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BRANCHED PROJECTIVE STRUCTURES, BRANCHED $\mathrm{SO}(3, \mathbb{C})$ -OPERS AND LOGARITHMIC CONNECTIONS ON JET BUNDLE

INDRANIL BISWAS AND SORIN DUMITRESCU

ABSTRACT. We study the branched holomorphic projective structures on a compact Riemann surface X with a fixed branching divisor $S = \sum_{i=1}^d x_i$, where $x_i \in X$ are distinct points. After defining branched $\mathrm{SO}(3, \mathbb{C})$ -opers, we show that the branched holomorphic projective structures on X are in a natural bijection with the branched $\mathrm{SO}(3, \mathbb{C})$ -opers singular at S . It is deduced that the branched holomorphic projective structures on X are also identified with a subset of the space of all logarithmic connections on $J^2((TX) \otimes \mathcal{O}_X(S))$ singular over S , satisfying certain natural geometric conditions.

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1. INTRODUCTION

The role of the space of holomorphic projective structures in the understanding of the uniformization theorem of Riemann surfaces was emphasized by many authors (see, for instance, [Gu, St] and references therein). Recall that a holomorphic projective structure on a Riemann surface X is given by (the equivalence class of) an atlas $\{(U_i, \phi_i)\}_{i \in I}$ with local charts $X \supset U_i \xrightarrow{\phi_i} \mathbb{CP}^1$, $i \in I$, such that all the transition maps $\phi_j \circ \phi_i^{-1}$ are restrictions of elements in the Möbius group $\mathrm{PGL}(2, \mathbb{C})$ of complex projective transformations (see Section 2.1 for more details).

Historically, the idea of using holomorphic projective structures to prove uniformization theorem for Riemann surfaces came from the study of second-order linear differential equations (see, [St, Chapter VIII]). The modern point of view summarizes this equivalence between holomorphic projective connections and second-order linear differential equations as an identification of the space of holomorphic projective connections with the space of $\mathrm{PGL}(2, \mathbb{C})$ -opers (see [De, Section I.5] and [BeDr]).

A more general notion is that of a branched holomorphic projective structure which was introduced and studied by Mandelbaum in [Ma1, Ma2]; more recently, a general notion of (non necessarily flat) branched Cartan geometry on complex manifolds was introduced and studied in [BiDu]. A branched holomorphic projective structure is defined by (the equivalence class of) a holomorphic atlas with the local charts being finite branched coverings of open subsets in \mathbb{CP}^1 , while the transition maps are restrictions of elements in the Möbius group $\mathrm{PGL}(2, \mathbb{C})$ (see Section 5.1 for details). The branching divisor S of a branched projective structure on X is the union of points in X where the local projective charts admit a ramification point. The Riemann surface $X \setminus S$ inherits a holomorphic projective structure in the classical sense. Branched holomorphic projective structures play an important role in the study of hyperbolic metrics with conical singularities on Riemann surfaces or in the study of codimension one transversally projective holomorphic foliations (see, for example, [CDF]).

From a global geometric point of view, a holomorphic projective structure over a Riemann surface X is known to give a flat \mathbb{CP}^1 -bundle over X together with a holomorphic section which is transverse to the horizontal distribution defining the flat connection on the bundle. For a branched holomorphic projective structure with branching divisor S , we have a similar description, but the section of the projective bundle fails to be transverse to the horizontal distribution (defining the flat structure) precisely at points in S (see, for example, [BDG, CDF, GKM, LM] or Section 5.1 here).

In this article we study the branched holomorphic projective structures on a Riemann surface X with fixed branching divisor S . Throughout, we assume that S is reduced, meaning that $S := \sum_{i=1}^d x_i$, with $x_i \in X$, distinct points.

The presentation is organized in the following way. Sections 2, 3 and 4 deal with the geometry of classical holomorphic projective connections and Sections 5, 6 and 7 are about branched holomorphic projective connections. More precisely, Section 2 presents the geometrical setting of a holomorphic projective structure on a Riemann surface X and proves that a projective structure induces a holomorphic connection on the rank three holomorphic 2-jet bundle $J^2(TX)$. Moreover this connection has a natural geometric behavior: it is an oper connection (see Definition 3.2) with respect to the canonical filtration of $J^2(TX)$ induced by the kernels of the natural forgetful projections of jet bundles $J^2(TX) \rightarrow J^1(TX)$ and $J^2(TX) \rightarrow TX$. In Section 3 we identify the space of holomorphic projective structures on the Riemann surface X with the space of oper connections on $J^2(TX)$ satisfying some extra natural geometrical properties (see Corollary 3.6). Section 4 translates the previous equivalence as an identification of the space of holomorphic projective connections with the space of $\mathrm{SO}(3, \mathbb{C})$ -opers.

Section 5 starts the study of branched holomorphic projective structures. It is shown that a branched projective structure on a Riemann surface X with branching divisor S gives rise to a logarithmic connection on the rank two 2-jet bundle $J^2((TX) \otimes \mathcal{O}_X(S))$ singular over S , with residues at S satisfying certain natural geometric conditions with respect to the canonical filtration of $J^2((TX) \otimes \mathcal{O}_X(S))$ (see Proposition 5.1).

The main results proved here are obtained in Section 6 and in Section 7.

In Section 6.1 we introduce the notion of a branched $\mathrm{SO}(3, \mathbb{C})$ -oper singular at the divisor S . We show that the space \mathcal{P}_S of branched holomorphic projective structures with fixed branching divisor S is naturally identified with the space of branched $\mathrm{SO}(3, \mathbb{C})$ -opers singular at S (see Theorem 6.7).

We deduce that this space \mathcal{P}_S also coincides with a subset of the set of all logarithmic connections with singular locus S , satisfying certain natural geometric conditions, on the rank three holomorphic 2-jet bundle $J^2((TX) \otimes \mathcal{O}_X(S))$ (see Proposition 6.8 and Theorem 7.1).

The above mentioned Theorem 7.1 generalizes the main result in [BDG] (Theorem 5.1) where, under the additional assumption that the degree d of S is even and such that $d \neq 2g - 2$ (with g the genus of X), it was proved that \mathcal{P}_S coincides with a subset of the set of all logarithmic connections with singular locus S , satisfying certain geometric conditions, on the rank two holomorphic jet bundle $J^1(Q)$, where Q is a fixed holomorphic line bundle on X such that $Q^{\otimes 2} = TX \otimes \mathcal{O}_X(S)$. It may be mentioned that for a branching divisor S of general degree d , the bundle Q considered in [BDG] does not exist.

Let us clarify an important improvement in the methods developed in this work with respect to those in [BDG]. The $\mathrm{PSL}(2, \mathbb{C})$ -monodromy of a (unbranched) holomorphic projective structure on a compact Riemann surface X always admits a lift to $\mathrm{SL}(2, \mathbb{C})$ (see [GKM, Lemma 1.3.1]). Geometrically this means that the associated flat $\mathbb{C}\mathbb{P}^1$ -bundle over X is the projectivization of a rank two vector bundle over X ; hence one can work in the

set-up of rank two vector bundles. This is not true anymore for branched holomorphic projective structures if the degree of the branching divisor is odd; for this reason the results and methods in [BDG] cannot be extended to the case where the branching divisor of the branched holomorphic projective structure is of odd degree.

Here we consider $\mathrm{SO}(3, \mathbb{C})$ -opers instead of the equivalent $\mathrm{PSL}(2, \mathbb{C})$ -opers and we develop the notion of branched $\mathrm{SO}(3, \mathbb{C})$ -opers. This enables us to investigate branched holomorphic projective structure in the framework of rank three holomorphic (2-jet) vector bundles instead of the projective bundles.

2. PROJECTIVE STRUCTURE AND SECOND JET OF TANGENT BUNDLE

2.1. Projective structure. The multiplicative group of nonzero complex numbers will be denoted by \mathbb{C}^* . Let \mathbb{V} be a complex vector space of dimension two. Let $\mathbb{P}(\mathbb{V})$ denote the projective line that parametrizes all one-dimensional subspaces of \mathbb{V} . Consider the projective linear group $\mathrm{PGL}(\mathbb{V}) := \mathrm{GL}(\mathbb{V})/(\mathbb{C}^* \cdot \mathrm{Id}_{\mathbb{V}}) = \mathrm{SL}(\mathbb{V})/(\pm \mathrm{Id}_{\mathbb{V}})$. The action of $\mathrm{GL}(\mathbb{V})$ on \mathbb{V} produces an action of $\mathrm{PGL}(\mathbb{V})$ on $\mathbb{P}(\mathbb{V})$. This way $\mathrm{PGL}(\mathbb{V})$ gets identified with the group of all holomorphic automorphisms of $\mathbb{P}(\mathbb{V})$.

Let \mathbb{X} be a connected Riemann surface. A holomorphic coordinate function on \mathbb{X} is a pair (U, ϕ) , where $U \subset \mathbb{X}$ is an open subset and

$$\phi : U \longrightarrow \mathbb{P}(\mathbb{V})$$

is a holomorphic embedding. A holomorphic coordinate atlas on \mathbb{X} is a family of holomorphic coordinate functions $\{(U_i, \phi_i)\}_{i \in I}$ such that $\bigcup_{i \in I} U_i = \mathbb{X}$. So $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$ is a biholomorphic map for every $i, j \in I$ with $U_i \cap U_j \neq \emptyset$.

A projective structure on \mathbb{X} is given by a holomorphic coordinate atlas $\{(U_i, \phi_i)\}_{i \in I}$ such that for all ordered pairs $i, j \in I$, with $U_i \cap U_j \neq \emptyset$, and every connected component $U \subset U_i \cap U_j$, there is an element $G_{j,i}^U \in \mathrm{PGL}(\mathbb{V})$ satisfying the condition that the biholomorphic map

$$\phi_j \circ \phi_i^{-1} : \phi_i(U) \longrightarrow \phi_j(U)$$

is the restriction, to $\phi_i(U)$, of the automorphism of $\mathbb{P}(\mathbb{V})$ given by of $G_{j,i}^U$. Note that $G_{j,i}^U$ is uniquely determined by $\phi_j \circ \phi_i^{-1}$.

Two holomorphic coordinate atlases $\{(U_i, \phi_i)\}_{i \in I}$ and $\{(U'_i, \phi'_i)\}_{i \in J}$ satisfying the above condition are called *equivalent* if their union $\{(U_i, \phi_i)\}_{i \in I} \cup \{(U'_i, \phi'_i)\}_{i \in J}$ also satisfies the above condition. A *projective structure* on \mathbb{X} is an equivalence class of atlases satisfying the above condition; see [Gu], [He], [GKM] for projective structures.

Let $\gamma : \mathbf{P} \rightarrow \mathbb{X}$ be a holomorphic $\mathbb{C}\mathbb{P}^1$ -bundle over \mathbb{X} . In other words, γ is a surjective holomorphic submersion such that each fiber of it is holomorphically isomorphic to the complex projective line $\mathbb{C}\mathbb{P}^1$. Let

$$T_\gamma := \ker(d\gamma) \subset T\mathbf{P} \tag{2.1}$$

be the relative holomorphic tangent bundle, where $T\mathbf{P}$ is the holomorphic tangent bundle of \mathbf{P} . A *holomorphic connection* on \mathbf{P} is a holomorphic line subbundle $\mathcal{H} \subset T\mathbf{P}$ such that

the natural homomorphism

$$T_\gamma \oplus \mathcal{H} \longrightarrow T\mathbf{P} \quad (2.2)$$

is an isomorphism; see [At].

Let $\mathcal{H} \subset T\mathbf{P}$ be a holomorphic connection on \mathbf{P} . Let $s : \mathbb{X} \longrightarrow \mathbf{P}$ be a holomorphic section of γ , meaning $\gamma \circ s = \mathrm{Id}_{\mathbb{X}}$. Consider the differential ds of s

$$T\mathbb{X} \longrightarrow s^*T\mathbf{P} = (s^*T_\gamma) \oplus (s^*\mathcal{H}),$$

where the decomposition is the pullback of the decomposition in (2.2). Let

$$\widehat{ds} : T\mathbb{X} \longrightarrow s^*T_\gamma \quad (2.3)$$

be the homomorphism obtained by composing it with the natural projection $(s^*T_\gamma) \oplus (s^*\mathcal{H}) \longrightarrow s^*T_\gamma$.

Giving a projective structure on \mathbb{X} is equivalent to giving a triple (γ, \mathcal{H}, s) , where

- $\gamma : \mathbf{P} \longrightarrow \mathbb{X}$ is a holomorphic $\mathbb{C}\mathbb{P}^1$ -bundle,
- $\mathcal{H} \subset T\mathbf{P}$ is a holomorphic connection, and
- $s : \mathbb{X} \longrightarrow \mathbf{P}$ is a holomorphic section of γ ,

such that the homomorphism \widehat{ds} in (2.3) is an isomorphism. More details on this can be found in [Gu].

2.2. Jet bundles. We briefly recall the definition of a jet bundle of a holomorphic vector bundle on \mathbb{X} . Let

$$p_j : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}, \quad j = 1, 2,$$

be the natural projection to the j -th factor. Let

$$\Delta := \{(x, x) \in \mathbb{X} \times \mathbb{X} \mid x \in \mathbb{X}\} \subset \mathbb{X} \times \mathbb{X}$$

be the reduced diagonal divisor. For a holomorphic vector bundle W on \mathbb{X} , and any integer $k \geq 0$, define the k -th order jet bundle

$$J^k(W) := p_{1*}((p_2^*W)/(p_2^*W \otimes \mathcal{O}_{\mathbb{X} \times \mathbb{X}}(-(k+1)\Delta))) \longrightarrow \mathbb{X}.$$

The natural inclusion of $\mathcal{O}_{\mathbb{X} \times \mathbb{X}}(-(k+1)\Delta)$ in $\mathcal{O}_{\mathbb{X} \times \mathbb{X}}(-k\Delta)$ produces a surjective homomorphism $J^k(W) \longrightarrow J^{k-1}(W)$. This way we obtain a short exact sequence of holomorphic vector bundles on \mathbb{X}

$$0 \longrightarrow K_{\mathbb{X}}^{\otimes k} \otimes W \longrightarrow J^k(W) \longrightarrow J^{k-1}(W) \longrightarrow 0, \quad (2.4)$$

where $K_{\mathbb{X}}$ is the holomorphic cotangent bundle of \mathbb{X} .

For holomorphic vector bundles W and W' on \mathbb{X} , any $\mathcal{O}_{\mathbb{X}}$ -linear homomorphism $W \longrightarrow W'$ induces a homomorphism

$$J^i(W) \longrightarrow J^i(W') \quad (2.5)$$

for every $i \geq 0$.

For holomorphic vector bundles W and W' on \mathbb{X} , any any integer $m \geq 0$, define the sheaf of differential operators of order m from W to W'

$$\mathrm{Diff}_{\mathbb{X}}^m(W, W') := \mathrm{Hom}(J^m(W), W') \longrightarrow \mathbb{X}. \quad (2.6)$$

Using the exact sequence in (2.4) we have the short exact sequence

$$0 \longrightarrow \text{Diff}_{\mathbb{X}}^{m-1}(W, W') \longrightarrow \text{Diff}_{\mathbb{X}}^m(W, W') \xrightarrow{\sigma} (T\mathbb{X})^{\otimes m} \otimes \text{Hom}(W, W') \longrightarrow 0, \quad (2.7)$$

where $T\mathbb{X}$ is the holomorphic tangent bundle of \mathbb{X} . The homomorphism σ in (2.7) is called the symbol map.

Remark 2.1. Consider the short exact sequences

$$0 \longrightarrow K_{\mathbb{X}} \otimes T\mathbb{X} = \mathcal{O}_{\mathbb{X}} \longrightarrow J^1(T\mathbb{X}) \longrightarrow T\mathbb{X} \longrightarrow 0$$

and

$$0 \longrightarrow K_{\mathbb{X}}^{\otimes 2} \otimes T\mathbb{X} = K_{\mathbb{X}} \longrightarrow J^2(T\mathbb{X}) \longrightarrow J^1(T\mathbb{X}) \longrightarrow 0$$

as in (2.4). These two together imply that $\bigwedge^3 J^2(T\mathbb{X}) = K_{\mathbb{X}} \otimes \bigwedge^2 J^1(T\mathbb{X}) = \mathcal{O}_{\mathbb{X}}$. It is straight-forward to check that any for biholomorphism $\beta : \mathbb{X} \longrightarrow \mathbb{Y}$, the homomorphism $J^2(T\mathbb{X}) \longrightarrow \beta^* J^2(T\mathbb{Y})$ corresponding to β takes the section of $\bigwedge^3 J^2(T\mathbb{X}) = \mathcal{O}_{\mathbb{X}}$ given by the constant function 1 on \mathbb{X} to the section of $\bigwedge^3 J^2(T\mathbb{Y}) = \mathcal{O}_{\mathbb{Y}}$ given by the constant function 1 on \mathbb{Y} .

2.3. A third order differential operator. We continue with the set-up of Section 2.1. Let

$$\mathcal{T} := \mathbb{P}(\mathbb{V}) \times H^0(\mathbb{P}(\mathbb{V}), T\mathbb{P}(\mathbb{V})) \longrightarrow \mathbb{P}(\mathbb{V})$$

be the trivial holomorphic vector bundle of rank three over $\mathbb{P}(\mathbb{V})$ with fiber $H^0(\mathbb{P}(\mathbb{V}), T\mathbb{P}(\mathbb{V}))$. For any integer $j \geq 1$, let

$$\psi_j : \mathcal{T} \longrightarrow J^j(T\mathbb{P}(\mathbb{V})) \quad (2.8)$$

be the holomorphic $\mathcal{O}_{\mathbb{P}(\mathbb{V})}$ -linear map that sends any $(x, s) \in \mathbb{P}(\mathbb{V}) \times H^0(\mathbb{P}(\mathbb{V}), T\mathbb{P}(\mathbb{V}))$ to the restriction of the section s to the j -th order infinitesimal neighborhood of the point $x \in \mathbb{P}(\mathbb{V})$.

Lemma 2.2. *The homomorphism ψ_2 in (2.8) is an isomorphism.*

Proof. If $(x, s) \in \text{kernel}(\psi_2(x))$, then

$$s \in H^0(\mathbb{P}(\mathbb{V}), \mathcal{O}_{\mathbb{P}(\mathbb{V})}(-3x) \otimes T\mathbb{P}(\mathbb{V})).$$

But $H^0(\mathbb{P}(\mathbb{V}), \mathcal{O}_{\mathbb{P}(\mathbb{V})}(-3x) \otimes T\mathbb{P}(\mathbb{V})) = 0$, because $\text{degree}(\mathcal{O}_{\mathbb{P}(\mathbb{V})}(-3x) \otimes T\mathbb{P}(\mathbb{V})) < 0$. So the homomorphism ψ_2 is fiberwise injective. This implies that ψ_2 is an isomorphism, because we have $\text{rank}(\mathcal{T}) = \text{rank}(J^2(T\mathbb{P}(\mathbb{V})))$. \square

Lemma 2.3. *There is a canonical holomorphic differential operator δ_0 of order three from $T\mathbb{P}(\mathbb{V})$ to $K_{\mathbb{P}(\mathbb{V})}^{\otimes 2}$. The symbol of δ_0 is the section of*

$$(T\mathbb{P}(\mathbb{V}))^{\otimes 3} \otimes \text{Hom}\left(T\mathbb{P}(\mathbb{V}), K_{\mathbb{P}(\mathbb{V})}^{\otimes 2}\right) = \mathcal{O}_{\mathbb{P}(\mathbb{V})}$$

given by the constant function 1 on $\mathbb{P}(\mathbb{V})$.

