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# Extended stress gradient elastodynamics: Wave dispersion and micro-macro identification of parameters 

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#### Abstract

In its original formulation by Forest \& Sab (Math. Mech. Solids, 2017), stress gradient elastodynamics incorporate two inner-lengths to account for size effects in continuum theory. Here, an extended one-dimensional stress gradient model is developed by means of Lagrangian formalism, incorporating an additional inner-length and a fourth-order space derivative in the wave equation. Dispersive properties are characterised and hyperbolicity and stability are proven. Group velocity remains bounded in both original and extended models, proving causality is satisfied for both contrary to a usually-accepted postulate. By means of two-scale asymptotic homogenization, the high-order wave equation satisfied by the stress gradient model is shown to stand for an effective description of heterogeneous materials in the low-frequency range. An upscaling method is developed to identify the stress gradient material parameters and bulk forces on the parameters of elastic micro-structures. Application of the micro-macro procedure to periodic multi-laminates demonstrates the accuracy of the stress gradient continuum to account for the dispersive features of wave propagation. Frequency and time-domain simulations illustrate these properties.


Keywords: Stress gradient media Generalized continua, Second-order homogenization, Wave dispersion, Hyperbolic system 2000 MSC: code, code

## 1. Introduction

Over the past decades, many extensions to the classical continuum mechanics theory have emerged [1], such as micro-polar [2] or more general micro-continuum mechanics theories $[3,4,5]$. As alternative approaches to atomistic models for micro-structured media, such extended continuum models have been developed to incorporate physical phenomena involving size effects beyond the reach of the classical theories. While many extended continuum models were developed in statics $[6,7]$, microstructural size effects are particularly significant in dynamics when the characteristic length $\lambda$ associated with the external excitation (typical wavelength and size of the domain of propagation) is not sufficiently large compared to the characteristic internal length $\ell$ so as to neglect the underlying micro-structural architecture. Neglected in classical theories, such characteristic inner-lengths usually result in additional high-order spatial or time derivatives of relevant state variables in the governing equations [8, 9, 10]. This leads to space and time behavior characterized through additional material parameters to estimate for numerical applications.

To justify such high-order derivatives, asymptotic homogenization techniques applied to micro-lattices or heterogeneous media were presented [11, $12,13,14,15]$. Advantageously, they bring micro-macro mechanical support to extended continuum models, and micro-macro relations to estimate ma-
terial parameters from underlying microscopic heterogeneities. For instance, long-wavelength Taylor expansions of transfer functions in analogous massspring discrete lattices have been used for that purpose [16]. Another example stands in two-scale asymptotic homogenization theory for periodic media $[17,18]$ which has proven very useful by including high-order terms in the asymptotic expansions [19, 20, 21]. In particular, microstructural size effects on wave dispersion, polarisation, attenuation, and mode conversions in threedimensional space were evidenced theoretically by means of two-scale asymptotic homogenization [20]. However, some continuum models developed by asymptotic approaches are unstable [22, 23], due to features developing at frequencies outside the initial range of the asymptotic expansions. This is a critical issue, especially for time-domain calculations involving broadband signals.

While many micro-continuum theories have been formulated by equipping material particles with enriched kinematics, including rotational degrees of freedom [2] or micro-deformations [3, 4, 5], another class of extended continuum model, called stress gradient elasticity, was formulated on the idea that stress rather than kinematics constitutes the driving variable in the material behavior. Tracing back to Eringen's constitutive relations [24] in the form of partial differential equations involving both the stress tensor and its Laplacian, stress gradient elasticity found further theoretical developments recently $[25,26,27,28]$ including its formulation in elastodynamics by Forest \& Sab [29]. These authors obtained a generalized wave equation for the displacement variable $U$ in one-dimensional (1-D) stress gradient elasticity media, which involved, in addition to the classical wave-operator, its fourth-
order time derivative $\partial_{t}^{4} U$ and mixed space-time derivative $\partial_{t}^{2} \partial_{x}^{2} U$. However, the dispersion of waves in such stress gradient elasticity material was not studied any further.

In the present work, our objective is fourfold: (i) to extend the elastodynamics stress gradient model developed by Forest \& Sab [29] in onedimensional space to include space derivative $\partial_{x}^{4} U$ in the generalized wave equation; (ii) to evidence the dispersive features of wave propagation in such medium; (iii) to identify all five material parameters involved in the extended stress gradient model applied to heterogeneous elastic materials ; and (iv) to compute the transient response of stress gradient media to transient bulk sources, to be compared with direct simulations in heterogeneous Cauchy media. In what follows, the original stress gradient model refers to the elastodynamics stress gradient model developed by Forest \& Sab [29].

The outline of this paper is the following. In Sec. 2, we propose an extended stress gradient model, which is obtained by considering an auxiliary elasticity. It is shown to be energetically consistent and stable and its governing equation includes a fourth-order space derivative that did not exist in the original stress gradient model. In Sec. 3, the wave dispersion, phase velocity and group velocity corresponding to this model are derived and discussed, in particular with respect to those of the original stress gradient model. In Sec. 4, a relation is drawn between the proposed model and higher-order homogenized models obtained from microstructured elastic materials. This relation yields a way to identify the parameters of the proposed stress gradient model for a given elastic micro-structured material, while ensuring stability and energy consistency. Finally, in Sec. 5, numerical
simulations are proposed to illustrate the behavior of the proposed model in time, in particular with respect to stability and in comparison with other similar models proposed in the literature.

## 2. Stress gradient model

Properties of stress-gradient materials are analysed theoretically: hyperbolicity, stability, existence and regularity of solutions. For the sake of generality, an extended stress gradient elasticity model with three inner length scales is considered. It generalizes the recent developments on stress gradient elastodynamics with two inner lengths [29].

### 2.1. Constitutive laws

Let $U$ be the particle displacement and $\zeta$ an auxiliary displacement which encapsulates the size effects. It follows the velocity $v$, strain e , auxiliary velocity $\varphi$ and auxiliary strain $\theta$

$$
\begin{equation*}
v=\partial_{t} U, \quad \mathrm{e}=\partial_{x} U, \quad \varphi=\partial_{t} \zeta, \quad \theta=\partial_{x} \zeta . \tag{1}
\end{equation*}
$$

One defines the kinetic and potential energy densities $\mathcal{K}$ and $\mathcal{W}$

$$
\begin{equation*}
\mathcal{K}=\varrho \frac{v^{2}}{2}+\mu \frac{\varphi^{2}}{2}, \quad \mathcal{W}=\frac{E}{2}(\mathrm{e}+\theta)^{2}+\kappa \frac{\theta^{2}}{2}+D \frac{\zeta^{2}}{2} \tag{2}
\end{equation*}
$$

where $\varrho$ is a density, $\mu$ is an auxiliary density, $E$ is an elasticity modulus, $\kappa$ is an auxiliary elasticity modulus and $D$ is a stress gradient modulus. The
energy densities (2) amount to the positive quadratic forms

$$
\begin{align*}
& \mathcal{K}=\frac{1}{2}\binom{v}{\varphi}^{T} \cdot \mathbb{M} \cdot\binom{v}{\varphi}  \tag{3a}\\
& \mathcal{W}=\frac{1}{2}\binom{\mathrm{e}}{\theta}^{T} \cdot \mathbb{E} \cdot\binom{\mathrm{e}}{\theta}+\frac{1}{2}\binom{U}{\zeta}^{T} \cdot \mathbb{D} \cdot\binom{U}{\zeta} \tag{3b}
\end{align*}
$$

98

99

101
where $\mathbb{M}, \mathbb{E}$ and $\mathbb{D}$ are symmetric and positive matrices

$$
\mathbb{M}=\left[\begin{array}{ll}
\varrho & 0  \tag{4}\\
0 & \mu
\end{array}\right], \quad \mathbb{E}=\left[\begin{array}{cc}
E & E \\
E & E+\kappa
\end{array}\right], \quad \mathbb{D}=\left[\begin{array}{ll}
0 & 0 \\
0 & D
\end{array}\right]
$$

Introducing the stress fields $\sigma$ and $\tau$ and stress gradient $R$

$$
\begin{equation*}
\sigma=\frac{\partial \mathcal{W}}{\partial \mathrm{e}}, \quad \tau=\frac{\partial \mathcal{W}}{\partial \theta}, \quad R=\frac{\partial \mathcal{W}}{\partial \zeta} \tag{5}
\end{equation*}
$$

the potential energy (2) provides the constitutive laws

$$
\begin{align*}
& \sigma=E \partial_{x}(U+\zeta)  \tag{6a}\\
& \tau=E \partial_{x}(U+\zeta)+\kappa \partial_{x} \zeta  \tag{6b}\\
& R=D \zeta \tag{6c}
\end{align*}
$$

102 The constitutive laws of original stress gradient model are recovered when
${ }^{103} \kappa=0[29]$.

104

### 2.2. Dynamic equations

Hamilton's stationary principle applied to the Lagrangian density $\mathcal{L}=$ $\mathcal{K}-\mathcal{W}$ gives the Euler-Lagrange equations

$$
\begin{align*}
& \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial v}\right)+\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \mathrm{e}}\right)-\frac{\partial \mathcal{L}}{\partial U}=0  \tag{7a}\\
& \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \varphi}\right)+\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \theta}\right)-\frac{\partial \mathcal{L}}{\partial \zeta}=0 \tag{7b}
\end{align*}
$$

Introducing the external bulk forces $F_{U}$ and $F_{\zeta}$, the dynamic equations follow from (2) and (7) in the form

$$
\begin{align*}
& \varrho \partial_{t}^{2} U=\partial_{x} \sigma+\varrho F_{U},  \tag{8a}\\
& \mu \partial_{t}^{2} \zeta=\partial_{x} \tau-R+\mu F_{\zeta}, \tag{8b}
\end{align*}
$$

Multiplication of (8a) and (8b) by $v$ and $\varphi$ and summation yield the conservation law for energy

$$
\begin{equation*}
\partial_{t}(\mathcal{K}+\mathcal{W})+\partial_{x}(\Pi)=\mathcal{P} \tag{9}
\end{equation*}
$$

where the power density of external bulk forces $\mathcal{P}$, and the Poynting 'vector' $\Pi$ are given by

$$
\begin{equation*}
\mathcal{P}=\varrho F_{U} v+\mu F_{\zeta} \varphi \quad \text { and } \quad \Pi=-(\sigma v+\tau \varphi) \tag{10}
\end{equation*}
$$

Equation (9) shows that the time variation of the energy $\mathcal{K}+\mathcal{W}$ corresponds to the power density of external bulk forces and energy fluxes described by the Poynting 'vector'.

### 2.3. Hyperbolic system

Equations (6) and (8) can be rewritten in the matrix form

$$
\begin{equation*}
\partial_{t} \mathbf{W}+\mathbb{A} \cdot \partial_{x} \mathbf{W}=\mathbb{S} \cdot \mathbf{W}+\mathbf{F}, \tag{11}
\end{equation*}
$$

with the state vector $\mathbf{W}$, the source $\mathbf{F}$ and the matrices $\mathbb{A}$ and $\mathbb{S}$

$$
\begin{align*}
& \mathbf{W}=\left\{\begin{array}{c}
v \\
\varphi \\
\sigma \\
\tau \\
R
\end{array}\right\}, \quad \mathbb{A}=-\left[\begin{array}{ccccc}
0 & 0 & 1 / \varrho & 0 & 0 \\
0 & 0 & 0 & 1 / \mu & 0 \\
E & E & 0 & 0 & 0 \\
E & E+\kappa & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right],  \tag{12a}\\
& \mathbf{F}=\left\{\begin{array}{c}
F_{U} \\
F_{\zeta} \\
0 \\
0 \\
0
\end{array}\right\}, \quad \mathbb{S}=\left[\begin{array}{lllll}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 / \mu \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & D & 0 & 0 & 0
\end{array}\right] . \tag{12b}
\end{align*}
$$

The eigenvalues of $\mathbb{A}$ and $\mathbb{S}$ are

$$
\begin{align*}
& \operatorname{Sp}(\mathbb{A})=\left\{0 ; \pm c_{\mathbb{A}}^{+} ; \pm c_{\mathbb{A}}^{-}\right\}  \tag{13a}\\
& \operatorname{Sp}(\mathbb{S})=\left\{0 ; 0 ; 0 ; \pm \mathrm{i} \omega_{\mathbb{S}}\right\} \tag{13b}
\end{align*}
$$

where the characteristic speeds $c_{\mathbb{A}}^{ \pm}$and frequency $\omega_{\mathbb{S}}$ read

$$
\begin{equation*}
c_{\mathbb{A}}^{ \pm}=c_{0} \sqrt{\frac{b_{m}^{2} \pm \sqrt{b_{m}^{4}-4 b_{t}^{2} b_{x}^{2}}}{2 b_{t}^{2}}} \quad \text { and } \quad \omega_{\mathbb{S}}=\sqrt{\frac{D}{\mu}} \tag{14}
\end{equation*}
$$

In (14), $c_{0}=\sqrt{E / \varrho}$ is the classical wave speed in the Cauchy medium, and the inner lengths are

$$
\begin{equation*}
b_{t}=\sqrt{\frac{\mu}{\varrho} B}, \quad b_{x}=\sqrt{\frac{\kappa}{E} B}, \quad b_{m}=\sqrt{b^{2}+b_{t}^{2}+b_{x}^{2}} \tag{15}
\end{equation*}
$$

with the characteristic area $B \equiv b^{2}=E / D$. The inequalities

$$
\begin{equation*}
b_{m}^{4} \geq b_{m}^{4}-4 b_{t}^{2} b_{x}^{2}=\left(b_{m}^{2}-2 b_{t}^{2}\right)^{2}+4 b_{t}^{2} \geq\left(b_{m}^{2}-2 b_{t}^{2}\right)^{2} \geq 0 \tag{16}
\end{equation*}
$$

${ }_{24}$ imply that the characteristic speeds $c_{\mathbb{A}}^{ \pm}$are real and satisfy

125

$$
\begin{equation*}
0 \leq c_{\mathbb{A}}^{-} \leq c_{0} \leq c_{\mathbb{A}}^{+} . \tag{17}
\end{equation*}
$$

When $\kappa \neq 0$, each eigenvalue of $\mathbb{A}$ is associated with one single eigenvector

127

$$
\mathbf{W}_{\mathbb{A}}(c \neq 0)=\left\{\begin{array}{c}
1  \tag{18}\\
\left(\frac{c^{2}}{c_{0}^{2}}-1\right) \\
\varrho c \\
\left(\frac{c^{2}}{c_{0}^{2}}-1\right) \mu c \\
0
\end{array}\right\}, \quad \mathbf{W}_{\mathbb{A}}(0)=\left\{\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
1
\end{array}\right\}
$$

${ }_{128}$ The matrix $\mathbb{A}$ is thus diagonalizable with real eigenvalues, which proves the 129 hyperbolicity of (11) [30]. Also, the eigenvalues of $\mathbb{S}$ are either null or purely ${ }_{130}$ imaginary, which implies that $\mathbf{W} \equiv \mathbf{0}$ is asymptotically stable [30]. Intro131 ducing the definite positive symmetric matrix
${ }^{132} \quad \mathbb{Q}=\left[\begin{array}{ccccc}\varrho & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 1 / E+1 / \kappa & -1 / \kappa & 0 \\ 0 & 0 & -1 / \kappa & 1 / \kappa & 0 \\ 0 & 0 & 0 & 0 & 1 / D\end{array}\right]$
133
the following relations hold
${ }^{134} \mathbb{Q} \mathbb{A}=-\left[\begin{array}{ccccc}0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0\end{array}\right], \mathbb{Q} \mathbb{S}=\left[\begin{array}{ccccc}0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0\end{array}\right]$.

These relations imply that matrix $\mathbb{Q}$ is a symmetrizer for the system (11), in the sense that $\mathbb{Q} \mathbb{A}$ is symmetric, and $\mathbb{Q} \mathbb{S}$ is skew-symmetric with null diagonal terms [32]. One has

$$
\begin{array}{ll}
\mathbf{W}^{T} \mathbb{Q} \partial_{t} \mathbf{W}=\partial_{t}(\mathcal{K}+\mathcal{W}), & \mathbf{W}^{T} \mathbb{Q} \mathbf{F}=\mathcal{P} \\
\mathbf{W}^{T}(\mathbb{Q} \mathbb{A}) \partial_{x} \mathbf{W}=\partial_{x}(\Pi), & \mathbf{W}^{T}(\mathbb{Q} \mathbb{S}) \mathbf{W}=0 \tag{21b}
\end{array}
$$

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As a result, multiplication of (11) by $\mathbf{W}^{T} \mathbb{Q}$ while using (21) recovers the equation of energy conservation (9). Moreover, the symmetrizer allows to use Friedrichs' theory on symmetric systems [31]: assuming sufficiently smooth initial data and excitation, it yields existence and regularity of the solution to (12). Details are given in section 3-2 and Theorem 3-1 of [40].

