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Geometric Regularity Results on $B_{\alpha,\beta}^k$ -Manifolds, I: Affine Connections

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Abstract

In this paper we consider the existence problem of affine connections on C^k -manifolds M whose coefficients are as regular as one needs. We show that if M admits a suitable subatlas, meaning a $\mathcal{B}_{\alpha,\beta}^k$ -structure for a certain presheaf of Fréchet spaces B and for certain functions α and β , then the existence of such regular connections can be established. It is also proved that if the $\mathcal{B}_{\alpha,\beta}^k$ -structure is actually nice (in the sense of [1]), then a multiplicity result can also be obtained by means of Thom's transversality arguments.

1 Introduction

As it is very well known, any paracompact C^k -manifold (M, \mathcal{A}) , with $k \geq 2$, admits many affine connections ∇ whose coefficients are of class C^{k-2} [2]. Recall that this very standard result is obtained as follows. Let $\text{Conn}_{k-2}(TM)$ be the set of affine connections of class C^{k-2} in M . One first shows that $\text{Conn}_{k-2}(T\mathbb{R}^n)$ is nonempty for every k . One then shows that each C^k -map $f : N \rightarrow M$ induces a pullback function $f^* : \text{Conn}_{k-2}(TM) \rightarrow \text{Conn}_{k-2}(f^*TN)$, from which one concludes that open sets of \mathbb{R}^n admit affine connections of class C^{k-2} , for every k . In particular sets C^k -diffeomorphic to open sets of \mathbb{R}^n admit C^{k-2} -connections. Then, for a given open covering \mathcal{U} by coordinate systems $\varphi_i : U_i \rightarrow \mathbb{R}^n$ of M one takes a subordinate partition of unity $\psi = (\psi_i)_i$ and one defines $\nabla = \sum_i \psi_i \cdot \nabla_i$, where $\nabla_i \in \text{Conn}_{k-2}(TU_i)$, so that $\text{Conn}_{k-2}(TM)$ is nonempty. Since the difference of C^{k-2} -connections is $C^k(M)$ -linear, one concludes that $\text{Conn}_{k-2}(TM)$ is an affine space of $\Omega^1(M; \text{End}(TM))$.

In many situations, e.g. in gauge theory and when we need certain uniform bounds on the curvature, it is desirable to work with connections ∇ whose coefficients are not only C^{k-2} , but actually L^p -integrable or even uniformly bounded [3, 4]. In other situations, e.g. in the study of holomorphic geometry, we consider holomorphic or hermitian connections over complex manifolds, whose coefficients are holomorphic [5, 6]. If M has a G -structure, we can also consider G -principal connections ω on the frame bundle FM , which induce an affine connection ∇_ω in M , whose

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coefficients $\omega_b^c(\partial_a) = \Gamma_{ab}^c$ are such that for every a , $\Gamma_a \in C^{k-2}(U) \otimes \mathfrak{g}$, where $(\Gamma_a)_{cb} = \Gamma_{ab}^c$ [7]. Interesting examples of these are the symplectic connections, which are specially important in formal deformation quantization of symplectic manifolds [8, 9]. As a final example, a torsion-free affine connection defines a L_3 -space structure in a smooth manifold M (in the sense of [10]) iff its coefficients satisfy a certain system of partial differential equations [11].

Thus, if ρ represents some “regularity condition”, e.g, one of the previous ones, let $\text{Conn}_\rho(TM) \subset \text{Conn}_{k-2}(TM)$ be the subset of those affine connections whose coefficients satisfy ρ . It is then natural to ask when $\text{Conn}_\rho(TM)$ is nonempty or when it remains an affine space. More precisely, it is natural to ask when the construction above can be applied *ipsis litteris* in order to prove that $\text{Conn}_\rho(TM)$ is an affine space. This is a nontrivial question, since the existence of such a regular connection may imply geometric and topological obstructions on M [4, 6]. In this article we attack this problem in the context of $B_{\alpha,\beta}^k$ -structures introduced in [1]. More precisely, for us the “regularity condition ρ ” is represented by sets $\Gamma \subset \mathbb{Z}_{\geq 0}$ and $S \subset \Gamma$, by integer functions $\alpha_0 : S \rightarrow \Gamma$ and $\beta_0 : S \rightarrow [2, k]$, and by a family $B = (B_i)_{i \in \Gamma}$ of presheaves of nuclear Fréchet spaces $B_i : \text{Open}(M)^{op} \rightarrow \mathbf{NFre}$ on M . Saying that an affine connection ∇ on (M, \mathcal{A}) satisfies the “regularity condition ρ ” described by these data means that there exists a subatlas $\mathcal{B} \subset \mathcal{A}$ and an intersection structure presheaf (ISP) \mathbb{X}_0 over M such that for every $\varphi : U \rightarrow \mathbb{R}^n$ in \mathcal{B}_ρ the induced coefficients $(\Gamma_\varphi)_{ab}^c$ are such that $\partial^\mu (\Gamma_\varphi)_{ab}^c \in B_{\alpha_0(i)}(U) \cap_{X_0(U)} C^{k-\beta_0(i)}(U)$ for every $i \in S$ and every μ such that $|\mu| = \beta_0(i)$. The space of these regular connections is denoted by $\text{Conn}_{\alpha_0, \beta_0}^k(M; \mathbb{X}_0) \subset \text{Conn}_{k-2}(M)$. All above examples of connections satisfying regularity conditions can be put in this language.

We prove that, given this “regularity condition” $\rho = (B, S, \alpha_0, \beta_0, \mathbb{X}_0)$, if the structural presheaf B is suitable and if (M, \mathcal{A}) admits a $(B_{\alpha,\beta}^k, \mathbb{X})$ -structure compatible with α_0 and β_0 , then there exists an ISP \mathbb{X}_0 , compatible with \mathbb{X} , such that for every (θ, ϑ) close enough to (α_0, β_0) the space $\text{Conn}_{\theta, \vartheta}^k(M; \mathbb{X}_0)$ is nonempty. In more details, we prove the following result:

Theorem A. *Let $(M, \mathcal{A}, \mathcal{B}_{\alpha,\beta}^k(\mathbb{X}))$ be a $(B_{\alpha,\beta}^k, \mathbb{X})$ -manifold, with $k \geq 2$, and suppose that the structural presheaf B is such that $\mathcal{B}_{\alpha,\beta}^k(\mathbb{X})$ admits a nice distributive $(\alpha'_0, \beta'_0; j)$ -connection of degree $r \geq 2$, say for a proper full ISP \mathbb{X}' . Then for every ordinary pair (θ, ϑ) and every $z \in \Gamma_{\geq 0; \beta'_0}$ there exists a $(B_{\theta, \vartheta}^k, \mathbb{X}'|_{\Gamma_{\geq 0; \beta'_0 - z}})$ -connection in M .*

The result above is about existence of regular connections. The second natural question is about multiplicity, i.e,

what is the size of their subspace. About this problem we prove the expected fact that the difference $D = \nabla - \overline{\nabla}$ of two $(B_{\alpha_0, \beta_0}^k, \mathbb{X}_0|_S)$ -connections is a $\text{End}(TM)$ -valued 1-form whose coefficients $D_{ab}^c = \Gamma_{ab}^c - \overline{\Gamma}_{ab}^c$ are such that $\partial^\mu D_{ab}^c \in B_{\alpha_0(i)}(U) \cap_{X_0(U)} C^{k-\beta_0(i)}$, i.e, $\text{Conn}_{\alpha_0, \beta_0}^k(M; \mathbb{X}_0)$ is an affine space over the \mathbb{R} -vector space $\Omega_{\alpha_0, \beta_0}^1(M; \mathbb{X}_0) \subset \Omega^1(M)$ of these 1-forms. This is not enough for ensuring the existence of many linearly independent connections, since a priori we have no information on the the space $\Omega_{\alpha_0, \beta_0}^1(M; \mathbb{X}_0)$. Nonetheless, we also prove a less obvious fact: in the smooth context, i.e, for $k = \infty$, if the structural presheaf B satisfies an additional closure condition, then there exists a map

$$F : \text{Conn}_{\theta, \vartheta}^\infty(M; \mathbb{X}_0) \rightarrow \text{Conn}_{\theta, \vartheta}^\infty(M; \mathbb{X}_0)$$

which has no local fixed points. More precisely, we say that two affine connections ∇ and $\overline{\nabla}$ are *locally different* and we write $\nabla \approx \overline{\nabla}$ if for every $\varphi : U \rightarrow \mathbb{R}^n$ in \mathcal{B} and every $a, b, c = 1, \dots, n$ there

exists $U_{abc} \subset U$ such that $\Gamma_{ab}^c(p) \neq \overline{\Gamma}_{ab}^c(p)$ for each $p \in U_{abc}$. We say that F has *no local fixed points* if $F(\nabla) \approx \nabla$ for every $\nabla \in \text{Conn}_{\theta, \vartheta}^\infty(M; \mathbb{X}_0)$. The existence of such F is consequence of the following theorem.

