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CHESSE BILLIARDS

ARNALDO NOGUEIRA AND SERGE TROUBETZKOY

ABSTRACT. We show the chess billiard map, which was introduced in [HM] in order to study a generalization of the n -Queens problem in chess, is a circle homeomorphism. We give a survey of some of the known results on circle homeomorphisms, and apply them to this map. We prove a number of new results which give answers to some of the open questions posed in [HM].

1. INTRODUCTION

The classical n -Queens problem asks in how many different ways n mutually non-attacking queens can be placed on an $n \times n$ chess board. In [HM] Hanusa and Manhankali studied a generalization of this problem, and they introduced a dynamical system which they describe as “reminiscent of billiards” to aid their study. We will call this dynamical system the chess billiard, and in this article we will study it from a purely dynamical point of view.

The chess billiard is defined as follows, consider a convex planar domain P and fix a direction θ_1 . Foliating the plane by lines in this direction yields an involution of T_1 of the boundary ∂P as follows: each line intersects ∂P in either no point, one point, two points, or a segment. In the cases when the intersection is non-empty, we define respectively the map T_1 to be the identity, to exchange the two points, or to be the central symmetry of the segment about its center. The chess billiard map is then the convolution of two such involutions defined by a pair of directions (θ_1, θ_2) . The chess billiard map turns out to be a circle homeomorphism.

The chess billiard map was already introduced in various other articles (without reference to chess), Arnold mentions this map as his motivation for the study of KAM theory [A]. The map was also studied in a special case by Khmelev [Kh] (see Section 4.4).

Our article has two purposes. We first collect various well known results on circle homeomorphisms and apply them to the chess billiard map to deduce certain interesting results, in particular answering some

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questions posted in [HM]. Then we go on to prove new results about the chess billiard. We prove some general results on periodic points, then we turn to the study of the chess billiard in a polygon. In particular we prove results about the chess billiard map in triangles, in centrally symmetric domains and in the square. Our results on the square are undoubtedly the most interesting of the article.

The ultimate goal of the study of chess billiards is to understand for which convex domains and directions the rotation number is rational and for which it is irrational.

1.1. Structure of the article. In Section 2 we give the formal definition of the chess billiard map and show that it is a circle homeomorphism (Proposition 1) and apply the theory of circle homeomorphisms, due to Poncaré and Denjoy to them (Corollary 2). Then in Section 3 we go on to illustrate this by analyzing the easiest case, the chess billiard map in the circle. In Section 4 we give a necessary and sufficient condition for S to have a fixed point (Proposition 5), this allows us to understand fixed points in strictly convex domains (Corollary 6) and to show that the chess billiard map is purely periodic in an arbitrary triangle (Corollary 7). In this section we also show that in a strictly convex domain the rotation number of the chess billiard map achieves all values in $[0, 1)$ (Theorem 10). In Section 5 we study the behavior of the chess billiard map in the square giving some elements of an answer to questions 7.5 and 7.6 of [HM]. We show that there exist directions for which the chess billiard map has no periodic orbit (Theorem 12) and that for an open dense set of directions (θ_1, θ_2) the chess billiard has a periodic orbit (Theorem 13), however the set of directions which have neutral periodic orbits (i.e., are “treacheries” in the language of [HM]) is small (Proposition 17). We also give a necessary and sufficient condition for a vertex to be a periodic point (Proposition 15) and we give a sufficient condition for the existence of neutral periodic orbits (Proposition 16). Finally in the short Section 6 we study centrally symmetric domains. Our results give complete or partial answers to several questions raised in [HM].

2. THE CHESS BILLIARD IS A CIRCLE HOMEOMORPHISM IN DISGUISE

We begin by describing the chess billiard slightly more formally. Consider a strictly convex planar domain P , and a foliation of P by a family of nonoriented parallel lines (see Figure 1). The choice of families of lines is parametrized by $\theta \in [0, \pi)$, i.e., the projective line. In some of our arguments it will be convenient to parametrize θ by $[0, 2\pi)$, this will be clear from the context. Since P is convex we can parametrize its boundary ∂P by arc length. Since P is strictly convex this foliation in direction θ_i induces a bijection $T_i : \partial P \rightarrow \partial P$ which is an orientation reversing homeomorphism with two fixed points. The boundary

of a convex set is always rectifiable, throughout the article we will express T_i in the arc-length parametrization of ∂P , points in ∂P will be denoted x or y .

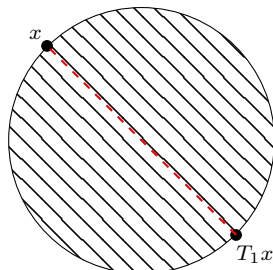


FIGURE 1. The circle foliated by a family of parallel lines, and the induced bijection.

Let $Q := \partial P \times \{1, 2\} =: Q_1 \cup Q_2$. We fix the same orientation on Q_1 and Q_2 .

Consider a pair of directions (θ_1, θ_2) , and the associated bijections $T_i : Q_i \rightarrow Q_i$. The pair (θ_1, θ_2) as well as each of its components will be referred to as a direction. Let

$$T = T_{\theta_1, \theta_2} : Q \rightarrow Q \quad \text{be defined by } T(x, i) = (T_i(x), i + 1 \pmod{2}).$$

The map T is an orientation reversing homeomorphism. We define the *chess billiard map*

$$S = S_{\theta_1, \theta_2} : Q \rightarrow Q \quad \text{by } S := T^2.$$

The map T is an orientation reversing homeomorphism since it is the composition of an orientation preserving homeomorphism $F(i) := i + 1 \pmod{2}$ with an orientation reversing homeomorphism T_i . From here on we will not write the mod 2 when referring to the two copies of Q . A degenerate case is when these two families coincide.

The first proposition summarizes several immediate properties of the map S .