Proof. Consider the short exact sequence

$$0 \longrightarrow K_{\mathbb{P}(\mathbb{V})}^{\otimes 3} \otimes T\mathbb{P}(\mathbb{V}) = K_{\mathbb{P}(\mathbb{V})}^{\otimes 2} \xrightarrow{\iota_0} J^3(T\mathbb{P}(\mathbb{V})) \longrightarrow J^2(T\mathbb{P}(\mathbb{V})) \longrightarrow 0 \quad (2.9)$$

in (2.4). Using Lemma 2.2, define the homomorphism

$$\psi_3 \circ (\psi_2)^{-1} : J^2(T\mathbb{P}(\mathbb{V})) \longrightarrow J^3(T\mathbb{P}(\mathbb{V})),$$

where the homomorphisms ψ_j are constructed in (2.8). This homomorphism $\psi_3 \circ (\psi_2)^{-1}$ is a holomorphic splitting of the exact sequence in (2.9). In other words, there is a unique surjective homomorphism

$$\delta_0 : J^3(T\mathbb{P}(\mathbb{V})) \longrightarrow K_{\mathbb{P}(\mathbb{V})}^{\otimes 2}$$

such that $\ker(\delta_0) = \mathrm{image}(\psi_3 \circ (\psi_2)^{-1})$ and

$$\delta_0 \circ \iota_0 = \mathrm{Id}_{K_{\mathbb{P}(\mathbb{V})}^{\otimes 2}}, \quad (2.10)$$

where ι_0 is the homomorphism in (2.9).

From the definition in (2.6) it follows that

$$\delta_0 \in H^0(\mathbb{P}(\mathbb{V}), \mathrm{Diff}_{\mathbb{P}(\mathbb{V})}^3(T\mathbb{P}(\mathbb{V}), K_{\mathbb{P}(\mathbb{V})}^{\otimes 2})).$$

Also, from (2.10) it follows immediately that $\sigma(\delta_0) = 1$, where σ is the symbol homomorphism in (2.7). \square

The trivialization of $J^2(T\mathbb{P}(\mathbb{V}))$ given by Lemma 2.2 produces a holomorphic connection on $J^2(T\mathbb{P}(\mathbb{V}))$; let

$$\mathbb{D}_0 : J^2(T\mathbb{P}(\mathbb{V})) \longrightarrow J^2(T\mathbb{P}(\mathbb{V})) \otimes K_{\mathbb{P}(\mathbb{V})} \quad (2.11)$$

be this holomorphic connection on $J^2(T\mathbb{P}(\mathbb{V}))$. Note that any holomorphic connection on a Riemann surface is automatically flat (see [At] for holomorphic connections).

Remark 2.4. Let $U \subset \mathbb{P}(\mathbb{V})$ be an open subset and

$$s \in H^0(U, J^2(T\mathbb{P}(\mathbb{V}))|_U) = H^0(U, J^2(TU))$$

a flat section for the connection \mathbb{D}_0 in (2.11). Since ψ_2 is an isomorphism (see Lemma 2.2), it follows that the section $s' \in H^0(U, TU)$ given by s using the natural projection $J^2(T\mathbb{P}(\mathbb{V})) \longrightarrow T\mathbb{P}(\mathbb{V})$ (see (2.4)) has the property that the section of $J^2(T\mathbb{P}(\mathbb{V}))|_U$ corresponding to s' coincides with s . If U is connected, then s' extends to a holomorphic section of $T\mathbb{P}(\mathbb{V})$ over $\mathbb{P}(\mathbb{V})$.

Let $\mathfrak{sl}(\mathbb{V})$ be the Lie algebra of $\mathrm{PGL}(\mathbb{V})$; it consists of endomorphisms of \mathbb{V} of trace zero. Using the action of $\mathrm{PGL}(\mathbb{V})$ on $\mathbb{P}(\mathbb{V})$ we get a homomorphism

$$\alpha_0 : \mathfrak{sl}(\mathbb{V}) \longrightarrow H^0(\mathbb{P}(\mathbb{V}), T\mathbb{P}(\mathbb{V})). \quad (2.12)$$

This α_0 is an isomorphism, because it is injective and $\dim \mathfrak{sl}(\mathbb{V}) = \dim H^0(\mathbb{P}(\mathbb{V}), T\mathbb{P}(\mathbb{V}))$. Note that $H^0(\mathbb{P}(\mathbb{V}), T\mathbb{P}(\mathbb{V}))$ has the structure of a Lie algebra given by the Lie bracket operation of vector fields. The homomorphism α_0 in (2.12) is in fact an isomorphism of Lie algebras. Therefore, from Lemma 2.2 it follows that the fibers of $J^2(T\mathbb{P}(\mathbb{V}))$ are identified with the Lie algebra $\mathfrak{sl}(\mathbb{V})$. In particular, the fibers of $J^2(T\mathbb{P}(\mathbb{V}))$ are Lie algebras. For any $x \in \mathbb{P}(\mathbb{V})$, and $v, w \in J^2(T\mathbb{P}(\mathbb{V}))_x$, let

$$[v, w]' \in J^2(T\mathbb{P}(\mathbb{V}))_x \quad (2.13)$$

be the Lie bracket operation on the fiber $J^2(T\mathbb{P}(\mathbb{V}))_x$.

Remark 2.5. Let s and t be two holomorphic vector fields on an open subset $U \subset \mathbb{P}(\mathbb{V})$. The holomorphic section of $J^2(T\mathbb{P}(\mathbb{V}))|_U$ defined by s and t will be denoted by \widehat{s} and \widehat{t} respectively. It should be clarified that the holomorphic section $\widehat{[s, t]}$ of $J^2(T\mathbb{P}(\mathbb{V}))|_U$ given by the Lie bracket $[s, t]$ of vector fields does not in general coincide with the section $[\widehat{s}, \widehat{t}]'$ defined by (2.13). The reason for it is that the operation in (2.13) is constructed using the finite dimensional space consisting of global holomorphic sections of $T\mathbb{P}(\mathbb{V})$, while the operation of Lie bracket of vector fields is constructed locally. However if s and t are such that \widehat{s} and \widehat{t} are flat sections for the holomorphic connection \mathbb{D}_0 on $J^2(T\mathbb{P}(\mathbb{V}))$ in (2.11), then the holomorphic section $\widehat{[s, t]}$ of $J^2(T\mathbb{P}(\mathbb{V}))|_U$ given by the Lie bracket of vector fields $[s, t]$ does coincide with the section $[\widehat{s}, \widehat{t}]'$ defined by (2.13). Indeed, this follows from the fact that these s and t are restrictions of global vector fields on $\mathbb{P}(\mathbb{V})$, if U is connected; see Remark 2.4.

Remark 2.6. The holomorphic connection \mathbb{D}_0 in (2.11) on $J^2(T\mathbb{P}(\mathbb{V}))$ preserves the Lie algebra structure on the fibers of $J^2(T\mathbb{P}(\mathbb{V}))$ given in (2.13). This means that

$$i_u \mathbb{D}_0([s, t]') = [i_u \mathbb{D}_0(s), t]' + [s, i_u \mathbb{D}_0(t)]'$$

for locally defined holomorphic sections s, t and u of $J^2(T\mathbb{P}(\mathbb{V}))$, where i_u is the contraction of 1-forms by u . In particular, the local system on $\mathbb{P}(\mathbb{V})$ given by the flat sections for the connection \mathbb{D}_0 is closed under the Lie bracket operation in (2.13). Note that using Remark 2.4 we may construct a Lie algebra structure on the fibers of $J^2(T\mathbb{P}(\mathbb{V}))$. Indeed, for $v, w \in J^2(T\mathbb{P}(\mathbb{V}))_x$, let \tilde{v}, \tilde{w} be the flat sections of $J^2(T\mathbb{P}(\mathbb{V}))$, for the connection \mathbb{D}_0 , defined around x such that $\tilde{v}(x) = v$ and $\tilde{w}(x) = w$. Let \tilde{v}' (respectively, \tilde{w}') be the holomorphic sections of $T\mathbb{P}(\mathbb{V})$ defined around x given by \tilde{v} (respectively, \tilde{w}) using the natural projection $J^2(T\mathbb{P}(\mathbb{V})) \rightarrow T\mathbb{P}(\mathbb{V})$ (see (2.4) and Remark 2.4). Now define $[v, w]$ to be the element of $J^2(T\mathbb{P}(\mathbb{V}))_x$ given by the locally defined section $[\tilde{v}', \tilde{w}']$ of $T\mathbb{P}(\mathbb{V})$. From Remark 2.5 it follows that this Lie algebra structure on the fibers of $J^2(T\mathbb{P}(\mathbb{V}))$ coincides with the one in (2.13).

Remark 2.7. Let \mathbb{L}_0 be the complex local system on $\mathbb{P}(\mathbb{V})$ given by the sheaf of solutions of the differential operator δ_0 in Lemma 2.3. From the construction of δ_0 it is straight-forward to deduce that \mathbb{L}_0 is identified with the local system given by the sheaf of flat sections of $J^2(T\mathbb{P}(\mathbb{V}))$ for the connection \mathbb{D}_0 in (2.11). Therefore, from Remark 2.6 we conclude that the stalks of the complex local system \mathbb{L}_0 are closed under the Lie bracket operation of vector fields. Moreover, the stalks of the local system \mathbb{L}_0 are identified with the Lie algebra $\mathfrak{sl}(\mathbb{V})$.

Proposition 2.8.

- (1) *Let \mathbb{X} be a connected Riemann surface equipped with a projective structure \mathcal{P} . Then \mathcal{P} produces a holomorphic connection, which will be called $\mathbb{D}(\mathcal{P})$, on $J^2(T\mathbb{X})$. For any open subset $U \subset \mathbb{X}$, and any section $s \in H^0(U, J^2(T\mathbb{X})|_U) = H^0(U, J^2(TU))$ flat for the connection $\mathbb{D}(\mathcal{P})$, there is a unique holomorphic section of TU that produces s . The space of section of TU given by the flat sections of $J^2(U)$ is closed under the usual Lie bracket operation of vector fields. The stalks for the local system on \mathbb{X} given by the sheaf of flat sections for $\mathbb{D}(\mathcal{P})$ are closed under the usual Lie bracket operation of vector fields, and moreover the stalks are isomorphic to the Lie algebra $\mathfrak{sl}(\mathbb{V})$.*

- (2) *The projective structure \mathcal{P} also produces a canonical holomorphic differential operator $\delta(\mathcal{P}) \in H^0(\mathbb{X}, \mathrm{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))$ whose symbol is the constant function 1 on \mathbb{X} .*
- (3) *The local system on \mathbb{X} gives by the sheaf of flat sections for $\mathbb{D}(\mathcal{P})$ is identified with the local system given by the sheaf of solutions of $\delta(\mathcal{P})$. This sheaf of solutions of $\delta(\mathcal{P})$ is closed under the Lie bracket operation of vector fields.*
- (4) *The connection on $\bigwedge^3 J^2(T\mathbb{X}) = \mathcal{O}_{\mathbb{X}}$ (see Remark 2.1) induced by $\mathbb{D}(\mathcal{P})$ coincides with the trivial connection on $\mathcal{O}_{\mathbb{X}}$ given by the de Rham differential d .*

Proof. The action of $\mathrm{PGL}(\mathbb{V})$ on $\mathbb{P}(\mathbb{V})$ produces actions of $\mathrm{PGL}(\mathbb{V})$ on $J^j(T\mathbb{P}(\mathbb{V}))$, $j \geq 0$, and $H^0(\mathbb{P}(\mathbb{V}), T\mathbb{P}(\mathbb{V}))$. The homomorphism ψ_j in (2.8) is clearly equivariant for the actions of $\mathrm{PGL}(\mathbb{V})$, with $\mathrm{PGL}(\mathbb{V})$ acting diagonally on $\mathbb{P}(\mathbb{V}) \times H^0(\mathbb{P}(\mathbb{V}), T\mathbb{P}(\mathbb{V}))$. Therefore, from Lemma 2.2 we get a holomorphic connection on $J^2(T\mathbb{X})$. To explain this with more details, take a holomorphic coordinate atlas $\{(U_i, \phi_i)\}_{i \in I}$ in the equivalence class defining \mathcal{P} . Using ϕ_i , the holomorphic connection $\mathbb{D}_0|_{\phi_i(U_i)}$ in (2.11) on $J^2(T\mathbb{P}(\mathbb{V}))|_{\phi_i(U_i)}$ produces a holomorphic connection on $J^2(T\mathbb{X})|_{U_i}$. Using the above $\mathrm{PGL}(\mathbb{V})$ -equivariance property, these locally defined connections on $J^2(T\mathbb{X})|_{U_i}$, $i \in I$, patch together compatibly on the intersections of the open subsets to produce a holomorphic connection $\mathbb{D}(\mathcal{P})$ on $J^2(T\mathbb{X})$.

Take any flat section $s \in H^0(U, J^2(T\mathbb{X})|_U) = H^0(U, J^2(TU))$ as in the first statement of the proposition. From Remark 2.4 we conclude that the section of TU given by s , using the natural projection $J^2(TU) \rightarrow TU$, actually produces s . From Remark 2.5 it follows that the space of section of TU given by the flat sections of $J^2(TU)$ is closed under the usual Lie bracket operation of vector fields. Consequently, the stalks for the local system on \mathbb{X} given by the sheaf of flat sections for $\mathbb{D}(\mathcal{P})$ are closed under the Lie bracket operation, and they are isomorphic to the Lie algebra $\mathfrak{sl}(\mathbb{V})$, because the connection \mathbb{D}_0 has these properties; see Remark 2.6.

Similarly, the second statement of the proposition follows from the fact that the differential operator δ_0 in Lemma 2.3 is $\mathrm{PGL}(\mathbb{V})$ -equivariant. Indeed, given a holomorphic coordinate atlas $\{(U_i, \phi_i)\}_{i \in I}$ as above, we have a differential operator on each U_i given by δ_0 using the coordinate function ϕ_i . These differential operator patch together compatibly to produce a differential operator

$$\delta(\mathcal{P}) \in H^0(\mathbb{X}, \mathrm{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2})).$$

The third statement follows from Remark 2.7 and the first statement of the proposition. For any $s \in H^0(U, TU)$, where $U \subset \mathbb{X}$ is an open subset, with $\delta(\mathcal{P})(s) = 0$, the section of $J^2(T\mathbb{X})|_U$ given by s is flat with respect to the connection $\mathbb{D}(\mathcal{P})$. Conversely, given a flat section s_1 of $J^2(T\mathbb{X})|_U$, the section $s_2 \in H^0(U, TU)$, given by s_1 using the natural projection $J^2(T\mathbb{X}) \rightarrow T\mathbb{X}$ (see (2.4)), satisfies the equation $\delta(\mathcal{P})(s_2) = 0$.

From Remark 2.1 we know that if $\phi : \mathbb{U}_1 \rightarrow \mathbb{U}_2$ is a biholomorphism between two open subsets of $\mathbb{P}(\mathbb{V})$, then the isomorphism $\bigwedge^3 J^2(T\mathbb{U}_1) \rightarrow \bigwedge^3 J^2(T\mathbb{U}_2)$ induced by ϕ takes the section of $\bigwedge^3 J^2(T\mathbb{U}_1)$ given by the constant function 1 on U_1 to the section of $\bigwedge^3 J^2(T\mathbb{U}_2)$ given by the constant function 1 on U_2 . In particular, this holds for $\phi \in \mathrm{PGL}(\mathbb{V}) = \mathrm{Aut}(\mathbb{P}(\mathbb{V}))$. The connection on $\bigwedge^3 J^2(T\mathbb{P}(\mathbb{V}))$ induced by the connection \mathbb{D}_0 on $J^2(T\mathbb{P}(\mathbb{V}))$

coincides with the trivial connection on $\bigwedge^3 J^2(T\mathbb{P}(\mathbb{V}))$, because $\mathbb{P}(\mathbb{V})$ is simply connected. The fourth statement of the proposition follows from these. \square

2.4. Killing form and holomorphic connection. Recall that the homomorphism α_0 in (2.12) is a Lie algebra isomorphism between $H^0(\mathbb{P}(\mathbb{V}), T\mathbb{P}(\mathbb{V}))$ and $\mathfrak{sl}(\mathbb{V})$. Consider the Killing form \widehat{B} on the Lie algebra $H^0(\mathbb{P}(\mathbb{V}), T\mathbb{P}(\mathbb{V}))$. Using the isomorphism ψ_2 in Lemma 2.2, this symmetric bilinear form \widehat{B} produces a fiberwise nondegenerate symmetric bilinear form

$$B_0 \in H^0(\mathbb{P}(\mathbb{V}), \text{Sym}^2(J^2(T\mathbb{P}(\mathbb{V})))^*) = \text{Sym}^2(H^0(\mathbb{P}(\mathbb{V}), T\mathbb{P}(\mathbb{V})))^*. \quad (2.14)$$

Recall that each fiber of $J^2(T\mathbb{P}(\mathbb{V}))$ is the Lie algebra $\mathfrak{sl}(\mathbb{V})$; the form B_0 in (2.14) is the fiberwise Killing form. The symmetric form B_0 is preserved by the holomorphic connection \mathbb{D}_0 on $J^2(T\mathbb{P}(\mathbb{V}))$ constructed in (2.11). Indeed, this follows immediately from the fact that both \mathbb{D}_0 and B_0 are constants with respect to the trivialization of $J^2(T\mathbb{P}(\mathbb{V}))$ given by ψ_2 in Lemma 2.2.

The vector bundle $J^2(T\mathbb{P}(\mathbb{V}))$ has a filtration of holomorphic subbundles

$$F_1^{\mathbb{P}} := K_{\mathbb{P}(\mathbb{V})} \subset F_2^{\mathbb{P}} \subset J^2(T\mathbb{P}(\mathbb{V})), \quad (2.15)$$

where $F_2^{\mathbb{P}}$ is the kernel of the composition

$$J^2(T\mathbb{P}(\mathbb{V})) \longrightarrow J^1(T\mathbb{P}(\mathbb{V})) \longrightarrow T\mathbb{P}(\mathbb{V})$$

of the two projections in the two short exact sequences in Remark 2.1; the subbundle $K_{\mathbb{P}(\mathbb{V})} \subset J^2(T\mathbb{P}(\mathbb{V}))$ in (2.15) is the one in the second of the two short exact sequences in Remark 2.1. In particular, we have $\text{rank}(F_j^{\mathbb{P}}) = j$. For any point $x \in \mathbb{P}(\mathbb{V})$, the fiber $(F_1^{\mathbb{P}})_x$ is a nilpotent subalgebra of the Lie algebra $J^2(T\mathbb{P}(\mathbb{V}))_x = \mathfrak{sl}(\mathbb{V})$. Moreover, the fiber $(F_2^{\mathbb{P}})_x$ is the unique Borel subalgebra of $J^2(T\mathbb{P}(\mathbb{V}))_x$ containing $(F_1^{\mathbb{P}})_x$. Consequently, we have

$$B_0(F_1^{\mathbb{P}} \otimes F_1^{\mathbb{P}}) = 0 \quad \text{and} \quad (F_1^{\mathbb{P}})^{\perp} = F_2^{\mathbb{P}}, \quad (2.16)$$

where $(F_1^{\mathbb{P}})^{\perp}$ denotes the orthogonal bundle for $F_1^{\mathbb{P}}$ with respect to the form B_0 in (2.14).