Theorem B. *In addition to the hypotheses of Theorem A, suppose that M is smooth and has countably many connected-components, that B admits a nice distributive $(\alpha'_0, \beta'_0; j)$ -connection of degree $r \geq 2$, say in a proper ISP \mathbb{X}' , and that B is $(\theta, \vartheta | \Gamma_{\geq 0; \beta'_0 - z})$ -strong in \mathbb{X}' for some ordinary (θ, ϑ) and some $z \in \Gamma_{\geq 0; \beta'_0}$. Then, for every $(B_{\theta, \vartheta}^\infty, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z})$ -connection ∇ in $(M, \mathcal{A}, \mathcal{B}_{\alpha, \beta}^\infty)$ there exists another $(B_{\theta, \vartheta}^\infty, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z})$ -connection $\overline{\nabla}$ such that for every $(U, \varphi) \in \mathcal{B}_{\alpha, \beta}^\infty$ and every $a', b', c = 1, \dots, n$, there exists $U_{a', b'} \subset U$ such that $(\Gamma_\varphi)_{a' b'}^c(p) \neq (\overline{\Gamma}_\varphi)_{a' b'}^c(p)$ for every $p \in U_{a', b'}$. In particular, $\nabla \approx \overline{\nabla}$.*

The paper is organized as follows. In Section 2 we quickly present the needed background for our main results, which are basically concepts introduced in [1]. These concepts can be understood in categorical terms by means of functors and commutative diagrams for natural transformations. Here, in order to make a more direct relation with the results of the next sections, these concepts will be revisited in the objectwise approach, via functions and equations. In Section 3 the notion of $(B_{\alpha_0, \beta_0}^k, \mathbb{X}_0 | S)$ -connection is formally introduced and Theorem A is proved. In Section 4 the additional conditions appearing in Theorem B are discussed and a proof of Theorem B is given.

2 Background

Let Γ be a set such that $\Gamma_{\geq 0} = \Gamma \cap \mathbb{Z}_{\geq 0}$ contains at least the number 0. Equivalently, let $\Gamma_{\geq 0}$ be a subset of $\mathbb{Z}_{\geq 0}$ containing $r = 0$. The *degree* of Γ is the biggest integer $\text{deg}(\Gamma) \in \mathbb{Z}_{\geq 0}$ such that the integer interval $[0, \text{deg}(\Gamma)]$ belongs to $\Gamma_{\geq 0}$. A (Γ, ϵ) -space is a sequence $B = (B_i)_{i \in \Gamma_{\geq 0}}$ of real nuclear Fréchet spaces endowed with a function $\epsilon : \Gamma_{\geq 0} \times \Gamma_{\geq 0} \rightarrow \Gamma_{\geq 0}$. The trivial example is that in which $B_i = 0$ for each i and $\epsilon(i, j) = 0$ for each i, j , where the first zero means the trivial real vector space. A *morphism* between (B, ϵ) and (B', ϵ') is a pair (f, μ) , where $\mu : \Gamma_{\geq 0} \rightarrow \Gamma_{\geq 0}$ is a function such that $\mu(\epsilon(i, j)) = \epsilon'(\mu(i), \mu(j))$, and $f = (f_i)_{i \in \Gamma_{\geq 0}}$ is a sequence of continuous linear maps $f_i : B_i \rightarrow B_{\mu(i)}$. Composition is well-defined and we have a category $\mathbf{NFre}_{\Gamma, \Sigma}$ of (Γ, ϵ) -spaces. A Γ -space is obtained from a (Γ, ϵ) -space by forgetting ϵ . Thus, they also constitute a category \mathbf{NFre}_Γ and we have a forgetful functor from (Γ, ϵ) -spaces to Γ -spaces. On the other hand, from any (Γ, ϵ) -space (B, ϵ) we get a $(\Gamma \times \Gamma)$ -space B_ϵ by defining $(B_\epsilon)_{ij} = B_{\epsilon(i, j)}$.

A *distributive Γ -space* consists of a Γ -space B endowed with two maps $\epsilon, \delta : \Gamma_{\geq 0} \times \Gamma_{\geq 0} \rightarrow \Gamma_{\geq 0}$ and continuous linear maps $*_{ij} : B_i \otimes B_j \rightarrow B_{\epsilon(i, j)}$ and $+_{ij} : B_i \otimes B_j \rightarrow B_{\delta(i, j)}$, where \otimes denotes the projective tensor product, which are both left-compatible and right-compatible, in the sense that for every $x \in B_i, y \in B_j$ and $z \in B_k$, and every $i, j, k \in \Gamma_{\geq 0}$ the following equations hold (notice that the first two makes sense only because of the last two):

$$\begin{aligned} x *_{i\delta(j, k)} (y +_{jk} z) &= (x *_{ij} y) + (x *_{ik} z) \\ (x +_{ij} y) *_{\delta(i, j)k} z &= (x *_{ik} z) + (y *_{jk} z) \\ \epsilon(i, \delta(j, k)) &= \delta(\epsilon(i, j), \epsilon(i, k)) \\ \epsilon(\delta(i, j), k) &= \delta(\epsilon(i, k), \epsilon(j, k)). \end{aligned}$$

We also require that $+_{ii}$ coincides with the sum in B_i . The pairs $(*, \epsilon)$ and $(+, \delta)$ define a *multiplicative structure* and an *additive structure* in B , respectively. We have a category $\mathbf{NFre}_{\Gamma, *, +}$ of

distributive Γ -space whose morphisms between $(B, +, *, \epsilon, \delta)$ and $(B', +', *', \epsilon', \delta')$ are pairs (f, μ) which are simultaneously a morphism of (Γ, ϵ) -spaces $(B, \epsilon) \rightarrow (B', \epsilon')$ and $(B, \delta) \rightarrow (B', \delta')$, in such a way that

$$f(x *_i j y) = f(x) *'_{\epsilon'(\mu(i), \mu(j))} f(y) \quad \text{and} \quad f(x +_i j y) = f(x) +'_{\delta'(\mu(i), \mu(j))} f(y).$$

A *full Γ -ambient* is a triple $(\mathbf{X}, \gamma, \gamma_\Sigma)$, where \mathbf{X} is a category with pullbacks and γ and γ_Σ are faithful functors $\gamma : \mathbf{NFre}_\Gamma \rightarrow \mathbf{X}$ and $\gamma_\Sigma : \mathbf{NFre}_{\Gamma \times \Gamma} \rightarrow \mathbf{X}$. The second functor extends to (B, ϵ) -spaces by defining $\gamma_\Sigma(B, \epsilon) = \gamma_\Sigma(B_\epsilon)$. Let $X, X_\epsilon \in \mathbf{X}$ and given spans of monomorphisms $\gamma(B) \hookrightarrow X \hookleftarrow \gamma(B')$ and $\gamma_\Sigma(B_\epsilon) \hookrightarrow X_\epsilon \hookleftarrow \gamma_\Sigma(B'_\epsilon)$, let $B \cap_{X, \gamma} B'$ and $B_\epsilon \cap_{X_\epsilon, \gamma_\Sigma} B'_\epsilon$ be the corresponding pullbacks. If B and B' are actually distributive Γ -spaces, we have additional spans for their multiplicative and additive structures, as presented below for the multiplicative case. The full data consisting of the full Γ -ambient and the spans above is called a *full intersection structure* between B and B' and denoted by \mathbb{X} . The object X is the *base object* of \mathbb{X} and the tuple $(X, X_\epsilon, X_*, X_+)$ is the *full base object* of \mathbb{X} .

$$\begin{array}{ccc}
\text{pb}(*, *'; X_*, \gamma_\Sigma) & \xrightarrow{\quad} & \gamma_\Sigma(B'_\epsilon \otimes B'_\epsilon) \\
\downarrow & & \downarrow \gamma_\Sigma(*') \\
& & B_\epsilon \cap_{X_\epsilon, \gamma_\Sigma} B'_\epsilon \xrightarrow{\quad} \gamma_\Sigma(B'_\epsilon) \\
& & \downarrow j' \\
\gamma_\Sigma(B_\epsilon \otimes B_\epsilon) & \xrightarrow{\gamma_\Sigma(*)} & \gamma_\Sigma(B_\epsilon) \xleftarrow{j} X_*
\end{array}$$

We say that a full Γ -ambient is *proper* if for every set of spans as above there are a Γ -space $B \cap_X B'$ and $(\Gamma \times \Gamma)$ -spaces $\text{pb}(*, *'; X_*)$ and $\text{pb}(+, +'; X_+)$ such that $\gamma(B \cap_X B') \simeq B \cap_{X, \gamma} B'$, $\gamma_\Sigma(\text{pb}(*, *'; X_*)) \simeq \text{pb}(*, *'; X_*, \gamma_\Sigma)$ and $\gamma_\Sigma(\text{pb}(+, +'; X_+)) \simeq \text{pb}(+, +'; X_+, \gamma_\Sigma)$. The triple consisting of those representing objects in $\mathbf{NFre}_\Gamma \times \mathbf{NFre}_{\Gamma \times \Gamma} \times \mathbf{NFre}_{\Gamma \times \Gamma}$ is the *intersection space between B and B' in \mathbb{X}* . We say that B and B' have *nontrivial intersection* in \mathbb{X} if both $B \cap_X B'$, $\text{pb}(*, *'; X_*)$ and $\text{pb}(+, +'; X_+)$ have positive real dimension. As established in [1], this is the case if \mathbb{X} is a standard vectorial intersection structure and $\gamma(B)$ and $\gamma(B')$ have nonempty intersection when regarded as sets (see there for more details).

