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1 **LOCAL TRANSPARENT BOUNDARY CONDITIONS FOR WAVE**
2 **PROPAGATION IN FRACTAL TREES (II). ERROR AND**
3 **COMPLEXITY ANALYSIS**

4 PATRICK JOLY * AND MARYNA KACHANOVSKA *

5 **Abstract.** This work is dedicated to a refined error analysis of the high-order transparent
6 boundary conditions introduced in the companion work [8] for the weighted wave equation on a
7 fractal tree. The construction of such boundary conditions relies on truncating a meromorphic
8 series that approximate the symbol of the Dirichlet-to-Neumann operator. The error induced by the
9 truncation depends on the behaviour eigenvalues and the eigenfunctions of the weighted Laplacian
10 on a self-similar metric tree. In this work we quantify this error by computing the asymptotics
11 for eigenvalues and bounds for Neumann traces of the eigenfunctions. We prove sharpness of the
12 obtained bounds for a class of self-similar trees.

13 **Key words.** Wave equation, fractal, metric graph, counting function, Weyl asymptotics

14 **AMS subject classifications.** 35P15, 34B45, 35L05

15 **1. Introduction.** In the field of medical diagnostics, sound propagation in a
16 human respiratory tract is used to detect various lung diseases, see e.g. the Audible
17 Human Project [1, 17]. The underlying physical process is highly complex, thus, for
18 the respective numerical simulations one often uses simplified models. One of such
19 models, which we study in this work, describes wave propagation inside a bronchiolar
20 tree, and is based on the following two assumptions: 1) the bronchiolar tree is self-
21 similar [15, 5, 16]; 2) the thickness of the bronchioles tends to zero. It is described
22 [10, 18] by the weighted 1D wave equation on a fractal tree with infinitely many edges.

23 To perform numerical simulations on such a tree, it is necessary to be able to
24 truncate the computational domain. To our knowledge, there exist three methods
25 of doing so, all of them based on approximating the Dirichlet-to-Neumann (DtN)
26 operator: local low-order approximate boundary conditions (BCs) [9, 18], convolution
27 quadrature based discrete transparent BCs [6], and local high-order approximate BCs
28 [8]. A refined error analysis of the latter method constitutes the subject of this article.

29 In [9], it was proven that the symbol (the Fourier-Laplace transform of the con-
30 volution kernel) of the DtN operator is a meromorphic function, which can be rep-
31 resented as a partial fraction expansion with infinitely many terms. Truncating this
32 expansion at finitely many terms results in a symbol of a convolution operator that is
33 local in the time domain, see [8]. While this process provides a numerically tractable
34 and stable realization of the DtN, it introduces an approximation error. This error is
35 controlled by a remainder of a convergent series, which, in turn, is related to the poles
36 and residues of the original partial fraction expansion [8]. These arguments prove
37 convergence of the method as the number of terms in the truncated series tends to
38 infinity, but do not provide information on the convergence rate.

39 The goal of this paper is to prove explicit bounds on the convergence rate of the
40 method of [8]. The principal idea is to exploit the connection of the poles and residues
41 of the partial fraction expansion of the symbol of the DtN with the eigenvalues and
42 eigenfunctions of a weighted Laplacian on a self-similar tree. This allows to relate the
43 convergence rate to the asymptotic estimates on the eigenvalues.

44 This article is organized as follows. Section 2 introduces the notations and the

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45 problem under consideration. In Section 3 we recall the method of approximated
 46 transparent boundary conditions from [8], and present the result about the conver-
 47 gence of the approximated BCs from [8]. The quantity controlling the error and
 48 depending on the remainder of the convergent series is referred to as the error indi-
 49 cator. Section 3.3 states the goals of the present work based on the results of the
 50 preceding sections. In Section 4 we present an upper bound for the error indicator,
 51 and in Section 5 a lower bound is studied. Section 6 is dedicated to conclusions.

52 **2. Problem setting.** This section is not novel and follows previous works [6, 8].

53 **2.1. Notation.** We consider an infinite (in terms of the number of edges) p -adic
 54 tree \mathcal{T} [9, Definiton 2.1]. Let us define it by construction. First of all, let, with $n \in \mathbb{N}$,
 55 \mathcal{G}^n be a set of edges, defined as follows: \mathcal{G}^0 contains a single edge ('root' edge), \mathcal{G}^1
 56 contains p edges (children of the root edge in \mathcal{G}^0 , i.e. edges which all share the same
 57 vertex with the root edge), and \mathcal{G}^{n+1} consists of all children edges (p^{n+1}) of all the
 58 edges $\Sigma \in \mathcal{G}^n$. The edges of \mathcal{T} are given by the collection $\bigcup_{n \in \mathbb{N}} \mathcal{G}^n$. The p^n edges
 59 belonging to \mathcal{G}^n will be denoted by

$$\S 0 \quad \Sigma_{n,k}, \quad k = 0, \dots, p^n - 1.$$

62 Each edge $\Sigma_{n,k}$ has p children

$$\S 3 \quad (2.1) \quad \Sigma_{n+1,pk+j} \in \mathcal{G}^{n+1}, \quad j = 0, \dots, p - 1.$$

65 The edges $\Sigma_{n+1,pk+j}$ share the vertex $M_{n,k}$ with the parent edge $\Sigma_{n,k}$ (see Figure 1).
 66 The root vertex is a vertex incident only to the root edge $\Sigma_{0,0}$, and is denoted by M^* .

67 We will study metric trees. This means that any edge $\Sigma_{n,k}$ can be identified with
 68 a segment of \mathbb{R} of length $\ell_{n,k}$; additionally, we assign to it a constant weight $\mu_{n,k} > 0$.
 All over the article, we assume that $\mu_{0,0} = 1$. In what follows, we will consider

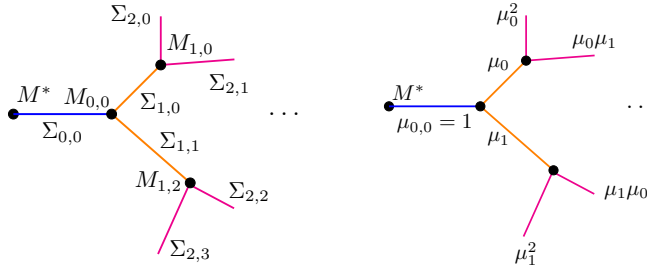


FIGURE 1. Left: A self-similar p -adic ($p = 2$) infinite tree. In blue we mark the edges that belong to \mathcal{G}^0 , in orange the edges of \mathcal{G}^1 , in magenta the edges of \mathcal{G}^2 . Right: Distribution of weights on the edges of a binary infinite self-similar tree.

69 self-similar (fractal) trees, see [9, Definition 2.3]. With such trees we associate

$$\S 1 \quad (2.2) \quad \boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{p-1}), \quad \boldsymbol{\mu} = (\mu_0, \dots, \mu_{p-1}), \quad \boldsymbol{\alpha}, \boldsymbol{\mu} \in \mathbb{R}_*^+.$$

73 The self-similarity means that the length/weight of the edge $\Sigma_{n+1,pk+j}$ is related to
 74 the length/weight of its parent edge $\Sigma_{n,k}$ according to the following:

$$\S 8 \quad \ell_{n+1,pk+j} = \alpha_j \ell_{n,k}, \quad \mu_{n+1,pk+j} = \mu_j \mu_{n,k}, \quad j = 0, \dots, p - 1.$$

77 We will assume that the tree is 'bounded', i.e.

$$78 \quad (2.3) \quad |\alpha|_\infty := \sup_{j=0, \dots, p-1} \alpha_j < 1.$$

79 Given $m \in \mathbb{N}_*$, by \mathcal{T}^m , we denote the subtree of \mathcal{T} made of the first m generations,
80 i.e., with an obvious abuse of notation:

$$81 \quad (2.4) \quad \mathcal{T} := \bigcup_{n=0}^{+\infty} \mathcal{G}^n, \quad \mathcal{T}^m := \bigcup_{\ell=0}^m \mathcal{G}^\ell.$$

82 Let us call a tree a 'reference tree' if the length of the root edge is $\ell_{0,0} = 1$. Unless
83 stated otherwise, we will always assume that \mathcal{T} is a **reference tree**.

84 **2.2. The weighted wave equation on fractal trees.** To write differential
85 equations on the tree \mathcal{T} , we need to introduce along each edge $\Sigma_{n,j} \in \mathcal{G}^n$ an abscissa
86 $s \in [0, \ell_{n,j}]$ in such a way that $\Sigma_{n,j}$ is oriented towards the generation \mathcal{G}^{n+1} . This
87 allows us, with an obvious abuse of notation, to write a function defined on \mathcal{T} as a
88 function of s , seen as the space variable.

89 We then define (formally) the weight function $s \mapsto \mu(s)$ on \mathcal{T} by $\mu(s) = \mu_{n,j}$,
90 $s \in \Sigma_{n,j}$. Denoting by t the time variable, and given a source term $f : \mathcal{T} \times \mathbb{R}^+ \rightarrow \mathbb{R}$,
91 we will look for $u : \mathcal{T} \times \mathbb{R}^+ \rightarrow \mathbb{R}$, defined as follows. With the notation $u_{n,j} = u|_{\Sigma_{n,j}}$,
92 u satisfies on each edge $\Sigma_{n,j}$ the 1D wave equation:

$$93 \quad (2.5) \quad \partial_t^2 u_{n,j} - \partial_s^2 u_{n,j} = f_{n,j} \quad \text{on } \Sigma_{n,j}, \quad j = 0, \dots, p^n - 1, \quad n \geq 0,$$

94 and the following continuity and Kirchoff conditions:

$$95 \quad (2.6) \quad \begin{aligned} u_{n,j}(M_{n,j}, t) &= u_{n+1, pj+k}(M_{n,j}, t), \quad k = 0, \dots, p-1, \\ \partial_s u_{n,j}(M_{n,j}, t) &= \sum_{k=0}^{p-1} \mu_k \partial_s u_{n+1, pj+k}(M_{n,j}, t), \quad j = 0, \dots, p^n - 1, \quad n \geq 0. \end{aligned}$$

97 Equations (2.5, 2.6) are completed by a homogeneous Dirichlet condition at the root
98 vertex M^* and vanishing initial conditions:

$$99 \quad (2.7) \quad u(M^*, t) = 0, \quad t > 0, \quad u(\cdot, 0) = \partial_t u(\cdot, 0) = 0, \quad \text{on } \mathcal{T}.$$

101 Additionally, we need to define the BCs at the 'infinite', fractal boundary of the tree.
102 This is the most delicate point of the model, which we will formalize in Section 2.3.

103 **2.3. Neumann and Dirichlet BVPs.** To provide a rigorous mathematical
104 formulation of the problem (2.5, 2.6, 2.7), we shall equip it with Neumann or Dirich-
105 let boundary conditions at the 'infinite' boundary of the tree. This is done in the
106 weak sense through the variational formulation on the problem, which requires the
107 introduction of appropriate weighted Sobolev spaces on \mathcal{T} .

108 **2.3.1. Sobolev Spaces.** For a function $v : \mathcal{T} \rightarrow \mathbb{C}$, we adopt the notation

$$109 \quad (2.8) \quad \int_{\mathcal{T}} \mu v := \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \mu_{n,k} \int_{\Sigma_{n,k}} v(s) ds$$

111 Let $C(\mathcal{T})$ be a space of continuous functions on \mathcal{T} , and

$$112 \quad C_0(\mathcal{T}) := \{v \in C(\mathcal{T}) : v = 0 \text{ on } \mathcal{T} \setminus \mathcal{T}^m, \quad \text{for some } m \in \mathbb{N}\}.$$

113 Let us introduce the following three spaces. First of all,

$$114 \quad L_\mu^2(\mathcal{T}) = \{v : v|_\Sigma \in L^2(\Sigma), \text{ for all } \Sigma \in \mathcal{T}; \|v\| < \infty\}, \quad \|v\|^2 = \|v\|_{L_\mu^2(\mathcal{T})}^2 := \int_{\mathcal{T}} \mu |v|^2.$$

115
116 We denote by (\cdot, \cdot) the corresponding Hermitian scalar product in $L_\mu^2(\mathcal{T})$. The weight-
117 ed Sobolev space H_μ^1 is defined as follows:

$$118 \quad H_\mu^1(\mathcal{T}) := \{v \in C(\mathcal{T}) \cap L_\mu^2(\mathcal{T}) : \|\partial_s v\| < \infty\}, \quad \|v\|_{H_\mu^1(\mathcal{T})}^2 = \|v\|^2 + \|\partial_s v\|^2.$$

120 Finally, the last space is an analogue of the classical H_0^1 -space:

$$121 \quad H_{\mu,0}^1(\mathcal{T}) := \overline{C_0(\mathcal{T}) \cap H_\mu^1(\mathcal{T})}^{\|\cdot\|_{H_\mu^1(\mathcal{T})}}.$$

123 Similarly, we define the corresponding spaces on a truncated tree \mathcal{T}^m . The associated
124 L_μ^2 -scalar product will be denoted by $(\cdot, \cdot)_{\mathcal{T}^m}$.

125 Finally, we will use the following result from [9].

126 **THEOREM 2.1.** *For $|\alpha|_\infty < 1$, the embedding of $H_\mu^1(\mathcal{T})$ in $L_\mu^2(\mathcal{T})$ is compact.*

127 **2.3.2. The BVP problems.** To define the Neumann (resp. Dirichlet) problem,
128 we introduce the space V_n (resp., V_∂) that differs from $H_\mu^1(\mathcal{T})$ (resp. $H_{\mu,0}^1(\mathcal{T})$) only
129 by the condition at the root vertex:

$$130 \quad V_n(\mathcal{T}) = \{v \in H_\mu^1(\mathcal{T}) : v(M^*) = 0\}, \quad V_\partial(\mathcal{T}) = \{v \in H_{\mu,0}^1(\mathcal{T}) : v(M^*) = 0\}.$$

132 In the sequel, we shall use the letter $\mathfrak{a} = \mathfrak{d}$ or \mathfrak{n} to distinguish between the Neumann
133 and Dirichlet problems, whose weak formulation is essentially the same, the only
134 difference lying in the choice of the trial and test space, i.e. $V_n(\mathcal{T})$ or $V_\partial(\mathcal{T})$.

135 **DEFINITION 2.2** (Neumann and Dirichlet problems).

136 *Find $u_\mathfrak{a} \in C^0(\mathbb{R}^+; V_\mathfrak{a}(\mathcal{T})) \cap C^1(\mathbb{R}^+; L_\mu^2(\mathcal{T}))$, s.t. $u_\mathfrak{a}(\cdot, 0) = \partial_t u_\mathfrak{a}(\cdot, 0) = 0$, and*

$$137 \quad (2.9) \quad (\partial_t^2 u_\mathfrak{a}, v) + (\partial_s u_\mathfrak{a}, \partial_s v) = (f, v), \quad \text{for all } v \in V_\mathfrak{a}(\mathcal{T}).$$

139 The above problems are well-posed whenever $f \in L_{loc}^1(\mathbb{R}_+^*; L_\mu^2(\mathcal{T}))$, [6, Theorem 2.1].
140 Surprisingly, in some cases the solutions u_n and u_∂ may coincide. To explain this
141 result in more detail, let us introduce

$$142 \quad (2.10) \quad \langle \mu \alpha \rangle := \sum_{i=0}^{p-1} \mu_i \alpha_i, \quad \langle \mu / \alpha \rangle := \sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i}, \quad (\text{since } |\alpha|_\infty < 1, \langle \mu \alpha \rangle < \langle \mu / \alpha \rangle).$$

144 **THEOREM 2.3** ([9]). *If $\langle \mu \alpha \rangle \geq 1$ or $\langle \mu / \alpha \rangle \leq 1$, the spaces $H_{\mu,0}^1(\mathcal{T})$ and $H_\mu^1(\mathcal{T})$
145 coincide, and thus $u_n = u_\partial$. Otherwise, $H_{\mu,0}^1(\mathcal{T}) \subsetneq H_\mu^1(\mathcal{T})$, and $u_n \neq u_\partial$.*

146 **2.4. Reduction to a finite tree.** In what follows, we assume that the source
147 term is supported on a finite number of generations:

$$148 \quad (2.11) \quad \exists m_0 > 1 \text{ such that } \forall t > 0, \quad \text{supp } f(\cdot, t) \subseteq \mathcal{T}^{m_0}.$$

149 To perform numerical simulations, we will truncate the tree \mathcal{T} to m generations with
150 $m > m_0$, in order to compute the restriction of the solution $u_\mathfrak{a}$ of (2.9) to \mathcal{T}^m . This
151 allows to consider (2.5, 2.6) only for $n \leq m$. The problem is then to find boundary
152 conditions at the end vertices of \mathcal{T}^m (except M^*), i.e. at points $\{M_{m,j}, 0 \leq j \leq$
153 $p^m - 1\}$, that should characterize the restriction of $u_\mathfrak{a}$ to \mathcal{T}^m . Such conditions are
154 called transparent boundary conditions.

155 **REMARK 2.4.** *In what follows, we consider the truncation parameter m to be fixed.*

156 For the problem (2.5, 2.6), the transparent BCs were constructed and characterized
157 in [9]. Before presenting them, let us introduce auxiliary notations.

158 **2.4.1. Notations.** We will denote by $\mathcal{K}(\partial_t)$ the following convolution operator:

$$159 \quad w(t) := (\mathcal{K}(\partial_t))v(t) = \int_0^t k(t-\tau)v(\tau)d\tau, \quad \text{where } v : t \in \mathbb{R}^+ \mapsto v(t) \in \mathbb{R}.$$

161 The integral in the above has to be understood in the sense of a convolution of
162 causal tempered distributions. The notation $\mathcal{K}(\partial_t)$ refers to the symbol $\mathcal{K}(\omega)$ of the
163 convolution operator (i.e. the Fourier-Laplace transform of the convolution kernel k):

$$164 \quad \mathcal{K}(\omega) = (\mathcal{F}k)(\omega) = \int_0^{+\infty} e^{i\omega t}k(t)dt, \quad \omega \in \mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im } z > 0\}.$$

166 Following the above convention, provided a scaling parameter $\delta > 0$, we will denote
167 by $\mathcal{K}(\delta \partial_t)$ the convolution operator with the symbol $\mathcal{K}(\delta\omega)$. In what follows we will
168 work with the space $H_{0,loc}^1(\mathbb{R}^+) := \{v \in H_{loc}^1(\mathbb{R}^+) : v(0) = 0\}$.