Proposition 1. *Consider a convex domain P and arbitrary θ_1, θ_2 .*

- (1) *For each $i \in \{1, 2\}$ the map $S|_{Q_i}$ is an orientation preserving circle homeomorphism.*
- (2) *The sum of the rotation numbers of $S|_{Q_1}$ and of $S|_{Q_2}$ is 1.*
- (3) *The map S is not topologically mixing.*

Item (1) allows us to slightly misuse terminology and to refer to S as a circle homeomorphism. Item (2) allows us to slightly abuse the definition of rotation number and to refer to the rotation number $\rho(S)$ of S .

Proof. Item (1) is immediate since S is the composition of an orientation reversing homeomorphism T with itself. Item (2) follows immediately from the commutation relation $FS^n(x, i) = S^{-n}F(x, i)$. Item

(3) follows by the construction, $T(Q_i) = T(Q_{i+1})$, and thus T is not topologically mixing. \square

For completeness we give the definition of rotation number introduced by Poincaré. Consider the natural projection $\pi : \mathbb{R} \rightarrow Q_i$, it provides a lift of $S|_{Q_i}$ to a homeomorphism $\tilde{S} : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $S \circ \pi = \pi \circ \tilde{S}$. Consider $\lim_{n \rightarrow \infty} \frac{\tilde{S}^n(x) - x}{n}$, this limit always exists, and its value does not depend on the point x nor on the lift, thus we call it the *rotation number* $\rho(S)$ of S .

If P is only convex, the exact same definition works in the case that the direction of each interval in ∂P is transverse to the two foliations; for example P is a convex polygon and neither of the two foliations is parallel to a side of P . In this case we will call the direction (θ_1, θ_2) *exceptional* if either θ_1 or θ_2 is parallel to a side of P . We can extend the definition to the exceptional case in the following way: we define T_i on any line segment in Q_i to be the central symmetry with respect to the center of this segment (see Figure 2). Again, the map T is an orientation reversing homeomorphism.

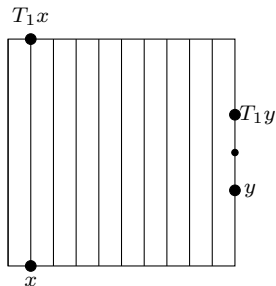


FIGURE 2. The square foliated by a family of parallel lines which are parallel to a side, and the induced bijection.

Our chess billiard map is defined on all of Q , while the map in [HM] is defined on $\partial P \setminus \{\text{the corners of } P\}$; the definitions agree where they are both defined. Furthermore in [HM] the authors only consider points in Q_i for which the direction θ_i points towards the interior of P , while we allow foliations tangent to an interval in ∂P .

If P is not convex, then for certain directions we have orbits which graze the boundary, any possible definition of the dynamical system will lead to a discontinuous map (see Figure 3). None the less, we can define a piecewise continuous chess map, for example by defining the map by one sided continuity at these points (answering part of Question 7.8 of [HM]). Such a definition leaves our nice framework, and thus in this article we will not consider such domains, although their study is certainly interesting. In the case of a non-convex polygon, the resulting

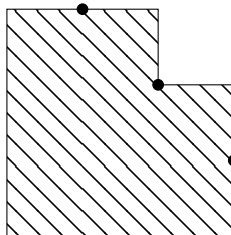


FIGURE 3. Any possible definition of the dynamics at the marked points will lead to a discontinuous map.

map is an affine interval exchange transformation, which have been studied for the last decade or so, see [BFG] and the references therein.

The study of the dynamics of orientation preserving circle homeomorphisms was initiated by Poincaré and developed by Denjoy, Herman, and others. Some of the main results are summarized in the next corollary. For the definitions and proof of the corollary see for example [KH][Propositions 11.2.2 and 11.2.5, Theorem 11.2.9].

Corollary 2. *If the rotation number of the chess billiard map S is rational, in reduced form p/q , then S has a periodic orbit, and all periodic orbits of S have period q . If S has exactly one periodic orbit then every other point is heteroclinic under S^q to two points on the periodic orbit. These points are different if the period is greater than 1. If S has more than one periodic orbit, then each nonperiodic point is heteroclinic under S^q to two points on different periodic orbits.*

Fix $i \in \{1, 2\}$. If the rotation number of S is irrational then a) the ω -limit set $\omega(x)$ is independent of $x \in Q_i$ and either $\omega(x) = Q_i$ or it is a perfect and nowhere dense subset of Q_i , and b) the map S is uniquely ergodic.

This corollary immediately gives answers to several of the questions posed in [HM]:

- 7.1, 7.2: if we interpret the words “predictable behavior” as having zero topological entropy then the answer is that the chess billiard is always predictable since circle homeomorphisms always have zero entropy.
- 7.3: for any (polygonal) board the directions which have a periodic orbit are those for which the rotation number is rational.
- 7.4: the map is ergodic if and only if its rotation number is irrational (and since it is then uniquely ergodic this does not depend on the starting point as asked in [HM]).
- 7.13: all close points have the same behavior in the sense of the corollary.

Questions 7.3 and 7.4 are quite general and the answers give quite a bit of information but are not definitive.

3. THE CIRCLE

The simplest case of chess billiards is that of the circle.