In the case of the original stress gradient model $\kappa=0$, then $\tau=\sigma$. It reduces the system in Eq. (12) to

$$
\begin{equation*}
\partial_{t} \mathbf{W}^{*}+\mathbb{A}^{*} \cdot \partial_{x} \mathbf{W}^{*}=\mathbb{S}^{*} \cdot \mathbf{W}^{*}+\mathbf{F}^{*} \tag{22}
\end{equation*}
$$

where the reduced vectors and matrices are

$$
\begin{array}{lc}
\mathbf{W}^{*}=\left\{\begin{array}{c}
v \\
\varphi \\
\sigma \\
R
\end{array}\right\}, & \mathbb{A}^{*}=-\left[\begin{array}{cccc}
0 & 0 & 1 / \varrho & 0 \\
0 & 0 & 1 / \mu & 0 \\
E & E & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right], \\
\mathbf{F}^{*}=\left\{\begin{array}{c}
F_{U} \\
F_{\zeta} \\
0 \\
0
\end{array}\right\}, \quad \mathbb{S}^{*}=\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 / \mu \\
0 & 0 & 0 & 0 \\
0 & D & 0 & 0
\end{array}\right] . \tag{23b}
\end{array}
$$

The eigenvalues of reduced matrices $\mathbb{A}^{*}$ and $\mathbb{S}^{*}$ are

$$
\begin{align*}
& \operatorname{Sp}\left(\mathbb{A}^{*}\right)=\left\{0 ; 0 ; \pm c_{\mathbb{A}}^{*}\right\}  \tag{24a}\\
& \operatorname{Sp}\left(\mathbb{S}^{*}\right)=\left\{0 ; 0 ; \pm \mathrm{i} \omega_{\mathbb{S}}\right\} \tag{24b}
\end{align*}
$$

143
${ }_{144} \pm c_{\mathbb{A}}^{-}$of $\mathbb{A}$ which is zero when $\kappa=0$ (14). The eigenvalue $c_{\mathbb{A}}^{*}$ is equal to $c_{\mathbb{A}}^{+}$ 145 calculated with the modulus $\kappa=0$,

147 The eigenvectors associated with the eigenvalues of $\mathbb{A}^{*}$ read
${ }^{148} \quad \mathbf{W}_{\mathbb{A}}^{*}\left( \pm c_{\mathbb{A}}^{*}\right)=\left\{\begin{array}{c}1 \\ \varrho / \mu \\ \pm \varrho c_{\mathbb{A}}^{*} \\ 0\end{array}\right\}, \quad \mathbf{W}_{\mathbb{A}}^{*}(0)=\operatorname{Vect}\left(\left\{\begin{array}{l}0 \\ 0 \\ 0 \\ 1\end{array}\right\},\left\{\begin{array}{c}1 \\ -1 \\ 0 \\ 0\end{array}\right\}\right)$.
The reduced matrix $\mathbb{A}^{*}$ is thus diagonalizable with real eigenvalues. A symmetrizer for the system (22) is

$$
\begin{equation*}
\mathbb{Q}^{*}=\operatorname{diag}(\varrho ; \mu ; 1 / E ; 1 / D) \tag{27}
\end{equation*}
$$

As a consequence, all the properties of the full system are still valid in the case of the reduced system (22).
2.4. High-order wave equations

Now, combination of Eqs. (6a) to (8a) yields

$$
\begin{align*}
& \varrho \partial_{t}^{2} U-E \partial_{x}^{2} U+\mathcal{H}(U)=\varrho F_{U}+\Psi_{U}  \tag{28a}\\
& \varrho \partial_{t}^{2} \zeta-E \partial_{x}^{2} \zeta+\mathcal{H}(\zeta)=\Psi_{\zeta} \tag{28b}
\end{align*}
$$

Here, the classical wave operator $\varrho \partial_{t}^{2} \bullet-E \partial_{x}^{2} \bullet$ is accompanied by higher order derivatives, described by the operator $\mathcal{H}(\bullet)$ and forcing terms $\Psi_{U}$ and $\Psi_{\zeta}$

$$
\begin{align*}
& \mathcal{H}(\bullet)=\frac{\varrho^{2} b_{t}^{2}}{E} \partial_{t}^{4}(\bullet)+E b_{x}^{2} \partial_{x}^{4}(\bullet)-\varrho b_{m}^{2} \partial_{t}^{2} \partial_{x}^{2}(\bullet),  \tag{29a}\\
& \Psi_{U}=\frac{\varrho^{2} b_{t}^{2}}{E} \partial_{t}^{2} F_{U}-\varrho \partial_{x}^{2}\left[\left(b_{m}^{2}-b_{t}^{2}\right) F_{U}-b_{t}^{2} F_{\zeta}\right],  \tag{29b}\\
& \Psi_{\zeta}=\frac{\varrho^{2} b_{t}^{2}}{E} \partial_{t}^{2} F_{\zeta}-\varrho \partial_{x}^{2}\left[b_{t}^{2} F_{\zeta}-b^{2} F_{U}\right], \tag{29c}
\end{align*}
$$

where the lengths $b_{t}, b_{x}$ and $b_{m}$ are given in (15). Derivatives $\partial_{x}^{4}, \partial_{x}^{2} \partial_{t}^{2}$ and $\partial_{t}^{4}$ of the displacements are present in (28), while only terms in $\partial_{x}^{2} \partial_{t}^{2}$ and $\partial_{t}^{4}$ are present in the original stress gradient model. The latter is shown to be causal in the next section: the postulated causality condition introduced by Metrikine [33], which states that leading-order derivatives in space and time should be of the same order, is therefore clearly not necessary.

Displacement $U$ and auxiliary displacement $\zeta$ are not forced by the same source in (28a) and (28b). In the absence of source, they satisfy the same high-order wave equation in the form $\varrho \partial_{t}^{2} \bullet-E \partial_{x}^{2} \bullet+\mathcal{H}(\bullet)=0$.

## 3. Wave dispersion in stress gradient media

In this section, wave dispersion with inner lengths is studied. Existence of supplementary modes when $\kappa \neq 0$ is examined. Lastly, causality is proven, whatever $\kappa$.

### 3.1. Dispersion relation

Plane wave of the form $U=e^{\mathrm{i}(k x-\omega t)}$ is considered, where $\omega$ is the angular frequency and $k$ is the wavenumber. Substitution into the high-order wave


Figure 1: (Color online) Dispersion relation in stress gradient media. Modes with (a) real positive and (b) imaginary positive normalised wavenumbers $k b$ versus normalised frequency $\omega b / c_{0}$. Here, filled or empty symbols stand for different modes. Hence, modes with real (propagative) and imaginary (exponentially-decaying) wavenumbers are distinct, and do not correspond to real and imaginary parts of a same wavenumber. Calculations with normalised inner lengths $b_{t} / b=1$ and various $b_{x} / b$. Grey zone is bandgap when $b_{x}=0$. Dashed grey line corresponds to elastodynamic relation $k b=\omega b / c_{0}$. Legend in (a) is the same as in (b).
equation leads to

$$
\begin{equation*}
-\left(1-\frac{\varrho \omega^{2}}{E} b_{t}^{2}\right) \frac{\varrho \omega^{2}}{E}+\left(1-\frac{\varrho \omega^{2}}{E} b_{m}^{2}\right) k^{2}+b_{x}^{2} k^{4}=0 \tag{30}
\end{equation*}
$$

In the limit case $B=0$, inner-lengths vanish $b_{x}=b_{m}=b_{t}=0$, and one recovers the Cauchy dispersion relation

$$
\begin{equation*}
k= \pm k_{0} \quad \text { with } \quad k_{0}=\omega \sqrt{\frac{\varrho}{E}}=\frac{\omega}{c_{0}} . \tag{31}
\end{equation*}
$$

Now, wave dispersion in stress gradient media with inner-lengths is studied. To illustrate the discussion, normalised dispersion curves and wave velocities are presented in Figs. 1 and 2. The angular frequencies $\omega_{t}$ and $\omega_{m}$, such that $k_{0} b_{t}=1$ and $k_{0} b_{m}=1$, are introduced. They write

$$
\begin{equation*}
\omega_{t}=\omega_{\mathbb{S}}=\sqrt{\frac{D}{\mu}}, \quad \omega_{m}=\sqrt{\frac{D}{\varrho\left(1+\frac{\kappa}{E}\right)+\mu}} \leq \omega_{t} \tag{32}
\end{equation*}
$$

where $\mathrm{i} \omega_{\mathbb{S}}$ is an eigenvalue of $\mathbb{S}$ (11). The biquadratic dispersion equation (30) admits four roots

$$
\begin{align*}
& k_{ \pm}^{\mathrm{I}}= \pm k_{0} \sqrt{\frac{2\left(1-\left(k_{0} b_{t}\right)^{2}\right)}{1-\left(k_{0} b_{m}\right)^{2}+\sqrt{\Delta}}},  \tag{33a}\\
& k_{ \pm}^{\mathrm{II}}= \pm k_{0} \sqrt{\frac{2\left(1-\left(k_{0} b_{t}\right)^{2}\right)}{1-\left(k_{0} b_{m}\right)^{2}-\sqrt{\Delta}}} . \tag{33b}
\end{align*}
$$

From (15), the discriminant $\Delta$ satisfies

$$
\begin{align*}
\Delta & =\left(1-\left(k_{0} b_{m}\right)^{2}\right)^{2}+4\left(k_{0} b_{x}\right)^{2}\left(1-\left(k_{0} b_{t}\right)^{2}\right) \\
& =\left(1-\left(k_{0} b_{m}\right)^{2}+2\left(k_{0} b_{x}\right)^{2}\right)^{2}+4\left(k_{0} b\right)^{2}\left(k_{0} b_{x}\right)^{2} \geq 0 \tag{34}
\end{align*}
$$

In the low frequency range $k_{0} b_{x} \ll 1, k_{0} b_{t} \ll 1$ and $k_{0} b_{m} \ll 1$, a Taylor expansion of (33) provides the approximates

$$
\begin{equation*}
k_{ \pm}^{\mathrm{I}} \sim \pm k_{0}\left(1+\frac{1}{2} k_{0}^{2} B\right) \quad \text { and } \quad k_{ \pm}^{\mathrm{II}} \sim \pm \frac{\mathrm{i}}{b_{x}} \tag{35}
\end{equation*}
$$

The first two branches with purely real wavenumbers $k_{ \pm}^{\mathrm{I}}$ are propagative. Their Taylor expansion is asymptotic to $\pm k_{0}= \pm \omega / c_{0}$, and deviation from this line is driven by the stress gradient parameter $B$. The two other branches with purely-imaginary wavenumbers $k_{ \pm}^{\mathrm{II}}$ in (35) correspond to non-oscillating exponentially-decaying displacements.

At high frequencies $k_{0} b_{x} \gg 1, k_{0} b_{t} \gg 1$ and $k_{0} b_{m} \gg 1$, the wavenumbers satisfy

$$
\begin{equation*}
k_{ \pm}^{\mathrm{I}} \sim \pm \frac{\omega}{c_{\mathbb{A}}^{-}} \quad \text { and } \quad k_{ \pm}^{\mathrm{II}} \sim \pm \frac{\omega}{c_{\mathbb{A}}^{+}}, \tag{36}
\end{equation*}
$$

where wave speeds $c_{\mathbb{A}}^{ \pm} \geq 0$ are eigenvalues of $\mathbb{A}(14)$. One recalls $c_{\mathbb{A}}^{-} \leq c_{0}$ and $c_{\mathbb{A}}^{+} \geq c_{0}$ in (17).

At the intermediate frequency $\omega_{t}=\omega_{\mathbb{S}}$ such that $k_{0} b_{t}=1$, the discriminant (34) yields $\sqrt{\Delta}=\left|1-\left(k_{0} b_{m}\right)^{2}\right|$, which provides $k_{ \pm}^{\mathrm{II}}= \pm 0$. This property holds for both the original stress gradient model $\left(b_{x}=0\right)$ and the extended one ( $b_{x} \neq 0$ ).

Finally, as inner length $b_{x}$ reaches zero in (33), the dispersion relation for the original stress gradient model is recovered

$$
\begin{array}{ll}
k_{ \pm}^{\mathrm{I}}= \pm k_{0} \sqrt{\frac{1-\left(k_{0} b_{t}\right)^{2}}{1-\left(k_{0} b_{m}\right)^{2}}}, \quad k_{ \pm}^{\mathrm{II}}= \pm \infty, & \text { if } \omega<\omega_{m}, \\
k_{ \pm}^{\mathrm{I}}= \pm \infty, \quad k_{ \pm}^{\mathrm{II}}= \pm k_{0} \sqrt{\frac{1-\left(k_{0} b_{t}\right)^{2}}{1-\left(k_{0} b_{m}\right)^{2}}}, & \text { if } \omega>\omega_{m} \tag{37b}
\end{array}
$$

While four modes are supported in stress gradient media in the extended model $\left(b_{x} \neq 0\right)$, only two of them exist in the original model $\left(b_{x}=0\right)$; the wavenumbers of the two others are sent to infinity. Finally, in the frequency range $\left[\omega_{m}, \omega_{t}\right]$, the wavenumber in the original stress gradient material becomes purely imaginary, see (37b), which leads to non-oscillating and
exponentially-decaying displacements with $\operatorname{Im}\left(k_{ \pm}^{\text {II }}\right)>0$.

### 3.2. Phase velocity and group velocity

Phase velocity $c_{\mathrm{p}}$ and group velocity $c_{\mathrm{g}}$ are defined by

$$
\begin{equation*}
c_{\mathrm{p}}\left(k_{\mathrm{R}}\right)=\frac{\bar{\omega}\left(k_{\mathrm{R}}\right)}{k_{\mathrm{R}}} \quad \text { and } \quad c_{\mathrm{g}}\left(k_{\mathrm{R}}\right)=\frac{\mathrm{d} \bar{\omega}\left(k_{\mathrm{R}}\right)}{\mathrm{d} k_{\mathrm{R}}} \tag{38}
\end{equation*}
$$

where the angular frequency $\bar{\omega}\left(k_{\mathrm{R}}\right)$ is solution to the dispersion equation (30) with the real-valued wavenumber $k=k_{\mathrm{R}}$ being prescribed. Whether in the original $\left(b_{x}=0\right)$ or the extended $\left(b_{x} \neq 0\right)$ stress gradient model, the $\omega$-biquadratic dispersion equation (30) admits the four solutions

$$
\begin{equation*}
\bar{\omega}_{ \pm}^{\mathrm{I}}\left(k_{\mathrm{R}}\right)= \pm k_{\mathrm{R}} c_{\mathrm{p}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right) \quad \text { and } \quad \bar{\omega}_{ \pm}^{\mathrm{II}}\left(k_{\mathrm{R}}\right)= \pm k_{\mathrm{R}} c_{\mathrm{p}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right) \tag{39}
\end{equation*}
$$

where the phase velocities $c_{\mathrm{p}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right)$ and $c_{\mathrm{p}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right)$ are given by

$$
\begin{align*}
& \frac{c_{\mathrm{p}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right)}{c_{0}}=\sqrt{\frac{1+\left(k_{\mathrm{R}} b_{m}\right)^{2}-\sqrt{\Delta_{c}}}{2\left(k_{\mathrm{R}} b_{t}\right)^{2}}}  \tag{40a}\\
& \frac{c_{\mathrm{p}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right)}{c_{0}}=\sqrt{\frac{1+\left(k_{\mathrm{R}} b_{m}\right)^{2}+\sqrt{\Delta_{c}}}{2\left(k_{\mathrm{R}} b_{t}\right)^{2}}} \tag{40b}
\end{align*}
$$

Here the discriminant $\Delta_{c}$ reads

$$
\begin{align*}
\Delta_{c} & =\left(1+\left(k_{\mathrm{R}} b_{m}^{2}\right)^{2}\right)^{2}-4\left(k_{\mathrm{R}} b_{t}\right)^{2}\left(1+\left(k_{\mathrm{R}} b_{x}\right)^{2}\right) \\
& =\left(1+k_{\mathrm{R}}^{2}\left(B+b_{x}^{2}-b_{t}^{2}\right)\right)^{2}+4 k_{\mathrm{R}}^{4} B b_{t}^{2} \tag{41}
\end{align*}
$$

Since $B \geq 0$, the following inequalities result from (41)

$$
\begin{equation*}
\Delta_{c} \geq\left(1+k_{\mathrm{R}}^{2}\left(b_{m}^{2}-2 b_{t}^{2}\right)\right)^{2}>0, \quad \sqrt{\Delta_{c}} \leq 1+\left(k_{\mathrm{R}} b_{m}\right)^{2} \tag{42}
\end{equation*}
$$

It implies that phase velocities $c_{\mathrm{p}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right)$ and $c_{\mathrm{p}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right)$ are real-valued and positive, and satisfy

$$
\begin{equation*}
0<c_{\mathrm{p}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right) \leq c_{0} \leq c_{\mathrm{p}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right) \tag{43}
\end{equation*}
$$



Figure 2: (Color online) Dispersion relation in stress gradient media: (a) normalised phase velocities $c_{\mathrm{p}} / c_{0}$ and ( b ) normalised group velocities $c_{\mathrm{g}} / c_{0}$ versus normalised wavenumber $k_{R} b$ for modes I and II. Here, filled or empty symbols stand for different modes. Symbols consistent with Fig. 1. Calculations with normalised inner lengths $b_{t} / b=1$ and various $b_{x} / b$. Dashed grey line corresponds to $c_{\mathrm{p}}=c_{\mathrm{g}}=c_{0}$. Legend in (b) is the same as in (a).

