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► **To cite this version:**

Yuri Ximenes Martins, Rodney Josué Biezuner. Towards Axiomatization and General Results on Strong Emergence Phenomena Between Lagrangian Field Theories. 2020. hal-02907897

HAL Id: hal-02907897

<https://hal.science/hal-02907897>

Preprint submitted on 27 Jul 2020

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Towards Axiomatization and General Results on Strong Emergence Phenomena Between Lagrangian Field Theories

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April 29, 2020

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Abstract

In this paper we propose a formal definition of what is a strong emergence phenomenon between two parameterized field theories and we present sufficient conditions ensuring the existence of such phenomena between two given parameterized Lagrangian field theories. More precisely, we prove that in an Euclidean background, typical parameterized kinetic theories emerge from any elliptic multivariate polynomial theories. Some concrete examples are given and a connection with the phenomenon of gravity emerging from noncommutativity is made.

1 Introduction

The term *emergence phenomenon* has been used for years in many different contexts. In each of them, Emergence Theory is the theory which studies those kinds of phenomena. E.g, we have versions of it in Philosophy, Art, Chemistry and Biology [1, 40, 13]. The term is also used many times in Physics, with different meanings (for a review on the subject, see [11, 16]). For an axiomatization approach, see [17]). This reveals that the concept of emergence phenomenon is very general and therefore difficult to formalize. Nevertheless, we have a clue of what it really is: when looking at all those presentations we see that each of them is about describing a system in terms of some other system, possibly in different *scales*. Thus, an emergence phenomenon is about a relation between two different systems, the *emergence relation*, and a system emerges from another when it (or at least part of it) can be recovered in terms of the other system, which is presumably more fundamental, at least in some scale. The different emergence phenomena in Biology, Philosophy, Physics, and so on, are obtained by fixing in the above abstract definition a meaning for system, scale, etc.

Notice that, in this approach, in order to talk about emergence we need to assume that to each system of interest we have assigned a *scale*. In Mathematics, scales are better known as *parameters*. So, emergence phenomena occur between some kinds of *parameterized systems*. This kind of assumption (that in order to fix a system we have to specify the scale in which we are

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considering it) is at the heart of the notion of effective field theory, where the scale (or parameter) is governed by Renormalization Group flows [10, 28, 24, 21]. Notice, in turn, that if a system emerges from some other, then the second one should be more fundamental, at least in the scale (or parameter) in which the emergence phenomenon is observed. This also puts Emergence Theory in the framework of searching for the fundamental theory of Physics (e.g Quantum Gravity), whose systems should be the minimal systems relative to the emergence relation [11, 16]. The main problem in this setting is then the existence problem for the minimum. A very related question is the general existence problem: *given two systems, is there some emergence relation between them?*

One can work on the existence problem is different levels of depth. Indeed, since the systems is question are parameterized one can ask if there exists a correspondence between them in *some scales* or in *all scales*. Obviously, by requiring a complete correspondence between them is much more strong than requiring a partial one. On the other hand, in order to attack the existence problem we also have to specify which kind of emergence relation we are looking for. Again, is it a full correspondence, in the sense that the emergent theory can be fully recovered from the fundamental one, or is it only a partial correspondence, through which only certain aspects can be recovered? Thus, we can say that we have the following four versions of the existence problem for emergence phenomena.

	<i>weak</i>	<i>weak-scale</i>	<i>weak-relation</i>	<i>strong</i>
<i>relation</i>	partial	full	partial	full
<i>scales</i>	some	some	all	all

Table 1: Types of Emergence

In Physics one usually works on finding weak emergence phenomena. Indeed, one typically shows that certain properties of a system can be described by some other system at some limit, corresponding to a certain regime of the parameter space. These emergence phenomena are strongly related with other kind of relation: the *physical duality*, where two different systems reveal the same physical properties. One typically builds emergence from duality. For example, AdS/CFT duality plays an important role in describing spacetime geometry (curvature) from mechanic statistical information (entanglement entropy) of dual strongly coupled systems [34, 31, 37, 6, 8, 7].

There are also some interesting examples of weak-scale emergence relations, following again from some duality. These typically occur when the action functional of two Lagrangian field theories are equal at some limit. The basic example is gravity emerging from noncommutativity following from the duality between commutative and noncommutative gauge theories established by the Seiberg-Witten maps [35]. Quickly, the idea was to consider a gauge theory $S[A]$ and modify it into two different ways:

1. by considering $S[A]$ coupled to some background field χ , i.e, $S_\chi[A; \chi]$;
2. by using the Seiberg-Witten map to get its noncommutative analogue $S_\theta[\hat{A}; \theta]$.

Both new theories can be regarded as parameterized theories: the parameter (or scale) of the first one is the background field χ , while that of the second one is the noncommutative parameter $\theta^{\mu\nu}$. By construction, the noncommutative theory $S_\theta[A; \theta]$ can be expanded in a power series on the noncommutative parameter, and we can also expand the other theory $S_\chi[A; \chi]$ on the background

field, i.e, one can write

$$S_\chi[A; \chi] = \sum_{i=0}^{\infty} S_i[A; \chi^i] = \lim_{n \rightarrow \infty} S_{(n)}[A; \chi] \quad \text{and} \quad S_\theta[\hat{A}; \theta] = \sum_{i=0}^{\infty} S_i[A; \theta^i] = \lim_{n \rightarrow \infty} S_{(n)}[A; \theta],$$

where $S_{(n)}[A; \chi] = \sum_{i=0}^n S_i[A; \chi^i]$ and $S_{(n)}[A; \theta] = \sum_{i=0}^n S_i[A; \theta^i]$ are partial sums. One then try to find solutions for the following question:

Question 1. *Given a gauge theory $S[A]$, is there a background version $S_\chi[A; \chi]$ of it and a number n such that for every given value $\theta^{\mu\nu}$ of the noncommutative parameter there exists a value of the background field $\chi(\theta)$, possibly depending on $\theta^{\mu\nu}$, such that for every gauge field A we have $S_{(n)}[A; \chi(\theta)] = S_{(n)}[A; \theta]$?*

Notice that if rephrased in terms of parameterized theories, the question above is precisely about the existence of a weak-scale emergence between S_χ and S_θ , at least up to order n . This can also be interpreted by saying that, in the context of the gauge theory $S[A]$, the background fields χ emerges in some regime from the noncommutativity of the spacetime coordinates. Since the noncommutative parameter $\theta^{\mu\nu}$ depends on two spacetime indexes, it is suggestive to consider background fields of the same type, i.e, $\chi^{\mu\nu}$. In this case, there is a natural choice: metric tensors $g^{\mu\nu}$. Thus, in this setup, the previous question is about proving that in the given gauge context, gravity emerges from noncommutativity at least up to a perturbation of order n . This was proved to be true for many classes of gauge theories and for many values of n [32, 43, 4, 27, 14]. On the other hand, this naturally leads to other two questions:

1. Can we find some emergence relation between gravity and noncommutativity in the nonperturbative setting? In other words, can we extend the weak-scale emergence relation above to a strong one?
2. Is it possible to generalize the construction of the cited works to other kind of background fields? In other words, is it possible to use the same idea in order to show that different fields emerge from spacetime noncommutativity?

