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Dynamic Portfolio Optimization with Liquidity Cost and Market Impact: A Simulation-and-Regression Approach

Rongju Zhang*, Nicolas Langrené†, Yu Tian‡, Zili Zhu§, Fima Klebaner¶ and Kais Hamza||

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Abstract

We present a simulation-and-regression method for solving dynamic portfolio allocation problems in the presence of general transaction costs, liquidity costs and market impacts. This method extends the classical least squares Monte Carlo algorithm to incorporate switching costs, corresponding to transaction costs and transient liquidity costs, as well as multiple endogenous state variables, namely the portfolio value and the asset prices subject to permanent market impacts. To do so, we improve the accuracy of the control randomization approach in the case of discrete controls, and propose a global iteration procedure to further improve the allocation estimates. We validate our numerical method by solving a realistic cash-and-stock portfolio with a power-law liquidity model. We quantify the certainty equivalent losses associated with ignoring liquidity effects, and illustrate how our dynamic allocation protects the investor's capital under illiquid market conditions. Lastly, we analyze, under different liquidity conditions, the sensitivities of certainty equivalent returns and optimal allocations with respect to trading volume, stock price volatility, initial investment amount, risk-aversion level and investment horizon.

Keywords: dynamic portfolio selection; portfolio optimization; transaction cost; liquidity cost; market impact; optimal stochastic control; switching cost; least squares Monte Carlo; simulation-and-regression

JEL Classification: G11; D81; C15; C44; C61

MSC Classification: 91G10; 93E20; 91B24; 65C05; 91G60; 91B06; 90C39; 93E24

*Corresponding author. Email: rongju.zhang@monash.edu. School of Mathematical Sciences, Monash University

†RiskLab, CSIRO

‡School of Mathematical Sciences, Monash University

§RiskLab, CSIRO

¶School of Mathematical Sciences, Monash University

||School of Mathematical Sciences, Monash University

1 Introduction

The effect of liquidity on the design of dynamic multi-period portfolio selection methods (a.k.a. asset allocation, portfolio optimization or portfolio management) has drawn great attention from academics and practitioners alike. Liquidity affects portfolio allocation in two main ways: temporary liquidity cost and permanent market impact. Liquidity cost, also known as implementation shortfall, temporary market impact or transitory market impact, is the difference between the realized transaction price and the pre-transaction price. Market impact is the permanent shift in the asset price after a transaction, due to the post-transaction “resilience” of the limit order book. These liquidity effects depend on several factors, such as the nature of the exchange platform, the duration of the trade execution, the transaction volume, the asset volatility and so on. Up to now, liquidity modeling for dynamic portfolio selection has been impeded by the intractability of analytical solutions and by the limited capability of numerical methods to handle endogenous stochastic prices. The purpose of the present paper is to introduce a new simulation-and-regression method capable of handling multivariate portfolio allocation problems under general transaction costs, liquidity costs and market impacts.

The original literature on dynamic portfolio selection started with simple problems without transaction costs. The seminal papers, [Mossin \(1968\)](#), [Samuelson \(1969\)](#), [Merton \(1969\)](#) and [Merton \(1971\)](#) provide closed-form solutions of optimal asset allocation strategies for long-term investors. In reality though, every transaction incurs commission fee (or brokerage cost), and several improvements have therefore been proposed to account for transaction cost. Examples of closed-form solutions are [Davis and Norman \(1990\)](#), [Shreve and Soner \(1994\)](#), [Liu \(2004\)](#) and [Gârleanu and Pedersen \(2013\)](#). Examples of numerical methods are [Lynch and Tan \(2010\)](#), [Muthuraman and Zha \(2008\)](#) and [Brown and Smith \(2011\)](#). Transient liquidity cost, viewed as another type of transaction cost, has also been studied by many researchers in the context of dynamic portfolio selection problems. [Çetin and Rogers \(2007\)](#) show the existence of optimal portfolios and how to turn the marginal price process under the optimal strategy into a martingale using the optimal terminal wealth as change of measure. We refer to [Ma, Song, and Zhang \(2013\)](#) and [Lim and Wimonkittiwat \(2014\)](#) for examples of solving the Hamilton–Jacobi–Bellman (HJB) equation. Other than liquidity cost, permanent market impact is also a crucial element when dealing with large transactions, as it affects portfolio valuation due to the shifts in asset prices. This effect has been widely incorporated in the studies of portfolio liquidation problems. For example, [Bertsimas and Lo \(1998\)](#), [Almgren and Chriss \(2000\)](#), [Obizhaeva and Wang \(2013\)](#) and [Tsoukalas, Wang, and Giesecke \(2015\)](#). These works, although restricted to either linear or linear-quadratic objective functions, provide a broad overview of trading modeling in illiquid markets. Dynamic portfolio selection under permanent market impact has been formulated in [Ly Vath, Mnif, and Pham \(2007\)](#) as an impulse control problem under state constraints, where the authors characterize the value function as the unique constrained viscosity solution to the associated quasi-variational HJB inequality. This framework has been extended to numerical approximation in [Gaigi, Ly Vath, Mnif, and Toumi \(2016\)](#). [Gârleanu and Pedersen \(2013\)](#) derive a closed-form optimal portfolio policy for the mean-variance framework with quadratic transaction costs such that liquidity cost and market impact are included. Following on this framework, many extensions have been proposed, for example [Collin-Dufresne, Daniel, Moallemi, and Sağlam \(2015\)](#) and [Mei, DeMiguel, and Nogales \(2016\)](#). However, due to the analytically intractable formulation, these methods are restricted in the range of applications when market impacts are present. To broaden the range of applications, the least-squares Monte Carlo (LSMC) algorithm is a possible

solution. The LSMC algorithm, originally developed by [Carriere \(1996\)](#), [Longstaff and Schwartz \(2001\)](#) and [Tsitsiklis and Van Roy \(2001\)](#) for the pricing of American options, has been extended to solve dynamic portfolio selection problems in [Brandt, Goyal, Santa-Clara, and Stroud \(2005\)](#), [Garlappi and Skoulakis \(2010\)](#) and [Cong and Oosterlee \(2016\)](#). [Brandt et al. \(2005\)](#) determine a semi-closed form by solving the first order condition of the Taylor series expansion of the future value function. [Garlappi and Skoulakis \(2010\)](#) claim that the convergence of [Brandt et al. \(2005\)](#)'s method is not stable and that it cannot handle problems where the control variable depends on the endogenous wealth variable. Instead, they introduce a state variable decomposition method to overcome this drawback. However, this decomposition relies on a linear separation between the observable component and stochastic deviation of returns, which cannot be applied to general return distributions. [Cong and Oosterlee \(2016\)](#) use a multi-stage strategy to perform forward simulation of control variables which are iteratively updated in the backward recursive program, where the admissible control sets are constructed as small neighborhoods of the solutions to the multi-stage strategy. Later, [Cong and Oosterlee \(2017\)](#) combine [Jain and Oosterlee \(2015\)](#)'s stochastic bundling technique with [Brandt et al. \(2005\)](#)'s method. To sum up, these three papers have opened the way to the use of the LSMC algorithm for solving dynamic portfolio selection problems, but are at this stage still limited and constrained in their possible formulations of transaction cost, liquidity cost and market impact.

In this paper, we make three contributions to this literature. Our first contribution is to propose a LSMC algorithm to solve dynamic portfolio selection problems with no restriction in the formulations of transaction cost, liquidity cost and market impact, and allowing for multiple assets with general dynamics in a computationally tractable way. Our method is the most general and versatile available in the literature, and can be easily adapted to other applications involving optimal multiple switching problems.

Our second contribution is to improve the numerical performance of [Kharroubi, Langrené, and Pham \(2014\)](#)'s control randomization algorithm in the case of discrete control. In [Kharroubi et al. \(2014\)](#), the randomized controls are part of the regression inputs, and the regression basis is extended accordingly. However, an inadequate regression basis for the control variable can slow down the convergence of this approach, all the more so for highly nonlinear payoffs. Moreover, finding an adequate basis for the controls can be problematic in practice. To avoid this difficulty, we account for the control information by discretizing the control space and performing one regression per control level. This discrete control approach extends the optimal switching approach ([Boogert and de Jong \(2008\)](#), [Aïd, Campi, Langrené, and Pham \(2014\)](#)) to the case with endogenous state variables. Finally, we iterate the whole algorithm by replacing the initial randomized controls by the optimal control estimates from the previous run. We show that these combined modifications improve the portfolio allocation estimates.

Our third contribution is to present an empirical study on how dynamic portfolio allocations are affected by transient and permanent liquidity effects. We apply our method to solve a realistic cash-and-stock portfolio allocation problem, for which we adopt the power-law liquidity model of [Almgren, Thum, Hauptmann, and Li \(2005\)](#). We measure the certainty equivalent losses associated with ignoring liquidity issues, and illustrate the ability of our dynamic allocation to protect the investor's capital in illiquid markets. Finally, based on different liquidity scenarios, we analyze the sensitivity of certainty equivalent returns and portfolio allocations with respect to trading volumes, stock price volatility, initial investment amount, risk-aversion level and investment horizon.

The outline of the paper is as follows. Section 2 formulates our dynamic portfolio selection problem with transaction cost, liquidity cost and market impact. Section 3 describes the LSMC algorithm developed to solve this problem. Section 4 describes the parametric liquidity model we used. Section 5 describes our numerical experiments and Section 6 concludes the paper.