Moreover, the phase velocities are smooth even functions of the wavenumber $k_{\mathrm{R}}$, and display the limits

$$
\begin{array}{lll}
c_{\mathrm{p}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right) \rightarrow c_{0}, & c_{\mathrm{p}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right) \sim \frac{c_{0}}{\left|k_{\mathrm{R}} b_{t}\right|}, & \text { as }\left|k_{\mathrm{R}}\right| \rightarrow 0 \\
c_{\mathrm{p}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right) \rightarrow c_{\mathbb{A}}^{-}, & c_{\mathrm{p}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right) \rightarrow c_{\mathbb{A}}^{+}, & \text {as }\left|k_{\mathrm{R}}\right| \rightarrow \infty \tag{44b}
\end{array}
$$

Derivation of (39) with respect to $k_{\mathrm{R}}$ provides

$$
\begin{equation*}
\frac{\mathrm{d} \bar{\omega}_{ \pm}^{\mathrm{I}}\left(k_{\mathrm{R}}\right)}{\mathrm{d} k_{\mathrm{R}}}= \pm c_{\mathrm{g}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right) \quad \text { and } \quad \frac{\mathrm{d} \bar{\omega}_{ \pm}^{\mathrm{II}}\left(k_{\mathrm{R}}\right)}{\mathrm{d} k_{\mathrm{R}}}= \pm c_{\mathrm{g}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right) \tag{45}
\end{equation*}
$$

where, accounting for Eq. (40),

$$
\begin{align*}
& c_{\mathrm{g}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right)=\left\{1-\frac{1}{\sqrt{\Delta_{c}}}\left(\left(\frac{c_{0}}{c^{\mathrm{I}}\left(k_{\mathrm{R}}\right)}\right)^{2}-1\right)\right\} c_{\mathrm{p}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right),  \tag{46a}\\
& c_{\mathrm{g}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right)=\left\{1-\frac{1}{\sqrt{\Delta_{c}}}\left(1-\left(\frac{c_{0}}{c^{\mathrm{II}}\left(k_{\mathrm{R}}\right)}\right)^{2}\right)\right\} c_{\mathrm{p}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right) . \tag{46b}
\end{align*}
$$

As a result of (43), one has

$$
\begin{equation*}
c_{\mathrm{g}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right) \leq c_{\mathrm{p}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right) \leq c_{0}, \quad \text { and } \quad c_{\mathrm{g}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right) \leq c_{\mathrm{p}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right) \tag{47}
\end{equation*}
$$

Equation (46) shows that group velocities are smooth and even functions of $k_{\mathrm{R}}$, and display the limits

$$
\begin{array}{lll}
c_{\mathrm{g}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right)=c_{0}, & c_{\mathrm{g}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right)=0, & \text { as }\left|k_{\mathrm{R}}\right|=0, \\
c_{\mathrm{g}}^{\mathrm{I}}\left(k_{\mathrm{R}}\right) \rightarrow c_{\mathbb{A}}^{-}, & c_{\mathrm{g}}^{\mathrm{II}}\left(k_{\mathrm{R}}\right) \rightarrow c_{\mathbb{A}}^{+}, & \text {as }\left|k_{\mathrm{R}}\right| \rightarrow \infty, \tag{48b}
\end{array}
$$

As a consequence, the group velocities are bounded whatever $b_{x}$, which complies with causality. The postulated causality condition introduced by Metrikine [33], which is not satisfied when $b_{x}=0$, is therefore clearly not necessary.

## 4. Micro-macro calculation of material parameters

It was shown in Sec. 2.2 that the displacement $U$ in the stress gradient model satisfies the following equation, according to Eqs. (28) and (29),

$$
\begin{align*}
& \varrho \partial_{t}^{2} U-E \partial_{x}^{2} U+\frac{\varrho^{2}}{E} b_{t}^{2} \partial_{t}^{4} U+E b_{x}^{2} \partial_{x}^{4} U-\varrho b_{m}^{2} \partial_{t}^{2} \partial_{x}^{2} U \\
& \quad=\varrho F_{U}+\frac{\varrho^{2} b_{t}^{2}}{E} \partial_{t}^{2} F_{U}-\varrho \partial_{x}^{2}\left[\left(b_{m}^{2}-b_{t}^{2}\right) F_{U}-b_{t}^{2} F_{\zeta}\right] . \tag{49}
\end{align*}
$$

Here we show that (49) stands for an effective description of micro-structured media in the low-frequency range. To this end, a micro-macro homogenization scheme is presented, which provides (i) the micro-mechanical background to (49); and (ii) a method to estimate the material parameters in the stress gradient model. It relies on the theory of two-scale asymptotic homogenization $[17,18]$ wherein high-order terms of the asymptotic expansions are included [19, 20, 21].

### 4.1. High-order asymptotic homogenization

The homogenization model starts from the equations of dynamic equilibrium and elasticity at the microstructural scale of the one-dimensional medium

$$
\begin{align*}
& \rho \partial_{t}^{2} u=\partial_{x} s+\rho \mathrm{f}  \tag{50a}\\
& s=a \partial_{x} u \tag{50b}
\end{align*}
$$

where $u$ and $s$ are the micro-structural displacement and stress fields, $\rho$ and $a$ the density and elastic modulus, and f is the bulk force in the microstructure. The heterogeneous medium consists of the 1-D periodic repetition of a Representative Elementary Volume (REV) $\Omega$ of length $\ell$.

A scale separation is assumed, whereby the characteristic macroscopic length $\lambda$ (related to the reduced wavelength $\mathcal{O}\left(1 / k_{0}\right)$ or size of the domain of propagation) is much larger than $\ell$. This condition is quantified by the scale parameter $\epsilon=\ell / \lambda \ll 1$ and enables the application of two-scale asymptotic homogenization. To account for both scales, the two space variables $x$ and $y=\epsilon^{-1} x$ are defined for macro- and microscopic description respectively. Material parameter $\rho(y)$ and $a(y)$ are set to depend on $y$, and are $\ell$-periodic

The homogenization of media with inner resonance due to high material contrasts is therefore out of the scope of the present study [34, 35]. Also, the propagative medium is supposed to be macroscopically homogeneous, which implies that $\rho(y)$ and $a(y)$ are independent of the variable $x$ of macroscopic description. The fields $u(x, y)$ and $s(x, y)$ can display variation at both microand macroscopic scales, and they are set to depend on both $x$ and $y$, while being $\ell$-periodic over $y$. This implies to modify spacial differential operator $\partial_{x}$ into $\partial_{x}+\epsilon^{-1} \partial_{y}$. Finally, the displacement and stress are expanded asymptotically into powers of the scale parameter $\epsilon$ according to

$$
\begin{align*}
& u(x, y)=u^{(0)}+\epsilon u^{(1)}+\epsilon^{2} u^{(2)}+\epsilon^{3} u^{(3)}+\mathcal{O}\left(\epsilon^{4}\right),  \tag{52a}\\
& s(x, y)=s^{(0)}+\epsilon s^{(1)}+\epsilon^{2} s^{(2)}+\mathcal{O}\left(\epsilon^{3}\right) \tag{52b}
\end{align*}
$$

over $y$. No high material contrast between the heterogeneities is considered: the local density $\rho(y)$ and elastic modulus $a(y)$ are assumed to vary only moderately around their mean values, $\rho(y)=\mathcal{O}(\langle\rho\rangle)$ and $a(y)=\mathcal{O}(\langle a\rangle)$, where the $y$-averaging operator $\langle\cdot\rangle$ is defined by

$$
\begin{equation*}
\langle\cdot\rangle=\frac{1}{\ell_{y}} \int_{\Omega} \cdot \mathrm{d} y \quad \text { with } \quad \ell_{y}=\int_{\Omega} 1 \mathrm{~d} y . \tag{51}
\end{equation*}
$$

路
where bracketed superscripts indicate the order of the terms and all terms $u^{(j)}(x, y)$ and $s^{(j)}(x, y)$ for $j \geq 0$ are two-scale fields that depend on both $x$ and $y$. The aim of high-order asymptotic homogenization is to determine not only the leading-order terms $u^{(0)}$ and $s^{(0)}$, but also higher-order terms $u^{(j)}(x, y)$ and $s^{(j)}(x, y)$ for $j \geq 1$. To do so, asymptotic expansions (52) are substituted into the governing equations (50) expressed with two-scale differential operator $\partial_{x}+\epsilon^{-1} \partial_{y}$. Terms of equal power of $\epsilon$ are collected to provide problems which are solved in increasing order of the power of $\epsilon$. Details of
the asymptotic homogenization procedure are provided in Appendix A and the main results with respect to the estimation of material parameters in the stress gradient model are discussed here.

The effective macroscopic description of the 1-D periodic elastic material relies on the displacements $\mathcal{U}^{(j)}(x)$ for $j \geq 0$, which correspond to the local mean-value of $u^{(j)}(x, y)$ in (52a)

$$
\begin{equation*}
\mathcal{U}^{(j)}(x)=\left\langle u^{(j)}(x, y)\right\rangle . \tag{53}
\end{equation*}
$$

They satisfy the governing equations

$$
\begin{align*}
& \mathcal{C}_{0}^{(0)}\left(\mathcal{U}^{(0)}\right)=\rho_{0} \mathrm{f}(x)  \tag{54a}\\
& \mathcal{C}_{0}^{(0)}\left(\mathcal{U}^{(1)}\right)=-\rho_{0} h^{(1)} \partial_{x} \mathrm{f}(x),  \tag{54b}\\
& \mathcal{C}_{0}^{(0)}\left(\mathcal{U}^{(2)}\right)=\rho_{0}\left(h^{(1)}\right)^{2} \partial_{x}^{2} \mathrm{f}(x)-\mathcal{C}_{2}^{(2)}\left(\mathcal{U}^{(0)}\right), \tag{54c}
\end{align*}
$$

where the operators $\mathcal{C}_{0}^{(0)}$ and $\mathcal{C}_{2}^{(2)}$ read

$$
\begin{align*}
& \mathcal{C}_{0}^{(0)}\left(\mathcal{U}^{(j)}\right)=\rho_{0} \partial_{t}^{2} \mathcal{U}^{(j)}-E_{0} \partial_{x}^{2} \mathcal{U}^{(j)}  \tag{55a}\\
& \mathcal{C}_{2}^{(2)}\left(\mathcal{U}^{(0)}\right)=-\rho_{0} \Gamma_{t}^{(2)} \partial_{x}^{2} \partial_{t}^{2} \mathcal{U}^{(0)}-E_{0} \Gamma_{x}^{(2)} \partial_{x}^{4} \mathcal{U}^{(0)} \tag{55b}
\end{align*}
$$

In (54) and (55), the density $\rho_{0}\left[\mathrm{~kg} / \mathrm{m}^{3}\right]$, elastic modulus $E_{0}$ [Pa], and characteristic length $h^{(1)}[\mathrm{m}]$ and surfaces $\Gamma_{t}^{(2)}$ and $\Gamma_{x}^{(2)}\left[\mathrm{m}^{2}\right]$ satisfy micro-macro relations involving volume averages of $\Omega$-periodic fields:

$$
\begin{align*}
& \rho_{0}=\langle\rho\rangle>0, \quad E_{0}=\left\langle q_{0}^{(0)}\right\rangle>0,  \tag{56a}\\
& h^{(1)}=\frac{\left\langle\rho \chi_{1}^{(1)}\right\rangle}{\langle\rho\rangle}=\frac{\left\langle q_{1}^{(1)}\right\rangle}{\left\langle q_{0}^{(0)}\right\rangle},  \tag{56b}\\
& \Gamma_{x}^{(2)}=\frac{\left\langle q_{2}^{(2)}\right\rangle}{\left\langle q_{0}^{(0)}\right\rangle}, \quad \Gamma_{t}^{(2)}=J_{1}^{(2)}-\frac{\left\langle\rho \chi_{2}^{(2)}\right\rangle}{\langle\rho\rangle},  \tag{56c}\\
& \text { with } \quad J_{1}^{(2)}=\left\langle\frac{\rho}{\rho_{0}}\left(\chi_{1}^{(1)}\right)^{2}\right\rangle-\left\langle\frac{\rho}{\rho_{0}} \chi_{1}^{(1)}\right\rangle^{2}>0 . \tag{56d}
\end{align*}
$$

The $\Omega$-periodic fields $\chi_{j}^{(j)}(y)\left[\mathrm{m}^{j}\right]$ and generalised stresses $q_{j}^{(j)}(y)\left[\mathrm{Pa} \cdot \mathrm{m}^{j-1}\right]$ satisfy recurrent cell problems for $j \geq 0$

$$
\begin{align*}
& q_{j}^{(j)}(y)=a\left(\chi_{j}^{(j)}(y)+\partial_{y} \chi_{j+1}^{(j+1)}\right)  \tag{57a}\\
& \partial_{y} q_{j}^{(j)}=\frac{\rho(y)}{\langle\rho\rangle}\left\langle q_{j-1}^{(j-1)}\right\rangle-q_{j-1}^{(j-1)}(y)  \tag{57b}\\
& \left\langle\chi_{j}^{(j)}\right\rangle=0 \quad \text { for } j \geq 1  \tag{57c}\\
& \chi_{0}^{(0)}(y) \equiv 1 \quad \text { and } \quad q_{-1}^{(-1)}(y) \equiv 0 . \tag{57d}
\end{align*}
$$

These cell problems (57) correspond to the static equilibrium of the periodic material under $\Omega$-periodic bulk excitation. Except for the effective density $\rho_{0}$ which depends only on $\rho(y)$, and the elasticity modulus $E_{0}$, which depends only on $a(y)$, all the other high-order material parameters $h^{(1)}, \Gamma_{t}^{(2)}$ and $\Gamma_{x}^{(2)}$ depend on both $\rho(y)$ and $a(y)$ through the cell problems. Hence, they cannot be distinguished as parameters related to either micro-inertia or micro-elasticity.

All macro-displacements $\mathcal{U}^{(j)}$ in (55) are governed by the classical wave operator encoded within $\mathcal{C}_{0}^{(0)}$ in (55a), but with different forcing sources on the right-hand-side. First, as the order $j$ of $\mathcal{U}^{(j)}$ increases, high-order space derivatives $\partial_{x}^{j} \mathrm{f}(x)$ of f are involved. Second, the leading-order term $\mathcal{U}^{(0)}$ and the corrector $\mathcal{U}^{(1)}$ are forced only by the bulk force f through its high-order space derivatives, while $\mathcal{U}^{(2)}$ is forced in addition by 4th-order space and time derivatives of the leading-order field $\mathcal{U}^{(0)}$. This is the clue to evidence the stress gradient dynamics operating in this low-frequency range. Third, the classical description of a Cauchy medium is recovered in the leading-order description $\mathcal{U}^{(0)}$, which can be sufficient when $\epsilon \lll 1$.

### 4.2. Identification of stress gradient parameters

Equations (54) are re-scaled to physical scale. To do so, (54b) and (54c) are multiplied by $\epsilon$ and $\epsilon^{2}$ respectively, and results are summed altogether with (54a) to provide

$$
\begin{equation*}
\mathcal{C}_{0}^{(0)}(\mathcal{U})=\rho_{0} \mathcal{F}-\epsilon^{2} \mathcal{C}_{2}^{(2)}\left(\mathcal{U}^{(0)}\right)+\mathcal{O}\left(\epsilon^{3}\right) \tag{58}
\end{equation*}
$$

where effective displacement $\mathcal{U}$ and bulk force $\mathcal{F}$ read

$$
\begin{align*}
\mathcal{U}(x) & =\mathcal{U}^{(0)}+\epsilon^{1} \mathcal{U}^{(1)}+\epsilon^{2} \mathcal{U}^{(2)}=\langle u\rangle+\mathcal{O}\left(\epsilon^{3}\right)  \tag{59a}\\
\mathcal{F}(x) & =\mathrm{f}(x)-h \partial_{x} \mathrm{f}(x)+h^{2} \partial_{x}^{2} \mathrm{f}(x) \tag{59b}
\end{align*}
$$

Here, (59b) represents the micro-macro relation between bulk force $\mathrm{f}(x)$ experienced at the micro-structural scale, and the resulting bulk force $\mathcal{F}(x)$ emerging from it at the macroscopic scale. In particular, this micro-macro relation involves the characteristic inner-length $h=\epsilon h^{(1)}$ related to the bulk force.

