



**HAL**  
open science

# Two Prime Number Objects and The Velucchi Numbers

César Aguilera

► **To cite this version:**

| César Aguilera. Two Prime Number Objects and The Velucchi Numbers. 2023. hal-02909691v2

**HAL Id: hal-02909691**

**<https://hal.science/hal-02909691v2>**

Preprint submitted on 7 Apr 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Two Prime Number Objects and The Velucchi Numbers

César Aguilera.

April 7, 2023

## **Abstract**

We explore two mathematical objects related to prime numbers, a sequence and a pythagorean triangle, we take the sequence and analyze thoroughly to find some properties -we talk about Twin Primes- then we connect these properties to the triangle and we pass to the Velucchi Numbers, we explore them and finally we get an algorithm to obtain their sequence.

## **Introduction.**

The mathematical objects that appear in this paper were performed by three legendary mathematicians, Artur Jasinski, Reinhard Zumkeller and Mario Velucchi, from the first we have a sequence related to prime numbers, from the second, a pythagorean triangle whose sides are related to the prime numbers and from the third we get some marvelous numbers.

## The Jasinski Sequence.

The Jasinski's sequence is formed by the terms:  $\frac{(p_{k-1})(p_k)(p_{k+1})}{24}$  or  $\frac{(p_{k-1})(p_k)(p_{k+1})}{4!}$ , being  $p_k$  any prime number larger or equal to 3.

The first terms of this sequence are:

1, 5, 14, 55, 91, 204, 285, 506, 1015, 1240, 2109, 2870, 3311, 4324, 6201...[1]

## Representation of Jasinski Sequence as $\sum_1^{\infty} \frac{1}{p_k^{2x+1}}$ .

The Jasinski Sequence can be represented in two forms, as a succession of single terms  $(n_k, n_{k+1}, n_{k+2}, n_{k+3} \dots)$  or as a succession of rational terms  $(\frac{n_{k+1}}{n_k}, \frac{n_{k+2}}{n_{k+1}}, \frac{n_{k+3}}{n_{k+2}}, \frac{n_{k+4}}{n_{k+3}} \dots)$ .

### Jasinski Sequence as a succession of single terms.

Given any prime number  $p_k$  then:  $\frac{(p_{k-1})(p_k)(p_{k+1})}{4!} = \sum_1^{\infty} \frac{1}{3^{2x+1}} \div \sum_1^{\infty} \frac{1}{p_k^{2x+1}}$ .

Examples:

$$\sum_1^{\infty} \frac{1}{3^{2x+1}} \div \sum_1^{\infty} \frac{1}{3^{2x+1}} = 1$$

$$\sum_1^{\infty} \frac{1}{3^{2x+1}} \div \sum_1^{\infty} \frac{1}{5^{2x+1}} = 5$$

$$\sum_1^{\infty} \frac{1}{3^{2x+1}} \div \sum_1^{\infty} \frac{1}{7^{2x+1}} = 14$$

$$\sum_1^{\infty} \frac{1}{3^{2x+1}} \div \sum_1^{\infty} \frac{1}{11^{2x+1}} = 55$$

### Jasinski Sequence as a succession of rational terms.

Given two consecutive prime numbers  $p_k$  and  $p_{k+1}$  we have:  $\sum_1^{\infty} \frac{1}{p_k^{2x+1}} \div \sum_1^{\infty} \frac{1}{p_{k+1}^{2x+1}}$ .

Examples:

$$\sum_1^{\infty} \frac{1}{3^{2x+1}} \div \sum_1^{\infty} \frac{1}{5^{2x+1}} = \frac{5}{1}$$

$$\sum_1^{\infty} \frac{1}{5^{2x+1}} \div \sum_1^{\infty} \frac{1}{7^{2x+1}} = \frac{14}{5}$$

$$\sum_1^{\infty} \frac{1}{7^{2x+1}} \div \sum_1^{\infty} \frac{1}{11^{2x+1}} = \frac{55}{14}$$

$$\sum_1^{\infty} \frac{1}{11^{2x+1}} \div \sum_1^{\infty} \frac{1}{13^{2x+1}} = \frac{91}{55}$$

That is:

$\frac{5}{1}, \frac{14}{5}, \frac{55}{14}, \frac{91}{55}, \frac{204}{91}, \frac{285}{204}, \frac{506}{285}, \frac{1015}{506} \dots$

we can now state a conjecture about the Twin Primes.