All the discussion above can be internalized in the presheaf category of presheaves in \mathbb{R}^n . For instance, we define a *presheaf of (Γ, ϵ) -spaces in \mathbb{R}^n* as a functor $B : \text{Open}^{op}(\mathbb{R}^n) \rightarrow \mathbf{NFre}_{\Gamma, \Sigma}$ and a *presheaf of distributive Γ -spaces* as a functor $B : \text{Open}^{op}(\mathbb{R}^n) \rightarrow \mathbf{NFre}_{\Gamma, *, +}$. The fundamental example is C_m^p . Here, $p : \Gamma_{\geq 0} \rightarrow [0, \infty]$ is some function and $C_m^p(U)_i = C^{p(i)}(U; \mathbb{R}^m)$ is the nuclear Fréchet space of \mathbb{R}^m -valued $C^{p(i)}$ -functions in $U \subset \mathbb{R}^n$, with the standard family of seminorms $\|f\|_{r, l} = \sum_j \sup_{|\mu|=r} \sup_{x \in K_l} |\partial^\mu f_j(x)|$ with $0 \leq r \leq p(i)$ and (K_l) some nice sequence of compact sets. The distributive structure is given by pointwise addition/multiplication, with $\epsilon(i, j) = \min(p(i), p(j)) = \delta(i, j)$. Of special interest are the cases $m = n$, $m = 1$ and $p = k - \beta$, where $k \in [0, \infty]$ is a fixed integer and $\beta : \Gamma_{\geq 0} \rightarrow [0, k]$ is a given function. For $m = 1$ we write simply C^p instead of C_1^p .

Let B be a presheaf of distributive Γ -spaces, $\alpha : \Gamma_{\geq 0} \rightarrow \Gamma_{\geq 0}$ a function and \mathbb{X} a proper full intersection structure presheaf (ISP) between B_α and $C^{k-\beta}$. We say that B is a $C_{\alpha, \beta}^k$ -*presheaf in \mathbb{X}* if B_α and $C^{k-\beta}$ have nontrivial intersection in \mathbb{X} and if there exists a sub-presheaf A of C^∞ such

that $f \cdot g \in B_{\alpha(i)}(U) \cap_{X(U)} C^{k-\beta(i)}(U)$ for every $f \in A(U)$ and $g \in B_{\alpha(i)}(U) \cap_{X(U)} C^{k-\beta(i)}(U)$, i.e, if the intersection is closed by pointwise multiplication by a certain space of smooth functions. If it is possible to choose A with nontrivial intersersection with the sub-presheaf C_b^∞ of bump functions, we say that B is a *nice* $C_{\alpha,\beta}^k$ -presheaf in \mathbb{X} . Given a $C_{\alpha,\beta}^k$ -presheaf B in \mathbb{X} , an integer $m > 0$ and a subset $S \subset \Gamma_{\geq 0}$, the corresponding *presheaf of $(B, k, \alpha, \beta|S)$ -functions in (\mathbb{X}, m)* is the presheaf assigning to each $U \subset \mathbb{R}^n$ the subspace $B_{\alpha,\beta}^{k,m}(U; X(U)|S) \subset C^0(U; \mathbb{R}^m)$ of continuous functions such that $\partial^\mu f_j \in B_{\alpha(i)}(U) \cap_{X(U)} C^{k-\beta(i)}(U)$ for every $i \in S$, $|\mu| = \beta(i)$ and $j = 1, \dots, m$. If $S = \Gamma_{\geq 0}$ we say that we have a presheaf of (B, k, α, β) -functions in (\mathbb{X}, m) and write $B_{\alpha,\beta}^k(U; X(U))$ instead of $B_{\alpha,\beta}^{k,m}(U; X(U)|S)$.

Remark 2.1. Throughou this paper we will work with presheaves of $(B, k, \alpha, \beta|S)$ -functions in (\mathbb{X}, m) for m being some multiple of n , i.e, $m = r \cdot n$ for some $r > 0$. Thus, when there is no risk of confusion we will omit m , leaving it implicit in each context.

A C^k -manifold is a paracompact Hausdorff topological space M endowed with a maximal atlas \mathcal{A} , say with charts $\varphi_i : U_i \rightarrow \mathbb{R}^n$, whose transition functions $\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_{ij}) \rightarrow \mathbb{R}^n$ are C^k , where $U_{ij} = U_i \cap U_j$. A $(B_{\alpha,\beta}^k, \mathbb{X}|S)$ -structure in a C^k -manifold (M, \mathcal{A}) is a subatlas $\mathcal{B}_{\alpha,\beta}^k(\mathbb{X}|S)$ such that $\varphi_{ji} \in B_{\alpha,\beta}^k(\varphi_i(U_{ij}); X(\varphi_i(U_{ij})|S))$. As proved in [1], under certain hypothesis on B , α and β , these additional structures exist in any C^k -manifold in the case $S = \Gamma_{\geq 0}$. A $(B_{\alpha,\beta}^k, \mathbb{X}|S)$ -manifold is a C^k manifold endowed with a $(B_{\alpha,\beta}^k, \mathbb{X}|S)$ -structure. A *nice* $(B_{\alpha,\beta}^k, \mathbb{X}|S)$ -manifold is one such that B is a nice $C_{\alpha,\beta}^k$ -presheaf. If (M, \mathcal{A}) and (M', \mathcal{A}') are C^k -manifolds, a $(B_{\alpha,\beta}^k, \mathbb{X}|S)$ -morphism between them is a C^k -map $f : M \rightarrow M'$ such that $\phi f \varphi$ is $(B, k, \alpha, \beta|S)$ -function in \mathbb{X} for every $\varphi \in \mathcal{A}$ and $\phi \in \mathcal{A}'$. If $S = \Gamma_{\geq 0}$ we talk about $(B_{\alpha,\beta}^k, \mathbb{X})$ -structures, $(B_{\alpha,\beta}^k, \mathbb{X})$ -manifolds and $(B_{\alpha,\beta}^k, \mathbb{X})$ -morphisms.

Remark 2.2. Throughou this paper we will always assume that sets Γ parametrizing the (Γ, ϵ) -spaces have a large enough degree, as required for making sense of the definitions and results of the next sections. Sometimes an estimative for the degree will be explicitly given; in other cases the degree will be implicit. This hypothesis on the degree, however, is not much restrictive. Indeed, if B is any Γ -space with degree $\deg(\Gamma)$ and $X \subset \mathbb{Z}_{\geq 0}$ is any subset of integers we can define another Γ_X -space $B_X = (B_{X,i})_{i \in \Gamma_X}$, where $\Gamma_X = \Gamma \cup X$. If $i \in \Gamma_{\geq 0} \cap X$, define $B_{X,i} = B_i$; otherwise, take $B_{X,i} = 0$. Any function ϵ in $\Gamma_{\geq 0}$ also extends to a function ϵ_X in $\Gamma_{\geq 0} \cup X$ defined by $\epsilon_X(i, j) = \epsilon(i, j)$ if $i, j \in \Gamma_{\geq 0}$ and $\epsilon_X(i, j) = 0$, otherwise. Notice that if $[0, r] \subset X$, then $\deg(B_X) \geq r$. Thus, given a (Γ, ϵ) -space we can always extend it by adding trivial (Γ, ϵ) -spaces with arbitrarily large degree.

3 Existence

- From now on we will assume that the $(B_{\alpha,\beta}^k, \mathbb{X})$ -structures $\mathcal{B}_{\alpha,\beta}^k(\mathbb{X}) \subset \mathcal{A}$ are *closed under restrictions* in the sense that if $(\varphi, U) \in \mathcal{B}_{\alpha,\beta}^k(\mathbb{X})$, then $(\varphi|_V, V) \in \mathcal{B}_{\alpha,\beta}^k(\mathbb{X})$ for every $V \subset U$.

Given a $(B_{\alpha,\beta}^k, \mathbb{X})$ -manifold $(M, \mathcal{A}, \mathcal{B}_{\alpha,\beta}^k(\mathbb{X}))$, let $D(\mathcal{B}_{\alpha,\beta}^k(\mathbb{X}))$ be the category whose objects are coordinate systems $(U, \varphi) \in \mathcal{B}_{\alpha,\beta}^k(\mathbb{X})$ and there exists a unique morphism $(U, \varphi) \rightarrow (U', \varphi')$ iff $U \subset U'$ and $\varphi = \varphi'|_U$. Let $C(\mathcal{B}_{\alpha,\beta}^k(\mathbb{X}))$ be the category whose objects are continuous functions $f : U \rightarrow \mathbb{R}$, where $(U, \varphi) \in D(\mathcal{B}_{\alpha,\beta}^k(\mathbb{X}))$. Similarly, there exists a morphism $g : f \Rightarrow f'$ between $f : U \rightarrow \mathbb{R}$ and $f' : U' \rightarrow \mathbb{R}$ iff $U \subset U'$ and $f = f'|_U$. Furthermore, given a subset $S \subset \Gamma_{\geq 0}$

and functions $\alpha_0 : S \rightarrow \Gamma_{\geq 0}$ and $\beta_0 : S \rightarrow [2, k]$, let $C_{\alpha_0, \beta_0}(\mathcal{B}_{\alpha, \beta}^k(\mathbb{X})|S) \subset C(\mathcal{B}_{\alpha, \beta}^k(\mathbb{X}))$ be the full subcategory whose objects are $(B_{\alpha_0, \beta_0}^k, \mathbb{X}|S)$ -morphisms. A *presheaf of functions in* $(M, \mathcal{A}, \mathcal{B}_{\alpha, \beta}^k)$ is a presheaf $\Omega : D(\mathcal{B}_{\alpha, \beta}^k(\mathbb{X}))^{op} \rightarrow C(\mathcal{B}_{\alpha, \beta}^k(\mathbb{X}))$. A *presheaf of* $(B_{\alpha_0, \beta_0}^k, \mathbb{X}|S)$ -*morphisms* is a presheaf of functions which takes values in $C_{\alpha_0, \beta_0}(\mathcal{B}_{\alpha, \beta}^k(\mathbb{X})|S)$. In the following we will work with a 3-parameter family Ω_{ab}^c of them, with $a, b, c = 1, \dots, n$. We will write $(\Omega_\varphi)_{ab}^c$ to denote $\Omega_{ab}^c(U, \varphi)$.