The effective macroscopic displacement $\mathcal{U}$ in (58) is forced not only by the effective bulk force $\mathcal{F}$ but also by fourth-order time and space derivatives $\partial_{x}^{2} \partial_{t}^{2} \mathcal{U}^{(0)}$ and $\partial_{x}^{4} \mathcal{U}^{(0)}$ of the leading-order displacement $\mathcal{U}^{(0)}$, see $\mathcal{C}_{2}^{(2)}$ in (55b). Meanwhile, $\mathcal{U}^{(0)}$ satisfies the classical wave equation (54a), which yields after $\partial_{t}^{2}-$ and $\partial_{x}^{2}-$ differentiation

$$
\begin{equation*}
\partial_{t}^{2} \partial_{x}^{2} \mathcal{U}^{(0)}=\frac{\rho_{0}}{E_{0}}\left\{\partial_{t}^{4} \mathcal{U}^{(0)}-\partial_{t}^{2} \mathrm{f}\right\}=\frac{E_{0}}{\rho_{0}} \partial_{x}^{4} \mathcal{U}^{(0)}+\partial_{x}^{2} \mathrm{f} \tag{60}
\end{equation*}
$$

Similar relations without bulk forces $\mathrm{f}(x)$ were used in early studies [20] to define the cell problems, and more recently $[11,14]$ to define families of effective high-order continuum media. Indeed, this capacity to transform

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arbitrarily any 4th-order time-space derivatives of $\mathcal{U}^{(0)}$ into any other ones results in the non-uniqueness of effective continuum models in the asymptotic framework [15]. The general relation is derived in Appendix A using (60)

$$
\begin{align*}
& \mathcal{C}_{2}^{(2)}\left(\mathcal{U}^{(0)}\right)=\frac{\rho_{0}^{2}}{E_{0}} \alpha_{t} \Gamma_{t}^{(2)} \partial_{t}^{4} \mathcal{U}^{(0)}+E_{0} \alpha_{x} \Gamma_{x}^{(2)} \partial_{x}^{4} \mathcal{U}^{(0)} \\
& \quad-\rho_{0}\left[\left(1+\alpha_{x}\right) \Gamma_{x}^{(2)}+\left(1+\alpha_{t}\right) \Gamma_{t}^{(2)}\right] \partial_{x}^{2} \partial_{t}^{2} \mathcal{U}^{(0)}  \tag{61}\\
& \quad+\rho_{0}\left(1+\alpha_{x}\right) \Gamma_{x}^{(2)} \partial_{x}^{2} \mathrm{f}-\frac{\rho_{0}^{2}}{E_{0}} \alpha_{t} \Gamma_{t}^{(2)} \partial_{t}^{2} \mathrm{f},
\end{align*}
$$

where $\alpha_{t}$ and $\alpha_{x}$ are arbitrary dimensionless numbers. However, constraints are imposed further to ensure hyperbolicity and stability of the resulting model. Such aspects have been overlooked in previous studies, see [11] and references therein.

To conclude, (61) is substituted into (58). The latter actually holds up to $\mathcal{O}\left(\epsilon^{3}\right)$, hence terms of order $\mathcal{O}\left(\epsilon^{3}\right)$ can be added to result in equivalent models. This property is used to identify $\epsilon^{2} \mathcal{U}^{(0)}$ with $\epsilon^{2} \mathcal{U}$ and $\epsilon^{2} \mathrm{f}(x)$ with $\epsilon^{2} \mathcal{F}$ in the expression of $\epsilon^{2} \mathcal{C}_{2}^{(2)}\left(\mathcal{U}^{(0)}\right)$. Then, characteristic surface areas are re-scaled according to $\Gamma_{x}=\epsilon^{2} \Gamma_{x}^{(2)}$ and $\Gamma_{t}=\epsilon^{2} \Gamma_{t}^{(2)}$, which finally leads to the following high-order wave equation

$$
\begin{align*}
& \rho_{0} \partial_{t}^{2} \mathcal{U}-E_{0} \partial_{x}^{2} \mathcal{U}+\frac{\rho_{0}^{2}}{E_{0}} \alpha_{t} \Gamma_{t} \partial_{t}^{4} \mathcal{U}+E_{0} \alpha_{x} \Gamma_{x} \partial_{x}^{4} \mathcal{U} \\
& \quad-\rho_{0}\left[\left(1+\alpha_{x}\right) \Gamma_{x}+\left(1+\alpha_{t}\right) \Gamma_{t}\right] \partial_{x}^{2} \partial_{t}^{2} \mathcal{U}  \tag{62}\\
& \quad=\rho_{0} \mathcal{F}+\frac{\rho_{0}^{2}}{E_{0}} \alpha_{t} \Gamma_{t} \partial_{t}^{2} \mathcal{F}-\rho_{0}\left(1+\alpha_{x}\right) \Gamma_{x} \partial_{x}^{2} \mathcal{F}
\end{align*}
$$

The high-order wave equation (62) is formally identical to that given by the stress gradient model in (49). This legitimates the stress gradient model as a candidate for the effective description of heterogeneous media in the low frequency range. Identification of the terms between (62) and (49) leads to
the following relations, with the superscript $\dagger$ on stress gradient parameters:

$$
\begin{align*}
& U^{\dagger}=\mathcal{U}, \quad F_{U}^{\dagger}=\mathcal{F}, \quad F_{\zeta}^{\dagger}=\frac{\mathcal{F}}{\alpha_{t}},  \tag{63a}\\
& \varrho^{\dagger}=\rho_{0}, \quad E^{\dagger}=E_{0},  \tag{63b}\\
& B^{\dagger}=\Gamma_{x}+\Gamma_{t}, \quad D^{\dagger}=\frac{E^{\dagger}}{B^{\dagger}}=\frac{E_{0}}{\Gamma_{x}+\Gamma_{t}},  \tag{63c}\\
& b_{x}^{\dagger 2}=\alpha_{x} \Gamma_{x}, \quad \kappa^{\dagger}=E^{\dagger} \frac{\hat{t}_{x}^{\dagger 2}}{B^{\dagger}}=E_{0} \frac{\alpha_{x} \Gamma_{x}}{\Gamma_{x}+\Gamma_{t}},  \tag{63d}\\
& b_{t}^{\dagger 2}=\alpha_{t} \Gamma_{t}, \quad \mu^{\dagger}=\varrho^{\dagger} \frac{t_{t}^{\dagger 2}}{B^{\dagger}}=\rho_{0} \frac{\alpha_{t} \Gamma_{t}}{\Gamma_{x}+\Gamma_{t}},  \tag{63e}\\
& b_{m}^{\dagger 2}=\left(1+\alpha_{x}\right) \Gamma_{x}+\left(1+\alpha_{t}\right) \Gamma_{t}=B^{\dagger}+b_{x}^{\dagger 2}+b_{t}^{\dagger 2} . \tag{63f}
\end{align*}
$$

All the families of models defined through (62) share the same degree of asymptotic accuracy. They all have the same density $\varrho^{\dagger}$, elasticity modulus $E^{\dagger}$ and stress gradient coefficient $B^{\dagger}$, defined without ambiguity by homogenization, independently of the coefficients $\alpha_{x}$ and $\alpha_{t}$. As a consequence of (35) where only $\varrho^{\dagger}, E^{\dagger}$ and $B^{\dagger}$ are involved, the dispersion relations of all models are therefore asymptotically the same at low frequencies. This implies that discrimination between auxiliary inertia (weighted by $\alpha_{t}$ ) and auxiliary elasticity (weighted by $\alpha_{x}$ ) in micro-structural effects can be made only at higher frequencies.

The orders of magnitude for coefficients $\alpha_{t}$ and $\alpha_{x}$ are constrained by asymptotic homogenization, which requires that $\epsilon^{2} \mathcal{C}_{2}^{(2)}\left(\mathcal{U}^{(0)}\right)$ remains a corrector of order $\epsilon^{2}$ in (58). It results in the orders of magnitude $\alpha_{t}=\mathcal{O}\left(\epsilon^{m}\right)$ and $\alpha_{x}=\mathcal{O}\left(\epsilon^{m}\right)$ with integer $m \geq 0$.

### 4.3. Closing criteria from Bloch-Floquet analysis

As analysed in Sec 2, the parameters $B^{\dagger}, b_{t}^{\dagger}$ and $b_{x}^{\dagger}$ are required to be positive. We assume that asymptotic homogenization theory provides $B^{\dagger}>$ 0 . This property is confirmed numerically through multiple configurations tested in Sec. 5, even if it remains to be proven rigorously.

Various closing criteria for $b_{t}^{\dagger}$ and $b_{x}^{\dagger}$ can be found in the literature in other contexts [36, 11, 14], mostly related to bilaminate micro-structures. Examples of such closing criteria are recalled in Sec. 5.1, where the study case of bilaminate micro-structures is revisited for comparison with the closing criteria that we choose here.

Our criteria follows the same aim of global fit as in [36, 11, 14] but is defined so that it can be applied to any 1-D periodic architecture, while positivity constraints $b_{t}^{\dagger}>0$ and $b_{x}^{\dagger}>0$ are satisfied: we choose to define $b_{t}^{\dagger}$ and $b_{x}^{\dagger}$ as the strictly-positive parameters which minimize the difference between the dispersion relations computed from the stress gradient model and Bloch-Floquet calculations on the heterogeneous medium.

In Bloch-Floquet calculations on the heterogeneous 1-D periodic medium, the dispersion relation is obtained numerically after substitution of the ansatz $u(x)=\eta(x) \exp (\mathrm{i} k x-\mathrm{i} \omega t)$ into the governing equations (50), yielding the eigenvalue problem,

$$
\begin{equation*}
\omega^{2} \rho \eta+\partial_{x}\left\{a \partial_{x} \eta+\mathrm{i} k a \eta\right\}+\mathrm{i} k a \partial_{x} \eta-k^{2} a \eta=0 \tag{64}
\end{equation*}
$$

where the cell function $\eta(x)$ is periodic. Then, the parameters $b_{t}^{\dagger}$ and $b_{x}^{\dagger}$ are obtained through the minimisation

$$
\begin{equation*}
\left(b_{t}^{\dagger}, b_{x}^{\dagger}\right)=\underset{b_{t}^{*}, b_{x}^{*}>0}{\operatorname{argmin}} \Phi\left(b_{t}^{*}, b_{x}^{*}\right), \tag{65}
\end{equation*}
$$

where the deviation function $\Phi$ reads

$$
\begin{equation*}
\Phi\left(b_{t}^{*}, b_{x}^{*}\right)=\frac{1}{N} \sqrt{\sum_{m=1}^{m=N}\left|\omega_{m}^{\mathrm{I}}-\bar{\omega}_{+}^{\mathrm{I}}\left(k_{m}, b_{t}^{*}, b_{x}^{*}\right)\right|^{2}} \tag{66}
\end{equation*}
$$

The angular frequencies $\omega_{m}^{I}$ denote the lowest eigenvalues of the problem stated in (64) for given values of the wavenumber $k=k_{m}=m \pi /(N \ell)$ with integers $1 \leq m \leq N$ and $N \geq 2$. The angular frequencies $\bar{\omega}_{+}^{\mathrm{I}}\left(k_{m}, b_{t}^{*}, b_{x}^{*}\right)$ defined in (39) are computed at wavenumber $k_{m}$ for the stress gradient material having inner-lengths $b_{t}^{*}$ and $b_{x}^{*}$, with parameters $\varrho^{\dagger}, E^{\dagger}$, and $B^{\dagger}$ given by asymptotic homogenization. In (65), the minimum of the deviation function is found by means of the Nelder-Mead Simplex Method [37] using the routine fminsearch in commercial software MatLab, with added bounds $\left.\left.b_{t}^{*}, b_{x}^{*} \in\right] 0,10 \ell\right]$.

This numerical procedure enforces the strict positivity of $b_{t}^{\dagger}$ and $b_{x}^{\dagger}$, ensuring the hyperbolicity and stability of the model. It can be applied to any 1-D unit cell $\Omega$, whether layered or continuously graded. Once the parameters $b_{t}^{\dagger}$ and $b_{x}^{\dagger}$ are obtained from the minimisation in (65), the coefficients $\alpha_{x}$ and $\alpha_{t}$ can be calculated according to (63d) and (63e), and the auxiliary bulk force $F_{\zeta}^{\dagger}$ is determined by (63a).

## 5. Numerical applications

In this section, the micro-macro procedure presented in previous Sec. 4 is applied. The objectives developed in the numerical applications are threefold. First, the influence of the choice in the closing criteria is illustrated in Sec. 5.1. To this aim, our closing criteria is compared to those proposed
by Wautier \& Guzina [11] or Cornaggia \& Guzina [14] in 1-D periodic bilaminate micro-structures. Second, a parametric study on values of parameters for multi-laminate materials with up to 10 layers in the unit cell is performed in Sec. 5.2. This led us to evidence the range of values for the stress gradient parameters, and their respective influence over dispersion relations. Finally, in Sec. 5.3, transient time-domain calculations are performed in response to oscillatory bulk forces in both multi-laminate periodic media and corresponding stress gradient media. They evidence the benefits of the equivalent continuum descriptions.

### 5.1. Comparison of closing criteria in bi-laminate materials

The micro-structure consists of the 1-D periodic repetition of the $\ell$-sized unit cell $\Omega$ made of the two homogeneous elastic layers $\Omega^{[1]}$ and $\Omega^{[2]}$ having the thickness $\ell^{[1]}=\phi^{[1]} \ell$ and $\ell^{[2]}=\phi^{[2]} \ell=\left(1-\phi^{[1]}\right) \ell$, densities $\rho^{[1]}$ and $\rho^{[2]}$, local elastic moduli $a^{[1]}$ and $a^{[2]}$, and wave speeds $c^{[1]}=\sqrt{a^{[1]} / \rho^{[1]}}$ and $c^{[2]}=\sqrt{a^{[2]} / \rho^{[2]}}$. As detailed in Appendix B.2, closed-form formula can be derived for the effective material properties in bilaminate structures. In particular, the stress gradient parameter is shown to be positive, $B^{\dagger}>0$, which is essential for energy consistency of the stress gradient model, see Sec. 2.

We consider the same micro-structure used by Wautier \& Guzina [11], with normalised parameters

$$
\begin{equation*}
\phi^{[1]}=\phi^{[2]}=\frac{1}{2}, \quad \frac{\rho^{[2]}}{\rho^{[1]}}=0.6, \quad \frac{a^{[2]}}{a^{[1]}}=0.4 \tag{67}
\end{equation*}
$$

which lead to the effective material parameters according to Appendix B. 2

$$
\begin{align*}
& \frac{\rho_{0}}{\rho^{[1]}}=0.8, \quad \frac{E_{0}}{a^{[1]}} \approx 0.571, \quad \frac{B^{\dagger}}{\ell^{2}} \approx 9.593 \times 10^{-3},  \tag{68a}\\
& \frac{\Gamma_{x}}{B^{\dagger}} \approx 0.233, \quad \frac{\Gamma_{t}}{B^{\dagger}} \approx 0.767 . \tag{68b}
\end{align*}
$$

We compare our results with those obtained in [14] and [11] with different closing criteria. Using the high-order continuum model (62) with $b_{x}^{\dagger} \equiv 0$ to describe a bilaminate without bulk force, Cornaggia \& Guzina [14] fixed the value of $b_{t}^{\dagger}$ by equating the terms up to $\left(k_{0} \ell\right)^{5}$ in the Taylor expansions of the dispersion relation in the high-order continuum model and in the bilaminate material. Doing so resulted into the closed-form formula for squared innerlengths, denoted with CG as superscript,

$$
\left\{\begin{array}{l}
\left(b_{x}^{\mathrm{CG}}\right)^{2}=0  \tag{69}\\
\left(b_{t}^{\mathrm{CG}}\right)^{2}=\left(\ell^{2}-6 B^{\dagger}-4 J^{\mathrm{CG}}\right) / 10 \\
\left(b_{m}^{\mathrm{CG}}\right)^{2}=B^{\dagger}+\left(b_{x}^{\mathrm{CG}}\right)^{2}+\left(b_{t}^{\mathrm{CG}}\right)^{2}
\end{array}\right.
$$

where the characteristic surface $J^{\mathrm{CG}}$ is given by

$$
\begin{equation*}
J^{\mathrm{CG}}=\frac{\ell^{2}}{12}\left(\left(\phi^{[1]} \frac{c_{0}}{c^{[1]}}\right)^{2}-\left(\phi^{[2]} \frac{c_{0}}{c^{[2]}}\right)^{2}\right)^{2} . \tag{70}
\end{equation*}
$$

The closing criteria (69) with $b_{x}^{\mathrm{CG}}=0$ results into the original stress gradient model. Using micro-structure described in Eq. (67) the values found for the squared inner-lengths and resulting material parameters are given in Tab. 1.

In another study on high-order continuum model (62) applied to describe the bilaminate micro-structure without bulk force, Wautier \& Guzina [11] used all three inner lengths $\left(b_{x}, b_{t}, b_{m}\right)$ along with the relation (15) and the following closing criteria: (i) the group velocity $d \omega / d(\operatorname{Re}(k))$ at the end of


Figure 3: Values of normalised inner-lengths issued from closing criterion (65) with the number $N$ of Bloch wavenumbers used for minimisation for 1-D periodic bilaminate materials described in (68).
the Brillouin zone $\operatorname{Re}(k \ell)=\pi$ is zero; and (ii) the frequency-jump in the dispersion relation between branches $\operatorname{Re}\left(k^{\mathrm{I}}\right)$ and $\operatorname{Re}\left(k^{\mathrm{II}}\right)$ of the stress gradient model at the end of the Brillouin zone $\operatorname{Re}(k \ell)=\pi$ equals the frequency bandwidth of first Bragg bandgap in bilaminate materials. The approach is numerical, and leads to values of inner-lengths $b_{x}^{\mathrm{WG}}, b_{t}^{\mathrm{WG}}, b_{m}^{\mathrm{WG}}$ and weighting factors $\alpha_{x}^{\mathrm{WG}}$ and $\alpha_{t}^{\mathrm{WG}}$ given in Tab. 1, and denoted with the superscript WG. However, such conditions led to the purely imaginary inner-length $b_{x}^{\mathrm{WG}} \approx 0.2252 \mathrm{i}$, which implies the negative auxiliary elasticity modulus $\kappa^{\dagger}=$ $E\left(b_{x}^{\mathrm{WG}}\right)^{2} / B^{\dagger}<0$, making the model energetically inconsistent and unstable, see Sec. 2.