Proposition 3. *For the circle, the map $S|_{Q_1}$ is the rotation of the circle by angle 2α where $\alpha = \theta_2 - \theta_1$ (and $S|_{Q_2}$ is the rotation by angle $2\pi - 2\alpha$).*

Corollary 4. *If $\frac{\theta_2 - \theta_1}{\pi} = p/q \in \mathbb{Q}$ with $p \geq 0$, $q \geq 1$, $\text{pgcd}(p, q) = 1$, then all orbits are periodic with period q , otherwise the map S is minimal and uniquely ergodic with respect to the length measure.*

Proof. By rotational symmetry the behavior of the chess billiard map in the circle depends only on $\alpha := \theta_2 - \theta_1$. It is convenient to use complex coordinates, $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. Suppose $\theta_1 = \pi/2$ then we have $T_1(z) = \bar{z}$. To compute $T_2(z)$ we rotate to make θ_2 vertical, take the complex conjugate, and then rotate back: $z \rightarrow e^{i\alpha}z \rightarrow \overline{e^{i\alpha}z} \rightarrow e^{-i\alpha}e^{i\alpha}\bar{z} = e^{-i2\alpha}\bar{z}$. Thus

$$S(z, 1) = (T_2 \circ T_1(z), 1) = (e^{-i2\alpha}z, 1) \text{ and}$$

$$S(z, 2) = (T_1 \circ T_2(z), 2) = (e^{i2\alpha}z, 2).$$

□

There are 4 special points, the fixed points ± 1 of T_1 and the fixed points $e^{\pm i\theta_2}$ of T_2 . They play a special role in the case of rational rotation number, if $\frac{\alpha}{\pi} = p/q$ with $\text{pgcd}(p, q) = 1$ and q even, then $S^p(\pm i) = \mp i$ and $S^p(\pm 1) = \mp 1$ (orbits shown in red and blue in Figure 4, while a generic orbit is shown in black), while if q is odd, then the orbit of ± 1 arrives at $\pm e^{i\theta_2}$ and then returns to itself (orbits show in red and blue in Figure 5). The role of these orbits will be investigated in the general setting in Section 4.2.

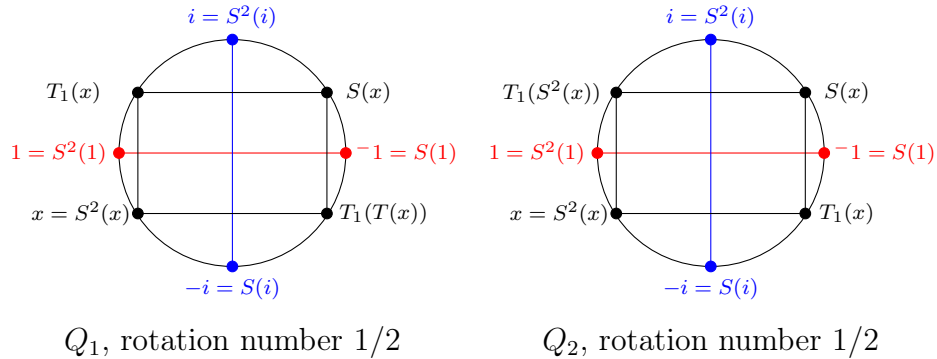


FIGURE 4. $\alpha = \pi/2$.

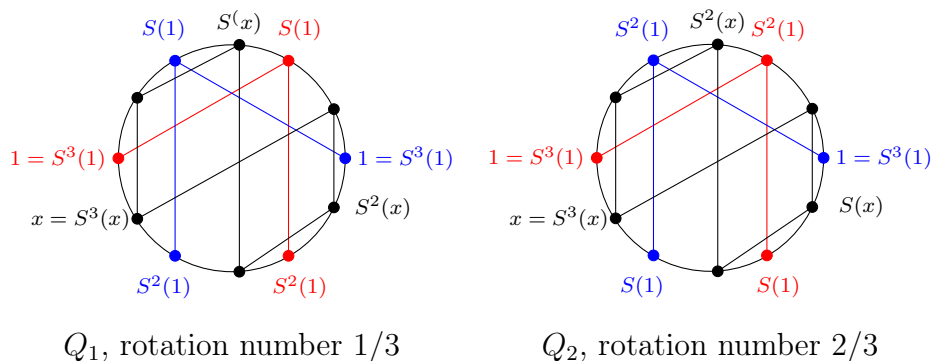


FIGURE 5. $\alpha = \pi/3$.

4. PERIODIC ORBITS

4.1. **Fixed points of S .** If the two foliations coincide, then S is the identity map. Now assume that the two foliations do not coincide, i.e., $\theta_1 \neq \theta_2$.

If P is strictly convex then the map T_i fixes the point x if and only if a line of the foliation is a supporting line¹ to Q_i at x . In the case P is convex but not strictly convex the necessary and sufficient condition is more complicated. The point x is a fixed point if and only if either i) a line of the foliation is a supporting line to Q_i at x , and x is isolated in this intersection; or ii) a line of the foliation is tangent to Q_i in an interval, and x is the center point of this interval.

If a point $x \in \partial P$ is fixed by both T_1 and T_2 , then it is a fixed point of S and thus the rotation number of $S|_{Q_i}$ is zero for $i \in \{1, 2\}$. The converse is also true. Suppose that $x \in Q_1$ is not fixed by T_1 , then since by assumption the two foliations are not parallel $T_1(x) \neq x$. Combining this with Corollary 2 we have shown

Proposition 5. *Suppose that $\theta_1 \neq \theta_2$. The map S_{θ_1, θ_2} has a fixed point if and only if there is a point $x \in \partial P$ such that the lines through x in these two directions are supporting lines at x . Moreover, each fixed point is semi stable, i.e., repelling, from one side, and attracting from the opposite side.*

Corollary 6. *A C^1 strictly convex domain can not have any fixed point (unless $\theta_1 = \theta_2$).*

Corollary 7. *If P is a triangle and θ_1, θ_2 are arbitrary, then S_{θ_1, θ_2} has a fixed point or a periodic point with period 2 or 3.*

Proof. If $\theta_1 = \theta_2$ then every point is fixed by S , thus we assume that they are not equal.

¹A supporting line L of a planar curve C is a line that contains at least one point of C , and C lies completely in one of the two closed half-planes defined by L .

Next consider the case when (θ_1, θ_2) is not exceptional. Consider the lines of the foliation in the direction θ_1 which intersect P , the extremal ones are supporting lines which pass through two distinct vertices of P . The same holds for θ_2 , since P has only three vertices there is a vertex for which both directions must have supporting lines, and thus a fixed point by Proposition 5.