169 **2.4.2. Transparent boundary conditions.** With the above, the transparent
170 boundary condition at the node $M_{m,j}$ takes the form (see Remark 2.5)

$$171 \quad (2.12) \quad -\mu_{m,j} \partial_s u_a(M_{m,j}, t) = \mathcal{B}_{m,j}^a(\partial_t)u_a(M_{m,j}, t),$$

173 where the operators $\mathcal{B}_{m,j}^a \in \mathcal{L}(H_{0,loc}^1(\mathbb{R}^+), L_{loc}^2(\mathbb{R}^+))$, cf. [7], are related to a single
174 convolution operator $\Lambda_a(\partial_t)$ (defined further) via

$$175 \quad (2.13) \quad \mathcal{B}_{m,j}^a(\partial_t) = \mu_{m,j} \alpha_{m,j}^{-1} \sum_{k=0}^{p-1} \frac{\mu_k}{\alpha_k} \Lambda_a(\alpha_k \alpha_{m,j} \partial_t).$$

177 The operator $\Lambda_a(\partial_t) \in \mathcal{L}(H_{0,loc}^1(\mathbb{R}^+), L_{loc}^2(\mathbb{R}^+))$ is the *reference DtN operator*.

178 **2.4.3. Reference DtN operator.** In this section, \mathcal{T} is the reference tree, as
179 defined previously. The reference DtN is defined as follows:

$$180 \quad (2.14) \quad (\Lambda_a(\partial_t)g)(t) := -\partial_s u_g^a(M^*, t), \quad g \in H_{0,loc}^1(\mathbb{R}^+),$$

182 where $u_g^a \in C^1(\mathbb{R}^+; L_\mu^2(\mathcal{T}_r)) \cap C^0(\mathbb{R}^+; V_a)$, $u_g^a(0) = \partial_t u_g^a(0) = 0$, and

$$183 \quad (2.15) \quad (\partial_t^2 u_g^a, v) + (\partial_s u_g^a, \partial_s v) = 0, \quad \text{for all } v \in V_a(\mathcal{T}), \quad u_g^a(M^*, t) = g(t).$$

185

186 **REMARK 2.5.** *Since understanding the construction of the transparent BCs is not*
187 *required for reading this paper, we refer the interested reader to [9] for more details.*

2.4.4. The truncated problem. We consider the problem consisting in solving
(2.5, 2.6) for $n \leq m$ completed with the transparent conditions (2.12). To formulate
it, let us introduce the space $\mathbf{V}_\mu(\mathcal{T}^m)$ of restrictions to \mathcal{T}^m of functions in $V_a(\mathcal{T})$:

$$\mathbf{V}_\mu(\mathcal{T}^m) := \{v \in H_\mu^1(\mathcal{T}^m) : v(M^*) = 0\},$$

188 and the vectorial trace operators on $\Gamma_m := \{M_{m,j}, 0 \leq j \leq p^{m-1}\} \equiv \mathbb{R}^{p^m}$:

189 (2.16) $\gamma_m : \mathbf{V}_\mu(\mathcal{T}^m) \rightarrow \Gamma_m, \quad \gamma_m v = (v(M_{m,0}), \dots, v(M_{m,p^m-1})).$

191 Accordingly, we introduce the matrix operator

192 (2.17) $\mathcal{B}_m^\alpha(\partial_t) = \text{diag}(\mathcal{B}_{m,j}^\alpha(\partial_t))_{0 \leq j \leq p^m-1} \in \mathcal{L}(H_{0,loc}^1(\mathbb{R}^+, \Gamma_m), L_{loc}^2(\mathbb{R}^+, \Gamma_m)).$

194 The weak formulation of the problem reduced to the finite tree \mathcal{T}^m reads:

195 Find $u_m^\alpha \in C(\mathbb{R}^+; \mathbf{V}_\mu(\mathcal{T}^m)) \cap C^1(\mathbb{R}^+; L_\mu^2(\mathcal{T}^m))$, s.t. $u_m^\alpha(\cdot, 0) = \partial_t u_m^\alpha(\cdot, 0) = 0$, and

196 (2.18) $(\partial_t^2 u_m^\alpha, v)_{\mathcal{T}^m} + (\partial_s u_m^\alpha, \partial_s v)_{\mathcal{T}^m} + \int_{\Gamma_m} \mathcal{B}_m^\alpha(\partial_t) \gamma_m u_m^\alpha \gamma_m v = (f, v)_{\mathcal{T}^m},$

197 for any $v \in \mathbf{V}_\mu(\mathcal{T}^m)$. Here, for $\varphi \in C^0(\mathbb{R}^+; \mathcal{T}^m)$ and $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$, we use the notation:

200 (2.19) $\int_{\Gamma_m} f_1(\boldsymbol{\mu}) f_2(\boldsymbol{\alpha}) \gamma_m \varphi := \sum_{j=0}^{p^m-1} f_1(\mu_{m,j}) f_2(\alpha_{m,j}) \varphi(M_{m,j}).$

203 The problems (2.9) and (2.18) are equivalent in the following sense.

204 **THEOREM 2.6** (Theorem 2.6 in [6]). For all $f \in L_{loc}^1(\mathbb{R}_*^+; L_\mu^2(\mathcal{T}))$ satisfying
 205 (2.11), the problem (2.18) has a unique solution u_m^α , and $u_m^\alpha = u_\alpha|_{\mathcal{T}^m}$, where u_α
 206 solves (2.9).

207 Unlike (2.9), the problem (2.18) is suitable for numerical simulations (provided that
 208 the boundary term is computable), as it is posed on a tree with finitely many edges.

209 2.5. Characterization and properties of the reference DtN operator.

210 The resolution of (2.18) relies on approximating the operator $\mathcal{B}_m^\alpha(\partial_t)$, in other words,
 211 through (2.13), on approximating the reference DtN operator $\Lambda_\alpha(\partial_t)$. The goal of this
 212 section is to describe an important property of the reference DtN, which will be used
 213 in the construction and analysis of the transparent boundary conditions. For this, let
 214 us define the following Hermitian non-negative sesquilinear form on $V_\alpha := V_\alpha(\mathcal{T})$:

215 (2.20) $\forall (u, v) \in V_\alpha, \quad a(u, v) = (\partial_s u, \partial_s v) \equiv \sum_{n=0}^{\infty} \sum_{k=0}^{p^n-1} \int_{\Sigma_{n,k}} \mu_{n,k} \partial_s u \partial_s \bar{v} ds.$

217 It allows to define two unbounded self-adjoint operators \mathcal{A}_α , $\alpha = \mathbf{n}$ or \mathfrak{d} , in $L_\mu^2(\mathcal{T})$:

218 (2.21) $\mathcal{A}_\alpha : D(\mathcal{A}_\alpha) \rightarrow L_\mu^2(\mathcal{T}), \quad (\mathcal{A}_\alpha u, v) = a(u, v),$

219 (2.22) $D(\mathcal{A}_\alpha) = \{v \in V_\alpha : \text{with } C(v) > 0, |a(v, g)| < C(v) \|g\|_{L_\mu^2(\mathcal{T})}, \text{ for all } g \in V_\alpha\}.$

220 **REMARK 2.7.** When $p = 1$, the tree \mathcal{T} can be identified with an interval of a finite
 224 length, and the operator \mathcal{A}_α is a weighted Laplacian: $\mathcal{A}_\alpha v = \mu^{-1}(s) \partial_s(\mu(s) \partial_s v)$.

225 From Theorem 2.1 it follows that the spectrum of these operators is a pure point
 226 spectrum. We define the eigenvalues and normalized eigenfunctions as

227 (2.23) $\mathcal{A}_\alpha \varphi_{\alpha,n} = \omega_{\alpha,n}^2 \varphi_{\alpha,n}, \quad \|\varphi_{\alpha,n}\|_{L_\mu^2(\mathcal{T})} = 1, \quad 0 < \omega_{\alpha,1}^2 \leq \omega_{\alpha,2}^2 \leq \dots \rightarrow +\infty.$

229 Remark that the eigenvalues do not vanish, as shown in [9, Remark 1.20]. A spectral
 230 representation of the operator \mathcal{A}_α yields the following result.

231 THEOREM 2.8 (Proposition 1.23, discussion after (144) in [9]). *The symbol of*
 232 *the reference DtN operator $\Lambda_{\mathbf{a}}$, $\mathbf{a} \in \{\mathbf{n}, \mathfrak{d}\}$, satisfies*

$$233 \quad (2.24) \quad \Lambda_{\mathbf{a}}(\omega) = \Lambda_{\mathbf{a}}(0) - \omega^2 \sum_{n=1}^{+\infty} \frac{a_{\mathbf{a},n}}{(\omega_{\mathbf{a},n})^2 - \omega^2}, \quad a_{\mathbf{a},n} = \omega_{\mathbf{a},n}^{-2} \partial_s \varphi_{\mathbf{a},n}(M^*)^2.$$

234
 235 *The above series converges uniformly on compact subsets of $\mathbb{C} \setminus \{\pm\omega_{\mathbf{a},n}, n \geq 1\}$.*

236 In (2.23), the eigenvalues are repeated with their multiplicities, and, as a conse-
 237 quence, the series (2.24) may have repeated poles. Moreover, the unique continuation
 238 principle for eigenfunctions no longer holds, and it may happen that $\varphi_{\mathbf{a},n} \neq 0$, but
 239 $\partial_s \varphi_{\mathbf{a},n}(M^*) = 0$ (hence $a_{\mathbf{a},n} = 0$). For this reason, we introduce the sets

$$240 \quad (2.25) \quad \begin{cases} \mathcal{S}_{\mathbf{a}}^{app} := \{\omega^2 / \exists n \geq 1 \text{ s. t. } \omega^2 = \omega_{\mathbf{a},n}^2\}, & (\Leftrightarrow \text{ apparent poles}) \\ \mathcal{S}_{\mathbf{a}} := \{\omega^2 \in \mathcal{S}_{\mathbf{a}}^{app} / \sum_{\omega_{\mathbf{a},n}^2 = \omega^2} \partial_s \varphi_{\mathbf{a},n}(M^*)^2 > 0\}, & (\Leftrightarrow \text{ poles}) \end{cases}$$

241 and recast the set $\mathcal{S}_{\mathbf{a}}$ as

$$242 \quad (2.26) \quad \mathcal{S}_{\mathbf{a}} := \{\Omega_{\mathbf{a},k}^2, k \geq 1, \text{ with } 0 < \Omega_{\mathbf{a},1} < \Omega_{\mathbf{a},2} < \dots \rightarrow +\infty\},$$

243 so that $\mathcal{P}_{\mathbf{a}} = \{\pm \Omega_{\mathbf{a},k}, k \geq 1\}$ is the set of poles of $\Lambda_{\mathbf{a}}(\omega)$. All the above allows to
 244 rewrite (2.24) as follows:

$$245 \quad (2.27) \quad \Lambda_{\mathbf{a}}(\omega) = \Lambda_{\mathbf{a}}(0) - \omega^2 \sum_{k=1}^{+\infty} \frac{A_{\mathbf{a},k}}{\Omega_{\mathbf{a},k}^2 - \omega^2}, \quad \text{with } A_{\mathbf{a},k} > 0.$$

246
 247
 248 REMARK 2.9. *Using Theorem 2.8, it is possible to show that the series (2.27)*
 249 *converges uniformly on the compact subsets of $\mathbb{C} \setminus \mathcal{P}_{\mathbf{a}}$.*

250 **3. Approximate transparent boundary conditions: formulation and er-**
 251 **ror analysis.** As pointed out in the end of Section 2.4.4, the numerical resolution of
 252 (2.18) requires approximating the operators $\mathcal{B}_m^{\mathbf{a}}$, or, with (2.13), the operators $\Lambda_{\mathbf{a}}(\partial_t)$.
 253 This is the subject of this section. All over this section we fix $m \geq 1$.

254 3.1. A formulation of the method.

255 **3.1.1. Approximating the reference DtN operator.** The idea is to realize
 256 a local approximation of the operator $\Lambda_{\mathbf{a}}(\partial_t)$ through a rational approximation of its
 257 symbol. Of course, the most natural idea is to truncate the series (2.27) at N terms
 258 (with N becoming an approximation parameter):

$$259 \quad (3.1) \quad \Lambda_{\mathbf{a}}^N(\omega) = \Lambda_{\mathbf{a}}(0) - \omega^2 \sum_{k=1}^N \frac{A_{\mathbf{a},k}}{\Omega_{\mathbf{a},k}^2 - \omega^2},$$

260
 261 and to define $\Lambda_{\mathbf{a}}^N(\partial_t)$ as the operator whose symbol is $\Lambda_{\mathbf{a}}^N(\omega)$. This operator can be
 262 realized through the introduction of N auxiliary unknowns which are coupled to the
 263 argument of $\Lambda_{\mathbf{a}}^N(\partial_t)$ via ODEs. More precisely

$$264 \quad (3.2) \quad (\Lambda_{\mathbf{a}}^N(\partial_t)g)(t) = \Lambda_{\mathbf{a}}(0)g(t) + \sum_{k=1}^N A_{\mathbf{a},k} \frac{d\lambda_k}{dt}, \quad \text{where}$$

$$265 \quad \frac{d^2\lambda_k}{dt^2} + \Omega_{\mathbf{a},k}^2 \lambda_k = \frac{dg}{dt}, \quad \lambda_k(0) = \frac{d\lambda_k}{dt}(0) = 0.$$

266

267 REMARK 3.1. *The coefficients $A_{\mathbf{a},k}$, $\Omega_{\mathbf{a},k}$ can be computed using the method pro-*
 268 *posed in [8].*

269 **3.1.2. Approximating the transparent boundary conditions.** According
 270 to the previous section, we introduce the truncated DtN operator $\mathcal{B}_m^{\mathbf{a},N}(\partial_t)$ by replac-
 271 ing in (2.13) $\Lambda_{\mathbf{a}}$ by $\Lambda_{\mathbf{a}}^N$, i.e.,

$$\begin{aligned} \mathcal{B}_m^{\mathbf{a},N}(\partial_t) &= \text{diag} \left(\mathcal{B}_{m,j}^{\mathbf{a},N}(\partial_t) \right)_{0 \leq j \leq p^m - 1}, \\ (3.3) \quad \mathcal{B}_{m,j}^{\mathbf{a},N}(\partial_t) &= \mu_{m,j} \alpha_{m,j}^{-1} \sum_{k=0}^{p-1} \frac{\mu_k}{\alpha_k} \Lambda_{\mathbf{a}}^N(\alpha_k \alpha_{m,j} \partial_t). \end{aligned}$$

273 **3.1.3. A truncated system.** Using the approximation (3.3) in the truncated
 274 system (2.18) yields the following problem:
 275 Find $u_m^{\mathbf{a},N} \in C^1(\mathbb{R}^+; \mathbf{L}_{\mu}^2(\mathcal{T}^m)) \cap C^0(\mathbb{R}^+; \mathbf{V}_{\mu}(\mathcal{T}^m))$ s.t. $u_m^{\mathbf{a},N}(0) = \partial_t u_m^{\mathbf{a},N}(0) = 0$,
 276 and s.t. for any $v \in \mathbf{V}_{\mu}(\mathcal{T}^m)$,

$$(3.4) \quad (\partial_t^2 u_m^{\mathbf{a},N}, v)_{\mathcal{T}^m} + (\partial_s u_m^{\mathbf{a},N}, \partial_s v)_{\mathcal{T}^m} + \int_{\Gamma_m} \mathcal{B}_m^{\mathbf{a},N}(\partial_t) \gamma_m u_m^{\mathbf{a},N} \gamma_m v = (f, v)_{\mathcal{T}^m}.$$

279 The well-posedness and stability with respect to N of (3.4) was proven in [8]: it relies
 280 on the non-negativity of $\Lambda_{\mathbf{a}}(0)$ and the coefficients $A_{\mathbf{a},k}$, $k \in \mathbb{N}_*$ in (3.1).

281 **3.2. Error analysis.** The convergence of $u_m^{\mathbf{a},N}$ towards $u_m^{\mathbf{a}}$ was established in
 282 [8] through an (abstract) error estimate. To state this result, let us introduce the
 283 following notation for the error:

$$(3.5) \quad \varepsilon_m^{\mathbf{a},N} := u_m^{\mathbf{a},N} - u_m^{\mathbf{a}} \quad (\text{recall that } u_m^{\mathbf{a}} = u_{\mathbf{a}}|_{\mathcal{T}^m}, \text{ cf. Theorem 2.6}).$$

285 We will need an additional regularity assumption on the source term f , namely

$$(3.6) \quad f \in W^{4,1}(\mathbb{R}^+; \mathbf{L}_{\mu}^2(\mathcal{T}^m)) \text{ and } f(0) = \dots = f^{(3)}(0) = 0.$$

287 Additionally, let us introduce the following parameter that will control the error

$$(3.7) \quad r_{\mathbf{a},N} := \sum_{k=N+1}^{\infty} A_{\mathbf{a},k} \Omega_{\mathbf{a},k}^{-2}, \quad \text{with } S_{\mathbf{a}} := \sum_{k=1}^{\infty} A_{\mathbf{a},k} \Omega_{\mathbf{a},k}^{-2} < +\infty.$$

290 The convergence of the series $S_{\mathbf{a}}$ was proven in [8]. Let us introduce the energy norm
 291 of $v \in C^1(\mathbb{R}^+; \mathbf{L}_{\mu}^2(\mathcal{T}^m)) \cap C^0(\mathbb{R}^+; \mathbf{V}_{\mu}(\mathcal{T}^m))$, with $v(0) = 0$:

$$(3.8) \quad \|v\|_{[0,T]; \mathcal{T}^m} := \sup_{t \leq T} \|\partial_t v(\cdot, t)\|_{\mathbf{L}_{\mu}^2(\mathcal{T}^m)} + \sup_{t \leq T} \|\partial_s v(\cdot, t)\|_{\mathbf{L}_{\mu}^2(\mathcal{T}^m)}.$$

293 We then have the following abstract error estimate.

294 **THEOREM 3.2 ([8]).** *Let $m, N \geq 1$. Let f satisfy (2.11) and (3.6).*
 295 *With $r_{\mathbf{a},N}$ defined in (3.7), the error (3.5) satisfies, with some $C_m > 0$,*

$$(3.9) \quad \|\varepsilon_m^{\mathbf{a},N}\|_{[0,T]; \mathcal{T}^m} \leq C_m r_{\mathbf{a},N} T \|\partial_t^4 \partial_s u_{\mathbf{a}}\|_{L^1(0,T; \mathbf{L}_{\mu}^2(\mathcal{T}))}, \text{ for all } T > 0.$$

298 *Therefore, for fixed T, m , for all $0 < t < T$, $\|\varepsilon_m^{\mathbf{a},N}(t)\|_{\mathbf{L}_{\mu}^2(\mathcal{T}^m)} \rightarrow 0$, as $N \rightarrow +\infty$.*

299 The error estimate (3.9) shows that the error is governed by the quantity $r_{\mathbf{a},N}$ (which
 300 we will refer to as to an *error indicator*) in the rest of the paper). It however remains
 301 *abstract* because we do not provide any explicit bound for $r_{\mathbf{a},N}$.