A $(B_{\alpha_0, \beta_0}^k, \mathbb{X}|S)$ -*connection* on $(M, \mathcal{A}, \mathcal{B}_{\alpha, \beta}^k(\mathbb{X}))$ is an affine connection ∇ on (M, \mathcal{A}) whose corresponding family of presheaves of functions $\Omega_{ab}^c(U, \varphi) = (\Gamma_\varphi)_{ab}^c$, where $(\Gamma_\varphi)_{ab}^c : U \rightarrow \mathbb{R}$ is the coefficients of ∇ in the local basis ∂_c^φ , is actually a family of presheaves of $(B_{\alpha_0, \beta_0}^k, \mathbb{X}|S)$ -morphisms. If (\ast, ϵ) is the multiplicative structure of B , let $\epsilon^r(l, m)$ denote $\epsilon(l, \epsilon(l, \epsilon(l, \dots \epsilon(l, m) \dots)))$.

Theorem 3.1. *Let \mathbb{X} be a proper full ISP and let $(M, \mathcal{A}, \mathcal{B}_{\alpha, \beta}^k(\mathbb{X}))$ be a nice $(B_{\alpha, \beta}^k, \mathbb{X})$ -manifold, with $k \geq 2$. Given functions α_0 and β_0 as above, there exists a connection ∇ in M whose coefficients in each $(\varphi, U) \in \mathcal{B}_{\alpha, \beta}^k$ belongs to*

$$B_{\alpha'_0}((\varphi(U')) \cap_{X(\varphi(U'))} C^{k-\beta'_0}(\varphi(U'))) \quad (1)$$

where $U' \subset U$ is a nonempty open set,

$$\alpha'_0 = \delta(\epsilon^3(\alpha(1), \alpha_0(0)), \epsilon(\alpha(2), \alpha(1))), \quad \beta'_0 = \max(\beta(1), \beta(2), \beta_0(0)) \quad (2)$$

and (ϵ, δ) is the distributive structure of B .

Proof. Recall that any family of C^2 -functions $f_{ab}^c : \mathbb{R}^n \rightarrow \mathbb{R}$ defines a connection in \mathbb{R}^n [2]. Thus, any family of C^2 -functions $f_{ab}^c : V \rightarrow \mathbb{R}$ in an open set $V \subset \mathbb{R}^n$ defines a connection ∇_0 in V . If $\varphi : U \rightarrow \mathbb{R}^n$ is a chart in M with $\varphi(U) \subset V$, pulling back ∇_0 we get a connection ∇_φ in U whose coefficients are $f_{ab}^c \circ \varphi$. Notice that if f_{ab}^c are $(B, k, \alpha_0, \beta_0)$ -functions in \mathbb{X} , then ∇_φ is a $(B_{\alpha, \beta}^k, \mathbb{X})$ -connection in U (independently of if $\varphi \in \mathcal{B}_{\alpha, \beta}^k(\mathbb{X})$ or not), so that the existence of ∇ is locally established. In order to globalize it, let $(\varphi_s, U_s)_s$ be an open covering of M by coordinate systems (not yet in $\mathcal{B}_{\alpha, \beta}^k(\mathbb{X})$) and let ∇_s be a connection in U_s with coefficients $(f_s)_{ab}^c$. Let (ψ_s) be a partition of unity subordinate to (φ_s, U_s) and recall that $\nabla = \sum_s \psi_s \cdot \nabla_s$ is a globally defined connection in M . We assert that if φ_s are in $\mathcal{B}_{\alpha, \beta}^k(\mathbb{X})$, then ∇ is a well-defined connection whose coefficients belong to (1) for certain U' . Notice that the coefficients $(\Gamma_s)_{ab}^c$ of ∇ in a fixed φ_s are obtained in the following way: let $N(s)$ be the finite set of every s' such that $U_{ss'} \equiv U_s \cap U_{s'} \neq \emptyset$. For each $s' \in N(s)$ we can do a change of coordinates and rewrite $(f_{s'})_{ab}^c$ in the coordinates φ_s as given by the functions

$$(f_{s';s})_{ab}^c = \sum_{l,m,o} \frac{\partial \varphi_s^c}{\partial \varphi_{s'}^l} \frac{\partial \varphi_{s'}^m}{\partial \varphi_s^a} \frac{\partial \varphi_{s'}^o}{\partial \varphi_s^b} (f_{s'})_{mo}^l + \sum_l \frac{\partial^2 \varphi_{s'}^l}{\partial \varphi_s^a \partial \varphi_s^b} \frac{\partial \varphi_s^c}{\partial \varphi_{s'}^l}, \quad (3)$$

where by abuse of notation $\varphi_s^c = (\varphi_s \circ \varphi_{s'}^{-1})^c$. We then have

$$(\Gamma_s)_{ab}^c = \sum_{s' \in N(s)} \psi_{s'} (f_{s';s})_{ab}^c. \quad (4)$$

By hypothesis $(f_{s'})_{mn}^l$ is a $(B, k, \alpha_0, \beta_0)$ -function in \mathbb{X} . Furthermore, since $\varphi_s \in \mathcal{B}_{\alpha, \beta}^k(\mathbb{X})$ we have

$$\frac{\partial \varphi_s^c}{\partial \varphi_{s'}^l} \in (B_{\alpha(1)} \cap_X C^{k-\beta(1)})(U_{ss'}) \quad \text{and} \quad \frac{\partial^2 \varphi_{s'}^l}{\varphi_s^a \partial \varphi_s^b} \in (B_{\alpha(2)} \cap_X C^{k-\beta(2)})(U_{ss'})$$

where $U_{ss'}^{s'} = \varphi_{s'}(U_{ss'})$. This is well-defined since $\beta(2) \leq k$. Due to the compatibility between the additive/multiplicative structures of B_α and $C^{k-\beta}$, the first and the second terms of the right-hand side of (3) belong to

$$(B_{\epsilon^3(\alpha(1), \alpha_0(0))} \cap_X C^{k-\max(\beta(1), \beta_0(0))})(U_{ss'}^{s'}) \quad \text{and} \quad (B_{\epsilon(\alpha(2), \alpha(1))} \cap_X C^{k-\max(\beta(2), \beta(1))})(U_{ss'}^{s'}),$$

respectively, so that the left-hand side of (3) belongs to

$$(B_{\tau(\epsilon^3(\alpha(1), \alpha_0(0)), \epsilon(\alpha(2), \alpha(1)))} \cap_X C^{k-\max(\beta(1), \beta(2), \beta_0(0))})(U_{N(s)}) = (B_{\alpha'_0} \cap_X C^{k-\beta'_0})(U_{N(s)}),$$

where $U_{N(s)} = \cap_{s' \in N(s)} U_{ss'}$. Since B is a nice $C_{\alpha, \beta}^k$ -presheaf in \mathbb{X} , the sum (4) belongs to the same space. Repeating the process for all coverings in $\mathcal{B}_{\alpha, \beta}^k(\mathbb{X})$ and noticing that α' and β' do not depend on s, s' , we get the desired result. \square

We would like to extend the connection ∇ in the last theorem in order to get a genuine $(B_{\alpha', \beta'}, \mathbb{X}|S)$ -connection, where α' and β' are functions such that $\alpha'(0) = \alpha'_0$ and $\beta'(0) = \beta'_0$. We will prove that this can be done under certain proper hypotheses on the underlying nice $C_{n, \alpha}^{k, \beta}$ -presheaf at the cost of making a change in the ambient presheaf \mathbb{X} .