The first of these questions is about finding a strong emergence phenomena and it has a positive answer in some cases [42, 5, 36, 33]. The second one, in turn, is about working on finding systematic and general conditions ensuring the existence (or nonexistence) of emergence phenomena. At least to the authors knowledge, there are no such studies, specially focused on the strong emergence between field theories. It is precisely this point that is the focus of the present work. Indeed we will:

1. based on Question 1, propose an axiomatization for the notion of *strong emergence* between field theories;
2. establish sufficient conditions ensuring that a given Lagrangian field theory emerges from each theory belonging to a certain class of theories;
3. show that the given sufficient conditions are not necessary conditions.

In the remainder of this Introduction, let us be a bit more explicit about our aim. For us, a *field theory* over a oriented n -dimensional manifold M (regarded as the spacetime) is defined by an action functional $S[\varphi] = \int_M \mathcal{L}(x, \varphi, \partial\varphi) d^n x$, where \mathcal{L} is the Lagrangian density and φ is some generic field (section of some real or complex vector bundle $E \rightarrow M$, the *field bundle*). A *parameterized field theory* consists of another bundle $P \rightarrow M$ (the *parameter bundle*), a subset $\text{Par}(P) \subset \Gamma(P)$ of global sections (the *parameters*) and a collection $S_\varepsilon[\varphi] = \int_M \mathcal{L}_\varepsilon(x, \varphi, \partial\varphi) d^n x$ of field theories, one for each parameter $\varepsilon \in \text{Par}(P)$. A more suggestive notation should be $S[\varphi; \varepsilon]$ and $\mathcal{L}(x, \varphi, \partial\varphi; \varepsilon)$. So, e.g, for the trivial parameter bundle $P \simeq M \times \mathbb{K}$ we have $\Gamma(P) \simeq C^\infty(M; \mathbb{K})$ and in this case we say that we have *scalar parameters*. If we consider only scalar parameters which are constant functions, then a parameterized theory becomes the same thing as a 1-parameter family of field theories. Here, and throughout the paper, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether the field bundle in consideration is real or complex.

We will think of a parameter ε as some kind of “physical scale”, so that for two given parameters ε and ε' , we regard $S[\varphi; \varepsilon]$ and $S[\varphi; \varepsilon']$ as *the same theory* in two different physical scales. Notice that if P has rank l , then we can locally write $\varepsilon = \sum \varepsilon^i e_i$, with $i = 1, \dots, l$, where e_i is a local basis for $\Gamma(P)$. Thus, locally each physical scale is completely determined by l scalar parameters ε^i which are the fundamental ones. In terms of these definitions, Question 1 has a natural generalization:

Question 2. *Let $S_1[\varphi; \varepsilon]$ and $S_2[\psi; \delta]$ be two parameterized theories defined on the same spacetime M , but possibly with different field bundles E_1 and E_2 , and different parameter bundles P_1 and P_2 . Arbitrarily giving a field $\varphi \in \Gamma(E_1)$ and a parameter $\varepsilon \in \text{Par}_1(P_1)$, can we find some field $\psi(\varphi) \in \Gamma(E_2)$ and some parameter $\delta(\varepsilon) \in \text{Par}_2(P_2)$ such that $S_1[\varphi; \varepsilon] = S_2[\psi(\varphi); \delta(\varepsilon)]$? In more concise terms, are there functions $F : \text{Par}_1(P_1) \rightarrow \text{Par}_2(P_2)$ and $G : \Gamma(E_1) \rightarrow \Gamma(E_2)$ such that $S_1[\varphi; \varepsilon] = S_2[G(\varphi); F(\varepsilon)]$?*

We say that the theory $S_1[\varphi; \varepsilon]$ *emerges from the theory* $S_2[\psi; \delta]$ if the problem above has a positive solution, i.e, if we can fully describe S_1 in terms of S_2 . Notice, however, that as stated the emergence problem is fairly general. Indeed, if P_1 and P_2 have different ranks, then, by the previous discussion, this means that the parameterized theories S_1 and S_2 have a different number of fundamental scales, so that we should not expect an emergence relation between them. This leads us to think of considering only the case in which $P_1 = P_2$. However, we could also consider the situations in which $P_1 \neq P_2$, but $P_2 = f(P_1)$ is some nice function of P_1 , e.g, $P_2 = P_1 \times P_1 \times \dots \times P_1$. In these cases the fundamental scales remain only those of P_1 , since from them we can generate those in the product. Throughout this paper we will also work with different theories defined on the same fields, i.e, $E_1 = E_2$. This will allow us to search for emergence relations in which G is the identity map $G(\varphi) = \varphi$.

Hence, after these hypotheses, we can rewrite our main problem, whose affirmative solutions axiomatize the notion of strong emergence we are searching for:

Question 3. *Let $S_1[\varphi; \varepsilon]$ and $S_2[\varphi; \delta]$ be two parametrized theories defined on the same spacetime M , on the same field bundle E and on the parameter bundles P_1 and $P_2 = f(P_1)$, respectively. Does there exists some map $F : \text{Par}_1(P_1) \rightarrow \text{Par}_2(f(P_1))$ such that $S_1[\varphi; \varepsilon] = S_2[\varphi, F(\varepsilon)]$?*

Our plan is to show that the problem in Question 3 has an affirmative solution in some interesting cases. We begin by noticing that when working in a spacetime without boundary, after integration by parts and using Stoke’s theorem, many of the typical field theories can be stated, at least locally, in the form $\mathcal{L}(x, \varphi, \partial\varphi) = \langle \varphi, D\varphi \rangle$, where $\langle \varphi, \varphi' \rangle$ is a nondegenerate pairing on the space of fields

$\Gamma(E)$ and $D : \Gamma(E) \rightarrow \Gamma(E)$ is a differential operator of degree d , which means that it can be locally written as $D\varphi(x) = \sum_{|\alpha| \leq d} a_\alpha(x) \partial^\alpha \varphi$, where $\alpha = (\alpha_1, \dots, \alpha_r)$ is some mult-index, $|\alpha| = \alpha_1 + \dots + \alpha_r$ is its degree and $\partial^\alpha = \partial_1^{\alpha_1} \circ \dots \circ \partial_r^{\alpha_r}$, with $\partial_i^l = \partial^l / \partial^l x_i$. Let $\text{Diff}^d(E; E)$ denote the space of differential operators of degree l . This is the case, e.g., of φ^3 and φ^4 scalar field theories, the standard spinorial field theories as well as Yang-Mills theories. More generally, recall that the first step in building the Feynman rules of a field theory is to find the (kinematic par of the) operator D and take its “propagator” .

