{\mathfrak{n},0}(\lambda;\mathcal{T})$.
Just like in the previous case, we remark that (with an abuse of notation)

$$\widetilde{V}_{\mathfrak{n},0} = H_0^1(\Sigma_{0,0}) \oplus \bigoplus_{j=0}^{p-1} V_{\mathfrak{n}}(\mathcal{T}_{1,j}).$$

Repeating the same arguments as in Step 2, we get the recursive equation for $\widetilde{\rho}_{\mathfrak{n},0}$:

$$\widetilde{\rho}_{\mathfrak{n},0}(\lambda) = \left\lceil \pi^{-1}\lambda^{\frac{1}{2}} \right\rceil - 1 + \sum \rho_{\mathfrak{n}}(\alpha_j^2\lambda). \tag{4.28}$$

Step 4. Proof of (4.20). Inserting (4.22) into (4.27) gives:

$$\rho_{\mathfrak{n}}(\lambda) \le \left\lceil \pi^{-1}\lambda^{\frac{1}{2}} + 1/2 \right\rceil - 1 + \sum \rho_N(\alpha_j^2\lambda) \le \left\lceil \pi^{-1}\lambda^{\frac{1}{2}} \right\rceil + \sum \rho_{\mathfrak{n}}(\alpha_j^2\lambda) + p, \tag{4.29}$$

where the last bound was obtained from (4.21) and the inequality $\lceil x+1/2 \rceil \le \lceil x \rceil + 1$.

The lower bound is obtained by combining (4.23) and (4.28):

$$\rho_{\mathfrak{n}}(\lambda) \ge \left\lceil \pi^{-1}\lambda^{\frac{1}{2}} \right\rceil - 1 + \sum \rho_{\mathfrak{n}}(\alpha_j^2\lambda). \tag{4.30}$$

Combining (4.29) and (4.30) yields (4.20).

Step 5. 'Causality' of $\rho_{\mathfrak{n}}(\lambda)$, $r_{\mathfrak{n}}(\lambda)$. $\rho_{\mathfrak{n}}(\lambda) = 0$ for $\lambda < \omega_{1,\mathfrak{n}}^2$; the same holds for $\rho_{\mathfrak{n}}(\alpha_j^2\lambda)$, for $j = 0, \dots, p-1$, since $\alpha_j < 1$. Thus, $r_{\mathfrak{n}}(\lambda) = 0$ for $\lambda < \lambda_{\mathfrak{n}}^* := \omega_{1,\mathfrak{n}}^2$. ∎

4.4.2. Renewal theory. In order to solve the recursive equation (4.18), we will rewrite it in a more convenient form. In general we are interested in equations of the type: provided Φ piecewise-continuous, find φ s.t. for all $\lambda > 0$,

$$\varphi(\lambda) = \Phi(\lambda) + \sum \varphi(\alpha_j^2\lambda). \tag{4.31}$$

The change of variables $x := \frac{1}{2}\log\lambda$ transforms (4.31) into an equation with delays:

$$\varphi(\mathrm{e}^{2x}) = \Phi(\mathrm{e}^{2x}) + \sum \varphi(\alpha_j^2\mathrm{e}^{2x}) = \Phi(\mathrm{e}^{2x}) + \sum \varphi(\mathrm{e}^{2(x-\log\alpha_j^{-1})}), \quad x \in \mathbb{R}. \tag{4.32}$$

To handle the equations of type (4.32), we will use the renewal theorem, cf. [4, p. 358] or [12, Appendix B.4]; we will exploit its version suggested by Levitin, Vassiliev [14], whose statement is the most suitable for our needs.

THEOREM 4.10 (Renewal theorem, [14]). *Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a piecewise-continuous function that satisfies the following bound: there exist $C, \alpha \ge 0$, s.t. $|\Psi(x)| < C\mathrm{e}^{-\alpha|x|}$ on $\mathbb{R}$. Let $\psi : \mathbb{R} \to \mathbb{R}$ solve*

$$\psi(x) = \Psi(x) + \sum c_j\psi(x-\gamma_j), \quad c_j, \gamma_j > 0, \quad \sum c_j = 1. \tag{4.33}$$

Assume additionally that $\lim_{x\to-\infty} \psi(x) = 0$.

Then ψ as defined above is unique; it is uniformly bounded on $\mathbb{R}$ and

$$\psi(x) = \psi_\infty(x) + o(1), \text{ as } x \to +\infty,$$

where ψ_∞ is a periodic function defined as follows:

(1) if the set $\{\gamma_j,\, j=0,\dots,p-1\}$ is arithmetic, then, with γ being its span,

$$\psi_\infty(x) = \gamma J^{-1} \sum_{k=-\infty}^{\infty} \Psi(x-k\gamma), \quad J = \sum c_j\gamma_j.$$

(2) otherwise, $\psi_\infty(x) = \text{const} = J^{-1} \int\limits_{-\infty}^{\infty} \Psi(x)dx$.

Remark 4.11. *If Ψ is non-negative, and for all x sufficiently large $\Psi(x)$ is strictly positive, then from the explicit expressions for ψ_∞ in Theorem 4.10 and periodicity of ψ_∞ it follows that $\psi_\infty(x) > \psi_- > 0$ for all $x \in \mathbb{R}$.*

We will use a direct consequence of Theorem 4.10, formulated below.

Lemma 4.12. *Let $\varphi : \mathbb{R}_*^+ \to \mathbb{R}$ satisfy*

$$\varphi(\lambda) = \Phi(\lambda) + \sum \varphi(\alpha_j^2\lambda), \qquad \text{and } \varphi(\lambda) = 0, \text{ for } \lambda \le \lambda_*,\ \lambda_* > 0, \tag{4.34}$$

where $\Phi : \mathbb{R}^+ \to \mathbb{R}$ is piecewise-continuous, s.t., with some $C_* > 0$,

$$|\Phi(\lambda)| \le C_* \qquad \text{for all } \lambda > 0. \tag{4.35}$$

Then, there exists $C > 0$, s.t. for all $\lambda > 0$, $|\varphi(\lambda)| \le C\lambda^{\frac{d_s}{2}}$.

Proof. First of all, remark that

$$\Phi(\lambda) = 0, \text{ for all } \lambda < \lambda_*. \tag{4.36}$$

This is seen by rewriting $\Phi(\lambda) = \varphi(\lambda) - \sum \varphi(\alpha_j^2\lambda)$ and recalling that for $\lambda \le \lambda_*$, $\varphi(\lambda) = \varphi(\alpha_j^2\lambda) = 0$, by (4.34).

The rest of the proof follows closely the proof of Lemma 2.2 in [14]. We will rewrite (4.34) in the form required by Theorem 4.10, by a change of variables and rescaling. As explained in the beginning of Section 4.4.2, we transform (4.34) by a change of variables $x = \frac{1}{2}\log\lambda$, $\lambda > 0$ (so that $\lambda = \mathrm{e}^{2x}$), into

$$\varphi(\mathrm{e}^{2x}) = \Phi(\mathrm{e}^{2x}) + \sum \varphi(\alpha_j^2\mathrm{e}^{2x}) = \Phi(\mathrm{e}^{2x}) + \sum \varphi(\mathrm{e}^{2(x-\log\alpha_j^{-1})}). \tag{4.37}$$

It remains to make appear the weights c_j s.t. $\sum c_j = 1$, required in (4.33). For this we multiply (4.37) by $\mathrm{e}^{-d_s x}$ and set $\psi(x) := \varphi(\mathrm{e}^{2x})\mathrm{e}^{-d_s x}$ (so that $\varphi(\lambda) = \lambda^{\frac{d_s}{2}}\psi(\frac{1}{2}\log\lambda)$). After some computations (4.37) rewrites, with $\gamma_j = \log\alpha_j^{-1}$,

$$\psi(x) = \Psi(x) + \sum \alpha_j^{d_s}\psi(x-\gamma_j), \qquad \text{with } \Psi(x) := \Phi(\mathrm{e}^{2x})\mathrm{e}^{-d_s x}, \tag{4.38}$$

where, with the definition (4.1) of d_s, we recognize (4.33). Let us verify the conditions of Theorem 4.10:

- Ψ is piecewise-continuous, because Φ is piecewise-continuous;
- the bound $|\Psi(x)| \le C\mathrm{e}^{-\alpha|x|}$ holds true (with $\alpha = d_s > 0$) because
  - for $x < \frac{1}{2}\log\lambda_*$, $\Psi(x) = 0$ by 'causality' of Φ (4.36);
  - for $x > 0$, $|\Psi(x)| \le C_*\mathrm{e}^{-d_s x}$ as a direct consequence of (4.35);
  - for $x \in \left[\frac{1}{2}\log\lambda_*,\, 0\right]$ (if $\lambda_* < 1$), by (4.35), $|\Psi(x)| \le C_*\mathrm{e}^{-d_s x} \le C_*\lambda_*^{-\frac{d_s}{2}}$.
- $\lim\limits_{x\to-\infty}\psi(x) = 0$, because $\varphi(\mathrm{e}^{2x}) = 0$ for $x < \frac{1}{2}\log\lambda_*$.