Turning to the case when (θ_1, θ_2) is exceptional, we begin by treating the case when only one direction is parallel to a side, say θ_1 , it is also a supporting line of the vertex opposite to this side. If θ_2 is a supporting line of this vertex then again applying Proposition 5 we conclude that this vertex is fixed (see Figure 7). Otherwise θ_2 is a supporting line of the two endpoints of the side parallel to θ_1 ; these endpoints are fixed by \bar{T}_2 and exchanged by \bar{T}_1 , thus they are exchanged by S .

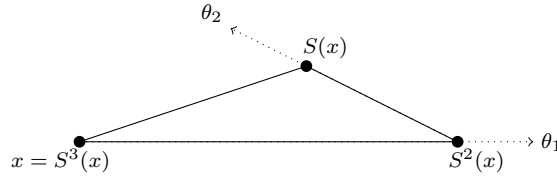


FIGURE 6. Period 3 orbit in a triangle.

Finally in the case when both directions are parallel to a side, the map S cyclically exchanges the vertices of the triangle (Figure 6). \square

If P is a square and both foliation directions are in the same quadrant, then S has two fixed points, both semi-stable, (this is essentially contained in [HM], however they have not defined the dynamics at the two fixed points). More generally, any convex polygon or even any convex table with a corner has an open set of pairs of directions for which S has a fixed point. Examples of strictly convex domains and of convex polygons with exactly one fixed point exist (see Figure 7). This point is repelling on one side and attracting on the other side.



FIGURE 7. A strictly convex domain and a triangle each having a single semi-attracting fixed point.

4.2. Connections. We call an S -orbit segment starting and ending at fixed points of the maps T_i a *connection*. We can think of a connection

as a broken line, the *length* of the connection is the number of segments in this line (so a fixed point has length 0). The following result generalizes what we showed for the circle and for fixed points.

Proposition 8. *If there is a connection then the map S has a periodic point.*

Proof. Proposition 5 is a special case of this result for fixed points. Consider the set $F := \{x : \text{either } T_1(x) = x \text{ or } T_2(x) = x\}$.

Suppose $x \in F$. Then the S orbit which arrives at x reverses its direction and retraces the orbit in the opposite direction. Thus if it arrives at another point $y \in F$, then it is a periodic orbit. \square

4.3. The rotation number achieves all values. If the direction θ_2 is not parallel to a segment in ∂P , then the map $(\theta_2, x) \mapsto T_2(x)$ is a monotone continuous function of θ_2 . Thus the chess billiard map S_{θ_1, θ_2} is a continuous function of θ_2 . Since the rotation number depends continuously on the map we have shown

Proposition 9. *For each fixed θ_2 , if the direction θ_1 is not parallel to a segment in ∂P , then*

- 1) *the rotation number map $\theta_1 \mapsto \rho(T_{\theta_1, \theta_2})$ is a monotone function of θ_1 and*
- 2) *the point θ_1 is a point of continuity of the rotation number map.*

Theorem 10. *Fix a strictly convex table and a direction θ_2 , then as we vary $\theta_1 \in [0, \pi)$ the rotation number of S achieves all values in $[0, 1)$.*

Proof. In the proof we think of θ_1 and θ_2 as oriented vectors. For $\theta_2 = \theta_1$ and for $\theta_2 = \theta_1 + \pi$ we have $S = id$. In both cases the rotation number is 0. By Corollary 6, for fixed θ_1 the rotation number is non zero for all $\theta_2 \notin \{\theta_1, \theta_1 + \pi\}$. Furthermore the rotation number is monotonic and continuous (by Proposition 9) in θ_2 . Combining these facts implies that the rotation number varies from 0 to 1 as θ_2 varies from θ_1 to $\theta_1 + \pi$, in the sense that $\lim_{\theta_2 \nearrow \theta_1 + \pi} \rho(S_{\theta_1, \theta_2}) = 1$. \square

4.4. Khmelev result. Khmelev [Kh] showed that if P is convex and is sufficiently smooth everywhere except one point where the first derivative has a jump discontinuity then the rotation number $\rho(S)$ is rational for almost all values of θ_1, θ_2 (see his article for the precise smoothness assumptions).

4.5. Periodic orbits in polygons. Fix P , and suppose that the rotation number associated to a pair of directions θ_1, θ_2 is rational, p/q in reduced form. Let I be an interval contained in a side of P , perhaps degenerate to a point, such that $S^q I = I$ and $S^q J \neq J$ for any $J \supset I$; we call $C(I) := \cup_{j=0}^{q-1} S^j I$ a *periodic cylinder*. By continuity a

periodic cylinder $C(I)$ is always a closed set. In the case I degenerates to a point, a periodic cylinder is simply a periodic orbit of period q . If the interval I is not degenerate then we will call each $x \in I$ a *neutral periodic orbit* and $C(I)$ a *neutral cylinder*, in the language of [HM] a neutral cylinder is called a *treachery* (see Example 5.3 of [HM] to understand this connection).

Proposition 11. *Suppose that P is a convex polygon with k sides, and (θ_1, θ_2) is such that the rotation number $\rho(S) = p/q$ is rational, then the number of periodic cylinders for $S|_{Q_i}$ is at most $3k - 4$ ($i = 1, 2$).*

Proof. Consider P foliated by lines in the direction θ_1 . There are 2 lines of this foliation which are supporting lines, and (at most) $k - 2$ other lines which pass through a vertex of P . Consider the intersection of these lines with ∂P , this intersection consists of the k corners plus (at most) $k - 2$ other points, so (at most) $2k - 2$ points. These points partition Q_1 into (at most) $2k - 2$ intervals on which T_1 is affine.