Typically, the pairing $\langle \varphi, \varphi' \rangle$ is symmetric (resp. skew-symmetric) and the operator D is formally self-adjoint (resp. formally anti-self-adjoint) relative to that pairing. Furthermore, $\langle \varphi, \varphi' \rangle$ is usually a L^2 -pairing induced by a semi-Riemannian metric g on the field bundle E and/or on the spacetime M , while D is usually a generalized Laplacian or a Dirac-type operator relative to g [15]. For example, this holds for the concrete field theories (scalar, spinorial and Yang-Mills) above. The skew-symmetric case generally arises in gauge theories (BV-BRST quantization) after introducing the Faddeev-Popov ghosts/anti-ghosts and it depends on the grading introduced by the ghost number [15].

Another remark, still having in mind the concrete situations above, is that if the metric g inducing the pairing $\langle \varphi, \varphi' \rangle$ is actually Riemannian (which means that the gravitational background is Euclidean), then $\langle \varphi, \varphi' \rangle$ becomes a genuine L^2 -inner product and D is elliptic and extends to a bounded self adjoint operator between Sobolev spaces [18, 22]. Working with elliptic operators is very useful, since they always admits parametrices (which in this Euclidean cases are the propagators) and for generalized Laplacians the heat kernel not only exists, but also has a well-known asymptotic behavior [41], which is very nice in the Dirac-type case [9].

From the discussion above, it is natural to focus on parameterized theories whose parameterized Lagrangian densities are of the form $\mathcal{L}(x, \varphi, \partial\varphi; \varepsilon) = \langle \varphi, D_\varepsilon \varphi \rangle$, i.e., are determined by a single nondegenerate pairing $\langle \varphi, \varphi' \rangle$ in $\Gamma(E)$, fixed a priori by the nature of M and E , and by a family of differential operators $D_\varepsilon \in \text{Diff}(E)$, one for each parameter $\varepsilon \in \text{Par}(P)$, where $\text{Diff}(E) = \bigoplus_d \text{Diff}^d(E; E)$ denotes the space of all differential operators in E . We will assume that the pairing extends to some space $\mathcal{H}(E)$, containing $\Gamma(E)$ as a dense subspace, such that the corresponding map $\ll \varphi, \varphi' \gg = \int_M \langle \varphi, \varphi' \rangle d^n x$ turns $\mathcal{H}(E)$ into a Hilbert space (typically a Sobolev space). We will further assume that each action functional $S_\varepsilon : \Gamma(E) \rightarrow \mathbb{R}$ is continuous and that each operator D_ε extends to a bounded operator $\hat{D}_\varepsilon : \mathcal{H}(E) \rightarrow \mathcal{H}(E)$. The denseness of $\Gamma(E)$ in $\mathcal{H}(E)$ then implies that each S_ε has a continuous extension $\hat{S}_\varepsilon \mathcal{H}(E) \rightarrow \mathbb{R}$, given by $\hat{S}[\varphi; \varepsilon] = \int_M \langle \varphi, \hat{D}_\varepsilon \varphi \rangle d^n x$. We will call these theories *differential parameterized theories* (DPT) *defined by the differential operators* D_ε .

Notice that in building the “propagator” of D we are actually finding some kind of “quasi-inverse” D^{-1} . For elliptic operators (where parametrices are propagators) the situation becomes more clear. Indeed, a parametrix is an inverse (up to compact operators) for the extended bounded linear map $\hat{D} : \mathcal{H}(E) \rightarrow \mathcal{H}(E)$. It would be very useful if the quasi-inverse D^{-1} could exist as a differential operator, i.e., if $D^{-1} = \hat{Q}$ for some differential operator Q . However, this is not the case. But D^{-1} may exist as a more generalized class of objects: *pseudo-differential operators* [23]. Since every differential operator is a pseudo-differential operator, we see that the process of building propagators (at least for elliptic operators in an Euclidean background) is more workable in the language of pseudo-differential operators.

Even so, as we will see, for many purposes it is better to work with theories defined in a suitable extension $\text{Nice}(E) \supset \text{Diff}(E)$ of the space of differential operators, consisting of certain

nice objects Ψ_ε (e.g, pseudo-differential operators), which can also be regarded as bounded operators $\tilde{\Psi}_\varepsilon : \mathcal{H}(E) \rightarrow \mathcal{H}(E)$. For example, some useful operations are not defined for arbitrary differential operators (like taking the inverse), but they are in some better-behaved context. Let us call theories defined in $\text{Nice}(E)$ as *nice parametrized theory (NPT)*. We will also need some additional structure on the parameter space $\text{Par}(P)$. Indeed, we will need to sum distinct parameters ($\varepsilon + \varepsilon'$, with $\varepsilon \neq \varepsilon'$) and multiply two arbitrary parameters ($\varepsilon * \varepsilon'$, with possibly $\varepsilon = \varepsilon'$). This fits into a structure that we call *nowhere vanishing algebra*. We also require the existence of square roots in $\text{Par}(P)$, meaning that for every ε there exists certain $\sqrt{\varepsilon}$ such that $\sqrt{\varepsilon} * \sqrt{\varepsilon} = \varepsilon$. If a NPT S_ε is such that $\text{Nice}(E)$ is a \mathbb{K} -algebra with the composition operation, $\text{Par}(P)$ is a nowhere vanishing algebra and the rule $\varepsilon \mapsto \Psi_\varepsilon$ preserves sum and multiplication, then we say that S_ε is *partially homomorphic on parameters*.