302 **3.3. The goal of this article.** Because the convergence of the approximate
 303 transparent boundary conditions is defined by $r_{\mathbf{a},N}$, the goal of the present article is
 304 to provide an explicit upper bound on $r_{\mathbf{a},N}$ in terms of N .

305 Moreover, while Theorem 3.2 provides only an upper bound for the error induced
 306 by approximation (3.3), in practical computations (see the numerical experiments in
 307 [8]) this bound often appears to be tight (i.e. the error $\varepsilon_m^{\mathbf{a},N}$ converges with N like
 308 $r_{\mathbf{a},N}$). Therefore, we will prove as well a lower bound on $r_{\mathbf{a},N}$ explicit in N .

309 **REMARK 3.3.** *As seen from (3.2), in the time domain, the use of the approxi-*
 310 *mation (3.1) requires introduction $O(N)$ auxiliary unknowns coupled to the original*
 311 *unknown $u_m^{\mathbf{a},N}$ through N ODEs. The numerical resolution of (3.4) discretized as*
 312 *suggested in [8] is then of $O(N)$ complexity.*

313 4. Explicit upper bounds for the error indicator.

314 **4.1. Main results.** Here we summarize the principal results of Section 4, while
 315 the proofs will be provided in the sections that follow.

316 **REMARK 4.1.** *In what follows we will use the notation \sum instead of $\sum_{j=0}^{p-1}$.*

317 To formulate these results, let us introduce the Minkowski dimension of \mathcal{T} , cf. [14]:

$$318 \quad (4.1) \quad d_s > 0 \text{ is a unique number s.t. } \sum \alpha_j^{d_s} = 1.$$

320 To prove the existence and uniqueness of $d_s > 0$, we remark that the function $d \mapsto$
 321 $\sum \alpha_j^d$ is continuous strictly monotonically decaying on \mathbb{R}^+ , and takes values from p
 322 to 0. In what follows we will also use the following notation:

$$323 \quad (4.2) \quad \langle \boldsymbol{\alpha} \rangle := \sum \alpha_j.$$

325 It is clear that $d_s < 1$ if and only if $\langle \boldsymbol{\alpha} \rangle < 1$. This case corresponds to the fact that
 326 the total length of all the branches of the tree \mathcal{T} is finite. We then have

327 **THEOREM 4.2 (Convergence rate).** *There exists $c_{\mathbf{a}}^+ > 0$, depending only on $\boldsymbol{\mu}, \boldsymbol{\alpha}$,*
 328 *such that, for all $N \geq 2$, $r_{\mathbf{a},N}$ defined in (3.7) satisfies:*

- 330 • if $\langle \boldsymbol{\alpha} \rangle < 1$ ($d_s < 1$), $r_{\mathbf{a},N} \leq c_{\mathbf{a}}^+ N^{-1}$.
- 331
- 332 • if $\langle \boldsymbol{\alpha} \rangle = 1$ ($d_s = 1$), $r_{\mathbf{a},N} \leq c_{\mathbf{a}}^+ N^{-1} \log N$.
- 333
- 334 • if $\langle \boldsymbol{\alpha} \rangle > 1$ ($d_s > 1$), $r_{\mathbf{a},N} \leq c_{\mathbf{a}}^+ N^{-\frac{1}{d_s}}$.

335 Let us remark that the constraint $N \geq 2$ is taken to ensure the validity of the state-
 336 ment (which is proven for N sufficiently large) when $\langle \boldsymbol{\alpha} \rangle = 1$ (so that $\log N \neq 0$).

337 Alternatively, we can reformulate the above statement as a result for

$$338 \quad (4.3) \quad N_{\mathbf{a},\varepsilon} := \inf\{N \in \mathbb{N}_* : r_{\mathbf{a},N} < \varepsilon\}.$$

340 This quantity shows how many terms should be taken in the approximation (3.3) in
 341 order to ensure that $\|\varepsilon_m^{\mathbf{a},N}\|_{[0,T];\mathcal{T}^m} < C\varepsilon$, with some $C > 0$, and is important for
 342 complexity estimates, cf. Remark 3.3.

343 **THEOREM 4.3.** *There exists $C_{\mathbf{a}}^+ > 0$, depending only on $\boldsymbol{\mu}, \boldsymbol{\alpha}$, such that, for all*
 344 *$0 < \varepsilon < 1/2$, $N_{\mathbf{a},\varepsilon}$ defined in (4.3) satisfies:*

- 346 • if $\langle \boldsymbol{\alpha} \rangle < 1$ ($d_s < 1$), $N_{\mathbf{a},\varepsilon} \leq C_{\mathbf{a}}^+ \varepsilon^{-1}$.
- 347

- 348 • if $\langle \alpha \rangle = 1$ ($d_s = 1$), $N_{\mathbf{a}, \varepsilon} \leq C_{\mathbf{a}}^+ \varepsilon^{-1} \log \varepsilon^{-1}$.
 349
 350 • if $\langle \alpha \rangle > 1$ ($d_s > 1$), $N_{\mathbf{a}, \varepsilon} \leq C_{\mathbf{a}}^+ \varepsilon^{-d_s}$.

351 The constraint $\varepsilon < 1/2$ is taken to ensure the validity of the formulation (which is
 352 proven for ε sufficiently small) when $\langle \alpha \rangle = 1$ (so that $\log \varepsilon^{-1} \neq 0$).

353 **4.2. Motivation and the plan of the section.** Proving Theorem 4.2 requires
 354 finding an explicit bound on $r_{\mathbf{a}, N}$ in terms of N . For this we remark that comparing
 355 the two series representations (2.24) and (2.27) of $\mathbf{\Lambda}_{\mathbf{a}}(\omega)$, it is possible to re-express
 356 $r_{\mathbf{a}, N}$ using the eigenvalues $\omega_{\mathbf{a}, n}^2$ of $\mathcal{A}_{\mathbf{a}}$ and the coefficients $a_{\mathbf{a}, n}$ (as defined by (2.23)).
 357 This yields two definitions of $r_{\mathbf{a}, N}$ (the first definition below is the definition (3.7)):

358 (4.4) $r_{\mathbf{a}, N} = \sum_{k=N+1}^{\infty} \frac{A_{\mathbf{a}, k}}{\Omega_{\mathbf{a}, k}^2}, \quad r_{\mathbf{a}, N} \equiv \sum_{\omega_{\mathbf{a}, n} > \Omega_{\mathbf{a}, N}} \frac{a_{\mathbf{a}, n}}{\omega_{\mathbf{a}, n}^2}, \quad a_{\mathbf{a}, n} = \frac{\partial_s \varphi_{\mathbf{a}, n}(M^*)^2}{\omega_{\mathbf{a}, n}^2}.$
 359

360 From the above it is clear that estimating $r_{\mathbf{a}, N}$ requires

- 361 (i) an estimate on the asymptotic (as $n \rightarrow +\infty$) growth of the eigenvalues $\omega_{\mathbf{a}, n}^2$;
 362 (ii) an estimate on the coefficients $a_{\mathbf{a}, n}$, which, in turn, requires an estimate on
 363 $\partial_s \varphi_{\mathbf{a}, n}(M^*)$.

364 To obtain (i), it is sufficient to find the asymptotic behaviour of the counting function

365 (4.5) $\rho_{\mathbf{a}} : \mathbb{R}^+ \rightarrow \mathbb{N}, \quad \rho_{\mathbf{a}}(\lambda) = \#\{\ell : \omega_{\mathbf{a}, \ell}^2 < \lambda\},$

of the operator $\mathcal{A}_{\mathbf{a}}$, as $\lambda \rightarrow +\infty$. For addressing (ii), the most natural approach would
 be to estimate individually each of the coefficients $a_{\mathbf{a}, n}$. It is not difficult to prove
 that there exists $C_a > 0$, s.t. $a_{\mathbf{a}, n} \leq C_a$ for all $n \in \mathbb{N}_*$. Combining this upper bound
 with the asymptotic estimate provided by (i), i.e. $\omega_{\mathbf{a}, n}^2 = c(n) + o(c(n))$ as $n \rightarrow +\infty$,
 yields an upper bound on $r_{\mathbf{a}, N}$:

$$r_{\mathbf{a}, N} \leq CC_a \sum_{\omega_{\mathbf{a}, n} > \Omega_{\mathbf{a}, N}} c(n)^{-1}, \quad \text{with some } C > 0.$$

367 However, depending on the geometry of the tree \mathcal{T} , the series in the right-hand side
 368 may not converge. Because the numerical experiments indicate that the uniform
 369 bound $a_{\mathbf{a}, n} \leq C_a$ is optimal (we conjecture that there exists $c_a > 0$, s.t. for all $n \in \mathbb{N}$
 370 there exists $n^* > n$ s.t. $a_{\mathbf{a}, n^*} > c_a$), a different strategy for estimating $r_{\mathbf{a}, N}$ is needed.

371 *An alternative strategy.* To obtain an estimate on $r_{\mathbf{a}, N}$, we replace an individual
 372 estimate of $a_{\mathbf{a}, n}$ by an estimate on a sum of $a_{\mathbf{a}, n}$ lying in a certain frequency window.
 373 This idea was inspired by the work of Barnett and Hassell [3], where the authors prove
 374 a certain quasi-orthogonality property of the Neumann traces of the eigenfunctions of
 375 the Dirichlet Laplacian $-\Delta$ in a open set in \mathbb{R}^d .

376 As we will see, for this it is more advantageous to work with the second definition
 377 of $r_{\mathbf{a}, N}$ in (4.4). The price to pay when using this strategy is that the obtained bound
 378 is of the type $r_{\mathbf{a}, N} \leq C\Omega_{\mathbf{a}, N}^{-1}$, which is not explicit in N . Nonetheless, it is possible to
 379 make this bound explicit in N by exploiting estimates on the counting function (4.5).

380 *Plan of the section.* The rest of this section is organized as follows. In Section
 381 4.3, we prove the bound $r_{\mathbf{a}, N} \leq C\Omega_{\mathbf{a}, N}^{-1}$. Section 4.4 is dedicated to the study of
 382 the asymptotics of the counting function (4.5). In Section 4.5 we combine these two
 383 results to prove Theorem 4.3. We could have started by the proof of Theorem 4.2,
 384 but this seemed somewhat less natural to us. In Section 4.6, we prove Theorem 4.2.

385 **4.3. An implicit bound on $r_{\mathbf{a},N}$.** Based on the second expression in (4.4), let
 386 us introduce the following quantity

$$387 \quad (4.6) \quad R_{\mathbf{a},\omega} := \sum_{\omega_{\mathbf{a},n} > \omega} \frac{a_{\mathbf{a},n}}{\omega_{\mathbf{a},n}^2}, \text{ so that } r_{\mathbf{a},N} \equiv R_{\mathbf{a},\Omega_{\mathbf{a},N}}.$$

388
 389 We then have the first result (cf. the discussion in the end of Section 4.2).

390 **PROPOSITION 4.4.** *There exists $\tilde{C}_{\mathbf{a}} > 0$, that depends on $\boldsymbol{\alpha}, \boldsymbol{\mu}$, s.t.*

$$391 \quad (4.7) \quad R_{\mathbf{a},\omega} < \tilde{C}_{\mathbf{a}} \omega^{-1}, \text{ for all } \omega > 0.$$

393 As discussed in the end of Section 4.2, the proof of Proposition 4.4 relies on an estimate
 394 on the Neumann traces of 'packages' of eigenfunctions. We need a corresponding
 395 auxiliary result, that is summarized and proven in the section that follows.

396 **4.3.1. An upper bound for the Neumann traces of eigenfunctions.**

397 **LEMMA 4.5.** *For any $\eta > 0$, there exists $C_{\eta}^+ > 0$ (that depends on $\boldsymbol{\mu}, \boldsymbol{\alpha}, \mathbf{a}$) s.t.*

$$398 \quad (4.8) \quad \sum_{j: |\omega_{\mathbf{a},j} - \omega| < \eta} (\partial_s \varphi_{\mathbf{a},j}(M^*))^2 \omega_{\mathbf{a},j}^{-2} \equiv \sum_{j: |\omega_{\mathbf{a},j} - \omega| < \eta} a_{\mathbf{a},j} \leq C_{\eta}^+, \quad \text{for all } \omega > 0.$$

400 The above lemma shows that the sum of the coefficients $a_{\mathbf{a},j}$ corresponding to the
 401 eigenfrequencies lying in the frequency window $(\omega - \eta, \omega + \eta)$ is bounded independently
 402 of ω . This result is stronger than the simple uniform estimate $|a_{\mathbf{a},j}| \leq C_{\mathbf{a}}$ because it
 403 is independent on the number of the eigenfrequencies on the interval $(\omega - \eta, \omega + \eta)$.

404 The proof of this result is adapted from the proof of a similar estimate for the
 405 Laplacian in a bounded domain in \mathbb{R}^d by Barnett, Hassell [3]. The only modification
 406 compared to [3] is required in the proof of Lemma 2.1 [3], where we choose a smooth
 407 multiplier $\chi(s)$ (in the notation of [3], $a(s)$) supported on $\Sigma_{0,0}$, s.t. $\chi(M^*) = -1$. We
 408 will nonetheless present its proof here, for the sake of self-consistency of the paper.
 409 To prove Lemma 4.5, we start with the following auxiliary lemma.

410 **LEMMA 4.6.** *Let $\mathcal{J}_{\omega,\eta}^{\mathbf{a}} := \{j : |\omega_{\mathbf{a},j} - \omega| < \eta\}$, where $\omega \geq 1$, $\eta > 0$. Let*

$$411 \quad \forall \mathbf{c} = (c_j)_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}} \in \mathbb{R}^{K_{\mathbf{a}}} \quad (\text{where } K_{\mathbf{a}} = \#\mathcal{J}_{\omega,\eta}^{\mathbf{a}}), \quad \Phi_{\mathbf{a},\mathbf{c}} = \sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}} c_j \varphi_{\mathbf{a},j}.$$

412
 413 *Then, with some $C_{\eta} > 0$,*

$$414 \quad (4.9) \quad |\Phi'_{\mathbf{a},\mathbf{c}}(M^*)| \leq C_{\eta} \omega \|\mathbf{c}\|_{\mathbb{R}^{K_{\mathbf{a}}}}.$$

416 *Proof of Lemma 4.6.* Let χ be a (sufficiently smooth) function supported on the
 417 root branch $\Sigma_{0,0}$ of the reference tree, with $\chi(M^*) = 1$ and $\chi(M_{0,0}) = 0$. Identifying
 418 $\Sigma_{0,0}$ with $[0, 1]$ ($M^* = 0$), with an abuse of notation $\Phi_{\mathbf{a},\mathbf{c}}(s) \equiv \Phi_{\mathbf{a},\mathbf{c}}|_{\Sigma_{0,0}}$, we have

$$419 \quad (4.10) \quad \Phi'_{\mathbf{a},\mathbf{c}}(0)^2 = - \int_0^1 \frac{d}{ds} (\chi(s) \Phi'_{\mathbf{a},\mathbf{c}}(s))^2 ds = I_1 + I_2,$$

$$420 \quad I_1 = - \int_0^1 \chi'(s) (\Phi'_{\mathbf{a},\mathbf{c}}(s))^2 ds, \quad I_2 = -2 \int_0^1 \chi(s) \Phi'_{\mathbf{a},\mathbf{c}}(s) \Phi''_{\mathbf{a},\mathbf{c}}(s) ds.$$

Our goal is to bound I_1, I_2 . First of all, from the definition (2.23) it follows that

$$(4.11) \quad \|\varphi_{\mathbf{a},j}\| = 1, \quad \|\partial_s \varphi_{\mathbf{a},j}\|^2 = \omega_{\mathbf{a},j}^2, \quad (\varphi_{\mathbf{a},j}, \varphi_{\mathbf{a},k}) = (\partial_s \varphi_{\mathbf{a},j}, \partial_s \varphi_{\mathbf{a},k}) = 0, \quad \text{for } j \neq k.$$

The above with the definition of $\mathcal{J}_{\omega,\eta}^{\mathbf{a}}$ yields the bound:

$$(4.12) \quad \|\Phi_{\mathbf{a},\mathbf{c}}\|_{L_\mu^2} = \|\mathbf{c}\|, \quad \|\Phi'_{\mathbf{a},\mathbf{c}}\|^2 = \sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}} \omega_{\mathbf{a},j}^2 c_j^2 \leq (\omega + \eta)^2 \|\mathbf{c}\|^2.$$