The same construction yields (at most) $2k - 2$ points in Q_2 for the direction θ_2 . Take the preimage T_1^{-1} of these points, yields (at most) $2k - 2$ points in Q_1 , however k of these points (namely T_1^{-1} of the vertices) are in the previously defined collection of points in Q_1 . Thus in total we have (at most) $3k - 4$ points in Q_1 , which define (at most) $3k - 4$ intervals on which S is affine. Therefore the map $S^q|_{Q_i}$ has at most $q(3k - 4)$ intervals of affinity. But each piece of affinity can intersect the diagonal at most one time. \square

5. THE SQUARE

Suppose the square is $[0, 1]^2$.

Theorem 12. *There exists a direction (θ_1, θ_2) such that the chess billiard map S_{θ_1, θ_2} in the square has an irrational rotation number (and thus is aperiodic).*

Proof. If $\theta_1 = \pi/4$ and $\theta_2 = 3\pi/4$ then all S -orbits in the square have period 2. On the other hand if $\tan \theta'_1 = 1/3$ and $\tan \theta'_2 = -2/3$ then a simple geometric exercise shows that the orbit of the point $(1, 1/2)$ is a period 3 orbit (see [HM], Figure 9). Consider the line segment $L \subset \mathbb{R}^2$ with endpoints $(\pi/4, 3\pi/4)$ and $(\arctan(1/3), \arctan(-2/3))$. The function $(\theta''_1, \theta''_2) \mapsto S_{\theta''_1, \theta''_2}$ is a continuous function when $(\theta''_1, \theta''_2) \in L$ since L does not intersect the set of exceptional directions.. Therefore the rotation number $\rho(S_{\theta''_1, \theta''_2})$ is a continuous function of (θ''_1, θ''_2) , and thus it takes all values between $1/2$ and $1/3$. \square

Theorem 13. *The chess billiard map S_{θ_1, θ_2} in the square has a periodic point for an open dense set of $(\theta_1, \theta_2) \in \mathbb{S}^1 \times \mathbb{S}^1$.*

The proof uses another cross-section to the chess billiard flow, which relies on the symmetries of the square. Throughout the proof we suppose that the directions θ_1 and θ_2 are not exceptional, and furthermore we suppose that they are in different quadrants, since if they are in the same quadrant the map has a fixed point. It suffices to treat the case $\theta_1 \in (0, \pi/2)$ and $\theta_2 \in (\pi/2, \pi)$.

Let D denote the diagonal $x + y = 1$ of the square. We define a map $F : D \rightarrow D$. We give two different descriptions of this map. Start at a point in D , flow in the direction θ_1 (towards the right) until we reach the boundary of the square, then flow in the direction θ_2 until we return to the boundary of the square, and finally again flow in the direction θ_1 until we return to D . The point we have returned to is in D , but we can be flowing either to the right or to the left depending on if the flow in the direction θ_2 had crossed the diagonal or not; if we are flowing to the right call this point $F(x)$ while if we are flowing to the left we apply a central symmetric to obtain $F(x)$.

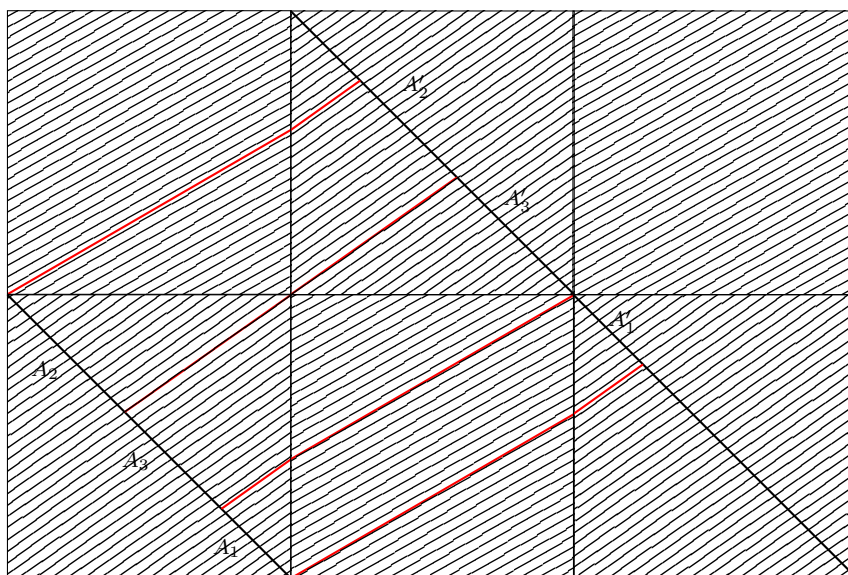
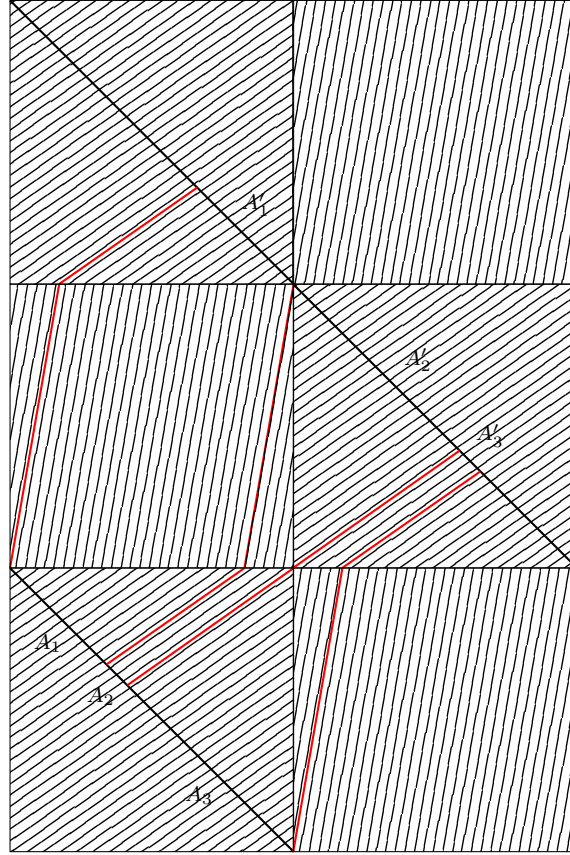
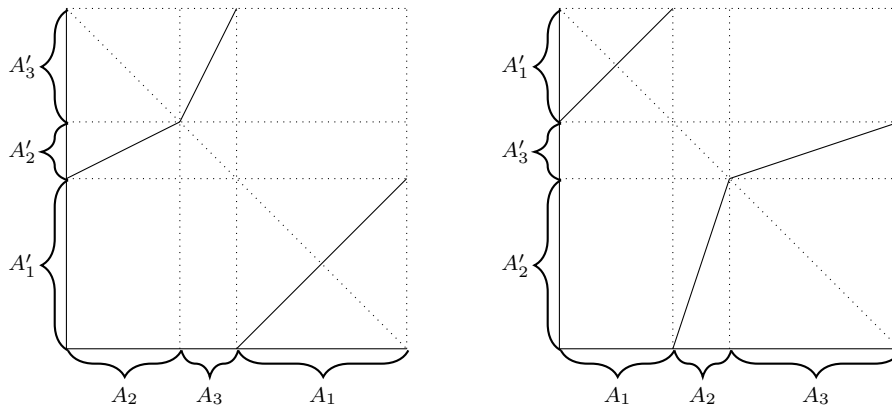