**Twin Primes Conjecture.**

Given three consecutive numbers  $t_{p_1}, n, t_{p_2}$  such that  $t_{p_1}$  and  $t_{p_2}$  are primes, and let  $(f)$  be the biggest prime factor of  $(n)$  then:

$$\sum_1^{\infty} \frac{1}{3^{2x+1}} \div \sum_1^{\infty} \frac{1}{n^{2x+1}} = \frac{a}{b} \text{ where } a = f^t \cdot t_{p_1} \cdot t_{p_2}; \text{ for } n \geq 18.$$

In other words, the Twin Primes are the biggest prime factors of the numerator of the rational form of  $\sum_1^{\infty} \frac{1}{3^{2x+1}} \div \sum_1^{\infty} \frac{1}{n^{2x+1}}$ .

**Reinhard Zumkeller Pythagorean Triangles.**

[2] These triangles can be summarized in the next table:

Table 1

Primes	$(p_{k+1})^2 + (p_k)^2$	$(p_{k+1})^2 - (p_k)^2$	$(2p_k)(p_{k+1})$	Area
(2,3)	13	5	12	30
(3,5)	34	16	30	240
(5,7)	74	24	70	840
(7,11)	170	72	154	5544
(11,13)	290	48	286	6864
(13,17)	458	120	442	26520

And we have:

$$Area = \frac{[(p_{k+1})^2 - (p_k)^2][(2p_k)(p_{k+1})]}{2} = Det \begin{vmatrix} p_k & 24b \\ p_{k+1} & 24a \end{vmatrix}$$

Where  $\frac{a}{b} = \sum_1^{\infty} \frac{1}{p_k^{2x+1}} \div \sum_1^{\infty} \frac{1}{p_{k+1}^{2x+1}}$

In particular, if the Area is generated by Twin Primes, then:

$$Area = f^t \cdot t_{p_1} \cdot t_{p_2} \cdot [2^j \cdot 3^z]$$

We can see them in the next table.

Table 2

<b>Twin Primes</b>	$(p_{k+1})^2 + (p_k)^2$	$(p_{k+1})^2 - (p_k)^2$	$(2p_k)(p_{k+1})$	<b>Area</b>	$f^t \cdot t_{p1} \cdot t_{p2} \cdot [2^j \cdot 3^z]$
(17,19)	650	72	646	23256	$3 \cdot 17 \cdot 19 \cdot [2^3 \cdot 3]$
(29,31)	1802	120	1798	107880	$5 \cdot 29 \cdot 31 \cdot [2^3 \cdot 3]$
(41,43)	3530	168	3526	296184	$7 \cdot 41 \cdot 43 \cdot [2^3 \cdot 3]$
(59,61)	7202	240	7198	863760	$5 \cdot 59 \cdot 61 \cdot [2^4 \cdot 3]$
(71,73)	10370	288	10366	1492704	$3 \cdot 71 \cdot 73 \cdot [2^5 \cdot 3]$
(101,103)	20810	408	20806	4244424	$17 \cdot 101 \cdot 103 \cdot [2^3 \cdot 3]$
(107,109)	23330	432	23326	5038416	$3 \cdot 107 \cdot 109 \cdot [2^4 \cdot 3^2]$
(137,139)	38090	552	38086	10511736	$23 \cdot 137 \cdot 139 \cdot [2^3 \cdot 3]$

## The Velucchi Numbers.

The Velucchi Numbers[3], arranged and composed by Mario Velucchi have the next form:

$$Velucchi\ Number(v) = a \cdot b = \sum_a^b n$$

The next table shows some of them.