Given a holomorphic vector bundle W on a Riemann surface \mathbb{X} , a holomorphic connection \mathcal{D} on W , and a holomorphic subbundle $W' \subset W$, the composition of homomorphisms

$$W \xrightarrow{\mathcal{D}} W \otimes K_{\mathbb{X}} \xrightarrow{q_{W'} \otimes \text{Id}} (W/W') \otimes K_{\mathbb{X}},$$

where $q_{W'} : W \rightarrow W/W'$ is the natural quotient map, defines a holomorphic section of $\text{Hom}(W', (W/W') \otimes K_{\mathbb{X}})$. This element of $H^0(\mathbb{X}, \text{Hom}(W', (W/W') \otimes K_{\mathbb{X}}))$ is called the *second fundamental form* of W' for the connection \mathcal{D} . If $\mathcal{D}(W') \subset W'' \otimes K_{\mathbb{X}}$, where W'' is a holomorphic subbundle of W containing W' , then the second fundamental form of W' for \mathcal{D} is clearly given by a holomorphic section

$$\zeta_1 \in H^0(\mathbb{X}, \text{Hom}(W', W''/W') \otimes K_{\mathbb{X}}) \quad (2.17)$$

using the natural inclusion of $\text{Hom}(W', W''/W') \otimes K_{\mathbb{X}}$ in $\text{Hom}(W', W/W') \otimes K_{\mathbb{X}}$. Also, in this case, the second fundamental form of W'' for \mathcal{D} is given by a holomorphic section

$$\zeta_2 \in H^0(\mathbb{X}, \text{Hom}(W''/W', W/W'') \otimes K_{\mathbb{X}}) \quad (2.18)$$

through the natural inclusion map

$$H^0(\mathbb{X}, \text{Hom}(W''/W', W/W'') \otimes K_{\mathbb{X}}) \hookrightarrow H^0(\mathbb{X}, \text{Hom}(W'', W/W'') \otimes K_{\mathbb{X}}).$$

For the filtration in (2.15) of $J^2(T\mathbb{P}(\mathbb{V}))$ equipped with the holomorphic connection \mathbb{D}_0 in (2.11), we have

$$\mathbb{D}_0(F_1^{\mathbb{P}}) = F_2^{\mathbb{P}} \otimes K_{\mathbb{P}(\mathbb{V})} \quad \text{and} \quad \mathbb{D}_0(F_2^{\mathbb{P}}) = J^2(T\mathbb{P}(\mathbb{V})) \otimes K_{\mathbb{P}(\mathbb{V})}.$$

These follow from a straight-forward computation.

Let

$$S(F_1^{\mathbb{P}}, \mathbb{D}_0) \in H^0(\mathbb{P}(\mathbb{V}), \text{Hom}(F_1^{\mathbb{P}}, F_2^{\mathbb{P}}/F_1^{\mathbb{P}}) \otimes K_{\mathbb{P}(\mathbb{V})}) = H^0(\mathbb{P}(\mathbb{V}), \mathcal{O}_{\mathbb{P}(\mathbb{V})})$$

be the second fundamental form of $F_1^{\mathbb{P}}$ for the connection \mathbb{D}_0 (see (2.17)). Similarly, let

$$S(F_2^{\mathbb{P}}, \mathbb{D}_0) \in H^0(\mathbb{P}(\mathbb{V}), \text{Hom}(F_2^{\mathbb{P}}/F_1^{\mathbb{P}}, J^2(T\mathbb{P}(\mathbb{V}))/F_2^{\mathbb{P}}) \otimes K_{\mathbb{P}(\mathbb{V})}) = H^0(\mathbb{P}(\mathbb{V}), \mathcal{O}_{\mathbb{P}(\mathbb{V})})$$

be the section that gives the second fundamental form of $F_2^{\mathbb{P}}$ for the connection \mathbb{D}_0 (see (2.18)). It is straight-forward to check that both $S(F_1^{\mathbb{P}}, \mathbb{D}_0)$ and $S(F_2^{\mathbb{P}}, \mathbb{D}_0)$ coincide with the element of $H^0(\mathbb{P}(\mathbb{V}), \mathcal{O}_{\mathbb{P}(\mathbb{V})})$ given by the constant function 1 on $\mathbb{P}(\mathbb{V})$.

3. DIFFERENTIAL OPERATORS, CONNECTIONS AND PROJECTIVE STRUCTURES

3.1. Differential operators and connections. For a holomorphic vector bundle W on a Riemann surface \mathbb{X} , there is a tautological fiberwise injective holomorphic homomorphism

$$J^{i+j}(W) \longrightarrow J^i(J^j(W)) \quad (3.1)$$

for every $i, j \geq 0$. On \mathbb{X} , we have the commutative diagram of holomorphic homomorphisms

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_{\mathbb{X}}^{\otimes 2} & \xrightarrow{\iota} & J^3(T\mathbb{X}) & \xrightarrow{q} & J^2(T\mathbb{X}) \longrightarrow 0 \\ & & \downarrow l & & \downarrow \lambda & & \parallel \\ 0 & \longrightarrow & J^2(T\mathbb{X}) \otimes K_{\mathbb{X}} & \xrightarrow{\iota'} & J^1(J^2(T\mathbb{X})) & \xrightarrow{q'} & J^2(T\mathbb{X}) \longrightarrow 0 \\ & & \downarrow & & \downarrow \mu & & \\ & & J^1(T\mathbb{X}) \otimes K_{\mathbb{X}} & \xrightarrow{\cong} & J^1(T\mathbb{X}) \otimes K_{\mathbb{X}} & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \quad (3.2)$$

where the horizontal short exact sequences are as in (2.4), the vertical short exact sequence in the left is the short exact sequence in (2.4) tensored with $K_{\mathbb{X}}$, and λ is the homomorphism in (3.1); the homomorphism μ in (3.2) is described below.

The projection $J^2(T\mathbb{X}) \longrightarrow J^1(T\mathbb{X})$ in (2.4) induces a homomorphism

$$f_1 : J^1(J^2(T\mathbb{X})) \longrightarrow J^1(J^1(T\mathbb{X})) \quad (3.3)$$

(see (2.5)); set W and W' in (2.5) to be $J^2(T\mathbb{X})$ and $J^1(T\mathbb{X})$ respectively to get f_1 . On the other hand, let

$$f_2' : J^1(J^2(T\mathbb{X})) \longrightarrow J^2(T\mathbb{X})$$

be the projection in (2.4). Composing f'_2 with the homomorphism $J^2(T\mathbb{X}) \longrightarrow J^1(J^1(T\mathbb{X}))$ in (3.1) we obtain a homomorphism

$$f_2 : J^1(J^2(T\mathbb{X})) \longrightarrow J^1(J^1(T\mathbb{X})).$$

The composition of homomorphisms

$$J^1(J^2(T\mathbb{X})) \xrightarrow{f_2} J^1(J^1(T\mathbb{X})) \xrightarrow{f_3} J^1(T\mathbb{X}),$$

where f_3 is projection in (2.4), coincides with the composition of homomorphisms

$$J^1(J^2(T\mathbb{X})) \xrightarrow{f_1} J^1(J^1(T\mathbb{X})) \xrightarrow{f_3} J^1(T\mathbb{X}),$$

where f_1 is the homomorphism in (3.3). Therefore, from (2.4) we have the homomorphism

$$\mu := f_1 - f_2 : J^1(J^2(T\mathbb{X})) \longrightarrow J^1(T\mathbb{X}) \otimes K_{\mathbb{X}},$$

where f_1 and f_2 are constructed above. This homomorphism μ is the one in (3.2).

Let

$$\eta \in H^0(\mathbb{X}, \text{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2})) = H^0(\mathbb{X}, K_{\mathbb{X}}^{\otimes 2} \otimes J^3(T\mathbb{X})^*) \quad (3.4)$$

be a differential operator whose symbol is the constant function 1 on \mathbb{X} . This means that η gives a holomorphic splitting of the top horizontal exact sequence in (3.2). Let

$$\hat{\eta} : J^2(T\mathbb{X}) \longrightarrow J^3(T\mathbb{X}) \quad (3.5)$$

be the corresponding splitting homomorphism, meaning

- $\hat{\eta}(J^2(T\mathbb{X})) = \text{kernel}(J^3(T\mathbb{X}) \xrightarrow{\eta} K_{\mathbb{X}}^{\otimes 2})$, and
- $q \circ \hat{\eta} = \text{Id}_{J^2(T\mathbb{X})}$, where q is the projection in (3.2).

From the commutativity of (3.2) we conclude that the homomorphism

$$\lambda \circ \hat{\eta} : J^2(T\mathbb{X}) \longrightarrow J^1(J^2(T\mathbb{X})), \quad (3.6)$$

where λ is the homomorphism in (3.2), satisfies the equation

$$q' \circ (\lambda \circ \hat{\eta}) = \text{Id}_{J^2(T\mathbb{X})},$$

where q' is the projection in (3.2). Consequently, the homomorphism $\lambda \circ \hat{\eta}$ defines a holomorphic connection on $J^2(T\mathbb{X})$ (see [At]).

Let $\text{Conn}(J^2(T\mathbb{X}))$ denote the space of all holomorphic connections on $J^2(T\mathbb{X})$.

We summarize the above construction in the following lemma.

Lemma 3.1. *Consider the subset*

$$H^0(\mathbb{X}, \text{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_0 \subset H^0(\mathbb{X}, \text{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))$$

defined by the differential operators whose symbol is the constant function 1 on \mathbb{X} . There is a natural map

$$\varpi : H^0(\mathbb{X}, \text{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_0 \longrightarrow \text{Conn}(J^2(T\mathbb{X}))$$

that sends any η as in (3.4) to the connection $\lambda \circ \hat{\eta}$ in (3.6).

We shall describe the image of the map ϖ in Lemma 3.1.

Consider the two short exact sequences in Remark 2.1. Let

$$K_{\mathbb{X}} := \ker(\mu_0) \subset F_2 := \mu_0^{-1}(\mathcal{O}_{\mathbb{X}}) \subset J^2(T\mathbb{X}) \quad (3.7)$$

be the filtration of holomorphic subbundles given by them, where $\mu_0 : J^2(T\mathbb{X}) \rightarrow J^1(T\mathbb{X})$ is the projection in (2.4); see (2.15).

As before, $\mathrm{Conn}(J^2(T\mathbb{X}))$ denotes the space of all holomorphic connections on $J^2(T\mathbb{X})$.

Definition 3.2. A holomorphic connection $\mathcal{D} \in \mathrm{Conn}(J^2(T\mathbb{X}))$ will be called an *oper connection* if the following three conditions hold:

- $\mathcal{D}(K_{\mathbb{X}}) = F_2 \otimes K_{\mathbb{X}}$ (see (3.7)),
- the second fundamental form of $K_{\mathbb{X}}$ for \mathcal{D} , which, by the first condition, is a holomorphic section of $\mathrm{Hom}(K_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}) \otimes K_{\mathbb{X}} = \mathcal{O}_{\mathbb{X}}$ (see (2.17)), coincides with the constant function 1 on \mathbb{X} , and
- the holomorphic section of $\mathrm{Hom}(F_2/K_{\mathbb{X}}, J^2(T\mathbb{X})/F_2) \otimes K_{\mathbb{X}} = \mathcal{O}_{\mathbb{X}}$ that gives the second fundamental form of F_2 for \mathcal{D} — see (2.18) — coincides with the constant function 1 on \mathbb{X} .

See [BeDr] for general opers; the oper connections in Definition 3.2 are $\mathrm{GL}(3, \mathbb{C})$ -opers on \mathbb{X} .

Lemma 3.3. Take any $\eta \in H^0(\mathbb{X}, \mathrm{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_0$ (see Lemma 3.1), and let

$$\varpi(\eta) \in \mathrm{Conn}(J^2(T\mathbb{X}))$$

be the holomorphic connection on $J^2(T\mathbb{X})$ given by η in Lemma 3.1. Then $\varpi(\eta)$ is an oper connection.

Proof. Take a projective structure \mathcal{P} on \mathbb{X} . Let

$$\delta(\mathcal{P}) \in H^0(\mathbb{X}, \mathrm{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_0$$

be the differential operator corresponding to \mathcal{P} in Proposition 2.8(2). Let

$$\mathbb{D}(\mathcal{P}) \in \mathrm{Conn}(J^2(T\mathbb{X}))$$

be the connection in Proposition 2.8(1) corresponding to \mathcal{P} . It can be shown that

- $\varpi(\delta(\mathcal{P})) = \mathbb{D}(\mathcal{P})$, and
- $\mathbb{D}(\mathcal{P})$ is an oper connection.

Indeed, from the proof of Proposition 2.8 we know that it suffices to prove this for the (unique) standard projective structure on $\mathbb{P}(\mathbb{V})$. Now, both these statements are straightforward for the unique projective structure on $\mathbb{P}(\mathbb{V})$.

Next note that $H^0(\mathbb{X}, \mathrm{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_0$ is an affine space for the complex vector space $H^0(\mathbb{X}, \mathrm{Hom}(J^2(T\mathbb{X}), K_{\mathbb{X}}^{\otimes 2}))$. So η in the statement of the lemma is of the form

$$\eta = \delta(\mathcal{P}) + \theta, \quad (3.8)$$

where $\theta \in H^0(\mathbb{X}, \text{Hom}(J^2(T\mathbb{X}), K_{\mathbb{X}}^{\otimes 2}))$. The composition of homomorphisms

$$J^2(T\mathbb{X}) \xrightarrow{\theta} K_{\mathbb{X}}^{\otimes 2} \xrightarrow{l} J^2(T\mathbb{X}) \otimes K_{\mathbb{X}},$$

where l is the homomorphism in (3.2), will be denoted by $\tilde{\theta}$. From (3.8) we have

$$\varpi(\eta) - \varpi(\delta(\mathcal{P})) = \varpi(\eta) - \mathbb{D}(\mathcal{P}) = \tilde{\theta}. \quad (3.9)$$

Since $\mathbb{D}(\mathcal{P})$ is an oper connection, from (3.9) it is straight-forward to deduce that $\varpi(\eta)$ is also an oper connection. \square

Take any $\mathcal{D} \in \text{Conn}(J^2(T\mathbb{X}))$. Using \mathcal{D} we shall construct an endomorphism of the vector bundle $J^2(T\mathbb{X})$. For that, let

$$p_0 : J^2(T\mathbb{X}) \longrightarrow T\mathbb{X} \quad (3.10)$$

be the composition $J^2(T\mathbb{X}) \longrightarrow J^1(T\mathbb{X}) \longrightarrow T\mathbb{X}$ of the projections in the two short exact sequences in Remark 2.1. For any $x \in \mathbb{X}$, and $v \in J^2(T\mathbb{X})_x$, let \tilde{v} be the unique section of $J^2(T\mathbb{X})$ defined on a simply connected open neighborhood of x such that

- \tilde{v} is flat for the connection \mathcal{D} , and
- $\tilde{v}(x) = v$.

Now we have a holomorphic homomorphism

$$F_{\mathcal{D}} : J^2(T\mathbb{X}) \longrightarrow J^2(T\mathbb{X}) \quad (3.11)$$

that sends any $v \in J^2(T\mathbb{X})_x$, $x \in \mathbb{X}$, to the element of $J^2(T\mathbb{X})_x$ defined by the section $p_0(\tilde{v})$, where p_0 is the projection in (3.10), and \tilde{v} is constructed as above using v and \mathcal{D} .

Lemma 3.4. *A holomorphic connection $\mathcal{D} \in \text{Conn}(J^2(T\mathbb{X}))$ lies in $\text{image}(\varpi)$ (see Lemma 3.1) if and only if*

- \mathcal{D} is an oper connection, and
- $F_{\mathcal{D}} = \text{Id}_{J^2(T\mathbb{X})}$, where $F_{\mathcal{D}}$ is constructed in (3.11).

Proof. As in the proof of Lemma 3.3, first take a projective structure \mathcal{P} on \mathbb{X} . Let $\delta(\mathcal{P})$ (respectively, $\mathbb{D}(\mathcal{P})$) be the differential operator (respectively, holomorphic connection) corresponding to \mathcal{P} as in the proof of Lemma 3.3. We saw that $\varpi(\delta(\mathcal{P})) = \mathbb{D}(\mathcal{P})$ and $\mathbb{D}(\mathcal{P})$ is an oper connection. It can be shown that $F_{\mathbb{D}(\mathcal{P})} = \text{Id}_{J^2(T\mathbb{X})}$, where $F_{\mathbb{D}(\mathcal{P})}$ is constructed as in (3.11). Indeed, it suffices to prove this for the unique projective structure on $\mathbb{P}(\mathbb{V})$, which is actually straight-forward.

Now take $\eta = \delta(\mathcal{P}) + \theta$ as in (3.8). Since $\varpi(\delta(\mathcal{P})) = \mathbb{D}(\mathcal{P})$ is an oper connection, and $F_{\mathbb{D}(\mathcal{P})} = \text{Id}_{J^2(T\mathbb{X})}$, from (3.9) it follows that $F_{\varpi(\eta)} = \text{Id}_{J^2(T\mathbb{X})}$, where $F_{\varpi(\eta)}$ is constructed as in (3.11) for the connection $\varpi(\eta)$; it was shown in Lemma 3.3 that $\varpi(\eta)$ is an oper connection.

To prove the converse, take any $\mathcal{D} \in \text{Conn}(J^2(T\mathbb{X}))$. Then

$$\mathcal{D} = \mathbb{D}(\mathcal{P}) + \beta, \quad (3.12)$$

where $\beta \in H^0(\mathbb{X}, \text{End}(J^2(T\mathbb{X})) \otimes K_{\mathbb{X}})$.

Now assume that

- \mathcal{D} is an oper connection, and

- $F_{\mathcal{D}} = \mathrm{Id}_{J^2(T\mathbb{X})}$, where $F_{\mathcal{D}}$ is constructed in (3.11).

Since $\mathbb{D}(\mathcal{P})$ also satisfies these two conditions, it follows that there is a unique section

$$\tilde{\beta} \in H^0(\mathbb{X}, \mathrm{Hom}(J^2(T\mathbb{X}), K_{\mathbb{X}}^{\otimes 2}))$$

such that β in (3.12) coincides with the composition of homomorphisms

$$J^2(T\mathbb{X}) \xrightarrow{\tilde{\beta}} K_{\mathbb{X}}^{\otimes 2} \xrightarrow{l} J^2(T\mathbb{X}) \otimes K_{\mathbb{X}},$$

where l is the homomorphism in (3.2). Consequently, we have

$$\varpi(\delta(\mathcal{P}) + \tilde{\beta}) = \mathcal{D}.$$

This proves the lemma. \square

3.2. Differential operator given by projective structures. Given a projective structure \mathcal{P} on a Riemann surface \mathbb{X} , recall that in Proposition 2.8(2) we constructed an element of $H^0(\mathbb{X}, \mathrm{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_0$.

Proposition 3.5. *The space of all projective structures on \mathbb{X} is in a natural bijection with the subspace of $H^0(\mathbb{X}, \mathrm{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_0$ consisting all differential operators δ satisfying the following two conditions:*

- (1) *The connection on $\bigwedge^3 J^2(T\mathbb{X})$ induced by the connection $\varpi(\delta)$ on $J^2(T\mathbb{X})$ (see Lemma 3.1) coincides with the trivial connection on $\bigwedge^3 J^2(T\mathbb{X})$ (see Remark 2.1).*
- (2) *If s and t are locally defined holomorphic sections of $T\mathbb{X}$ such that $\delta(s) = 0 = \delta(t)$, then $\delta([s, t]) = 0$, where $[s, t]$ is the usual Lie bracket of the vector fields s and t .*

Proof. Let $\mathbf{P}(\mathbb{X})$ denote the space of all projective structures on \mathbb{X} . Let

$$\mathbf{D}(\mathbb{X}) \subset H^0(\mathbb{X}, \mathrm{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_0$$

be the subset consisting of all differential operators δ satisfying the following conditions:

- (1) The connection on $\bigwedge^3 J^2(T\mathbb{X})$ induced by the connection $\varpi(\delta)$ on $J^2(T\mathbb{X})$ coincides with the trivial connection on $\bigwedge^3 J^2(T\mathbb{X})$.
- (2) If s and t are locally defined holomorphic sections of $T\mathbb{X}$ such that $\delta(s) = 0 = \delta(t)$, then $\delta([s, t]) = 0$.