Finally, we will need to multiply operators $\text{Nice}(E)$ by families of parameters. This means we will need actions $\cdot^\ell : \text{Par}(P)^\ell \times \text{Nice}(E) \rightarrow \text{Nice}(E)$. These are bilinear maps which are compatible with the composition of nice operators, in the sense that

$$(\varepsilon(\ell) \cdot^\ell \Psi) \circ \Psi' = \varepsilon(\ell) \cdot^\ell (\Psi \circ \Psi'). \quad (1)$$

Here, given a non-negative integer $\ell \geq 0$, $\text{Par}(P)^\ell = \text{Par}(P) \times \dots \times \text{Par}(P)$ denotes the ℓ -power of the parameter set. For short, if $\ell > 0$ we will denote an element of the product by $\varepsilon(\ell)$, so that $\varepsilon(\ell) = (\varepsilon_1, \dots, \varepsilon_\ell)$, with $\varepsilon_i \in \text{Par}(P)$. We will also use the convention that $\text{Par}(P)^0$ is a singleton, whose element we denote by $\varepsilon(0)$. For a fixed Ψ we get a map

$$r_\Psi^\ell : \text{Par}(P)^\ell \rightarrow \text{Nice}(E) \quad \text{given by} \quad r_\Psi^\ell(\varepsilon(\ell)) = \varepsilon(\ell) \cdot^\ell \Psi, \quad (2)$$

i.e, the right-hand side multiplication by Ψ . We will demand that for every ℓ the map r_Ψ^ℓ is injective if we fix the identity operator $\Psi = I$. Thus, for every $\varepsilon(\ell), \delta(\ell)$ we have $\varepsilon(\ell) \cdot^\ell I = \delta(\ell) \cdot^\ell I$ iff $\varepsilon(\ell) = \delta(\ell)$, i.e, iff $\varepsilon_i = \delta_i$, with $i = 1, \dots, \ell$. A NPT S_ε whose operator algebra $\text{Nice}(E)$ is endowed with the actions \cdot^ℓ is called a *NPT with action by parameters of degree ℓ* .

The fundamental property which will be required on the actions \cdot^ℓ is that they must allow a nice functional calculus in $\text{Nice}(E)$. In order to be more precise, let $\text{Map}(\text{Par}(P)^\ell; \mathbb{K})$ denote the set of all functions $f : \text{Par}(P)^\ell \rightarrow \mathbb{K}$. Thus, if $\ell = 0$, functions $f : \text{Par}(P)^0 \rightarrow \mathbb{K}$ are identified with the number $f(\varepsilon(0))$, so that $\text{Map}(\text{Par}(P)^0; \mathbb{K}) \simeq \mathbb{K}$. Notice that the scalar multiplication $\cdot : \mathbb{K} \times \text{Nice}(E) \rightarrow \text{Nice}(E)$, which exists since $\text{Nice}(E)$ is a \mathbb{K} -algebra, induces maps

$$\cdot \Psi : \text{Map}(\text{Par}(P)^\ell; \mathbb{K}) \rightarrow \text{Map}(\text{Par}(P)^\ell; \text{Nice}(E)) \quad \text{given by} \quad [f \cdot \Psi](\varepsilon(\ell)) = f(\varepsilon(\ell))\Psi.$$

Thus, in a similar way, the actions \cdot^ℓ above induce

$$R_\Psi^\ell : \text{Map}(\text{Par}(P)^\ell; \text{Par}(P)^\ell) \rightarrow \text{Map}(\text{Par}(P)^\ell; \text{Nice}(E)) \quad \text{given by} \quad [R_\Psi^\ell F](\varepsilon(\ell)) = F(\varepsilon(\ell)) \cdot^\ell \Psi$$

A *functional calculus in $\text{Nice}(E)$ compatible with the action \cdot^ℓ* consists of a set $C_\ell(P; \mathbb{K})$ of functions $f : \text{Par}(P)^\ell \rightarrow \mathbb{K}$ for each $\ell \geq 0$, together with a map $\Psi_-^\ell : C_\ell(P; \mathbb{K}) \rightarrow \text{Nice}(E)$ assigning to each function f a corresponding operator Ψ_f^ℓ with the property that $\Psi_f^\ell \circ (f \cdot \Psi) = id \cdot^\ell \Psi$ for every nice operator Ψ^1 . Explicitly, this means that

$$\Psi_f^\ell \circ [f(\varepsilon(\ell))\Psi] = \varepsilon(\ell) \cdot^\ell \Psi. \quad (3)$$

¹This condition could be weakened by requiring $\Psi_f^\ell \circ (f \cdot \Psi) = id \cdot^\ell \Psi$ only for Ψ belonging to a subalgebra $\text{Nice}_0(E) \subset \text{Nice}(E)$. This would produce a slight generalization of some steps in the proof of our main theorem. However, for simplicity we will assume the existence of a functional calculus in the whole algebra $\text{Nice}(E)$.

Let $S_{\varepsilon(\ell)}$ be a NPT with action by parameters of degree ℓ . If its operator algebra is endowed with a functional calculus compatible with \cdot^ℓ , say by a set of functions $C_\ell(P; \mathbb{K})$, we will say that $S_{\varepsilon(\ell)}$ is a *NPT with functional calculus defined on $C(\text{Par}(P)^\ell; \mathbb{K})$* . The following example reveals some interest in nowhere vanishing functions.

Example 1.1. Every NPT $S_{\varepsilon(\ell)}$ with action \cdot^ℓ admits a unique structure of NPT with functional calculus defined on the set $\text{Map}_{\neq 0}(\text{Par}(P)^\ell; \mathbb{K})$ of nowhere vanishing functions. In order to refer to this canonical structure we will say simply that $S_{\varepsilon(\ell)}$ is a *NPT with nowhere vanishing functional calculus*. Assume that it exists. Then $\Psi_f^\ell \circ [f(\varepsilon(\ell))\Psi] = \varepsilon(\ell) \cdot^\ell \Psi$ for every f, Ψ and $\varepsilon(\ell)$, so that $\Psi_f^\ell \circ \Psi = (\varepsilon(\ell) \cdot^\ell \Psi)/f(\varepsilon(\ell))$. In particular, for $\Psi = I$, we get $\Psi_f^\ell = (\varepsilon(\ell) \cdot^\ell I)/f(\varepsilon(\ell))$. In order to prove uniqueness, define $\Psi_f^\ell = (\varepsilon(\ell) \cdot^\ell I)/f(\varepsilon(\ell))$, so that

$$\Psi_f^\ell \circ [f(\varepsilon(\ell))\Psi] = (\varepsilon(\ell) \cdot^\ell I) \circ \Psi = \varepsilon(\ell) \cdot^\ell (I \circ \Psi) = \varepsilon(\ell) \cdot^\ell \Psi,$$

where in the last step we used the compatibility between \cdot^ℓ and \circ , as described in (1).