2 Problem Description

In this section, we provide the detailed mathematical description of the portfolio allocation problem we aim to solve. Consider a dynamic portfolio selection problem over a finite time horizon T . Suppose there are d risky assets available for investment. Denote r^f as the risk-free rate. Let $\{\mathbf{r}_t\}_{0 \leq t \leq T} = \{r_t^i\}_{0 \leq t \leq T}^{1 \leq i \leq d}$ and $\{\mathbf{S}_t\}_{0 \leq t \leq T} = \{S_t^i\}_{0 \leq t \leq T}^{1 \leq i \leq d}$ respectively denote the asset returns and prices. Denote $\{\mathbf{Z}_t\}_{0 \leq t \leq T}$ as the vector of return predictors. This vector $\{\mathbf{Z}_t\}_{0 \leq t \leq T}$ is used to construct the dynamics of the assets. Let $\boldsymbol{\alpha}_t = (\alpha_t^i)_{1 \leq i \leq d}$ be the portfolio allocation in each risky asset at time t ; the allocation in the risk-free asset is then given by $\alpha_t^f = 1 - \sum_{1 \leq i \leq d} \alpha_t^i$. In a similar manner, let $\mathbf{q}_t = (q_t^i)_{0 \leq t \leq T}^{1 \leq i \leq d}$ describes the number of units held in each risky asset and let $\{q_t^f\}_{0 \leq t \leq T}$ denote the amount allocated in the risk-free cash. Define $\Delta q_t^i := q_t^i - q_{t-}^i$ as the transaction volume for the i^{th} risky asset at time t . Let $\mathcal{A} \subseteq \mathbb{R}^d$ be the set of admissible portfolio strategies. These sets may include constraints defined by the investor, such as weight limits in each individual asset for example. Finally, let $\{W_t\}_{0 \leq t \leq T}$ denote the portfolio value (or wealth) process.

For every transaction, due to transaction cost, liquidity cost and market impact, there are immediate shifts to the endogenous asset prices and portfolio value. Let $\mathbf{TC}(\Delta \mathbf{q}_t) = \{\text{TC}^i(\Delta q_t^i)\}_{1 \leq i \leq d}$, $\mathbf{LC}(\Delta \mathbf{q}_t) = \{\text{LC}^i(\Delta q_t^i)\}_{1 \leq i \leq d}$ and $\mathbf{MI}(\Delta \mathbf{q}_t) = \{\text{MI}^i(\Delta q_t^i)\}_{1 \leq i \leq d}$ respectively denote the vector of transaction costs, liquidity costs and market impacts generated by the transaction Δq_t^i for each risky asset $i = 1, \dots, d$. In general, we write these quantities as deterministic functions of transaction volume: $\text{TC}^i : \mathbb{R} \rightarrow \mathbb{R}$, $\text{LC}^i : \mathbb{R} \rightarrow \mathbb{R}$ and $\text{MI}^i : \mathbb{R} \rightarrow \mathbb{R}$, and thus $\mathbf{TC} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\mathbf{LC} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\mathbf{MI} : \mathbb{R}^d \rightarrow \mathbb{R}^d$. Given a transaction $\{\Delta q_t^i\}_{1 \leq i \leq d}$ at time t , the following immediate changes occur:

$$\begin{aligned} \mathbf{S}_t &= \mathbf{S}_{t-} + \mathbf{MI}(\Delta \mathbf{q}_t), \\ W_t &= W_{t-} - \mathbf{TC}(\Delta \mathbf{q}_t) \cdot \vec{\mathbf{1}}_d - \mathbf{LC}(\Delta \mathbf{q}_t) \cdot \vec{\mathbf{1}}_d + \mathbf{MI}(\Delta \mathbf{q}_t) \cdot \mathbf{q}_t. \end{aligned} \quad (2.1)$$

where $\vec{\mathbf{1}}_d$ is a vector of size d with all the entries equal to 1.

It is important to note that there are two possible descriptions of the portfolio positions: *absolute positions* using the quantity (number of units) in each asset \mathbf{q}_t and *relative positions* using the proportions of wealth in each asset $\boldsymbol{\alpha}_t$. We describe our portfolio allocation decisions using $\boldsymbol{\alpha}_t$, while transaction cost, liquidity cost and market impact depend on \mathbf{q}_t . Fortunately, there is a natural one-to-one correspondence between these two descriptions, namely,

$$\boldsymbol{\alpha}_t \times W_t = \mathbf{q}_t \times \mathbf{S}_t, \quad (2.2)$$

where “ \times ” denotes the element-wise multiplication and we also denote “ \div ” as element-wise division. Suppose that at time t , one wants to rebalance the portfolio from the absolute position \mathbf{q}_{t-1} to the relative weight $\boldsymbol{\alpha}_t \in \mathcal{A}$. Then, using the dynamics (2.1) and the relation (2.2), the following system of

equations holds:

$$\boldsymbol{\alpha}_t \times \left(W_{t^-} - \mathbf{TC}(\Delta \mathbf{q}_t) \cdot \vec{\mathbf{1}}_d - \mathbf{LC}(\Delta \mathbf{q}_t) \cdot \vec{\mathbf{1}}_d + \mathbf{MI}(\Delta \mathbf{q}_t) \cdot \mathbf{q}_t \right) = \mathbf{q}_t \times (\mathbf{S}_{t^-} + \mathbf{MI}(\Delta \mathbf{q}_t)). \quad (2.3)$$

This is a system of nonlinear equations coupled by the wealth variable. Solving these equations enables us to simultaneously update $\boldsymbol{\alpha}_t$ and \mathbf{q}_t , and thus avoid the potential mismatch between actual allocation and target allocation. To solve it numerically (i.e. being given $\boldsymbol{\alpha}_t$ and \mathbf{q}_{t^-} , find \mathbf{q}_t) we use a fixed-point argument as described by Algorithm 1. Based on our numerical experiment, a stable solution can be reached within three iterations for a tolerance set to $\text{tol} = 10^{-4}$. This algorithm ensures that the post-transaction portfolio holdings, accounting for immediate transaction costs, liquidity costs and market impacts, match exactly the required portfolio allocation $\boldsymbol{\alpha}_t$. Ignoring this actual rebalancing could result in a large mismatch between the actual post-transaction allocation and the initial target allocation. We denote the transaction volume as a function of $\boldsymbol{\alpha}_t$, \mathbf{S}_{t^-} and W_{t^-} , i.e., $\Delta \mathbf{q}_t = \mathcal{Q}(\boldsymbol{\alpha}_t, \mathbf{S}_{t^-}, W_{t^-})$ where $\mathcal{Q} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$.

Algorithm 1 Compute \mathbf{q}_t and q_t^f

- 1: **Input:** \mathbf{q}_{t^-} , \mathbf{S}_{t^-} , W_{t^-} and $\boldsymbol{\alpha}_t$
 - 2: **Result:** \mathbf{q}_t , q_t^f , \mathbf{S}_t and W_t
 - 3: Set tol
 - 4: Initial guess: $\mathbf{q}_t = \boldsymbol{\alpha}_t \times W_{t^-} \div \mathbf{S}_{t^-}$
 - 5: **while** $\text{dist} > \text{tol}$ **do**
 - 6: $\mathbf{S}_t = \mathbf{S}_{t^-} + \mathbf{MI}(\Delta \mathbf{q}_t)$
 - 7: $W_t = W_{t^-} - \mathbf{TC}(\Delta \mathbf{q}_t) \cdot \vec{\mathbf{1}}_d - \mathbf{LC}(\Delta \mathbf{q}_t) \cdot \vec{\mathbf{1}}_d + \mathbf{MI}(\Delta \mathbf{q}_t) \cdot \mathbf{q}_t$
 - 8: $\mathbf{q}_t^{\text{aux}} = \boldsymbol{\alpha}_t \times W_t \div \mathbf{S}_t$
 - 9: $\text{dist} = \sum |\mathbf{q}_t^{\text{aux}} - \mathbf{q}_t| / \mathbf{q}_t^{\text{aux}}$
 - 10: $\mathbf{q}_t = \mathbf{q}_t^{\text{aux}}$
 - 11: **end while**
 - 12: $q_t^f = W_t - \mathbf{q}_t \cdot \mathbf{S}_t$
-

The dynamic portfolio allocation is chosen to maximize the investor's expected utility of final wealth $\mathbb{E}[U(W_T)]$ over all the possible strategies $\{\boldsymbol{\alpha}_t \in \mathcal{A}\}_{0 \leq t \leq T}$. Let $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ be the filtration generated by all the state variables. At any time $t \in [0, T]$, the objective function reads

$$v_t(z, s, w) = \sup_{\{\boldsymbol{\alpha}_\tau \in \mathcal{A}\}_{t \leq \tau \leq T}} \mathbb{E}[U(W_T) | \mathbf{Z}_t = z, \mathbf{S}_{t^-} = s, W_{t^-} = w], \quad (2.4)$$

where $V_t = v_t(\mathbf{Z}_t, \mathbf{S}_{t^-}, W_{t^-})$ and $\boldsymbol{\alpha}_t$ are \mathcal{F}_t -adapted. The state variables of the problem are:

1. Exogenous state variables: the return predictors \mathbf{Z}_t
2. Endogenous state variables: the relative portfolio weights $\boldsymbol{\alpha}_t$, the absolute portfolio holdings \mathbf{q}_t , the asset prices \mathbf{S}_t and the portfolio value W_t

Henceforth, we restrict the rebalancing times to an equally-spaced discrete grid $0 = t_0 < \dots < t_N = T$. The asset price processes evolve as

$$\mathbf{S}_{t_{n+1}} = \mathbf{S}_{t_n} \times \exp(\mathbf{r}_{t_{n+1}}) + \mathbf{MI}(\Delta \mathbf{q}_{t_n}), \quad (2.5)$$

and the wealth process evolves as

$$W_{t_{n+1}} = W_{t_n} + r^f q_{t_n}^f + \mathbf{q}_{t_n} \cdot (\mathbf{S}_{t_n} \times \mathbf{r}_{t_{n+1}}) - \mathbf{TC}(\Delta \mathbf{q}_{t_{n+1}}) \cdot \vec{\mathbf{1}}_d - \mathbf{LC}(\Delta \mathbf{q}_{t_{n+1}}) \cdot \vec{\mathbf{1}}_d + \mathbf{MI}(\Delta \mathbf{q}_{t_{n+1}}) \cdot \mathbf{q}_{t_{n+1}}, \quad (2.6)$$

where \mathbf{q} in (2.5)-(2.6) satisfy the relation (2.3). The value function satisfies the following discrete dynamic programming principle

$$\begin{aligned} v_{t_n}(z, s, w) &= \sup_{\boldsymbol{\alpha}_{t_n} \in \mathcal{A}} \mathbb{E} [v_{t_{n+1}}(\mathbf{Z}_{t_{n+1}}, \mathbf{S}_{t_{n+1}}, W_{t_{n+1}}) | \mathbf{Z}_{t_n} = z, \mathbf{S}_{t_n} = s, W_{t_n} = w] \\ v_{t_N}(z, s, w) &= U(w) \end{aligned} \quad (2.7)$$

and we assume that the investor begins with 100% holding in the cash account and liquidate all the risky assets at the terminal time, i.e., $\boldsymbol{\alpha}_{(t_0)^-} = \boldsymbol{\alpha}_{t_N} = 0$.