Given B and B' presheaves of Γ -spaces and given functions $\theta, \vartheta, \theta', \vartheta' : \Gamma_{\geq 0} \rightarrow \Gamma_{\geq 0}$, let \mathbb{X} and \mathbb{Y} be proper ISP which are both between B_θ and $B'_{\theta'}$, and between B'_ϑ and $B'_{\vartheta'}$. We say that \mathbb{X} and \mathbb{Y} are *compatible* if they are defined over the same Γ -ambient. A $(\theta, \vartheta; \theta', \vartheta')$ -connection between B and B' in \mathbb{X}, \mathbb{Y} is a morphism $\xi_{\theta', \vartheta'}^{\theta, \vartheta} : B_\theta \cap_X B'_\vartheta \Rightarrow B_{\theta'} \cap_Y B'_{\vartheta'}$. We are interested in certain “universal connections”, as follows. Let $(\mathbf{X}, \gamma, \gamma_\Sigma)$ be a full Γ -ambient and let $\cap_{\mathbf{X}, \gamma, \gamma_\Sigma}(B, B')$ be the presheaf category of presheaves of base spaces of proper ISP between B and B' which are compatible and defined over $(\mathbf{X}, \gamma, \gamma_\Sigma)$. Furthermore, let $\cap_{\mathbf{X}, \gamma, \gamma_\Sigma}(B_\mathcal{O}, B'_\mathcal{Q}) = \coprod_{\theta \in \mathcal{O}} \coprod_{\vartheta \in \mathcal{Q}} \cap_{\mathbf{X}, \gamma, \gamma_\Sigma}(B_\theta, B'_\vartheta)$, where $\mathcal{O}, \mathcal{Q} \subset \text{Mor}(\Gamma_{\geq 0}; \Gamma_{\geq 0})$ are set of functions. Given maps $D_\mathcal{O}, D_\mathcal{Q}$, defined in $\mathcal{O} \times \mathcal{O}$ and in $\mathcal{Q} \times \mathcal{Q}'$, respectively, and assigning to each pair $\theta, \theta' \in \mathcal{O}$ (resp. $\vartheta, \vartheta' \in \mathcal{Q}$) a morphism of presheaves of Γ -spaces $D_\mathcal{O}(\theta, \theta') : B_\theta \Rightarrow B_{\theta'}$ (resp. $D_\mathcal{Q}(\vartheta, \vartheta') : B'_\vartheta \Rightarrow B'_{\vartheta'}$), define a *semi* $(D_\mathcal{O}, D_\mathcal{Q})$ -connection between B and B' as a functor $\mathcal{D} : \cap_{\mathbf{X}, \gamma, \gamma_\Sigma}(B_\mathcal{O}, B'_\mathcal{Q}) \rightarrow \text{Ar}(\cap_{\mathbf{X}, \gamma, \gamma_\Sigma}(B_\mathcal{O}, B'_\mathcal{Q}))$ such that $s(\mathcal{D}(X)) = X$ and such that the diagram below commutes for every $\mathfrak{f} = (\theta, \theta', \vartheta, \vartheta') \in \mathcal{O} \times \mathcal{Q}$. Thus, the dotted arrow exists (compare with the definition of connection between ISP given in [1]). If this universal arrow is created by the functor γ (e.g. if γ create pullbacks), then there exists $\xi : B_\theta \cap_X B'_\vartheta \Rightarrow B_{\theta'} \cap_Y B'_{\vartheta'}$ such that $\gamma \circ \xi \simeq \xi_\gamma$ and ξ is a $(\theta, \theta', \vartheta, \vartheta')$ -connection between B and B' in $\mathbb{X}, \mathcal{D}(\mathbb{X})$ for every $\theta, \theta' \in \mathcal{O}$ and every $\vartheta, \vartheta' \in \mathcal{Q}$. A $(D_\mathcal{O}, D_\mathcal{Q})$ -connection is a semi $(D_\mathcal{O}, D_\mathcal{Q})$ -connection for which these universal arrows are created.

$$\begin{array}{ccc}
 B_\theta \cap_{X, \gamma} B'_\vartheta & \xrightarrow{\quad \quad \quad} & \gamma \circ B'_\vartheta \\
 \downarrow & \searrow \xi_\gamma & \downarrow D_\mathcal{Q}(\vartheta, \vartheta') \\
 & & B_{\theta'} \cap_{Y, \gamma} B'_{\vartheta'} \xrightarrow{\quad \quad \quad} \gamma \circ B'_{\vartheta'} \\
 & & \downarrow \ell' \\
 \gamma \circ B_\theta & \xrightarrow{D_\mathcal{O}(\theta, \theta')} & \gamma \circ B_{\theta'} \xrightarrow{\ell} & Y \\
 & \searrow j & & \uparrow \mathcal{D}(X) \\
 & & & X
 \end{array} \tag{5}$$

Suppose now that B is a $C_{\alpha,\beta}^k$ -presheaf in some ISP \mathbb{X} whose underlying full ambient space is $(\mathbf{X}, \gamma, \gamma_\Sigma)$. Let $\alpha_0 : \Gamma_{\geq 0} \rightarrow \Gamma_{\geq 0}$ and $\beta_0 : \Gamma_{\geq 0} \rightarrow [2, k]$ be given functions and for a fixed $j \in \Gamma_{\geq 0}$, let $\Gamma_{\geq 0}(\beta_0; j)$ be the set of all $i \in \Gamma_{\geq 0}$ such that $0 \leq \beta_0(j) - i \leq k$, i.e, the set of every $i \leq k - \beta_0(j)$. A $(\alpha_0, \beta_0; j)$ -connection in B , is a $(D_{\mathcal{O}}, D_{\mathcal{Q}})$ -connection between B and C^{k-} , both regarded as presheaves of $\Gamma_{\geq 0}(\beta_0; j)$ -spaces, in which the functions $\alpha_{0,j}$ and $\beta_{\#,j}$, given by $\alpha_{0,j}(i) = \alpha_0(j)$ and $\beta_{\#,j}(i) = \beta_0(j) - i$, belongs to \mathcal{O} and \mathcal{Q} , respectively. Furthermore, for every $\vartheta \in \mathcal{Q}$ such that $\vartheta(i) \leq \beta_0(i)$ the transformation $D_{\mathcal{Q}}(\beta_0, \vartheta) : C^{k-\beta_0} \Rightarrow C^{k-\vartheta}$ is required to be the canonical inclusion.

We say that a subset $W \subset \Gamma_{\geq 0}(\beta_0; j) \times \Gamma_{\geq 0}$ is a (β_0, j) -ideal if the sum operation induces a map $+ : W \rightarrow \Gamma_{\geq 0}(\beta_0; j)$, i.e, if there exists the dotted arrow making commutative the diagram below. Let $i \in \text{pr}_r(W)$, with $r = 1, 2$, be the image of the projection on the r th factor. For a fixed $z \in \text{pr}_1(W)$, let $W[z]$ be set of all $l \in \Gamma_{\geq 0}$ such that $(z, l) \in W$, i.e, such that $\text{pr}_1(\text{pr}_2^{-1}(W[i])) = z$. We say that a sequence of pairs of functions $(\theta_l, \vartheta_l)_{l \in W[z]}$ is *ordinary* if the corresponding pair (θ_*, ϑ_*) , given by $\theta_*(l) = \theta_l(z)$ and $\vartheta_*(l) = \vartheta_l(z)$ belongs to $\mathcal{O} \times \mathcal{Q}$. Finally, we say that $(\theta, \vartheta) \in \mathcal{O} \times \mathcal{Q}$ is *ordinary* if $\theta = \theta'_*$ and $\vartheta = \vartheta'_*$ for some θ'_* and ϑ'_* , i.e, if it is the induced pair of an ordinary sequence of pairs.

$$\begin{array}{ccc} \Gamma_{\geq 0}(\beta_0; j) \times \Gamma_{\geq 0} & \hookrightarrow & \Gamma_{\geq 0} \times \Gamma_{\geq 0} \xrightarrow{+} \mathbb{Z}_{\geq 0} \\ \uparrow & & \uparrow \\ W & \dashrightarrow & \Gamma_{\geq 0}(\beta_0; j) \end{array}$$

Lemma 3.1 (Key Lemma). *Suppose that a $(B_{\alpha,\beta}^k, \mathbb{X})$ -manifold is such that B admits a $(\alpha_0, \beta_0; j)$ -connection $(\mathcal{D}, D_{\mathcal{O}}, D_{\mathcal{Q}})$ and let W be a (β_0, j) -ideal. For every $U \subset \mathbb{R}^n$ and every $z \in \text{pr}_1(W)$, each element of $B_{\alpha_0(j)}(U) \cap_{X(U)} C^{k-\beta_0(j)-z}(U)$ can be regarded as a $(B, k, \theta, \vartheta|W[z])$ -function in some ISP compatible with \mathbb{X} , for every ordinary pair $(\theta, \vartheta) \in \mathcal{O} \times \mathcal{Q}$.*

Proof. Since the B admits a connection $(\mathcal{D}, D_{\mathcal{O}}, D_{\mathcal{Q}})$, by definition it follows that there exists a ISP \mathbb{Y} , compatible with \mathbb{X} , such that for every $\theta, \theta' \in \mathcal{O}$ and $\vartheta, \vartheta' \in \mathcal{Q}$ we have morphisms $\xi_{U,i} : B_{\theta(i)}(U) \cap_{X(U)} C^{k-\vartheta(i)} \rightarrow B_{\theta'(i)}(U) \cap_{Y(U)} C^{k-\vartheta'(i)}(U)$ making commutative the diagram (5). Since the connection is actually a $(\alpha_0, \beta_0; j)$ -connection we have $\alpha_{0,j} \in \mathcal{O}$ and $\beta_{\#,j} \in \mathcal{Q}$. Thus, for $\theta = \alpha_{0,j}$ and $\vartheta = \beta_{\#,j}$ we see that for every $\theta' \in \mathcal{O}$, $\vartheta' \in \mathcal{Q}$, and $i \in \Gamma_{\geq 0}(\beta_0; j)$ we have a morphism from $B_{\alpha_0(j)}(U) \cap_{X(U)} C^{k-\beta_0(j)-i}(U)$ to $B_{\theta'(i)}(U) \cap_{Y(U)} C^{k-\vartheta'(i)}(U)$. Notice that if $f \in B_{\alpha_0(j)}(U) \cap_{X(U)} C^{k-\beta_0(j)-i}(U)$, then $\partial^\mu f \in C^{k-\beta_0(j)-(i+l)}(U)$ for every μ such that $|\mu| = l$ and for every $l \in \Gamma_{\geq 0}$ such that $i+l \in \Gamma_{\geq 0}(\beta_0; j)$. Thus, by universality, for every such l we can regard $\partial^\mu f \in B_{\theta'_l(i)}(U) \cap_{Y(U)} C^{k-\vartheta'_l(i)}(U)$, where $\theta'_l \in \mathcal{O}$ and $\vartheta'_l \in \mathcal{Q}$. Notice that if $z \in \text{pr}_1(W)$, then for every $l \in W[z]$ we have $z+l \in \Gamma_{\geq 0}(\beta_0; j)$, so that the functions $\theta(l) = \theta'_l(z)$ and $\vartheta(l) = \vartheta'_l(z)$ are defined on $W[z]$. Suppose that this pair (θ, ϑ) belongs to $\mathcal{O} \times \mathcal{Q}$, i.e, suppose that it is an ordinary pair. Thus, $\partial^\mu f \in B_{\theta(l)}(U) \cap_{Y(U)} C^{k-\vartheta(l)}(U)$, meaning that f can be regarded as a $(B, k, \theta, \vartheta|W[z])$ in \mathbb{Y} . \square