Finally, the criteria (65) proposed here is used. The eigenvalue problem (64) related to Bloch-Floquet analysis is solved numerically by the Plane Wave Method presented in Appendix C. Fig. 3 shows the values of normalised inner-lengths issued from closing criterion using different number $N$ of Bloch


Figure 4: (Color online) Dispersion relation with (a) real and (b) imaginary parts of normalised wavenumbers versus normalised frequency in 1-D periodic bilaminate materials described in Eq. (68) and its stress gradient continuum representations using closing criteria WG, CG, and $\dagger$, defined by Wautier \& Guzina [11], Cornaggia \& Guzina [14], and in the present work (with $N=10^{5}$ Bloch wavenumbers used in minimisation) respectively. Here, filled or empty symbols stand for different modes. Dashed grey line corresponds to elastodynamic relation $k=\omega / c_{0}$. For clarity, branches which satisfy $\operatorname{Im}(k)=0$ have not been represented in Fig. (b), except for WG calculations.

|  | WG | CG | $\dagger\left(N=10^{5}\right)$ | $\dagger(N=20)$ |
| :---: | :---: | :---: | :---: | :---: |
| $b_{x}^{*} / \ell$ | 0.2252 i | 0 | $4.93 \times 10^{-9}$ | $4.41 \times 10^{-9}$ |
| $b_{t}^{*} / \ell$ | 0.2032 | 0.3066 | 0.3314 | 0.3340 |
| $b_{m}^{*} / \ell$ | 0.0139 | 0.3218 | 0.3455 | 0.3481 |
| $\alpha_{x}^{*}$ | -22.71 | 0 | $1.09 \times 10^{-14}$ | $8.72 \times 10^{-15}$ |
| $\alpha_{t}^{*}$ | 5.61 | 12.77 | 14.9174 | 15.1558 |

Table 1: Values of normalised inner-lengths $\left(b_{x}^{*}, b_{t}^{*}, b_{m}^{*}\right) / \ell$, and weighting factors $\left(\alpha_{x}^{*}, \alpha_{t}^{*}\right)$ following closing criteria WG, CG, and $\dagger$, defined by Wautier \& Guzina [11], Cornaggia \& Guzina [14], and in the present work (with $N=10^{5}$ or $N=20$ Bloch wavenumbers used in minimisation) respectively. Calculations performed for micro-structure given in Eq. (67).
wavenumbers in minimisation. For $N=20$, the inner lengths $b_{t}^{\dagger}$ and $b_{m}^{\dagger}$ already reached the value obtained for $N=10^{5}$ with less than $1 \%$ error. On the other hand, the smallness of the inner length $b_{x}^{\dagger}$ compared to the two other inner-length, and its difficulty to converge towards a definite value, are noticeable. This difficulty might be due to the low sensitivity of the acoustic branch of the dispersion relation (on which is performed the minimisation) on $b_{x}^{\dagger}$. Values of inner-lengths and weighting factors for $N=10^{5}$ and $N=20$ are reported in Tab. 1.

Fig. 4 displays the dispersion relations for the 1-D periodic bilaminate material, and for the stress gradient media with parameters issued from WG and CG criteria, and from our closing criterion (65). Focusing only on the real part of the wavenumbers in Fig. 4(a), all stress gradient representation are in excellent agreement with the acoustic branch of the dispersion relation of the bilaminate material in the first Brillouin zone $\operatorname{Re}(k \ell) \leq \pi$. However,
for larger wavenumbers and higher frequencies outside this zone, the stress gradient continuum is inaccurate in its representation of the bilaminate material (for all closing criteria), and remains therefore a long-wavelength/lowfrequency effective medium.

Despite their irrelevance at high frequencies, the continuum models must remain stable. Here, the imaginary part of the wavenumbers represented in Fig. 4(b) reveals clearly that stress gradient medium using the WG closing criterion is unstable: in the vicinity of the Bragg limit $\operatorname{Re}(k \ell) \equiv \pi$, the imaginary part of the wavenumber becomes negative, which results in displacements that increase exponentially as they propagate. Finally, using $N=10^{5}$ or $N=20$ Bloch wavenumbers in our closing criterion (65) produces no significant effects upon the dispersion relation (graphically superimposed) in the long-wavelength/low-frequency range wherein stress gradient continuum is relevant.

### 5.2. Parametric study on multi-laminate materials

The proposed closing criterion (65) can be applied to any 1-D periodic micro-structure to retrieve both positive inner-lengths $b_{t}^{\dagger}$ and $b_{x}^{\dagger}$. An heuristic study on 1-D periodic multi-laminate materials is then conducted so as to evidence ranges of values for effective stress gradient parameters.

The 1-D periodic multi-laminate material consists of the $\ell$-sized unit cell $\Omega$ made of the $n=1 \ldots \mathcal{N}$ homogeneous elastic layers $\Omega^{[n]}=\left[L^{[n]}, L^{[n+1]}\right]$ having the densities $\rho^{[n]}$, local elastic moduli $a^{[n]}$, and thickness $\ell^{[n]}=L^{[n+1]}-L^{[n]}$. Closed-form formula for effective material parameters $\rho_{0}, E_{0}, h, \Gamma_{x}, \Gamma_{t}$ and $B^{\dagger}$ in 1-D periodic multi-laminate materials are derived in Appendix B according to (56) by solving the cell problems (57).

The micro-structural parameters $\ell^{[n]}, \rho^{[n]}$, and $a^{[n]}$ are chosen in the form $10^{v}$, where $v$ is a number picked randomly in the interval $[-1.2,1.2]$ according to the uniform law. The effective material parameters $\rho_{0}, E_{0}, h, \Gamma_{x}, \Gamma_{t}$ and $B^{\dagger}$ are computed according to Appendix B. The closing criterion (65) with 20 Bloch wavenumbers is used to derive the stress gradient continuum corresponding to the multi-laminate material. Such systematic procedure is applied for micro-structures with $\mathcal{N}=2,3,5$, and 10 layers in the unit cell, and 10000 random realisations of micro-structures is considered for each number $\mathcal{N}$. Figure 5 shows the histogram (or distribution density) of values reached by the parameters once normalised by $\ell, \rho_{0}$ or $E_{0}$.

Values reached by micro-structural parameters $\ell^{[n]} / \ell, \rho^{[n]} / \rho_{0}$, and $a^{[n]} / E_{0}$, cover nearly 3 orders of magnitudes each. As for material parameters issued from two-scale asymptotic homogenization, $\Gamma_{x} / \ell^{2}$ and $h / \ell$ can reach positive and negative values, while $\Gamma_{t} / \ell^{2}$ and $B^{\dagger} / \ell^{2}$ are always positive. As for inner-lengths, results show that the approximation $b_{t} / \ell \approx 1 / 3$ is relevant for all realisations of micro-structures considered here, while $b_{x}$ remains much smaller than $b_{t}$ with $b_{x} / \ell \sim 5 \times 10^{-9}$ and the approximation $b_{x} / \ell=0$ is tantalizing. The inner-length $b_{m}=\sqrt{B+b_{x}^{2}+b_{t}^{2}}$ and weighting coefficients $\alpha_{x}=b_{x}^{2} / \Gamma_{x}$ and $\alpha_{t}=b_{t}^{2} / \Gamma_{t}$ are then obtained from the other parameters, and their range of values is given for illustration.

### 5.3. Time-domain simulations

Finally, time-domain simulations are performed on a 1-D periodic trilaminate material and its equivalent stress gradient continuum. Micro-structural


Figure 5: (Color online) Histogram (or distribution density) of characteristic parameters for random realisations of 1-D periodic multi-laminate materials with $\mathcal{N}=2,3,5$ and 10 layers in the unit cell. For each number $\mathcal{N}$ of layers, 10000 random realisations are considered.
parameters for the tri-laminate material are chosen according to

$$
\left.\begin{array}{lll}
\ell^{[1]}=0.3 \ell, & \ell^{[2]}=0.3 \ell, & \ell^{[3]}=0.4 \ell  \tag{71}\\
a^{[1]}=0.45 E_{0}, & a^{[2]}=3.90 E_{0}, & a^{[3]}=1.56 E_{0} \\
\rho^{[1]}=0.34 \rho_{0}, & \rho^{[2]}=0.14 \rho_{0}, & \rho^{[3]}=2.14 \rho_{0}
\end{array}\right\}
$$

which lead to the normalised parameters

$$
\left.\begin{array}{lll}
\Gamma_{x} / \ell^{2} \approx 0.009, & \Gamma_{t} / \ell^{2} \approx 0.022, & B^{\dagger} / \ell^{2} \approx 0.032 \\
b_{x}^{\dagger} / \ell \approx 5 \times 10^{-9}, & b_{t}^{\dagger} / \ell \approx 0.33, & b_{m}^{\dagger} / \ell \approx 0.38  \tag{72}\\
\alpha_{x} \approx 3 \times 10^{-15}, & \alpha_{t} \approx 4.92, & h / \ell \approx-0.07
\end{array}\right\}
$$

Dimensional parameters are used, with the period size $\ell=40 \mathrm{~m}$, the density $\varrho=\rho_{0}=2980 \mathrm{~kg} / \mathrm{m}^{3}$, and the elasticity modulus $E=E_{0}=2.33 \times 10^{9} \mathrm{~Pa}$. Using (63), it yields the auxiliary density $\mu \approx 3.5 \rho_{0} \approx 10358 \mathrm{~kg} / \mathrm{m}^{3}$, the auxiliary elasticity modulus $\kappa \approx 2.2 \times 10^{-6} \mathrm{~Pa} \ll E_{0}$, and the stress gradient modulus $D \approx 4.6 \times 10^{7} \mathrm{~Pa} / \mathrm{m}^{2}$. The dispersion relations in the 1-D periodic tri-laminate material and its equivalent stress gradient continuum are shown in Fig. 6. The straight line $k=\omega / c_{0}$ corresponding to the non-dispersive Cauchy medium with density $\varrho$ and elasticity modulus $E$ is also plotted.

The medium $x \in[0, L]$ is $L=2000 \mathrm{~m}$ long and consists of 50 unit cells, with abscissa $x=L / 2$ corresponding to the interface between $\Omega^{[3]}$ at $x=(L / 2)^{-}$and $\Omega^{[1]}$ at $x=(L / 2)^{+}$.

Whether in the tri-laminate materials, or its equivalent stress gradient and Cauchy continua, a uniform Cartesian grid with mesh size $\delta x$ and time step $\delta t$ is introduced. A fourth-order ADER scheme is implemented to integrate the evolution equations [38]. This explicit two-step and single-grid finitedifference scheme is stable under the usual CFL condition:

$$
\begin{equation*}
\nu=\max \left(c_{i}\right) \frac{\delta t}{\delta x} \leq 1 \tag{73}
\end{equation*}
$$



Figure 6: (Color on-line) Complex dispersion relation in the 1-D tri-laminate material described in Eq. (71), and in its equivalent stress gradient and Cauchy media characterised by parameters $(\varrho, E, \mu, \kappa, D)$ and $(\varrho, E)$ respectively. See main text for values.
where $c_{i}$ is the maximal eigenvalue of $\mathbb{A}$ at node $i$. In multi-laminate materials, a large number of interfaces is involved. To ensure a correct discretisation of the jump conditions, and hence to provide reliable reference solutions, we implement an immersed interface method [39]. While the relaxation matrix $\mathbb{S}$ is null in Cauchy media, $\mathbb{S}$ is non-null in equivalent stress gradient media. Its contribution is then discretised by a splitting approach [40]. In what follows, the spatial mesh size is $\delta x=1 \mathrm{~m}$ and the time step is deduced from the CFL condition (73) where $\nu=0.95$ is chosen.

The following external force is considered

$$
\begin{equation*}
\mathrm{f}(x, t)=X_{\mathrm{f}}(x) \mathcal{T}_{\mathrm{f}}(t) \tag{74}
\end{equation*}
$$

The time and space functions $\mathcal{T}_{\mathrm{f}}(t)$ and $X_{\mathrm{f}}(x)$ read

$$
\begin{align*}
& \mathcal{T}_{\mathrm{f}}(t)=\sum_{m=1}^{4} \mathbb{1}_{\left[0, T_{c}\right]} \hat{\mathrm{f}}_{m} \sin \left(2^{m-1} \omega_{c} t\right),  \tag{75a}\\
& X_{\mathrm{f}}(x)=\exp \left(-x^{2} /\left(2 d^{2}\right)\right) . \tag{75b}
\end{align*}
$$



Figure 7: (Color on-line) External bulk force applied to materials in time-domain calculations. (a) Time signals $\mathcal{T}_{\mathrm{f}}(t)$ for three central frequencies $f_{c}$; (b) Fourier Transform (F.T.) of time signals $\mathcal{T}_{\mathrm{f}}(t)$; and (c) space function $X_{\mathrm{f}}(x)$ applied in the phase $\Omega^{[2]}$ in vicinity of central abscissa $x_{s}=978 \mathrm{~m}$. In figure (b), the shapes of dispersion relations for 1-D periodic tri-laminate material and its equivalent Cauchy medium are plotted in thin continuous and dashed grey lines respectively.

In (75a), the time function $\mathcal{T}_{\mathrm{f}}(t)$ is a combination of sinusoids with support bounded to the time interval $t \in\left[0, T_{c}\right]$ by the indicator function $\mathbb{1}_{\left[0, T_{c}\right]}$ equal to 1 over $\left[0, T_{c}\right]$ and 0 elsewhere. The amplitudes of the sinusoids are $\hat{\mathrm{f}}_{1}=1, \hat{\mathrm{f}}_{2}=-21 / 32, \hat{\mathrm{f}}_{3}=63 / 768$ and $\hat{\mathrm{f}}_{4}=-1 / 512$. It makes the differentiability class of time function $\mathcal{T}_{\mathrm{f}}(t)$ of order 6 , which enable the use of numerical scheme ADER 4 [41] for time-domain calculations. The time evolution of $\mathcal{T}_{\mathrm{f}}(t)$ and its frequency spectrum are displayed in Fig. 7(a) and (b). In particular, Fig. 7(b) shows that $\mathcal{T}_{\mathfrak{f}}(t)$ is a broadband signal with decreasing spectral lobes, the first principal lobe having the central frequency $f_{c}=1 / T_{c}=\omega_{c} / 2 \pi$. The values $f_{c}=\{1 ; 2 ; 3\} \mathrm{Hz}$ are considered, corresponding to scale factors $\epsilon_{c}=\ell \omega_{c} / c_{0}=\{0.28 ; 0.57 ; 0.85\}$. While most frequency content of signal with central frequency $f_{c}=1 \mathrm{~Hz}$ is within the low frequency
range, the whole acoustic branch of the dispersion relation related to the trilaminate material is excited for central frequency $f_{c}=3 \mathrm{~Hz}$ (and its second lobe is within first Bragg bandgap).

In (75b), the space function $X_{\mathrm{f}}(x)$ is a Gaussian function centred at $x_{s}=$ 978 m (phase $\Omega^{[2]}$ in the tri-laminate material) and with standard deviation $d=4 / \sqrt{2} \approx 2.83 \mathrm{~m}$, see Fig. 7(c). The Gaussian is actually cancelled at grid nodes $x_{i}$ such that $\left|x_{i}-x_{s}\right| \geq 2 d$, which makes the source terms belong only to the phase $\Omega^{[2]}$ in the tri-laminate material. Using micro-macro relations (63a) with (59b), the external bulk force $F_{U}^{\dagger}$ applied in stress gradient medium and corresponding to (74) reads

$$
\begin{equation*}
F_{U}^{\dagger}(x, t)=\left[1+\frac{h}{d} \frac{x}{d}+\frac{h^{2}}{d^{2}}\left(\frac{x^{2}}{d^{2}}-1\right)\right] \mathrm{f}(x, t) \tag{76}
\end{equation*}
$$

The forces $F_{U}^{\dagger}$ and $F_{\zeta}^{\dagger}$ in the equivalent stress gradient medium are applied at the same nodes $x_{i}$ as in the tri-laminate material.

Figure 8 shows the particle velocity $\partial_{t} u$ in the tri-laminate material, and $v$ in its equivalent stress gradient and Cauchy media, at $t=1 \mathrm{~s}$. For central frequency $f_{c}=1 \mathrm{~Hz}$, the dispersive effects are negligible, and the agreement of both stress gradient and Cauchy models (leading-order homogenization) with reference field in tri-laminate material is excellent. The interest of the stress gradient model occurs at higher frequencies, when the dispersive effects are solicited: at $f_{c}=2 \mathrm{~Hz}$, the oscillations are well captured by the stress gradient model and the amplitude of the waves is also reproduced. At $f_{c}=3 \mathrm{~Hz}$, the stress gradient model remains still in good agreement with the reference signal in tri-laminate material, while severe discrepancies in terms of wave dispersion and amplitudes can be observed for the Cauchy medium. Near the source, at $f_{c}=2 \mathrm{~Hz}$ and $f_{c}=3 \mathrm{~Hz}$, a sharp peak in the velocity is
due to the small support of the source, included within one micro-structural layer: it disappears when we spread the source by increasing $d$ (not shown here).