FIGURE 8. The map $F : D \rightarrow D$, case $\phi_2 \in (0, \pi/4)$

Another way to define F is via unfolding, this is shown in Figures 8 and 9. The direction in the bottom left square and in every other square is θ_1 . The direction in the other squares is unfolded, thus the angle is $\pi - \theta_2$. For conveniences we use the notation $\phi_1 = \theta_1, \phi_2 = \pi - \theta_2$. We remark that the points in the interval A_1 have crossed the diagonal D during the flow in the direction θ_2 , and arrive to D with the same orientation; while the points in $A_2 \cup A_3$ do not cross D . In the original chess billiard flow when they return to D we need to apply the central symmetry to define F , but this is not needed in the unfolded picture.

The graph of the map $F : D \rightarrow D$ has two possible forms, they are shown in Figure 10. The set D decomposes into three segments

FIGURE 9. The map $F : D \rightarrow D$, case $\phi_2 \in (\pi/4, \pi/2)$ FIGURE 10. The map F for $\phi_2 \in (0, \pi/4)$ and $(\pi/4, \pi/2)$.

A_1, A_2, A_3 such that the derivative $F'|_{A_i}$ is constant for each i ; we call their images $A'_i = F(A_i)$. Let $a_i := |A_i|$, where $|\cdot|$ denotes the length of a segment. In the case $\pi/4 < \phi_2 < \pi/2$ the central symmetry of Figure 9 about the point $(1/2, 3/2)$ implies $|A'_1| = |A_1|$; while the central symmetry of the figure about the point $(1, 1)$ yields $|A'_2| = |A_3|$;

$|A'_3| = |A_2|$ and thus

$$\frac{|A'_2|}{|A_2|} = \frac{|A_3|}{|A'_3|}.$$

Notice that these symmetries imply that the point $(a_1 + a_2, F(a_1 + a_2))$ of the graph of F lies on the anti-diagonal marked in dots in Figure 10, i.e., $a_1 + a_2 + F(a_1 + a_2) = 1$. (Similar symmetries arise in the case $\phi_2 \in (0, \pi/4)$).

The length of D is $\sqrt{2}$. We parametrize D with arclength and note that $F(0) = F(\sqrt{2})$, thus we think of D as a circle of length $\sqrt{2}$ which we do not normalize. Elementary plane geometry (see Figures 8 and 9) yields

$$a_1 = (1 - \tan(\phi_2)) \frac{\sin(\pi/2 - \phi_1)}{\sin(\pi/4 + \phi_1)} \quad \text{if } 0 < \phi_2 < \frac{\pi}{4}$$

$$a_1 = (1 - \cot(\phi_2)) \frac{\sin(\phi_1)}{\sin(3\pi/4 - \phi_1)} \quad \text{if } \frac{\pi}{4} < \phi_2 < \frac{\pi}{2}.$$

Remark: if $\phi_2 = \pi/4$ the interval A_1 disappears, and there are only two intervals; on the other hand if $\phi_1 = \phi_2$, then F is a circle rotation by the length $a_1 \pmod{\sqrt{2}}$.

It is not hard to check that if we increase ϕ_2 (i.e., decrease θ_2) then the graphs of the resulting maps $F = F_{\phi_1, \phi_2}$ and $F_h = F_{\phi_1, \phi_2 + h}$ do not intersect (see Figure 11).

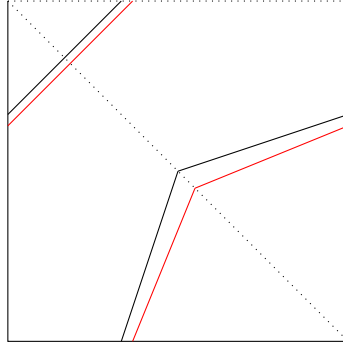


FIGURE 11. The original map is in black, and the red map arises by increasing ϕ_2 a bit.