Corollary 3.1. *In the same notations of Theorem 3.1, if B admits a $(\alpha'_0, \beta'_0; j)$ -connection, then for every $z \in \Gamma_{\geq 0; \beta'_0}$, the elements of (1) can be regarded as $(B, k, \theta, \vartheta|_{\Gamma_{\geq 0; \beta'_0-z}})$ -functions in some ISP compatible with \mathbb{X} , for every ordinary pair (θ, ϑ) , where for every $0 \leq r \leq k$, $\Gamma_{\geq 0; r}$ is the set of $i \in \Gamma_{\geq 0}$ such that $0 \leq r - i \leq k$.*

Proof. First of all, notice that $\Gamma_{\geq 0}(\beta'_0; j) = \Gamma_{\geq 0; \beta'_0}$ for every j . Furthermore, let $W \subset \Gamma_{\geq 0; \beta'_0} \times \Gamma_{\geq 0}$ be the subset of each (z, l) such that $z+l \in \Gamma_{\geq 0; k}$, so that $W[z]$ is the set of every $0 \leq l \leq k - \beta'_0 - z$, i.e, $W[z] = \Gamma_{\geq 0; \beta'_0 - z}$. Finally, apply Lemma 3.1. \square

Suppose now that B and B' are actually presheaves of distributive Γ -spaces and \mathbb{X}, \mathbb{Y} are proper full ISP. In this case, define a *distributive* $(\theta, \vartheta; \theta', \vartheta')$ -connection as one which preserves the multiplicative and additive structures, in the sense that there are objectwise monomorphisms $\gamma_* : B_{\epsilon(\theta, \theta)} \Rightarrow B_{\epsilon(\theta', \theta')}$ and $\gamma'_* : B'_{\epsilon'(\vartheta, \vartheta)} \Rightarrow B'_{\epsilon'(\vartheta', \vartheta')}$ making commutative the diagram below, and also objectwise monomorphisms $\gamma_+ : B_{\delta(\theta, \theta)} \Rightarrow B_{\delta(\theta', \theta')}$ and $\gamma'_+ : B'_{\delta'(\vartheta, \vartheta)} \Rightarrow B'_{\delta'(\vartheta', \vartheta')}$ making commutative the analogous diagram for the additive structures. A *distributive* $(D_{\mathcal{O}}, D_{\mathcal{Q}})$ -connection between distributive B and B' is a $(D_{\mathcal{O}}, D_{\mathcal{Q}})$ -connection such that for every $\theta, \theta', \vartheta, \vartheta'$, the induced $(\theta, \vartheta; \theta', \vartheta')$ -connection is distributive.

$$\begin{array}{ccccccc}
B'_{\epsilon'(\vartheta, \vartheta)} & \xleftarrow{*\prime} & B'_{\vartheta} \otimes B'_{\vartheta} & \xleftarrow{\quad} & (B_{\theta} \cap_X B'_{\vartheta}) \otimes (B_{\theta} \cap_X B'_{\vartheta}) & \xrightarrow{\quad} & B_{\theta} \otimes B_{\theta} \xrightarrow{*\prime} B_{\epsilon(\theta, \theta)} \\
\gamma'_* \Downarrow & & & & \xi_{\theta', \vartheta'}^{\theta, \vartheta} \otimes \xi_{\theta', \vartheta'}^{\theta, \vartheta} \Downarrow & & \Downarrow \gamma_* \\
B'_{\epsilon'(\vartheta', \vartheta')} & \xleftarrow{*\prime} & B'_{\vartheta'} \otimes B'_{\vartheta'} & \xleftarrow{\quad} & (B_{\theta'} \cap_Y B'_{\vartheta'}) \otimes (B_{\theta'} \cap_Y B'_{\vartheta'}) & \xrightarrow{\quad} & B_{\theta'} \otimes B_{\theta'} \xrightarrow{*\prime} B_{\epsilon(\theta', \theta')}
\end{array}$$

A *distributive* $(\alpha_0, \beta_0; j)$ -connection in B is a $(\alpha_0, \beta_0; j)$ -connection in B which is distributive when regarded as a $(D_{\mathcal{O}}, D_{\mathcal{Q}})$ -connection. We say that a distributive $(\alpha_0, \beta_0; j)$ -connection is *nice* if in the diagram below the dotted arrows, arising as natural factorizations of the upper and lower paths in the diagram, are equal for every $\theta \in \mathcal{O}$ and $\vartheta \in \mathcal{Q}$, where u arises from universality of pullbacks (the non-identified arrows are inclusions). In other words, if the $(\alpha_0, \beta_0; j)$ -connection preserves bump-functions. Let $(M, \mathcal{A}, \mathcal{B}_{\alpha, \beta}^k(\mathbb{X}))$ be a $(B_{\alpha, \beta}^k, \mathbb{X})$ -manifold. A $(\alpha_0, \beta_0; j)$ -connection of degree r in the $(B_{\alpha, \beta}^k, \mathbb{X})$ -structure $\mathcal{B}_{\alpha, \beta}^k(\mathbb{X})$ is a $(\alpha_0, \beta_0; j)$ -connection in the presheaf B which preserves the transition functions φ_{ji} in $\mathcal{B}_{\alpha, \beta}^k(\mathbb{X})$ and also their derivatives $\partial^{\mu} \varphi_{ji}$, with $|\mu| \leq r$, in the same sense that it preserves bump functions. A *nice* $(\alpha_0, \beta_0; j)$ -connection in $\mathcal{B}_{\alpha, \beta}^k(\mathbb{X})$ is one whose underlying $(\alpha_0, \beta_0; j)$ -connection in B is nice.

$$\begin{array}{ccc}
B_{\alpha_0} \cap_X C^{k-\beta_0} & \xrightarrow{\quad} & C^{k-\beta_0} \\
\downarrow \xi & \swarrow u & \swarrow D_{\mathcal{Q}}(\beta_0, \infty) \\
& B_{\alpha_0} \cap_X C_b^{\infty} \xrightarrow{\quad} & C_b^{\infty} \\
& & \swarrow D_{\mathcal{Q}}(\vartheta, \infty) \\
B_{\theta} \cap_Y C^{k-\vartheta} & \xrightarrow{\quad} & C^{k-\vartheta}
\end{array}$$

We can finally establish the relation between $(\alpha'_0, \beta'_0; j)$ -connections on the structural presheaf B and $(B_{\theta, \vartheta}^k, \mathbb{Y})$ -connections on $(B_{\alpha, \beta}^k, \mathbb{X})$ -manifolds.

Proposition 3.1. *Let $(M, \mathcal{A}, \mathcal{B}_{\alpha, \beta}^k(\mathbb{X}))$ be a $(B_{\alpha, \beta}^k, \mathbb{X})$ -manifold, with $k \geq 2$, and suppose that the structural presheaf B is such that $\mathcal{B}_{\alpha, \beta}^k(\mathbb{X})$ admits a nice distributive $(\alpha'_0, \beta'_0; j)$ -connection of degree $r \geq 2$, say for a proper full ISP \mathbb{X}' . Then for every ordinary pair (θ, ϑ) and every $z \in \Gamma_{\geq 0; \beta'_0}$ there exists a $(B_{\theta, \vartheta}^k, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z})$ -connection in M .*

Proof. From Theorem 3.1 there exists an affine connection ∇ whose coefficients (4) belong to (1). By the hypothesis we have a $(\alpha'_0, \beta'_0; j)$ -connection ξ , so that from Corollary 3.1 for any ordinary pair (θ, ϑ) , the image $(\Gamma_\varphi^\xi)_{ab}^c$ of the coefficients (4) by $\xi_{\theta\vartheta}$ belong to

$$B_{\theta\vartheta}^k(\varphi(U'), X'(\varphi(U')) | \Gamma_{\geq 0; \beta'_0 - z}). \quad (6)$$