By Theorem 4.10, ψ is uniformly bounded on $\mathbb{R}$, and so is $\varphi(\lambda)\lambda^{-\frac{d_s}{2}} \equiv \psi(\frac{1}{2}\log\lambda)$. ∎

4.4.3. Proof of Theorem 4.8. Before stating an actual proof, let us first explain the principal idea. One could try making an ansatz for the asymptotic behaviour of the counting function by generalizing the known asymptotics of the counting function for the Neumann/Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbb{R}^d$. I.e.

$$\rho_{\mathfrak{a}}(\lambda) = C\lambda^{\kappa} + o(\lambda^{\kappa}), \text{ as } \lambda \to +\infty, \quad C > 0,$$

where $\kappa \geq 0$ is a power to be determined; substituting the above into (4.18) yields

$$C\lambda^{\kappa} + o(\lambda^{\kappa}) = \frac{\sqrt{\lambda}}{\pi} + O(1) + C\sum \alpha_j^{2\kappa}\lambda^{\kappa} + o(\lambda^{\kappa}),$$

or, alternatively,

$$C\left(1 - \sum \alpha_j^{2\kappa}\right)\lambda^{\kappa} = \frac{\sqrt{\lambda}}{\pi} + o(\lambda^{\kappa}), \text{ as } \lambda \to +\infty. \tag{4.39}$$

The above shows that we should expect $\kappa \geq 1/2$. Two cases are possible:

- $\kappa > 1/2$. Necessarily, by power matching in (4.39), $\sum \alpha_j^{2\kappa} = 1$. By (4.1), $2\kappa = d_s > 1$, which is possible only if $\langle \boldsymbol{\alpha} \rangle > 1$.
- $\kappa = 1/2$. By power matching in (4.39), $C = \pi^{-1}(1 - \langle \boldsymbol{\alpha} \rangle)^{-1}$ (which is positive iff $\langle \boldsymbol{\alpha} \rangle < 1$).

Let us now prove the above rigorously. Let us remark that the proofs of the cases $\langle \boldsymbol{\alpha} \rangle \leq 1$ and $\langle \boldsymbol{\alpha} \rangle > 1$ differ slightly.

Proof of Theorem 4.8. Case $\langle \boldsymbol{\alpha} \rangle < 1$. Let us remark that a similar problem was treated in [14]. We present the proof here in a way that is somewhat different from the one in the aforementioned work, which allows to extend it easily to the particular case $\langle \boldsymbol{\alpha} \rangle = 1$, not studied in [14]. Let, with $C_{\boldsymbol{\alpha}} = \pi^{-1}\left(1 - \langle \boldsymbol{\alpha} \rangle\right)^{-1}$,

$$\varphi_{\mathfrak{a}}(\lambda) := \rho_{\mathfrak{a}}(\lambda) - \left\lfloor C_{\boldsymbol{\alpha}}\lambda^{\frac{1}{2}} \right\rfloor. \tag{4.40}$$

Our goal is to prove that $\varphi_{\mathfrak{a}}(\lambda) = O(\lambda^{\frac{d_s}{2}})$. For this we will apply Lemma 4.12 with $\varphi = \varphi_{\mathfrak{a}}$. Let us show that $\varphi_{\mathfrak{a}}(\lambda)$ satisfies the conditions of this lemma. Let

$$\Phi_{\mathfrak{a}}(\lambda) := \varphi_{\mathfrak{a}}(\lambda) - \sum \varphi_{\mathfrak{a}}(\alpha_j^2\lambda). \tag{4.41}$$

Evidently, $\varphi_{\mathfrak{a}}(\lambda)$ is piecewise-continuous and vanishes for $\lambda < \min(C_{\boldsymbol{\alpha}}^{-2}, \omega_{\mathfrak{a},1}^2)$ (same is true for $\Phi_{\mathfrak{a}}$). It remains to show that, with some $C_* > 0$, $|\Phi_{\mathfrak{a}}(\lambda)| \leq C_*$.

Let us prove first the lower bound $\Phi_{\mathfrak{a}}(\lambda) \geq -C_*$. Replacing the first term in (4.41) by using the recursive equation (4.18) yields

$$\Phi_{\mathfrak{a}}(\lambda) = r_{\mathfrak{a}}(\lambda) + \sum \rho_{\mathfrak{a}}(\alpha_j^2\lambda) - \left\lfloor C_{\boldsymbol{\alpha}}\lambda^{\frac{1}{2}} \right\rfloor - \sum \left(\rho_{\mathfrak{a}}(\alpha_j^2\lambda) - \left\lfloor C_{\boldsymbol{\alpha}}\alpha_j\lambda^{\frac{1}{2}} \right\rfloor\right).$$

Using $x \geq \lfloor x \rfloor \geq x - 1$, the lower bound (4.19) and $\lceil x \rceil \geq x$, the above rewrites

$$\Phi_{\mathfrak{a}}(\lambda) \geq \pi^{-1}\sqrt{\lambda} - 1 - C_{\boldsymbol{\alpha}}\lambda^{\frac{1}{2}} + \sum \alpha_j C_{\boldsymbol{\alpha}}\lambda^{\frac{1}{2}} - p.$$

Thus, with $C_{\boldsymbol{\alpha}} = \pi^{-1}(1 - \langle \boldsymbol{\alpha} \rangle)^{-1}$, $\Phi_{\mathfrak{a}}(\lambda) \geq -1 - p$. The upper bound can be obtained repeating the same arguments almost verbatim.

Applying Lemma 4.12, we deduce that $\varphi_{\mathfrak{a}}(\lambda) = O(\lambda^{\frac{d_s}{2}})$, $d_s < 1$.
Case $\langle \boldsymbol{\alpha} \rangle = 1$. The proof mimics the proof of the case $\langle \boldsymbol{\alpha} \rangle < 1$ almost verbatim, with the only difference being that the function under consideration is

$$\varphi_{\mathfrak{a}}(\lambda) = \rho_{\mathfrak{a}}(\lambda) - \Big\lfloor \pi^{-1} \Big(\sum \alpha_j \log \alpha_j^{-1} \Big)^{-1} \lambda^{\frac{1}{2}} \log \lambda^{\frac{1}{2}} \Big\rfloor.$$

We leave the details to the reader.
Case $\langle \boldsymbol{\alpha} \rangle > 1$. In this case we will not apply Lemma 4.12, but rather use the refined result stated in the end of Theorem 4.10.
For this we rewrite the recursive relation (4.18) like in the proof of Lemma 4.12, cf. (4.38). With $\psi_{\mathfrak{a}}(x) = \rho_{\mathfrak{a}}(\mathrm{e}^{2x})\mathrm{e}^{-d_s x}$ (so that $\rho_{\mathfrak{a}}(\lambda) = \psi_{\mathfrak{a}}\left(\frac{1}{2}\log\lambda\right)\lambda^{\frac{d_s}{2}}$), (4.18) gives

$$\psi_{\mathfrak{a}}(x) = \Psi_{\mathfrak{a}}(x) + \sum \psi_{\mathfrak{a}}(x - \log \alpha_j^{-1}) \alpha_j^{d_s}, \qquad \Psi_{\mathfrak{a}}(x) := r_{\mathfrak{a}}(\mathrm{e}^{2x})\mathrm{e}^{-d_s x}.$$

The above equation satisfies the conditions of Theorem 4.10 (this can be shown using the same arguments as in the proof of Lemma 4.12, using the bound $r_{\mathfrak{a}}(\mathrm{e}^{2x}) \le C\mathrm{e}^{x}$, $x > 0$, cf. (4.19), 'causality' of $r_{\mathfrak{a}}$ and the fact that and $d_s > 1$). Thus we can apply Theorem 4.10, which shows that, as $x \to +\infty$, $\psi_{\mathfrak{a}}(x) = \psi_{\mathfrak{a}}^{\infty}(x) + o(1)$. Therefore,

$$\rho_{\mathfrak{a}}(\lambda) = \psi_{\mathfrak{a}}\left(\frac{1}{2}\log\lambda\right)\lambda^{\frac{d_s}{2}} = \rho_{\mathfrak{a}}^{\infty}(\lambda)\,\lambda^{\frac{d_s}{2}} + o(\lambda^{\frac{d_s}{2}}), \qquad \text{as } \lambda \to +\infty,$$

where $\rho_{\mathfrak{a}}^{\infty}(\lambda) = \psi_{\mathfrak{a}}^{\infty}\left(\frac{1}{2}\log\lambda\right)$. Let us remark that $\rho_{\mathfrak{a}}^{\infty}$ is bounded by Theorem 4.10. Moreover, by Remark 4.11 and observation that $\Psi_{\mathfrak{a}}(x)$ is strictly positive for large x, cf. an explicit expression for $r_{\mathfrak{a}}$ in (4.19), it follows that $\rho_{\mathfrak{a}}^{\infty}$ is bounded from below by a positive constant. ∎

4.5. Proof of Theorem 4.3. To prove Theorem 4.3, let us start by the following corollary of Theorem 4.8. Similarly to $\rho_{\mathfrak{a}}(\lambda)$, let us introduce the related quantity, namely the number of positive poles of $\boldsymbol{\Lambda}_{\mathfrak{a}}$ smaller than λ:

$$P_{\mathfrak{a}}(\lambda) = \#\{n : \Omega_{\mathfrak{a},n} < \lambda\}. \tag{4.42}$$