3 Solution

In this section, we describe our method for solving the recursive dynamic programming problem (2.7). Our algorithm can be decomposed into three main parts:

1. First, a forward simulation of all the state variables of the problem, including the endogenous state variables, following the control randomization method of [Kharroubi et al. \(2014\)](#), described in Section 3.1;
2. Then, a backward recursive dynamic programming where the conditional expectations are approximated by least squares regressions, and the optimal allocation obtained by exhaustive search, described in Section 3.2;
3. Finally, an iteration procedure for updating the simulated control variables of the first step by the estimates generated from the second step, described in Section 3.3.

3.1 Step 1: Monte Carlo simulations

The first main part consists in simulating a large sample of all the stochastic state variables. The return predictors \mathbf{Z}_t and the asset excess returns \mathbf{r}_t are exogenous risk factors, and therefore easy to simulate. By contrast, the asset prices \mathbf{S}_t and the portfolio value W_t are endogenous risk factors, i.e. their dynamics depend on the control $\boldsymbol{\alpha}_t$. In order to simulate \mathbf{S}_t , W_t , which is necessary to initiate the algorithm, we rely on the control randomization technique of [Kharroubi et al. \(2014\)](#). In summary, we first simulate a sample of return predictors $\{\mathbf{Z}_{t_n}^m\}_{0 \leq n \leq N}^{1 \leq m \leq M}$, asset excess returns $\{\mathbf{r}_{t_n}^m\}_{0 \leq n \leq N}^{1 \leq m \leq M}$ and random portfolio weights $\{\tilde{\boldsymbol{\alpha}}_{t_n}^m\}_{0 \leq n \leq N}^{1 \leq m \leq M}$, then compute the corresponding absolute holdings $\{\tilde{\mathbf{q}}_{t_n}^m\}_{0 \leq n \leq N}^{1 \leq m \leq M}$, asset prices $\{\tilde{\mathbf{S}}_{(t_n)^-}^m\}_{0 \leq n \leq N}^{1 \leq m \leq M}$ and portfolio values $\{\tilde{W}_{(t_n)^-}^m\}_{0 \leq n \leq N}^{1 \leq m \leq M}$ according to Algorithm 1. The next subsection explains how these initial random weights will be turned into estimates of the optimal allocation.

3.2 Step 2: discretization, regression and maximization

The second part of our LSMC algorithm is the regression and maximization by exhaustive search. We discretize the control space as $\mathcal{A} \approx \mathcal{A}^d = \{\mathbf{a}_1, \dots, \mathbf{a}_J\}$. According to the dynamic programming principle (2.7), at time t_N , the objective function (2.4) is equal to $\hat{v}_{t_N}(z, s, w) = U(w)$. At time t_n , assume that the mapping $\hat{v}_{t_{n+1}} : (z, s, w) \mapsto \hat{v}_{t_{n+1}}(z, s, w)$ has been estimated, one obtains

$$\begin{aligned} & v_{t_n}(z, s, w) \\ &= \sup_{\boldsymbol{\alpha}_{t_n} \in \mathcal{A}} \mathbb{E} \left[\hat{v}_{t_{n+1}} \left(\mathbf{Z}_{t_{n+1}}, \mathbf{S}_{(t_{n+1})^-}, W_{(t_{n+1})^-} \right) \middle| \mathbf{Z}_{t_n} = z, \mathbf{S}_{(t_n)^-} = s, W_{(t_n)^-} = w \right] \\ &\approx \max_{\mathbf{a}_j \in \mathcal{A}^d} \mathbb{E} \left[\hat{v}_{t_{n+1}} \left(\mathbf{Z}_{t_{n+1}}, \mathbf{S}_{(t_{n+1})^-}, W_{(t_{n+1})^-} \right) \middle| \mathbf{Z}_{t_n} = z, \boldsymbol{\alpha}_{t_n} = \mathbf{a}_j, \mathbf{S}_{(t_n)^-} = s, W_{(t_n)^-} = w \right]. \end{aligned}$$

By taking the decision $\boldsymbol{\alpha}_{t_n} = \mathbf{a}_j$, the endogenous state variables at time $(t_n)^-$ can be updated to their post-transaction values at time t_n :

$$\begin{aligned} & v_{t_n}(z, s, w) \\ &= \max_{\mathbf{a}_j \in \mathcal{A}^d} \mathbb{E} \left[\hat{v}_{t_{n+1}} \left(\mathbf{Z}_{t_{n+1}}, \mathbf{S}_{(t_{n+1})^-}, W_{(t_{n+1})^-} \right) \middle| \begin{array}{l} \mathbf{Z}_{t_n} = \mathbf{Z}'_{t_n}, \boldsymbol{\alpha}_{t_n} = \boldsymbol{\alpha}'_{t_n}, \mathbf{q}_{t_n} = \mathbf{q}'_{t_n}, \\ \mathbf{S}_{t_n} = \mathbf{S}'_{t_n}, W_{t_n} = W'_{t_n} \end{array} \right] \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} \mathbf{Z}'_{t_n} &= z \\ \boldsymbol{\alpha}'_{t_n} &= \mathbf{a}_j \\ \mathbf{q}'_{t_n} &= \mathbf{q}_{t_{n-1}} + \mathcal{Q}(\mathbf{a}_j, s, w) \\ \mathbf{S}'_{t_n} &= s + \mathbf{MI}(\mathcal{Q}(\mathbf{a}_j, s, w)) \\ W'_{t_n} &= w - \mathbf{TC}(\mathcal{Q}(\mathbf{a}_j, s, w)) \cdot \vec{\mathbf{I}}_d - \mathbf{LC}(\mathcal{Q}(\mathbf{a}_j, s, w)) \cdot \vec{\mathbf{I}}_d + \mathbf{MI}(\mathcal{Q}(\mathbf{a}_j, s, w)) \cdot \mathbf{q}'_{t_n} \end{aligned}$$

Therefore, for each Monte Carlo path $m = 1, \dots, M$, we update the decisions $\boldsymbol{\alpha}_{t_n}^m$ to \mathbf{a}_j and recompute the corresponding endogenous variables at time t_n

$$\begin{aligned} \Delta \hat{\mathbf{q}}_{t_n}^m &= \mathcal{Q} \left(\mathbf{a}_j, \tilde{\mathbf{S}}_{(t_n)^-}^m, \tilde{W}_{(t_n)^-}^m \right) \\ \hat{\mathbf{q}}_{t_n}^m &= \tilde{\mathbf{q}}_{t_{n-1}}^m + \Delta \hat{\mathbf{q}}_{t_n}^m \\ \hat{\mathbf{S}}_{t_n}^m &= \tilde{\mathbf{S}}_{(t_n)^-}^m + \mathbf{MI}(\Delta \hat{\mathbf{q}}_{t_n}^m) \\ \hat{W}_{t_n}^m &= \tilde{W}_{(t_n)^-}^m - \mathbf{TC}(\Delta \hat{\mathbf{q}}_{t_n}^m) \cdot \vec{\mathbf{I}}_d - \mathbf{LC}(\Delta \hat{\mathbf{q}}_{t_n}^m) \cdot \vec{\mathbf{I}}_d + \mathbf{MI}(\Delta \hat{\mathbf{q}}_{t_n}^m) \cdot \hat{\mathbf{q}}_{t_n}^m, \end{aligned}$$

then recompute the endogenous state variables one time-step forward at time $t_{(n+1)^-}$, i.e.,

$$\begin{aligned} \hat{\mathbf{S}}_{(t_{n+1})^-}^m &= \hat{\mathbf{S}}_{t_n}^m \times \exp(\mathbf{r}_{t_{n+1}}^m) \\ \hat{W}_{(t_{n+1})^-}^m &= \hat{W}_{t_n}^m + r^f q_{t_n}^{f,m} + \hat{\mathbf{q}}_{t_n}^m \cdot \left(\hat{\mathbf{S}}_{t_n}^m \times \mathbf{r}_{t_{n+1}}^m \right). \end{aligned}$$

Finally, set $\{L_k(z, s, w)\}_{1 \leq k \leq K}$ to be a vector of basis functions of state variables. We estimate the ‘‘continuation values’’ (the conditional expectations in equation (3.1)) by least squares minimization,

i.e.,

$$\left\{ \hat{\beta}_{k,t_n}^j \right\}_{1 \leq k \leq K} = \arg \min_{\beta \in \mathbb{R}^K} \sum_{m=1}^M \left(\begin{array}{c} \sum_{k=1}^K \beta_k L_k \left(\mathbf{Z}_{t_n}^m, \hat{\mathbf{S}}_{t_n}^m, \hat{W}_{t_n}^m \mid \boldsymbol{\alpha}_{t_n}^m = \mathbf{a}_j \right) \\ - \hat{v}_{t_{n+1}} \left(\mathbf{Z}_{t_{n+1}}^m, \hat{\mathbf{S}}_{(t_{n+1})^-}^m, \hat{W}_{(t_{n+1})^-}^m \mid \boldsymbol{\alpha}_{t_n}^m = \mathbf{a}_j \right) \end{array} \right)^2. \quad (3.2)$$

Therefore the ‘‘continuation value’’ at time t_n for $\mathbf{a}_j \in \mathcal{A}^d$ is formulated as

$$\hat{C}\hat{V}_{t_n}^j(z, s, w) = \sum_{k=1}^K \hat{\beta}_{k,t_n}^j L_k(z, s, w),$$

and the mappings $\hat{\boldsymbol{\alpha}}_{t_n} : (z, s, w) \mapsto \hat{\boldsymbol{\alpha}}_{t_n}(z, s, w)$ and $\hat{v}_{t_n} : (z, s, w) \mapsto \hat{v}_{t_n}(z, s, w)$ are estimated by

$$\hat{\boldsymbol{\alpha}}_{t_n}(z, s, w) = \arg \max_{\mathbf{a}_j \in \mathcal{A}^d} \hat{C}\hat{V}_{t_n}^j(z, s, w) \quad \text{or} \quad \hat{v}_{t_n}(z, s, w) = \max_{\mathbf{a}_j \in \mathcal{A}^d} \hat{C}\hat{V}_{t_n}^j(z, s, w).$$

It is important to remark that the discretization of the control allowed us to substitute the extended control regression of [Kharroubi et al. \(2014\)](#) by one regression (3.2) for each control level.