Table 3

<b>a</b>	3	15	85	493	2871	16731	97513	568345
<b>b</b>	6	35	204	1189	6930	40391	235416	1372105
<b>v</b>	18	525	17340	586177	19896030	675781821	22956120408	779829016225

At first glance we can notice some properties:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \approx \lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} \approx 3 + 2\sqrt{2}$$

$$\lim_{n \rightarrow \infty} \frac{v_{n+1}}{v_n} \approx 17 + 12\sqrt{2}$$

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} \approx \lim_{k \rightarrow \infty} \left[ \sum_{x=1}^{\infty} \frac{1}{a_k^{2x-1}} \div \sum_{x=1}^{\infty} \frac{1}{b_k^{2x-1}} \right] \approx 1 + \sqrt{2}$$

Being  $(1 + \sqrt{2})$  the Silver Ratio.

## Velucchi Numbers and the Jasinski Sequence.

If we build ordered groups of 2, with initial values  $[a_1 = 3; b_1 = 6]$  or  $b_k \text{ even}$ .

$$\left| \begin{array}{cc} a_1 & a_2 \\ b_1 & b_2 \end{array} \right| \left| \begin{array}{cc} a_3 & a_4 \\ b_3 & b_4 \end{array} \right| \left| \begin{array}{cc} a_5 & a_6 \\ b_5 & b_6 \end{array} \right| \dots \left| \begin{array}{cc} a_k & a_{k+1} \\ b_k & b_{k+1} \end{array} \right|$$

We will find that:

$$\begin{aligned} \sum_{x=1}^{\infty} \frac{1}{a_1^{2x+1}} \div \sum_{x=1}^{\infty} \frac{1}{b_1^{2x+1}} &= \frac{(b_1-1)(b_1)(b_1+1)}{(a_1-1)(a_1)(a_1+1)} = \frac{b_2}{a_1+1} \\ \sum_{x=1}^{\infty} \frac{1}{a_3^{2x+1}} \div \sum_{x=1}^{\infty} \frac{1}{b_3^{2x+1}} &= \frac{(b_3-1)(b_3)(b_3+1)}{(a_3-1)(a_3)(a_3+1)} = \frac{b_4}{a_3+1} \\ \sum_{x=1}^{\infty} \frac{1}{a_5^{2x+1}} \div \sum_{x=1}^{\infty} \frac{1}{b_5^{2x+1}} &= \frac{(b_5-1)(b_5)(b_5+1)}{(a_5-1)(a_5)(a_5+1)} = \frac{b_6}{a_5+1} \\ \dots \\ \sum_{x=1}^{\infty} \frac{1}{a_k^{2x+1}} \div \sum_{x=1}^{\infty} \frac{1}{b_k^{2x+1}} &= \frac{(b_k-1)(b_k)(b_k+1)}{(a_k-1)(a_k)(a_k+1)} = \frac{b_{k+1}}{a_k+1} \end{aligned}$$

Now, if:  $\text{Det} \begin{vmatrix} a_k & a_{k+1} \\ b_k & b_{k+1} \end{vmatrix} = a_{k+1}$

we can build an algorithm to calculate the Velucchi Numbers sequence.

$$\begin{cases} a_k & a_{k+1} \\ b_k & b_{k+1} \end{cases} \rightarrow \begin{cases} 2b_k + a_k = a_{k+1} & (1) \\ \frac{a_{k+1} \cdot (b_k + 1)}{a_k} = b_{k+1} & (2) \end{cases}$$

### gcd of Velucchi Numbers.

From values  $\mathbf{a}_k$  and  $\mathbf{b}_k$  (table 3) we find the Greatest common divisor (gcd) to get

$\mathbf{a}'_{k+1} = (\text{gcd of } \mathbf{a}_k \text{ and } \mathbf{b}_k)$  and then  $\mathbf{b}'_k$ , with initial values  $\mathbf{a}'_{k+0} = 1$  and  $\mathbf{b}'_{k+0} = 2$ .