Following the discussion preceding the derivation of (2.27), let us recall that the positive poles of $\boldsymbol{\Lambda}_{\mathfrak{a}}$ constitute a subset of $(\omega_{\mathfrak{a},n})_{n\in\mathbb{N}}$ (we assume $\omega_{\mathfrak{a},n} > 0$ for all n), and the eigenvalues, unlike the poles, are counted with their multiplicities. Therefore,

$$P_{\mathfrak{a}}(\lambda) \le \#\{k : \omega_{\mathfrak{a},k} < \lambda\} \equiv \rho_{\mathfrak{a}}(\lambda^2). \tag{4.43}$$

Combining the above relation with Theorem 4.8 yields

Corollary 4.13. *Let $P_{\mathfrak{a}} : \mathbb{R}^+ \to \mathbb{N}$ be defined in (4.42). Then, with $C_{\mathfrak{a}} > 0$, $\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$, depending on $\boldsymbol{\alpha}$, $\boldsymbol{\mu}$, it holds, for all $\lambda > 2$,*

1. *if $\langle \boldsymbol{\alpha} \rangle < 1$ ($d_s < 1$), then $P_{\mathfrak{a}}(\lambda) \le C_{\mathfrak{a}}\lambda$.*
2. *if $\langle \boldsymbol{\alpha} \rangle = 1$ ($d_s = 1$), then $P_{\mathfrak{a}}(\lambda) \le C_{\mathfrak{a}}\lambda\log\lambda$.*
3. *if $\langle \boldsymbol{\alpha} \rangle > 1$ ($d_s > 1$), then $P_{\mathfrak{a}}(\lambda) \le C_{\mathfrak{a}}\lambda^{d_s}$.*

We now have all the necessary ingredients required to prove Theorem 4.3.

Proof of Theorem 4.3. Recall that $N_{\mathfrak{a},\varepsilon}$ (cf. (4.2)) is defined as

$$N_{\mathfrak{a},\varepsilon} = \min\{N \in \mathbb{N}_* : r_{\mathfrak{a},N} < \varepsilon\}, \quad r_{\mathfrak{a},N} = \sum_{\omega_{\mathfrak{a},n} > \Omega_{\mathfrak{a},N}} \frac{a_{\mathfrak{a},n}}{\omega_{\mathfrak{a},n}^2}. \tag{4.44}$$

By Proposition 4.4, with (4.5),

$$r_{\mathfrak{a},N} < \tilde{c}_{\mathfrak{a}}^{+}\,\Omega_{\mathfrak{a},N}^{-1}, \quad \text{and thus} \quad N_{\mathfrak{a},\varepsilon} \le \min\{N \in \mathbb{N}_* : \tilde{c}_{\mathfrak{a}}^{+}\,\Omega_{\mathfrak{a},N}^{-1} \le \varepsilon\}, \tag{4.45}$$

or, in other words,

$$N_{\mathfrak{a},\varepsilon} \le \min\{N \in \mathbb{N}_* : \Omega_{\mathfrak{a},N} \ge \tilde{c}_{\mathfrak{a}}^{+}\,\varepsilon^{-1}\} \equiv \#\{n : \Omega_{\mathfrak{a},n} < \tilde{c}_{\mathfrak{a}}^{+}\,\varepsilon^{-1}\} + 1.$$

From the above and (4.42) we have

$$N_{\mathfrak{a},\varepsilon} \le P_{\mathfrak{a}}(\tilde{c}_{\mathfrak{a}}^{+}\varepsilon^{-1}) + 1, \tag{4.46}$$

and the result of Theorem 4.3 follows from the above bound and Corollary 4.13. ∎

4.6. Proof of Theorem 4.2. We will prove this result as a corollary of Theorem 4.3. Let us show the result for the case $\langle \boldsymbol{\alpha} \rangle < 1$. Evidently, it suffices to show this bound for all N sufficiently large.

First, let us fix $0 < \varepsilon < 1/2$. By definition (4.44) of $N_{\mathfrak{a},\varepsilon}$, we have $r_{\mathfrak{a},N} < \varepsilon$ for all $N \ge N_{\mathfrak{a},\varepsilon}$. In particular, by Theorem 4.3, we have that $N_{\mathfrak{a},\varepsilon} < N_* := \lceil C_{\mathfrak{a}}^{+}\varepsilon^{-1} \rceil$, and therefore $r_{\mathfrak{a},N_*} < \varepsilon$. In other words, for all $0 < \varepsilon < 1/2$,

$$r_{\mathfrak{a},\lceil C_{\mathfrak{a}}^{+}\varepsilon^{-1}\rceil} < \varepsilon. \tag{4.47}$$

For any $N \in \mathbb{N}$ sufficiently large, there exists $0 < \varepsilon < 1/2$ s.t. $N = \lceil C_{\mathfrak{a}}^{+}\varepsilon^{-1} \rceil$, and in this case $\varepsilon \le C_{\mathfrak{a}}^{+}(N-1)^{-1}$. This with (4.47) yields the desired bound $r_{\mathfrak{a},N} \le c_{\mathfrak{a}}^{+}N^{-1}$.

The result for the cases $\langle \boldsymbol{\alpha} \rangle \ge 1$ follows similarly (with the case $\langle \boldsymbol{\alpha} \rangle = 1$ being somewhat less trivial); the details are left to the reader.

5. Explicit lower bounds for the error indicator. The goal of this section is to prove the lower bounds on $r_{\mathfrak{a},N}$, as discussed in Section 3.3.

5.1. Main results. The two main results of this section read.

THEOREM 5.1. *There exists $c_{\mathfrak{a}}^{-} > 0$, depending only on $\boldsymbol{\mu}$, $\boldsymbol{\alpha}$, such that, for all $N \ge 1$, $r_{\mathfrak{a},N}$ defined in (3.7) satisfies:*

$$r_{\mathfrak{a},N} > c_{\mathfrak{a}}^{-}N^{-1}.$$

This result can be re-formulated in terms of the quantity $N_{\mathfrak{a},\varepsilon}$.

THEOREM 5.2. *There exists $C_{\mathfrak{a}}^{-} > 0$, depending only on $\boldsymbol{\mu}$, $\boldsymbol{\alpha}$, such that, for all $0 < \varepsilon < 1$, $N_{\mathfrak{a},\varepsilon}$ defined in (4.2) satisfies:*

$$N_{\mathfrak{a},\varepsilon} > C_{\mathfrak{a}}^{-}\varepsilon^{-1}.$$

Comparing the statements of Theorems 4.2 and Theorem 5.1, we see that the upper bound on $r_{\mathfrak{a},N}$ when $\langle \boldsymbol{\alpha} \rangle < 1$ is sharp. This however does not seem to be the case when $\langle \boldsymbol{\alpha} \rangle \ge 1$. In particular, when $\langle \boldsymbol{\alpha} \rangle > 1$, the upper bound reads $r_{\mathfrak{a},N} < c_{\mathfrak{a}}^{+}N^{-1/d_s}$, and for large N evidently $N^{-1/d_s} \gg N^{-1}$.

It is natural to ask whether in this case one of the bounds (the upper or the lower one) is not sharp. The answer to this question depends on the tree $\mathcal{T}$ and is two-fold:

- when there exists $i \ne j$ s.t. $\alpha_i = \alpha_j$ (i.e. in the presence of symmetries), it may happen that $\langle \boldsymbol{\alpha} \rangle \ge 1$, and yet $r_{\mathfrak{a},N} \le CN^{-1}$, with some $C > 0$. This is clarified in Appendix A.
- in the case when $\alpha_i \ne \alpha_j$ for $i \ne j$, our numerical experiments (cf. [8]) indicate that the upper bound provided by Theorem 4.2 is likely to be sharp.

5.2. Plan of the proof and of this section. The proofs of Theorems 5.1 and 5.2 will be similar in their ideas to the proofs of Theorems 4.2, 4.3. Recall in particular that the proof of Theorem 4.3 relies on the following ingredients:

1. an upper bound $r_{\mathfrak{a},N} \le \tilde{c}^+_{\mathfrak{a}} \Omega^{-1}_{\mathfrak{a},N}$, which shows that to guarantee that $r_{\mathfrak{a},N} < \varepsilon$, one takes $N_{\mathfrak{a},\varepsilon} \le N_*$ where $N_* = \min\{n : \tilde{c}^+_{\mathfrak{a}} \varepsilon^{-1} \le \Omega_{\mathfrak{a},N_*}\}$;
2. an observation that $N_* = P_{\mathfrak{a}}(\tilde{c}^+_{\mathfrak{a}} \varepsilon^{-1}) + 1$;
3. an upper bound on $P_{\mathfrak{a}}(\tilde{c}^+_{\mathfrak{a}} \varepsilon^{-1})$ (cf. Corollary 4.13).

In the proof of the lower bound we will repeat this reasoning but with inverse inequalities. We will make use of the auxiliary results that are counterparts of Proposition 4.4 and Theorem 4.8.