Let $\mathcal{P} \in \mathbf{P}(\mathbb{X})$ be a projective structure on \mathbb{X} . Let

$$\delta \in H^0(\mathbb{X}, \mathrm{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_0$$

be the differential operator given by \mathcal{P} (see Proposition 2.8(2)). In view of Proposition 2.8(4), the first one of the above two conditions on δ is satisfied; also, the second condition is satisfied because of Proposition 2.8(3). Therefore, we get a map

$$\Theta : \mathbf{P}(\mathbb{X}) \longrightarrow \mathbf{D}(\mathbb{X}) \tag{3.13}$$

that sends any \mathcal{P} to the corresponding differential operator δ .

There is a natural map

$$\Psi : \mathbf{D}(\mathbb{X}) \longrightarrow \mathbf{P}(\mathbb{X}) \tag{3.14}$$

(see [Bi, p. 14, (3.7)]). To clarify, set $n = 2$ in the definition of \mathcal{B} in [Bi, p. 13]. Then \mathcal{B}_0 in [Bi, p. 13] coincides with the subset of $H^0(\mathbb{X}, \text{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))$ consisting of all δ' such that

- the symbol of δ' is the constant function 1, and
- the holomorphic connection on $\bigwedge^3 J^2(T\mathbb{X})$ induced by the connection $\varpi(\delta')$ on $J^2(T\mathbb{X})$ coincides with the trivial connection on $\bigwedge^3 J^2(T\mathbb{X}) = \mathcal{O}_{\mathbb{X}}$.

For the maps Θ and Ψ constructed in (3.13) and (3.14) respectively, we have

$$\Psi \circ \Theta = \text{Id}_{\mathbf{P}(\mathbb{X})}; \quad (3.15)$$

this follows from the combination of the facts that

- the map F in [Bi, p. 19, (5.4)] is a bijection,
- the map Ψ in (3.14) coincides with the composition of the map F in [Bi, p. 19, (5.4)] with the natural projection $\mathbf{P}(\mathbb{X}) \times H^0(\mathbb{X}, K_{\mathbb{X}}^{\otimes 3}) \rightarrow H^0(\mathbb{X}, K_{\mathbb{X}}^{\otimes 3})$, and
- $F^{-1}(\mathcal{P}, 0) = \Theta(\mathcal{P})$ for all $\mathcal{P} \in \mathbf{P}(\mathbb{X})$, where Θ is the map in (3.13).

From (3.15) we conclude that the map Θ in (3.13) is injective. We will prove that the map Θ is surjective as well.

Let

$$H^0(\mathbb{X}, \text{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_1 \subset H^0(\mathbb{X}, \text{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_0$$

be the subset consisting of all $\eta \in H^0(\mathbb{X}, \text{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_0$ such that the connection on $\bigwedge^3 J^2(T\mathbb{X})$ induced by the connection $\varpi(\eta)$ on $J^2(T\mathbb{X})$ (see Lemma 3.1) coincides with the trivial connection on $\bigwedge^3 J^2(T\mathbb{X})$ (see Remark 2.1). Let

$$\Psi' : H^0(\mathbb{X}, \text{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_1 \rightarrow \mathbf{P}(\mathbb{X}) \quad (3.16)$$

be the map in [Bi, p. 14, (3.7)]; recall that the map Ψ in (3.14) is the restriction of this map Ψ' to the subset $\mathbf{D}(\mathbb{X})$ of $H^0(\mathbb{X}, \text{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_1$.

Take any

$$\eta \in H^0(\mathbb{X}, \text{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_1. \quad (3.17)$$

From the isomorphism F in [Bi, p. 19, (5.4)] we know that there is a holomorphic section

$$\xi \in H^0(\mathbb{X}, K_{\mathbb{X}}^{\otimes 3})$$

such that

$$\eta = \Theta(\Psi'(\eta)) + \xi, \quad (3.18)$$

where Ψ' and Θ are the maps in (3.16) and (3.13) respectively.

We now impose the following condition on η in (3.17):

If s and t are locally defined holomorphic sections of $T\mathbb{X}$ such that $\eta(s) = 0 = \eta(t)$, then $\eta([s, t]) = 0$.

Since $\Theta(\Psi'(\eta)) \in \mathbf{D}(\mathbb{X})$, in particular, $\Theta(\Psi'(\eta))$ satisfies the above condition, from the above condition on η it follows that the section ξ in (3.18) vanishes identically. So, we have $\eta = \Theta(\Psi'(\eta))$. This implies that the map Θ is surjective. \square

As before, $\text{Conn}(J^2(T\mathbb{X}))$ denotes the space of all holomorphic connections on $J^2(T\mathbb{X})$. Lemma 3.4 and Proposition 3.5 combine together to give the following:

Corollary 3.6. *The space of all projective structures on \mathbb{X} is in a natural bijection with the subset of $\mathrm{Conn}(J^2(T\mathbb{X}))$ defined by all connections \mathcal{D} satisfying the following four conditions:*

- (1) \mathcal{D} is an oper connection,
- (2) $F_{\mathcal{D}} = \mathrm{Id}_{J^2(T\mathbb{X})}$, where $F_{\mathcal{D}}$ is constructed in (3.11),
- (3) the connection on $\bigwedge^3 J^2(T\mathbb{X})$ induced by \mathcal{D} , coincides with the trivial connection on $\bigwedge^3 J^2(T\mathbb{X})$, and
- (4) if s and t are locally defined holomorphic sections of $T\mathbb{X}$ such that the sections of $J^2(T\mathbb{X})$ corresponding to s and t are flat with respect to \mathcal{D} , then the section of $J^2(T\mathbb{X})$ corresponding to $[s, t]$ is also flat with respect to \mathcal{D} .

Proof. In Proposition 2.8 we constructed a map from the projective structures on \mathbb{X} to the holomorphic connections on $J^2(T\mathbb{X})$. The holomorphic connections on $J^2(T\mathbb{X})$ obtained this way satisfy all the four conditions in the statement of the corollary. See Proposition 2.8 for conditions (3) and (4); see the proof of Lemma 3.3 for condition (1); see the proof of Lemma 3.4 for (2).

Conversely, let \mathcal{D} be a holomorphic connection on $J^2(T\mathbb{X})$ satisfying the four conditions. In view of Lemma 3.4, from the first two conditions we conclude that $\mathcal{D} = \varpi(\delta)$ for some $\delta \in H^0(\mathbb{X}, \mathrm{Diff}_{\mathbb{X}}^3(T\mathbb{X}, K_{\mathbb{X}}^{\otimes 2}))_0$. In view of Proposition 3.5, from the third and fourth conditions we conclude that \mathcal{D} corresponds to a projective structures on \mathbb{X} . \square

4. PROJECTIVE STRUCTURES AND ORTHOGONAL OPERS

Projective structures on a Riemann surface \mathbb{X} are precisely the $\mathrm{PSL}(2, \mathbb{C})$ -opers on \mathbb{X} . On the other hand, we have the isomorphism $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SO}(3, \mathbb{C})$. This isomorphism is obtained by identifying \mathbb{C}^3 equipped with the standard nondegenerate symmetric form and $\mathrm{Sym}^2(\mathbb{C}^2)$ equipped with the nondegenerate symmetric form constructed using the standard anti-symmetric form on \mathbb{C}^2 ; this identification produces a homomorphism from $\mathrm{SL}(2, \mathbb{C})$ to $\mathrm{SO}(3, \mathbb{C})$, which factors through $\mathrm{PSL}(2, \mathbb{C})$, producing an isomorphism of $\mathrm{PSL}(2, \mathbb{C})$ with $\mathrm{SO}(3, \mathbb{C})$. Therefore, projective structures on \mathbb{X} are precisely the $\mathrm{SO}(3, \mathbb{C})$ -opers on \mathbb{X} . In this subsection we shall elaborate this point of view.

A holomorphic $\mathrm{SO}(3, \mathbb{C})$ -bundle on a Riemann surface \mathbb{X} consists of a holomorphic vector bundle W of rank three on \mathbb{X} together with a holomorphic section $B_W \in H^0(\mathbb{X}, \mathrm{Sym}^2(W^*))$, such that

- $\bigwedge^3 W$ is identified with $\mathcal{O}_{\mathbb{X}}$ by a given isomorphism,
- B_W is fiberwise nondegenerate, and
- the given isomorphism of $\bigwedge^3 W$ with $\mathcal{O}_{\mathbb{X}}$ takes the bilinear form on $\bigwedge^3 W$ induced by B_W to the standard constant bilinear form on $\mathcal{O}_{\mathbb{X}}$ (the corresponding quadratic form takes the section of $\mathcal{O}_{\mathbb{X}}$ given by the constant function 1 on \mathbb{X} to the function 1).

A *filtered* $\mathrm{SO}(3, \mathbb{C})$ -bundle on \mathbb{X} is a holomorphic $\mathrm{SO}(3, \mathbb{C})$ -bundle (W, B_W) together with a filtration of holomorphic subbundles

$$F_1^W \subset F_2^W \subset W$$

such that

- (1) F_1^W is holomorphically identified with $K_{\mathbb{X}}$ by a given isomorphism,
- (2) $B_W(F_1^W \otimes F_1^W) = 0$,
- (3) F_2^W/F_1^W is holomorphically identified with $\mathcal{O}_{\mathbb{X}}$ by a given isomorphism,
- (4) $B_W(F_1^W \otimes F_2^W) = 0$; in other words, $(F_1^W)^\perp = F_2^W$.

Note that the first and third conditions together imply that W/F_2^W is holomorphically identified with $\bigwedge^3 W \otimes (K_{\mathbb{X}})^* = T\mathbb{X}$.

A *holomorphic connection* on a filtered $\mathrm{SO}(3, \mathbb{C})$ -bundle $(W, B_W, \{F_i^W\}_{i=1}^2)$ is a holomorphic connection D_W on W such that

- the holomorphic connection D_W preserves the bilinear form B_W on W ,
- the holomorphic connection on $\bigwedge^3 W = \mathcal{O}_{\mathbb{X}}$ induced by D_W coincides with the holomorphic connection on $\mathcal{O}_{\mathbb{X}}$ given by the de Rham differential d ,
- $D_W(F_1^W) = F_2^W \otimes K_{\mathbb{X}}$, and
- the second fundamental form of F_1^W for D_W , which is a holomorphic section of $\mathrm{Hom}(K_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}) \otimes K_{\mathbb{X}} = \mathcal{O}_{\mathbb{X}}$ (see (2.17)), coincides with the section of $\mathcal{O}_{\mathbb{X}}$ given by the constant function 1 on \mathbb{X} .

An $\mathrm{SO}(3, \mathbb{C})$ -oper on \mathbb{X} is a filtered $\mathrm{SO}(3, \mathbb{C})$ -bundle $(W, B_W, \{F_i^W\}_{i=1}^2)$ equipped with a holomorphic connection D_W .

The above conditions imply that the holomorphic section of $\mathrm{Hom}(\mathcal{O}_{\mathbb{X}}, T\mathbb{X}) \otimes K_{\mathbb{X}} = \mathcal{O}_{\mathbb{X}}$ that gives the second fundamental form of F_2^W for D_W — see (2.18) — coincides with the one given by the constant function 1. See [BeDr] for more on $\mathrm{SO}(3, \mathbb{C})$ -opers.

Recall from (2.14) that $J^2(T\mathbb{P}(\mathbb{V}))$ is equipped with the bilinear form B_0 , and it has the filtration $\{F_i^{\mathbb{P}}\}_{i=1}^2$ constructed in (2.15). From Section 2.4 we conclude that

$$(J^2(T\mathbb{P}(\mathbb{V})), B_0, \{F_i^{\mathbb{P}}\}_{i=1}^2, \mathbb{D}_0)$$

is an $\mathrm{SO}(3, \mathbb{C})$ -oper on $\mathbb{P}(\mathbb{V})$, where \mathbb{D}_0 is the holomorphic connection on $J^2(T\mathbb{P}(\mathbb{V}))$ constructed in (2.11). Consider the action of $\mathrm{PGL}(\mathbb{V})$ on $J^2(T\mathbb{P}(\mathbb{V}))$ given by the action of $\mathrm{PGL}(\mathbb{V})$ on $\mathbb{P}(\mathbb{V})$. Both B_0 and the filtration $\{F_i^{\mathbb{P}}\}_{i=1}^2$ are clearly preserved by this action of $\mathrm{PGL}(\mathbb{V})$ on $J^2(T\mathbb{P}(\mathbb{V}))$. As noted in the proof of Proposition 2.8(1), the connection \mathbb{D}_0 is preserved by the action of $\mathrm{PGL}(\mathbb{V})$ on $J^2(T\mathbb{P}(\mathbb{V}))$.

Let $(W, B_W, \{F_i^W\}_{i=1}^2, D_W)$ be an $\mathrm{SO}(3, \mathbb{C})$ -oper on \mathbb{X} . Denote by $P(W)$ the projective bundle on \mathbb{X} that parametrizes the lines in the fibers of W . Let

$$\mathbf{P}_W \subset P(W) \tag{4.1}$$

be the $\mathbb{C}\mathbb{P}^1$ -bundle on \mathbb{X} that parametrizes the isotropic lines for B_W (the lines on which the corresponding quadratic form vanishes). So the given condition $B_W(F_1^W \otimes F_1^W) = 0$ implies that the line subbundle $F_1^W \subset W$ produces a holomorphic section

$$s_W : \mathbb{X} \longrightarrow \mathbf{P}_W$$

of the $\mathbb{C}\mathbb{P}^1$ -bundle in (4.1). The connection D_W produces a holomorphic connection on $P(W)$, which, in turn, induces a holomorphic connection on \mathbf{P}_W ; this holomorphic connection on \mathbf{P}_W will be denoted by \mathcal{H}_W . It is straight-forward to check that the triple $(\mathbf{P}_W, \mathcal{H}_W, s_W)$

defines a projective structure on \mathbb{X} (the condition for the triple to define a projective structure is recalled in Section 2.1).

Conversely, let \mathcal{P} be a projective structure on \mathbb{X} . Take a holomorphic coordinate atlas $\{(U_i, \phi_i)\}_{i \in I}$ in the equivalence class defining \mathcal{P} . Using ϕ_i ,

- the bilinear form B_0 on $J^2(T\mathbb{P}(\mathbb{V}))|_{\phi_i(U_i)}$ constructed in (2.14) produces a bilinear form $B_{\mathcal{P}}(i)$ on $J^2(T\mathbb{X})|_{U_i}$,
- the filtration $\{F_j^{\mathbb{P}}\}_{j=1}^2$ of $J^2(T\mathbb{P}(\mathbb{V}))|_{\phi_i(U_i)}$ constructed in (2.15) produces a filtration $\{F_j^{\mathcal{P}}(i)\}_{j=1}^2$ of $J^2(T\mathbb{X})|_{U_i}$, and
- the holomorphic connection $\mathbb{D}_0|_{\phi_i(U_i)}$ in (2.11) on $J^2(T\mathbb{P}(\mathbb{V}))|_{\phi_i(U_i)}$ produces a holomorphic connection $\mathbb{D}_{\mathcal{P}}(i)$ on $J^2(T\mathbb{X})|_{U_i}$.

Since B_0 , $\{F_j^{\mathbb{P}}\}_{j=1}^2$ and \mathbb{D}_0 are all $\mathrm{PGL}(\mathbb{V})$ -equivariant, each of the locally defined structures $\{B_{\mathcal{P}}(i)\}_{i \in I}$, $\{\{F_j^{\mathcal{P}}(i)\}_{j=1}^2\}_{i \in I}$ and $\{\mathbb{D}_{\mathcal{P}}(i)\}_{i \in I}$ patch together compatibly to define

- a holomorphic nondegenerate symmetric bilinear form $B_{\mathcal{P}}$ on $J^2(T\mathbb{X})$,
- a filtration $\{F_j^{\mathcal{P}}\}_{j=1}^2$ of holomorphic subbundles of $J^2(T\mathbb{X})$, and
- a holomorphic connection $\mathbb{D}_{\mathcal{P}}$ on $J^2(T\mathbb{X})$.

Since $(J^2(T\mathbb{P}(\mathbb{V})), B_0, \{F_i^{\mathbb{P}}\}_{i=1}^2, \mathbb{D}_0)$ is an $\mathrm{SO}(3, \mathbb{C})$ -oper on $\mathbb{P}(\mathbb{V})$, we conclude that

$$(J^2(T\mathbb{X}), B_{\mathcal{P}}, \{F_j^{\mathcal{P}}\}_{j=1}^2, \mathbb{D}_{\mathcal{P}})$$

is an $\mathrm{SO}(3, \mathbb{C})$ -oper on \mathbb{X} .

It is straight-forward to check that the above two constructions, namely from $\mathrm{SO}(3, \mathbb{C})$ -opers on \mathbb{X} to projective structures on \mathbb{X} and vice versa, are inverses of each other.

The above construction of an $\mathrm{SO}(3, \mathbb{C})$ -oper on \mathbb{X} from \mathcal{P} has the following alternative description.

Let $\gamma : \mathbf{P}_{\mathcal{P}} \rightarrow \mathbb{X}$ be a holomorphic $\mathbb{C}\mathbb{P}^1$ -bundle, $\mathcal{H}_{\mathcal{P}}$ a holomorphic connection on $\mathbf{P}_{\mathcal{P}}$ and $s_{\mathcal{P}}$ a holomorphic section of γ , such that the triple

$$(\gamma, \mathcal{H}_{\mathcal{P}}, s_{\mathcal{P}})$$

gives the projective structure \mathcal{P} (see Section 2.1). Let

$$W := \gamma_* T_{\gamma} \rightarrow \mathbb{X}$$

be the direct image, where T_{γ} is defined in (2.1). For any $x \in \mathbb{X}$, the fiber W_x is identified with $H^0(\gamma^{-1}(x), T(\gamma^{-1}(x)))$, so W_x is a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$; the Lie algebra structure is given by the Lie bracket of vector fields. Let B_W denote the Killing form on W .

If \mathbb{V} is a rank two holomorphic vector bundle on \mathbb{X} such that $\mathbb{P}(\mathbb{V}) = \mathbf{P}_{\mathcal{P}}$, then $W = \mathrm{ad}(\mathbb{V}) \subset \mathrm{End}(\mathbb{V})$ (the subalgebra bundle of trace zero endomorphisms). It should be clarified, that although \mathbb{V} is not uniquely determined by $\mathbf{P}_{\mathcal{P}}$, any two choices of \mathbb{V} with $\mathbb{P}(\mathbb{V}) = \mathbf{P}_{\mathcal{P}}$ differ by tensoring with a holomorphic line bundle on \mathbb{X} . As a consequence, $\mathrm{End}(\mathbb{V})$ and $\mathrm{ad}(\mathbb{V})$ are uniquely determined by $\mathbf{P}_{\mathcal{P}}$. The bilinear form B_W coincides with the bilinear form on $\mathrm{ad}(\mathbb{V})$ defined by the Killing bilinear form of the endomorphism algebra (defined from the trace).