3.3 Step 3: control iteration

In the forward simulation, the endogenous state variables are generated using the randomized controls $\{\tilde{\boldsymbol{\alpha}}_{t_n}^m\}_{0 \leq n \leq N}^{1 \leq m \leq M}$. Although the endogenous state variables will be updated and corrected backwards during Step 2, the evaluation of

$$v_{t_n}(z, s, w) = \sup_{\boldsymbol{\alpha}_{t_n} \in \mathcal{A}} \mathbb{E} \left[\hat{v}_{t_{n+1}} \left(\mathbf{Z}_{t_{n+1}}, \mathbf{S}_{(t_{n+1})^-}, W_{(t_{n+1})^-} \right) \mid \mathbf{Z}_{t_n} = z, \mathbf{S}_{(t_n)^-} = s, W_{(t_n)^-} = w \right]$$

is made on the sample of path-dependent variables $\left\{ \mathbf{S}_{(t_n)^-}^m, W_{(t_n)^-}^m \right\}_{1 \leq m \leq M}$ which still depend on the historical randomized controls $\{\tilde{\boldsymbol{\alpha}}_{t_n}^m\}_{0 \leq n' \leq n-1}^{1 \leq m \leq M}$. In theory, this fact does not affect the optimality of the allocation estimates, as the regression provides an estimate of $v_{t_n}(z, s, w)$ everywhere, including the region where the optimally controlled endogenous variables $\mathbf{S}_{(t_n)^-}$ and $W_{(t_n)^-}$ will eventually lie. In practice, it may lead to possibly large numerical errors if the regression is numerically inaccurate in the optimal region, due to an insufficiently large sample size or inadequate regression basis for example. To mitigate this possibility, we propose to iterate the whole algorithm, with the initial randomized controls replaced by the estimated optimal controls produced by the previous run. This iteration procedure will bring the whole sample $\left\{ \mathbf{S}_{(t_n)^-}^m, W_{(t_n)^-}^m \right\}_{1 \leq m \leq M}$ closer to the optimal region, and thus improve the overall portfolio allocation estimates. Our numerical experiments in Section 5 show that this iteration procedure does improve accuracy, especially for small sample sizes and highly nonlinear utility functions, and that most of the improvements occur after one single additional iteration.

3.4 Summary and remarks

Finally, this subsection provides a detailed description of the backward iterations, followed by a few additional implementation details.

Summary of algorithm Being given the “continuation values” at time t_{N-1} , the detailed implementation of one backward iteration (cf. Section 3.2) is summarized in Algorithm 2, where we set $\{\hat{\alpha}_{(t_0)^-}^m\}_{1 \leq m \leq M} = 0$, $\{\hat{q}_{(t_0)^-}^m\}_{1 \leq m \leq M} = 0$ and $\hat{q}_{(t_0)^-}^{f,m} = W_{(t_0)^-} =$ initial investment amount. Additional implementation details are discussed below.

Algorithm 2 Backward Dynamic Programming

1: **Input:** $(\mathbf{Z}_{t_n}^m, \mathbf{r}_{t_n}^m, \hat{\alpha}_{t_n}^m, \hat{q}_{t_n}^m, \hat{\mathbf{S}}_{(t_n)^-}^m, \hat{W}_{(t_n)^-}^m)_{0 \leq n \leq N}^{1 \leq m \leq M}$, $(\hat{C}\hat{V}_{t_{N-1}}^j, \hat{\beta}_{t_{N-1}}^j)_{1 \leq j \leq J}$

2: **Result:** $\hat{\alpha}_{t_0}$

3: **for all** rebalancing time $t_n = t_{N-1}, \dots, t_0$ **do**

4: **for all** decision $\mathbf{a}_j \in \mathcal{A}^d$ **do**

5: **for all** Monte Carlo path $m = 1, \dots, M$ **do**

6: Compute $(\hat{q}_{t_n}^m, \hat{\mathbf{S}}_{t_n}^m, \hat{W}_{t_n}^m)$ from $(\hat{q}_{t_{n-1}}^m, \hat{\mathbf{S}}_{(t_n)^-}^m, \hat{W}_{(t_n)^-}^m, \alpha_{t_n}^m = \mathbf{a}_j)$ using Algorithm 1

7: Compute $\hat{\mathbf{S}}_{(t_{n+1})^-}^m = \hat{\mathbf{S}}_{t_n}^m \times \exp(\mathbf{r}_{t_{n+1}}^m)$ and $\hat{W}_{(t_{n+1})^-}^m = \hat{W}_{t_n}^m + r^f q_{t_n}^{f,m} + \hat{q}_{t_n}^m \cdot (\hat{\mathbf{S}}_{t_n}^m \times \mathbf{r}_{t_{n+1}}^m)$

8: **for all** rebalancing time $t_{n'} = t_{n+1}, \dots, t_{N-1}$ **do**

9: **for all** decision $\mathbf{a}_l \in \mathcal{A}^d$ **do**

10: Compute $(\hat{q}_{t_{n'}}^m, \hat{\mathbf{S}}_{t_{n'}}^m, \hat{W}_{t_{n'}}^m)$ from $(\hat{q}_{t_{n'-1}}^m, \hat{\mathbf{S}}_{(t_{n'})^-}^m, \hat{W}_{(t_{n'})^-}^m, \alpha_{t_{n'}}^m = \mathbf{a}_l)$ using Algorithm 1

11: Compute $\hat{C}\hat{V}_{t_{n'}}^l(\mathbf{Z}_{t_{n'}}^m, \hat{\mathbf{S}}_{t_{n'}}^m, \hat{W}_{t_{n'}}^m) = \sum_{k=1}^K \hat{\beta}_{k,t_{n'}}^l L_k(\mathbf{Z}_{t_{n'}}^m, \hat{\mathbf{S}}_{t_{n'}}^m, \hat{W}_{t_{n'}}^m | \alpha_{t_{n'}}^m = \mathbf{a}_l)$

12: **end for**

13: Update $(\hat{q}_{t_{n'}}^m, \hat{\mathbf{S}}_{t_{n'}}^m, \hat{W}_{t_{n'}}^m)$ with $\alpha_{t_{n'}}^m = \arg \max_{\mathbf{a}_l \in \mathcal{A}^d} \hat{C}\hat{V}_{t_{n'}}^l(\mathbf{Z}_{t_{n'}}^m, \hat{\mathbf{S}}_{t_{n'}}^m, \hat{W}_{t_{n'}}^m)$

14: Compute $\hat{\mathbf{S}}_{(t_{n'+1})^-}^m = \hat{\mathbf{S}}_{t_{n'}}^m \times \exp(\mathbf{r}_{t_{n'+1}}^m)$ and $\hat{W}_{(t_{n'+1})^-}^m = \hat{W}_{t_{n'}}^m + r^f q_{t_{n'}}^{f,m} + \hat{q}_{t_{n'}}^m \cdot (\hat{\mathbf{S}}_{t_{n'}}^m \times \mathbf{r}_{t_{n'+1}}^m)$

15: **end for**

16: Compute $\hat{W}_{t_N}^m$ from $(\hat{q}_{t_{N-1}}^m, \hat{\mathbf{S}}_{(t_N)^-}^m, \hat{W}_{(t_N)^-}^m, \alpha_{t_N}^m = 0)$ using Algorithm 1

17: **end for**

18: **if** $t_n > t_0$ **then**

19: Least-squares approximation with basis functions of state variables, $\{L_k(z, s, w)\}_{1 \leq k \leq K}^2$:

$$\{\hat{\beta}_{k,t_n}^j\}_{1 \leq k \leq K} = \arg \min_{\beta \in \mathbb{R}^K} \sum_{m=1}^M \left(\sum_{k=1}^K \beta_k L_k(\mathbf{Z}_{t_n}^m, \hat{\mathbf{S}}_{t_n}^m, \hat{W}_{t_n}^m | \alpha_{t_n}^m = \mathbf{a}_j) - U(\hat{W}_{t_n}^m) \right)$$

20: Formulate: $\hat{C}\hat{V}_{t_n}^j(z, s, w) = \sum_{k=1}^K \hat{\beta}_{k,t_n}^j \cdot L_k(z, s, w)$

21: **else**

22: Compute: $\hat{C}\hat{V}_{t_0}^j = \frac{1}{M} \sum_{m=1}^M U(\hat{W}_{t_N}^m)$

23: **end if**

24: **end for**

25: **end for**

26: Initial optimal control: $\hat{\alpha}_{t_0} = \arg \max_{\mathbf{a}_j \in \mathcal{A}^d} \hat{C}\hat{V}_{t_0}^j$

VFI versus PFI Two alternative implementations of the LSMC algorithm can be used: value function iteration (VFI, [Carriere \(1996\)](#), [Tsitsiklis and Van Roy \(2001\)](#), a.k.a. regression surface value iteration), and performance function iteration (PFI, [Longstaff and Schwartz \(2001\)](#), a.k.a. realized value iteration, or portfolio weight iteration). The difference lies in the t_{n+1} -response in the least squares regressions (3.2): the VFI scheme regresses the estimated continuation value function from the previous regression, while the PFI scheme regresses the realized paths under the estimated optimal policy. The PFI scheme

produces more accurate results, as it avoids the compounding of regression errors of the VFI scheme. However, when some state variables are endogenous, the PFI scheme requires to recompute all the endogenous state variables until the end of the horizon, which increases the computational complexity from linear to quadratic in time. By contrast, the computational complexity of the VFI is linear in the number of time steps. More discussions on VFI versus PFI are available in [Van Binsbergen and Brandt \(2007\)](#), [Garlappi and Skoulakis \(2009\)](#) and [Denault and Simonato \(2017\)](#). In this paper, we choose to implement the PFI scheme for its greater accuracy and stability.