Plan of the section. This section is organized as follows. In Section 5.3, we prove the bound $r_{\mathfrak{a},N} \ge c\Omega^{-1}_{\mathfrak{a},N}$. Section 5.4 is dedicated to the proof of the lower bound for the number of the poles of $\mathbf{\Lambda}_{\mathfrak{a}}$ on an interval. In Section 5.5 we combine these two results to prove Theorem 5.2. In Section 5.6, we prove Theorem 5.1.

5.3. An implicit lower bound for $r_{\mathfrak{a},N}$. Recall that, cf. (4.5), $R_{\mathfrak{a},\omega} = \sum_{\omega_{\mathfrak{a},n} > \omega} \frac{a_{\mathfrak{a},n}}{\omega^2_{\mathfrak{a},n}}$. The following result shows that Proposition 4.4 is sharp.

PROPOSITION 5.3. *There exists $\tilde{c}^-_{\mathfrak{a}} > 0$, which depends on $\boldsymbol{\mu}, \boldsymbol{\alpha}$, s.t.*

$$R_{\mathfrak{a},\omega} \ge \tilde{c}^-_{\mathfrak{a}}\, \omega^{-1}, \text{ for all } \omega \ge 1. \tag{5.1}$$

Just like Proposition 4.4, the above result relies on the estimates on the Neumann traces of the eigenfunctions.

5.3.1. A lower bound for the Neumann traces of eigenfunctions. The result that follows shows the sharpness of the estimate of Lemma 4.5.

LEMMA 5.4. *There exists $\eta_* > 0$, s.t. for all $\eta > \eta_*$, there exists $C^-_\eta > 0$, s.t.*

$$\sum_{j:\, |\omega_{\mathfrak{a},j} - \omega| < \eta} (\partial_s \varphi_{\mathfrak{a},j}(M^*))^2 \omega^{-2}_{\mathfrak{a},j} \equiv \sum_{j:\, |\omega_{\mathfrak{a},j} - \omega| < \eta} a_{\mathfrak{a},j} \ge C^-_\eta, \text{ for all } \omega \ge 1. \tag{5.2}$$

The proof is based on ideas different from the ones used in the proof of Lemma 4.5.

Proof. First, let us remark the following. By (2.21, 2.22), see also Remark 2.8, because μ is piecewise-constant, the eigenfunctions of $\mathcal{A}_{\mathfrak{a}}$ satisfy $-\partial_s^2 \varphi_{\mathfrak{a},n} = \omega^2_{\mathfrak{a},n} \varphi_{\mathfrak{a},n}$ on $\Sigma_{0,0}$ (identified with $[0,1]$). Since $\varphi_{\mathfrak{a}}(M^*) = 0$, we have on $\Sigma_{0,0}$,

$$\varphi_{\mathfrak{a},n}(s) = C_{\mathfrak{a},n} \sin \omega_{\mathfrak{a},n} s, \quad C_{\mathfrak{a},n} \ge 0.$$

We can easily relate $C_{\mathfrak{a},n}$ to the coefficients $a_{\mathfrak{a},n} = (\partial_s \varphi_{\mathfrak{a},n}(0))^2 \omega^{-2}_{\mathfrak{a},n}$:

$$\varphi_{\mathfrak{a},n}(s) = a^{\frac{1}{2}}_{\mathfrak{a},n} \sin \omega_{\mathfrak{a},n} s. \tag{5.3}$$

Let us now prove the desired result. Let $\omega \ge 1$, and let $v_\omega : \mathcal{T} \to \mathbb{R}$ be s.t. $v_\omega(s) = \sin(\omega s)$ on $\Sigma_{0,0}$ and $v_\omega(s) = 0$ otherwise. Because $v_\omega \in \mathrm{L}^2_\mu(\mathcal{T})$, and $\{\varphi_{\mathfrak{a},n}\}_{n \in \mathbb{N}_*}$ is a Schauder basis in $\mathrm{L}^2_\mu(\mathcal{T})$, the following series converges in $\mathrm{L}^2_\mu(\mathcal{T})$:

$$v_\omega = \sum_{n=0}^{\infty} \gamma^\omega_{\mathfrak{a},n} \varphi_{\mathfrak{a},n}, \quad \gamma^\omega_{\mathfrak{a},n} = \int_0^1 v_\omega(s) \varphi_{\mathfrak{a},n}(s) ds. \tag{5.4}$$

Moreover, for all $\eta > 0$,

$$\begin{aligned} \|v_\omega\|^2_{\mathrm{L}^2_\mu(\mathcal{T})} &= \sum_{n=0}^{\infty} |\gamma^\omega_{\mathfrak{a},n}|^2 = \mathcal{S}^{\mathfrak{a}}_{\omega,\eta} + \mathcal{I}^{\mathfrak{a}}_{\omega,\eta}, \\ \mathcal{S}^{\mathfrak{a}}_{\omega,\eta} &= \sum_{n:\,|\omega_{\mathfrak{a},n}-\omega|<\eta} |\gamma^\omega_{\mathfrak{a},n}|^2, \quad \mathcal{I}^{\mathfrak{a}}_{\omega,\eta} = \sum_{n:\,|\omega_{\mathfrak{a},n}-\omega|\geq\eta} |\gamma^\omega_{\mathfrak{a},n}|^2. \end{aligned} \tag{5.5}$$

To prove (5.2), we will prove the three inequalities (with constants uniform in η, ω):

$$(a)\ |\mathcal{S}^{\mathfrak{a}}_{\omega,\eta}| \leq \sum_{|\omega_{\mathfrak{a},n}-\omega|<\eta} a_{\mathfrak{a},n}, \qquad (b)\ |\mathcal{I}^{\mathfrak{a}}_{\omega,\eta}| \leq C_I \eta^{-1}, \qquad (c)\ \|v_\omega\|^2 \geq \frac{1}{4}. \tag{5.6}$$

Combining these bounds in (5.5) then yields

$$\frac{1}{4} \leq \sum_{|\omega_{\mathfrak{a},n}-\omega|<\eta} a_{\mathfrak{a},n} + C_I \eta^{-1},$$

and the lower bound (5.2) follows by taking $\eta > \eta_*$, with η_* sufficiently large. It remains to prove (5.6). We start by rewriting $\gamma_{\mathfrak{a},n}$ using (5.3):

$$\gamma^\omega_{\mathfrak{a},n} = a^{\frac{1}{2}}_{\mathfrak{a},n} \int_0^1 \sin(\omega s)\sin(\omega_{\mathfrak{a},n} s)ds = \frac{a^{\frac{1}{2}}_{\mathfrak{a},n}}{2}\left(-\frac{\sin(\omega-\omega_{\mathfrak{a},n})}{\omega-\omega_{\mathfrak{a},n}} + \frac{\sin(\omega+\omega_{\mathfrak{a},n})}{\omega+\omega_{\mathfrak{a},n}}\right). \tag{5.7}$$

Because $|\sin(x)/x| \leq 1$, and also $\omega + \omega_{\mathfrak{a},n} > |\omega_{\mathfrak{a},n} - \omega|$, the above yields

$$|\gamma^\omega_{\mathfrak{a},n}| \leq a^{\frac{1}{2}}_{\mathfrak{a},n} \min\left(\frac{1}{|\omega-\omega_{\mathfrak{a},n}|}, 1\right). \tag{5.8}$$

Proof of (5.6)-(a). By (5.8), $|\gamma^\omega_{\mathfrak{a},n}| \leq a^{\frac{1}{2}}_{\mathfrak{a},n}$, hence the desired bound is immediate from the definition (5.5) of $\mathcal{S}^{\mathfrak{a}}_{\omega,\eta}$.
Proof of (5.6)-(b). Let us first prove the desired bound for $\omega = W \in \mathbb{N}_*$, and $\eta = k \in \mathbb{N}_*$, and next argue about extending it to the general case. With the definition (5.5) of $\mathcal{I}^{\mathfrak{a}}_{\omega,\eta}$,

$$\begin{aligned} \mathcal{I}^{\mathfrak{a}}_{W,k} &= \sum_{\omega_{\mathfrak{a},n}\leq W-k} |\gamma^\omega_{\mathfrak{a},n}|^2 + \sum_{\omega_{\mathfrak{a},n}\geq W+k} |\gamma^\omega_{\mathfrak{a},n}|^2 \overset{(5.8)}{\leq} I^-_{W,k} + I^+_{W,k}, \text{ where} \\ I^-_{W,k} &= \sum_{\omega_{\mathfrak{a},n}\leq W-k} \frac{a_{\mathfrak{a},n}}{(W-\omega_{\mathfrak{a},n})^2}, \quad I^+_{W,k} = \sum_{\omega_{\mathfrak{a},n}\geq W+k} \frac{a_{\mathfrak{a},n}}{(\omega_{\mathfrak{a},n}-W)^2}. \end{aligned} \tag{5.9}$$

To bound each of the above terms, we will make use of Lemma 4.5. The first term in (5.9) vanishes when $W \leq k$; otherwise, it can be rewritten as follows:

$$I^-_{W,k} = \sum_{\ell=0}^{W-k-1} \sum_{\ell\leq\omega_{\mathfrak{a},n}<\ell+1} \frac{a_{\mathfrak{a},n}}{(W-\omega_{\mathfrak{a},n})^2} \leq \sum_{\ell=0}^{W-k-1} \frac{1}{(W-\ell-1)^2} \sum_{\ell\leq\omega_{\mathfrak{a},n}<\ell+1} a_{\mathfrak{a},n}.$$