Table 4

$\mathbf{a}'$	1	3	5	17	29	99	169	577
$\mathbf{b}'$	2	7	12	41	70	239	408	1393

In this table we find that  $\frac{b_k}{b'_k} = a'_{k+1}$ ,  $\text{Det} \begin{vmatrix} a'_k & a'_{k+1} \\ b'_k & b'_{k+1} \end{vmatrix} = 1$

$$(b'_k \text{ even}) - a'_k = b'_{k-1} \forall \{b'_k \text{ even} > 2\}, (b'_k \text{ even}) + a'_k = a'_{k+1}$$

For any values  $\{a'_k, (b'_k \text{ even})\}$ , then:  $\text{Det} \begin{vmatrix} a'_k & \dots & a'_{k+n} \\ b'_k \text{ even} & \dots & b'_{k \text{ even} + n} \end{vmatrix} = \{a'_{k+0}, b'_{k+0}, b'_{k+1}, b'_{k+2}, \dots, b'_{k+n}\}$

Notice that:

$$(2 \cdot 2) + 1 = 5, (2 \cdot 5) + 2 = 12, (2 \cdot 12) + 5 = 29, (2 \cdot 29) + 12 = 70, \\ (2 \cdot 70) + 29 = 169, (2 \cdot 169) + 70 = 408$$

this is:

$$2b'_k + a'_k = a'_{k+2}$$

$$2a'_k + b'_{k-2} = b'_k \quad \forall \{b'_k > 7, a'_k > 3\}$$

Also, if:

$$\frac{a'_{k-1} + a'_k + a'_{k+1}}{a'_k} = \frac{b'_{k-1} + b'_k + b'_{k+1}}{b'_k} = 3,$$

then:

$$\frac{a'_k + a'_{k+1} + a'_{k+2}}{a'_{k+1}} = \frac{b'_k + b'_{k+1} + b'_{k+2}}{b'_{k+1}} = 5$$

$$\text{For any values } \{a'_k, (b'_{k \text{ odd}})\}, \text{ then: } \text{Det} \begin{vmatrix} a'_k & \cdots & a'_{k+n} \\ b'_{k \text{ odd}} & \cdots & b'_{k \text{ odd}+n} \end{vmatrix} =$$

$$\{b'_{k+0} - a'_{k+0}, b'_{k+1} - a'_{k+1}, \dots, b'_{k+n} - a'_{k+n}\}$$

$$3 + \left( \frac{a'_k}{b'_{k \text{ odd}}} \right) = \frac{b'_{k+2} - a'_{k+2}}{b'_{k \text{ odd}}}; \lim_{k \rightarrow \infty} \frac{b'_{k \text{ odd}} - a'_k}{b'_{k \text{ odd}-1} - a'_{k-1}} \approx 2 + \sqrt{2}$$

$$1 + \left( \frac{a'_k}{2b'_{k \text{ odd}-1}} \right) = \frac{b'_{k \text{ odd}}}{2b'_{k \text{ odd}-1}}; \lim_{k \rightarrow \infty} \frac{b'_{k \text{ odd}}}{2b'_{k \text{ odd}-1}} \approx \frac{1 + \sqrt{2}}{\sqrt{2}}$$

From this table: [4]

Table 5

<b>n</b>	1	2	3	4	5	6	7	8
$(1 + \sqrt{2})^n \approx$	2	6	14	34	82	198	478	1154
$\frac{(1 + \sqrt{2})^n}{\sqrt{2}} \approx$	2	4	10	24	58	140	338	816

We divide by 2:  $(1 + \sqrt{2})^{n \text{ odd}}$  and  $\frac{(1 + \sqrt{2})^{n \text{ even}}}{\sqrt{2}}$ , to get:  $\frac{(1 + \sqrt{2})^{n \text{ odd}}}{2}$  and  $\frac{\sqrt{2}(1 + \sqrt{2})^{n \text{ even}}}{4}$

We find that:

$$\left\{ \frac{(1 + \sqrt{2})^1}{2}, \frac{\sqrt{2}(1 + \sqrt{2})^2}{4}, \frac{(1 + \sqrt{2})^3}{2}, \frac{\sqrt{2}(1 + \sqrt{2})^4}{4} \dots \frac{(1 + \sqrt{2})^{n \text{ odd}}}{2}, \frac{\sqrt{2}(1 + \sqrt{2})^{n \text{ even}}}{4} \right\} \approx \{a'_{k+0}, b'_{k+0}, b'_{k+1}, b'_{k+2}, \dots, b'_{k+n}\}$$