Given integers $l, \ell \geq 0$ and $r > 0$, let $p_\ell^l[x_1, \dots, x_r] = \sum_{|\alpha| \leq l} f_\alpha \cdot x^\alpha$ be some multivariable polynomial of degree l whose coefficients are functions $f_\alpha : \text{Par}(P)^\ell \rightarrow \mathbb{K}$. Thus, e.g, the polynomials $p_0^l[x_1, \dots, x_r]$ are precisely the classical polynomials with coefficients in \mathbb{K} . If $\Psi_1, \dots, \Psi_r \in \text{Nice}(E)$ are fixed operators and $p_\ell^l[x_1, \dots, x_r]$ is a polynomial as above, then by means of replacing the formal variables x_i with the operators Ψ_i we get another operator $p_\ell^l[\Psi_1, \dots, \Psi_r] = \sum_{|\alpha| \leq l} f_\alpha \cdot \Psi^\alpha$, now depending on ℓ parameters. Indeed, for every $\varepsilon(\ell) \in \text{Par}(P)$ we get $p_\ell^l[\Psi_1, \dots, \Psi_r]_{\varepsilon(\ell)} \in \text{Nice}(E)$ such that $p_\ell^l[\Psi_1, \dots, \Psi_r]_{\varepsilon(\ell)} = \sum_{|\alpha| \leq l} f_\alpha(\varepsilon(\ell))\Psi^\alpha$. Let us say that a NPT is a *polynomial theory of degree (l, ℓ) in r variables* if its parameterized operator is of the form $p_\ell^l[\Psi_1, \dots, \Psi_r]$ for certain polynomial $p_\ell^l[x_1, \dots, x_r]$ and certain operators Ψ_1, \dots, Ψ_r .

We can now state the main result of this paper. It says that typical parameterized theories S_ε emerges from any suitably multivariate polynomial theory.

Main Theorem. *Let $S_{1, \varepsilon(\ell)}$ be a NPT which is partially homomorphic on parameters, whose parameter algebra $\text{Par}(P)^\ell$ has square roots. Then $S_{1, \varepsilon(\ell)}$ emerges from any NPT $S_{2, \delta(\ell')}$, with functional calculus $C_{\ell'}(P; \mathbb{K})$, which is a polynomial theory of degree (l, ℓ') in r variables, whose defining polynomial $p_{\ell'}^l[\Psi_1, \dots, \Psi_r]$ has coefficients given by functions f_α belonging to $C_{\ell'}(P; \mathbb{K})$ and whose operators Ψ_1, \dots, Ψ_r are right-invertible.*

The proof will be done in several steps.

1. We first prove the case $p_{\ell'}^1[\Psi]_{\delta(\ell')} = g(\delta(\ell'))\Psi$.
2. Then we obtain an additivity result for the emergence problem.
3. Next we obtain a multiplicativity result for the emergence problem.
4. We use the previous steps and some additional hypotheses on the functional calculus in order to prove the case $p_{\ell'}^l[\Psi]_{\delta_1, \dots, \delta_l}$ of an arbitrary univariate polynomial.
5. Then we prove a recurrence result for the emergence problem.
6. Finally use all the steps above to prove the general multivariate case $p_{\ell'}^l[\Psi_1, \dots, \Psi_r]$.

The paper is organized as follows. In Section 2 the previous discussion is reviewed, now in more precise and rigorous terms. In Sections 3-6 we prove the first five steps described above, while in Section 7 the main theorem is restated in a more concise form and then proved. Some concrete examples of our methods are given in Section 8, where we also try to emphasize the true range of our results.

2 Definitions, Notations and Remarks

Let us begin by recalling (and presenting in more details) some definitions briefly presented at the introduction. A *classical background for doing emergence theory*, denoted by \mathcal{CB} , is given by the following data:

- a compact oriented² smooth manifold M ;
- a real/complex vector bundle $E \rightarrow M$ (the field bundle);
- a vector space $\mathcal{H}(E)$ containing $\Gamma(E)$ as a dense subspace;
- a pairing $\langle \varphi, \varphi' \rangle$ in $\mathcal{H}(E)$ such that $\ll \varphi, \varphi' \gg = \int \langle \varphi, \varphi' \rangle d^n x$ turns it into a Hilbert space;
- a subset $\text{Diff}_0(E) \subset \text{Diff}(E)$;
- an injective linear map $\hat{\cdot} : \text{Diff}(E) \rightarrow B(\mathcal{H}(E))$;
- a parameter bundle $P \rightarrow M$ and a set of parameters $\text{Par}(P) \subset \Gamma(P)$.

A *differential parameterized theory (DPT)* in that classical background \mathcal{CB} is given by

- an integer $\ell \geq 0$, called the *parameter degree*;
- for each parameter $\varepsilon(\ell) \in \text{Par}(P)^\ell$ a differential operator $D_{\varepsilon(\ell)} \in \text{Diff}_0(E)$.

The *action functional* and the *extended action functional* are then defined by $S_{\varepsilon(\ell)}[\varphi; \varepsilon(\ell)] = \int \langle \varphi, D_{\varepsilon(\ell)} \varphi \rangle d^n x$ and $\hat{S}_{\varepsilon(\ell)}[\varphi] = \int \langle \varphi, \hat{D}_{\varepsilon(\ell)} \varphi \rangle d^n x$. The linearity of the map $\hat{\cdot} : \text{Diff}(E) \rightarrow B(\mathcal{H}(E))$ means that $\widehat{D_1 + D_2} = \hat{D}_1 + \hat{D}_2$ and that $\widehat{c\hat{D}} = c\hat{D}$, where $c \in \mathbb{C}$ is a scalar. In the following we need to embed the space of differential operators into a more interesting space than $B(\mathcal{H}(E))$: the space $RB(\mathcal{H}(E))$ of bounded operators T which have a bounded right-inverse, i.e, which admit a retraction R such that $T \circ R = I$. The problem is that given $\text{Diff}_0(E) \subset \text{Diff}(E)$ in general there is no injective linear map $\hat{\cdot} : \text{Diff}(E) \rightarrow B(\mathcal{H}(E))$ factoring through $RB(\mathcal{H}(E))$, as pictured below.

$$\begin{array}{ccc} \text{Diff}(E) & \xrightarrow{\hat{\cdot}} & B(\mathcal{H}(E)) \\ \uparrow & & \uparrow \\ \text{Diff}_0(E) & \dashrightarrow & RB(\mathcal{H}(E)) \end{array}$$

The solution is to modify the strategy: instead of considering parameterized theories defined by linear maps $D_\varepsilon : \Gamma(E) \rightarrow \Gamma(E)$ which belongs to a *rigid space* $\text{Diff}_0(E) \subset \text{Diff}(E)$ of *well-behaved*

²Most of the results hold without compactness and orientability hypotheses. Instead, we need only assume integrability conditions on global sections and consider Lagrangians as taking values on general densities.

objects, the idea is to consider theories defined by maps in a *well-behaved space* $\text{Nice}(E) \supset \text{Diff}(E)$ of *nice objects* Ψ_ε . This leads us to define a *nice background for doing emergence theory*, denoted by \mathcal{NC} , as being given by:

- a vector bundle $E \rightarrow M$ (the field bundle);
- a vector space $\mathcal{H}(E)$ containing $\Gamma(E)$ as a dense subspace;
- a pairing $\langle \varphi, \varphi' \rangle$ in $\mathcal{H}(E)$ such that $\ll \varphi, \varphi' \gg = \int \langle \varphi, \varphi' \rangle d^n x$ turns it into a Hilbert space;
- an algebra $\text{Nice}(E) \subset \text{End}(\Gamma(E))$;
- an injective algebra homomorphism $\tilde{\cdot} : \text{Nice}(E) \rightarrow B(\mathcal{H}(E))$;
- a parameter bundle $P \rightarrow M$ and a set of parameters $\text{Par}(P) \subset \Gamma(P)$.