Hence, even under poor condition of scale separation $\left(\epsilon_{c}=0.85\right.$ for central frequency $f_{c}=3 \mathrm{~Hz}$ ) and for an external bulk source with micro-structural support, the stress gradient model remains robust to capture the dispersive size effects within the micro-structured material, at small distance away from the zone where the force is applied. All the results presented in Fig. 8 are given with the same number of nodes for both the 1-D periodic tri-laminate material and the homogeneous equivalent media. In the low-frequency range, this number of nodes is mostly driven by the discretisation of the microstructural layers in the tri-laminate material. This constraint is obviously not present in the homogeneous equivalent media and a $1 / \epsilon$-coarser mesh would be sufficient to compute fields with an accuracy nearly as good. In addition, a coarser mesh lowers also the number of time-steps through the CFL condition (73). These aspects emphasise the high interest of homogenization models in dynamic simulations.

Figure 9 shows the snapshots of the elastic and auxiliary velocities $v$ and $\varphi$ in the case $f_{c}=1 \mathrm{~Hz}$. It evidences that (i) in the low-frequency range where the stress gradient model is relevant, the auxiliary velocity $\varphi$ is much smaller than elastic velocity $v$, by two orders of magnitude here; and (ii) $\varphi$ is a long-wavelength field with spatial variations as large as $v$, and is not a micro-structural field with period-wise variations.


Figure 8: (Color online) Snapshot at $t=1 \mathrm{~s}$ of the elastic velocity emitted by a source at $x_{s}=978 \mathrm{~m}$ (phase $\Omega^{[2]}$ in the tri-laminate material) for various central frequencies $f_{c}: 1 \mathrm{~Hz}\left(\epsilon_{c}=0.28\right), 2 \mathrm{~Hz}\left(\epsilon_{c}=0.57\right)$ and $3 \mathrm{~Hz}\left(\epsilon_{c}=0.85\right)$. Left: Elastic velocity in the 1-D periodic tri-laminate material. Right: comparison between fields in 1-D periodic tri-laminate material and its equivalent stress gradient Cauchy homogeneous media.


Figure 9: (Color online) Snapshot at $t=1 \mathrm{~s}$ of the elastic velocity $v$ in (a) and auxiliary velocity $\varphi$ in (b) emitted by a source at $x_{s}=978 \mathrm{~m}$ for $f_{c}=1 \mathrm{~Hz}$.

## 6. Conclusions

This paper focused on the dynamic study of stress gradient materials in 1D. For generality, an extended stress-gradient model was considered, introducing an additional inner-length to the original stress gradient model and a term in $\partial_{x}^{4}$ in the wave equation. On the one hand, we studied theoretically the properties of waves in the extended model: hyperbolicity, stability, dispersion and causality. On the other hand, we proposed a numerical method to identify the parameters of this model to represent the dynamic behavior of a microstructured Cauchy medium. The numerical experiments showed that the dispersive effects were then well captured by the homogeneous stress gradient material. A systematic numerical study showed that the additional parameter of the extended model was very close to 0 when it comes to represent classical heterogeneous elastic media, and could very well be neglected in that case. Contrary to a usually-accepted postulate, this parameter is not
necessary for causality. An interesting perspective of this work concerns the study (theoretical and numerical) of stress gradient media in higher spatial dimensions.

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## Appendix A. Two-scale asymptotic homogenization

This appendix describes the two-scale asymptotic homogenization highorder terms. The approach follows the procedure developed initially in 3-D elastodynamics without bulk force [20] or 3-D elastostatics with the bulk force $[19,21]$, and applied in more recent work $[13,11,14]$ in 1-D elastodynamics without bulk force f. Here, the problem in elastodynamics with the mass density of external bulk force $f$ is tackled. After substitution of asymptotic expansions (52) into governing equations (50) expressed with two-scale differential operator $\partial_{x}+\epsilon^{-1} \partial_{y}$, terms of equal power of $\epsilon$ are collected to
provide

$$
\begin{align*}
& \partial_{y} u^{(0)}=0  \tag{A.1a}\\
& \partial_{y} s^{(0)}=0  \tag{A.1b}\\
& s^{(0)}=a\left(\partial_{y} u^{(1)}+\partial_{x} u^{(0)}\right),  \tag{A.1c}\\
& \partial_{y} s^{(1)}+\partial_{x} s^{(0)}=\rho\left(\partial_{t}^{2} u^{(0)}-\mathrm{f}\right),  \tag{A.1d}\\
& s^{(1)}=a\left(\partial_{y} u^{(2)}+\partial_{x} u^{(1)}\right),  \tag{A.1e}\\
& \partial_{y} s^{(2)}+\partial_{x} s^{(1)}=\rho \partial_{t}^{2} u^{(1)},  \tag{A.1f}\\
& s^{(2)}=a\left(\partial_{y} u^{(3)}+\partial_{x} u^{(2)}\right),  \tag{A.1g}\\
& \partial_{y} s^{(3)}+\partial_{x} s^{(2)}=\rho \partial_{t}^{2} u^{(2)} . \tag{A.1h}
\end{align*}
$$

First, $y$-averaged values $\mathcal{U}^{(j)}(x)$ and $\Sigma^{(j)}(x)$ of two-scale fields $u^{(j)}(x, y)$ and $s^{(j)}(x, y)$ are defined for $j \geq 0$,

$$
\begin{equation*}
\mathcal{U}^{(j)}(x)=\left\langle u^{(j)}\right\rangle \quad \text { and } \quad \Sigma^{(j)}(x)=\left\langle s^{(j)}\right\rangle . \tag{A.2}
\end{equation*}
$$

The local $\Omega$-periodicity of stress fields $s^{(j)}(x, y)$ over the variable $y$ implies the following relation, that is useful when Eqs. (A.1b), (A.1d), (A.1f) and (A.1h) is $y$-averaged

$$
\begin{equation*}
\left\langle\partial_{y} s^{(j)}\right\rangle=0 \tag{A.3}
\end{equation*}
$$

Equation (A.1a) shows that $u^{(0)}(x, y)$ is independent of $y$. It is therefore purely macroscopic, and reads

$$
\begin{equation*}
u^{(0)}(x, y) \equiv \mathcal{U}^{(0)}(x) \tag{A.4}
\end{equation*}
$$

Equations (A.1b) and (A.1c) show that $u^{(1)}(x, y)$ and $s^{(0)}(x, y)$ are forced at the macroscopic scale by $\partial_{x} \mathcal{U}^{(0)}$. Thus they write

$$
\begin{align*}
& u^{(1)}(x, y)=\mathcal{U}^{(1)}(x)+\chi_{1}^{(1)}(y) \partial_{x}^{1} \mathcal{U}^{(0)}  \tag{A.5a}\\
& s^{(0)}(x, y)=q_{0}^{(0)}(y) \partial_{x}^{1} \mathcal{U}^{(0)} \tag{A.5b}
\end{align*}
$$

where cell field $\chi_{1}^{(1)}(y)$ and local modulus $q_{0}^{(0)}(y)$ are $\Omega$-periodic and satisfy the cell problem

$$
\begin{align*}
& \partial_{y} q_{0}^{(0)}=0  \tag{A.6a}\\
& q_{0}^{(0)}=a\left(1+\partial_{y} \chi_{1}^{(1)}\right), \tag{A.6b}
\end{align*}
$$

with the condition $\left\langle\chi_{1}^{(1)}\right\rangle=0$ so that $\mathcal{U}^{(1)}=\left\langle u^{(1)}\right\rangle$. Then, the equations governing the macroscopic field $\mathcal{U}^{(0)}$ are obtained by $y$-averaging (A.5b) and (A.1d),

$$
\begin{align*}
& \Sigma^{(0)}=E_{0} \partial_{x}^{1} \mathcal{U}^{(0)}  \tag{A.7a}\\
& \partial_{x} \Sigma^{(0)}=\rho_{0}\left(\partial_{t}^{2} \mathcal{U}^{(0)}-\mathrm{f}(x)\right), \tag{A.7b}
\end{align*}
$$

where effective material parameters are given by

$$
\begin{equation*}
E_{0}=\left\langle q_{0}^{(0)}\right\rangle=\left\langle a\left(1+\partial_{y} \chi_{1}^{(1)}\right)^{2}\right\rangle, \quad \rho_{0}=\langle\rho\rangle . \tag{A.8}
\end{equation*}
$$

Hence, both $E_{0}$ and $\rho_{0}$ are positive. The second expression for $E_{0}$ in Eq. (A.8) follows from the weak formulation of cell problem (A.6) with $\chi_{1}^{(1)}$ as the testfield [17]

$$
\begin{equation*}
\left\langle\chi_{1}^{(1)} \partial_{y} q_{0}^{(0)}\right\rangle=-\left\langle\left(\partial_{y} \chi_{1}^{(1)}\right) a\left(1+\partial_{y} \chi_{1}^{(1)}\right)\right\rangle=0 \tag{A.9}
\end{equation*}
$$

Combination of (A.7a) and (A.7b) yields the equation for the leading-order displacement $\mathcal{U}^{(0)}$

$$
\begin{equation*}
\mathcal{C}_{0}\left(\mathcal{U}^{(0)}\right) \stackrel{\text { def }}{=} \rho_{0} \partial_{t}^{2} \mathcal{U}^{(0)}-E_{0} \partial_{x}^{2} \mathcal{U}^{(0)}=\rho_{0} \mathrm{f}(x), \tag{A.10}
\end{equation*}
$$

Expressions for $\partial_{t}^{2} \mathcal{U}^{(0)}-\mathrm{f}(x)$ given in (A.7a) and $u^{(1)}$ in (A.5a) are substituted into (A.1d) and (A.1e) to yield:

$$
\begin{align*}
& \partial_{y} s^{(1)}=r_{0}^{(0)}(y) \partial_{x}^{2} \mathcal{U}^{(0)},  \tag{A.12a}\\
& s^{(1)}=a\left(\partial_{y} u^{(2)}+\partial_{x} \mathcal{U}^{(1)}+\chi_{1}^{(1)}(y) \partial_{x}^{2} \mathcal{U}^{(0)}\right) . \tag{A.12b}
\end{align*}
$$

${ }^{37}$ Due to (A.5b) and (A.7a), the source term $r_{0}^{(0)}$ in (A.12a) writes

$$
\begin{equation*}
r_{0}^{(0)}(y)=\frac{\rho(y)}{\rho_{0}^{(0)}} E_{0}^{(0)}-q_{0}^{(0)}(y)=\left(\frac{\rho(y)}{\rho_{0}^{(0)}}-1\right) E_{0}^{(0)} . \tag{A.13}
\end{equation*}
$$

Equations (A.12a) to (A.13) show that $u^{(2)}$ and $s^{(1)}$ are forced by both $\partial_{x} \mathcal{U}^{(1)}$ and $\partial_{x}^{2} \mathcal{U}^{(0)}$ according to

$$
\begin{align*}
& u^{(2)}(x, y)=\mathcal{U}^{(2)}(x)+\chi_{1}^{(1)} \partial_{x}^{1} \mathcal{U}^{(1)}+\chi_{2}^{(2)} \partial_{x}^{2} \mathcal{U}^{(0)},  \tag{A.14a}\\
& s^{(1)}(x, y)=q_{0}^{(0)} \partial_{x}^{1} \mathcal{U}^{(1)}+q_{1}^{(1)} \partial_{x}^{2} \mathcal{U}^{(0)} . \tag{A.14b}
\end{align*}
$$

The cell field $\chi_{2}^{(2)}(y)$ and local modulus $q_{1}^{(1)}(y)$ are $\Omega$-periodic and satisfy the cell problem

$$
\begin{align*}
& \partial_{y} q_{1}^{(1)}=r_{0}^{(0)}(y)  \tag{A.15a}\\
& q_{1}^{(1)}=a\left(\chi_{1}^{(1)}(y)+\partial_{y} \chi_{2}^{(2)}\right) \tag{A.15b}
\end{align*}
$$

with $\left\langle\chi_{2}^{(2)}\right\rangle=0$, so that $\mathcal{U}^{(2)}=\left\langle u^{(2)}\right\rangle$. Averaging $q_{1}^{(1)} / a$ from (A.15b) with $\left\langle\partial_{y} \chi_{2}^{(2)}\right\rangle=0$ due to $\Omega$-periodicity leads to

$$
\begin{equation*}
\left\langle\frac{q_{1}^{(1)}(y)}{a(y)}\right\rangle=\left\langle\chi_{1}^{(1)}(y)+\partial_{y} \chi_{2}^{(2)}\right\rangle \equiv 0 . \tag{A.16}
\end{equation*}
$$

Then, the equations governing the macroscopic field $\mathcal{U}^{(1)}$ are obtained by $y$-averaging (A.14b) and (A.1f),

$$
\begin{align*}
& \Sigma^{(1)}=E_{0}^{(0)} \partial_{x}^{1} \mathcal{U}^{(1)}+E_{1}^{(1)} \partial_{x}^{2} \mathcal{U}^{(0)}  \tag{A.17a}\\
& \partial_{x} \Sigma^{(1)}=\rho_{0}^{(0)} \partial_{t}^{2} \mathcal{U}^{(1)}+\rho_{1}^{(1)} \partial_{t}^{2} \partial_{x} \mathcal{U}^{(0)} \tag{A.17b}
\end{align*}
$$

where effective material parameters are given by

$$
\begin{equation*}
E_{1}^{(1)}=\left\langle q_{1}^{(1)}\right\rangle, \quad \rho_{1}^{(1)}=\left\langle\rho \chi_{1}^{(1)}\right\rangle \tag{A.18}
\end{equation*}
$$

The weak formulation of cell problems (A.6) and (A.15) with $\chi_{1}^{(1)}$ and $\chi_{2}^{(2)}$ as test-fields yields [20]

$$
\begin{align*}
E_{1}^{(1)} & =\left\langle\chi_{1}^{(1)} \partial_{y} q_{1}^{(1)}-\chi_{2}^{(2)} \partial_{y} q_{0}^{(0)}+\chi_{1}^{(1)} q_{0}^{(0)}\right\rangle  \tag{A.19}\\
& =\left\langle\chi_{1}^{(1)} r_{0}^{(0)}+\chi_{1}^{(1)} q_{0}^{(0)}\right\rangle=\rho_{1}^{(1)} E_{0} / \rho_{0}
\end{align*}
$$

Combination of (A.17a) and (A.17b) leads to

$$
\begin{equation*}
\mathcal{C}_{0}\left(\mathcal{U}^{(1)}\right)=-\partial_{x}\left\{\rho_{1}^{(1)} \partial_{t}^{2} \mathcal{U}^{(0)}-E_{1}^{(1)} \partial_{x}^{2} \mathcal{U}^{(0)}\right\} \tag{A.20}
\end{equation*}
$$

Multiplication of (A.10) by $\rho_{1}^{(1)} / \rho_{0}$ and use of (A.19) gives

$$
\begin{equation*}
\rho_{1}^{(1)} \partial_{t}^{2} \mathcal{U}^{(0)}-E_{1}^{(1)} \partial_{x}^{2} \mathcal{U}^{(0)}=\rho_{1}^{(1)} \mathrm{f}(x), \tag{A.21}
\end{equation*}
$$

which is analogous to classical elastodynamics in Cauchy media, with $\rho_{1}^{(1)}$ and $E_{1}^{(1)}$ as density and elasticity modulus. Substitution of (A.21) into (A.20) provides

$$
\begin{equation*}
\mathcal{C}_{0}\left(\mathcal{U}^{(1)}\right)=-\rho_{0} h^{(1)} \partial_{x} \mathrm{f}(x), \quad h^{(1)}=\rho_{1}^{(1)} / \rho_{0} \tag{A.22}
\end{equation*}
$$

${ }_{657}$ (A.19) to give
which shows that $\mathcal{U}^{(1)}$ is actually forced by the gradient of the bulk force, $\partial_{x} \mathrm{f}(x)$. As previously, (A.22) is multiplied by $\rho_{1}^{(1)} / \rho_{0}$ and combined with

$$
\begin{equation*}
\rho_{1}^{(1)} \partial_{t}^{2} \mathcal{U}^{(1)}-E_{1}^{(1)} \partial_{x}^{2} \mathcal{U}^{(1)}=-\rho_{0}\left(h^{(1)}\right)^{2} \partial_{x} \mathrm{f}(x) \tag{A.23}
\end{equation*}
$$

Substitution of (A.5a) and (A.14b) into (A.1f), while using (A.17a) and (A.17b) to express $\partial_{t}^{2} \mathcal{U}^{(1)}$ and $\Sigma^{(1)}$ provides

$$
\begin{equation*}
\partial_{y} s^{(2)}=r_{0}^{(0)} \partial_{x}^{2} \mathcal{U}^{(1)}+r_{1}^{(1)} \partial_{x}^{3} \mathcal{U}^{(0)}+\rho_{0} \widetilde{r}_{1}^{(1)} \partial_{t}^{2} \partial_{x} \mathcal{U}^{(0)} \tag{A.24}
\end{equation*}
$$

where the source terms $r_{1}^{(1)}(y)$ and $\widetilde{r}_{1}^{(1)}(y)$ read

$$
\begin{align*}
& r_{1}^{(1)}(y)=\frac{\rho(y)}{\rho_{0}} E_{1}^{(1)}-q_{1}^{(1)}(y)  \tag{A.25a}\\
& \widetilde{r}_{1}^{(1)}(y)=\frac{\rho(y)}{\rho_{0}}\left(\chi_{1}^{(1)}(y)-\frac{\rho_{1}^{(1)}}{\rho_{0}}\right) \tag{A.25b}
\end{align*}
$$