The first term in (4.10) thus satisfies

$$(4.13) \quad I_1 \leq |\chi'|_\infty (\omega + \eta)^2 \|\mathbf{c}\|^2.$$

The second term can be rewritten as follows

$$(4.14) \quad I_2 := 2 \int_0^1 \chi(s) \Phi'_{\mathbf{a},\mathbf{c}}(s) \sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}} \omega_{\mathbf{a},j}^2 c_j \varphi_{\mathbf{a},j}(s) ds.$$

Bounding the above term with the Cauchy-Schwarz inequality would result in

$$I_2 \leq C \|\Phi'_{\mathbf{a},\mathbf{c}}\| \left\| \sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}} \omega_{\mathbf{a},j}^2 c_j \varphi_{\mathbf{a},j} \right\| \stackrel{(4.12)}{\leq} C (\omega + \eta)^3 \|\mathbf{c}\|^2,$$

which is, for large ω , ω times more than what we would like to have. To overcome this problem, we will use two observations: 1) $\omega_{\mathbf{a},j}$ in the above sum are 'close' to ω ; 2) the orthogonality of eigenfunctions (cf. the first equation of (4.12)). We rewrite

$$\sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}} \omega_{\mathbf{a},j}^2 c_j \varphi_{\mathbf{a},j}(s) = \sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}} (\omega_{\mathbf{a},j}^2 - \omega^2) c_j \varphi_{\mathbf{a},j}(s) + \omega^2 \Phi_{\mathbf{a},\mathbf{c}}(s) = \Psi_{\mathbf{a},\mathbf{c}}(s) + \omega^2 \Phi_{\mathbf{a},\mathbf{c}}(s).$$

Then (4.14) gives

$$(4.15) \quad \begin{aligned} I_2 &= 2 \int_0^1 \Psi_{\mathbf{a},\mathbf{c}}(s) \chi(s) \Phi'_{\mathbf{a},\mathbf{c}}(s) ds + 2\omega^2 \int_0^1 \chi(s) \Phi'_{\mathbf{a},\mathbf{c}}(s) \Phi_{\mathbf{a},\mathbf{c}}(s) \\ &= 2 \int_0^1 \Psi_{\mathbf{a},\mathbf{c}}(s) \chi(s) \Phi'_{\mathbf{a},\mathbf{c}}(s) ds - \omega^2 \int_0^1 \Phi_{\mathbf{a},\mathbf{c}}^2(s) \chi'(s) ds, \end{aligned}$$

where we integrated by parts and used $\chi(1) = 0$ and $\Phi_{\mathbf{a},\mathbf{c}}(0) = 0$. With (4.12) and $|\omega_{\mathbf{a},j}^2 - \omega^2| \leq \eta(2\eta + \omega)$, $j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}$, we remark that

$$(4.16) \quad \|\Psi_{\mathbf{a},\mathbf{c}}\| = \sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}} (\omega_{\mathbf{a},j}^2 - \omega^2)^2 c_j^2 \leq (2\eta\omega + \eta^2) \|\mathbf{c}\|.$$

Therefore, by the Cauchy-Schwarz inequality, using the bounds (4.12) and (4.16), applied to (4.15), we have

$$I_2 \leq 2|\chi|_\infty (2\eta\omega + \eta^2) (\omega + \eta) \|\mathbf{c}\|^2 + |\chi'|_\infty (\omega + \eta)^2 \|\mathbf{c}\|^2.$$

Combining the above bound and (4.13) in (4.10) results in

$$|\Phi'_{\mathbf{a},\mathbf{c}}(0)|^2 \leq C(\eta)\omega^2 \|\mathbf{c}\|^2,$$

which is the desired statement (with an abuse of notation $M^* \equiv 0$). \square

454 Lemma 4.5 is then almost an immediate corollary of Lemma 4.6.

455 *Proof of Lemma 4.5.* Clearly, it suffices to prove the bound for ω large enough.
 456 Take in Lemma 4.6 $\mathbf{c} := (\partial_s \varphi_{\mathbf{a},j}(M^*))_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}}$. Then (4.9) rewrites

$$457 \quad \sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}} \partial_s \varphi_{\mathbf{a},j}(M^*)^2 \leq C_\eta \omega \left(\sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}} \partial_s \varphi_{\mathbf{a},j}(M^*)^2 \right)^{\frac{1}{2}}.$$

459 This yields $\sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}} \partial_s \varphi_{\mathbf{a},j}(M^*)^2 \leq C_\eta^2 \omega^2$. For ω large enough, for all $j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}$, one has
 460 that $\omega_{\mathbf{a},j} \geq \omega - \eta > 0$, and therefore,

$$461 \quad \sum_{j \in \mathcal{J}_{\omega,\eta}^{\mathbf{a}}} (\partial_s \varphi_{\mathbf{a},j}(M^*))^2 \omega_{\mathbf{a},j}^{-2} \leq C_\eta \frac{\omega^2}{|\omega - \eta|^2} \leq \tilde{C}_\eta,$$

463 for all ω large enough. This proves Lemma 4.5. \square

464 **4.3.2. Proof of Proposition 4.4.** Because $\omega \mapsto R_{\mathbf{a},\omega}$ is non-increasing and
 465 piecewise-constant, it suffices to prove Proposition 4.4 for $\omega = M \in \mathbb{N}_*$. We start by
 466 rewriting (4.6) in the following form

$$467 \quad R_{\mathbf{a},M} = \sum_{n: \omega_{\mathbf{a},n} \geq M} \frac{a_{\mathbf{a},n}}{\omega_{\mathbf{a},n}^2} = \sum_{n=M}^{\infty} \sum_{n \leq \omega_{\mathbf{a},j} < n+1} \frac{a_{\mathbf{a},j}}{\omega_{\mathbf{a},j}^2},$$

469 where, in order to apply Lemma 4.5, we split the interval $[m, \infty)$ into subintervals
 470 $[k, k+1)$, $k \geq m$, and sum over the eigenvalues belonging to these intervals. Obviously,

$$471 \quad R_{\mathbf{a},M} \leq \sum_{n=M}^{\infty} \frac{1}{n^2} \sum_{n \leq \omega_{\mathbf{a},j} < n+1} a_{\mathbf{a},j} < \sum_{n=M}^{\infty} \frac{1}{n^2} \sum_{|\omega_{\mathbf{a},j} - n| \leq 1} a_{\mathbf{a},j} \leq C_1^+ \sum_{n=M}^{\infty} \frac{1}{n^2},$$

473 where the last bound follows from Lemma 4.5 applied with $\eta = 1$. The above remain-
 474 der is then $O(M^{-1})$, which proves the desired statement.

475 **4.4. Asymptotics of the counting function.** The second result, necessary
 476 for the proof of Theorem 4.2, cf. Section 4.2, summarizes estimates on the counting
 477 function $\rho_{\mathbf{a}}$. We shall need the following definition:

478 **DEFINITION 4.7** (Arithmetic set, cf. [14]). *A set $E = \{\gamma_j, 0 \leq j \leq p-1\} \subset \mathbb{R}_*^+$
 479 is called arithmetic if $\gamma_j/\gamma_k \in \mathbb{Q}$ for all j, k . The span of E is the largest $\gamma \in \mathbb{R}_*^+$, s.t.
 480 $\gamma_j/\gamma \in \mathbb{N}$ for all j .*

481 **THEOREM 4.8.** *The counting function $\rho_{\mathbf{a}}$, cf. (4.5), satisfies, as $\lambda \rightarrow +\infty$,*

- 482 1. *if $\langle \boldsymbol{\alpha} \rangle < 1$ ($d_s < 1$), then $\rho_{\mathbf{a}}(\lambda) = \pi^{-1} (1 - \langle \boldsymbol{\alpha} \rangle)^{-1} \lambda^{\frac{1}{2}} + O(\lambda^{\frac{d_s}{2}})$.*
- 483 2. *if $\langle \boldsymbol{\alpha} \rangle = 1$ ($d_s = 1$), then $\rho_{\mathbf{a}}(\lambda) = \pi^{-1} C_{\boldsymbol{\alpha}} \lambda^{\frac{1}{2}} \log \lambda^{\frac{1}{2}} + O(\lambda^{\frac{1}{2}})$, where*

$$484 \quad C_{\boldsymbol{\alpha}} = \left(\sum \alpha_j \log \alpha_j^{-1} \right)^{-1}.$$

- 485 3. *if $\langle \boldsymbol{\alpha} \rangle > 1$ ($d_s > 1$), there exists a bounded function $\rho_{\mathbf{a}}^\infty(\lambda)$, s.t.*

$$486 \quad (4.17) \quad \rho_{\mathbf{a}} = \rho_{\mathbf{a}}^\infty(\lambda) \lambda^{\frac{d_s}{2}} + o(\lambda^{\frac{d_s}{2}}), \quad \mathbf{a} \in \{\mathfrak{d}, \mathfrak{n}\},$$

488 where, with $E := \{\log \alpha_j^{-1}, j = 0, \dots, p-1\}$, the function $\rho_{\mathbf{a}}^\infty$ satisfies:
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- if the set E is arithmetic with the span γ , then $\rho_{\mathfrak{a}}^{\infty}(\lambda) = \psi_{\mathfrak{a}}^{\infty}(\log \lambda)$ with $\psi_{\mathfrak{a}}^{\infty}$ piecewise-continuous and γ -periodic, s.t. $0 < \psi_{-} < \psi_{\mathfrak{a}}^{\infty} < \psi_{+}$.
- otherwise, $\rho_{\mathfrak{a}}^{\infty}(\lambda) = \rho_{\mathfrak{a}}^{\infty}(\text{const} > 0)$.

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The goal of this section is to prove Theorem 4.8. Asymptotics of counting functions of the discrete and continuous Laplacian on various types of fractals were obtained in e.g. [13, 14, 2], see as well [11] and references therein. Our geometric setting bears some similarities to the one from the work by Levitin, Vassiliev [14], however, we do not use the same geometrical and boundary conditions assumptions.

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We will make use of the ideas from the seminal article by Kigami and Lapidus [13], whether the authors study the counting function for the discrete Laplacian on the post-critically finite self-similar fractals (the geometry considered in the present paper does not belong to this class). In particular, like for the classical Weyl's estimates (for the Laplacian in bounded domains of \mathbb{R}^d for instance), the study of $\rho_{\mathfrak{a}}(\lambda)$ relies on the min-max principle and the so-called Dirichlet-Neumann bracketing technique. Using properties of the fractal geometry, it is possible to write a recursive equation for $\rho_{\mathfrak{a}}(\lambda)$, which next is investigated using the renewal theorem from [14].

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4.4.1. A recursive equation for $\rho_{\mathfrak{a}}(\lambda)$. This section is dedicated to the proof of the recursive equation for $\rho_{\mathfrak{a}}(\lambda)$. For this, we will extensively use the well-known min-max characterization of the eigenvalues:

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$$(4.18) \quad \omega_{\mathfrak{a},n}^2 = \inf_{Q \in \mathcal{F}^n(V_{\mathfrak{a}})} \sup_{v \in Q} \frac{a(v,v)}{\|v\|_{L_{\mu}^2(\mathcal{T})}^2} \quad \text{where } \mathcal{F}^n(V_{\mathfrak{a}}) := \{Q : Q \subset V_{\mathfrak{a}}, \dim Q = n\}.$$

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LEMMA 4.9 (A recursive equation). *The functions $\rho_{\mathfrak{a}}$, $\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$ satisfy*

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$$(4.19) \quad \rho_{\mathfrak{a}}(\lambda) = r_{\mathfrak{a}}(\lambda) + \sum \rho_{\mathfrak{a}}(\alpha_j^2 \lambda), \quad \lambda \geq 0,$$

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where $r_{\mathfrak{a}}(\lambda)$ is a piecewise-continuous function, s.t.

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$$(4.20) \quad \lfloor \pi^{-1} \sqrt{\lambda} \rfloor \leq r_{\mathfrak{a}}(\lambda) \leq \lfloor \pi^{-1} \sqrt{\lambda} \rfloor + p + 1.$$

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Moreover, there exists $\lambda_{\mathfrak{a}}^* > 0$, s.t. $r_{\mathfrak{a}}(\lambda) = 0$ and $\rho_{\mathfrak{a}}(\lambda) = 0$ for $\lambda < \lambda_{\mathfrak{a}}^*$.

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Proof. Note that showing (4.19, 4.20) amounts to proving the double inequality

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$$(4.21) \quad \sum \rho_{\mathfrak{a}}(\alpha_j^2 \lambda) + \lfloor \pi^{-1} \sqrt{\lambda} \rfloor \leq \rho_{\mathfrak{a}}(\lambda) \leq \sum \rho_{\mathfrak{a}}(\alpha_j^2 \lambda) + \lfloor \pi^{-1} \sqrt{\lambda} \rfloor + p + 1.$$

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so that, setting $r_{\mathfrak{a}}(\lambda) := \rho_{\mathfrak{a}}(\lambda) - \sum \rho_{\mathfrak{a}}(\alpha_j^2 \lambda)$, $r_{\mathfrak{a}}$ inherits piecewise-continuity from $\rho_{\mathfrak{a}}$

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and satisfies (4.20). We will prove (4.21) for $\mathfrak{a} = \mathfrak{n}$, the case $\mathfrak{a} = \mathfrak{d}$ being almost

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verbatim the same. The proof is based on two ideas: the classical Dirichlet-Neumann

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bracketing technique [13] and a rescaling argument.

527

Step 1. Dirichlet-Neumann bracketing.

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Step 1.1. Let \mathcal{A}_N be the self-adjoint operator in $L_{\mu}^2(\mathcal{T})$ associated to the sesquilinear

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form $a(u,v)$ (2.20), defined like \mathcal{A}_n in (2.21), but with the following domain (remark

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that V_n is replaced by $H_{\mu}^1(\mathcal{T})$, i. e. the Dirichlet condition in M^* is removed):

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$$\mathcal{D}(\mathcal{A}_N; \mathcal{T}) = \{u \in H_{\mu}^1(\mathcal{T}) : |a(u,v)| \leq C(u)\|v\|_{L_{\mu}^2(\mathcal{T})}, \quad \forall v \in H_{\mu}^1(\mathcal{T})\}.$$

In a strong form, this operator corresponds to the weighted Laplacian on \mathcal{T} , with the Neumann conditions at M^* and at the fractal boundary of the tree \mathcal{T} . By Theorem

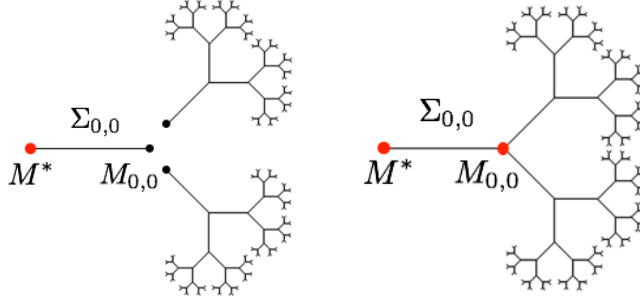


FIGURE 2. *Left: we illustrate the space $\tilde{V}_n(\mathcal{T})$, which includes functions that may be discontinuous in the point $M_{0,0}$. Right: an illustration to the space $\tilde{V}_{n,0}(\mathcal{T})$, where all functions vanish in the point $M_{0,0}$. By red we mark the points where the functions vanish.*

2.1, it has a compact resolvent. Its spectrum is denoted by $\sigma_N(\mathcal{T})$ and the associated counting function by $\rho_N(\lambda; \mathcal{T})$.

Let $\mathcal{T}_{1,j}$ be the self-similar p -adic subtree of \mathcal{T} whose root edge is $\Sigma_{1,j}$. Let us introduce a broken space that contains $V_n(\mathcal{T})$ (see Figure 2 for an illustration)

$$\tilde{V}_n(\mathcal{T}) := \{u \in L^2_\mu(\mathcal{T}) : u|_{\Sigma_{0,0}} \in H^1(\Sigma_{0,0}), u(M^*) = 0, u|_{\mathcal{T}_{1,j}} \in H^1_\mu(\mathcal{T}_{1,j})\}.$$

533 Let us define the operator $\tilde{\mathcal{A}}_n$ like in (2.21), with the domain

$$534 \quad \mathcal{D}(\tilde{\mathcal{A}}_n) = \{u \in \tilde{V}_n(\mathcal{T}) : |a(u, v)| \leq C(u)\|v\|_{L^2_\mu(\mathcal{T})}, \quad \forall v \in \tilde{V}_n(\mathcal{T})\}.$$

536 Again, by Theorem 2.1, it has a compact resolvent. We denote by $\tilde{\sigma}_n(\mathcal{T})$ and by
537 $\tilde{\rho}_n(\lambda; \mathcal{T})$ the spectrum and counting function of the operator $\tilde{\mathcal{A}}_n$.

538 Finally, let $\tilde{V}_{n,0}$ be a subspace of V_n of functions vanishing in $M_{0,0}$,

$$539 \quad \tilde{V}_{n,0}(\mathcal{T}) = \{v \in V_n(\mathcal{T}) : v(M_{0,0}) = 0\},$$

541 see Figure 2 for an illustration. The spectrum of the associated operator $\tilde{\mathcal{A}}_{n,0}$ (defined
542 like above) will be denoted by $\tilde{\sigma}_{n,0}(\mathcal{T})$, and the counting function by $\tilde{\rho}_{n,0}(\lambda; \mathcal{T})$.

543

544 Let us remark that we will need to use the counting functions for different trees,
545 and hence the notation $\rho(\lambda; \mathcal{T})$, where the tree is made explicit. The notation $\rho(\lambda)$ is
546 reserved for the counting functions for the operators on the original tree \mathcal{T} .