Note that for  $\frac{\sqrt{2}(1 + \sqrt{2})^{n \text{ even}}}{4}$  values, we obtain:

$$\frac{3\sqrt{2} + 4}{4} \approx 2; \frac{17\sqrt{2} + 24}{4} \approx 12; \frac{99\sqrt{2} + 140}{4} \approx 70, \text{ this is: } \frac{a'_{k(n-1)}\sqrt{2} + 2b'_{k(n-2)}}{4} \approx b'_{k(n-2) \text{ even}}$$

For the sequence  $\{b'_{k+0} - a'_{k+0}, b'_{k+1} - a'_{k+1}, \dots, b'_{k+n} - a'_{k+n}\}$  we have:

$$\frac{(1 + \sqrt{2})^{n \text{ odd}}}{2} \approx b'_{k \text{ even}} - a'_k, \quad \frac{(1 + \sqrt{2})^{n \text{ even}}}{\sqrt{2}} \approx b'_{k \text{ odd}} - a'_k$$

And finally:

$$b'_{k \text{ even}} + b'_{k+2} \approx (1 + \sqrt{2})^{k \text{ even}+3}$$

$$b'_{k \text{ odd}} - b'_{k-2} \approx (1 + \sqrt{2})^{k \text{ odd}+1} \quad \text{for } b'_{k \text{ odd}} > 7$$

$$(1 + \sqrt{2})^n - \frac{(1+\sqrt{2})^n}{\sqrt{2}} \approx \frac{(1+\sqrt{2})^{n-1}}{\sqrt{2}}$$

Note that:

$$(1 + \sqrt{2})^1 \cdot (1 + \sqrt{2})^2 + 2 \approx (1 + \sqrt{2})^3$$

$$(1 + \sqrt{2})^2 \cdot (1 + \sqrt{2})^3 - 2 \approx (1 + \sqrt{2})^5$$

$$(1 + \sqrt{2})^3 \cdot (1 + \sqrt{2})^4 + 2 \approx (1 + \sqrt{2})^7$$

$$(1 + \sqrt{2})^{n \text{ odd}} \cdot (1 + \sqrt{2})^{n \text{ odd}+1} + 2 \approx (1 + \sqrt{2})^{2n \text{ odd}+1}$$

$$(1 + \sqrt{2})^{n \text{ even}} \cdot (1 + \sqrt{2})^{n \text{ even}+1} - 2 \approx (1 + \sqrt{2})^{2n \text{ even}+1}$$

In the specific case of Velucchi Numbers.

$$(1 + \sqrt{2})^{n \text{ odd}} + a_{n-5} \approx a_{n-3}$$

$$(1 + \sqrt{2})^{n \text{ odd}+1} + b_{n-6} \approx a_{n-4} \quad \text{for } n \geq 5$$

Example:

$$(1 + \sqrt{2})^5 \approx 82 \rightarrow 82 + 3 = 85$$

$$(1 + \sqrt{2})^6 \approx 198 \rightarrow 198 + 6 = 204$$

$$(1 + \sqrt{2})^7 \approx 478 \rightarrow 478 + 15 = 493$$

$$(1 + \sqrt{2})^8 \approx 1154 \rightarrow 1154 + 35 = 1189$$



## References

- [1] *OEIS Foundation Inc. (2020), The On-Line Encyclopedia of Integer Sequences, <http://oeis.org/A127922>*
- [2] *OEIS Foundation Inc. (2020), The On-Line Encyclopedia of Integer Sequences, <http://oeis.org/A069487>*
- [3] *Mario Velucchi*  
*From the desk of Mario Velucchi. In Mathematics and Informatics quarterly. volume 7 - 2/1997, p. 81.*
- [4] *<https://mathcircle.berkeley.edu/sites/default/files/archivedocs/2015/lecture/1+sqrt2plus.pdf>, page 3.*