Dimension reduction of state vector Although in theory all the risk factors need to be included in the regression so as to take all the available information into account when making decisions, in practice the bias-variance tradeoff suggests to omit the variables that bring little additional information. In portfolio allocation problems, the portfolio wealth is a linear combination of the asset prices, determined by $W_{t_n} = \mathbf{S}_{t_n} \cdot \mathbf{q}_{t_n}$, such that most of the relevant price changes can be reflected in a single wealth variable. Moreover, our objective is to maximize the expected utility of final wealth, thus the wealth variable plays a much more crucial role when approximating such objective function than the price variables. After testing and comparing different subsets of regression inputs, we decided to remove the endogenous price variables in the regressions and only regress on (\mathbf{Z}, W) . Doing so improves the out-of-sample quality of regression estimates, and has the advantage that the number of assets does not increase the numerical complexity of each least-squares regression in the LSMC algorithm.

Regressing on post- versus pre-transaction variables The evolution of the endogenous state variables from time t_{n-} to t_N can be decomposed into an immediate deterministic component depending on the switching costs $\mathbf{TC}(\Delta \mathbf{q}_{t_n})$, $\mathbf{LC}(\Delta \mathbf{q}_{t_n})$ and $\mathbf{MI}(\Delta \mathbf{q}_{t_n})$, and a stochastic component depending on the dynamics of the state variables from time t_n to t_N . For demonstration purposes, we use in this paragraph the wealth variable W as one single regressor in the regression, use a simple linear utility function $\mathcal{U}(w) = w$, denote $\text{SC}(\Delta \mathbf{q}_{t_n})$ as the corresponding overall switching cost which is the immediate deterministic component at time t_n , and denote the stochastic component evolving from time t_n to t_N as Δ_{t_n, t_N} . Then the two alternative regressions are given by

$$\text{regression on } W_{(t_n)-} : \quad \mathbb{E} \left[W_{(t_n)-} - \text{SC}(\Delta \mathbf{q}_{t_n}) + \Delta_{t_n, t_N} \mid W_{(t_n)-} \right] \approx \beta W_{(t_n)-} \quad (3.3)$$

$$\text{regression on } W_{t_n} : \quad \mathbb{E} \left[W_{(t_n)-} - \text{SC}(\Delta \mathbf{q}_{t_n}) + \Delta_{t_n, t_N} \mid W_{t_n} \right] \approx \beta \left(W_{(t_n)-} - \text{SC}(\Delta \mathbf{q}_{t_n}) \right) \quad (3.4)$$

Here, the pre-transaction regression (3.3) accounts for both the deterministic and stochastic evolutions, while the post-transaction regression (3.4) accounts for the stochastic evolution only. We favor the regression on post-transaction variables for several reasons. Firstly, the deterministic component $\text{SC}(\Delta \mathbf{q}_{t_n})$ is \mathcal{F}_{t_n} -adapted, and thus not necessary for the regression. Secondly, the switching costs $\text{SC}(\Delta \mathbf{q}_{t_n})$ are, at this stage of the algorithm, computed from randomized portfolio positions $\tilde{\mathbf{q}}_{t_{n-1}}$, and thus also randomized. Consequently, the switching costs $\{\text{SC}(\Delta \mathbf{q}_{t_n}^m)\}_{1 \leq m \leq M}$ are not smooth w.r.t the regressor $\left\{ W_{(t_n)-}^m \right\}_{1 \leq m \leq M}$, which may lead to a substantial information loss w.r.t. the switching cost by anchoring the unsmoothness around $\mathbb{E} \left[\text{SC}(\Delta \mathbf{q}_{t_n}) \mid W_{(t_n)-} \right]$ (overestimation of the conditional expectation (3.3) for large $\text{SC}(\Delta \mathbf{q}_{t_n})$ realizations, and underestimation of the conditional expectation (3.3) for small $\text{SC}(\Delta \mathbf{q}_{t_n})$ realizations). Therefore, subtracting the switching costs from the regressor will avoid this

problem by removing this auxiliary randomness from the regression.

Finally, from a practical decision point of view, an investor would consider the known, immediate transaction cost, liquidity cost and market impact when making a portfolio rebalancing decision.

4 Power law liquidity function

One key feature of the presented portfolio allocation algorithm is its flexibility to accommodate general transaction cost, liquidity cost and market impact. This is the reason why the presented algorithm has so far involved general costs **TC**, **LC** and **MI**. In this section, we now specify a realistic model for these costs in view of implementation and testing in the next Section 5.

Transaction cost refers to the commission fee charged by the broker, usually a fixed amount or a fixed proportional rate, and therefore easy to account for. The focus of the paper will be liquidity cost and market impact. During a transaction, the following are the key observables:

$$\begin{aligned} S_{t-} &= \text{market price before the transaction begins} \\ S_t &= \text{market price immediately after the transaction is completed} \\ \bar{S}_t &= \text{trading volume-weighted average price on the transaction} \end{aligned}$$

In our framework, the post-transaction price S_t captures the (permanent) market impact, i.e., $\text{MI} = S_t - S_{t-}$ and the average price captures the (temporary) liquidity cost, i.e., $\text{LC} = |\bar{S}_t - S_{t-}|$.

In reality, the shape of the limit order book differs by the characteristics of the portfolio assets. A power law of both liquidity cost and market impact for the U.S. stock markets has been found in [Almgren et al. \(2005\)](#). [Obizhaeva and Wang \(2013\)](#) assume a linear price shift and uses a negative exponential function to model the resilience of the limit order book. [Tian, Rood, and Oosterlee \(2013\)](#) found a ‘square-root’ relation between the price and the available market orders and for large or medium-cap equities and a ‘square’ relation for small-cap equities in the European market. A different type of ‘square-root’ relation is shown in [Cont, Kukanov, and Stoikov \(2014\)](#) for the stocks listed on NYSE.

In this paper, we adopt the calibrated power law functions of [Almgren et al. \(2005\)](#) to analyze the impact of the market illiquidity on the dynamic portfolio selection problem. These power law functions are given by

$$\text{MI}(\Delta q) = 0.314 \cdot \sigma_{\text{day}} \cdot \frac{\Delta q}{\text{Vol}_{\text{day}}} \cdot \left(\frac{\Theta}{\text{Vol}_{\text{day}}} \right)^{1/4} \quad (4.1)$$

$$\text{LC}(\Delta q) = \left| \frac{\text{MI}(\Delta q)}{2} + 0.142 \cdot \text{sign}(\Delta q) \cdot \sigma_{\text{day}} \cdot \left| \frac{\Delta q}{\delta \cdot \text{Vol}_{\text{day}}} \right|^{3/5} \right| \quad (4.2)$$

where Vol^{day} is the daily trading volume of the stock, σ^{day} is the daily volatility of the stock price, δ is the time length of trade execution, Θ is the number of outstanding shares. In the following numerical section, we will fix the values of δ and Θ and focus on the impact of σ^{day} and Vol^{day} on the portfolio selection problem.

5 Numerical experiments

In this section, we test our algorithm on a cash and stock portfolio. The outline of this numerical section is as follows:

1. Subection 5.1 validates the Monte Carlo convergence of our method with different risk aversion levels, investment horizon and liquidity settings.
2. Subection 5.2 discusses the time evolution of the distribution of portfolio value and the percentage allocation under different liquidity settings.
3. Subection 5.3 identifies the certainty equivalent losses associated with ignoring liquidity effects.
4. Finally, subsection 5.4 provides sensitivity analyzes of the portfolio performance and allocation with respect to liquidity settings.

But first, we detail the numerical settings used to perform the numerical experiments reported in this section.

Data and modeling Table 1 summarizes the financial instruments considered for return predictors. We calibrate a first order vector autoregressive model to monthly log-returns (i.e., $\log S_t - \log S_{t-1}$) from October 2007 to January 2016¹. We assume the annual interest rate on the cash account is 1.2% and use SPDR S&P500 index ETF as the proxy for the stock return.

Switching costs To focus on the liquidity effects, we assume for simplicity no fixed or proportional transaction cost in the numerical study. Regarding liquidity cost and market impact modeling, we use the power law functions (4.2), where we assume the number of outstanding share $\Theta = 988m$ and the trading duration $\delta = 5$ min. We will analyze the liquidity effects characterized by different levels of $(\sigma_{\text{day}}, \text{Vol}_{\text{day}})$ and we follow the usual U.S. equity markets such that $\sigma_{\text{day}} \in [2, 13]$ and $\text{Vol}_{\text{day}} \in [10m, 120m]$.

Certainty equivalent return For all the numerical tests, we report the portfolio performances in terms of monthly adjusted certainty equivalent returns (CER) calculated by

$$\text{CER} = U^{-1}(\mathbb{E}[U(W_T)])^{\frac{1}{T}} - 1 \approx U^{-1}\left(\frac{1}{M} \sum_{m=1}^M U(W_T^m)\right)^{\frac{1}{T}} - 1.$$

The magnitude of monthly returns is usually less than one percent, thus we display the certainty equivalent returns in basis points (0.01%) to make comparisons easier.

LSMC settings We use $M = 10^5$ Monte Carlo simulations and $N = 12$ monthly time steps (one year horizon and 12 rebalancing periods), except when we test the numerical sensitivity to these two parameters (subsection 5.1). After the LSMC algorithm is completed, we generate another sample of $M = 10^5$ to calculate the CER. We denote I as the number of additional control iterations of the whole LSMC algorithm (subsection 3.3), $I = 0$ meaning only one LSMC run and no additional iterations.

¹These data are obtained from Yahoo Finance.

Portfolio weight We denote α as the percentage allocation to the stock component, and $1 - \alpha$ as the allocation to the cash component. We assume a discrete set of admissible controls with step size 0.01, i.e., $\alpha \in \{0.01, 0.02, \dots, 0.99, 1.00\} = \mathcal{A}^d$.

Basis function and regression We first scale all the exogenous risk factors (in our case the log-returns) by dividing by their unconditional mean. For the endogenous risk factor (the portfolio wealth W), we transform it as $U(W/W_0)$, where W_0 is the initial portfolio wealth and $U(\cdot)$ is the CRRA utility function. These transformed quantities form the inputs of our regression basis. For the regression basis, we use a simple second order multivariate polynomial basis. We chose this basis and its order by observing the plots of the objective function w.r.t. the regression bases at various intermediate times. The surface shape was found to be close to linear but slightly curved, suggesting that polynomials of order two could be sufficient.