547 *Step 1.2. Relations between the counting functions.*

548 *Relating ρ_n and ρ_N .* We will need an upper bound for $\rho_N(\lambda; \mathcal{T})$ in terms of $\rho_n(\lambda) \equiv$
549 $\rho_n(\lambda; \mathcal{T})$. It can be obtained from the same arguments as [13, Corollary 4.7] because
550 the co-dimension of $V_n(\mathcal{T})$, as a subspace of $H^1_\mu(\mathcal{T})$, is finite. Namely,

$$551 \quad (4.22) \quad \rho_N(\lambda; \mathcal{T}) \leq \rho_n(\lambda; \mathcal{T}) + \dim(H^1_\mu(\mathcal{T}) \setminus V_n(\mathcal{T})) = \rho_n(\lambda; \mathcal{T}) + 1.$$

553 *Relating ρ_n and $\tilde{\rho}_n$.* The min-max principle (4.18), since $V_n(\mathcal{T}) \subset \tilde{V}_n(\mathcal{T})$, yields

$$554 \quad (4.23) \quad \rho_n(\lambda; \mathcal{T}) \leq \tilde{\rho}_n(\lambda; \mathcal{T}).$$

555

556 Relating ρ_n and $\tilde{\rho}_{n,0}$. As $\tilde{V}_{n,0}(\mathcal{T}) \subset V_n(\mathcal{T})$, we obtain, by the min-max principle,

$$557 \quad (4.24) \quad \tilde{\rho}_{n,0}(\lambda; \mathcal{T}) \leq \rho_n(\lambda; \mathcal{T}).$$

559 *Step 2. An equation for $\tilde{\rho}_n(\lambda; \mathcal{T})$.*

560 *Step 2.1. An immediate identity.* Because, with an obvious abuse of notation,

$$561 \quad \tilde{V}_n = V_n(\Sigma_{0,0}) \oplus \bigoplus_{j=0}^{p-1} H_\mu^1(\mathcal{T}_{1,j}), \quad V_n(\Sigma_{0,0}) := \{u \in H^1(\Sigma_{0,0}) : u(M^*) = 0\},$$

563 the spectrum $\tilde{\sigma}_n$ of $\tilde{\mathcal{A}}_n$ consists of the eigenvalues of $-\Delta$ on $V_n(\Sigma_{0,0})$ (the associated
564 counting function is denoted by $\rho_n(\lambda; \Sigma_{0,0})$ and is known explicitly) and the union
565 $\bigcup_{j=0}^{p-1} \sigma_N(\mathcal{T}_{1,j})$. The counting function $\tilde{\rho}_n$ is a sum of the respective counting functions:

$$566 \quad (4.25) \quad \tilde{\rho}_n(\lambda; \mathcal{T}) = \rho_n(\lambda; \Sigma_{0,0}) + \sum_{j=0}^{p-1} \rho_N(\lambda; \mathcal{T}_{1,j}), \quad \rho_n(\lambda; \Sigma_{0,0}) = \left\lfloor \pi^{-1} \lambda^{\frac{1}{2}} + \frac{1}{2} \right\rfloor.$$

568 *Step 2.2. Scaling argument.* The goal of this step is to show that

$$569 \quad (4.26) \quad \rho_N(\lambda; \mathcal{T}_{1,j}) = \rho_N(\alpha_j^2 \lambda; \mathcal{T}).$$

571 To see this, we get back to the min-max characterization of eigenvalues (4.18). Since
572 $\mathcal{T}_{1,j}$ is obtained from \mathcal{T} by a similitude transformation γ_j of ratio α_j (cf. [9]), setting
573 formally $\tilde{s} = \gamma_j(s)$, we obtain the following identities, valid for any $v \in H_\mu^1(\mathcal{T})$,

$$574 \quad \int_{\mathcal{T}} |v(s)|^2 \mu(s) ds = \mu_j^{-1} \int_{\mathcal{T}_{1,j}} |v(\gamma_j^{-1}(\tilde{s}))|^2 \alpha_j^{-1} \mu(\tilde{s}) d\tilde{s},$$

$$575 \quad \int_{\mathcal{T}} |\partial_s v(s)|^2 \mu(s) ds = \mu_j^{-1} \int_{\mathcal{T}_{1,j}} |\partial_{\tilde{s}} v(\gamma_j^{-1}(\tilde{s}))|^2 \alpha_j \mu(\tilde{s}) d\tilde{s},$$

577 and thus, see the min-max principle (4.18),

$$578 \quad (4.27) \quad \omega_{N,\ell}^2 \in \sigma_N(\mathcal{T}) \iff \alpha_j^{-2} \omega_{N,\ell}^2 \in \sigma_N(\mathcal{T}_{1,j}).$$

580 Therefore, $\rho_N(\lambda; \mathcal{T}_{1,j}) = \#\{\ell : \alpha_j^{-2} \omega_{N,\ell}^2 < \lambda\} = \rho_N(\alpha_j^2 \lambda; \mathcal{T})$, i.e. (4.26).

581 *Step 2.3. Summary.* Combining (4.26) with (4.25), we obtain

$$582 \quad (4.28) \quad \tilde{\rho}_n(\lambda) = \left\lfloor \pi^{-1} \lambda^{\frac{1}{2}} + \frac{1}{2} \right\rfloor + \sum_{j=0}^{p-1} \rho_N(\alpha_j^2 \lambda).$$

584 *Step 3. An equation for $\tilde{\rho}_{n,0}(\lambda; \mathcal{T})$.*

585 Just like in the previous case, we remark that (with an abuse of notation)

$$586 \quad \tilde{V}_{n,0} = H_0^1(\Sigma_{0,0}) \oplus \bigoplus_{j=0}^{p-1} V_n(\mathcal{T}_{1,j}).$$

588 Repeating the same arguments as in Step 2, we get the recursive equation for $\tilde{\rho}_{n,0}$:

$$589 \quad (4.29) \quad \tilde{\rho}_{n,0}(\lambda) = \left\lfloor \pi^{-1} \lambda^{\frac{1}{2}} \right\rfloor + \sum_{j=0}^{p-1} \rho_n(\alpha_j^2 \lambda).$$

590

591 *Step 4. Proof of (4.21).* Inequality (4.23) with (4.28) yields

$$592 \quad (4.30) \quad \rho_n(\lambda) \leq \left\lfloor \pi^{-1} \lambda^{\frac{1}{2}} + \frac{1}{2} \right\rfloor + \sum_{j=0}^{p-1} \rho_N(\alpha_j^2 \lambda) \leq \left\lfloor \pi^{-1} \lambda^{\frac{1}{2}} + \frac{1}{2} \right\rfloor + \sum_{j=0}^{p-1} \rho_n(\alpha_j^2 \lambda) + p,$$

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594 where the last bound was obtained from (4.22). To get the lower bound, we combine
595 the inequality $\rho_n \geq \rho_{n,0}$, cf. (4.24), and (4.29):

$$596 \quad (4.31) \quad \rho_n(\lambda) \geq \left\lfloor \pi^{-1} \lambda^{\frac{1}{2}} \right\rfloor + \sum_{j=0}^{p-1} \rho_n(\alpha_j^2 \lambda).$$

597

598 Combining (4.30) and (4.31) yields (4.21).

599 *Step 5. 'Causality' of $\rho_n(\lambda)$, $r_n(\lambda)$.* $\rho_n(\lambda) = 0$ for $\lambda < \omega_{1,n}^2$; the same holds for
600 $\rho_n(\alpha_j^2 \lambda)$, for $j = 0, \dots, p-1$, since $\alpha_j < 1$. Thus, $r_n(\lambda) = 0$ for $\lambda < \lambda_n^* := \omega_{1,n}^2$. \square

601 **4.4.2. Renewal theory.** In order to solve the recursive equation (4.19), we will
602 rewrite it in a more convenient form. In general we are interested in equations of the
603 type: provided Φ piecewise-continuous, find φ s.t. for all $\lambda > 0$,

$$604 \quad (4.32) \quad \varphi(\lambda) = \Phi(\lambda) + \sum \varphi(\alpha_j^2 \lambda).$$

605

606 The change of variables $x := \frac{1}{2} \log \lambda$ transforms (4.32) into an equation with delays
607 $\log \alpha_j^{-1}$:

$$608 \quad (4.33) \quad \varphi(e^{2x}) = \Phi(e^{2x}) + \sum \varphi(\alpha_j^2 e^{2x}) = \Phi(e^{2x}) + \sum \varphi(e^{2(x - \log \alpha_j^{-1})}), \quad x \in \mathbb{R}.$$

609

610 To handle the equations of type (4.33), we will use the renewal theorem, cf. [4, p.
611 358] or [12, Appendix B.4]; we will exploit its version suggested by Levitin, Vassiliev
612 [14], whose statement is the most suitable for our needs.

613 **THEOREM 4.10 (Renewal theorem, [14]).** *Let $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ be a piecewise-*
614 *continuous function that satisfies the following bound: there exist $C, \alpha \geq 0$, s.t.*
615 *$|\Psi(x)| < Ce^{-\alpha|x|}$ on \mathbb{R} . Provided such Ψ , let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ solve*

$$616 \quad (4.34) \quad \psi(x) = \Psi(x) + \sum c_j \psi(x - \gamma_j), \quad c_j, \gamma_j > 0, \quad \sum c_j = 1.$$

617

618 *Assume additionally that $\lim_{x \rightarrow -\infty} \psi(x) = 0$.*

619 *Then ψ as defined above is unique; it is uniformly bounded on \mathbb{R} and*

$$620 \quad \psi(x) = \psi_\infty(x) + o(1), \quad \text{as } x \rightarrow +\infty,$$

622 *where ψ_∞ is a periodic function defined as follows:*

(1) *if the set $\{\gamma_j\}$ is arithmetic, then, with γ being its span,*

$$\psi_\infty(x) = \gamma J^{-1} \sum_{k=-\infty}^{\infty} \Psi(x - k\gamma), \quad J = \sum_{j=0}^{p-1} c_j \gamma_j.$$

623 (2) *otherwise, $\psi_\infty(x) = \text{const} = J^{-1} \int_{-\infty}^{\infty} \Psi(x) dx$.*

624 We will use a direct consequence of Theorem 4.10, formulated below.

625 LEMMA 4.11. *Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfy*

$$626 \quad (4.35) \quad \varphi(\lambda) = \Phi(\lambda) + \sum \varphi(\alpha_j^2 \lambda), \quad \text{and } \varphi(\lambda) = 0, \text{ for } \lambda \leq \lambda_*, \lambda_* > 0,$$

628 where $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ is piecewise-continuous, s.t., with some $C_* > 0$,

$$630 \quad (4.36) \quad |\Phi(\lambda)| \leq C_* \quad \text{for all } \lambda > 0.$$

631 Then, there exists $C > 0$, s.t. for all $\lambda > 0$, $|\varphi(\lambda)| \leq C\lambda^{\frac{d_s}{2}}$.

632 *Proof.* First of all, remark that

$$633 \quad (4.37) \quad \Phi(\lambda) = 0, \text{ for all } \lambda < \lambda_*.$$

635 This is seen by rewriting $\Phi(\lambda) = \varphi(\lambda) - \sum \varphi(\alpha_j^2 \lambda)$. Indeed, for $\lambda < 0$, $\varphi(\lambda) =$
636 $\varphi(\alpha_j^2 \lambda) = 0$, by (4.35). For $0 < \lambda \leq \lambda_*$, this also follows from (4.35) as $\alpha_j < 1$.

637 The rest of the proof follows closely the proof of Lemma 2.2 in [14]. We will
638 rewrite (4.35) in the form required by Theorem 4.10, by a change of variables and
639 rescaling. As explained in the beginning of Section 4.4.2, we transform (4.35) by a
640 change of variables $x = \frac{1}{2} \log \lambda$, $\lambda > 0$ (so that $\lambda = e^{2x}$), into

$$641 \quad (4.38) \quad \varphi(e^{2x}) = \Phi(e^{2x}) + \sum \varphi(\alpha_j^2 e^{2x}) = \Phi(e^{2x}) + \sum \varphi(e^{2(x - \log \alpha_j^{-1})}).$$

643 It remains to make appear the weights c_j required in (4.34). For this we multiply
644 (4.38) by $e^{-d_s x}$ and set $\psi(x) := \varphi(e^{2x})e^{-d_s x}$ (so that $\varphi(\lambda) = \lambda^{-\frac{d_s}{2}} \psi(\frac{1}{2} \log \lambda)$). After
645 some computations this gives, with $\gamma_j = \log \alpha_j^{-1}$,

$$646 \quad (4.39) \quad \psi(x) = \Psi(x) + \sum \alpha_j^{d_s} \psi(x - \gamma_j), \quad \text{with } \Psi(x) := \Phi(e^{2x})e^{-d_s x},$$

648 where, with the definition (4.1) of d_s , we recognize (4.34). Let us verify the conditions
649 of Theorem 4.10:

- 651 • Ψ is piecewise-continuous, because Φ is piecewise-continuous;
- 652
- 653 • the bound $|\Psi(x)| \leq Ce^{-\alpha|x|}$ holds true (with $\alpha = d_s > 0$) because
- 654 – for $x < \frac{1}{2} \log \lambda_*$, $\Psi(x) = 0$ by 'causality' of Φ (4.37);
- 655 – for $x > 0$, $|\Psi(x)| \leq C_* e^{-d_s x}$ as a direct consequence of (4.36);
- 656 – for $x \in [\frac{1}{2} \log \lambda_*, 0]$ (if $\lambda_* < 1$), by (4.36), $|\Psi(x)| \leq C_* e^{-d_s x} \leq C_* \lambda_*^{-\frac{d_s}{2}}$.
- 658 • $\lim_{x \rightarrow -\infty} \psi(x) = 0$, because $\varphi(e^{2x}) = 0$ for $x < \frac{1}{2} \log \lambda_*$.

661 By Theorem 4.10, $\psi(x)$ is uniformly bounded on \mathbb{R} , and so is $\varphi(\lambda)\lambda^{-\frac{d_s}{2}}$. \square

4.4.3. Proof of Theorem 4.8. Before stating an actual proof, let us first explain the principal idea. One could try making an ansatz for the asymptotic behaviour of the counting function by generalizing the known asymptotics of the counting function for the Neumann/Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbb{R}^d$. I.e.

$$\rho_{\alpha}(\lambda) = C\lambda^{\kappa} + o(\lambda^{\kappa}), \text{ as } \lambda \rightarrow +\infty, \quad C > 0,$$

662 where κ is a power to be determined; substituting the above into (4.19) yields

$$663 \quad C\lambda^\kappa + o(\lambda^\kappa) = \frac{\sqrt{\lambda}}{\pi} + o(1) + C \sum \alpha_j^{2\kappa} \lambda^\kappa + o(\lambda^\kappa),$$

664 or, alternatively,

$$666 \quad (4.40) \quad C \left(1 - \sum \alpha_j^{2\kappa}\right) \lambda^\kappa = \frac{\sqrt{\lambda}}{\pi} + o(\lambda^\kappa), \text{ as } \lambda \rightarrow +\infty.$$

668 The above shows that we should expect $\kappa \geq 1/2$. Two cases are possible:

- 669 • $\kappa > 1/2$. Necessarily, by power matching in (4.40), $\sum \alpha_j^{2\kappa} = 1$. By (4.1),
670 $2\kappa = d_s > 1$, which is possible only if $\langle \alpha \rangle > 1$.
- 671 • $\kappa = 1/2$. By power matching in (4.40), $C = \pi^{-1}(1 - \langle \alpha \rangle)^{-1}$ (which is positive
672 iff $\langle \alpha \rangle < 1$).

676 Let us now prove the above rigorously. Let us remark that the proofs of the cases
677 $\langle \alpha \rangle \leq 1$ and $\langle \alpha \rangle > 1$ slightly differ.

678 *Proof of Theorem 4.8. Case $\langle \alpha \rangle < 1$.* Let us remark that this case, though in a
679 different setting, had been treated by Levitin and Vassiliev, cf. [14]. We present the
680 proof here in a way that is somewhat different from [14], which allows to extend it
681 easily to the particular case $\langle \alpha \rangle = 1$ (not studied in [14]).

682 Let, with $C_\alpha = \pi^{-1}(1 - \langle \alpha \rangle)^{-1}$,

$$683 \quad (4.41) \quad \varphi_\alpha(\lambda) := \rho_\alpha(\lambda) - \left\lfloor C_\alpha \lambda^{\frac{1}{2}} \right\rfloor.$$

685 Let us show that $\varphi_\alpha(\lambda)$ satisfies the conditions of Lemma 4.11 (with $\varphi = \varphi_\alpha$ in Lemma
686 4.11). For this let us examine the difference

$$687 \quad (4.42) \quad \Phi_\alpha(\lambda) := \varphi_\alpha(\lambda) - \sum \varphi_\alpha(\alpha_j^2 \lambda).$$

689 Evidently, $\varphi_\alpha(\lambda)$ is piecewise-continuous and vanishes for $\lambda < \min(C_\alpha^{-1}, \omega_{\alpha,1}^2)$ (same
690 is true for Φ_α). It remains to show that, with some $C_* > 0$, $|\Phi_\alpha(\lambda)| \leq C_*$.

691 Let us prove first the lower bound $\Phi_\alpha \geq -C_*$. Replacing the first term in (4.42) by
692 using the recursive equation (4.19) yields

$$693 \quad \Phi_\alpha(\lambda) = r_\alpha(\lambda) + \sum \rho_\alpha(\alpha_j^2 \lambda) - \left\lfloor C_\alpha \lambda^{\frac{1}{2}} \right\rfloor - \sum \left(\rho_\alpha(\alpha_j^2 \lambda) - \left\lfloor C_\alpha \alpha_j \lambda^{\frac{1}{2}} \right\rfloor \right).$$

695 Using $x \geq \lfloor x \rfloor \geq x - 1$ and the lower bound (4.20) the above rewrites

$$696 \quad \Phi_\alpha(\lambda) \geq \pi^{-1} \sqrt{\lambda} - 1 - C_\alpha \lambda^{\frac{1}{2}} + \sum \alpha_j C_\alpha \lambda^{\frac{1}{2}} - p.$$

Thus, with $C_\alpha = \pi^{-1}(1 - \langle \alpha \rangle)^{-1}$, $\Phi_\alpha(\lambda) \geq -1 - p$. The upper bound can be obtained
repeating the same arguments almost verbatim.

Applying Lemma 4.11, we deduce that $\varphi_\alpha(\lambda)$ is bounded, and hence the conclusion
about $\varphi_\alpha(\lambda) = O(\lambda^{\frac{d_s}{2}})$, $d_s < 1$.