Lemma 14. *If $\phi_1 \in (0, \pi/2)$ and $z \in D$, then*

$$\frac{\partial F(z)}{\partial \phi_2} \leq \frac{-\sin(\phi_1)}{\sin(3\pi/4 - \phi_1)}.$$

Proof. The constant a_1 defined above varies with h , we denote this dependence by $a_1(h)$. Suppose h is such that $\phi_2 + h \in (\pi/4, \pi/2)$, then

$$\begin{aligned} F_h(0) - F(0) &= a_1 - a_1(h) \\ &= \left(\cot(\phi_2 + h) - \cot(\phi_2) \right) \frac{\sin(\phi_1)}{\sin(3\pi/4 - \phi_1)}, \end{aligned}$$

and thus

$$\begin{aligned} \frac{\partial F_{\phi_1, \phi_2}}{\partial \phi_2}(0) &= \lim_{h \rightarrow 0} \frac{\cot(\phi_2 + h) - \cot(\phi_2)}{h} \frac{\sin(\phi_1)}{\sin(3\pi/4 - \phi_1)} \\ &= -\csc^2(\phi_2) \frac{\sin(\phi_1)}{\sin(3\pi/4 - \phi_1)} \\ &\leq \frac{-\sin(\phi_1)}{\sin(3\pi/4 - \phi_1)}. \end{aligned}$$

However

$$\begin{aligned} F_h(z) - F(z) &\leq F_h(0) - F(0) \text{ for } h > 0 \\ F_h(z) - F(z) &\geq F_h(0) - F(0) \text{ for } h < 0 \end{aligned}$$

for any $z \in D$, and thus

$$\frac{\partial F_{\phi_1, \phi_2}}{\partial \phi_2}(z) \leq \frac{\partial F_{\phi_1, \phi_2}}{\partial \phi_2}(0).$$

□

Proof. Denseness: Suppose that the rotation number of F is irrational. Fix $\varepsilon > 0$ satisfying $\varepsilon < \min(\pi/2 - \phi_2, \phi_2 - \pi/4) < \sqrt{2}/2$. Choose a point $z \in D$ such that $z \in \omega(z)$ (the ω -limit set of the orbit z) and fix an $n > 0$ so that $|F^n(z) - z| < \varepsilon$.

Remember that D is a circle, we define $z_2 -_D z_1$ the signed distance between points $z_1, z_2 \in D$ as follows. If $0 \leq z_1 \leq z_2 \leq \sqrt{2}$ and $z_2 - z_1 < \sqrt{2}/2$ then $z_2 -_D z_1 = z_2 - z_1$ while if $0 \leq z_2 \leq z_1 \leq \sqrt{2}$ and $\sqrt{2} - (z_2 - z_1) < \sqrt{2}/2$ the $z_2 -_D z_1 = \sqrt{2} - (z_2 - z_1) < \sqrt{2}/2$ and then extend to negative distances by setting $z_1 -_D z_2 = -(z_2 -_D z_1)$. Throughout the rest of this section we will simply write $-$ instead of $-_D$.

Let $(h_0^{(0)}, h_1^{(0)})$ be the maximal interval containing 0 such that the points z and $F^n(z)$ are in the same semicircle, which allows us to define the continuous function $e : (h_0^{(0)}, h_1^{(0)}) \rightarrow \mathbb{R}$ by $e(h) := F_h^n(z) - z$.

Similarly let $(h_0^{(i)}, h_1^{(i)})$ ($i = 1, 2$) be the maximal intervals containing 0 where the functions $P_1(h) := F_h^n(z) - F_h(F^{n-1}(z))$ and $P_2(h) := F_h(F^{n-1}(z)) - F^n(z)$ are respectively defined.

Consider the interval

$$(h_0^{(3)}, h_1^{(3)}) := \bigcap_{i=0}^2 (h_0^{(i)}, h_1^{(i)}).$$

Note that $(h_0^{(3)}, h_1^{(3)})$ contains the points 0 and depends on ϕ_1, ϕ_2 and on z which are fixed throughout the proof but does not depend on the choice of ε . For all $h \in (h_0^{(3)}, h_1^{(3)})$, we have

$$e(h) = P_1(h) + P_2(h) + (F^n(z) - z).$$

We need to estimate each of these terms. We have already supposed that $|(F^n(z) - z)| < \varepsilon$.

Note that $P_1(0) = 0$. The function $F_h^{n-1}(z)$ is a decreasing function of h , thus since F_h is an increasing function of z , one obtains

$$\begin{aligned} P_1(h) &\leq 0 \text{ if } h \geq 0 \\ P_1(h) &\geq 0 \text{ if } h \leq 0. \end{aligned}$$

We also have $P_2(0) = 0$, and applying the Lemma yields

$$\begin{aligned} P_2(h) &\leq Ch \text{ for } h \geq 0 \\ P_2(h) &\geq -Ch \text{ for } h \leq 0 \end{aligned}$$

where $C := \frac{-\sin(\phi_1)}{\sin(3\pi/4-\phi_1)}$ is a negative constant.

First suppose $F^n(z) - z$ is positive. The functions P_1 and P_2 are both negative and continuous on the open interval $(0, h_1^{(3)})$. Furthermore $P_1(h_1^{(3)}) > 0$ and $P_2(h_1^{(3)}) > 0$. Since $h_1^{(3)}$ does not depend on ε , it follows that if $0 < \varepsilon < P_1(h_1^{(3)}) + P_2(h_1^{(3)})$, then there is an $h' \in (0, h_1^{(3)})$ such that $e(h') = 0$, and thus $F_{h'}^n(z) = z$.

The case $F^n(z) - z < 0$ is similar, varying $h \in (h_0^{(3)}, 0)$.

Choosing $\varepsilon > 0$ arbitrarily small and remembering that $h_0^{(3)}$ and $h_1^{(3)}$ do not depend on ε , yields h' arbitrarily close to 0; showing that the periodic directions are dense.

Openess: Suppose that z_0 is a periodic point of period n for the map F_{ϕ_1, ϕ_2} where the directions additionally satisfy $\phi_1 \notin \{0, \pi/2\}$ and $\phi_2 \notin \{0, \pi/4, \pi/2\}$. The graph of $y = F^n(z)$ intersects the diagonal at the point (z_0, z_0) , if this intersection is transverse then since the graph of F (and thus also the graph of F^n) changes continuously with (ϕ_1, ϕ_2) (and thus with (θ_1, θ_2)) the intersection persists for a non-empty open set of parameters.

Now suppose that the intersection is not transverse, then either (i) $(F^n)'(z_0)$ does not exist and thus the orbit of the periodic point $z_0 = F^n(z_0)$ must pass through a corner of the polygon; or (ii) $(F^n)'(z_0) = 1$, in this case there is an interval $J := (z_-, z_+)$ containing z_0 such that $F^n|_J$ is the identity map.