Using Lemma 4.5 with $\eta = 1$ we obtain the following bound

$$I^-_{W,k} \leq C_1^+ \sum_{\ell=0}^{W-k-1} (W-\ell-1)^{-2} = C_1^+ \sum_{\ell=k}^{W-1} \frac{1}{\ell^2} \leq \frac{C_1}{k},$$

where C_1 does not depend on W or k. Altogether, we have

$$I^-_{W,k} = \begin{cases} 0, & \text{if } k \geq W, \\ \leq C_1 k^{-1}, & \text{otherwise.} \end{cases} \tag{5.10}$$

It remains to bound the second term in (5.9); we use the same ideas:

$$I^+_{W,k} = \sum_{\ell=W+k}^{\infty} \sum_{\ell \leq \omega_{\mathfrak{a},n} < \ell+1} \frac{a_{\mathfrak{a},n}}{(\omega_{\mathfrak{a},n} - W)^2} \leq \sum_{\ell=W+k}^{\infty} \frac{1}{(\ell - W)^2} \sum_{\ell \leq \omega_{\mathfrak{a},n} < \ell+1} a_{\mathfrak{a},n}$$
$$\leq C_1^+ \sum_{\ell=W+k}^{\infty} \frac{1}{(\ell - W)^2}, \text{ by Lemma 4.5.}$$

We then obtain a uniform in W bound, with C_2 independent of W and k:

$$I^+_{W,k} < C_2 k^{-1}. \tag{5.11}$$

Combining (5.10) and (5.11) into (5.9) yields the desired bound

$$|\mathcal{I}^{\mathfrak{a}}_{W,k}| \leq (C_1 + C_2) k^{-1}. \tag{5.12}$$

When $\omega, \eta \notin \mathbb{N}$, and $\omega, \eta > 1$, one has a bound $\mathcal{I}^{\mathfrak{a}}_{\omega,\eta} \leq I^-_{\lceil\omega\rceil,\lfloor\eta\rfloor} + I^+_{\lfloor\omega\rfloor,\lfloor\eta\rfloor}$, which, with (5.10) and (5.11), yields the desired bound (5.6)-(b).
Proof of (5.6)-(c). The proof is straightforward:

$$\|v\|^2_{\mathrm{L}^2_\mu(\mathcal{T})} \equiv \int_0^1 \sin^2(\omega s) ds = \frac{1}{2} - \frac{\sin(2\omega s)}{4\omega} \geq \frac{1}{4}, \text{ when } \omega \geq 1. \qquad \square$$

5.3.2. Proof of Proposition 5.3. Let us fix $\eta := \ell \geq 1$ integer and large enough, so that the conditions of Lemma 5.4 hold true. Because $\omega \mapsto R_{\mathfrak{a},\omega}$ is decreasing and piecewise-constant, it suffices to prove the result for $\omega = M \in \mathbb{N}$ sufficiently large. Let us assume that $M > \ell$. Recall that $R_{\mathfrak{a},M} = \sum_{n:\, \omega_{\mathfrak{a},n} > M} \frac{a_{\mathfrak{a},n}}{\omega^2_{\mathfrak{a},n}}$.

Because we would like to apply the result of Lemma 5.4, we use the same trick as in the proof of Proposition 4.4, namely, we split the interval $[M, \infty)$ into smaller intervals of the width 2ℓ that do not intersect:

$$[M, \infty) = [(M+\ell) - \ell, (M+\ell) + \ell) \cup [(M+3\ell) - \ell,\ (M+3\ell) + \ell) \cup \dots$$

With a shortened notation, $I_k := [(M + k\ell) - \ell,\ (M + k\ell) + \ell)$, we rewrite

$$R_{\mathfrak{a},M} = \sum_{k=1}^{\infty} \sum_{\omega_{\mathfrak{a},n} \in I_{2k-1}} \frac{a_{\mathfrak{a},n}}{\omega^2_{\mathfrak{a},n}} \geq \sum_{k=1}^{\infty} \frac{1}{\left((M + (2k-1)\ell) - \ell\right)^2} \sum_{\omega_{\mathfrak{a},n} \in I_{2k-1}} a_{\mathfrak{a},n}. \tag{5.13}$$

By Lemma 5.4, applied with $\omega = M + (2k-1)\ell$ and $\eta = \ell$,

$$\sum_{\omega_{\mathfrak{a},n} \in I_{2k-1}} a_{\mathfrak{a},n} \geq C^-_\ell > 0 \text{ for all } k \in \mathbb{N}_*.$$

Inserting the above bound into (5.13) yields the desired bound:

$$R_{\mathfrak{a},M} \geq C^-_\ell \sum_{k=1}^{\infty} \frac{1}{\left(M + (2k-1)\ell\right)^2} \geq C^-_\ell \sum_{k:\, (2k-1)\ell \geq M} \frac{1}{\left(M + (2k-1)\ell\right)^2}$$
$$\geq C^-_\ell \sum_{k:\, (2k-1)\ell \geq M} \frac{1}{4(2k-1)^2} \geq \frac{\widetilde{c}}{M}, \qquad \text{with some } \widetilde{c} > 0.$$

5.4. A lower bound for the number of poles on an interval. In this section we will state a lower bound for $P_{\mathfrak{a}}(\lambda) = \#\{n : \Omega_{\mathfrak{a},n} < \lambda\}$.

THEOREM 5.5. *$P_{\mathfrak{a}}(\lambda)$ defined in (4.42) satisfies*

$$P_{\mathfrak{a}}(\lambda) > \lfloor (2\pi)^{-1}\lambda \rfloor, \quad \text{for all } \lambda > 0. \tag{5.14}$$

The above result does not follow from the asymptotics of Theorem 4.8, since it also encodes some information about the eigenvalue multiplicity and the Neumann traces of the eigenfunctions, see the discussion in Section 2.5. It shows that inside the interval $(0, \lambda)$ there are at least $O\left(\lambda^{\frac{1}{2}}\right)$ distinct eigenvalues, s.t. the corresponding eigenfunctions $\varphi_{\mathfrak{a},n}$ satisfy $\partial_s \varphi_{\mathfrak{a},n}(M^*) \neq 0$. The proof of this result is simple, and relies on Lemma 5.3 from [9], which shows that $\mathbf{\Lambda}_{\mathfrak{a}}(\omega)$ satisfies a certain non-linear equation.

LEMMA 5.6 (Lemma 5.3 from [9]). *For any $\omega \in \mathbb{C} \setminus \mathbb{R}$,*

$$\mathbf{\Lambda}_{\mathfrak{a}}(\omega) = \omega \frac{\cos\omega \mathbf{F}_{\mathfrak{a}}(\omega) - \omega \sin\omega}{\sin\omega \mathbf{F}_{\mathfrak{a}}(\omega) + \omega\cos\omega}, \qquad \mathbf{F}_{\mathfrak{a}}(\omega) = \sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i} \mathbf{\Lambda}_{\mathfrak{a}}(\alpha_i \omega), \quad \mathfrak{a} \in \{\mathfrak{d}, \mathfrak{n}\}. \tag{5.15}$$

We will also need the following result which is an expression of a general monotonicity property of Herglotz functions.

LEMMA 5.7. *Let $\mathbf{F}_{\mathfrak{a}}$ be defined in (5.15). Then $\mathbf{F}'_{\mathfrak{a}}(\omega) \leq 0$ for all $\omega > 0$ s.t. ω is not a pole of $\mathbf{F}_{\mathfrak{a}}(\omega)$.*

Proof. Evidently it is sufficient to show that for $\omega > 0$, $\mathbf{\Lambda}'_{\mathfrak{a}}(\omega) < 0$. Using the representation (2.27), one proves that $\mathbf{\Lambda}'_{\mathfrak{a}}(\omega)$ equals to the series (which converges uniformly on compact subsets of $\mathbb{C}$ not containing the poles of $\mathbf{\Lambda}_{\mathfrak{a}}$ (this can be proven by comparing the series to the uniformly convergent series (2.27))):

$$\mathbf{\Lambda}'_{\mathfrak{a}}(\omega) = -\sum_{n=1}^{+\infty} \frac{2A_{\mathfrak{a},n}\Omega^2_{\mathfrak{a},n}\omega}{(\Omega^2_{\mathfrak{a},n} - \omega^2)^2}.$$

The above quantity is negative for $\omega > 0$, since $A_{\mathfrak{a},n} > 0$ for all $n \geq 1$. ∎

The above two results suffice for the proof of Theorem 5.5.

Proof of Theorem 5.5. To prove the desired result, we will show that on each interval $I_m := [m\pi, (m+1)\pi]$, $m \in \mathbb{N}_*$, the function $\mathbf{\Lambda}_{\mathfrak{a}}(\omega)$ has at least one pole. For this we will use the expression (5.15). Since $\mathbf{\Lambda}_{\mathfrak{a}}$ is meromorphic, it is also valid for $\omega \notin \mathbb{C} \setminus \mathbb{R}$, i.e. $\omega \in \mathbb{R} \setminus \mathcal{P}_{\mathfrak{a}}$ (outside of poles).