Case $\langle \alpha \rangle = 1$. The proof mimics the proof of the case $\langle \alpha \rangle < 1$ almost verbatim,
with the only difference being that the function under consideration is

$$\varphi_\alpha(\lambda) = \rho_\alpha(\lambda) - \left\lfloor \pi^{-1} \left(\sum \alpha_j \log \alpha_j^{-1} \right)^{-1} \lambda^{\frac{1}{2}} \log \lambda^{\frac{1}{2}} \right\rfloor.$$

698 We leave the details to the reader.

699 *Case $\langle \alpha \rangle > 1$.* In this case we will not apply Lemma 4.11, but rather use a refined
700 result stated in the end of Theorem 4.10.

701 For this we rewrite the recursive relation (4.19) like in the proof of Lemma 4.11, cf.
702 (4.39). With $\psi_{\mathbf{a}}(x) = \rho_{\mathbf{a}}(e^{2x})e^{-d_s x}$ (so that $\rho_{\mathbf{a}}(\lambda) = \psi_{\mathbf{a}}(\frac{1}{2} \log \lambda) \lambda^{\frac{d_s}{2}}$), (4.19) gives

$$703 \quad \psi_{\mathbf{a}}(x) = \Psi_{\mathbf{a}}(x) + \sum \psi_{\mathbf{a}}(x - \log \alpha_j^{-1}) \alpha_j^{d_s}, \quad \Psi_{\mathbf{a}}(x) := r_{\mathbf{a}}(e^{2x})e^{-d_s x}.$$

705 The above equation satisfies the conditions of Theorem 4.10 (this can be shown using
706 the same arguments as in the proof of Lemma 4.11, using $r_{\mathbf{a}}(e^{2x}) \leq Ce^x$ and $d_s > 1$.
707 cf. (4.20)). Thus we can apply Theorem 4.10, which shows that, as $x \rightarrow +\infty$,
708 $\psi_{\mathbf{a}}(x) = \psi_{\mathbf{a}}^{\infty}(x) + o(1)$. Therefore,

$$709 \quad \rho_{\mathbf{a}}(\lambda) = \psi_{\mathbf{a}}\left(\frac{1}{2} \log \lambda\right) \lambda^{\frac{d_s}{2}} = \rho_{\mathbf{a}}^{\infty}(\log \lambda) \lambda^{\frac{d_s}{2}} + o(\lambda^{\frac{d_s}{2}}), \quad \text{as } \lambda \rightarrow +\infty,$$

711 where $\rho_{\mathbf{a}}^{\infty}(x) = \psi_{\mathbf{a}}^{\infty}\left(\frac{x}{2}\right)$. Let us remark that $\rho_{\mathbf{a}}^{\infty}$ is bounded by Theorem 4.10. More-
712 over, $\rho_{\mathbf{a}}^{\infty}$ is strictly positive: in the non-arithmetic case this follows from its explicit
713 expression; in the arithmetic case we recall that (with the notation of Theorem 4.10)

$$714 \quad \rho_{\mathbf{a}}^{\infty}(x) = \gamma J^{-1} \sum_{k=-\infty}^{\infty} r_{\mathbf{a}}(e^{x-2k\gamma}) e^{-d_s(x/2-k\gamma)} \geq \gamma J^{-1} \sum_{k=-\infty}^0 r_{\mathbf{a}}(e^{x-2k\gamma}) e^{-d_s(x/2-k\gamma)},$$

716 where the last inequality holds true because $r_{\mathbf{a}}$ is non-negative. From the explicit
717 expression for $r_{\mathbf{a}}$ in (4.20) we see that for all sufficiently large x the above is strictly
718 positive, and therefore, by periodicity of $\rho_{\mathbf{a}}^{\infty}$, $\rho_{\mathbf{a}}^{\infty}(x) > 0$ for all $x > 0$. \square

719 **4.5. Proof of Theorem 4.3.** To prove Theorem 4.3, let us start by the following
720 corollary of Theorem 4.8. Similarly to $\rho_{\mathbf{a}}(\lambda)$, let us introduce the related quantity,
721 namely the number of positive poles pf $\Lambda_{\mathbf{a}}$ smaller than λ :

$$722 \quad (4.43) \quad P_{\mathbf{a}}(\lambda) = \#\{n : \Omega_{\mathbf{a},n} < \lambda\}.$$

724 Following the discussion preceding the derivation of (2.27), let us recall that the
725 positive poles of $\Lambda_{\mathbf{a}}$ constitute a subset of $(\omega_{\mathbf{a},n})_{n \in \mathbb{N}}$, and the eigenvalues, unlike the
726 poles, are counted with their multiplicities. Therefore,

$$727 \quad (4.44) \quad P_{\mathbf{a}}(\lambda) \leq \#\{k : \omega_{\mathbf{a},k} < \lambda\} \equiv \rho_{\mathbf{a}}(\lambda^2).$$

729 Combining the above relation with Theorem 4.8 yields

730 **COROLLARY 4.12.** *Let $P_{\mathbf{a}} : \mathbb{R}^+ \rightarrow \mathbb{N}$ be defined in (4.43). Then, with $C_{\mathbf{a}} > 0$,
731 $\mathbf{a} \in \{\mathfrak{n}, \mathfrak{d}\}$, depending on α , μ , it holds, for all $\lambda > 2$,*

733 1. *if $\langle \alpha \rangle < 1$ ($d_s < 1$), then $P_{\mathbf{a}}(\lambda) \leq C_{\mathbf{a}} \lambda$.*

735 2. *if $\langle \alpha \rangle = 1$ ($d_s = 1$), then $P_{\mathbf{a}}(\lambda) \leq C_{\mathbf{a}} \lambda \log \lambda$.*

737 3. *if $\langle \alpha \rangle > 1$ ($d_s > 1$), then $P_{\mathbf{a}}(\lambda) \leq C_{\mathbf{a}} \lambda^{d_s}$.*

738 We now have all the necessary ingredients required to prove Theorem 4.3.

739 *Proof of Theorem 4.3.* Recall that $N_{\mathbf{a},\varepsilon}$ (cf. Definition 4.3) is defined as

$$740 \quad (4.45) \quad N_{\mathbf{a},\varepsilon} = \min\{N \in \mathbb{N}_* : r_{\mathbf{a},N} < \varepsilon\}, \quad r_{\mathbf{a},N} = \sum_{\omega_{\mathbf{a},n} > \Omega_{\mathbf{a},N}} \frac{a_{\mathbf{a},n}}{\omega_{\mathbf{a},n}^2}.$$

741

742 By Proposition 4.4, with (4.6),

$$743 \quad (4.46) \quad r_{\mathbf{a},N} < \tilde{C}_{\mathbf{a}} \Omega_{\mathbf{a},N}^{-1}, \quad \text{and thus} \quad N_{\mathbf{a},\varepsilon} \leq \min\{N \in \mathbb{N}_* : \tilde{C}_{\mathbf{a}} \Omega_{\mathbf{a},N}^{-1} \leq \varepsilon\},$$

745 or, in other words,

$$746 \quad N_{\mathbf{a},\varepsilon} \leq \min\{N \in \mathbb{N}_* : \Omega_{\mathbf{a},N} \geq \tilde{C}_{\mathbf{a}} \varepsilon^{-1}\} \equiv \#\{n : \Omega_{\mathbf{a},n} < \tilde{C}_{\mathbf{a}} \varepsilon^{-1}\} + 1.$$

748 From the above and (4.43) we have

$$749 \quad (4.47) \quad N_{\mathbf{a},\varepsilon} \leq P_{\mathbf{a}}(\tilde{C}_{\mathbf{a}} \varepsilon^{-1}) + 1,$$

751 and the result of Theorem 4.3 follows from the above bound and Corollary 4.12. \square

752 **4.6. Proof of Theorem 4.2.** We will prove this result as a corollary of Theorem
 753 4.3. Let us show the result for the case $\langle \alpha \rangle < 1$. Evidently, it suffices to show this
 754 bound for all N sufficiently large.

755 First, let us fix $0 < \varepsilon < 1/2$. By definition (4.45) of $N_{\mathbf{a},\varepsilon}$, we have $r_{\mathbf{a},N} < \varepsilon$ for all
 756 $N \geq N_{\mathbf{a},\varepsilon}$. In particular, by Theorem 4.3, we have that $N_{\mathbf{a},\varepsilon} < N_* := \lceil C_{\mathbf{a}}^+ \varepsilon^{-1} \rceil$, and
 757 therefore $r_{\mathbf{a},N_*} < \varepsilon$. In other words, for all $0 < \varepsilon < 1/2$,

$$758 \quad (4.48) \quad r_{\mathbf{a},\lceil C_{\mathbf{a}}^+ \varepsilon^{-1} \rceil} < \varepsilon.$$

760 For any $N \in \mathbb{N}$ sufficiently large, there exists $0 < \varepsilon < 1/2$ s.t. $N = \lceil C_{\mathbf{a}}^+ \varepsilon^{-1} \rceil$, and in
 761 this case $\varepsilon \leq C_{\mathbf{a}}^+(N-1)^{-1}$. This with (4.48) yields the desired bound $r_{\mathbf{a},N} \leq c_{\mathbf{a}}^+ N^{-1}$.

762 The result for the cases $\langle \alpha \rangle \geq 1$ follows similarly (with the case $\langle \alpha \rangle = 1$ being
 763 somewhat less trivial); the details are left to the reader.

764 **5. Explicit lower bounds for the error indicator.** The goal of this section
 765 is to prove the lower bounds on $r_{\mathbf{a},N}$, as discussed in Section 3.3.

766 **5.1. Main results.** The two main results of this section read.

767 **THEOREM 5.1.** *There exists $c_{\mathbf{a}}^- > 0$, depending only on μ, α , such that, for all*
 768 *$N \geq 1$, $r_{\mathbf{a},N}$ defined in (3.7) satisfies:*

$$769 \quad r_{\mathbf{a},N} \geq c_{\mathbf{a}}^- N^{-1}.$$

771 This result can be re-formulated in terms of (4.3).

772 **THEOREM 5.2.** *There exists $C_{\mathbf{a}}^- > 0$, depending only on μ, α , such that, for all*
 773 *$0 < \varepsilon < 1$, $N_{\mathbf{a},\varepsilon}$ defined in (4.3) satisfies:*

$$774 \quad N_{\mathbf{a},\varepsilon} \geq C_{\mathbf{a}}^- \varepsilon^{-1}.$$

776 Comparing the statements of Theorems 4.2 and Theorem 5.1, we see that the upper
 777 bound on $r_{\mathbf{a},N}$ when $\langle \alpha \rangle < 1$ is sharp. This however does not seem to be the case
 778 when $\langle \alpha \rangle \geq 1$. In particular, when $\langle \alpha \rangle > 1$, we have the bound $r_{\mathbf{a},N} < c_{\mathbf{a}}^+ N^{-1/d_s}$.

779 It is natural to ask whether in this case one of the bounds (the upper or the lower
 780 one) is not sharp. The answer to this question depends on the tree \mathcal{T} and is two-fold:

- 781 \bullet when there exists $i \neq j$ s.t. $\alpha_i = \alpha_j$ (i.e. in the presence of symmetries), it
 782 may happen that $\langle \alpha \rangle \geq 1$, and yet $r_{\mathbf{a},N} \leq CN^{-1}$, with some $C > 0$. This is
 783 clarified in Appendix A.
- 784 \bullet in the case when $\alpha_i \neq \alpha_j$ for $i \neq j$, our numerical experiments (cf. [8])
 785 indicate that the upper bound provided by Theorem 4.2 is likely to be sharp.

786 **5.2. Plan of the proof and of this section.** The proofs of Theorems 5.1 and
 787 5.2 will be similar in their ideas to the proofs of Theorems 4.2, 4.3. Recall in particular
 788 that the proof of Theorem 4.3 relies on the following ingredients:

- 789 1. a lower bound on $r_{\mathbf{a},N} \leq C\Omega_{\mathbf{a},N}^{-1}$, which shows that to guarantee that $r_{\mathbf{a},N} < \varepsilon$,
 790 one takes $N \leq N_*$ where N_* is the smallest number s.t. $C\varepsilon^{-1} < \Omega_{\mathbf{a},N_*}$;
 791 2. an observation that $N_* = P_{\mathbf{a}}(C\varepsilon^{-1}) + 1$;
 792 3. an upper bound on $P_{\mathbf{a}}(C\varepsilon^{-1})$ (cf. Corollary 4.12).

793 In the proof of the lower bound we will repeat this reasoning but with inverse inequal-
 794 ities. We will make use of the auxiliary results that are counterparts of Proposition
 795 4.4 and Theorem 4.8.

796 *Plan of the section.* This section is organized as follows. In Section 5.3, we prove
 797 the bound $r_{\mathbf{a},N} \geq c\Omega_{\mathbf{a},N}^{-1}$. Section 4.4 is dedicated to the proof of the lower bound for
 798 the number of the poles of $\Lambda_{\mathbf{a}}$ on an interval. In Section ?? we combine these two
 799 results to prove Theorem 5.2. In Section ??, we prove Theorem 5.1.

800 **5.3. An implicit lower bound for $r_{\mathbf{a},N}$.** Recall that according to the definition
 801 (4.6), $R_{\mathbf{a},\omega} = \sum_{\omega_{\mathbf{a},n} > \omega} \frac{a_{\mathbf{a},n}}{\omega_{\mathbf{a},n}^2}$. This quantity satisfies the counterpart of Proposition 4.4:

802 PROPOSITION 5.3. *There exists $\tilde{c} > 0$, s.t.*

803 (5.1)
$$R_{\mathbf{a},\omega} \geq \tilde{c}\omega^{-1}, \text{ for all } \omega \geq 1.$$

805 Just like Proposition 4.4, the above result relies on the estimates on the Neumann
 806 traces of the eigenfunctions.

807 **5.3.1. A lower bound for the Neumann traces of eigenfunctions.** The
 808 result that follows shows the sharpness of the estimate of Lemma 4.5.

809 LEMMA 5.4. *There exists $\eta_* > 0$, s.t. for all $\eta > \eta_*$, there exists $C_{\eta}^- > 0$, s.t.*

810 (5.2)
$$\sum_{j: |\omega_{\mathbf{a},j} - \omega| < \eta} (\partial_s \varphi_{\mathbf{a},j}(M^*))^2 \omega_{\mathbf{a},j}^{-2} \equiv \sum_{j: |\omega_{\mathbf{a},j} - \omega| < \eta} a_{\mathbf{a},j} \geq C_{\eta}^-, \text{ for all } \omega \geq 1.$$

811

812 The proof is based on the ideas different from the ones from the proof of Lemma 4.5.

813 *Proof.* First, we need an auxiliary observation. By (2.21, 2.22), see also Remark
 814 2.7, because μ is piecewise-constant, the eigenfunctions of $\mathcal{A}_{\mathbf{a}}$ satisfy $-\partial_s^2 \varphi_{\mathbf{a},n} =$
 815 $\omega_{\mathbf{a},n}^2 \varphi_{\mathbf{a},n}$ on $\Sigma_{0,0}$ (identified with $[0, 1]$). Since $\varphi_{\mathbf{a}}(M^*) = 0$ on $\Sigma_{0,0}$,

816 (5.2)
$$\varphi_{\mathbf{a},n}(s) = C_{\mathbf{a},n} \sin \omega_{\mathbf{a},n} s, \quad C_{\mathbf{a},n} \geq 0.$$

818 We can easily relate $C_{\mathbf{a},n}$ to the coefficients $a_{\mathbf{a},n} = (\partial_s \varphi_{\mathbf{a},n}(0))^2 \omega_{\mathbf{a},n}^{-2}$:

819 (5.3)
$$\varphi_{\mathbf{a},n}(s) = a_{\mathbf{a},n}^{\frac{1}{2}} \sin \omega_{\mathbf{a},n} s.$$

820

821 Let us now prove the desired result. Let $\omega \geq 1$, and let $v_{\omega} : \mathcal{T} \rightarrow \mathbb{R}$ be s.t. $v_{\omega}(s) =$
 822 $\sin(\omega s)$ on $\Sigma_{0,0}$ and $v_{\omega}(s) = 0$ otherwise. Because $v_{\omega} \in L_{\mu}^2(\mathcal{T})$, and $\{\varphi_{\mathbf{a},n}\}_{n \in \mathbb{N}_*}$ is a
 824 Schauder basis in $L_{\mu}^2(\mathcal{T})$, the following series converges in $L_{\mu}^2(\mathcal{T})$:

825 (5.4)
$$v_{\omega} = \sum_{n=0}^{\infty} \gamma_{\mathbf{a},n}^{\omega} \varphi_{\mathbf{a},n}, \quad \gamma_{\mathbf{a},n}^{\omega} = \int_0^1 v_{\omega}(s) \varphi_{\mathbf{a},n}(s) ds.$$

826

827 Moreover, for all $\eta > 0$,

$$828 \quad (5.5) \quad \|v_\omega\|_{L^2_\mu(\mathcal{T})}^2 = \sum_{n=0}^{\infty} |\gamma_{\mathbf{a},n}^\omega|^2 = \mathcal{S}_{\omega,\eta}^{\mathbf{a}} + \mathcal{I}_{\eta,\omega}^{\mathbf{a}},$$

$$829 \quad \mathcal{S}_{\omega,\eta}^{\mathbf{a}} = \sum_{n: |\omega_{\mathbf{a},n} - \omega| < \eta} |\gamma_{\mathbf{a},n}^\omega|^2, \quad \mathcal{I}_{\eta,\omega}^{\mathbf{a}} = \sum_{n: |\omega_{\mathbf{a},n} - \omega| \geq \eta} |\gamma_{\mathbf{a},n}^\omega|^2.$$

830 To prove (5.2), we will prove the three inequalities (with constants uniform in η, ω):

$$831 \quad (5.6) \quad (a) |\mathcal{S}_{\omega,\eta}^{\mathbf{a}}| \leq \sum_{|\omega_{\mathbf{a},n} - \omega| < \eta} a_{\mathbf{a},n}, \quad (b) |\mathcal{I}_{\omega,\eta}^{\mathbf{a}}| \leq C_I \eta^{-1}, \quad (c) \|v_\omega\|^2 \geq \frac{1}{4}.$$

832

833 Combining these bounds in (5.5) then yields

$$834 \quad \frac{1}{4} \leq \sum_{|\omega_{\mathbf{a},n} - \omega| < \eta} a_{\mathbf{a},n} + C_I \eta^{-1},$$

835

836 and the lower bound (5.2) follows by taking $\eta > \eta_*$, with η_* sufficiently large. It
837 remains to prove (5.6). For this, first of all, we rewrite $\gamma_{\mathbf{a},n}$ using (5.3):

$$838 \quad (5.7) \quad \gamma_{\mathbf{a},n}^\omega = a_{\mathbf{a},n}^{\frac{1}{2}} \int_0^1 \sin(\omega s) \sin(\omega_{\mathbf{a},n} s) ds = \frac{a_{\mathbf{a},n}^{\frac{1}{2}}}{2} \left(-\frac{\sin(\omega - \omega_{\mathbf{a},n})}{\omega - \omega_{\mathbf{a},n}} + \frac{\sin(\omega + \omega_{\mathbf{a},n})}{\omega + \omega_{\mathbf{a},n}} \right).$$

839

840 Because $|\sin(x)/x| \leq 1$, and also $\omega + \omega_{\mathbf{a},n} > |\omega_{\mathbf{a},n} - \omega|$, the above yields

$$841 \quad (5.8) \quad |\gamma_{\mathbf{a},n}^\omega| \leq a_{\mathbf{a},n}^{\frac{1}{2}} \min\left(\frac{1}{|\omega - \omega_{\mathbf{a},n}|}, 1\right).$$

842

843 *Proof of (5.6)-(a).* By (5.8), $|\gamma_{\mathbf{a},n}^\omega| \leq a_{\mathbf{a},n}^{\frac{1}{2}}$, hence the desired bound is immediate from
844 the definition (5.5) of $\mathcal{S}_{\omega,\eta}^{\mathbf{a}}$.