Suppose that the graph of F^n stays below the diagonal except for the tangency at the point (z_0, z_0) , respectively on the segment $\{(z, z) : z \in J\}$. Then since F_h^n is decreasing, for all sufficiently small negative h the graph of F_h^n will cross the diagonal transversely at a point near (z_0, z_0) , respectively at two points near (z_-, z_-) and (z_+, z_+) . The case when the graph of F^n is above the diagonal is treated similarly using positive h . \square

Proposition 15. *In the square for a non-exceptional direction an orbit passing through a vertex is periodic if and only if it is a connection.*

Proof. Proposition 8 yields the converse assertion of the lemma since in the square connections must connect vertices.

Vertices which are fixed points are connections. Now consider the case when the period of an orbit of a vertex a is at least 2 and thus by Proposition 5 the directions must be in different quadrants; so one of the directions is a supporting line at a . Thus a acts as a reflector in the sense that after hitting this corner the orbit retraces itself backwards. The orbit going through the corner a is periodic (and is not a fixed point), thus it must make its way back to a . Since this is the only mechanism for retracing an orbit, the only way this can happen is by retracing the orbit once again: the orbit must hit a different corner, i.e., it is a connection. \square

Proposition 16. *In the square, if there is a neutral periodic orbit in a non-exceptional direction, then there is a connection in this direction.*

Proof. If θ_1 and θ_2 are parallel, then all points are fixed by S , thus each side of P is a neutral cylinder and each vertex of P is a connection.

Suppose now that θ_1 and θ_2 are not parallel and (θ_1, θ_2) is non-exceptional. Let q denote the period of the neutral cylinder and consider a maximal interval I defining the neutral periodic cylinder. By definition $C(I) = C(S(I)) = \dots = C(S^{q-1}(I))$, thus we can choose $0 \leq j < q$ such that $S^j(I) = [a, b]$, where a is a vertex of the polygon, otherwise we could extend I to a larger interval. Since $C(I)$ is closed, the orbit of the vertex a is periodic and thus a connection by Proposition 15. \square

The proof shows a bit more. If we consider the other side of the cylinder it also passes through a vertex, and repeating the proof shows that the orbit of this vertex is also a connection. Thus either there are two connections, or a single connection which bounds both sides of the cylinder.

Proposition 17. *For the square, the set*

$$\left\{ (\theta_1, \theta_2) : S \text{ has a neutral periodic orbit} \right\}$$

is a union of at most countably many one-dimensional sets.

Proof. Using the previous Proposition, it suffices to prove that the set

$$\left\{ (\theta_1, \theta_2) : S \text{ has a connection in this direction} \right\}$$

is a union of at most countably many one-dimensional sets.

We will use the following implication of a strengthening of the implicit function theorem (IFT) due to Kumagai [Ku]:

Consider a continuous function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and a point (θ_1^0, θ_2^0) such that $f(\theta_1^0, \theta_2^0) = 0$. If there exist open neighborhoods $C \subset \mathbb{R}$ and $E \subset \mathbb{R}$ of θ_1^0 and θ_2^0 , respectively, such that, for all $\theta_1 \in C$, $f(\theta_1, \cdot) : E \rightarrow \mathbb{R}$ is locally one-to-one then there exist open neighborhoods $C_0 \subset \mathbb{R}$ and $E_0 \subset \mathbb{R}$ of θ_1^0 and θ_2^0 , such that, for every $\theta_1 \in C_0$, the equation $f(\theta_1, \theta_2) = 0$ has a unique solution $\theta_2 = g(\theta_1) \in E_0$, where g is a continuous function from C_0 into E_0 .

Consider a neutral periodic orbit in the direction (θ_1^0, θ_2^0) , and one of the associated connections given by the Proposition 16. Suppose that this saddle connection starts at a vertex a . We use the representation $F : D \rightarrow D$ given in Theorem 13 and by a slight misuse of notation we will also denote the point in D on this saddle connection by a , so $a = F_{\theta_1^0, \theta_2^0}^n(a)$. This point depends on θ_1 , in the proof θ_1 is fixed, and θ_2 varies, thus the identification of the vertex a and with a point in the diagonal remains valid throughout the proof.

Consider a lift $\tilde{F} : \mathbb{R} \rightarrow \mathbb{R}$ of F . Then there is an $m \in \mathbb{Z}$ such that $a + m = \tilde{F}_{\theta_1^0, \theta_2^0}^n(a)$. The proposition follows immediately if we can apply Kumagai's IFT to the function

$$f(\theta_1, \theta_2) := \tilde{F}_{\theta_1, \theta_2}^n(a) - (a + m).$$

The Proof of Theorem 13 shows that there is an interval E such that for each θ_1 the function $F^n(\theta_1, \cdot)$ is a strictly monotonic map of $\theta_2 \in E$. This immediately implies that $F^n(\theta_1, \cdot)$ is a strictly monotonic map of $\theta_2 \in E$, and thus so is $\tilde{F}^n(\theta_1, \cdot)$. Thus $f(\theta_1, \cdot)|_E$ is locally one to one and we can apply Kumagai's theorem. \square

6. CENTRALLY SYMMETRIC DOMAINS.

Proposition 18. *Suppose that P is centrally symmetric, for example a circular or square table and suppose that θ_1, θ_2 are such that the rotation number of S is irrational, then for any $x \in Q_i$ the ω -limit set $\omega(x) \subset Q_i$ is centrally symmetric ($i = 1, 2$).*

Proof. Consider two points $x^\pm \in \partial P$ such that the vector x^-x^+ passes through the center of symmetry of P and is in the direction θ_1 . The ω -limit sets of these two points are centrally symmetric to each other, however from Corollary 2 we have $\omega(x)$ does not depend on x . \square

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