Step 1. Necessary and sufficient conditions for $\mathbf{\Lambda}_{\mathfrak{a}}(\Omega_0) = \infty$. Examining (5.15) reveals that $\omega = \Omega_0 \in \mathbb{R}^+_*$ is a pole of $\mathbf{\Lambda}_{\mathfrak{a}}(\omega)$ iff one of the following holds true:

(C1) $\mathbf{F}_{\mathfrak{a}}(\omega)$ has a pole in Ω_0, and $\lim_{\omega\to\Omega_0} \mathbf{\Lambda}_{\mathfrak{a}}(\omega) = \infty$, in other words, with (5.15),

$$\lim_{\omega\to\Omega_0} \omega \frac{\cos\omega - \omega\sin\omega \mathbf{F}^{-1}_{\mathfrak{a}}(\omega)}{\sin\omega + \omega\cos\omega \mathbf{F}^{-1}_{\mathfrak{a}}(\omega)} = \lim_{\omega\to\Omega_0} \omega \frac{\cos\omega}{\sin\omega} = \infty.$$

This is possible, if and only if, for some $k \in \mathbb{N}_*$,

$$\Omega_0 = k\pi \text{ and } \mathbf{F}_{\mathfrak{a}}(\Omega_0) = \infty. \tag{5.16}$$

(C2) $\mathbf{F}_{\mathfrak{a}}(\Omega_0) \neq \infty$, the denominator of (5.15) vanishes and the numerator does not:

$$(5.17) \quad (a)\ \sin\Omega_0 \mathbf{F}_{\mathfrak{a}}(\Omega_0) = -\Omega_0 \cos\Omega_0, \qquad (b)\ \cos\Omega_0 \mathbf{F}_{\mathfrak{a}}(\Omega_0) \neq \Omega_0 \sin\Omega_0.$$

Remark that $(a) \Rightarrow (b)$. To see this, assume the opposite: $(a) \not\Rightarrow (b)$, i.e.

$$(a)\ \sin\Omega_0 \mathbf{F}_{\mathfrak{a}}(\Omega_0) = -\Omega_0 \cos\Omega_0, \quad \text{and} \quad (b')\ \cos\Omega_0 \mathbf{F}_{\mathfrak{a}}(\Omega_0) = \Omega_0 \sin\Omega_0.$$

Evidently, as $\Omega_0 \neq 0$, and $\mathbf{F}_{\mathfrak{a}}(\Omega_0) \neq \infty$, for (a) to hold it is necessary that $\sin\Omega_0 \neq 0$. With (a), (b') is equivalent to $-\cos^2\Omega_0 = \sin^2\Omega_0$, hence a contradiction. Therefore, (5.17) is equivalent to

$$(5.18) \qquad \mathbf{F}_{\mathfrak{a}}(\Omega_0) = -\Omega_0 \left(\tan\Omega_0\right)^{-1}, \quad \Omega_0 \neq k\pi,\ k \in \mathbb{N}_*.$$

Step 2. Proof that on I_m $\mathbf{\Lambda}_{\mathfrak{a}}$ has at least one pole. We will prove the result by contradiction. Let us assume that $\mathbf{\Lambda}_{\mathfrak{a}}$ has no poles in I_m. In particular, this is possible iff (5.16) and (5.18) do not hold. I.e. $\mathbf{\Lambda}_{\mathfrak{a}}$ has no poles in I_m iff

$$(5.19) \qquad \mathbf{F}_{\mathfrak{a}}(m\pi) \neq \infty, \qquad \mathbf{F}_{\mathfrak{a}}((m+1)\pi) \neq \infty, \text{ and}$$

$$(5.20) \qquad \mathbf{F}_{\mathfrak{a}}(\omega) \neq f(\omega), \quad f(\omega) := -\omega(\tan\omega)^{-1} \text{ on } I_m.$$

It remains to consider two possibilities:
Case 1. $\mathbf{F}_{\mathfrak{a}}$ has no poles in I_m. By continuity of $\mathbf{F}_{\mathfrak{a}}$ and f, (5.20) holds if and only if $\mathbf{F}_{\mathfrak{a}} > f$ or $\mathbf{F}_{\mathfrak{a}} < f$ on $(m\pi, (m+1)\pi)$.

However, f is strictly growing on I_m, and

$$(5.21) \qquad \lim_{\omega \to m\pi+} f(\omega) = -\infty, \qquad \lim_{\omega \to (m+1)\pi-} f(\omega) = +\infty.$$

Hence $\mathbf{F}_{\mathfrak{a}} > f$ on I_m would mean that $\mathbf{F}_{\mathfrak{a}}(\omega)$ has a pole in $\omega = (m+1)\pi$, while $\mathbf{F}_{\mathfrak{a}} < f$ on I_m would imply that $\mathbf{F}_{\mathfrak{a}}(\omega)$ has a pole in $\omega = m\pi$. This contradicts (5.19).
Case 2. $\mathbf{F}_{\mathfrak{a}}$ has at least one pole inside I_m. We will consider the case when it has a single pole, while the case with multiple poles can be studied similarly.
With (5.19), we assume that $\mathbf{F}_{\mathfrak{a}}$ has a pole in Ω_0 with $\Omega_0 \in (m\pi, (m+1)\pi)$. With (5.20), on the interval $(m\pi, \Omega_0)$, $\mathbf{F}_{\mathfrak{a}} \neq f$. This is possible iff on $(m\pi, \Omega_0)$ either $\mathbf{F}_{\mathfrak{a}} > f$ or $\mathbf{F}_{\mathfrak{a}} < f$. Let us show that considering both possibilities leads to a contradiction:

- let us assume that $(m\pi, \Omega_0)$, $\mathbf{F}_{\mathfrak{a}} > f$. By Lemma 5.7, $\mathbf{F}'_{\mathfrak{a}} \leq 0$, and thus on the interval $(m\pi, \Omega_0)$, $\mathbf{F}_{\mathfrak{a}}$ is a continuous monotonically decreasing function that changes its value from $\mathbf{F}_{\mathfrak{a}}(m\pi)$ to $-\infty$. This implies that f has a pole in $\Omega_0 \in (m\pi, (m+1)\pi)$, and we arrive at the contradiction.
- let us assume that $(m\pi, \Omega_0)$, $\mathbf{F}_{\mathfrak{a}} < f$. By (5.21) $\lim_{\omega \to m\pi+} f(\omega) = -\infty$, thus $\mathbf{F}_{\mathfrak{a}}$ has a pole in $m\pi$, but this is impossible by (5.20).

Thus, inside each interval $I_m = [m\pi, (m+1)\pi]$ $\mathbf{\Lambda}_{\mathfrak{a}}(\omega)$ has at least one pole; hence, on each half-open interval $[m\pi, (m+2)\pi)$, $m \geq 0$, $\mathbf{\Lambda}_{\mathfrak{a}}(\omega)$ has at least one pole. Because the intervals of such form do not intersect, and the interval $(0, \lambda)$ contains at least $\lfloor \frac{\lambda}{2\pi} \rfloor$ such intervals (and thus poles), we conclude that $P_{\mathfrak{a}}(\lambda) \geq \lfloor \frac{\lambda}{2\pi} \rfloor$. ∎

5.5. Proof of Theorem 5.2. Evidently, it suffices to prove the desired result for all $\varepsilon > 0$ sufficiently small. We proceed just like in the proof of Theorem 4.3. First of all, recall that $r_{\mathfrak{a},N} = R_{\mathfrak{a},\Omega_{\mathfrak{a},N}}$, cf. (4.5), and, by Proposition 5.3, we have the following bound for all $\Omega_{\mathfrak{a},N} > 1$:

$$(5.22) \qquad r_{\mathfrak{a},N} > \widetilde{c}^{-}_{\mathfrak{a}}\, \Omega^{-1}_{\mathfrak{a},N}, \text{ with some } \widetilde{c}^{-}_{\mathfrak{a}} > 0.$$

By definition $N_{\mathfrak{a},\varepsilon} = \min\{N : r_{\mathfrak{a},N} < \varepsilon\}$, and therefore, for all ε sufficiently small,

$$\begin{aligned} N_{\mathfrak{a},\varepsilon} &\geq \min\{N : \widetilde{c}_{\mathfrak{a}}^{-}\, \Omega_{\mathfrak{a},N}^{-1} \leq \varepsilon\} = \min\{N : \Omega_{\mathfrak{a},N} \geq \widetilde{c}_{\mathfrak{a}}^{-}\, \varepsilon^{-1}\} \\ &\equiv \#\{n : \Omega_{\mathfrak{a},n} < \widetilde{c}_{\mathfrak{a}}^{-}\, \varepsilon^{-1}\} + 1 = P_{\mathfrak{a}}(\widetilde{c}_{\mathfrak{a}}^{-}\, \varepsilon^{-1}) + 1. \end{aligned}$$

By Theorem 5.5, for all ε sufficiently small, $N_{\mathfrak{a},\varepsilon} > C_{\mathfrak{a}}^{-}\varepsilon^{-1}$, with $C_{\mathfrak{a}}^{-} > 0$.