We assert that $(\Gamma_\varphi^\xi)_{ab}^c$ also define a connection. This follows the fact that ξ is nice and has degree $r \geq 2$. Indeed, we have

$$\begin{aligned} \xi_{\theta\vartheta}((f_{s';s})_{ab}^c) &= \xi_{\theta\vartheta} \left(\sum_{l,m,o} \frac{\partial \varphi_s^c}{\partial \varphi_{s'}^l} \frac{\partial \varphi_{s'}^m}{\partial \varphi_s^a} \frac{\partial \varphi_{s'}^o}{\partial \varphi_s^b} (f_{s'})_{mo}^l + \sum_l \frac{\partial^2 \varphi_{s'}^l}{\partial \varphi_s^a \partial \varphi_s^b} \frac{\partial \varphi_s^c}{\partial \varphi_{s'}^l} \right) \\ &= \sum_{l,m,o} \xi_{\theta\vartheta} \left(\frac{\partial \varphi_s^c}{\partial \varphi_{s'}^l} \right) \xi_{\theta\vartheta} \left(\frac{\partial \varphi_{s'}^m}{\partial \varphi_s^a} \right) \xi_{\theta\vartheta} \left(\frac{\partial \varphi_{s'}^o}{\partial \varphi_s^b} \right) \xi_{\theta\vartheta}((f_{s'})_{mo}^l) + \sum_l \xi_{\theta\vartheta} \left(\frac{\partial^2 \varphi_{s'}^l}{\partial \varphi_s^a \partial \varphi_s^b} \right) \xi_{\theta\vartheta} \left(\frac{\partial \varphi_s^c}{\partial \varphi_{s'}^l} \right) \\ &= \sum_{l,m,o} \frac{\partial \varphi_s^c}{\partial \varphi_{s'}^l} \frac{\partial \varphi_{s'}^m}{\partial \varphi_s^a} \frac{\partial \varphi_{s'}^o}{\partial \varphi_s^b} \xi_{\theta\vartheta}((f_{s'})_{mo}^l) + \sum_l \frac{\partial^2 \varphi_{s'}^l}{\partial \varphi_s^a \partial \varphi_s^b} \frac{\partial \varphi_s^c}{\partial \varphi_{s'}^l}, \end{aligned}$$

where in the first step we used (3), in the second we used compatibility between the distributive structures of B and C^{k-} and also the distributive properties of ξ , and in the third step we used that ξ has degree $r \geq 2$. Thus, by the same kind of arguments,

$$\begin{aligned} (\Gamma_s^\xi)_{ab}^c &= \xi_{\theta\vartheta} \left(\sum_{s' \in N(s)} \psi_{s'} \cdot (f_{s';s})_{ab}^c \right) \\ &= \sum_{s' \in N(s)} \psi_{s'} \left(\sum_{l,m,o} \frac{\partial \varphi_s^c}{\partial \varphi_{s'}^l} \frac{\partial \varphi_{s'}^m}{\partial \varphi_s^a} \frac{\partial \varphi_{s'}^o}{\partial \varphi_s^b} \xi_{\theta\vartheta}((f_{s'})_{mo}^l) + \sum_l \frac{\partial^2 \varphi_{s'}^l}{\partial \varphi_s^a \partial \varphi_s^b} \frac{\partial \varphi_s^c}{\partial \varphi_{s'}^l} \right), \end{aligned}$$

which clearly define an affine connection $\xi_{\theta\vartheta}(\nabla)$ in M . Since they belong to (6), it follows that $\xi_{\theta\vartheta}(\nabla)$ is actually a $(B_{\theta,\vartheta}^k, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z})$ -connection. \square

4 Multiplicity

The last result was about the existence of $(B_{\theta\vartheta}^k, \mathbb{X}')$ -connections. Now we will consider the multiplicity problem. Let $(M, \mathcal{A}, \mathcal{B}_{\alpha,\beta}^k(\mathbb{X}))$ be a $(B_{\alpha,\beta}^k, \mathbb{X})$ -manifold and let $D_{\text{End}}(\mathcal{B}_{\alpha,\beta}^k(\mathbb{X})) \subset D(\mathcal{B}_{\alpha,\beta}^k(\mathbb{X}))$ be the full subcategory of those coordinate systems $(\varphi, U) \in \mathcal{B}_{\alpha,\beta}^k(\mathbb{X})$ which trivialize $\text{End}(TM)$. Notice that every $\text{End}(TM)$ -valued 1-form $\omega \in \Omega^1(M; \text{End}(TM))$ induces a 3-parameter family presheaf $\omega_{ab}^c : D_{\text{End}}(\mathcal{B}_{\alpha,\beta}^k(\mathbb{X}))^{op} \rightarrow C(\mathcal{B}_{\alpha,\beta}^k(\mathbb{X}))$ of functions. We say that ω is a $\text{End}(TM)$ -valued form $(B_{\alpha_0,\beta_0}^k, \mathbb{X}_0 | S)$ -form of degree one if ω_{ab}^c are actually presheaves of $(B_{\alpha_0,\beta_0}^k, \mathbb{X}_0 | S)$ -morphisms. Let

$$\Omega_{\alpha_0,\beta_0}^1(M; \text{End}(TM), \mathbb{X}_0 | S)$$

be the set of all of them. This is clearly a real vector subspace of $\Omega^1(M; \text{End}(TM))$. Indeed, recall that in a distributive Γ -space $B = (B_i)$ the additive structure $+_{ij} : B_i \otimes B_j \rightarrow B_{\delta(i,j)}$ is such that $\delta(i,i) = i$ and $+_{ii}$ is the sum of B_i . On the other hand, if ∇ and $\overline{\nabla}$ are two affine connections in M , then $\omega = \nabla - \overline{\nabla}$ belongs to $\Omega^1(M; \text{End}(TM))$. In terms of presheaves of coefficients, we

have $\omega_{ab}^c = \Gamma_{ab}^c - \bar{\Gamma}_{ab}^c$. Thus, again by the fact that $+_{ii} = +_i$, it follows that if ∇ and $\bar{\nabla}$ are $(B_{\alpha_0, \beta_0}^k, \mathbb{X}_0|S)$ -connections, then ω is a $\text{End}(TM)$ -valued form $(B_{\alpha_0, \beta_0}^k, \mathbb{X}_0|S)$ -form of degree one. In other words, we have proved:

Proposition 4.1. *Let $(M, \mathcal{A}, \mathcal{B}_{\alpha, \beta}^k(\mathbb{X}))$ be a $(B_{\alpha, \beta}^k, \mathbb{X})$ -manifold. For every $S \subset \Gamma_{\geq 0}$, every integer functions $\alpha_0 : S \rightarrow \Gamma_{\geq 0}$ and $\beta_0 : S \rightarrow [2, k]$ and every ISP \mathbb{X}_0 the set $\text{Conn}_{\alpha_0, \beta_0}^k(M, \mathbb{X}_0|S)$ is an affine space of $\Omega_{\alpha_0, \beta_0}^1(M; \text{End}(TM), \mathbb{X}_0|S)$.*

As commented in Section 1, this is not enough to conclude the existence of many linearly independent $(B_{\alpha_0, \beta_0}^k, \mathbb{X}_0|S)$ -connections. Indeed, we need to analyze the dimension of the space of $(B_{\alpha_0, \beta_0}^k, \mathbb{X}_0|S)$ -forms of degree one. Instead of doing this, with eyes in future applications, we will follow a more direct approach.

Notice that under the assumption of a distributive nice $(\alpha'_0, \beta'_0; j)$ -connection for B , the proof of Theorem 3.1 suggests that a nice $(B_{\alpha, \beta}^k, \mathbb{X})$ -manifold has not only one, but actually many of those $B_{\theta, \vartheta}^k$ -connections, for every ordinary (θ, ϑ) . Indeed, for each choice of $(B, k, \alpha_0, \beta_0)$ -functions f_{ac}^c covering M by charts in $\mathcal{B}_{\alpha, \beta}^k(\mathbb{X})$, the expression (4) defines a affine connection in M , which is really a $(B_{\theta, \vartheta}^k, \mathbb{X}')$ -connection, due to Proposition 3.1. But different families of functions could a priori define the same $(B_{\theta, \vartheta}^k, \mathbb{X}')$ -connection. In other words, Theorem 3.1 and Proposition 3.1 give us the functions in the diagram below, where the first set is that of all 3-parameter families f_{ab}^c of presheaves of $(B, k, \alpha_0, \beta_0)$ -functions in \mathbb{X} and the second one is the set of affine connections in M whose coefficients belong to (1). It happens that the composition map usually does not have a trivial kernel (each choice of a $(\alpha'_0, \beta'_0; j)$ -connection induces one of these composition maps).

$$\text{Psh}_{\alpha_0, \beta_0; 3}^k(\mathbb{R}^n; \mathbb{X}) \xrightarrow[\text{Thm}]{} \text{Conn}_{\alpha'_0, \beta'_0}^k(M, \mathcal{A}; \mathbb{X}) \xrightarrow[\text{Prop}]{} \text{Conn}_{\theta, \vartheta}^k(M, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z}) \quad (7)$$

$\xrightarrow{\text{Prop} \circ \text{Thm}}$

The next result reveals that under certain additional assumptions every $(B_{\theta, \vartheta}^k, \mathbb{X}')$ -connection ∇ can be deformed into another $(B_{\theta, \vartheta}^k, \mathbb{X}')$ -connection $\bar{\nabla}$ which differs from ∇ in at least one nonempty open set. In order to be more precise, notice that diagram (7) can be completed to the following noncommutative diagram, where the first vertical arrow is the inclusion and the other two take a connection ∇ and give the 3-parameter family presheaf of local expressions. Furthermore, $\text{Psh}_3(\mathbb{R}^n)$ is just the set of 3-parameter family of presheaves of continuous maps.

$$\begin{array}{ccccc} \text{Psh}_{\alpha_0, \beta_0; 3}^k(\mathbb{R}^n; \mathbb{X}) & \xrightarrow{\text{Thm}} & \text{Conn}_{\alpha'_0, \beta'_0}^k(M, \mathcal{A}; \mathbb{X}) & \xrightarrow{\text{Prop}} & \text{Conn}_{\theta, \vartheta}^k(M, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z}) \\ & \searrow \iota & \downarrow j & \swarrow j & \\ & & \text{Psh}_3^k(\mathbb{R}^n) & & \end{array}$$