Further, substitution of (A.14a) into (A.1g) yields

$$
\begin{equation*}
s^{(2)}=a\left(\partial_{y} u^{(3)}+\partial_{x} \mathcal{U}^{(2)}+\chi_{1}^{(1)} \partial_{x}^{2} \mathcal{U}^{(1)}+\chi_{2}^{(2)} \partial_{x}^{3} \mathcal{U}^{(0)}\right) \tag{A.26}
\end{equation*}
$$

Equations (A.24) and (A.26) evidence that $u^{(3)}(x, y)$ and $s^{(2)}(x, y)$ are forced at the macroscopic scale by $\partial_{x} \mathcal{U}^{(2)}, \partial_{x}^{2} \mathcal{U}^{(1)}, \partial_{x}^{3} \mathcal{U}^{(0)}$, and $\partial_{t}^{2} \partial_{x} \mathcal{U}^{(0)}$ according to

$$
\begin{gather*}
u^{(3)}(x, y)=\mathcal{U}^{(3)}+\chi_{1}^{(1)} \partial_{x}^{1} \mathcal{U}^{(2)}+\chi_{2}^{(2)} \partial_{x}^{2} \mathcal{U}^{(1)}  \tag{A.27a}\\
+\chi_{3}^{(3)} \partial_{x}^{3} \mathcal{U}^{(0)}+\rho_{0} \psi^{(3)} \partial_{t}^{2} \partial_{x} \mathcal{U}^{(0)}, \\
s^{(2)}(x, y)=q_{0}^{(0)} \partial_{x}^{1} \mathcal{U}^{(2)}+q_{1}^{(1)} \partial_{x}^{2} \mathcal{U}^{(1)}  \tag{A.27b}\\
\quad+q_{2}^{(2)} \partial_{x}^{3} \mathcal{U}^{(0)}+\rho_{0} p^{(2)} \partial_{t}^{2} \partial_{x} \mathcal{U}^{(0)} .
\end{gather*}
$$

The cell fields $\chi_{3}^{(3)}(y)$ and $\psi^{(3)}(y)$, and associated local moduli $q_{2}^{(2)}(y)$ and $p^{(2)}(y)$ are $\Omega$-periodic and satisfy the cell problems

$$
\begin{array}{ll}
\partial_{y} q_{2}^{(2)}=r_{1}^{(1)}, & q_{2}^{(2)}=a\left(\chi_{2}^{(2)}(y)+\partial_{y} \chi_{3}^{(3)}\right) \\
\partial_{y} p^{(2)}=\widetilde{r}_{1}^{(1)}, & p^{(2)}=a \partial_{y} \psi^{(3)} \tag{A.28b}
\end{array}
$$

with the condition $\left\langle\chi_{3}^{(3)}\right\rangle=0$ and $\left\langle\psi^{(3)}\right\rangle=0$, so that $\mathcal{U}^{(3)}=\left\langle u^{(3)}\right\rangle$. Averaging $q_{2}^{(2)} / a$ and $p^{(2)} / a$ with $\left\langle\partial_{y} \chi_{3}^{(3)}\right\rangle=0$ and $\left\langle\partial_{y} \psi^{(3)}\right\rangle=0$ due to $\Omega$-periodicity yields

$$
\begin{equation*}
\left\langle\frac{q_{2}^{(2)}(y)}{a(y)}\right\rangle=0 \quad \text { and } \quad\left\langle\frac{p^{(2)}(y)}{a(y)}\right\rangle=0 \tag{A.29}
\end{equation*}
$$

In (A.27b), we choose to involve the derivative $\partial_{t}^{2} \partial_{x} \mathcal{U}^{(0)}$ in the stress field. Using the wave equation (A.10) at dominant order, this derivative is transformed into $\rho_{0} \partial_{t}^{2} \partial_{x} \mathcal{U}^{(0)}=E_{0} \partial_{x}^{3} \mathcal{U}^{(0)}+\rho_{0} \partial_{x} \mathrm{f}$. In the absence of bulk force $\mathrm{f}=0$, this procedure is usually applied, so as to keep only high-order space derivatives $\partial_{x}^{3} \mathcal{U}^{(0)}$ in the expression of the stress field [20, 13, 11, 14]. With bulk force, we prefer to keep the space-time derivative $\rho_{0} \partial_{t}^{2} \partial_{x} \mathcal{U}^{(0)}$ in the effective constitutive relation (A.27b) rather than involving the bulk force in it. This choice leads to define the cell problems according to (A.28).

Finally, the equations governing the macroscopic field $\mathcal{U}^{(2)}$ are obtained by $y$-averaging (A.27b) and (A.1h),

$$
\begin{align*}
\Sigma^{(2)}= & E_{0} \partial_{x} \mathcal{U}^{(2)}+E_{1}^{(1)} \partial_{x}^{2} \mathcal{U}^{(1)} \\
& +E_{2}^{(2)} \partial_{x}^{3} \mathcal{U}^{(0)}+\rho_{0} J_{1}^{(2)} \partial_{t}^{2} \partial_{x} \mathcal{U}^{(0)}  \tag{A.30a}\\
\partial_{x} \Sigma^{(2)}= & \rho_{0} \partial_{t}^{2} \mathcal{U}^{(2)}+\rho_{1}^{(1)} \partial_{x} \partial_{t}^{2} \mathcal{U}^{(1)}  \tag{A.30b}\\
& \quad+\rho_{2}^{(2)} \partial_{x}^{2} \partial_{t}^{2} \mathcal{U}^{(0)},
\end{align*}
$$

From (A.10) and for any $\alpha_{t}$ and $\alpha_{x}$, one gets

$$
\begin{align*}
& \left(1+\alpha_{t}-\alpha_{t}\right) \partial_{x}^{2} \partial_{t}^{2} \mathcal{U}^{(0)}=\frac{\rho_{0}}{E_{0}}\left(\partial_{t}^{4} \mathcal{U}^{(0)}-\partial_{t}^{2} \mathrm{f}\right),  \tag{A.35a}\\
& \left(1+\alpha_{x}-\alpha_{x}\right) \partial_{x}^{4} \mathcal{U}^{(0)}=\frac{\rho_{0}}{E_{0}}\left(\partial_{x}^{2} \partial_{t}^{2} \mathcal{U}^{(0)}-\partial_{x}^{2} \mathrm{f}\right) . \tag{A.35b}
\end{align*}
$$

It results into

$$
\begin{align*}
& \partial_{x}^{2} \partial_{t}^{2} \mathcal{U}^{(0)}=\left(1+\alpha_{t}\right) \partial_{x}^{2} \partial_{t}^{2} \mathcal{U}^{(0)}-\alpha_{t} \frac{\rho_{0}}{E_{0}}\left(\partial_{t}^{4} \mathcal{U}^{(0)}-\partial_{t}^{2} \mathrm{f}\right),  \tag{A.36a}\\
& \partial_{x}^{4} \mathcal{U}^{(0)}=\left(1+\alpha_{x}\right) \frac{\rho_{0}}{E_{0}}\left(\partial_{x}^{2} \partial_{t}^{2} \mathcal{U}^{(0)}-\partial_{x}^{2} \mathrm{f}\right)-\alpha_{x} \partial_{x}^{4} \mathcal{U}^{(0)} \tag{A.36b}
\end{align*}
$$

which, once substituted into (A.33) leads to (61).

## Appendix B. Cell functions and effective properties in periodic multilaminate materials

Here, cell functions and effective material parameters defined through two-scale asymptotic homogenization in Appendix A are given for the periodic multilaminate material. It consists of the $\ell$-sized unit cell $\Omega$ made of the $n=1 . . \mathcal{N}$ homogeneous elastic layers $\Omega^{[n]}=\left[L^{[n]}, L^{[n+1]}\right]$ having the densities $\rho^{[n]}$, local elastic moduli $a^{[n]}$, and thickness $\ell^{[n]}=\phi^{[n]} \ell=L^{[n+1]}-L^{[n]}$, with filling fractions $\phi^{[n]}$ and boundaries at abscissa $L^{[n]}$ so that

$$
\begin{equation*}
\sum_{n=1}^{\mathcal{N}} \phi^{[n]}=1, \quad L^{[n]}=\sum_{m=1}^{n-1} \ell^{[m]}=\left(\sum_{m=1}^{n-1} \phi^{[m]}\right) \ell \tag{B.1}
\end{equation*}
$$

with $L^{[1]}=0$ and $L^{[\mathcal{N}+1]}=\ell$. Here and in what follows, the usual convention that the sum $\sum_{a}^{b}$ is zero if $b<a$ is adopted. Further, space, fields and parameters have been re-scaled to physical scale, with $x=\epsilon y$ and $q_{j}=\epsilon^{j} q_{j}^{(j)}$ for instance. The local normalised coordinate $\xi^{[n]}$ holding on layer $\Omega^{[n]}$ is also defined as follows, for $n \geq 1$,

$$
\begin{equation*}
\forall x \in\left[L^{[n]}, L^{[n+1]}\right], \quad \xi^{[n]}=\frac{x-L^{[n]}}{\ell} \in\left[0, \phi^{[n]}\right] \tag{B.2}
\end{equation*}
$$

Explicit resolution of the cell problems is provided in the general case of the multilaminate materials; then, formula are provided in the simple case of the bilaminate materials with $\mathcal{N}=2$.

## Appendix B.1. General formulation

First, equations (A.8) and (A.11) provide

$$
\begin{equation*}
\rho_{0}=\sum_{n=1}^{\mathcal{N}} \phi^{[n]} \rho^{[n]}, \quad q_{0} \equiv E_{0}=\left(\sum_{n=1}^{\mathcal{N}} \frac{\phi^{[n]}}{a^{[n]}}\right)^{-1} \tag{B.3}
\end{equation*}
$$

Then, successive integrations of Eqs. (A.6b) and (A.15a) to solve for $\chi_{1}$ and $q_{1}$, yield the cell functions in the following form of linear functions on layer $\Omega^{[n]}$,

$$
\begin{align*}
& \frac{\chi_{1}\left(\xi^{[n]}\right)}{\ell}=\vartheta_{E}^{[n]} \xi^{[n]}+\mathcal{X}_{1}^{[n]}-\widetilde{\chi}_{1},  \tag{B.4a}\\
& \frac{q_{1}\left(\xi^{[n]}\right)}{E_{0} \ell}=\vartheta_{\rho}^{[n]} \xi^{[n]}+Q_{1}^{[n]}-\widetilde{q}_{1} . \tag{B.4b}
\end{align*}
$$

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Here, the dimensionless factors $\vartheta_{E}^{[n]}$ and $\vartheta_{\rho}^{[n]}$ read

$$
\begin{equation*}
\vartheta_{E}^{[n]}=\frac{E_{0}}{a^{[n]}}-1 \quad \text { and } \quad \vartheta_{\rho}^{[n]}=\frac{\rho^{[n]}}{\rho_{0}}-1 \tag{B.5}
\end{equation*}
$$

while factors $\mathcal{X}_{1}^{[n]}$ and $Q_{1}^{[n]}$ are defined to ensure continuity and periodicity of the cell functions,

$$
\begin{equation*}
\mathcal{X}_{1}^{[n]}=\sum_{m=1}^{n-1} \vartheta_{E}^{[m]} \phi^{[m]}, \quad Q_{1}^{[n]}=\sum_{m=1}^{n-1} \vartheta_{\rho}^{[m]} \phi^{[m]}, \tag{B.6}
\end{equation*}
$$

and dimensionless parameters $\widetilde{\chi}_{1}$ and $\widetilde{q}_{1}$ are prescribed by conditions of zero mean values $\left\langle\chi_{1}\right\rangle=0$ and $\left\langle q_{1} / a\right\rangle=0$,

$$
\begin{align*}
& \widetilde{\chi}_{1}=\sum_{n=1}^{\mathcal{N}}\left[\vartheta_{E}^{[n]} \frac{\left(\phi^{[n]}\right)^{2}}{2}+\mathcal{X}_{1}^{[n]} \phi^{[n]}\right],  \tag{B.7a}\\
& \widetilde{q}_{1}=\sum_{n=1}^{\mathcal{N}} \frac{E_{0}}{a^{[n]}}\left[\vartheta_{\rho}^{[n]} \frac{\left(\phi^{[n]}\right)^{2}}{2}+Q_{1}^{[n]} \phi^{[n]}\right] . \tag{B.7b}
\end{align*}
$$

For efficient calculations of $\mathcal{X}_{1}^{[n]}$ and $Q_{1}^{[n]}$, it is worth noting the following recurrence relations for $n \geq 1$,

$$
\begin{align*}
& \mathcal{X}_{1}^{[n+1]}=\mathcal{X}_{1}^{[n]}+\vartheta_{E}^{[n]} \phi^{[n]}  \tag{B.8a}\\
& Q_{1}^{[n+1]}=Q_{1}^{[n]}+\vartheta_{\rho}^{[n]} \phi^{[n]} \tag{B.8b}
\end{align*}
$$

and following relations due to $\ell$-periodicity,

$$
\begin{align*}
& \mathcal{X}_{1}^{[\mathcal{N}+1]}=\sum_{m=1}^{\mathcal{N}} \vartheta_{E}^{[m]} \phi^{[m]}=0=\mathcal{X}_{1}^{[1]}  \tag{B.9a}\\
& Q_{1}^{[\mathcal{N}+1]}=\sum_{m=1}^{\mathcal{N}} \vartheta_{\rho}^{[m]} \phi^{[m]}=0=Q_{1}^{[1]} \tag{B.9b}
\end{align*}
$$

As a result of cell functions in Eq. (B.4), the material parameters $\rho_{1}$ and $E_{1}$ defined in Eq. (A.18) are given by

$$
\begin{array}{ll}
\frac{\rho_{1}}{\rho_{0} l}= & \sum_{n=1}^{\mathcal{N}} \vartheta_{\rho}^{[n]}\left[\vartheta_{E}^{[n]} \frac{\left(\phi^{[n]}\right)^{2}}{2}+\mathcal{X}_{1}^{[n]} \phi^{[n]}\right], \\
\frac{E_{1}}{E_{0} l}= & -\sum_{n=1}^{\mathcal{N}} \vartheta_{E}^{[n]}\left[\vartheta_{\rho}^{[n]} \frac{\left(\phi^{[n]}\right)^{2}}{2}+Q_{1}^{[n]} \phi^{[n]}\right] . \tag{B.10b}
\end{array}
$$

At order $\epsilon^{2}$, successive integrations of (A.15b), (A.28a) and (A.28b) to solve for $\chi_{2}, q_{2}$, and $p$ yield the cell functions in the form of second-order polynomials on layer $\Omega^{[n]}$,

$$
\begin{align*}
& \frac{\chi_{2}\left(\xi^{[n]}\right)}{\ell^{2}}=\vartheta_{\mathrm{d}}^{[n]} \frac{\left(\xi^{[n]}\right)^{2}}{2}+\mathcal{Z}^{[n]} \xi^{[n]}+\mathcal{X}_{2}^{[n]}-\widetilde{\chi}_{2}  \tag{B.11a}\\
& \frac{q_{2}\left(\xi^{[n]}\right)}{E_{0} \ell^{2}}=-\vartheta_{\rho}^{[n]} \frac{\left(\xi^{[n]}\right)^{2}}{2}+\mathcal{Y}^{[n]} \xi^{[n]}+Q_{2}^{[n]}-\widetilde{q}_{2}  \tag{B.11b}\\
& \frac{p\left(\xi^{[n]}\right)}{\ell^{2}}=\frac{\rho^{[n]}}{\rho_{0}} \vartheta_{E}^{[n]} \frac{\left(\xi^{[n]}\right)^{2}}{2}+\mathcal{R}^{[n]} \xi^{[n]}+P^{[n]}-\widetilde{p} . \tag{B.11c}
\end{align*}
$$