A *nice parameterized theory* in a nice background \mathcal{NB} consists of

- an integer $\ell \geq 0$, called the *parameter degree*;
- for each parameter $\varepsilon(\ell) \in \text{Par}(P)^\ell$ an element $\Psi_{\varepsilon(\ell)} \in \text{Nice}(E)$.

The *action functional* and the *extended action functional* for a NPT are defined analogously by $S_{\varepsilon(\ell)}[\varphi; \varepsilon(\ell)] = \int \langle \varphi, \Psi_{\varepsilon(\ell)} \varphi \rangle d^n x$ and $\tilde{S}_{\varepsilon(\ell)}[\varphi] = \int \langle \varphi, \tilde{\Psi}_{\varepsilon(\ell)} \varphi \rangle d^n x$. The *right-invertible operators* in a nice background are those Ψ such that $\tilde{\Psi} \in RB(\mathcal{H}(E))$, i.e., such that there exists $R_{\tilde{\Psi}}$ with $\tilde{\Psi} \circ R_{\tilde{\Psi}} = I$. In this case, there also exists $R_\Psi \in \text{Nice}(E)$ such that $\widetilde{R_\Psi} = R_{\tilde{\Psi}}$.

Example 2.1. Every classical background in which $\text{Diff}_0(E)$ is an algebra can be regarded as a nice background by taking $\text{Nice}(E) = \text{Diff}_0(E)$. However, they generally do not have right-inverses.

Example 2.2. The main setup in building nice backgrounds which admit right-invertible operators is to take M a compact Riemannian manifold, $E = M \times \mathbb{C}$ a scalar field bundle, $\mathcal{H}(E) = \bigoplus_k H^k(M)$ a sum of Sobolev spaces, $\text{Nice}(E)$ the algebra of scalar pseudo-differential operators (as suggested by the notation Ψ) and the map $\tilde{\cdot}$ as the canonical extensions of a pseudo-differential operator as bounded operators between Sobolev spaces. We then look for ellipticity conditions to ensure right-inverses [25, 12, 38, 39, 2, 26, 3].

Remark 2.1. Throughout this paper we will consider NPT defined in a fixed nice background. Thus, instead of saying “let $S_{1,\varepsilon(\ell)}$ and $S_{2,\delta(\ell')}$ be two NPT, with parameter degrees ℓ and ℓ' , defined in a nice background \mathcal{NB} ” we will say “let $S_{1,\varepsilon(\ell)}$ and $S_{2,\delta(\ell')}$ be two NPT with parameter degrees ℓ and ℓ' ”, leaving the fixed nice background implicit. Actually, many results will be independent of the parameter degree of the emerging theory. In order to emphasize this fact we will omit the parameter degree when its value does not matter. Thus, in such situations we will simply say “let $S_{1,\varepsilon}$ and $S_{2,\delta(\ell')}$ be two NPT”, being implicit that the parameter degree of the emerging theory is arbitrary, while the parameter degree of the ambient theory is constrained.

Remark 2.2. The constraining on the parameter degree of the ambient theory $S_{2,\delta(\ell')}$ remarked above is basically due to the fact that in order to prove that $S_{1,\varepsilon}$ emerges from $S_{2,\delta(\ell')}$ we will need a functional calculus for sets of functions $C_\ell(P; \mathbb{K}) \subset \text{Map}(\text{Par}(P)^\ell; \mathbb{K})$, where the latter ℓ is the parameter degree of $S_{2,\delta(\ell')}$. This is a constrain on the nice background in which both NPT $S_{1,\varepsilon}$

and $S_{2,\delta(\ell)}$ are defined. More precisely, given $\ell \geq 0$, we say that a nice background \mathcal{NB} is *actioned by degree ℓ parameters* if it is endowed with an action $\cdot^\ell : \text{Par}(P)^\ell \times \text{Nice}(E) \rightarrow \text{Nice}(E)$ such that (1) is satisfied and which becomes injective in the first variable when we fix the identity the second one, i.e., $\varepsilon(\ell) \cdot^\ell I = \varepsilon'(\ell) \cdot^\ell I$ implies $\varepsilon(\ell) = \varepsilon'(\ell)$. We write \mathcal{NB}_ℓ to denote a nice background which is actioned by degree ℓ parameters. We say that $C_\ell(P; \mathbb{K})$ is the *domain of a functional calculus* in \mathcal{NB}_ℓ if it becomes endowed with a map $\Psi_-^\ell : C_\ell(P; \mathbb{K}) \rightarrow \text{Nice}(E)$, assigning to each function f a corresponding operator Ψ_f^ℓ , such that (3) is satisfied.

Remark 2.3. Consider the function $r_I^\ell : \text{Par}(P)^\ell \rightarrow \text{Nice}(E)$ given by (2), i.e., $r_I^\ell(\varepsilon(\ell)) = \varepsilon(\ell) \cdot^\ell I$. The injectivity condition above means precisely that r_I^ℓ is injective, so that it can actually be regarded as an isomorphism $\text{Par}(P)^\ell \simeq \text{Par}(P)^\ell \cdot^\ell I$ between its domain and its image.

Remark 2.4. If a nice background \mathcal{NB} is actioned by degree ℓ parameters, then it is also actioned by degree ℓ' parameters, for every $0 \leq \ell' \leq \ell$. In other words, if it is of type \mathcal{NB}_ℓ , then it also is of type $\mathcal{NB}_{\ell'}$, with $0 \leq \ell' \leq \ell$. This is done by induction and noticing $\text{Par}(P)^{\ell'}$ can be embedded in $\text{Par}(P)^\ell$ by making constants the first $\ell - \ell'$ variables. Furthermore, using the same arguments, every functional calculus in \mathcal{NB}_ℓ with domain $C_\ell(P; \mathbb{K})$ induces a functional calculus in $\mathcal{NB}_{\ell'}$ with domain $C_{\ell'}^{\ell'}(P; \mathbb{K})$ given by the restriction to $\text{Par}(P)^{\ell'}$ of the functions $f : \text{Par}(P)^\ell \rightarrow \mathbb{K}$ in $C_\ell(P; \mathbb{K})$.

The following lemma will be the starting point for each step in the proof of our main theorem. It says that in order to prove emergence between two NPT it is enough to analyze emergence between the corresponding extended operators.