845 *Proof of (5.6)-(b).* Let us first prove the desired bound for $\omega = W \in \mathbb{N}$, and $\eta = k \in \mathbb{N}$,
846 and next argue on extending it to the general case. With the definition (5.5) of $\mathcal{I}_{\omega,\eta}^{\mathbf{a}}$,

$$848 \quad \mathcal{I}_{W,k}^{\mathbf{a}} = \sum_{\omega_{\mathbf{a},n} \leq W-k} |\gamma_{\mathbf{a},n}^\omega|^2 + \sum_{\omega_{\mathbf{a},n} \geq W+k} |\gamma_{\mathbf{a},n}^\omega|^2 \stackrel{(5.8)}{\leq} I_1 + I_2,$$

$$849 \quad (5.9) \quad I_1 = \sum_{\omega_{\mathbf{a},n} \leq W-k} \frac{a_{\mathbf{a},n}}{(W - \omega_{\mathbf{a},n})^2}, \quad I_2 = \sum_{\omega_{\mathbf{a},n} \geq W+k} \frac{a_{\mathbf{a},n}}{(\omega_{\mathbf{a},n} - W)^2}.$$

850

851 To bound each of the above terms, we will make use of Lemma 4.5. The first term in
852 (5.9) vanishes when $W \leq k$; otherwise, it can be rewritten as follows:

$$853 \quad I_1 = \sum_{\ell=0}^{W-k-1} \sum_{\ell \leq \omega_{\mathbf{a},n} < \ell+1} \frac{a_{\mathbf{a},n}}{(W - \omega_{\mathbf{a},n})^2} \leq \sum_{\ell=0}^{W-k-1} \frac{1}{(W - \ell - 1)^2} \sum_{\ell \leq \omega_{\mathbf{a},n} < \ell+1} a_{\mathbf{a},n}.$$

854

855 Using Lemma 4.5 with $\eta = 1$ we obtain the following bound

$$856 \quad I_1 \leq C_1^+ \sum_{\ell=0}^{W-k-1} (W - \ell - 1)^{-2} = C_1^+ \sum_{\ell=k}^{W-1} \frac{1}{\ell^2} \leq \frac{C_1}{k},$$

857

858 where C_1 does not depend on W or k . Altogether, we have

$$859 \quad (5.10) \quad I_1 = \begin{cases} 0, & \text{if } k \geq W, \\ \leq C_1 k^{-1}, & \text{otherwise.} \end{cases}$$

861 It remains to bound the second term in (5.9); we use the same ideas:

$$862 \quad I_2 = \sum_{\ell=W+k}^{\infty} \sum_{\ell \leq \omega_{a,n} < \ell+1} \frac{a_{a,n}}{(\omega_{a,n} - W)^2} \leq \sum_{\ell=W+k}^{\infty} \frac{1}{(\ell - W)^2} \sum_{\ell \leq \omega_{a,n} < \ell+1} a_{a,n}$$

$$863 \quad \leq C_1^+ \sum_{\ell=W+k}^{\infty} \frac{1}{(\ell - W)^2}, \text{ by Lemma 4.5.}$$

865 We thus easily obtain a uniform in W bound, with C_2 independent of W and k :

$$866 \quad (5.11) \quad I_2 < C_2 k^{-1}.$$

868 Combining (5.10) and (5.11) into (5.9) yields the desired bound

$$869 \quad (5.12) \quad |\mathcal{I}_{W,k}^a| \leq (C_1 + C_2)k^{-1}.$$

871 When $\omega, \eta \notin \mathbb{N}$, and $\omega, \eta > 1$, one has a very rough bound $\mathcal{I}_{\omega,\eta} \leq \mathcal{I}_{[\omega],[\eta]} + \mathcal{I}_{[\omega],[\eta]}$,

872 which, with (5.12), yields the desired bound (5.6)-(b).

873 *Proof of (5.6)-(c).* The proof is straightforward:

$$874 \quad \|v\|_{L^2_{\mu}(\mathcal{T})}^2 \equiv \int_0^1 \sin^2(\omega s) ds = \frac{1}{2} - \frac{\sin(2\omega s)}{4\omega} \geq \frac{1}{4}, \text{ when } \omega \geq 1. \quad \square$$

876 **5.3.2. Proof of Proposition 5.3.** Let us fix $\eta := \ell \geq 1$ integer and large
 877 enough, so that the conditions of Lemma 5.4 hold true. Because $\omega \mapsto R_{a,\omega}$ is decreas-
 878 ing and piecewise-constant, it suffices to prove the result for $\omega = M \in \mathbb{N}$ sufficiently
 879 large. Let us assume that $M > \ell$. Recall that $R_{a,M} = \sum_{n:\omega_{a,n} > M} \frac{a_{a,n}}{\omega_{a,n}^2}$.

880 Because we would like to apply the result of Lemma 5.4, we use the same trick
 881 like in the proof of Proposition 4.4, namely, we split the interval $[M, \infty)$ into smaller
 882 intervals of the width 2ℓ that do not intersect:

$$883 \quad [M, \infty) = [(M + \ell) - \ell, (M + \ell) + \ell) \cup [(M + 3\ell) - \ell, (M + 3\ell) + \ell) \cup \dots$$

885 With a shortened notation, $I_k := [(M + k\ell) - \ell, (M + k\ell) + \ell)$, we rewrite

$$886 \quad (5.13) \quad R_{a,M} = \sum_{k=1}^{\infty} \sum_{\omega_{a,n} \in I_{2k-1}} \frac{a_{a,n}}{\omega_{a,n}^2} \geq \sum_{k=1}^{\infty} \frac{1}{((M + (2k-1)\ell) - \ell)^2} \sum_{\omega_{a,n} \in I_{2k-1}} a_{a,n}.$$

888 By Lemma 5.4, applied with $\omega = M + (2k-1)\ell$ and $\eta = \ell$,

$$889 \quad \sum_{\omega_{a,n} \in I_{2k-1}} a_{a,n} \geq C_{\ell}^{-} > 0 \text{ for all } k \in \mathbb{N}_*.$$

891 Inserting the above bound into (5.13) yields the desired bound:

$$892 \quad R_{a,M} \geq C_{\ell}^{-} \sum_{k=1}^{\infty} \frac{1}{(M + (2k-1)\ell)^2} \geq C_{\ell}^{-} \sum_{k:(2k-1)\ell \geq M} \frac{1}{(M + (2k-1)\ell)^2}$$

$$893 \quad \geq C_{\ell}^{-} \sum_{k:(2k-1)\ell \geq M} \frac{1}{4(2k-1)^2} \geq \frac{\tilde{c}}{M}, \text{ with some } \tilde{c} > 0.$$

894

895 **5.4. A lower bound for the number of poles on an interval.** In this section
 896 we will state a lower bound for $P_a(\lambda) = \#\{n : \Omega_{a,n} < \lambda\}$.

897 **THEOREM 5.5.** $P_a(\lambda)$ defined in (4.43) satisfies

898 (5.14)
$$P_a(\lambda) > \lfloor (2\pi)^{-1}\lambda \rfloor, \quad \text{for all } \lambda > 0.$$

900 The above result does not follow from the asymptotics of Theorem 4.8, since it also
 901 encodes some information about the eigenvalue multiplicity and the Neumann traces
 902 of the eigenfunctions, see the discussion in Section 2.5. It shows that inside the interval
 903 $(0, \lambda)$ there are at least $O\left(\lambda^{\frac{1}{2}}\right)$ distinct eigenvalues, with corresponding eigenfunctions
 904 $\varphi_{a,n}$ satisfying $\partial_s \varphi_{a,n}(M^*) \neq 0$. The proof of this result is simple, and relies on
 905 Lemma 5.3 from [9], which shows that $\Lambda_a(\omega)$ satisfies a certain non-linear equation.

906 **LEMMA 5.6** (Lemma 5.3 from [9]). *For any $\omega \in \mathbb{C} \setminus \mathbb{R}$,*

907 (5.15)
$$\Lambda_a(\omega) = \omega \frac{\cos \omega \mathbf{F}_a(\omega) - \omega \sin \omega}{\sin \omega \mathbf{F}_a(\omega) + \omega \cos \omega}, \quad \mathbf{F}_a(\omega) = \sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i} \Lambda_a(\alpha_i \omega), \quad a \in \{\mathfrak{d}, \mathfrak{n}\}.$$

908

909 We will also need the following result which is an expression of a general monotonicity
 910 property of Herglotz functions.

911 **LEMMA 5.7.** *Let \mathbf{F}_a be defined in (5.15). Then $\mathbf{F}'_a(\omega) \leq 0$ for all $\omega > 0$ s.t. ω is
 912 not a pole of $\mathbf{F}_a(\omega)$.*

913 *Proof.* Evidently it is sufficient to show that $\omega > 0$, $\Lambda'_a(\omega) < 0$. Using the
 914 representation (2.27), one proves that $\Lambda'_a(\omega)$ equals to the series (which converges
 915 uniformly on compact subsets of \mathbb{C} not containing the poles of Λ_a (this is easy to see
 916 by comparing the series to the uniformly convergent series (2.27)):

917
$$\Lambda'_a(\omega) = - \sum_{n=1}^{+\infty} \frac{2A_{a,n} \Omega_{a,n}^2 \omega}{(\Omega_{a,n}^2 - \omega^2)^2}.$$

918

919 The above quantity is negative for $\omega > 0$, since $A_{a,n} > 0$ for all $n \geq 1$. □

920 The above two results suffice for the proof of Theorem 5.5.

921 *Proof of Theorem 5.5.* To prove the desired result, we will show that on each
 922 interval $I_m := [m\pi, (m+1)\pi]$, $m \in \mathbb{N}_*$, the function $\Lambda_a(\omega)$ has at least one pole.
 923 For this we will use the expression (5.15). As Λ_a is meromorphic, it is also valid for
 924 $\omega \in \mathbb{R} \setminus \mathcal{P}_a$ (outside of poles).

925 *Step 1. Necessary and sufficient conditions for $\Lambda_a(\Omega_0) = \infty$.* Examining (5.15)
 926 reveals that $\omega = \Omega_0 \in \mathbb{R}_*^+$ is a pole of $\Lambda_a(\omega)$ iff one of the following holds true:

927 (C1) $\mathbf{F}_a(\omega)$ has a pole in Ω_0 , and $\lim_{\omega \rightarrow \Omega_0} \Lambda_a(\omega) = \infty$, in other words, with (5.15),

928

929
$$\lim_{\omega \rightarrow \Omega_0} \omega \frac{\cos \omega - \omega \sin \omega \mathbf{F}_a^{-1}(\omega)}{\sin \omega + \omega \cos \omega \mathbf{F}_a^{-1}(\omega)} = \lim_{\omega \rightarrow \Omega_0} \omega \frac{\cos \omega}{\sin \omega} = \infty.$$

930

931 This is possible, if and only if, for some $k \in \mathbb{N}_*$,

932 (5.16)
$$\Omega_0 = k\pi \text{ and } \mathbf{F}_a(\Omega_0) = \infty.$$

934 (C2) $\mathbf{F}_a(\Omega_0) \neq \infty$, the denominator of (5.15) vanishes and the numerator does not:

935 (5.17) (a) $\sin \Omega_0 \mathbf{F}_a(\Omega_0) = -\Omega_0 \cos \Omega_0$, (b) $\cos \Omega_0 \mathbf{F}_a(\Omega_0) \neq \Omega_0 \sin \Omega_0$.

937 Remark that $(a) \Rightarrow (b)$. To see this, assume the opposite: $(a) \not\Rightarrow (b)$, i.e.

$$938 \quad (a) \sin \Omega_0 \mathbf{F}_a(\Omega_0) = -\Omega_0 \cos \Omega_0, \quad (b') \cos \Omega_0 \mathbf{F}_a(\Omega_0) = \Omega_0 \sin \Omega_0.$$

940 Evidently, as $\Omega_0 \neq 0$, and $\mathbf{F}_a(\Omega_0) \neq \infty$, for (a) to hold it is necessary that
 941 $\sin \Omega_0 \neq 0$. With (a), (b') is equivalent to $-\cos^2 \Omega_0 = \sin^2 \Omega_0$, and as $\Omega_0 \in$
 942 \mathbb{R}_* , we arrive at the contradiction. Therefore, (5.17) is equivalent to

$$943 \quad (5.18) \quad \mathbf{F}_a(\Omega_0) = -\Omega_0 (\tan \Omega_0)^{-1}, \quad \Omega_0 \neq k\pi, k \in \mathbb{N}_*.$$

945 *Step 2. Proof that on I_m Λ_a has at least one pole.* We will prove the result by
 946 contradiction. Let us assume that Λ_a has no poles in I_m . In particular, this is
 947 possible iff (5.16) and (5.18) do not hold. I.e. Λ_a has no poles in I_m iff

$$948 \quad (5.19) \quad \mathbf{F}_a(m\pi) \neq \infty, \quad \mathbf{F}_a((m+1)\pi) \neq \infty, \text{ and}$$

$$949 \quad (5.20) \quad \mathbf{F}_a(\omega) \neq f(\omega), \quad f(\omega) := -\omega(\tan \omega)^{-1} \text{ on } I_m.$$

951 It remains to consider two possibilities:

952 **Case 1. \mathbf{F}_a has no poles on the interval I_m .** By continuity of \mathbf{F}_a and f , (5.20)
 953 holds if and only if $\mathbf{F}_a > f$ or $\mathbf{F}_a < f$ on $(m\pi, (m+1)\pi)$.

954 However, f is strictly growing on I_m , and

$$955 \quad (5.21) \quad \lim_{\omega \rightarrow m\pi^+} f(\omega) = -\infty, \quad \lim_{\omega \rightarrow (m+1)\pi^-} f(\omega) = +\infty.$$

957 Hence $\mathbf{F}_a > f$ on I_m would mean that $\mathbf{F}_a(\omega)$ has a pole in $\omega = (m+1)\pi$, while
 958 $\mathbf{F}_a < f$ on I_m would imply that $\mathbf{F}_a(\omega)$ has a pole in $\omega = m\pi$. This contradicts (5.19).

959 **Case 2. \mathbf{F}_a has at least one pole inside I_m .** We will consider the case when it
 960 has a single pole, while the case with multiple poles can be studied similarly.

961 With (5.19), we assume that \mathbf{F}_a has a pole in Ω_0 with $\Omega_0 \in (m\pi, (m+1)\pi)$. With
 962 (5.20), on the interval $(m\pi, \Omega_0)$, $\mathbf{F}_a \neq f$. This is possible iff on $(m\pi, \Omega_0)$ either $\mathbf{F}_a > f$
 963 or $\mathbf{F}_a < f$. Let us show that considering both possibilities leads to a contradiction:

965 • let us assume that $(m\pi, \Omega_0)$, $\mathbf{F}_a > f$. By Lemma 5.7, $\mathbf{F}'_a \leq 0$, and thus on
 966 the interval $(m\pi, \Omega_0)$, \mathbf{F}_a is a continuous monotonically decreasing function
 967 that changes its value from $\mathbf{F}_a(m\pi)$ to $-\infty$. This implies that f has a pole
 968 in $\Omega_0 \in (m\pi, (m+1)\pi)$, and we arrive at the contradiction.