5.1 Monte Carlo convergence

Table 2 reports the Monte Carlo convergence of the portfolio allocation algorithm 2 described in Section 3 on a simple cash and stock allocation problem with CARA utility $U(w) = -\exp(-\gamma w)$, risk-free rate 0.012, and a stock annual return with mean 0.03 and volatility 0.15. These convergence results are compared to the original control randomization algorithm of Kharroubi et al. (2014) (KLP) for which we include the portfolio allocation into the same second-order global polynomial basis. The main observation is that Algorithm 2 uniformly improves the accuracy of the KLP algorithm with second-order basis. The improvement is more substantial with long maturities ($N = 15$), large risk-aversion ($\gamma = 15$) or small sample size ($M = 10^3$). In these three cases, the benefit of using control iteration (subsection 3.3) is noticeable, and most of the improvement is achieved after one single additional control iteration ($I = 1$).

A similar result can be observed in Table 3 where the Monte Carlo convergence of CER is reported for different liquidity settings characterized by daily volatility σ_{day} and daily trading volume Vol_{day} . Once again, Algorithm 2 with one additional control iteration ($I = 1$) is superior to both KLP and Algorithm 2 with no additional iteration ($I = 0$). Adding further control iterations ($I = 2$ and more) does not bring significant improvement over $I = 1$. When the market liquidity effects are small, e.g., small σ_{day} or large Vol_{day} , a small Monte Carlo sample size is enough for convergence, while a large sample size is needed for large market liquidity effects, e.g., large σ_{day} or small Vol_{day} . For the rest of this numerical section, we will use $M = 10^5$ with $I = 1$ to ensure convergence and accuracy.

A final remark is that, for large enough sample size ($M \geq 10^4$), our LSMC method with $I = 0$ greatly outperforms the KLP algorithm for large risk-aversion levels (Table 2), but makes little difference for large switching costs but low risk-aversion levels (Table 3), indicating that the nonlinearity of the final payoff function plays a more crucial role than the size of switching costs in the accuracy of simulation-and-regression approximating dynamic programming schemes.

5.2 Time-evolution of distribution of control and wealth

Figure 1 shows the time evolution of the portfolio allocation distribution and the wealth distribution. When the market liquidity effects are small ($\sigma_{\text{day}} = 2.5, \text{Vol}_{\text{day}} = 120m$, left-hand side column), both

the portfolio allocation and the wealth are widely spread, along with large portfolio turnovers at the beginning and at the end of the investment horizon. By contrast, for large market liquidity effects ($\sigma_{\text{day}} = 12.5, \text{Vol}_{\text{day}} = 12m$, right-hand side column) transactions become very costly and the algorithm disallows large portfolio turnovers. As a consequence, the portfolio allocation distribution is tightened at a relatively low level ($\alpha \simeq 0.2$) and its time evolution is smooth. Regarding the wealth distribution, as expected, the less liquid the market, the lower the CER (as was shown in Table 3) and the lower the dispersion of the wealth distribution. This is due to the lower and more stable allocation in stock.

5.3 Certainty equivalent losses associated with ignoring liquidity effects

Table 4 compares the CER of an investor who takes heed of liquidity effects when making allocation decisions to the CER of an investor who ignores liquidity effects. For the investor who ignores liquidity effects, we set $\text{LC} = \text{MI} = 0$ in the LSMC algorithm, then reset $\text{LC} = \text{LC}(\Delta q)$ and $\text{MI} = \text{MI}(\Delta q)$ for calculating CER. Unsurprisingly, the liquidity-aware investor always has a positive CER, while the CER of the liquidity-blind investor can reach negative territory in illiquid markets. The massive gain in CER of liquidity-aware portfolio allocation over liquidity-blind portfolio allocation illustrates how taking these costs into account is vital for reaching one's performance target in real life situations where these liquidity effects do occur. It also illustrates the ability of our algorithm to properly cope with intermediate costs.

5.4 Sensitivity analysis

Table 5 reports the sensitivity of CER and of the initial stock allocation α_0 with respect to the daily trading volume. The effect of increasing daily trading volume (and therefore increasing liquidity, cf. equations (4.1)-(4.2)) on CER and α_0 is consistent under different levels of daily volatility: CER and α_0 increase at diminishing rates. Similarly, Table 6 shows that increasing daily volatility (and therefore decreasing liquidity, cf. equations (4.1)-(4.2)) decreases CER and α_0 at diminishing rates.

Table 7 reports the sensitivity of CER and the initial stock allocation α_0 with respect to the initial investment amount W_{0-} . As expected, the CER and α_0 decrease with respect to W_{0-} due to the larger liquidity effects on bigger portfolios. Under extreme liquidity effects $(\sigma_{\text{day}}, \text{Vol}_{\text{day}}) = (12.5, 12m)$, α_0 remains zero for $W_{0-} > 600m$, meaning that the quasi impossibility to rebalance the portfolio makes a full risk-free allocation the best initial investment in terms of expected utility.

Table 8 reports the sensitivity of CER and α_0 with respect to the investment horizon. The initial allocations quickly converge to a certain level when the time horizon is increased: they decrease towards this limit when liquidity effect is small ($\sigma_{\text{day}} = 2.5, \text{Vol}_{\text{day}} = 120m$) while they increase towards this limit in the two other cases ($\sigma_{\text{day}} = 7.5, \text{Vol}_{\text{day}} = 55m$ and $\sigma_{\text{day}} = 12.5, \text{Vol}_{\text{day}} = 12m$). The CER first increases then decreases with time horizon for the two cases $\sigma_{\text{day}} = 2.5, \text{Vol}_{\text{day}} = 120m$ and $\sigma_{\text{day}} = 7.5, \text{Vol}_{\text{day}} = 55m$ while monotonically increases when liquidity effect is large ($\sigma_{\text{day}} = 12.5, \text{Vol}_{\text{day}} = 12m$).

Finally, Table 9 reports the sensitivity of CER and α_0 with respect to the risk-aversion level γ of the CRRA utility. As expected, CER and α_0 both decrease under every liquidity situation when risk-aversion is increased.

To conclude this numerical section, we can emphasize that a key feature of the general portfolio allocation algorithm proposed in this paper is the ability to measure and account for the effect of imperfect liquidity on dynamic portfolio allocation. After having validated the stability and convergence of the algorithm, we were able to compute and report the sensitivities of portfolio allocation and portfolio performance with respect to various parameters. Such analyzes can be adapted to different models, markets and investment styles, and bring insights into the most advantageous way to adjust dynamic portfolio allocation in less liquid markets.

6 Conclusion

This paper describes a simulation-and-regression method for solving portfolio allocation problems with general transaction costs, temporary liquidity costs and permanent market impacts. To deal with permanent market impacts, we model the price dynamics as endogenous state variables which are separate from the exogenous return dynamics, while maintain the same computational complexity of the algorithm with these additional endogenous variables. The simulation nature of the chosen algorithm makes it suitable for multivariate portfolios with realistic asset dynamics and realistic liquidity effects. The algorithm adapts [Kharroubi et al. \(2014\)](#)'s control randomization approach to the discrete portfolio allocation. For each allocation level, the endogenous state variables are correspondingly updated and are used to estimate the value function by a simple linear least-squares regression. We iterate the whole algorithm by using the optimal control estimates of the first run as the initial controls of the second run. Our numerical tests show that, with second-order polynomial basis, the proposed control discretization combined with global control iteration outperforms the control regression approach of [Kharroubi et al. \(2014\)](#), all the more so with highly nonlinear utility functions (high risk-aversion).

We apply our method to solve a realistic cash-and-stock portfolio with the power-law liquidity model of [Almgren et al. \(2005\)](#). We show that the losses associated with ignoring liquidity effects can be substantial, indicating the necessity to account for liquidity effects when making portfolio allocation decisions in real markets. Most importantly, our algorithm is able to protect the portfolio value in illiquid markets. Going further, we analyze the sensitivities of certainty equivalent returns and optimal allocations with respect to trading volume, stock price volatility, initial capital, risk-aversion level and investment horizon.

The flexibility of the algorithm motivates future studies to investigate alternative portfolio performance measures beyond expected utility, alternative liquidity models, or to incorporate additional features such as cross-asset price impact. It could also be easily adapted to the problems of optimal portfolio liquidation and more general optimal switching problems with endogenous uncertainty.

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Table 1: Return predictors (exogenous state variables)

Return Predictors	ETF Name	ETF Ticker
U.S. stock	SPDR S&P 500 ETF	SPY
U.S. bond	Vanguard Total Bond Market ETF	BND
International stock	iShares MSCI EAFE ETF	EFA
Emerging market stock	iShares MSCI Emerging Markets ETF	EEM
Gold	SPDR Gold Shares ETF	GLD
International bond	SPDR Barclays International Treasury Bond ETF	BWX
Silver	iShares Silver Trust ETF	SLV
Crude oil	U.S. Oil ETF	USO
U.S. dollar	PowerShares Deutsche Bank U.S. Dollar Bullish ETF	UUP
Euro	CurrencyShares Euro ETF	FXE
Japanese Yen	CurrencyShares Japanese Yen ETF	FXJ
Australian dollar	CurrencyShares Australian dollar ETF	FXA

Table 2: Monte Carlo convergence with respect to risk aversion level and time horizon

		$\gamma = 5$					$\gamma = 15$				
		KLP	$I = 0$	$I = 1$	$I = 2$	$I = 3$	KLP	$I = 0$	$I = 1$	$I = 2$	$I = 3$
$N = 5$	$M=10^3$	1.45	1.54	1.55	1.58	1.58	1.02	1.19	1.25	1.31	1.34
	$M=10^4$	1.54	1.57	1.58	1.58	1.58	1.15	1.36	1.38	1.39	1.39
	$M=10^5$	1.54	1.58	1.58	1.58	1.58	1.18	1.38	1.39	1.39	1.39
	$M=10^6$	1.55	1.58	1.58	1.58	1.58	1.24	1.38	1.39	1.39	1.39
						CER* = 1.58					CER* = 1.39
$N = 15$	$M=10^3$	0.85	1.37	1.40	1.42	1.43	0.03	0.59	0.92	1.06	1.13
	$M=10^4$	1.33	1.49	1.51	1.52	1.53	0.45	1.01	1.13	1.20	1.25
	$M=10^5$	1.47	1.51	1.52	1.53	1.53	0.84	1.18	1.31	1.33	1.35
	$M=10^6$	1.48	1.52	1.53	1.53	1.53	1.04	1.32	1.36	1.37	1.37
						CER* = 1.53					CER* = 1.37