Lemma 2.1. *Let $S_{1,\varepsilon}$ and $S_{2,\delta}$ be two NPT of arbitrary parameter degrees and defined by operators $\Psi_{1,\varepsilon}$ and $\Psi_{2,\delta}$, respectively. If there exists a function F such that $\tilde{\Psi}_{1,\varepsilon} = \tilde{\Psi}_{2,F(\varepsilon)}$ for every ε , then $S_{1,\varepsilon}$ emerges from $S_{2,\delta}$*

Proof. Indeed, since $\tilde{\cdot}$ is injective, this condition implies $\Psi_{1,\varepsilon} = \Psi_{2,F(\varepsilon)}$. But the theories are defined on the same nice background, so that $\ll \varphi, \Psi_{1,\varepsilon} \varphi \gg = \ll \varphi, \Psi_{2,F(\varepsilon)} \varphi \gg$ for every $\varphi \in \Gamma(E)$, which means precisely that $S_1[\varphi; \varepsilon] = S_2[\varphi; F(\varepsilon)]$ for every φ , i.e., that $S_{1,\varepsilon} = S_{2,F(\varepsilon)}$. \square

Remark 2.5. Due to the last lemma, if $\Psi_{2,\delta}$ is a parameterized operator, in some cases we will say “ $S_{1,\varepsilon}$ emerges from $\Psi_{2,\delta}$ ”, meaning that the parameterized operator $\Psi_{1,\varepsilon}$ emerges from $\Psi_{2,\delta}$ and, consequently, that $S_{1,\varepsilon}$ emerges from the NPT defined by $\Psi_{2,\delta}$.

We close this section by showing that under coerciveness or self-adjointness hypothesis on the parameterized operators, the reciprocal of Lemma 2.1 is true. This follows from the following general fact. Let \mathcal{H} be a \mathbb{K} -Hilbert space, $T : \mathcal{H} \rightarrow \mathcal{H}$ a bounded linear operator and consider the bilinear map $B_T : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{K}$ given by $B_T(v, w) = \langle v, T(w) \rangle$, which is bounded, since T is. For every T we have a corresponding quadratic form $q_T : \mathcal{H} \rightarrow \mathbb{K}$ such that $q_T(v) = \langle v, T(v) \rangle$. Recall that T is *coercive* if the induced quadratic form $q_T(v) = B_T(v, v)$ is coercive in the classical sense, i.e., if there exists $K > 0$ such that $K\|v\|^2 \leq |q_T(v)|$ for every $v \in \mathcal{H}$.

Lemma 2.2. *Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator which is self-adjoint or coercive. Then $q_T \equiv 0$ for every $v \in \mathcal{H}$ iff $T \equiv 0$.*

Proof. Assume T coercive, so that exists $K > 0$ such that $K\|v\|^2 \leq |B_T(v, v)|$. In this case, we have $B_T(v, v) = 0$ for every $v \in \mathcal{H}$ iff $T = 0$. The “if” part is obvious. For the “only if” part, assume $q_T(v) = B_T(v, v) = 0$ for every v and that $T \neq 0$. From Lax-Milgran theorem, for each $f \in \mathcal{H}^*$ there exists a unique $u \in \mathcal{H}$ such that $B_T(u, T(v)) = f(v)$ for every v . In particular, taking

$f = 0$ we see that there exists a unique u such that $B_T(u, v) = 0$ for every u . Since $T \neq 0$, this is a contradiction to the hypothesis $q_T \equiv 0$. For the self-ajoint case, from the spectral theorem it follows that T is unitarily equivalent to a multiplication operator T_λ . But $q_{T_\lambda}(v) = \lambda \|v\|^2$, so that if $q_{T_\lambda} \equiv 0$, then $\lambda = 0$, i.e, $T \equiv 0$. \square

3 First Step

We are now ready to prove the first step, which is also starting point lemma, in the sense that it will be used as the basis of many inductions.

Lemma 3.1. *Let $S_{1,\varepsilon}$ be a NPT in $\mathcal{NB}_{\ell'}$, defined by the parameterized operators $\Psi_{1,\varepsilon}$. Let $C_{\ell'}(P; \mathbb{K})$ be a domain of a functional calculus in $\mathcal{NB}_{\ell'}$. Then $S_{1,\varepsilon}$ emerges from every $\Psi_{2,\delta(\ell')}^l = g(\delta(\ell'))\Psi^l$, with $l \geq 0$, where $\Psi \in \text{Nice}(E)$ is right-invertible, $g \in C_{\ell'}(P; \mathbb{K})$ and $\Psi^l = \Psi \circ \dots \circ \Psi$, with $\Psi^0 = I$.*

Proof. By Remark 2.1 it suffices to analyze emergence of the extended operators. Since I is right-invertible, notice that the case $l = 0$ is a particular setup of case $l = 1$. Furthermore, if $l > 1$ and Ψ is right-invertible, then $\Xi = \Psi^l$ is right-invertible too, so that the case $l > 1$ also follows from the $l = 1$ case. Thus, we will work with $l = 1$. Suppose $\tilde{\Psi}_{2,\delta(\ell')} = \tilde{\Psi}_{1,\varepsilon}$, i.e $g(\delta(\ell'))I \circ \tilde{\Psi} = \tilde{\Psi}_{1,\varepsilon}$. Applying the right-inverse of $\tilde{\Psi}$ in both sides and using that $\tilde{\cdot}$ is an algebra homomorphism, we find that $g(\delta(\ell'))I = \tilde{\Psi}_{1,\varepsilon} \circ \tilde{R}_\Psi = \Psi_{1,\varepsilon} \circ R_\Psi$. Since $\tilde{\cdot}$ is also injective, we get $g(\delta(\ell'))I = \Psi_{1,\varepsilon} \circ R_\Psi$. Because $g \in C(\text{Par}(P)^{\ell'}; \mathbb{K})$ and since $S_{2,\delta(\ell')}$ has functional calculus, from (3) it follows that there exists $\Psi_g^{\ell'}$ such that $\Psi_g^{\ell'} \circ (g(\delta(\ell'))I) = \delta(\ell') \cdot^{\ell'} I$, so that $\Psi_g^{\ell'} \circ (\Psi_{1,\varepsilon} \circ R_\Psi) = \delta(\ell') \cdot^{\ell'} I$. Thus, we have a function $F : \text{Par}(P)^\ell \rightarrow \text{Nice}(E)$, given by $F(\varepsilon) = \Psi_g^{\ell'} \circ (\Psi_{1,\varepsilon} \circ R_\Psi)$, which by the above is exactly $r_I^{\ell'}(\delta(\ell'))$. Thus, due Remark 2.3 F can be regarded as a map $F : \text{Par}(P)^\ell \rightarrow \text{Par}(P)^{\ell'}$ which by construction is such that $\tilde{\Psi}_{1,\varepsilon} = \tilde{\Psi}_{2,F(\varepsilon)}$, as desired. \square