Here, dimensionless coefficients read

$$
\begin{align*}
\vartheta_{\mathrm{d}}^{[n]} & =\frac{E_{0}}{a^{[n]}} \vartheta_{\rho}^{[n]}-\vartheta_{E}^{[n]}  \tag{B.12a}\\
\mathcal{Z}^{[n]} & =\frac{E_{0}}{a^{[n]}}\left(Q_{1}^{[n]}-\widetilde{q}_{1}\right)-\left(\mathcal{X}_{1}^{[n]}-\widetilde{\chi}_{1}\right),  \tag{B.12b}\\
\mathcal{Y}^{[n]} & =\frac{\rho^{[n]}}{\rho_{0}} \frac{E_{1}}{E_{0} \ell}-\left(Q_{1}^{[n]}-\widetilde{q}_{1}\right)  \tag{B.12c}\\
\mathcal{R}^{[n]} & =\frac{\rho^{[n]}}{\rho_{0}}\left(\mathcal{X}_{1}^{[n]}-\widetilde{\chi}_{1}-\frac{\rho_{1}}{\rho_{0} \ell}\right) \tag{B.12d}
\end{align*}
$$

while factors $\mathcal{X}_{2}^{[n]}, Q_{2}^{[n]}$ and $P^{[n]}$ are defined to ensure continuity and periodicity of the cell functions,

$$
\begin{align*}
& \mathcal{X}_{2}^{[n]}=\sum_{m=1}^{n-1} \vartheta_{\mathrm{d}}^{[m]} \frac{\left(\phi^{[m]}\right)^{2}}{2}+\mathcal{Z}^{[m]} \phi^{[m]}  \tag{B.13a}\\
& Q_{2}^{[n]}=\sum_{m=1}^{n-1}-\vartheta_{\rho}^{[m]} \frac{\left(\phi^{[m]}\right)^{2}}{2}+\mathcal{Y}^{[m]} \phi^{[m]}  \tag{B.13b}\\
& P^{[n]}=\sum_{m=1}^{n-1} \frac{\rho^{[m]}}{\rho_{0}} \vartheta_{E}^{[m]} \frac{\left(\phi^{[m]}\right)^{2}}{2}+\mathcal{R}^{[m]} \phi^{[m]} \tag{B.13c}
\end{align*}
$$

and parameters $\widetilde{\chi}_{2}, \widetilde{q}_{2}$ and $\widetilde{p}$ are prescribed by conditions of zero mean values $\left\langle\chi_{2}\right\rangle=0,\left\langle q_{2} / a\right\rangle=0$ and $\langle p / a\rangle=0$,

$$
\begin{align*}
& \widetilde{\chi}_{2}=\sum_{n=1}^{\mathcal{N}} \vartheta_{\mathrm{d}}^{[n]} \frac{\left(\phi^{[n]}\right)^{3}}{6}+\mathcal{Z}^{[n]} \frac{\left(\phi^{[n]}\right)^{2}}{2}+\mathcal{X}_{2}^{[n]} \phi^{[n]}  \tag{B.14a}\\
& \widetilde{q}_{2}=\sum_{n=1}^{\mathcal{N}} \frac{E_{0}}{a^{[n]}}\left[-\vartheta_{\rho}^{[n]} \frac{\left(\phi^{[n]}\right)^{3}}{6}+\mathcal{Y}^{[n]} \frac{\left(\phi^{[n]}\right)^{2}}{2}+Q_{2}^{[n]} \phi^{[n]}\right],  \tag{B.14b}\\
& \widetilde{p}=\sum_{n=1}^{\mathcal{N}} \frac{E_{0}}{a^{[n]}}\left[\frac{\rho^{[n]}}{\rho_{0}} \vartheta_{E}^{[n]} \frac{\left(\phi^{[n]}\right)^{3}}{6}+\mathcal{R}^{[n]} \frac{\left(\phi^{[n]}\right)^{2}}{2}+P^{[n]} \phi^{[n]}\right] . \tag{B.14c}
\end{align*}
$$

For efficient calculations of $\mathcal{X}_{2}^{[n]}, Q_{2}^{[n]}$ and $P^{[n]}$, one notices

$$
\begin{align*}
& \mathcal{X}_{2}^{[n+1]}=\mathcal{X}_{2}^{[n]}+\vartheta_{\mathrm{d}}^{[n]} \frac{\left(\phi^{[n]}\right)^{2}}{2}+\mathcal{Z}^{[n]} \phi^{[n]}  \tag{B.15a}\\
& Q_{2}^{[n+1]}=Q_{2}^{[n]}-\vartheta_{\rho}^{[n]} \frac{\left.\phi^{[n]}\right)^{2}}{2}+\mathcal{Y}^{[n]} \phi^{[n]}  \tag{B.15b}\\
& P^{[n+1]}=P^{[n]}+\frac{\rho^{[n]}}{\rho_{0}} \vartheta_{E}^{[n]} \frac{\left(\phi^{[n]}\right)^{2}}{2}+\mathcal{R}^{[n]} \phi^{[n]} \tag{B.15c}
\end{align*}
$$

and following relations due to $\ell$-periodicity,

$$
\begin{align*}
\mathcal{X}_{2}^{[\mathcal{N}+1]} & =\sum_{m=1}^{\mathcal{N}} \vartheta_{\mathrm{d}}^{[m]} \frac{\left(\phi^{[m]}\right)^{2}}{2}+\mathcal{Z}^{[m]} \phi^{[m]}=0=\mathcal{X}_{2}^{[1]}  \tag{B.16a}\\
Q_{2}^{[\mathcal{N}+1]} & =\sum_{m=1}^{\mathcal{N}}-\vartheta_{\rho}^{[m]} \frac{\left(\phi^{[m]}\right)^{2}}{2}+\mathcal{Y}^{[m]} \phi^{[m]}=0=Q_{2}^{[1]}  \tag{B.16b}\\
P^{[\mathcal{N}+1]} & =\sum_{m=1}^{\mathcal{N}} \frac{\rho^{[m]}}{\rho_{0}} \vartheta_{E}^{[m]} \frac{\left(\phi^{[m]}\right)^{2}}{2}+\mathcal{R}^{[m]} \phi^{[m]}=0=P^{[1]} . \tag{B.16c}
\end{align*}
$$

From (B.11), the parameters $\rho_{2}, E_{2}$ and $J_{1}$ in (A.31) read

$$
\begin{align*}
& \frac{\rho_{2}}{\rho_{0} \ell^{2}}=\sum_{n=1}^{\mathcal{N}} \vartheta_{\rho}^{[n]}\left[\vartheta_{\mathrm{d}}^{[n]} \frac{\left(\phi^{[n]}\right)^{3}}{6}+\mathcal{Z}^{[n]} \frac{\left(\phi^{[n]}\right)^{2}}{2}+\mathcal{X}_{2}^{[n]} \phi^{[n]}\right],  \tag{B.17a}\\
& \frac{E_{2}}{E_{0} \ell^{2}}=-\sum_{n=1}^{\mathcal{N}} \vartheta_{E}^{[n]}\left[-\vartheta_{\rho}^{[n]} \frac{\left(\phi^{[n]}\right)^{3}}{6}+\mathcal{Y}^{[n]} \frac{\left(\phi^{[n]}\right)^{2}}{2}+Q_{2}^{[n]} \phi^{[n]}\right],  \tag{B.17b}\\
& \frac{J_{1}}{\ell^{2}}=-\sum_{n=1}^{\mathcal{N}} \vartheta_{E}^{[n]}\left[\frac{\rho^{[n]}}{\rho_{0}} \vartheta_{E}^{[n]} \frac{\left(\phi^{[n]}\right)^{3}}{6}+\mathcal{R}^{[n]} \frac{\left(\phi^{[n]}\right)^{2}}{2}+P^{[n]} \phi^{[n]}\right] . \tag{B.17c}
\end{align*}
$$

while $\Gamma_{x}, \Gamma_{t}$ and $B$ are calculated according to

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$$
\begin{equation*}
\frac{\Gamma_{x}}{\ell^{2}}=\frac{E_{2}}{E_{0} \ell^{2}}, \quad \frac{\Gamma_{t}}{\ell^{2}}=\frac{J_{1}}{\ell^{2}}-\frac{\rho_{2}}{\rho_{0} \ell^{2}}, \quad \frac{B}{\ell^{2}}=\frac{\Gamma_{x}+\Gamma_{t}}{\ell^{2}} \tag{B.18}
\end{equation*}
$$

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Appendix B.2. Case of the bilaminate materials
In the case of the periodic bilaminate material $(\mathcal{N}=2)$, cell functions result in the following effective material parameters,

$$
\begin{align*}
& \rho_{0}=\phi^{[1]} \rho^{[1]}+\phi^{[2]} \rho^{[2]}, \quad \frac{1}{E_{0}}=\frac{\phi^{[1]}}{a_{[1]}}+\frac{\phi^{[2]}}{a_{[2]}},  \tag{B.19a}\\
& \rho_{1}=0, \quad E_{1}=\rho_{1} E_{0} / \rho_{0}=0,  \tag{B.19b}\\
& \frac{\rho_{2}}{\rho_{0}}=-\frac{\ell^{2}}{12}\left(\vartheta_{\rho}^{[1]} \vartheta_{\mathrm{d}}^{[1]}\left(\phi^{[1]}\right)^{3}+\vartheta_{\rho}^{[2]} \vartheta_{\mathrm{d}}^{[2]}\left(\phi^{[2]}\right)^{3}\right),  \tag{B.19c}\\
& \frac{E_{2}}{E_{0}}=-\frac{\ell^{2}}{12}\left(\vartheta_{\rho}^{[1]} \vartheta_{E}^{[1]}\left(\phi^{[1]}\right)^{3}+\vartheta_{\rho}^{[2]} \vartheta_{E}^{[2]}\left(\phi^{[2]}\right)^{3}\right),  \tag{B.19d}\\
& J_{1}=\frac{\ell^{2}}{12}\left(\frac{\rho^{[1]}}{\rho_{0}}\left(\vartheta_{E}^{[1]}\right)^{2}\left(\phi^{[1]}\right)^{3}+\frac{\rho^{[2]}}{\rho_{0}}\left(\vartheta_{E}^{[2]}\right)^{2}\left(\phi^{[2]}\right)^{3}\right),  \tag{B.19e}\\
& J_{2}=\frac{\ell^{2}}{12}\left(\frac{E_{0}}{a_{[1]}}\left(\vartheta_{\rho}^{[1]}\right)^{2}\left(\phi^{[1]}\right)^{3}+\frac{E_{0}}{a_{[2]}}\left(\vartheta_{\rho}^{[2]}\right)^{2}\left(\phi^{[2]}\right)^{3}\right), \tag{B.19f}
\end{align*}
$$

$$
\begin{equation*}
\left(\phi^{[1]}\right)^{2} \vartheta_{\rho}^{[1]} \vartheta_{E}^{[1]}=\left(\phi^{[2]}\right)^{2} \vartheta_{\rho}^{[2]} \vartheta_{E}^{[2]} \tag{B.20}
\end{equation*}
$$

the following relations are found,

$$
\begin{align*}
& J_{1}=\frac{\left(\ell^{[1]}\right)^{2}}{12}\left(\vartheta_{E}^{[1]}\right)^{2}=\frac{\left(\ell^{[2]}\right)^{2}}{12}\left(\vartheta_{E}^{[2]}\right)^{2} \geq 0,  \tag{B.21a}\\
& J_{2}=\frac{\left(\ell^{[1]}\right)^{2}}{12}\left(\vartheta_{\rho}^{[1]}\right)^{2}=\frac{\left(\ell^{[2]}\right)^{2}}{12}\left(\vartheta_{\rho}^{[2]}\right)^{2} \geq 0,  \tag{B.21b}\\
& \frac{E_{2}}{E_{0}}=-\frac{\left(\ell^{[1]}\right)^{2}}{12} \vartheta_{\rho}^{[1]} \vartheta_{E}^{[1]}=-\frac{\left(\ell^{[2]}\right)^{2}}{12} \vartheta_{\rho}^{[2]} \vartheta_{E}^{[2]} . \tag{B.21c}
\end{align*}
$$

719 The characteristic surface $B=\Gamma_{x}+\Gamma_{t}=J_{1}+J_{2}+2 E_{2} / E_{0}$ reads

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$$
\begin{equation*}
B=\frac{\left(\ell^{[1]}\right)^{2}}{12}\left(\vartheta_{\rho}^{[1]}-\vartheta_{E}^{[1]}\right)^{2}=\frac{\left(\ell^{[2]}\right)^{2}}{12}\left(\vartheta_{\rho}^{[2]}-\vartheta_{E}^{[2]}\right)^{2} \tag{B.22}
\end{equation*}
$$

## Appendix C. Plane Wave Method for dispersion relation in periodic laminates

The dispersion relation in 1-D periodic multi-laminate materials can be computed according to the Plane Wave Method. A medium with $\ell$-sized unit cell $\Omega$ made of $n=1$.. $\mathcal{N}$ homogeneous elastic layers $\Omega^{[n]}=\left[x^{[n]}, x^{[n+1]}\right]$ of thickness $\ell^{[n]}=x^{[n+1]}-x^{[n]}=\phi^{[n]} \ell$, densities $\rho^{[n]}$, local elastic moduli $a^{[n]}$, and wave speed $c^{[n]}=\sqrt{a^{[n]} / \rho^{[n]}}$ is considered. In the layer $\Omega^{[n]}$, the displacement $u^{[n]}(x) e^{-\mathrm{i} \omega t}$ writes

$$
\begin{equation*}
u^{[n]}(x)=\mathbb{U}_{1}^{[n]} e^{\mathrm{i} k^{[n]}\left(x-x^{[n]}\right)}+\mathbb{U}_{2}^{[n]} e^{-\mathrm{i} k^{[n]}\left(x-x^{[n]}\right)}, \tag{C.1}
\end{equation*}
$$

where $\mathbb{U}_{1}^{[n]}$ and $\mathbb{U}_{2}^{[n]}$ are complex amplitudes and $k^{[n]}=\omega / c^{[n]}$ is the wavenumber in the layer $\Omega^{[n]}$. Continuity on the displacement $u^{[n]}=u^{[n+1]}$, and stress $a^{[n]} \partial_{x} u^{[n]}=a^{[n+1]} \partial_{x} u^{[n+1]}$ at point $x^{[n+1]}$ yield the conditions on the complex amplitudes

$$
\left\{\begin{array}{l}
\mathbb{U}_{1}^{[n+1]}  \tag{C.2}\\
\mathbb{U}_{2}^{[n+1]}
\end{array}\right\}=\mathbb{T}^{[n]} \cdot\left\{\begin{array}{l}
\mathbb{U}_{1}^{[n]} \\
\mathbb{U}_{2}^{[n]}
\end{array}\right\}
$$

where the transfer matrix $\mathbb{T}^{[n]}$ reads

$$
\mathbb{T}^{[n]}=\frac{1}{2}\left[\begin{array}{ll}
\left(1+z^{[n]}\right) e^{i k^{[n]} \ell^{[n]}} & \left(1-z^{[n]}\right) e^{-\mathrm{i} k^{[n]} \ell^{[n]}}  \tag{C.3}\\
\left(1-z^{[n]}\right) e^{\mathrm{i} k^{[n]} \ell^{[n]}} & \left(1+z^{[n]}\right) e^{-\mathrm{i} k^{[n]} \ell^{[n]}}
\end{array}\right]
$$

with impedance ratio $z^{[n]}=\rho^{[n]} c^{[n]} / \rho^{[n+1]} c^{[n+1]}$. Relation (C.2) yields

$$
\left\{\begin{array}{l}
\mathbb{U}_{1}^{[\mathcal{N}+1]}  \tag{C.4}\\
\mathbb{U}_{2}^{[\mathcal{N}+1]}
\end{array}\right\}=\mathbb{T} \cdot\left\{\begin{array}{l}
\mathbb{U}_{1}^{[1]} \\
\mathbb{U}_{2}^{[1]}
\end{array}\right\}, \quad \mathbb{T}=\mathbb{T}^{[\mathcal{N}]} \cdot \ldots \cdot \mathbb{T}^{[1]}
$$

Then, the condition of quasi-periodicity with wavenumber $k$ is prescribed in layers $\Omega^{[\mathcal{N}+1]}$ and $\Omega^{[1]}$,

$$
\left\{\begin{array}{l}
\mathbb{U}_{1}^{[\mathcal{N}+1]}  \tag{C.5}\\
\mathbb{U}_{2}^{[\mathcal{N}+1]}
\end{array}\right\}=\mathbb{T} \cdot\left\{\begin{array}{l}
\mathbb{U}_{1}^{[1]} \\
\mathbb{U}_{2}^{[1]}
\end{array}\right\}=e^{i k \ell}\left\{\begin{array}{c}
\mathbb{U}_{1}^{[1]} \\
\mathbb{U}_{2}^{[1]}
\end{array}\right\}
$$

Equation (C.5) shows that $e^{i k \ell}=\operatorname{eig}(\mathbb{T})$ is an eigenvalue eig $(\mathbb{T})$ of $2 \times 2$-matrix $\mathbb{T}$, which leads to the relation

$$
\begin{equation*}
\operatorname{det}(\mathbb{T})-\operatorname{tr}(\mathbb{T}) e^{\mathrm{i} k \ell}+e^{2 \mathrm{i} k \ell}=0 \tag{C.6}
\end{equation*}
$$

The wavenumber $k$ takes the form, where $\nu$ is an integer:

$$
\begin{equation*}
k=\left(\frac{1}{\ell} \operatorname{Arg}(\operatorname{eig}(\mathbb{T}))+2 \pi \nu\right)-\mathrm{i}\left(\frac{1}{\ell} \operatorname{Ln}(|\operatorname{eig}(\mathbb{T})|)\right) \tag{C.7}
\end{equation*}
$$

Here, matrix $\mathbb{T}$ and its eigenvalues depend on the frequency $\omega$, and the calculated wavenumber $k$ can be complex-valued. In the dual approach used in closing criterion based in Bloch-analysis, the wavenumber $k$ can be prescribed, and the corresponding frequencies $\omega$ are found numerically by solving for $\omega$ in Eq. (C.6).

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