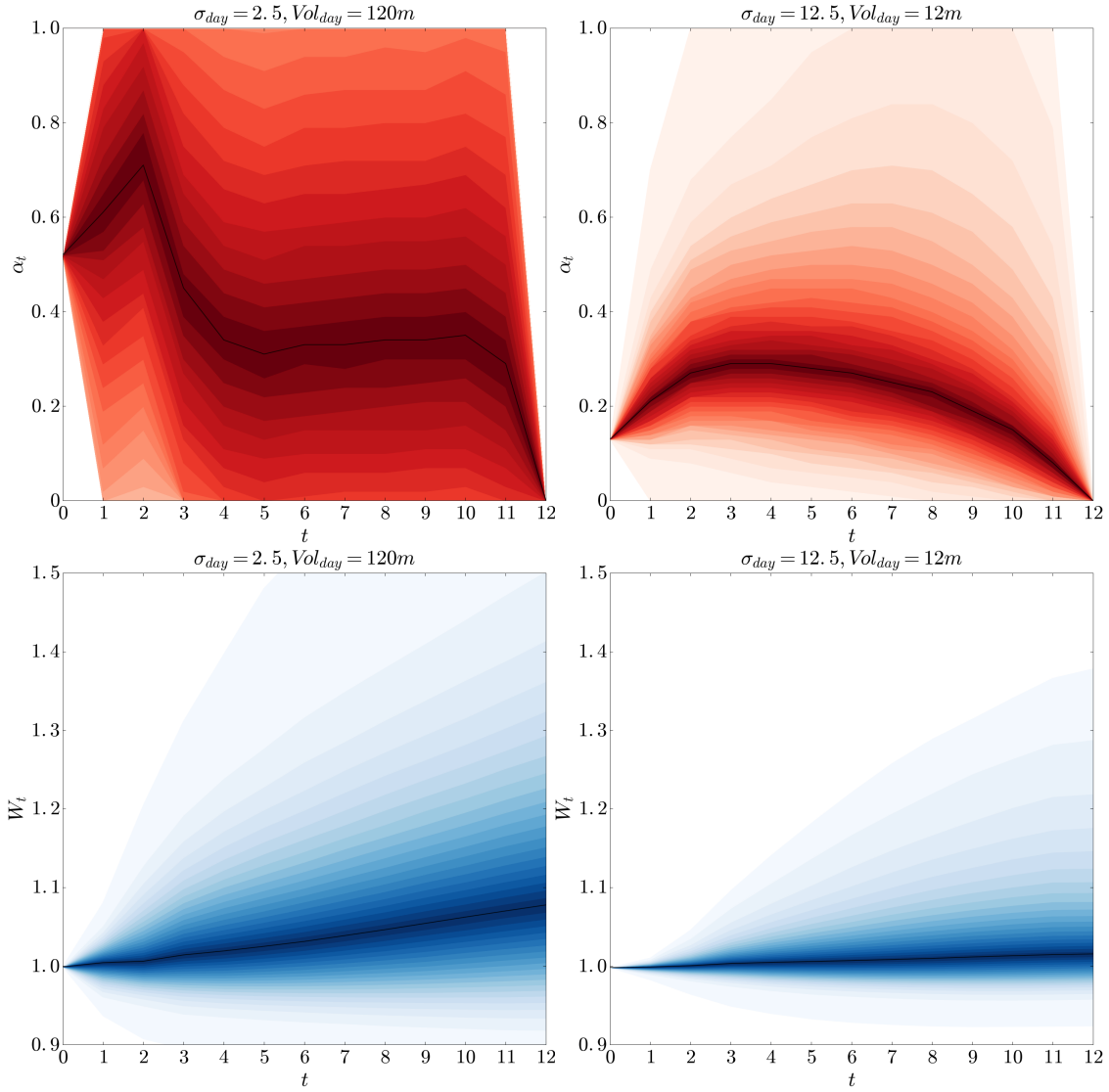
This table compares the Monte Carlo convergence of Algorithm 2 with the control regression algorithm of [Kharroubi et al. \(2014\)](#) ('KLP') with second-order polynomial basis. $I = 0, 1, 2, 3$ denotes the number of additional control iterations, and 'CER*' stands for the best estimate. We assume a CARA utility investor, i.e., $U(w) = -\exp(-\gamma w)$, a cash and stock portfolio with no switching costs, risk-free rate 0.012, and stock annual return with mean 0.03 and volatility 0.15. The annually adjusted certainty equivalent returns (in percentage points) are reported for different Monte Carlo sample sizes ($M = 10^3, 10^4, 10^5, 10^6$), different investment horizons ($N = 5\text{yrs}, 15\text{yrs}$) and different risk-aversion parameters ($\gamma = 5, 15$).

Table 3: Monte Carlo convergence under liquidity cost and market impact

		Vol _{day} = 120m					Vol _{day} = 55m					Vol _{day} = 12m				
		KLP	I = 0	I = 1	I = 2	I = 3	KLP	I = 0	I = 1	I = 2	I = 3	KLP	I = 0	I = 1	I = 2	I = 3
$\sigma_{\text{day}} = 2.5$	M=10 ³	33.3	55.0	56.9	57.7	57.8	31.7	56.9	58.7	58.7	58.5	21.9	41.6	42.6	43.6	43.6
	M=10 ⁴	55.1	56.5	57.5	57.7	57.7	51.9	54.8	54.9	54.9	54.9	42.0	42.7	44.0	44.0	44.0
	M=10 ⁵	57.0	57.5	57.8	57.8	57.8	53.5	54.1	54.2	54.2	54.2	42.7	43.5	44.2	44.2	44.2
	M=10 ⁶	57.5	57.5	57.9	57.9	57.9	54.1	54.2	54.5	54.5	54.5	43.7	44.1	44.5	44.5	44.5
$\sigma_{\text{day}} = 7.5$	M=10 ³	21.3	47.0	47.6	47.6	47.6	20.3	39.7	41.5	41.5	41.5	17.9	28.2	29.4	29.4	29.5
	M=10 ⁴	45.4	47.2	47.6	47.6	47.6	39.7	40.2	41.7	41.7	41.7	28.1	28.6	29.6	29.6	29.6
	M=10 ⁵	47.1	47.4	47.6	47.6	47.6	40.4	41.0	41.8	41.8	41.8	28.6	29.0	29.9	29.9	29.9
	M=10 ⁶	47.4	47.5	47.9	47.9	47.9	41.0	41.6	42.0	42.0	42.0	29.0	29.2	30.0	30.0	30.0
$\sigma_{\text{day}} = 12.5$	M=10 ³	20.0	39.9	40.4	40.9	41.2	20.4	31.0	33.5	33.7	34.1	12.8	19.0	23.5	23.7	23.7
	M=10 ⁴	39.2	40.4	41.1	41.1	41.2	33.0	32.2	33.9	34.4	35.1	20.3	20.7	23.6	23.7	23.8
	M=10 ⁵	40.5	40.8	41.2	41.2	41.2	33.5	33.6	35.1	35.1	35.1	21.6	21.9	24.0	24.1	24.1
	M=10 ⁶	41.0	41.1	41.3	41.3	41.3	33.9	33.9	35.1	35.2	35.2	22.0	22.1	24.1	24.2	24.2

This table compares the Monte Carlo convergence of the monthly adjusted certainty equivalent return (in basis points) for the CRRA utility with $\gamma = 5$ and investment horizon $N = 12$ months, under different daily volatilities ($\sigma_{\text{day}} = 2.5, 7.5, 12.5$) and different daily trading volumes (Vol_{day} = 120m, 55m, 12m), where the Monte Carlo procedure is performed under different sizes of Monte Carlo sample ($M = 10^3, 10^4, 10^5, 10^6$) and different iterations ($I = 0, 1, 2, 3$). A portfolio of cash and SPDR S&P 500 ETF is investigated, with annual risk free rate $r^f = 0.012$, portfolio weight increment 0.01 and initial investment amount $W_{0-} = \$100m$.

Figure 1: Time evolution of the distribution of the control and wealth



This figure shows the time evolution of the distribution of the **portfolio allocation (top panel)** and **wealth (bottom panel)** for a CRRA utility with $\gamma = 5$ and investment horizon $N = 12$ months under different liquidity effects $(\sigma_{day}, Vol_{day}) = (2.5, 120m), (12.5, 12m)$, using $M = 10^5$ Monte Carlo simulations with one control iteration $I = 1$. A portfolio of cash and SPDR S&P 500 ETF is investigated, with annual risk free rate $r^f = 0.012$, portfolio weight increment 0.01 and initial investment amount $W_{0-} = \$100m$.