970 • let us assume that $(m\pi, \Omega_0)$, $\mathbf{F}_a < f$. By (5.21) $\lim_{\omega \rightarrow m\pi^+} f(\omega) = -\infty$, thus \mathbf{F}_a
 971 has a pole in $m\pi$, but this is impossible by (5.20).
 972

973 Thus, inside each interval $I_m = [m\pi, (m+1)\pi]$ $\Lambda_a(\omega)$ has at least one pole;
 974 hence, on each half-open interval $[m\pi, (m+2)\pi)$, $m \geq 0$, $\Lambda_a(\omega)$ has at least one pole.
 975 Because the intervals of such form do not intersect, and the interval $(0, \lambda)$ contains at
 976 least $\lfloor \frac{\lambda}{2\pi} \rfloor$ such intervals (and thus poles), we conclude that $P_a(\lambda) \geq \lfloor \frac{\lambda}{2\pi} \rfloor$. \square

977 **5.5. Proof of Theorem 5.2.** Evidently, it suffices to prove the desired result
 978 for all $\varepsilon > 0$ sufficiently small. We proceed just like in the proof of Theorem 4.3.
 979 First of all, recall that $r_{a,N} = R_{a,\Omega_{a,N}}$, cf. (4.6), and, by Proposition 5.3, we have the
 980 following bound for all $\Omega_{a,N} > 1$:

$$981 \quad (5.22) \quad r_{a,N} > \tilde{c} \Omega_{a,N}^{-1}, \text{ with some } \tilde{c} > 0.$$

983 By definition $N_{\mathbf{a},\varepsilon} = \min\{N : r_{\mathbf{a},N} < \varepsilon\}$, and therefore, for all ε sufficiently small,

$$\begin{aligned} 984 \quad N_{\mathbf{a},\varepsilon} &\geq \min\{N : \tilde{c}\Omega_{\mathbf{a},N}^{-1} \leq \varepsilon\} = \min\{N : \Omega_{\mathbf{a},N} \geq \tilde{c}\varepsilon^{-1}\} \\ 985 \quad &\equiv \#\{n : \Omega_{\mathbf{a},n} < \tilde{c}\varepsilon^{-1}\} + 1 = P_{\mathbf{a}}(\tilde{c}\varepsilon^{-1}) + 1. \end{aligned}$$

987 By Theorem 5.5, for all ε sufficiently small, $N_{\mathbf{a},\varepsilon} > c\varepsilon^{-1}$.

988 **5.6. Proof of Theorem 5.1.** We proceed just like in the proof of Theorem 4.2.

989 Let $\varepsilon > 0$. From the definition of $N_{\mathbf{a},\varepsilon}$ (4.3), it follows that $r_{\mathbf{a},N} > \varepsilon$ for all
990 $N < N_{\mathbf{a},\varepsilon}$. From Theorem 5.2, we know that $N_{\mathbf{a},\varepsilon} > C_{\mathbf{a}}^{-}\varepsilon^{-1}$ (which is > 1 for ε small
991 enough), and therefore $r_{\mathbf{a},\lfloor C_{\mathbf{a}}^{-}\varepsilon^{-1} \rfloor} > \varepsilon$. For all N sufficiently large there exists ε s.t.
992 $N = \lfloor C_{\mathbf{a}}^{-}\varepsilon^{-1} \rfloor$, and $\varepsilon > C_{\mathbf{a}}^{-}N^{-1}$. Therefore, $r_{\mathbf{a},N} > C_{\mathbf{a}}^{-}N^{-1}$.

993 **6. Conclusions and Open Questions.** In this work, we have presented a
994 refined error analysis for high-order approximated transparent boundary conditions
995 for the weighted wave equation on a self-similar one-dimensional fractal tree. This
996 approach is an alternative to the convolution quadrature [6] and is based on the
997 truncation of the meromorphic series representing the symbol of the DtN operator.
998 The complexity of the method depends on the number of poles in the truncated
999 series; we have presented estimates on the number of poles, required to achieve a
1000 desired accuracy ε , based on the analysis of the eigenfunctions and eigenvalues of
1001 the weighted Laplacian on the fractal tree. Our future efforts are directed towards
1002 improving the convergence of the technique, based on approximation of the remainder
1003 of the meromorphic series.

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1006 **Appendix A. Behaviour of $r_{\mathbf{a},N}$ in the presence of symmetries.** Let us
1007 consider the tree \mathcal{T} with the length ratios $\boldsymbol{\alpha}$ s.t. there exists $i \neq j$, s.t. $\alpha_i = \alpha_j$. We
1008 will call such a tree a symmetric tree.

1009 In particular, let us assume that the length ratios are given by

$$1010 \quad (\text{A.1}) \quad \boldsymbol{\alpha} = (\alpha_0^{(0)}, \alpha_0^{(1)}, \dots, \alpha_0^{(n_0)}, \alpha_1^{(0)}, \dots, \alpha_1^{(n_1)}, \dots, \alpha_{p-1}^{(0)}, \dots, \alpha_{p-1}^{(n_{p-1})}), \quad n_i \geq 0,$$

1012 such that for all $k = 0, \dots, p-1$, $i = 0, \dots, n_{p-1}$, it holds that

$$1013 \quad \alpha_k^{(i)} = \tilde{\alpha}_k, \quad \text{and} \quad \tilde{\alpha}_k \neq \tilde{\alpha}_j, \quad k \neq j.$$

1015 Similarly we define the vector $\boldsymbol{\mu} = (\mu_0^{(0)}, \mu_0^{(1)}, \dots, \mu_{p-1}^{(n_{p-1})})$ (without any constraints
1016 but the positivity of the coefficients). Temporary, in this appendix, the symbol of
1017 the reference DtN operator associated to the tree with the parameters $\boldsymbol{\alpha}, \boldsymbol{\mu}$ will be
1018 denoted by $\Lambda_{\mathbf{a}}^{\boldsymbol{\alpha}, \boldsymbol{\mu}}$. Similarly, we will use the notation $r_{\mathbf{a},N}^{\boldsymbol{\alpha}, \boldsymbol{\mu}}, N_{\mathbf{a},\varepsilon}^{\boldsymbol{\alpha}, \boldsymbol{\mu}}, \rho_{\mathbf{a},\varepsilon}^{\boldsymbol{\alpha}, \boldsymbol{\mu}}, P_{\mathbf{a}}^{\boldsymbol{\alpha}, \boldsymbol{\mu}}$.

1019 Given (A.1), let us define the vector of the values $\tilde{\alpha}_k$ and the associated \tilde{d}_s (4.1):

$$1020 \quad (\text{A.2}) \quad \tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_0, \dots, \tilde{\alpha}_{p-1}), \quad \text{and} \quad \tilde{d}_s \in \mathbb{R}_*^+ \text{ s.t. } \sum_j \tilde{\alpha}_j^{\tilde{d}_s} = 1.$$

1022 Remark that the above quantity does not coincide with the Minkowski dimension d_s
1023 of the tree \mathcal{T} , cf. (4.1). More precisely, because $\sum_j \sum_{k=0}^{n_j} \alpha_j^{(k)} > \sum_j \tilde{\alpha}_j$, we have

$$1024 \quad (\text{A.3}) \quad \langle \tilde{\boldsymbol{\alpha}} \rangle \leq \langle \boldsymbol{\alpha} \rangle, \quad \tilde{d}_s \leq d_s.$$

1026 **A.1. Two principal results.** As discussed in Section 5.1, we have the following
 1027 improvement of the upper bound of Theorem 4.2 for symmetric trees.

1028 PROPOSITION A.1. *For some c_a^+ depending only on α, μ , the following holds true:*
 1029 *for all $N \geq 2$, $r_{a,N}^{\alpha,\mu}$ satisfies:*

- 1031 \bullet *if $\langle \tilde{\alpha} \rangle < 1$ ($\tilde{d}_s < 1$), $r_{a,N}^{\alpha,\mu} \leq c_a^+ N^{-1}$.*
- 1032 \bullet *if $\langle \tilde{\alpha} \rangle = 1$ ($\tilde{d}_s = 1$), $r_{a,N}^{\alpha,\mu} \leq c_a^+ N^{-1} \log N$.*
- 1033 \bullet *if $\langle \tilde{\alpha} \rangle > 1$ ($\tilde{d}_s > 1$), $r_{a,N}^{\alpha,\mu} \leq c_a^+ N^{-1/\tilde{d}_s}$.*

1036 The above proposition is very similar to Theorem 4.2. The difference is that in Theo-
 1037 rem 4.2 it is the whole vector α that plays a role, while the above result involves only
 1038 the vector of the non-repeating values $\tilde{\alpha}$. With (A.3), we see that the error indicator
 1039 may decrease faster than predicted by Theorem 4.2 in the presence of symmetries.

1040 REMARK A.2. *The simplest illustration to the statement of Proposition A.1 is*
 1041 *given by \mathcal{T} s.t. $\alpha = (\alpha_0^{(0)}, \dots, \alpha_0^{(p-1)})$, i.e. $\tilde{\alpha} = (\tilde{\alpha}_0)$. In this case $d_s = \log p / \log \tilde{\alpha}_0^{-1}$,*
 1042 *while $\tilde{d}_s = 1 / \log \tilde{\alpha}_0^{-1} (< 1)$. Independently of the value of p , by Proposition A.1,*
 1043 *$r_{a,N}^{\alpha,\mu} \leq c_a^+ N^{-1}$.*

1044 In a similar manner, the counterpart of Theorem 4.3 in the symmetric case reads.

1045 PROPOSITION A.3. *There exists $C_a^+ > 0$, depending only on μ, α , such that, for*
 1046 *all $0 < \varepsilon < 1/2$, $N_{a,\varepsilon}^{\alpha,\mu}$ satisfies:*

- 1048 \bullet *if $\langle \tilde{\alpha} \rangle < 1$ ($\tilde{d}_s < 1$), $N_{a,\varepsilon}^{\alpha,\mu} \leq C_a^+ \varepsilon^{-1}$.*
- 1049 \bullet *if $\langle \tilde{\alpha} \rangle = 1$ ($\tilde{d}_s = 1$), $N_{a,\varepsilon}^{\alpha,\mu} \leq C_a^+ \varepsilon^{-1} \log \varepsilon^{-1}$.*
- 1050 \bullet *if $\langle \tilde{\alpha} \rangle > 1$ ($\tilde{d}_s > 1$), $N_{a,\varepsilon}^{\alpha,\mu} \leq C_a^+ \varepsilon^{-\tilde{d}_s}$.*

1053 **A.2. Main idea of the proof and auxiliary results.** Like in Section 4, we
 1054 first prove Proposition A.3, and next Proposition A.1. Recall that the proof of Theo-
 1055 rem 4.3 (whose refinement is given by Proposition A.3) relies on Corollary 4.12 about
 1056 the upper bound of the pole counting function $P_a(\lambda)$. The improvement in the bounds
 1057 of Proposition A.3 thus stems from the improved version of Corollary 4.12.

1058 PROPOSITION A.4. *With $C_a > 0$, $\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$, depending on α, μ , it holds:*

- 1060 \bullet *if $\langle \tilde{\alpha} \rangle < 1$ ($\tilde{d}_s < 1$), then $P_a^{\alpha,\mu}(\lambda) \leq C_a \lambda$.*
- 1061 \bullet *if $\langle \tilde{\alpha} \rangle = 1$ ($\tilde{d}_s = 1$), then $P_a^{\alpha,\mu}(\lambda) \leq C_a \lambda \log \lambda$.*
- 1062 \bullet *if $\langle \tilde{\alpha} \rangle > 1$ ($\tilde{d}_s > 1$), then $P_a^{\alpha,\mu}(\lambda) \leq C_a \lambda^{\tilde{d}_s}$.*

1065 To prove this proposition, we will need the following refinement of Lemma 5.6, which
 1066 we repeat for the convenience of the reader below.

1067 THEOREM A.5 (Lemma 5.3 in [9], Lemma 5.5, Corollary 5.6 in [9]). *The symbol*
 1068 *of the reference DtN operator $\Lambda(\omega) = \Lambda_{\mathfrak{a}}(\omega)$, $\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$, $\Lambda : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$, satisfies*

$$1069 \quad (A.4) \quad \Lambda(\omega) = -\omega \frac{\omega \tan \omega - \mathbf{F}_{\alpha,\mu}(\omega)}{\tan \omega \mathbf{F}_{\alpha,\mu}(\omega) + \omega}, \quad \mathbf{F}_{\alpha,\mu}(\omega) = \sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i} \Lambda(\alpha_i \omega).$$

1070 Moreover, $\Lambda_{\mathfrak{a}}$ is the unique even solution of (A.4) analytic in the origin that satisfies
 1071 (cf. Theorem 2.3)

- 1073 \bullet *if $\langle \mu/\alpha \rangle \leq 1$, $\Lambda(0) = 0$, for $\mathfrak{a} \in \{\mathfrak{d}, \mathfrak{n}\}$.*

1075 • if $\langle \mu/\alpha \rangle > 1$ and $\langle \mu\alpha \rangle < 1$, $\Lambda(0) = 1 - \langle \mu/\alpha \rangle^{-1}$ if $\mathfrak{a} = \mathfrak{d}$, and $\Lambda(0) = 0$ if
 1076 $\mathfrak{a} = \mathfrak{n}$.

1077
 1078 • if $\langle \mu\alpha \rangle \geq 1$, $\Lambda(0) = 1 - \langle \mu/\alpha \rangle^{-1}$, for $\mathfrak{a} \in \{\mathfrak{d}, \mathfrak{n}\}$.

1079 *Proof of Proposition A.4.* The proof of this proposition relies on the following
 1080 observation: $\Lambda_{\mathfrak{a}}^{\alpha, \mu} = \Lambda_{\mathfrak{a}}^{\tilde{\alpha}, \tilde{\mu}}$, with $\tilde{\alpha}$ defined in (A.2) and $\tilde{\mu}$ specified further.

1081 *Step 1.* Given α, μ (cf. (A.1)), let $\tilde{\alpha}$ be defined like in (A.2) and

$$1082 \quad \tilde{\mu} := (\tilde{\mu}_0, \dots, \tilde{\mu}_{p-1}), \quad \tilde{\mu}_i := \sum_{k=0}^{n_i} \mu_i^{(k)}.$$

1084 Let us prove that $\Lambda_{\mathfrak{a}}^{\alpha, \mu} = \Lambda_{\mathfrak{a}}^{\tilde{\alpha}, \tilde{\mu}}$ with the help of Theorem A.5. Without loss of
 1085 generality, we will consider the case $\langle \mu\alpha \rangle \geq 1$, while the remaining cases can be
 1086 proven similarly.

1087 *Step 1.1.* A problem satisfied by $\Lambda_{\mathfrak{a}}^{\alpha, \mu}$. First, remark that for all $\sigma \in \mathbb{R}$,

$$1088 \quad (A.5) \quad \sum_{i=0}^{p-1} \sum_{k=0}^{n_i} \mu_i^{(k)} \left(\alpha_i^{(k)} \right)^{\sigma} = \sum_{i=0}^{p-1} \tilde{\alpha}_i^{\sigma} \sum_{k=0}^{n_i} \mu_i^{(k)} = \sum_{i=0}^{p-1} \tilde{\alpha}_i^{\sigma} \tilde{\mu}_i.$$

1090 By Theorem A.5, $\Lambda_{\mathfrak{a}}^{\alpha, \mu} = \Lambda$, where $\Lambda(\omega)$ is the unique even solution, analytic in the
 1091 origin, of

$$1092 \quad (A.6) \quad \Lambda(\omega) = -\omega \frac{\omega \tan \omega - \mathbf{F}_{\alpha, \mu}(\omega)}{\tan \omega \mathbf{F}_{\alpha, \mu}(\omega) + \omega}, \quad \text{where}$$

$$1093 \quad \mathbf{F}_{\alpha, \mu}(\omega) = \sum_{i=0}^{p-1} \sum_{k=0}^{n_i} \frac{\mu_i^{(k)}}{\alpha_i^{(k)}} \Lambda(\alpha_i^{(k)} \omega) = \sum_{i=0}^{p-1} \frac{\tilde{\mu}_i}{\tilde{\alpha}_i} \Lambda(\tilde{\alpha}_i \omega),$$

1094 (where the last identity is derived like (A.5)), with the condition in the origin

$$1095 \quad (A.7) \quad \Lambda(0) = 1 - \langle \mu/\alpha \rangle^{-1} \stackrel{(A.5)}{=} 1 - \langle \tilde{\mu}/\tilde{\alpha} \rangle^{-1}.$$

1097 *Step 1.2.* A problem satisfied by $\Lambda_{\mathfrak{a}}^{\tilde{\alpha}, \tilde{\mu}}$. Let us remark that $\langle \mu\alpha \rangle = \langle \tilde{\mu}\tilde{\alpha} \rangle$ by (A.5).

1098 By Theorem A.5, $\Lambda_{\mathfrak{a}}^{\tilde{\alpha}, \tilde{\mu}}$ satisfies (A.6) with $\Lambda(0) = 1 - \langle \tilde{\mu}/\tilde{\alpha} \rangle^{-1}$, i.e. (A.7).

1099 Because the solution to (A.6) and (A.7) is unique, we conclude that $\Lambda_{\mathfrak{a}}^{\tilde{\alpha}, \tilde{\mu}} = \Lambda_{\mathfrak{a}}^{\alpha, \mu}$.

1100 *Step 2.* An inequality for $P_{\mathfrak{a}}^{\alpha, \mu}(\lambda)$. From the previous step it follows that $P_{\mathfrak{a}}^{\alpha, \mu} =$
 1101 $P_{\mathfrak{a}}^{\tilde{\alpha}, \tilde{\mu}}$. By (4.44), for all $\lambda > 0$, $P_{\mathfrak{a}}^{\tilde{\alpha}, \tilde{\mu}}(\lambda) \leq \rho_{\mathfrak{a}}^{\tilde{\alpha}, \tilde{\mu}}(\lambda^2)$, hence

$$1103 \quad P_{\mathfrak{a}}^{\alpha, \mu}(\lambda) \leq \rho_{\mathfrak{a}}^{\tilde{\alpha}, \tilde{\mu}}(\lambda^2), \quad \text{for all } \lambda > 0.$$

1104 It remains to use the asymptotic estimates of Theorem 4.8 to bound $\rho_{\mathfrak{a}}^{\tilde{\alpha}, \tilde{\mu}}$. \square

1105 A.3. Proof of Propositions A.3, A.1.

1106 *Proof of Proposition A.3.* We proceed like in the proof of Theorem 4.3 to obtain
 1107 (4.47), i.e. $N_{\mathfrak{a}, \varepsilon}^{\alpha, \mu} \leq P_{\mathfrak{a}}^{\alpha, \mu}(C\varepsilon^{-1}) + 1$, and next conclude by using Proposition A.4. \square

1108 The proof of Proposition A.1 is similar to the proof of Theorem 4.2, with the only
 1109 difference being that instead of Theorem 4.3, we use Proposition A.3, and hence we
 1110 leave it to the reader.

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