Since the image $s_{\mathcal{P}}(\mathbb{X}) \subset \mathbf{P}_{\mathcal{P}}$ of the section $s_{\mathcal{P}}$ is a divisor, the holomorphic vector bundle W has a filtration of holomorphic subbundles

$$F_1^{\mathbb{P}} := \gamma_*(T_{\gamma} \otimes \mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-2s_{\mathcal{P}}(\mathbb{X}))) \subset F_2^{\mathbb{P}} := \gamma_*(T_{\gamma} \otimes \mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-s_{\mathcal{P}}(\mathbb{X}))) \subset \gamma_*T_{\gamma} =: W. \quad (4.2)$$

Recall from Section 2.1 that the homomorphism $\widehat{ds_{\mathcal{P}}} : T\mathbb{X} \rightarrow (s_{\mathcal{P}})^*T_{\gamma}$ (see (2.3)) is an isomorphism. Therefore, $\widehat{ds_{\mathcal{P}}}$ gives a holomorphic isomorphism between $T\mathbb{X}$ and the normal bundle $\mathbf{N} = N_{s_{\mathcal{P}}(\mathbb{X})}$ of the divisor $s_{\mathcal{P}}(\mathbb{X}) \subset \mathbf{P}_{\mathcal{P}}$. On the other hand, by the Poincaré adjunction formula, \mathbf{N}^* is identified with the restriction $\mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-s_{\mathcal{P}}(\mathbb{X}))|_{s_{\mathcal{P}}(\mathbb{X})}$ of the holomorphic line bundle $\mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-s_{\mathcal{P}}(\mathbb{X}))$ to the divisor $s_{\mathcal{P}}(\mathbb{X})$; see [GH, p. 146] for the Poincaré adjunction formula. Therefore, the pulled back holomorphic line bundle

$$(s_{\mathcal{P}})^*(T_{\gamma} \otimes \mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-2s_{\mathcal{P}}(\mathbb{X}))) \quad (4.3)$$

is identified with $T\mathbb{X} \otimes K_{\mathbb{X}}^{\otimes 2} = K_{\mathbb{X}}$.

Since the line bundle in (4.3) is canonically identified with $F_1^{\mathbb{P}}$ in (4.2), we conclude that $F_1^{\mathbb{P}}$ is identified with $K_{\mathbb{X}}$.

The quotient line bundle $F_2^{\mathbb{P}}/F_1^{\mathbb{P}}$ in (4.2) is identified with

$$(s_{\mathcal{P}})^*(T_{\gamma} \otimes \mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-s_{\mathcal{P}}(\mathbb{X}))).$$

Since $(s_{\mathcal{P}})^*(\mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-s_{\mathcal{P}}(\mathbb{X})))$ is identified with $K_{\mathbb{X}}$, it follows that $F_2^{\mathbb{P}}/F_1^{\mathbb{P}}$ is identified with $\mathcal{O}_{\mathbb{X}}$.

For any $x \in \mathbb{X}$, the subspace $(F_1^{\mathbb{P}})_x \subset W_x$ is a nilpotent subalgebra, and $(F_2^{\mathbb{P}})_x \subset W_x$ is the unique Borel subalgebra containing $(F_1^{\mathbb{P}})_x$. Hence we have $B_W(F_1^{\mathbb{P}} \otimes F_1^{\mathbb{P}}) = 0$ and $(F_1^{\mathbb{P}})^{\perp} = F_2^{\mathbb{P}}$.

Consequently, $(W, B_W, \{F_i^{\mathbb{P}}\}_{i=1}^2)$ is a filtered $\mathrm{SO}(3, \mathbb{C})$ -bundle on \mathbb{X} . The holomorphic connection $\mathcal{H}_{\mathcal{P}}$ on $\mathbf{P}_{\mathcal{P}}$ produces a holomorphic connection

$$\mathbb{D}_{\mathcal{P}} \quad (4.4)$$

on W . Indeed, from the fact that $\mathrm{Aut}(\mathbb{C}\mathbb{P}^1) = \mathrm{PSL}(2, \mathbb{C})$ it follows that the $\mathbb{C}\mathbb{P}^1$ -bundle $\mathbf{P}_{\mathcal{P}}$ gives a holomorphic principal $\mathrm{PSL}(2, \mathbb{C})$ -bundle $\mathbf{P}_{\mathcal{P}}(\mathrm{PSL}(2, \mathbb{C}))$ on \mathbb{X} , and $\mathcal{H}_{\mathcal{P}}$ produces a holomorphic connection on $\mathbf{P}_{\mathcal{P}}(\mathrm{PSL}(2, \mathbb{C}))$. This holomorphic connection on $\mathbf{P}_{\mathcal{P}}(\mathrm{PSL}(2, \mathbb{C}))$ produces a holomorphic connection on the vector bundle W associated to $\mathbf{P}_{\mathcal{P}}(\mathrm{PSL}(2, \mathbb{C}))$ for the adjoint action of $\mathrm{PSL}(2, \mathbb{C})$ on its Lie algebra $\mathfrak{sl}(2, \mathbb{C})$; this connection on W is denoted by $\mathbb{D}_{\mathcal{P}}$.

The connection $\mathbb{D}_{\mathcal{P}}$ in (4.4) is in fact a holomorphic connection on the filtered $\mathrm{SO}(3, \mathbb{C})$ -bundle $(W, B_W, \{F_i^{\mathbb{P}}\}_{i=1}^2)$. The resulting $\mathrm{SO}(3, \mathbb{C})$ -oper $(W, B_W, \{F_i^{\mathbb{P}}\}_{i=1}^2, \mathbb{D}_{\mathcal{P}})$ coincides with the one constructed earlier from \mathcal{P} .

5. BRANCHED PROJECTIVE STRUCTURES AND LOGARITHMIC CONNECTIONS

5.1. Branched projective structure. Let X be a connected Riemann surface. Fix a nonempty finite subset

$$\mathbb{S} := \{x_1, \dots, x_d\} \subset X \quad (5.1)$$

of d distinct points. For each x_i , $1 \leq i \leq d$, fix an integer $n_i \geq 1$. Let

$$S := \sum_{i=1}^d n_i \cdot x_i \quad (5.2)$$

be the effective divisor on X .

The group of all holomorphic automorphisms of $\mathbb{C}\mathbb{P}^1$ is the Möbius group $\mathrm{PGL}(2, \mathbb{C})$. Any

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

acts on $\mathbb{C}\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ as $z \mapsto \frac{az+b}{cz+d}$; the center $\mathbb{Z}/2\mathbb{Z}$ of $\mathrm{SL}(2, \mathbb{C})$ acts trivially, thus producing an action of $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/(\mathbb{Z}/2\mathbb{Z})$ on $\mathbb{C}\mathbb{P}^1$.

A branched projective structure on X with branching type S (defined in (5.2)) is given by data

$$\{(U_j, \phi_j)\}_{j \in J}, \quad (5.3)$$

where

- (1) $U_j \subset X$ is a connected open subset with $\#(U_j \cap S_0) \leq 1$ such that $\bigcup_{j \in J} U_j = X$,
- (2) $\phi_j : U_j \rightarrow \mathbb{C}\mathbb{P}^1$ is a holomorphic map which is an immersion on the complement $U_j \setminus (U_j \cap S_0)$,
- (3) if $U_j \cap S_0 = x_i$, then ϕ_j is of degree $n_i + 1$ and totally ramified at x_i , while ϕ_j is an embedding if $U_j \cap S_0 = \emptyset$, and
- (4) for every $j, j' \in J$ with $U_j \cap U_{j'} \neq \emptyset$, and every connected component $U \subset U_j \cap U_{j'}$, there is an element $f_{j,j'}^U \in \mathrm{PGL}(2, \mathbb{C})$, such that $\phi_j = f_{j,j'}^U \circ \phi_{j'}$ on U .

Two such data $\{(U_j, \phi_j)\}_{j \in J}$ and $\{(U'_j, \phi'_j)\}_{j \in J'}$ satisfying the above conditions are called *equivalent* if their union $\{(U_j, \phi_j)\}_{j \in J} \cup \{(U'_j, \phi'_j)\}_{j \in J'}$ also satisfies the above conditions.

A *branched projective structure* on X with branching type S is an equivalence class of data $\{(U_j, \phi_j)\}_{j \in J}$ satisfying the above conditions. This definition was introduced in [Ma1], [Ma2] (see also [BiDu] for the more general notion of a branched Cartan geometry).

We will now describe an equivalent formulation of the definition of a branched projective structure.

Consider a triple (γ, \mathcal{H}, s) , where

- $\gamma : \mathbf{P} \rightarrow \mathbb{X}$ is a holomorphic $\mathbb{C}\mathbb{P}^1$ -bundle,
- $\mathcal{H} \subset T\mathbf{P}$ is a holomorphic connection (as before, $T\mathbf{P}$ is the holomorphic tangent bundle and $\mathcal{H} \oplus \ker(d\gamma) = T\mathbf{P}$), and
- $s : \mathbb{X} \rightarrow \mathbf{P}$ is a holomorphic section of γ ,

such that the divisor for the homomorphism \widehat{ds} in (2.3) coincides with S in (5.2). This triple (γ, \mathcal{H}, s) gives a branched projective structure on \mathbb{X} with branching type S .

Two such triples $(\gamma_1, \mathcal{H}_1, s_1)$ and $(\gamma_2, \mathcal{H}_2, s_2)$ are called equivalent if there is a holomorphic isomorphism

$$\mathbb{I} : \mathbf{P}_1 \rightarrow \mathbf{P}_2$$

such that

- $\gamma_1 = \gamma_2 \circ \mathbb{I}$,
- $d\mathbb{I}(\mathcal{H}_1) = \mathbb{I}^*\mathcal{H}_2$, where $d\mathbb{I} : TP_1 \rightarrow \mathbb{I}^*TP_2$ is the differential of the map \mathbb{I} , and
- $\mathbb{I} \circ s_1 = s_2$.

Two equivalent triples produce the same branched projective structure on \mathbb{X} . More precisely, this map from the equivalence classes of triples to the branched projective structures on \mathbb{X} with branching type S is both injective and surjective. More details on this can be found in [BDG, Section 2.1].

For the convenience of the exposition, we would assume that we are in the generic situation where the divisor S in (5.2) is reduced. In other words, $n_i = 1$ for all $1 \leq i \leq d$, and

$$S = \sum_{i=1}^d x_i. \quad (5.4)$$

5.2. Logarithmic connections. Let Y be a connected Riemann surface and $\mathcal{S} = \sum_{i=1}^m y_i$ be an effective divisor on Y , where y_1, \dots, y_m are m distinct points of Y . The holomorphic cotangent bundle of Y will be denoted by K_Y . For any point $y \in \mathcal{S}$, the fiber $(K_Y \otimes \mathcal{O}_Y(\mathcal{S}))_y$ is identified with \mathbb{C} by sending any meromorphic 1-form defined around y to its residue at y . In other words, for any holomorphic coordinate function z on Y defined around the point y , with $z(y) = 0$, consider the isomorphism

$$R_y : (K_Y \otimes \mathcal{O}_Y(\mathcal{S}))_y \rightarrow \mathbb{C}, \quad c \cdot \frac{dz}{z} \mapsto c. \quad (5.5)$$

This isomorphism is in fact independent of the choice of the above coordinate function z .

Let W be a holomorphic vector bundle on Y .

A *logarithmic connection* on W singular over \mathcal{S} is a holomorphic differential operator of order one

$$D : W \rightarrow W \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S})$$

such that $D(fs) = fD(s) + s \otimes df$ for all locally defined holomorphic function f on Y and all locally defined holomorphic section s of W . This means that

$$D \in H^0(Y, \text{Diff}_Y^1(W, W \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S})))$$

such that the symbol of D is the holomorphic section of $\text{End}(W) \otimes \mathcal{O}_Y(\mathcal{S})$ given by $\text{Id}_W \otimes 1$, where 1 is the constant function 1 on Y .

Note that a logarithmic connection D defines a holomorphic connection on the restriction of W to the complement $Y \setminus \mathcal{S}$. The logarithmic connection D is called an extension of this holomorphic connection on $W|_{Y \setminus \mathcal{S}}$.

For a logarithmic connection D_0 on W singular over \mathcal{S} , and a point $y \in \mathcal{S}$, consider the composition of homomorphisms

$$W \xrightarrow{D_0} W \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S}) \xrightarrow{\text{Id}_W \otimes R_y} W_y \otimes \mathbb{C} = W_y,$$

where R_y is the homomorphism in (5.5). This composition of homomorphism vanishes on the subsheaf $W \otimes \mathcal{O}_Y(-y) \subset W$, and hence it produces a homomorphism

$$\text{Res}(D_0, y) : W/(W \otimes \mathcal{O}_Y(-y)) = W_y \rightarrow W_y.$$

This endomorphism $\mathrm{Res}(D_0, y)$ of W_y is called the *residue* of the logarithmic connection D_0 at the point y ; see [De, p. 53].

The notion of second fundamental form in Section 2.4 extends to the set-up of logarithmic connections.

Let W be a holomorphic vector bundle on Y equipped with a logarithmic connection D singular over \mathcal{S} . Let $W' \subset W$ be a holomorphic subbundle. Then the composition of homomorphisms

$$W \xrightarrow{D} W \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S}) \xrightarrow{q_{W'} \otimes \mathrm{Id}} (W/W') \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S}),$$

where $q_{W'} : W \rightarrow W/W'$ is the natural quotient map, defines a holomorphic section of $\mathrm{Hom}(W', (W/W') \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S}))$, which is called the *second fundamental form* of W' for D . If $W'' \subset W$ is a holomorphic subbundle containing W' such that $D(W') \subset W'' \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S})$, then the second fundamental form of W' for D is given by a section

$$\zeta_1 \in H^0(Y, \mathrm{Hom}(W', W''/W') \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S})) \quad (5.6)$$

using the natural inclusion map

$$H^0(Y, \mathrm{Hom}(W', W''/W') \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S})) \hookrightarrow H^0(Y, \mathrm{Hom}(W', W/W') \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S})).$$

The second fundamental form of this subbundle W'' for D is given by a section

$$\zeta_2 \in H^0(Y, \mathrm{Hom}(W''/W', W/W'') \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S})) \quad (5.7)$$

through the natural inclusion map

$$H^0(Y, \mathrm{Hom}(W''/W', W/W'') \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S})) \hookrightarrow H^0(Y, \mathrm{Hom}(W'', W/W'') \otimes K_Y \otimes \mathcal{O}_Y(\mathcal{S})).$$

5.3. Logarithmic connection from a branched projective structure. We now use the notation of Section 5.1.

Let

$$\mathbb{T} := (TX) \otimes \mathcal{O}_X(S) \quad (5.8)$$

be the holomorphic line bundle on X , where S is the divisor in (5.4). From the isomorphism in (5.5) it follows that the fiber $\mathcal{O}_X(S)_y$ is identified with $T_y X$ for every $y \in \mathbb{S}$ in (5.1). For any point $y \in \mathbb{S}$, let

$$F_y^1 := (TX) \otimes \mathcal{O}_X(S) \otimes (T^*X)^{\otimes 2}_y = \mathbb{C} \subset F_y^2 \subset J^2(\mathbb{T})_y \quad (5.9)$$

be the filtration of subspaces of the fiber $J^2(\mathbb{T})_y$ over y constructed as in (3.7); more precisely, F_y^1 is the kernel of the projection $\alpha_y : J^2(\mathbb{T})_y \rightarrow J^1(\mathbb{T})_y$ (see (2.4)), and

$$F_y^2 = \alpha_y^{-1}(\mathbb{T}_y \otimes (K_X)_y) = \alpha_y^{-1}(\mathcal{O}(S)_y)$$

(see (2.4)). In other words, F_y^2 is the kernel of the composition of homomorphisms

$$J^2(\mathbb{T})_y \xrightarrow{\alpha_y} J^1(\mathbb{T})_y \rightarrow \mathbb{T}_y$$

(see (2.4)). Note that $(TX) \otimes \mathcal{O}_X(S) \otimes (T^*X)^{\otimes 2}_y = \mathbb{C}$, because $\mathcal{O}_X(S)_y = T_y X$.

The complement $X \setminus \mathbb{S}$ will be denoted by \mathbb{X} , where \mathbb{S} is the subset in (5.1).

Let P be a branched projective structure on X of branching type S , where S is the divisor in (5.4). So P gives a projective structure on $\mathbb{X} := X \setminus \mathbb{S}$; this projective structure on \mathbb{X}

will be denoted by \mathcal{P} . As shown in Proposition 2.8(1), the projective structure \mathcal{P} produces a holomorphic connection $\mathbb{D}(\mathcal{P})$ on the vector bundle $J^2(T\mathbb{X})$.

Proposition 5.1. *The above holomorphic connection $\mathbb{D}(\mathcal{P})$ on $J^2(T\mathbb{X})$ extends to a logarithmic connection $\mathbb{D}(P)$ on $J^2(\mathbb{T})$, where \mathbb{T} is the holomorphic line bundle in (5.8).*

For any $x_i \in \mathbb{S}$ in (5.1), the eigenvalues of the residue $\text{Res}(\mathbb{D}(P), x_i)$ are $\{-2, -1, 0\}$.

The eigenspace for the eigenvalue -2 of $\text{Res}(\mathbb{D}(P), x_i)$ is the line $F_{x_i}^1$ in (5.9). The eigenspace for the eigenvalue -1 of $\text{Res}(\mathbb{D}(P), x_i)$ is contained in the subspace $F_{x_i}^2$ in (5.9).

Proof. This proposition follows from some general properties of logarithmic connections which we shall explain first.

Let Y be a Riemann surface and $y_0 \in Y$ a point; let $\iota : y_0 \hookrightarrow Y$ be the inclusion map. Take a holomorphic vector bundle \mathcal{W} on Y , and let $\mathcal{W}_{y_0} \rightarrow Q \rightarrow 0$ be a quotient of the fiber of \mathcal{W} over the point y_0 . Let

$$0 \rightarrow V \xrightarrow{\beta} \mathcal{W} \rightarrow \mathcal{Q} := i_*Q \rightarrow 0 \quad (5.10)$$

be a short exact sequence of coherent analytic sheaves on Y ; so the sheaf \mathcal{Q} is supported on the reduced point y_0 . Let

$$0 \rightarrow \text{kernel}(\beta(y_0)) \rightarrow V_{y_0} \xrightarrow{\beta(y_0)} \mathcal{W}_{y_0} \rightarrow \text{cokernel}(\beta(y_0)) = Q \rightarrow 0 \quad (5.11)$$

be the exact sequence of vector spaces obtained by restricting the exact sequence in (5.10) to the point y_0 . It can be shown that there is a canonical isomorphism

$$\text{kernel}(\beta(y_0)) \xrightarrow{\sim} Q \otimes (T_{y_0}Y)^*. \quad (5.12)$$

To prove this, take any $v \in \text{kernel}(\beta(y_0))$, and let \tilde{v} be a holomorphic section of V defined around y_0 such that $\tilde{v}(y_0) = v$. The locally defined section $\beta(\tilde{v})$ of \mathcal{W} vanishes at y_0 , so its 1-jet at y_0 gives an element of $\mathcal{W}_{y_0} \otimes (T_{y_0}Y)^*$; this element of $\mathcal{W}_{y_0} \otimes (T_{y_0}Y)^*$ will be denoted by v_1 . Let $v' \in Q \otimes (T_{y_0}Y)^*$ denote the image of v_1 under the homomorphism $\beta(y_0) \times \text{Id}_{(T_{y_0}Y)^*}$. It is straightforward to check that v' does not depend on the choice of the above extension \tilde{v} of v . Consequently, we get a homomorphism as in (5.12). This homomorphism is in fact an isomorphism.