5.6. Proof of Theorem 5.1. We proceed just like in the proof of Theorem 4.2.

Let $\varepsilon > 0$. From the definition of $N_{\mathfrak{a},\varepsilon}$ (4.2), it follows that $r_{\mathfrak{a},N} > \varepsilon$ for all $N < N_{\mathfrak{a},\varepsilon}$. From Theorem 5.2, we know that $N_{\mathfrak{a},\varepsilon} > C_{\mathfrak{a}}^{-}\varepsilon^{-1}$ (which is > 1 for ε small enough), and therefore $r_{\mathfrak{a},\lfloor C_{\mathfrak{a}}^{-}\varepsilon^{-1}\rfloor} > \varepsilon$. For all N sufficiently large there exists ε s.t. $N = \lfloor C_{\mathfrak{a}}^{-}\varepsilon^{-1}\rfloor$, and $\varepsilon > C_{\mathfrak{a}}^{-}N^{-1}$. Therefore, $r_{\mathfrak{a},N} > C_{\mathfrak{a}}^{-}N^{-1}$.

6. Conclusions and Open Questions. In this work, we have presented a refined error analysis for high-order approximated transparent boundary conditions for the weighted wave equation on a self-similar one-dimensional fractal tree. This approach is an alternative to the convolution quadrature [6] and is based on the truncation of the meromorphic series representing the symbol of the DtN operator. The complexity of the method depends on the number of poles in the truncated series; we have presented estimates on the number of poles, required to achieve a desired accuracy ε, based on the analysis of the eigenfunctions and eigenvalues of the weighted Laplacian on the fractal tree. Our future efforts are directed towards improving the convergence of the technique, based on approximation of the remainder of the meromorphic series.

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Appendix A. Behaviour of $r_{\mathfrak{a},N}$ in the presence of symmetries. Let us consider the tree $\mathcal{T}$ with the length ratios $\boldsymbol{\alpha}$ s.t. there exists $i \neq j$, s.t. $\alpha_i = \alpha_j$. We will call such a tree a symmetric tree.

In particular, let us assume that the length ratios are given by

$$\boldsymbol{\alpha} = (\alpha_0^{(0)}, \alpha_0^{(1)}, \ldots, \alpha_0^{(n_0)}, \alpha_1^{(0)}, \ldots, \alpha_1^{(n_1)}, \ldots, \alpha_{p-1}^{(0)}, \ldots \alpha_{p-1}^{(n_{p-1})}), \quad n_i \geq 0, \tag{A.1}$$

such that for all $k = 0, \ldots, p-1$, $i = 0, \ldots, n_{p-1}$, it holds that

$$\alpha_k^{(i)} = \widetilde{\alpha}_k, \text{ and } \widetilde{\alpha}_k \neq \widetilde{\alpha}_j,\ k \neq j.$$

Similarly we define the vector $\boldsymbol{\mu} = (\mu_0^{(0)}, \mu_0^{(1)}, \ldots \mu_{p-1}^{(n_{p-1})})$ (without any constraints but the positivity of the coefficients). Temporary, in this appendix, the symbol of the reference DtN operator associated to the tree with the parameters $\boldsymbol{\alpha}, \boldsymbol{\mu}$ will be denoted by $\boldsymbol{\Lambda}_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}}$. Similarly, we will use the notation $r_{\mathfrak{a},N}^{\boldsymbol{\alpha},\boldsymbol{\mu}}$, $N_{\mathfrak{a},\varepsilon}^{\boldsymbol{\alpha},\boldsymbol{\mu}}$, $\rho_{\mathfrak{a},\varepsilon}^{\boldsymbol{\alpha},\boldsymbol{\mu}}$, $P_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}}$.

Given (A.1), let us define the vector of the values $\widetilde{\alpha}_k$ and the associated quantity $\widetilde{d}_s$:

$$\widetilde{\boldsymbol{\alpha}} = (\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_{p-1}), \quad \text{and } \widetilde{d}_s \in \mathbb{R}_*^+ \text{ s.t. } \sum \widetilde{\alpha}_j^{\widetilde{d}_s} = 1, \text{ cf. (4.1)}. \tag{A.2}$$

Remark that the above quantity does not coincide with the Minkowski dimension d_s of the tree $\mathcal{T}$. More precisely, because $\sum_j \sum_{k=0}^{n_j} \alpha_j^{(k)} = \sum (n_j+1)\widetilde{\alpha}_j \geq \sum \widetilde{\alpha}_j$, we have

$$\sum_{k=0}^{n_j} (\alpha_j^{(k)})^{\widetilde{d}_s} = \sum (n_j+1)^{\widetilde{d}_s} \widetilde{\alpha}_j^{\widetilde{d}_s} \geq \sum \widetilde{\alpha}_j^{\widetilde{d}_s} = 1.$$

Therefore,

$$\langle \widetilde{\boldsymbol{\alpha}} \rangle \leq \langle \boldsymbol{\alpha} \rangle, \qquad \text{and} \qquad \widetilde{d}_s \leq d_s. \tag{A.3}$$

A.1. Two principal results. As mentioned in Section 5.1, we have the following improvement of the upper bound of Theorem 4.2 for symmetric trees.

PROPOSITION A.1. *For some $c_{\mathfrak{a}}^+$ depending only on $\boldsymbol{\alpha}$, $\boldsymbol{\mu}$, the following holds true: for all $N \geq 2$, $r_{\mathfrak{a},N}^{\boldsymbol{\alpha},\boldsymbol{\mu}}$ satisfies:*

- *if $\langle \widetilde{\boldsymbol{\alpha}} \rangle < 1$ ($\widetilde{d}_s < 1$), $r_{\mathfrak{a},N}^{\boldsymbol{\alpha},\boldsymbol{\mu}} \leq c_{\mathfrak{a}}^+ \, N^{-1}$.*
- *if $\langle \widetilde{\boldsymbol{\alpha}} \rangle = 1$ ($\widetilde{d}_s = 1$), $r_{\mathfrak{a},N}^{\boldsymbol{\alpha},\boldsymbol{\mu}} \leq c_{\mathfrak{a}}^+ \, N^{-1} \log N$.*
- *if $\langle \widetilde{\boldsymbol{\alpha}} \rangle > 1$ ($\widetilde{d}_s > 1$), $r_{\mathfrak{a},N}^{\boldsymbol{\alpha},\boldsymbol{\mu}} \leq c_{\mathfrak{a}}^+ \, N^{-1/\widetilde{d}_s}$.*

The above proposition is very similar to Theorem 4.2. The difference is that in Theorem 4.2 it is the whole vector $\boldsymbol{\alpha}$ that plays a role, while the above result involves only the vector of the 'non-repeating; values $\widetilde{\boldsymbol{\alpha}}$. With (A.3), we see that the error indicator may decrease faster than predicted by Theorem 4.2 in the presence of symmetries.

REMARK A.2. *The simplest illustration to the statement of Proposition A.1 is given by $\mathcal{T}$ s.t. $\boldsymbol{\alpha} = (\alpha_0^{(0)}, \ldots, \alpha_0^{(p-1)})$, i.e. $\widetilde{\boldsymbol{\alpha}} = (\widetilde{\alpha}_0)$. In this case $d_s = \log p / \log \widetilde{\alpha}_0^{-1}$, while $\widetilde{d}_s = 1/\log \widetilde{\alpha}_0^{-1}$ (< 1). Independently of the value of p, by Proposition A.1, $r_{\mathfrak{a},N}^{\boldsymbol{\alpha},\boldsymbol{\mu}} \leq c_{\mathfrak{a}}^+ \, N^{-1}$.*

In a similar manner, the counterpart of Theorem 4.3 in the symmetric case reads.

PROPOSITION A.3. *There exists $C_{\mathfrak{a}}^+ > 0$, depending only on $\boldsymbol{\mu}$, $\boldsymbol{\alpha}$, such that, for all $0 < \varepsilon < 1/2$, $N_{\mathfrak{a},\varepsilon}^{\boldsymbol{\alpha},\boldsymbol{\mu}}$ satisfies:*

- *if $\langle \widetilde{\boldsymbol{\alpha}} \rangle < 1$ ($\widetilde{d}_s < 1$), $N_{\mathfrak{a},\varepsilon}^{\boldsymbol{\alpha},\boldsymbol{\mu}} \leq C_{\mathfrak{a}}^+ \, \varepsilon^{-1}$.*
- *if $\langle \widetilde{\boldsymbol{\alpha}} \rangle = 1$ ($\widetilde{d}_s = 1$), $N_{\mathfrak{a},\varepsilon}^{\boldsymbol{\alpha},\boldsymbol{\mu}} \leq C_{\mathfrak{a}}^+ \, \varepsilon^{-1} \log \varepsilon^{-1}$.*
- *if $\langle \widetilde{\boldsymbol{\alpha}} \rangle > 1$ ($\widetilde{d}_s > 1$), $N_{\mathfrak{a},\varepsilon}^{\boldsymbol{\alpha},\boldsymbol{\mu}} \leq C_{\mathfrak{a}}^+ \, \varepsilon^{-\widetilde{d}_s}$.*

A.2. Main idea of the proof and auxiliary results. Like in Section 4, we first prove Proposition A.3, and next Proposition A.1. Recall that the proof of Theorem 4.3 (whose refinement is given by Proposition A.3) relies on Corollary 4.13 about the upper bound for the pole counting function $P_{\mathfrak{a}}(\lambda)$. The improvement in the bounds of Proposition A.3 thus stems from the improved version of Corollary 4.13.