We say that $\Omega = (\Omega_{ab}^c)$ and $\Lambda = (\Lambda_{ab}^c)$ in $\text{Psh}_3(\mathbb{R}^n)$ are *locally different* (and we write $\Omega \approx \Lambda$) if for every $a, b, c = 1, \dots, n$ and every $U \subset \mathbb{R}^n$ there exists a nonempty open set $U_{a,b,c} \subset U$ such that $\Omega_{ab}^c(p) \neq \Lambda_{ab}^c(p)$ for every $p \in U_{a,b,c}$. We say that two $(B_{\theta, \vartheta}^k, \mathbb{X}'|S)$ -connections ∇ and $\bar{\nabla}$ are *locally different* if their underlying 3-parameter family presheaves are locally different, i.e, if $j(\nabla) \approx j(\bar{\nabla})$. A map F assigning to each $(B_{\theta, \vartheta}^k, \mathbb{X}'|S)$ -connection ∇ another $(B_{\theta, \vartheta}^k, \mathbb{X}'|S)$ -connection $F(\nabla)$ is *has no local fixed points* if for every ∇ we have $F(\nabla) \approx \nabla$. In the following we will show that under

certain addition assumptions there exists one map

$$\rho : \text{Conn}_{\theta, \vartheta}^k(M, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z}) \rightarrow \text{Conn}_{\theta, \vartheta}^k(M, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z})$$

which has no local fixed points. This will be done by making use of Thom's transversality, so that we will need to assume $k = \infty$. We will also need a closure condition. Indeed, given functions $\theta : S \rightarrow \Gamma_{\geq 0}$ and $\vartheta : S \rightarrow [0, k]$, with $S \subset \Gamma_{\geq 0}$, we say that a $B_{\alpha, \beta}^k$ -presheaf B is $(\theta, \vartheta|S)$ -strong in a proper ISP \mathbb{X}' between B_θ and $C^{k-\vartheta}$ if for every U and every $i \in S$, the image of the projection map $B_{\theta(i)}(U) \cap_{X'(U)} C^{k-\vartheta(i)}(U) \rightarrow C^{k-\vartheta(i)}(U)$ is a closed set.

Theorem 4.1. *In addition to the hypotheses of Theorem 3.1, assume that M is smooth and has countably many connected-components, that B admits a nice distributive $(\alpha'_0, \beta'_0; j)$ -connection of degree $r \geq 2$, say in a proper ISP \mathbb{X}' , and that B is $(\theta, \vartheta| \Gamma_{\geq 0; \beta'_0 - z})$ -strong in \mathbb{X}' for some ordinary (θ, ϑ) and some $z \in \Gamma_{\geq 0; \beta'_0}$. In this case, for any 3-parameter family Ω_{ab}^c of presheaves of $(B_{\theta, \vartheta}^\infty, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z})$ -morphisms there exists a $(B_{\theta, \vartheta}^\infty, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z})$ -connection ∇ in $(M, \mathcal{A}, \mathcal{B}_{\alpha, \beta}^\infty(\mathbb{X}))$ such that for each $(\varphi, U) \in \mathcal{B}_{\alpha, \beta}^\infty(\mathbb{X})$ the functions $\Gamma_\varphi^c = \sum_{a,b} [(\Gamma_\varphi)_{ab}^c - (\Omega_\varphi)_{ab}^c]$, are nonzero in some nonempty $\overline{U} \subset U$, where Γ_{ab}^c is the presheaf of coefficients of ∇ .*

Proof. First of all we notice that if $f, g : V \rightarrow \mathbb{R}$ are two transversal real smooth functions defined on an open set $V \subset \mathbb{R}^n$, then the set V_- of points $p \in V$ such that $f(p) = g(p)$ cannot be dense in V . Indeed, V_- is given by $f \cap_p g = f^{-1}(p) \cap g^{-1}(p)$. Since $f \pitchfork g$, this is a $(n-2)$ -submanifold of V and therefore of \mathbb{R}^n , but $n - (n-2) = 2 \geq 1$, so that $f \cap_p g$ cannot be dense in V and thus in \mathbb{R}^n . On the other hand, transversality is a generic property, so that if f is not transversal to g , we can modify it a bit in order to get \overline{f} such that $\overline{f} \pitchfork g$. Now, since we are assuming $k = \infty$ and since B admits a connection in which B is $(\theta, \vartheta| \Gamma_{\geq 0; \beta'_0 - z})$ -strong in \mathbb{X}' , each $B_{\theta(i)}(V) \cap_{X'(V)} C^\infty(V)$ can be regarded as a closed set in $C^\infty(V)$, so that any dense set X in $C^\infty(V)$ is also dense in $B_{\theta(i)}(V) \cap_{X'(V)} C^\infty(V)$. Thus, any f in that intersection can be regarded as a smooth real function and small modifications of it (regarded as a smooth function) remains in $B_{\theta(i)}(V) \cap_{X'(V)} C^\infty(V)$. Now, given a countable covering of M by charts $(U_s, \varphi_s) \in \mathcal{B}_{\alpha, \beta}^\infty(\mathbb{X})$, which exists since M is paracompact and has countably many connected-components, let $r_0 = \min \Gamma_{\geq 0; \beta'_0 - z}$ and consider $(\Omega_{\varphi_s})_{ab}^c \in B_{\theta(r_0)}(U_{N(s)}) \cap_{X'(N(s))} C^\infty(U_{N(s)})$, regard each of them as a smooth function and take the sum $g^c = \sum_{ab} (\Omega_{\varphi_s})_{ab}^c$. Let $(f_s)_{mn}^l$ be (α_0, β_0) -functions in \mathbb{X} defining a $(B_{\theta, \vartheta}^\infty, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z})$ -connection, whose existence is ensured by Corollary 3.1. Consider the corresponding functions $(\Gamma_{\varphi_s})_{ab}^c \in B_{\theta(r_0)}(U_{N(s)}) \cap_{X'(N(s))} C^\infty(U_{N(s)})$, regard them as smooth function and also take the sum $f_{\varphi_s}^c = \sum_{ab} (\Gamma_{\varphi_s})_{ab}^c$. By the above we can assume $f_{\varphi_s}^c \pitchfork g_{\varphi_s}^c$, otherwise we can modify $(f_s)_{mn}^l$ a bit in order to get this. Therefore, the set $U_{s,-}$ in which $f_{\varphi_s}^c(p) = g_{\varphi_s}^c(p)$ is not dense, so that there exists $p \in \varphi_s(U)$ and small neighborhood $B_{s,p} \ni p$ such that $B_{s,p} \cap U_- = \emptyset$. Let \overline{U}_c the union of all those $B_{s,p}$. Finally, take \overline{U} as the intersection of \overline{U}_c for all c . That this intersection is nonempty follows from the fact that $C^\infty(V)$ is a Baire space for every V and that the covering $(U_s, \varphi_s)_s$ is enumerable. \square

Corollary 4.1. *In the same conditions of the last theorem, for every $(B_{\theta, \vartheta}^\infty, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z})$ -connection ∇ in $(M, \mathcal{A}, \mathcal{B}_{\alpha, \beta}^\infty)$ there exists another $(B_{\theta, \vartheta}^\infty, \mathbb{X}' | \Gamma_{\geq 0; \beta'_0 - z})$ -connection $\overline{\nabla}$ such that for every $(U, \varphi) \in \mathcal{B}_{\alpha, \beta}^\infty$ and every $a', b', c = 1, \dots, n$, there exists $U_{a', b' c} \subset U$ such that $(\Gamma_\varphi)_{a' b' c}^c(p) \neq (\overline{\Gamma}_\varphi)_{a' b' c}^c(p)$ for every $p \in U_{a', b' c}$. In particular, $\nabla \approx \overline{\nabla}$.*

Proof. Take $\Gamma = \Omega$ in the last theorem, so that there exists such $\bar{\nabla}$ and for every φ there exists \bar{U} in which $\sum_{a,b}[(\Gamma_\varphi)_{ab}^c - (\bar{\Gamma}_\varphi)_{ab}^c] \neq 0$ for all c . Therefore, for every $p \in \bar{U}$ we have

$$0 < |\Gamma_\varphi^c(p)| \leq \sum_{a,b} |(\Gamma_\varphi)_{ab}^c(p) - (\bar{\Gamma}_\varphi)_{ab}^c(p)|,$$

implying the existence of functions $\bar{a}, \bar{b} : \bar{U} \rightarrow [n]$ such that

$$(\Gamma_\varphi)_{\bar{a}(p)\bar{b}(p)}^c(p) \neq (\bar{\Gamma}_\varphi)_{\bar{a}(p)\bar{b}(p)}^c(p).$$

Regard $[n]$ as a 0-manifold and notice that if two smooth functions $f, g : V \rightarrow [n]$ are transversal, then they coincide in an open set $V' \subset V$. Since $[n]$ is a 0-manifold, any smooth map $f : V \rightarrow [n]$ is a submersion and therefore transversal to each other. In particular, \bar{a} (resp. \bar{b}) is transversal to each constant map $a' \in [n]$ (resp. b'), so that for each a' (resp. b') there exists $U_{a'} \subset \bar{U}$ (resp. $U_{b'} \subset \bar{U}$) in which

$$(\Gamma_\varphi)_{a'\bar{b}(p)}^c(p) \neq (\bar{\Gamma}_\varphi)_{a'\bar{b}(p)}^c(p) \quad (\text{resp.} \quad (\Gamma_\varphi)_{\bar{a}(p)b'}^c(p) \neq (\bar{\Gamma}_\varphi)_{\bar{a}(p)b'}^c(p))$$

Taking $U_{a',b'} = U_{a'} \cap U_{b'}$ and noticing that this intersection is nonempty, the proof is done. \square

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