Let $\nabla^{\mathcal{W}}$ be a logarithmic connection on \mathcal{W} singular over y_0 . Then $\nabla^{\mathcal{W}}$ induces a logarithmic connection on the subsheaf V in (5.10) if and only if the residue $\text{Res}(\nabla^{\mathcal{W}}, y_0) \in \text{End}(\mathcal{W}_{y_0})$ preserves the subspace $\beta(y_0)(V_{y_0}) \subset \mathcal{W}_{y_0}$ in (5.11).

Now assume that $\nabla^{\mathcal{W}}$ induces a logarithmic connection ∇^1 on V . Then $\text{Res}(\nabla^1, y_0)$ preserves the subspace $\text{kernel}(\beta(y_0)) \subset V_{y_0}$ in (5.11), and the endomorphism of

$$V_{y_0}/\text{kernel}(\beta(y_0)) = \beta(y_0)(V_{y_0})$$

induced by $\text{Res}(\nabla^1, y_0)$ coincides with the restriction of $\text{Res}(\nabla^{\mathcal{W}}, y_0)$ to $\beta(y_0)(V_{y_0})$. Let $\text{Res}(\nabla^{\mathcal{W}}, y_0)_Q \in \text{End}(Q)$ be the endomorphism induced by $\text{Res}(\nabla^{\mathcal{W}}, y_0)$. The restriction of $\text{Res}(\nabla^1, y_0)$ to $\text{kernel}(\beta(y_0)) \subset V_{y_0}$ coincides with $\text{Id} + \text{Res}(\nabla^{\mathcal{W}}, y_0)_Q$; from (5.12) it follows that

$$\text{End}(\text{kernel}(\beta(y_0))) = \text{End}(Q),$$

so $\text{Res}(\nabla^{\mathcal{W}}, y_0)_Q$ gives an endomorphism of $\text{kernel}(\beta(y_0))$.

Conversely, let ∇^V be a logarithmic connection on V singular over y_0 . Then ∇^V induces a logarithmic connection on the holomorphic vector bundle \mathcal{W} in (5.10) if and only if the residue $\mathrm{Res}(\nabla^V, y_0) \in \mathrm{End}(V_{y_0})$ preserves the subspace $\mathrm{kernel}(\beta(y_0)) \subset V_{y_0}$ in (5.11).

Now assume that ∇^V induces a logarithmic connection ∇' on \mathcal{W} . Then ∇' gives the logarithmic connection ∇^V on V . Consequently, the residues of ∇^V and ∇' are related in the fashion described above.

Take a point $x_i \in \mathbb{S}$ (see (5.1)). Let $x_i \in U \subset X$ be a sufficiently small contractible open neighborhood of x_i in X , in particular, $U \cap \mathbb{S} = x_i$. Let

$$\mathbb{D}_1 := \{z \in \mathbb{C} \mid |z| < 1\} \subset \mathbb{C}$$

be the unit disk. Fix a biholomorphism

$$\tilde{\gamma} : \mathbb{D}_1 \longrightarrow U$$

such that $\tilde{\gamma}(0) = x_i$. Let

$$\gamma : U \longrightarrow \mathbb{D}_1, \quad x \longmapsto (\tilde{\gamma}^{-1}(x))^2 \tag{5.13}$$

be the branched covering map. Then using γ , the branched projective structure on U , given by the branched projective structure P on X of branching type S , produces an usual (unbranched) projective structure on \mathbb{D}_1 ; this projective structure on \mathbb{D}_1 will be denoted by P_1 .

Now substituting (\mathbb{D}_1, P_1) in place of $(\mathbb{X}, \mathcal{P})$ in Proposition 2.8(1), we get a holomorphic connection $\mathbb{D}(P_1)$ on $J^2(T\mathbb{D}_1)$. The holomorphic vector bundle $\gamma^*J^2(T\mathbb{D}_1)$ over U , where γ is the map in (5.13), is equipped with the holomorphic connection $\gamma^*\mathbb{D}(P_1)$.

The differential $d\gamma : TU \longrightarrow \gamma^*T\mathbb{D}_1$ induces an isomorphism $\mathbb{T}|_U \xrightarrow{\sim} \gamma^*T\mathbb{D}_1$, where \mathbb{T} is the line bundle in (5.8). This, in turn, produces an isomorphism

$$J^2(\mathbb{T})|_U \xrightarrow{\sim} J^2(\gamma^*T\mathbb{D}_1). \tag{5.14}$$

On the other hand, there is a natural homomorphism $\gamma^*J^2(T\mathbb{D}_1) \longrightarrow J^2(\gamma^*T\mathbb{D}_1)$. Combining this with the inverse of the isomorphism in (5.14) we get a homomorphism

$$\gamma^*J^2(T\mathbb{D}_1) \longrightarrow J^2(\mathbb{T})|_U. \tag{5.15}$$

This homomorphism is an isomorphism over $U \setminus \{x_i\}$.

Apply the above mentioned criterion, for the extension of a logarithmic connection on the vector bundle V in (5.10) to a logarithmic connection on the vector bundle \mathcal{W} , to the homomorphism in (5.15) and the holomorphic connection $\gamma^*\mathbb{D}(P_1)$ on $\gamma^*J^2(T\mathbb{D}_1)$. We conclude that $\gamma^*\mathbb{D}(P_1)$ induces a logarithmic connection on $J^2(\mathbb{T})|_U$. The first statement of the proposition follows from this.

The other two statements of the proposition follow from the earlier mentioned properties of residues of the logarithmic connections on the vector bundles V and \mathcal{W} in (5.10) that are induced by each other. \square

6. BRANCHED PROJECTIVE STRUCTURES AND BRANCHED $\mathrm{SO}(3, \mathbb{C})$ -OPERS

6.1. **Branched $\mathrm{SO}(3, \mathbb{C})$ -opers.** We shall now use the terminology in Section 4.

Definition 6.1. Let (W, B_W) be a holomorphic $\mathrm{SO}(3, \mathbb{C})$ -bundle on a connected Riemann surface X . A *branched filtration* on (W, B_W) , of type S (see (5.4)), is a filtration of holomorphic subbundles

$$F_1^W \subset F_2^W \subset W \quad (6.1)$$

such that

- F_1^W is holomorphically identified with $K_X \otimes \mathcal{O}_X(-S)$ by a given isomorphism,
- $B_W(F_1^W \otimes F_1^W) = 0$,
- F_2^W/F_1^W is holomorphically identified with \mathcal{O}_X by a given isomorphism,
- $B_W(F_1^W \otimes F_2^W) = 0$ (equivalently, $(F_1^W)^\perp = F_2^W$).

Compare the above definition with the definition of a filtration of (W, B_W) given in Section 4. They differ only at the first condition.

The above conditions imply that

$$W/F_2^W = (\bigwedge^3 W) \otimes (K_X \otimes \mathcal{O}_X(-S))^* = TX \otimes \mathcal{O}_X(S). \quad (6.2)$$

A *branched filtered $\mathrm{SO}(3, \mathbb{C})$ -bundle*, of type S , is a holomorphic $\mathrm{SO}(3, \mathbb{C})$ -bundle (W, B_W) equipped with a branched filtration $\{F_i^W\}_{i=1}^2$ of type S as in (6.1).

Definition 6.2. A *branched holomorphic connection* on a branched filtered $\mathrm{SO}(3, \mathbb{C})$ -bundle $(W, B_W, \{F_i^W\}_{i=1}^2)$, of type S , is a holomorphic connection D_W on W such that

- the holomorphic connection D_W preserves the bilinear form B_W on W ,
- the holomorphic connection on $\bigwedge^3 W = \mathcal{O}_X$ induced by D_W coincides with the holomorphic connection on \mathcal{O}_X given by the de Rham differential d ,
- $D_W(F_1^W) \subset F_2^W \otimes K_X$, and
- the second fundamental form of F_1^W for D_W , which is a holomorphic section of

$$\mathrm{Hom}(K_X \otimes \mathcal{O}_X(-S), \mathcal{O}_X) \otimes K_X = \mathcal{O}_X(S)$$

(see (2.17)), coincides with the section of $\mathcal{O}_X(S)$ given by the constant function 1 on X .

Definition 6.3. A *branched $\mathrm{SO}(3, \mathbb{C})$ -oper*, of type S , on X is a branched filtered $\mathrm{SO}(3, \mathbb{C})$ -bundle

$$(W, B_W, \{F_i^W\}_{i=1}^2),$$

of type S , equipped with a branched holomorphic connection D_W .

Since the divisor S is fixed, we would drop mentioning it explicitly.

The above conditions imply that the holomorphic section of

$$\mathrm{Hom}(\mathcal{O}_X, TX \otimes \mathcal{O}_X(S)) \otimes K_X = \mathcal{O}_X(S)$$

that gives the second fundamental form of F_2^W for D_W — see (2.18) — coincides with the section of $\mathcal{O}_X(S)$ given by the constant function 1 on X .

6.2. Branched projective structures are branched $\mathrm{SO}(3, \mathbb{C})$ -opers. Take a branched $\mathrm{SO}(3, \mathbb{C})$ -oper $(W, B_W, \{F_i^W\}_{i=1}^2, D_W)$ on X of type S . Let $P(W)$ be the projective bundle on X that parametrizes the lines in the fibers of W . As in (4.1),

$$P(W) \supset \mathbf{P}_W \xrightarrow{\gamma} X$$

is the $\mathbb{C}\mathbb{P}^1$ -bundle on X that parametrizes the isotropic lines for B_W . The line subbundle $F_1^W \subset W$, being isotropic, produces a holomorphic section

$$s_W : X \longrightarrow \mathbf{P}_W \tag{6.3}$$

of the projection γ . The connection D_W produces a holomorphic connection on $P(W)$, and this connection on $P(W)$ induces a holomorphic connection on \mathbf{P}_W ; recall that a holomorphic connection on \mathbf{P}_W is a holomorphic line subbundle of $T\mathbf{P}_W$ transversal to the relative tangent bundle T_γ for the projection γ . Let

$$\mathcal{H}_W \subset T\mathbf{P}_W \tag{6.4}$$

be the holomorphic connection on \mathbf{P}_W given by D_W .

Lemma 6.4. *The triple $(\mathbf{P}_W, \mathcal{H}_W, s_W)$ (see (6.3) and (6.4)) defines a branched projective structure on X of branching type S .*

Proof. Recall that the last condition in the definition of a branched holomorphic connection on $(W, B_W, \{F_i^W\}_{i=1}^2)$ (see Definition 6.2) says that the second fundamental form of F_1^W for D_W is the holomorphic section of

$$\mathrm{Hom}(K_X \otimes \mathcal{O}_X(-S), \mathcal{O}_X) \otimes K_X = \mathcal{O}_X(S)$$

given by the constant function 1 on X . On the other hand, the divisor for the second fundamental form of F_1^W for D_W coincides with the divisor for the homomorphism

$$\widehat{ds_W} : TX \longrightarrow (s_W)^*T_\gamma$$

(see (2.3)). Consequently, the divisor for the homomorphism $\widehat{ds_W}$ is S . Hence $(\mathbf{P}_W, \mathcal{H}_W, s_W)$ defines a branched projective structure on X of branching type S ; see Section 5.1. \square

Now let \mathcal{P} be a branched projective structure on X of branching type S . Let

$$\gamma : \mathbf{P}_\mathcal{P} \longrightarrow X$$

be a holomorphic $\mathbb{C}\mathbb{P}^1$ -bundle, $\mathcal{H}_\mathcal{P}$ a holomorphic connection on $\mathbf{P}_\mathcal{P}$ and $s_\mathcal{P}$ a holomorphic section of γ , such that the triple

$$(\gamma, \mathcal{H}_\mathcal{P}, s_\mathcal{P}) \tag{6.5}$$

gives the branched projective structure \mathcal{P} ; see Section 5.1. Define the holomorphic vector bundle of rank three on X

$$W := \gamma_*T_\gamma \longrightarrow X, \tag{6.6}$$

where $T_\gamma \subset T\mathbf{P}_\mathcal{P}$ is the relative holomorphic tangent for the projection γ (as in (2.1)). If s and t are locally defined holomorphic sections of $T_\gamma \subset T\mathbf{P}_\mathcal{P}$, then the Lie bracket $[s, t]$ is also a section of T_γ . Consequently, each fiber of $W = \gamma_*T_\gamma$ is a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Let

$$B_W \in H^0(X, \mathrm{Sym}^2(W^*)) \tag{6.7}$$

be the holomorphic section given by the fiberwise Killing form on W .

As shown in Section 4, the pair (W, B_W) can also be constructed by choosing a rank two holomorphic vector bundle \mathbb{V} on X such that $\mathbb{P}(\mathbb{V}) = \mathbf{P}_{\mathcal{P}}$; then $W = \text{ad}(\mathbb{V}) \subset \text{End}(\mathbb{V})$ (the subalgebra bundle of trace zero endomorphisms) and B_W is constructed using the trace map of endomorphisms.

Exactly as in (4.2), construct the filtration

$$F_1^{\mathbb{P}} := \gamma_*(T_\gamma \otimes \mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-2s_{\mathcal{P}}(X))) \subset F_2^{\mathbb{P}} := \gamma_*(T_\gamma \otimes \mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-s_{\mathcal{P}}(X))) \subset \gamma_*T_\gamma =: W \quad (6.8)$$

of W .

Lemma 6.5. *The triple $(W, B_W, \{F_i^{\mathbb{P}}\}_{i=1}^2)$ constructed in (6.6), (6.7), (6.8) is a branched filtered $\text{SO}(3, \mathbb{C})$ -bundle of type S (see (5.4)).*

Proof. The holomorphic line bundle $F_1^{\mathbb{P}}$ in (6.8) admits a canonical isomorphism

$$F_1^{\mathbb{P}} \xrightarrow{\sim} (s_{\mathcal{P}})^*(T_\gamma \otimes \mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-2s_{\mathcal{P}}(X))), \quad (6.9)$$

because the restriction of $T_\gamma \otimes \mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-2s_{\mathcal{P}}(X))$ to any fiber of γ is holomorphically trivializable, and the evaluation, at a given point, of the global sections of a holomorphically trivializable bundle is an isomorphism. The quotient bundle $F_2^{\mathbb{P}}/F_1^{\mathbb{P}}$ admits a canonical isomorphism

$$F_2^{\mathbb{P}}/F_1^{\mathbb{P}} \xrightarrow{\sim} (s_{\mathcal{P}})^*(T_\gamma \otimes \mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-s_{\mathcal{P}}(X))), \quad (6.10)$$

which is again constructed by evaluating the holomorphic sections of $T_\gamma \otimes \mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-s_{\mathcal{P}}(X))|_{\gamma^{-1}(x)}$ at the point $s_{\mathcal{P}}(x)$ for every $x \in X$.

Now, by the Poincaré adjunction formula, $(s_{\mathcal{P}})^*\mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-s_{\mathcal{P}}(X))$ is identified with the dual bundle $(s_{\mathcal{P}})^*\mathbf{N}^*$, where $\mathbf{N} = N_{s_{\mathcal{P}}(X)}$ is the normal bundle of the divisor $s_{\mathcal{P}}(X) \subset \mathbf{P}_{\mathcal{P}}$. On the other hand, \mathbf{N} is canonically identified with the restriction $T_\gamma|_{s_{\mathcal{P}}(X)}$, because the divisor $s_{\mathcal{P}}(X)$ is transversal to the fibration γ . Consequently, $(s_{\mathcal{P}})^*\mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-s_{\mathcal{P}}(X))$ is identified with $(s_{\mathcal{P}})^*T_\gamma^*$.

Recall from Section 5.1 that the divisor for the homomorphism $\widehat{ds_{\mathcal{P}}} : TX \rightarrow (s_{\mathcal{P}})^*T_\gamma$ (see (2.3)) coincides with S . Consequently, $\widehat{ds_{\mathcal{P}}}$ identifies $(TX) \otimes \mathcal{O}_X(S)$ with $(s_{\mathcal{P}})^*T_\gamma$. Therefore, the line bundle $(s_{\mathcal{P}})^*(T_\gamma \otimes \mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-2s_{\mathcal{P}}(X)))$ in (6.9) is identified with $K_X \otimes \mathcal{O}_X(-S)$, and the line bundle $(s_{\mathcal{P}})^*(T_\gamma \otimes \mathcal{O}_{\mathbf{P}_{\mathcal{P}}}(-s_{\mathcal{P}}(X)))$ in (6.10) is identified with \mathcal{O}_X .

From the above descriptions of $F_1^{\mathbb{P}}$ and $F_2^{\mathbb{P}}$ it follows that for each point $x \in X$, the fiber $(F_1^{\mathbb{P}})_x \subset W_x$ is a nilpotent subalgebra of the Lie algebra W_x , and $(F_2^{\mathbb{P}})_x \subset W_x$ is the unique Borel subalgebra of W_x containing $(F_1^{\mathbb{P}})_x$. These imply that $B_W(F_1^{\mathbb{P}} \otimes F_1^{\mathbb{P}}) = 0$, and $(F_1^{\mathbb{P}})^\perp = F_2^{\mathbb{P}}$. Hence $(W, B_W, \{F_i^{\mathbb{P}}\}_{i=1}^2)$ is branched filtered $\text{SO}(3, \mathbb{C})$ -bundle of type S . \square

Consider the holomorphic connection $\mathcal{H}_{\mathcal{P}}$ in (6.5) on the holomorphic $\mathbb{C}\mathbb{P}^1$ -bundle $\mathbf{P}_{\mathcal{P}}$. It produces a holomorphic connection on the direct image $W = \gamma_*T_\gamma$; see (4.4). This holomorphic connection on W will be denoted by $\mathbb{D}_{\mathcal{P}}$.

Lemma 6.6. *The above connection $\mathbb{D}_{\mathcal{P}}$ on W is a branched holomorphic connection on the branched filtered $\text{SO}(3, \mathbb{C})$ -bundle $(W, B_W, \{F_i^{\mathbb{P}}\}_{i=1}^2)$ in Lemma 6.5. In other words,*

$$(W, B_W, \{F_i^{\mathbb{P}}\}_{i=1}^2, \mathbb{D}_{\mathcal{P}})$$

is a $\mathrm{SO}(3, \mathbb{C})$ -oper.

Proof. The connection $\mathbb{D}_{\mathcal{P}}$ preserves the bilinear form B_W on W , because the action of $\mathrm{PSL}(2, \mathbb{C})$ on $\mathrm{Sym}^2(\mathbb{C}^2)$ preserves the symmetric bilinear form on $\mathrm{Sym}^2(\mathbb{C}^2)$ given by the standard symplectic form on \mathbb{C}^2 . Also, the holomorphic connection on $\bigwedge^3 W$ induced by $\mathbb{D}_{\mathcal{P}}$ coincides with the holomorphic connection on \mathcal{O}_X given by the de Rham differential d , because the action of $\mathrm{PSL}(2, \mathbb{C})$ on $\bigwedge^3 \mathrm{Sym}^2(\mathbb{C}^2)$ is the trivial action.

Since any holomorphic connection on a holomorphic bundle over a Riemann surface is automatically integrable, the $\mathbb{C}\mathbb{P}^1$ -bundle $\mathbf{P}_{\mathcal{P}}$ is locally isomorphic the trivial holomorphic $\mathbb{C}\mathbb{P}^1$ -bundle, and $\mathcal{H}_{\mathcal{P}}$ in (6.5) is locally holomorphically isomorphic to the trivial connection on the trivial holomorphic $\mathbb{C}\mathbb{P}^1$ -bundle. So W is locally holomorphically isomorphic to the trivial vector bundle whose fibers are quadratic polynomials in one variable, and $\mathbb{D}_{\mathcal{P}}$ is the trivial connection on this locally defined trivial holomorphic vector bundle. With respect to these trivialization, and a suitable pair (U, ϕ) as in (5.3) compatible with the projective structure \mathcal{P} , the section $s_{\mathcal{P}}$ in (6.5) around any point $x_i \in \mathbb{S}$ is of the form $z \mapsto (z, z^2)$, where z is a holomorphic function around x_i with $z(x_i) = 0$.