PROPOSITION A.4. *With $C_{\mathfrak{a}} > 0$, $\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$, depending on $\boldsymbol{\alpha}$, $\boldsymbol{\mu}$, it holds:*

- *if $\langle \widetilde{\boldsymbol{\alpha}} \rangle < 1$ ($\widetilde{d}_s < 1$), then $P_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}}(\lambda) \leq C_{\mathfrak{a}} \lambda$.*
- *if $\langle \widetilde{\boldsymbol{\alpha}} \rangle = 1$ ($\widetilde{d}_s = 1$), then $P_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}}(\lambda) \leq C_{\mathfrak{a}} \lambda \log \lambda$.*
- *if $\langle \widetilde{\boldsymbol{\alpha}} \rangle > 1$ ($\widetilde{d}_s > 1$), then $P_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}}(\lambda) \leq C_{\mathfrak{a}} \lambda^{\widetilde{d}_s}$.*

To prove this proposition, we will need the following refinement of Lemma 5.6, which we repeat for the convenience of the reader below.

THEOREM A.5 (Lemma 5.3 in [9], Lemma 5.5, Corollary 5.6 in [9]). *The symbol of the reference DtN operator $\boldsymbol{\Lambda}(\omega) = \boldsymbol{\Lambda}_{\mathfrak{a}}(\omega)$, $\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$, $\boldsymbol{\Lambda} : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$, satisfies*

$$\boldsymbol{\Lambda}(\omega) = -\,\omega \, \frac{\omega \tan \omega - \mathbf{F}_{\alpha,\mu}(\omega)}{\tan \omega \mathbf{F}_{\alpha,\mu}(\omega) + \omega}, \qquad \mathbf{F}_{\alpha,\mu}(\omega) = \sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i} \, \boldsymbol{\Lambda}(\alpha_i \omega). \tag{A.4}$$

Moreover, $\boldsymbol{\Lambda}_{\mathfrak{a}}$ is the unique even solution of (A.4) analytic in the origin that satisfies

- *if* $\langle \boldsymbol{\mu}/\boldsymbol{\alpha} \rangle \leq 1$, $\boldsymbol{\Lambda}(0) = 0$, *for* $\mathfrak{a} \in \{\mathfrak{d}, \mathfrak{n}\}$.
- *if* $\langle \boldsymbol{\mu}/\boldsymbol{\alpha} \rangle > 1$ *and* $\langle \boldsymbol{\mu}\boldsymbol{\alpha} \rangle < 1$, $\boldsymbol{\Lambda}(0) = 1 - \langle \boldsymbol{\mu}/\boldsymbol{\alpha} \rangle^{-1}$ *if* $\mathfrak{a} = \mathfrak{d}$, *and* $\boldsymbol{\Lambda}(0) = 0$ *if* $\mathfrak{a} = \mathfrak{n}$.
- *if* $\langle \boldsymbol{\mu}\boldsymbol{\alpha} \rangle \geq 1$, $\boldsymbol{\Lambda}(0) = 1 - \langle \boldsymbol{\mu}/\boldsymbol{\alpha} \rangle^{-1}$, *for* $\mathfrak{a} \in \{\mathfrak{d}, \mathfrak{n}\}$.

Proof of Proposition A.4. The proof of this proposition relies on the following observation: $\boldsymbol{\Lambda}_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}} = \boldsymbol{\Lambda}_{\mathfrak{a}}^{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\mu}}}$, with $\widetilde{\boldsymbol{\alpha}}$ defined in (A.2) and $\widetilde{\boldsymbol{\mu}}$ specified further.

Step 1. Given $\boldsymbol{\alpha}$, $\boldsymbol{\mu}$ (cf. (A.1)), let $\widetilde{\boldsymbol{\alpha}}$ be defined like in (A.2) and

$$\widetilde{\boldsymbol{\mu}} := (\widetilde{\mu}_0, \dots, \widetilde{\mu}_{p-1}), \quad \widetilde{\mu}_i := \sum_{k=0}^{n_i} \mu_i^{(k)}.$$

Let us prove that $\boldsymbol{\Lambda}_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}} = \boldsymbol{\Lambda}_{\mathfrak{a}}^{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\mu}}}$ with the help of Theorem A.5. Without loss of generality, we will consider the case $\langle \boldsymbol{\mu}\boldsymbol{\alpha} \rangle \geq 1$, while the remaining cases can be proven similarly.

Step 1.1. A problem satisfied by $\boldsymbol{\Lambda}_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}}$. First, remark that for all $\sigma \in \mathbb{R}$,

$$\sum_{i=0}^{p-1} \sum_{k=0}^{n_i} \mu_i^{(k)} \left(\alpha_i^{(k)}\right)^{\sigma} = \sum_{i=0}^{p-1} \widetilde{\alpha}_i^{\sigma} \sum_{k=0}^{n_i} \mu_i^{(k)} = \sum_{i=0}^{p-1} \widetilde{\alpha}_i^{\sigma} \widetilde{\mu}_i. \tag{A.5}$$

By Theorem A.5, $\boldsymbol{\Lambda}_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}} = \boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}(\omega)$ is the unique even solution, analytic in the origin, of

$$\begin{aligned} &\boldsymbol{\Lambda}(\omega) = -\,\omega\, \frac{\omega \tan\omega - \mathbf{F}_{\alpha,\mu}(\omega)}{\tan\omega \mathbf{F}_{\alpha,\mu}(\omega) + \omega}, \text{ where} \\ &\mathbf{F}_{\alpha,\mu}(\omega) = \sum_{i=0}^{p-1} \sum_{k=0}^{n_i} \frac{\mu_i^{(k)}}{\alpha_i^{(k)}} \boldsymbol{\Lambda}(\alpha_i^{(k)}\omega) = \sum_{i=0}^{p-1} \frac{\widetilde{\mu}_i}{\widetilde{\alpha}_i} \boldsymbol{\Lambda}(\widetilde{\alpha}_i \omega), \end{aligned} \tag{A.6}$$

(where the last identity is derived like (A.5)), with the condition in the origin

$$\boldsymbol{\Lambda}(0) = 1 - \langle \boldsymbol{\mu}/\boldsymbol{\alpha} \rangle^{-1} \overset{(A.5)}{=} 1 - \langle \widetilde{\boldsymbol{\mu}}/\widetilde{\boldsymbol{\alpha}} \rangle^{-1}. \tag{A.7}$$

Step 1.2. A problem satisfied by $\boldsymbol{\Lambda}_{\mathfrak{a}}^{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\mu}}}$. Let us remark that $\langle \boldsymbol{\mu}\boldsymbol{\alpha} \rangle = \langle \widetilde{\boldsymbol{\mu}}\widetilde{\boldsymbol{\alpha}} \rangle$ by (A.5).

By Theorem A.5, $\boldsymbol{\Lambda}_{\mathfrak{a}}^{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\mu}}}$ satisfies (A.6) with $\boldsymbol{\Lambda}(0) = 1 - \langle \widetilde{\boldsymbol{\mu}}/\widetilde{\boldsymbol{\alpha}} \rangle^{-1}$, i.e. (A.7). Because the solution to (A.6) and (A.7) is unique, we conclude that $\boldsymbol{\Lambda}_{\mathfrak{a}}^{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\mu}}} = \boldsymbol{\Lambda}_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}}$.

Step 2. An inequality for $P_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}}(\lambda)$. From the previous step it follows that $P_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}} = P_{\mathfrak{a}}^{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\mu}}}$. By (4.43), for all $\lambda > 0$, $P_{\mathfrak{a}}^{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\mu}}}(\lambda) \leq \rho_{\mathfrak{a}}^{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\mu}}}(\lambda^2)$, hence

$$P_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}}(\lambda) \leq \rho_{\mathfrak{a}}^{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\mu}}}(\lambda^2), \qquad \text{for all } \lambda > 0.$$

It remains to use the asymptotic estimates of Theorem 4.8 to bound $\rho_{\mathfrak{a}}^{\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\mu}}}$. ∎

A.3. Proof of Propositions A.3, A.1.

Proof of Proposition A.3. We proceed like in the proof of Theorem 4.3 to obtain (4.46), i.e. $N_{\mathfrak{a},\varepsilon}^{\boldsymbol{\alpha},\boldsymbol{\mu}} \leq P_{\mathfrak{a}}^{\boldsymbol{\alpha},\boldsymbol{\mu}}(C\varepsilon^{-1}) + 1$, and next conclude by using Proposition A.4. ∎

The proof of Proposition A.1 is similar to the proof of Theorem 4.2, the only difference being that instead of Theorem 4.3, we use Proposition A.3. We leave it to the reader.

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