Table 4: Certainty equivalent losses with ignoring liquidity effects

		$\gamma = 5$						$\gamma = 10$					
		Liquidity-aware			Liquidity-blind			Liquidity-aware			Liquidity-blind		
Vol_{day}	W_{0-}	$\sigma_{\text{day}}=2.5$	$\sigma_{\text{day}}=7.5$	$\sigma_{\text{day}}=12.5$	$\sigma_{\text{day}}=2.5$	$\sigma_{\text{day}}=7.5$	$\sigma_{\text{day}}=12.5$	$\sigma_{\text{day}}=2.5$	$\sigma_{\text{day}}=7.5$	$\sigma_{\text{day}}=12.5$	$\sigma_{\text{day}}=2.5$	$\sigma_{\text{day}}=7.5$	$\sigma_{\text{day}}=12.5$
120m	0.1b	57.8	47.6	41.2	56.6	38.5	21.2	37.7	31.6	27.8	36.2	25.4	15.3
	0.5b	49.2	35.3	28.5	41.8	-2.4	-41.0	32.6	24.2	20.1	27.4	1.7	-19.9
	1.0b	44.2	29.9	24.0	29.9	-32.7	-82.6	29.6	20.9	16.4	20.4	-15.3	-42.9
55m	0.1b	54.2	41.8	35.1	51.1	23.0	-3.2	35.5	28.2	24.1	32.9	16.3	1.2
	0.5b	43.5	29.3	23.5	28.1	-37.2	-88.4	29.2	20.4	16.0	19.3	-17.8	-46.1
	1.0b	38.1	24.5	19.5	-2.0	-77.9	-137.3	25.9	16.8	5.7	8.7	-40.3	-73.9
12m	0.1b	44.2	29.9	24.0	29.9	-32.7	-82.6	29.6	20.9	16.4	20.4	-15.3	-42.9
	0.5b	31.6	19.6	14.1	-22.0	-135.6	-197.3	21.9	12.6	9.1	-9.3	-72.9	-112.1
	1.0b	26.5	15.6	7.3	-58.7	-186.0	-239.8	18.5	7.4	2.1	-29.7	-104.3	-150.9

This table compares the monthly adjusted certainty equivalent return (in basis points) for two CRRA investor with $\gamma = 5, 10$ and investment horizon $N = 12$ months: the first one takes heed of liquidity effects (liquidity-aware) while the second one ignores liquidity effects (liquidity-blind). The results are compared under different daily volatilities ($\sigma_{\text{day}} = 2.5, 7.5, 12.5$), different daily trading volumes ($\text{Vol}_{\text{day}} = 120m, 55m, 12m$), and different initial investment amount ($W_{0-} = \$100m, 500m, 1b$), using $M = 10^5$ Monte Carlo simulations with one control iteration $I = 1$. A portfolio of cash and SPDR S&P 500 ETF is investigated, with annual risk free rate $r^f = 0.012$, and portfolio weight increment 0.01.

Table 5: Sensitivity to daily trading volatility

Vol _{day}	CER			Initial allocation α_0		
	$\sigma_{\text{day}} = 2.5$	$\sigma_{\text{day}} = 7.5$	$\sigma_{\text{day}} = 12.5$	$\sigma_{\text{day}} = 2.5$	$\sigma_{\text{day}} = 7.5$	$\sigma_{\text{day}} = 12.5$
10m	42.8	28.6	22.9	0.31	0.18	0.12
20m	47.9	33.8	27.4	0.36	0.23	0.16
30m	50.6	37.0	30.4	0.41	0.26	0.19
40m	52.4	39.3	32.6	0.42	0.27	0.21
50m	53.7	41.1	34.3	0.45	0.30	0.23
60m	54.7	42.5	35.8	0.45	0.31	0.25
70m	55.4	43.7	37.0	0.46	0.32	0.26
80m	56.1	44.7	38.0	0.48	0.33	0.27
90m	56.6	45.5	39.0	0.50	0.34	0.27
100m	57.1	46.3	39.0	0.51	0.34	0.28
110m	57.5	47.0	40.6	0.52	0.35	0.29
120m	57.8	47.6	41.1	0.53	0.37	0.29

This table reports the sensitivity of the monthly adjusted certainty equivalent return (in basis points) and the initial stock allocation with respect to the daily trading volume $\text{Vol}_{\text{day}} = 10m, 20m, \dots, 120m$, for a CRRA utility with $\gamma = 5$ and investment horizon $N = 12$ months under different daily volatilities ($\sigma_{\text{day}} = 2.5, 7.5, 12.5$), using $M = 10^5$ Monte Carlo simulations with one control iteration $I = 1$. A portfolio of cash and SPDR S&P 500 ETF is investigated, with annual risk free rate $r^f = 0.012$, portfolio weight increment 0.01 and initial investment amount $W_{0-} = \$100m$.

Table 6: Sensitivity to daily volatility

σ_{day}	CER			Initial allocation α_0		
	Vol _{day} = 120m	Vol _{day} = 55m	Vol _{day} = 12m	Vol _{day} = 120m	Vol _{day} = 55m	Vol _{day} = 12m
2	59.2	56.1	46.9	0.55	0.48	0.35
3	56.5	52.5	41.8	0.50	0.44	0.30
4	54.2	49.5	38.1	0.45	0.39	0.25
5	52.1	46.9	35.1	0.42	0.36	0.24
6	50.2	44.6	32.7	0.40	0.33	0.22
7	48.4	42.7	30.8	0.39	0.31	0.20
8	46.9	41.0	29.1	0.35	0.29	0.18
9	45.4	39.4	27.7	0.34	0.28	0.17
10	44.1	38.0	26.5	0.33	0.27	0.16
11	42.9	36.7	25.4	0.31	0.26	0.14
12	41.8	35.6	24.5	0.31	0.25	0.14
13	40.7	34.6	23.6	0.30	0.23	0.12

This table reports the sensitivity of the monthly adjusted certainty equivalent return (in basis points) and the initial stock allocation with respect to the daily volatility $\sigma_{\text{day}} = 2, 3, \dots, 13$, for a CRRA utility with $\gamma = 5$ and investment horizon $N = 12$ months under different daily trading volume ($\text{Vol}_{\text{day}} = 120m, 55m, 12m$), using $M = 10^5$ Monte Carlo simulations with one control iteration $I = 1$. A portfolio of cash and SPDR S&P 500 ETF is investigated, with annual risk free rate $r^f = 0.012$, portfolio weight increment 0.01 and initial investment amount $W_{0-} = \$100m$.

Table 7: Sensitivity to Investment Amount

W_{0-}	σ_{day} Vol_{day}	CER			Initial allocation α_0		
		2.5	7.5	12.5	2.5	7.5	12.5
		120m	55m	12m	120m	55m	12m
\$100m		57.8	41.8	24.0	0.52	0.31	0.13
\$200m		54.7	36.3	19.9	0.47	0.25	0.09
\$300m		52.4	33.1	17.8	0.43	0.22	0.07
\$400m		50.6	30.9	16.1	0.40	0.20	0.04
\$500m		49.2	29.3	14.1	0.38	0.19	0.02
\$600m		47.9	28.0	12.0	0.36	0.18	0.01
\$700m		46.8	26.9	10.8	0.35	0.16	0.00
\$800m		45.8	25.9	10.3	0.34	0.15	0.00
\$900m		45.0	25.2	8.3	0.33	0.14	0.00
\$1b		44.2	24.5	7.3	0.32	0.13	0.00

This table reports the sensitivity of the monthly adjusted certainty equivalent return (in basis points) and the initial stock allocation with respect to the initial investment amount $W_{0-} = \$100, 200, \dots, 1000m$, for a CRRA utility with $\gamma = 5$ and investment horizon $N = 12$ months under different liquidity settings $(\sigma_{\text{day}}, \text{Vol}_{\text{day}}) = (2.5, 120m), (7.5, 55m), (12.5, 12m)$, using $M = 10^5$ Monte Carlo simulations with one control iteration $I = 1$. A portfolio of cash and SPDR S&P 500 ETF is investigated, with annual risk free rate $r^f = 0.012$, portfolio weight increment 0.01.

Table 8: Sensitivity to investment horizon

N	σ_{day} Vol_{day}	CER			Initial allocation α_0		
		2.5	7.5	12.5	2.5	7.5	12.5
		120m	55m	12m	120m	55m	12m
2		54.0	26.3	12.3	0.60	0.24	0.04
3		67.0	39.8	16.0	0.58	0.30	0.07
4		65.8	43.3	18.6	0.57	0.31	0.09
5		63.7	43.8	20.4	0.55	0.31	0.11
6		62.0	43.5	21.6	0.55	0.31	0.12
7		60.8	43.3	22.4	0.54	0.31	0.13
8		60.1	42.9	23.0	0.54	0.31	0.13
9		59.4	42.6	23.4	0.53	0.31	0.13
10		58.7	42.3	23.6	0.53	0.31	0.13
11		58.2	42.0	23.8	0.53	0.31	0.13
12		57.8	41.8	24.0	0.53	0.31	0.13

This table reports the sensitivity of the monthly adjusted certainty equivalent return (in basis points) and the initial stock allocation with respect to the investment horizon $N = 2, 3, \dots, 12$ months, for a CRRA utility with $\gamma = 5$ under different liquidity settings $(\sigma_{\text{day}}, \text{Vol}_{\text{day}}) = (2.5, 120m), (7.5, 55m), (12.5, 12m)$, using $M = 10^5$ Monte Carlo simulations with one control iteration $I = 1$. A portfolio of cash and SPDR S&P 500 ETF is investigated, with annual risk free rate $r^f = 0.012$, portfolio weight increment 0.01 and initial investment amount $W_{0-} = \$100m$.

Table 9: Sensitivity to risk-aversion level

γ	σ_{day} Vol_{day}	CER			Initial allocation α_0		
		2.5 120m	7.5 55m	12.5 12m	2.5 120m	7.5 55m	12.5 12m
2		82.2	61.8	34.0	1.00	0.52	0.23
3		72.6	53.4	29.5	0.81	0.42	0.19
4		64.5	46.8	26.4	0.66	0.35	0.16
5		57.8	41.8	24.0	0.53	0.31	0.13
6		52.3	37.9	22.1	0.45	0.27	0.11
7		47.6	34.7	20.4	0.38	0.23	0.10
8		43.8	32.1	18.9	0.33	0.22	0.09
9		40.5	30.0	17.8	0.29	0.20	0.08
10		37.7	28.2	16.4	0.27	0.18	0.08

This table reports the sensitivity of the monthly adjusted certainty equivalent return (in basis points) and the initial stock allocation with respect to the risk-aversion level $\gamma = 2, 3, \dots, 10$ for a CRRA utility for investment horizon $N = 12$ months, under different liquidity settings $(\sigma_{\text{day}}, \text{Vol}_{\text{day}}) = (2.5, 120m), (7.5, 55m), (12.5, 12m)$, using $M = 10^5$ Monte Carlo simulations with one control iteration $I = 1$. A portfolio of cash and SPDR S&P 500 ETF is investigated, with annual risk free rate $r^f = 0.012$, portfolio weight increment 0.01 and initial investment amount $W_{0-} = \$100m$.