In view of the above observations, from a straight-forward computation it follows that

- $\mathbb{D}_{\mathcal{P}}(F_1^{\mathbb{P}}) \subset F_2^{\mathbb{P}} \otimes K_X$, and
- the second fundamental form of $F_1^{\mathbb{P}}$ for $\mathbb{D}_{\mathcal{P}}$, which is a holomorphic section of

$$\mathrm{Hom}(K_X \otimes \mathcal{O}_X(-S), \mathcal{O}_X) \otimes K_X = \mathcal{O}_X(S),$$

coincides with the section of $\mathcal{O}_X(S)$ given by the constant function 1 on X .

This completes the proof. \square

The above construction of a branched projective structure on X from a branched $\mathrm{SO}(3, \mathbb{C})$ -oper (see Lemma 6.4), and the construction of a branched $\mathrm{SO}(3, \mathbb{C})$ -oper from a branched projective structure (see Lemma 6.6), are clearly inverses of each other.

We summarize the constructions done in this subsection in the following theorem.

Theorem 6.7. *There is a natural bijective correspondence between the branched projective structures on X with branching type S and the branched $\mathrm{SO}(3, \mathbb{C})$ -opers on X of type S .*

6.3. Logarithmic connection from branched $\mathrm{SO}(3, \mathbb{C})$ -opers. In Proposition 5.1 we constructed a logarithmic connection on $J^2(\mathbb{T})$ from a branched projective structure on X , where $\mathbb{T} = (TX) \otimes \mathcal{O}_X(S)$ (see (5.8)). On the other hand, Theorem 6.7 identifies branched projective structure on X with branched $\mathrm{SO}(3, \mathbb{C})$ -opers on X . Thus a branched $\mathrm{SO}(3, \mathbb{C})$ -oper gives a logarithmic connection on $J^2(\mathbb{T})$. Now we shall give a direct construction of the logarithmic connection on $J^2(\mathbb{T})$ associated to a branched $\mathrm{SO}(3, \mathbb{C})$ -oper on X .

Let

$$F_{\mathbb{T}}^1 \subset F_{\mathbb{T}}^2 \subset J^2(\mathbb{T}) \tag{6.11}$$

be the filtration of holomorphic subbundles constructed as in (3.7); so, $F_{\mathbb{T}}^1$ is the kernel of the natural projection

$$\mathbf{c}_1 : J^2(\mathbb{T}) \longrightarrow J^1(\mathbb{T}) \tag{6.12}$$

(see (2.4)), and $F_{\mathbb{T}}^2$ is the kernel of the homomorphism

$$\mathbf{c}_2 : J^2(\mathbb{T}) \longrightarrow \mathbb{T} \quad (6.13)$$

obtained by composing $J^2(\mathbb{T}) \xrightarrow{\mathbf{c}_1} J^1(\mathbb{T})$ with the natural homomorphism $J^1(\mathbb{T}) \longrightarrow \mathbb{T}$ in (2.4). Note that the restriction of the filtration in (6.11) to any point $y \in \mathbb{S}$ coincides with the filtration in (5.9). We have

$$F_{\mathbb{T}}^1 = \mathbb{T} \otimes K_X^2 = K_X \otimes \mathcal{O}_X(S), \quad F_{\mathbb{T}}^2/F_{\mathbb{T}}^1 = \mathbb{T} \otimes K_X = \mathcal{O}_X(S), \quad J^2(\mathbb{T})/F_{\mathbb{T}}^2 = \mathbb{T}. \quad (6.14)$$

Let

$$(W, B_W, \{F_i^W\}_{i=1}^2, D_W) \quad (6.15)$$

be a branched $\mathrm{SO}(3, \mathbb{C})$ -oper on X of type S . Recall from (6.2) and (5.8) that $W/F_2^W = TX \otimes \mathcal{O}_X(S) = \mathbb{T}$. Let

$$q_0 : W \longrightarrow \mathbb{T} = W/F_2^W \quad (6.16)$$

be the quotient map.

Take any point $x \in X$ and any $w \in W_x$. Let \tilde{w} be the unique holomorphic section of W , defined on a simply connected open neighborhood of x , such that \tilde{w} is flat with respect to the holomorphic connection D_W in (6.15), and

$$\tilde{w}(x) = w. \quad (6.17)$$

Let

$$\Phi : W \longrightarrow J^2(\mathbb{T}) \quad (6.18)$$

be the homomorphism that sends any $w \in W_x$, $x \in X$, to the element of $J^2(\mathbb{T})_x$ given by the restriction of the section $q_0(\tilde{w})$ to the second order infinitesimal neighborhood of x , where q_0 is the homomorphism in (6.16) and \tilde{w} is constructed as above from w . This construction is similar to the construction of the homomorphisms ψ_j in (2.8).

Proposition 6.8.

(1) For the homomorphism Φ in (6.18),

$$\Phi(F_1^W) \subset F_{\mathbb{T}}^1 \quad \text{and} \quad \Phi(F_2^W) \subset F_{\mathbb{T}}^2$$

where $\{F_{\mathbb{T}}^i\}_{i=1}^2$ and $\{F_i^W\}_{i=1}^2$ are the filtrations in (6.11) and (6.15) respectively.

(2) The homomorphism Φ takes the holomorphic connection D_W in (6.15) to a logarithmic connection on $J^2(\mathbb{T})$ whose singular locus is \mathbb{S} in (5.1). The logarithmic connection on $J^2(\mathbb{T})$ induced by D_W will be denoted by \mathbb{D}_J .

(3) For any $x_i \in \mathbb{S}$ (see (5.1)), the residue $\mathrm{Res}(\mathbb{D}_J, x_i)$ of \mathbb{D}_J at x_i has eigenvalues $\{-2, -1, 0\}$.

(4) The eigenspace of $\mathrm{Res}(\mathbb{D}_J, x_i)$ for the eigenvalue -2 is the line $(F_{\mathbb{T}}^1)_{x_i} \subset J^2(\mathbb{T})_{x_i}$ in (6.11). The eigenspace of $\mathrm{Res}(\mathbb{D}_J, x_i)$ for the eigenvalue -1 is contained in the subspace $(F_{\mathbb{T}}^2)_{x_i} \subset J^2(\mathbb{T})_{x_i}$ in (6.11).

Proof. To prove statement (1), first take any $x \in X$ and any $w \in (F_2^W)_x$. Since $q_0(w) = 0$, where q_0 is the projection in (6.16), from (6.17) it follows immediately that $\mathbf{c}_2 \circ \Phi(w) = 0$, where \mathbf{c}_2 is the homomorphism in (6.13). This implies that $\Phi(F_2^W) \subset F_{\mathbb{T}}^2$. To prove statement (1), we need to show that $\Phi(F_1^W) \subset F_{\mathbb{T}}^1$.

Take any $x \in X$ and any $w \in (F_1^W)_x$. From the given condition that $D_W(F_1^W) \subset F_2^W \otimes K_X$ it follows that $\mathbf{c}_1 \circ \Phi(w) = 0$, where \mathbf{c}_1 is the homomorphism in (6.12). More precisely, the restriction $\Phi|_{F_1^W}$ coincides with the natural inclusion homomorphism

$$F_1^W = K_X \otimes \mathcal{O}_X(-S) \hookrightarrow K_X \otimes \mathcal{O}_X(S) = F_{\mathbb{T}}^1;$$

see (6.14) for $K_X \otimes \mathcal{O}_X(S) = F_{\mathbb{T}}^1$ and Definition 6.1 for $F_1^W = K_X \otimes \mathcal{O}_X(-S)$. Consequently, we have $\Phi(F_1^W) \subset F_{\mathbb{T}}^1$. This proves (1).

Denote $X \setminus \mathbb{S}$ by \mathbb{X} . We note that the restriction

$$\Phi|_{\mathbb{X}} : W|_{\mathbb{X}} \longrightarrow J^2(\mathbb{T})|_{\mathbb{X}} = J^2(T\mathbb{X})$$

is a holomorphic isomorphism.

From Theorem 6.7 we know that the branched $\mathrm{SO}(3, \mathbb{C})$ -oper $(W, B_W, \{F_i^W\}_{i=1}^2, D_W)$ in (6.15) defines a branched projective structure on X of type S . Let \mathcal{P} denote the projective structure on $\mathbb{X} = X \setminus \mathbb{S}$ given by this branched projective structure on X . Using Proposition 2.8(1), the projective structure \mathcal{P} yields a holomorphic connection $\mathbb{D}(\mathcal{P})$ on $J^2(T\mathbb{X})$. This holomorphic connection $\mathbb{D}(\mathcal{P})$ evidently coincides with the holomorphic connection $\mathbb{D}_J|_{\mathbb{X}}$ on $J^2(\mathbb{T})|_{\mathbb{X}} = J^2(T\mathbb{X})$ given by the connection D_W using the isomorphism $\Phi|_{\mathbb{X}}$. Consequently, the statements (2), (3) and (4) in the proposition follow from Proposition 5.1 and statement (1). \square

Proposition 6.8 yields the following:

Corollary 6.9. *For the logarithmic connection \mathbb{D}_J on $J^2(\mathbb{T})$ in Proposition 6.8(2),*

$$\mathbb{D}_J(F_{\mathbb{T}}^1) = F_{\mathbb{T}}^2 \otimes K_X \otimes \mathcal{O}_X(S),$$

and

$$\mathbb{D}_J(F_{\mathbb{T}}^2) = J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(S),$$

where $\{F_{\mathbb{T}}^i\}_{i=1}^2$ is the filtration in (6.11).

Proof. From Proposition 6.8(2) we know that the logarithmic connection \mathbb{D}_J is given by the connection D_W using Φ . Consequently, the corollary follows from Proposition 6.8(1) and the properties, given in Proposition 6.8(3) and Proposition 6.8(4), of the residue of the logarithmic connection \mathbb{D}_J . \square

Recall the second fundamental form for a logarithmic connection defined in Section 5.2. Consider the logarithmic connection \mathbb{D}_J on $J^2(\mathbb{T})$ in Proposition 6.8(2) and the subbundles $F_{\mathbb{T}}^1, F_{\mathbb{T}}^2$ in (6.11). Let

$$\mathbf{S}(\mathbb{D}_J, F_{\mathbb{T}}^1) \quad \text{and} \quad \mathbf{S}(\mathbb{D}_J, F_{\mathbb{T}}^2)$$

be the second fundamental forms of $F_{\mathbb{T}}^1$ and $F_{\mathbb{T}}^2$ respectively for \mathbb{D}_J .

From Corollary 6.9 and (5.6) we know that

$$\mathbf{S}(\mathbb{D}_J, F_{\mathbb{T}}^1) \in H^0(X, \mathrm{Hom}(F_{\mathbb{T}}^1, F_{\mathbb{T}}^2/F_{\mathbb{T}}^1) \otimes K_X \otimes \mathcal{O}_X(S)) = H^0(X, \mathcal{O}_X(S)); \quad (6.19)$$

see (6.14). From Corollary 6.9 and (5.7) we have

$$\mathbf{S}(\mathbb{D}_J, F_{\mathbb{T}}^2) \in H^0(X, \mathrm{Hom}(F_{\mathbb{T}}^2/F_{\mathbb{T}}^1, J^2(\mathbb{T})/F_{\mathbb{T}}^2) \otimes K_X \otimes \mathcal{O}_X(S)) = H^0(X, \mathcal{O}_X(S)); \quad (6.20)$$

see (6.14).

Lemma 6.10. *The second fundamental forms $\mathbf{S}(\mathbb{D}_J, F_{\mathbb{T}}^1)$ and $\mathbf{S}(\mathbb{D}_J, F_{\mathbb{T}}^2)$, in (6.19) and (6.20) respectively, coincide with the section of $\mathcal{O}_X(S)$ given by the constant function 1 on X .*

Proof. As in the proof of Proposition 6.8, consider the branched projective structure on X of type S given by the $\mathrm{SO}(3, \mathbb{C})$ -oper $(W, B_W, \{F_i^W\}_{i=1}^2, D_W)$ in (6.15) using Theorem 6.7. It defines a projective structure on $\mathbb{X} = X \setminus \mathbb{S}$. Now from the statement (1) in Corollary 3.6 we conclude that the restrictions to \mathbb{X} of both $\mathbf{S}(\mathbb{D}_J, F_{\mathbb{T}}^1)$ and $\mathbf{S}(\mathbb{D}_J, F_{\mathbb{T}}^2)$ coincide with the section of $\mathcal{O}_{\mathbb{X}}$ given by the constant function 1 on \mathbb{X} . Hence the second fundamental forms $\mathbf{S}(\mathbb{D}_J, F_{\mathbb{T}}^1)$ and $\mathbf{S}(\mathbb{D}_J, F_{\mathbb{T}}^2)$ coincide with the section of $\mathcal{O}_X(S)$ given by the constant function 1 on X , because \mathbb{X} is a dense open subset of X . \square

6.4. A twisted symmetric form. We continue with the set-up of Section 6.3.

Using the homomorphism Φ in (6.18), the nondegenerate symmetric form B_W on W in (6.15) produces a nondegenerate symmetric form

$$\mathbb{B}_J \in H^0(X, \mathrm{Sym}^2(J^2(\mathbb{T})^*) \otimes \mathcal{O}_X(2S)). \quad (6.21)$$

This follows from the fact that the image of the following composition of homomorphisms

$$J^2(\mathbb{T})^* \xrightarrow{\Phi^*} W^* \xrightarrow{\sim} W \xrightarrow{\Phi} J^2(\mathbb{T})$$

coincides with the subsheaf $J^2(\mathbb{T}) \otimes \mathcal{O}_X(-2S) \subset J^2(\mathbb{T})$; the above isomorphism $W^* \xrightarrow{\sim} W$ is given by the nondegenerate symmetric form B_W .

The logarithmic connection \mathbb{D}_J on $J^2(\mathbb{T})$ in Proposition 6.8(2), and the canonical logarithmic connection on $\mathcal{O}_X(2S)$ defined by the de Rham differential, together define a logarithmic connection on the vector bundle $\mathrm{Sym}^2(J^2(\mathbb{T})^*) \otimes \mathcal{O}_X(2S)$. This logarithmic connection on $\mathrm{Sym}^2(J^2(\mathbb{T})^*) \otimes \mathcal{O}_X(2S)$ will be denoted by $\mathrm{Sym}^2(\mathbb{D}_J)'$.

Proposition 6.11.

- (1) *The form \mathbb{B}_J in (6.21) is covariant constant with respect to the above logarithmic connection $\mathrm{Sym}^2(\mathbb{D}_J)'$ on the vector bundle $\mathrm{Sym}^2(J^2(\mathbb{T})^*) \otimes \mathcal{O}_X(2S)$.*
- (2) *For the subbundles $F_{\mathbb{T}}^1, F_{\mathbb{T}}^2$ in (6.11),*

$$\mathbb{B}_J(F_{\mathbb{T}}^1 \otimes F_{\mathbb{T}}^1) = 0 \quad \text{and} \quad (F_{\mathbb{T}}^1)^\perp = F_{\mathbb{T}}^2,$$

where $(F_{\mathbb{T}}^1)^\perp \subset J^2(\mathbb{T})$ is the subbundle orthogonal to $F_{\mathbb{T}}^1$ with respect to \mathbb{B}_J .

Proof. This proposition can be proved exactly as Lemma 6.10 is proved. Indeed, from the statement (1) in Corollary 3.6 we know that both the statements in the proposition holds over $\mathbb{X} = X \setminus \mathbb{S}$. Hence the proposition follows. \square

Consider the logarithmic connection \mathbb{D}_J in Proposition 6.8(2) and the twisted holomorphic symmetric bilinear form \mathbb{B}_J in (6.21) on the vector bundle $J^2(\mathbb{T})$ constructed from the branched $\mathrm{SO}(3, \mathbb{C})$ -oper $(W, B_W, \{F_i^W\}_{i=1}^2, D_W)$ in (6.15). We will now show that $(W, B_W, \{F_i^W\}_{i=1}^2, D_W)$ can be reconstructed back from this pair

$$(\mathbb{B}_J, \mathbb{D}_J). \quad (6.22)$$

For each point $x_i \in \mathbb{S}$ (see (5.1)), let $L_i \subset J^2(\mathbb{T})_{x_i}$ be the eigenspace for the eigenvalue 0 of the residue $\mathrm{Res}(\mathbb{D}_J, x_i)$. Let \mathcal{F} be the holomorphic vector bundle on X that fits in the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow J^2(\mathbb{T}) \longrightarrow \bigoplus_{i=1}^d J^2(\mathbb{T})_{x_i}/L_i \longrightarrow 0 \quad (6.23)$$

of coherent analytic sheaves on X . Recall the criterion for a logarithmic connection on the vector bundle \mathcal{W} in (5.10) to induce a logarithmic connection on the vector bundle V in (5.10). Applying this criterion to (6.23) it follows that \mathbb{D}_J induces a logarithmic connection \mathbb{D}'_J on \mathcal{F} . Moreover, the eigenvalues of the residue $\mathrm{Res}(\mathbb{D}'_J, x_i)$ at $x_i \in \mathbb{S}$ are $(-1, 0, 0)$; this again follows from the expression of the residue, of the logarithmic connection on the vector bundle V in (5.10) induced by a logarithmic connection on \mathcal{W} , in terms of the residue of the logarithmic connection on \mathcal{W} .

For each point $x_i \in \mathbb{S}$, let $M_i \subset \mathcal{F}_{x_i}$ be the eigenspace for the eigenvalue 0 of the residue $\mathrm{Res}(\mathbb{D}'_J, x_i)$. Let \mathcal{E} be the holomorphic vector bundle on X that fits in the short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{i=1}^d \mathcal{F}_{x_i}/M_i \longrightarrow 0 \quad (6.24)$$

of coherent analytic sheaves on X . From the above mentioned criterion it follows that the logarithmic connection \mathbb{D}'_J induces a logarithmic connection \mathbb{D}''_J on \mathcal{E} . Moreover, the eigenvalues of the residue $\mathrm{Res}(\mathbb{D}''_J, x_i)$ at any $x_i \in \mathbb{S}$ are $(0, 0, 0)$.

Using the homomorphism Φ in (6.18), consider W as a subsheaf of $J^2(\mathbb{T})$. On the other hand, from (6.23) and (6.24) we have $\mathcal{E} \subset \mathcal{F} \subset J^2(\mathbb{T})$, using which \mathcal{E} will be considered as a subsheaf of $J^2(\mathbb{T})$. It is straightforward to check that $\mathcal{E} \subset J^2(\mathbb{T})$ coincides with the subsheaf W of $J^2(\mathbb{T})$. This identification between \mathcal{E} and W takes \mathbb{D}''_J to the holomorphic connection D_W in (6.15). In particular, the logarithmic connection \mathbb{D}''_J is actually a nonsingular connection on \mathcal{E} .

The nondegenerate twisted symmetric form \mathbb{B}_J on $J^2(\mathbb{T})$ in (6.21) produces a twisted symmetric form on the subsheaf $\mathcal{E} \subset J^2(\mathbb{T})$. The above identification between \mathcal{E} and W takes this twisted symmetric form on \mathcal{E} given by \mathbb{B}_J to B_W in (6.15). In particular, the twisted symmetric form on \mathcal{E} given by \mathbb{B}_J is nondegenerate and there is no nontrivial twisting.

The filtration $\{F_i^W\}_{i=1}^2$ of $W = \mathcal{E}$ in (6.15) is given by the filtration $\{F_{\mathbb{T}}^i\}_{i=1}^2$ of $J^2(\mathbb{T})$ in (6.11). In other words, F_i^W is the unique holomorphic subbundle of \mathcal{E} such that the space of holomorphic sections of F_i^W over any open subset $U \subset X$ is the space of holomorphic sections s of $\mathcal{E}|_U$ such that $s|_{U \cap (X \setminus \mathbb{S})}$ is a section of $F_{\mathbb{T}}^i$ over $U \cap (X \setminus \mathbb{S})$.

This way we recover the branched $\mathrm{SO}(3, \mathbb{C})$ -oper $(W, B_W, \{F_i^W\}_{i=1}^2, D_W)$ in (6.15) from the pair $(\mathbb{B}_J, \mathbb{D}_J)$ in (6.22) constructed from it.

7. A CHARACTERIZATION OF BRANCHED $\mathrm{SO}(3, \mathbb{C})$ -OPERS

Let $\mathfrak{g} = \mathrm{Lie}(\mathrm{SO}(3, \mathbb{C}))$ be the Lie algebra of $\mathrm{SO}(3, \mathbb{C})$. We will need a property of \mathfrak{g} which is formulated below.

Up to conjugacy, there is only one nonzero nilpotent element in \mathfrak{g} . Indeed, this follows immediately from the fact that $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let

$$A \in \mathfrak{g} = \text{Lie}(\text{SO}(3, \mathbb{C}))$$

be a nonzero nilpotent element. From the above observation we know that $\dim A(\mathbb{C}^3) = 2$. Therefore, if $B \in \mathfrak{g}$ is a nilpotent element such that $B(V_0) = 0$, where $V_0 \subset \mathbb{C}^3$ is some subspace of dimension two, then

$$B = 0. \quad (7.1)$$

Take a pair

$$(\mathbb{B}, \mathbf{D}), \quad (7.2)$$

where

- $\mathbb{B} \in H^0(X, \text{Sym}^2(J^2(\mathbb{T})^*) \otimes \mathcal{O}_X(2S))$ is fiberwise nondegenerate bilinear form on $J^2(\mathbb{T})$ with values in $\mathcal{O}_X(2S)$, and
- \mathbf{D} is a logarithmic connection on $J^2(\mathbb{T})$ singular over S ,

such that the following five conditions hold:

- (1) For the subbundles $F_{\mathbb{T}}^1, F_{\mathbb{T}}^2$ in (6.11),

$$\mathbb{B}(F_{\mathbb{T}}^1 \otimes F_{\mathbb{T}}^1) = 0 \quad \text{and} \quad (F_{\mathbb{T}}^1)^\perp = F_{\mathbb{T}}^2,$$

where $(F_{\mathbb{T}}^1)^\perp \subset J^2(\mathbb{T})$ is the subbundle orthogonal to $F_{\mathbb{T}}^1$ with respect to \mathbb{B} .

- (2) The section \mathbb{B} is covariant constant with respect to the logarithmic connection on the vector bundle $\text{Sym}^2(J^2(\mathbb{T})^*) \otimes \mathcal{O}_X(2S)$ induced by \mathbf{D} and the logarithmic connection on $\mathcal{O}_X(2S)$ given by the de Rham differential d .
- (3) $\mathbf{D}(F_{\mathbb{T}}^1) = F_{\mathbb{T}}^2 \otimes K_X \otimes \mathcal{O}_X(S)$ and $\mathbf{D}(F_{\mathbb{T}}^2) = J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(S)$.
- (4) For any $x_i \in \mathbb{S}$ (see (5.1)), the residue $\text{Res}(\mathbf{D}, x_i)$ of \mathbf{D} at x_i has eigenvalues $\{-2, -1, 0\}$.
- (5) The eigenspace of $\text{Res}(\mathbf{D}, x_i)$ for the eigenvalue -2 is the line $(F_{\mathbb{T}}^1)_{x_i} \subset J^2(\mathbb{T})_{x_i}$ in (6.11). The eigenspace of $\text{Res}(\mathbf{D}, x_i)$ for the eigenvalue -1 is contained in the subspace $(F_{\mathbb{T}}^2)_{x_i} \subset J^2(\mathbb{T})_{x_i}$.

In other words, the pair (\mathbb{B}, \mathbf{D}) satisfies all properties obtained in Proposition 6.8, Corollary 6.9, Lemma 6.10 and Proposition 6.11 for the pair in (6.22) corresponding to the branched $\text{SO}(3, \mathbb{C})$ -oper $(W, B_W, \{F_i^W\}_{i=1}^2, D_W)$ in (6.15). However this does not ensure that (\mathbb{B}, \mathbf{D}) defines a branched $\text{SO}(3, \mathbb{C})$ -oper. The reason for this is that the logarithmic connection \mathbf{D} might possess local monodromy around some points of the subset \mathbb{S} in (5.1). On the other hand, if the local monodromy of \mathbf{D} around every point of \mathbb{S} is trivial, then it can be shown that (\mathbb{B}, \mathbf{D}) defines a branched $\text{SO}(3, \mathbb{C})$ -oper (see Remark 7.2).

Take a point $x_i \in \mathbb{S}$. Since the eigenvalues of $\text{Res}(\mathbf{D}, x_i)$ are $\{-2, -1, 0\}$ (the fourth one of the five conditions above), the local monodromy of \mathbf{D} around x_i is unipotent (meaning 1 is the only eigenvalue). Let

$$L_2^i, L_1^i, L_0^i \subset J^2(\mathbb{T})_{x_i} \quad (7.3)$$

be the eigenspaces of $\text{Res}(\mathbf{D}, x_i)$ for the eigenvalues $-2, -1, 0$ respectively, so

$$L_2^i \oplus L_1^i \oplus L_0^i = J^2(\mathbb{T})_{x_i}. \quad (7.4)$$

Using the logarithmic connection \mathbf{D} , we will construct a homomorphism

$$\varphi_i \in \mathrm{Hom}(L_1^i \otimes (K_X)_{x_i}, L_2^i \otimes (K_X^{\otimes 2})_{x_i}) = \mathrm{Hom}(L_1^i, L_2^i \otimes (K_X)_{x_i}). \quad (7.5)$$

Take any

$$v \in L_1^i \otimes (K_X)_{x_i} \subset (J^2(\mathbb{T}) \otimes K_X)_{x_i}$$

(see (7.3)). Note that $\mathcal{O}_X(-x_i)_{x_i} = (K_X)_{x_i}$ (see (5.5)). Let \tilde{v} be a holomorphic section of $J^2(\mathbb{T}) \otimes \mathcal{O}_X(-x_i)$ defined on some open neighborhood U of x_i such that

$$\tilde{v}(x_i) = v; \quad (7.6)$$

here the identification $\mathcal{O}_X(-x_i)_{x_i} = (K_X)_{x_i}$ is used. Fix the open subset U such that $U \cap \mathbb{S} = x_i$.

In particular \tilde{v} is a holomorphic section of $J^2(\mathbb{T})$ over U and we have

$$\mathbf{D}(\tilde{v}) \in H^0(U, J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(S)) = H^0(U, J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(x_i)).$$

We will show that $\mathbf{D}(\tilde{v})$ lies in the image of the natural inclusion map

$$H^0(U, J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(-x_i)) \hookrightarrow H^0(U, J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(x_i)).$$

For that, first note that the section \tilde{v} can be expressed as

$$\tilde{v} = f \cdot s_1 + s_2,$$

where

- f is a holomorphic function on U with $f(x_i) = 0$,
- $s_1 \in H^0(U, J^2(\mathbb{T}))$ with $s_1(x_i) \in L_1^i$ (see (7.3)), and
- $s_2 \in H^0(U, J^2(\mathbb{T}))$, and it vanishes at x_i of order at least two.

Now consider the section

$$\mathbf{D}(\tilde{v}) = \mathbf{D}(f s_1) + \mathbf{D}(s_2).$$

Since s_2 vanishes at x_i of order at least two, it follows that

$$\mathbf{D}(s_2) \in H^0(U, J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(-x_i)).$$

Consequently, to prove that

$$\mathbf{D}(\tilde{v}) \in H^0(U, J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(-x_i)) \hookrightarrow H^0(U, J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(x_i)), \quad (7.7)$$

it suffices to show that $\mathbf{D}(f s_1) \in H^0(U, J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(-x_i))$.

The Leibniz rule for \mathbf{D} says that

$$\mathbf{D}(f s_1) = f \mathbf{D}(s_1) + df \otimes s_1. \quad (7.8)$$

Since $s_1(x_i) \in L_1^i$, and L_1^i is the eigenspace of $\mathrm{Res}(\mathbf{D}, x_i)$ for the eigenvalue -1 , from (7.8) it follows that

$$\mathbf{D}(f s_1) \in H^0(U, J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(-x_i)).$$

Indeed, if we consider $f \mathbf{D}(s_1)$ and $df \otimes s_1$ as sections of $(J^2(\mathbb{T}) \otimes K_X)|_U$, then $f \mathbf{D}(s_1)(x_i) = -v$ by the residue condition, and $(df \otimes s_1)(x_i) = v$ by (7.6). These imply that the section

$$\mathbf{D}(f s_1) \in H^0(U, J^2(\mathbb{T}) \otimes K_X)$$

vanishes at x_i , making it a section of $(J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(-x_i))|_U$.

Since $\mathbf{D}(fs_1) \in H^0(U, J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(-x_i))$, we conclude that (7.7) holds.

Using the decomposition in (7.4), the fiber $(J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(-S))_{x_i}$ decomposes as

$$\begin{aligned} & (J^2(\mathbb{T}) \otimes K_X \otimes \mathcal{O}_X(-S))_{x_i} \\ &= ((L_2^i \otimes K_X \otimes \mathcal{O}_X(-S))_{x_i}) \oplus ((L_1^i \otimes K_X \otimes \mathcal{O}_X(-S))_{x_i}) \oplus ((L_0^i \otimes K_X \otimes \mathcal{O}_X(-S))_{x_i}). \end{aligned} \quad (7.9)$$

For the section $\mathbf{D}(\tilde{v})$ in (7.7), let

$$\mathbf{D}(\tilde{v})_2^i \in (L_2^i \otimes K_X \otimes \mathcal{O}_X(-S))_{x_i} = L_2^i \otimes (K_X^{\otimes 2})_{x_i} \quad (7.10)$$

be the component of $\mathbf{D}(\tilde{v})(x_i)$ for the decomposition in (7.9); recall from (5.5) that we have $\mathcal{O}_X(-S)_{x_i} = (K_X)_{x_i}$.

We will now show that the element $\mathbf{D}(\tilde{v})_2^i \in L_2^i \otimes (K_X^{\otimes 2})_{x_i}$ in (7.10) is independent of the choice of the section \tilde{v} in (7.6) that extends v . To prove this, take any

$$\hat{v} \in H^0(U, J^2(\mathbb{T}) \otimes \mathcal{O}_X(-x_i))$$

such that $\hat{v}(x_i) = v$. So the section $\tilde{v} - \hat{v}$ of $J^2(\mathbb{T})|_U$ vanishes at x_i of order at least two. Therefore, we can write

$$\tilde{v} - \hat{v} = f_2 s_2 + f_1 s_1 + f_0 s_0,$$

where f_0, f_1, f_2 are holomorphic functions on U vanishing at x_i of order at least two, and $s_2(x_i) \in L_2^i, s_1(x_i) \in L_1^i, s_0(x_i) \in L_0^i$.

It is straightforward to check that $\mathbf{D}(f_1 s_1)$ and $\mathbf{D}(f_0 s_0)$ do not contribute to the component $L_2^i \otimes (K_X^{\otimes 2})_{x_i}$ in (7.9); as before, $\mathcal{O}_X(-S)_{x_i}$ is identified with $(K_X)_{x_i}$. Therefore, to prove that $\mathbf{D}(\tilde{v})_2^i$ in (7.10) is independent of the choice of the section \tilde{v} , it suffices to show that $\mathbf{D}(f_2 s_2)$ also does not contribute to the component $L_2^i \otimes (K_X^{\otimes 2})_{x_i}$ in (7.9). But this follows from the facts that $s_2(x_i) \in L_2^i$, and L_2^i is the eigenspace of $\text{Res}(\mathbf{D}, x_i)$ for the eigenvalue -2 . Hence we conclude that $\mathbf{D}(\tilde{v})_2^i$ is independent of the choice of \tilde{v} .

Now we construct the homomorphism in φ_i in (7.5) by sending any $v \in L_1^i \otimes (K_X)_{x_i}$ (as in (7.6)) to $\mathbf{D}(\tilde{v})_2^i$ in (7.10) constructed from v .

Theorem 7.1. *The pair (\mathbb{B}, \mathbf{D}) in (7.2) defines a branched $\text{SO}(3, \mathbb{C})$ -oper if and only if $\varphi_i = 0$ for every $x_i \in \mathbb{S}$, where φ_i is the homomorphism in (7.5).*

Proof. We first invoke the algorithm, described at the end of Section 6.4, to recover a branched $\text{SO}(3, \mathbb{C})$ -oper from the corresponding logarithmic connection and the twisted bilinear form on $J^2(\mathbb{T})$.

Let \mathcal{F} be the holomorphic vector bundle on X that fits in the short exact sequence of coherent analytic sheaves on X

$$0 \longrightarrow \mathcal{F} \longrightarrow J^2(\mathbb{T}) \longrightarrow \bigoplus_{i=1}^d J^2(\mathbb{T})_{x_i}/L_0^i \longrightarrow 0, \quad (7.11)$$

where L_0^i is the eigenspace in (7.3). Applying the criterion for a logarithmic connection on the vector bundle \mathcal{W} in (5.10) to induce a logarithmic connection on the vector bundle V in (5.10), the logarithmic connection \mathbf{D} on $J^2(\mathbb{T})$ induces a logarithmic connection \mathbf{D}' on \mathcal{F} ; the eigenvalues of the residue $\text{Res}(\mathbf{D}', x_i)$ of \mathbf{D}' at any $x_i \in \mathbb{S}$ are $(-1, 0, 0)$. For each point

$x_i \in \mathbb{S}$, let $M_i \subset \mathcal{F}_{x_i}$ be the eigenspace for the eigenvalue 0 of the residue $\mathrm{Res}(\mathbf{D}', x_i)$. Let \mathcal{E} be the holomorphic vector bundle on X that fits in the short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{i=1}^d \mathcal{F}_{x_i}/M_i \longrightarrow 0 \quad (7.12)$$

of coherent analytic sheaves on X . Applying the above mentioned criterion we conclude that the logarithmic connection \mathbf{D}' induces a logarithmic connection \mathbf{D}'' on \mathcal{E} ; the eigenvalues of the residue $\mathrm{Res}(\mathbf{D}'', x_i)$ of \mathbf{D}'' at $x_i \in \mathbb{S}$ are $(0, 0, 0)$.

Although all the eigenvalues of $\mathrm{Res}(\mathbf{D}'', x_i)$ are zero, the residue $\mathrm{Res}(\mathbf{D}'', x_i)$ need not vanish in general; it can be a nilpotent endomorphism. We shall investigate the residue $\mathrm{Res}(\mathbf{D}'', x_i)$.

Let

$$\iota : \mathcal{E} \hookrightarrow J^2(\mathbb{T})$$

be the inclusion map obtained from the injective homomorphisms in (7.11) and (7.12). Since ι is an isomorphism over $\mathbb{X} = X \setminus \mathbb{S}$, any holomorphic subbundle V of $J^2(\mathbb{T})$ generates a holomorphic subbundle \tilde{V} of \mathcal{E} . This \tilde{V} is uniquely determined by the condition that the space of holomorphic sections of \tilde{V} over any open subset $U \subset X$ is the space of holomorphic sections s of $\mathcal{E}|_U$ such that $s|_{U \cap (X \setminus \mathbb{S})}$ is a section of V over $U \cap (X \setminus \mathbb{S})$.

Let $\tilde{F}_{\mathbb{T}}^1$ and $\tilde{F}_{\mathbb{T}}^2$ be the holomorphic subbundles of \mathcal{E} corresponding to the holomorphic subbundles $F_{\mathbb{T}}^1$ and $F_{\mathbb{T}}^2$ of $J^2(\mathbb{T})$ in (6.11).

It is straightforward to check that for any point $x \in \mathbb{S}$,

$$\mathrm{Res}(\mathbf{D}'', x_i)((\tilde{F}_{\mathbb{T}}^1)_{x_i}) = 0 \quad \text{and} \quad \mathrm{Res}(\mathbf{D}'', x_i)((\tilde{F}_{\mathbb{T}}^2)_{x_i}) \subseteq (\tilde{F}_{\mathbb{T}}^1)_{x_i}. \quad (7.13)$$

Moreover, the homomorphism $(\tilde{F}_{\mathbb{T}}^2)_{x_i}/(\tilde{F}_{\mathbb{T}}^1)_{x_i} \longrightarrow (\tilde{F}_{\mathbb{T}}^1)_{x_i}$ induced by $\mathrm{Res}(\mathbf{D}'', x_i)$ coincides with the homomorphism φ_i in (7.5).

If (\mathbb{B}, \mathbf{D}) in (7.2) defines a branched $\mathrm{SO}(3, \mathbb{C})$ -oper, then \mathbf{D}'' is a nonsingular connection, meaning $\mathrm{Res}(\mathbf{D}'', x_i) = 0$ for every $x_i \in \mathbb{S}$, and consequently, $\varphi_i = 0$ for all $x_i \in \mathbb{S}$.

Conversely, if $\varphi_i = 0$ for all $x_i \in \mathbb{S}$, then using (7.13) it follows that $\mathrm{Res}(\mathbf{D}'', x_i)((\tilde{F}_{\mathbb{T}}^2)_{x_i}) = 0$. Now from (7.1) we conclude that $\mathrm{Res}(\mathbf{D}'', x_i) = 0$. Hence the logarithmic connection \mathbf{D}'' is nonsingular. Therefore, (\mathbb{B}, \mathbf{D}) defines a branched $\mathrm{SO}(3, \mathbb{C})$ -oper. This completes the proof. \square

Remark 7.2. Assume that the local monodromy of the logarithmic connection \mathbf{D} around every point of \mathbb{S} is trivial. Then the local monodromy of the logarithmic connection \mathbf{D}'' around every point of \mathbb{S} is trivial, because the monodromies of \mathbf{D} and \mathbf{D}'' coincide. Consider the following four facts:

- (1) For every point $x_i \in \mathbb{S}$, the eigenvalues of $\mathrm{Res}(\mathbf{D}'', x_i)$ are $(0, 0, 0)$.
- (2) The local monodromy of \mathbf{D}'' around every $x_i \in \mathbb{S}$ is trivial.
- (3) The local monodromy, around $x_i \in \mathbb{S}$, of any logarithmic connection D^0 lies in the conjugacy class of $\exp(-2\pi\sqrt{-1}\mathrm{Res}(D^0, x_i))$.
- (4) $\exp(-2\pi\sqrt{-1}A) \neq I$ for any nonzero nilpotent complex matrix A .

These together imply that $\text{Res}(\mathbf{D}'', x_i) = 0$ for every point $x_i \in \mathbb{S}$. Hence \mathbf{D}'' is a nonsingular connection, and (\mathbb{B}, \mathbf{D}) in (7.2) defines a branched $\text{SO}(3, \mathbb{C})$ -oper.

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SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI 400005, INDIA

Email address: `indranil@math.tifr.res.in`

UNIVERSITÉ CÔTE D’AZUR, CNRS, LJAD, FRANCE

Email address: `dumitres@unice.fr`