4 Second and Third Steps

The next two steps are additivity and multiplicativity results for emergence phenomena. In order to prove them, we have to add hypothesis on the set of parameters instead of on the shape of the operators. A *nowhere vanishing space* is a subset $W \subset V$ of a \mathbb{K} -vector space V such that $v + v' \in W$ for every $v, v' \in W$ with $v' \neq -v$. In particular, $0 \notin W$. We also require that $c \cdot v \in W$ for every scalar $c \neq 0$ and for every nonzero vector $v \in W$. Let W be a nowhere vanishing space and let Z be a vector space or nowhere vanishing space. A function $T : W \rightarrow Z$ is *linear* if it preserves sum and scalar multiplication. Notice that T is a nowhere vanishing function. Bilinear maps are defined analogously. A *nowhere vanishing algebra* is a nowhere vanishing space $A \subset V$ endowed with a bilinear multiplication $*$: $W \times W \rightarrow W$. Let A be nowhere vanishing algebra and let B be another nowhere vanishing algebra or an algebra in the classical sense. An *homomorphism* between them is a linear map $T : A \rightarrow B$ preserving the multiplication.

Given $\ell \geq 0$, we say that a nice background \mathcal{NB} has a *degree ℓ nowhere vanishing space of parameters* (resp. *degree ℓ vector space of parameters*) if $\text{Par}(P)^\ell$ is a nowhere vanishing space (resp. vector space). Similarly, we say that \mathcal{NB} has a *degree ℓ nowhere vanishing algebra of parameters* (resp. *degree ℓ algebra of parameters*) if $\text{Par}(P)^\ell$ is a nowhere vanishing algebra (resp. algebra). We say that a NPT $S_{\varepsilon(\ell)}$ of parameter degree ℓ and parameterized operators $\Psi_{\varepsilon(\ell)}$ is

partially additive on parameters (resp. *additive on parameters*) if the underlying nice background \mathcal{NB} has degree ℓ nowhere vanishing space of parameters (resp. degree ℓ vector space of parameters) and if the rule $\varepsilon(\ell) \mapsto \Psi_{\varepsilon(\ell)}$ is linear relative to that structure, i.e., $\Psi_{\varepsilon(\ell)+\varepsilon(\ell')} = \Psi_{\varepsilon(\ell)} + \Psi_{\varepsilon(\ell')}$ and $\Psi_{c\varepsilon(\ell)} = c\Psi_{\varepsilon(\ell)}$. In an analogous way, we say that $S_{\varepsilon(\ell)}$ is *partially multiplicative on parameters* (resp. *multiplicative on parameters*) if \mathcal{NB} has a degree ℓ nowhere vanishing algebra of parameters (resp. a degree ℓ algebra of parameters) and the rule $\varepsilon(\ell) \mapsto \Psi_{\varepsilon(\ell)}$ is not necessarily linear, but preserves the multiplication, i.e., $\Psi_{\varepsilon(\ell)*\varepsilon'(\ell)} = \Psi_{\varepsilon(\ell)} \circ \Psi_{\varepsilon'(\ell)}$. Finally, we say that $S_{\varepsilon(\ell)}$ is *partially homomorphic on parameters* (resp. *homomorphic on parameters*) if it is both partially additive and partially multiplicative (resp. additive and multiplicative).

Example 4.1. It is straightforward to check that if \mathcal{NB} has a degree ℓ nowhere vanishing space of parameters (resp. degree ℓ vector space of parameters), then it also has for every $\ell' = n\ell$, where $n \geq 1$ is another integer. The structure is obtained by noticing that

$$\text{Par}(P)^{n\ell} = \text{Par}(P)^\ell \times \dots \times \text{Par}(P)^\ell$$

and then defining sum and multiplication componentwise. A similar argument applies to the case of degree ℓ nowhere vanishing algebra of parameters (resp. degree ℓ algebra of parameters).

Example 4.2. In general P is a vector bundle or an algebra bundle, so that $\Gamma(P)$ is a vector space or an algebra and $\text{Par}(P) \subset \Gamma(P)$ is some nowhere vanishing subspace, vector subspace, nowhere vanishing algebra or subalgebra. By the last example it then follows that each of these structures can be lifted to $\text{Par}(P)^n$ for any $n \geq 1$.

Given two NPT theories $S_{2,\delta(\ell')}$ and $S_{3,\kappa(\ell')}$, define their *sum* as the theory $S_{2,\delta(\ell')} + S_{3,\kappa(\ell')}$ whose parameter and whose parameterized operator is $\Psi_{\delta(\ell'),\kappa(\ell')} = \Psi_{2,\delta(\ell')} + \Psi_{3,\kappa(\ell')}$. In an analogous way we define the *composition theory* $S_{2,\delta(\ell')} \circ S_{3,\kappa(\ell')}$. We say that $S_{1,\varepsilon(\ell)}$ *emerges* from $S_{2,\delta(\ell')} + S_{3,\kappa(\ell')}$ (resp. $S_{2,\delta(\ell')} \circ S_{3,\kappa(\ell')}$) if there exists $F_+ : \text{Par}(P)^\ell \rightarrow \text{Par}(P)^{\ell'} \times \text{Par}(P)^{\ell''}$ (resp. F_\circ) such that

$$S_1[\varphi; \varepsilon(\ell)] = (S_2 + S_3)[\varphi; F_+(\varepsilon(\ell))] \quad (\text{resp. } S_1[\varphi; \varepsilon(\ell)] = (S_2 \circ S_3)[\varphi; F_\circ(\varepsilon(\ell))]),$$

exactly as in the previous situations. The next lemmas are independent of the parameter degrees of the theories $S_{2,\delta(\ell')}$ and $S_{3,\delta(\ell'')}$. Thus, following Remark 2.1, they will be omitted.

Lemma 4.1. *Let $S_{1,\varepsilon(\ell)}$, $S_{2,\delta}$ and $S_{3,\kappa}$ be three NPT, defined by $\Psi_{1,\varepsilon(\ell)}$, $\Psi_{2,\delta}$ and $\Psi_{3,\kappa}$. Assume that:*

1. *the theory $S_{1,\varepsilon(\ell)}$ is additive or partially additive on parameters;*
2. *the theory $S_{1,\varepsilon(\ell)}$ emerges from both $S_{2,\delta}$ and $S_{3,\kappa}$, while theory $S_{2,\delta}$ emerges from $S_{3,\kappa}$;*

Then $S_{1,\varepsilon(\ell)}$ emerges from the sum $S_{2,\delta} + S_{3,\kappa}$. If, in addition, $S_{3,\kappa}$ is also additive or partially additive on parameters, then the emergence of $S_{1,\varepsilon(\ell)}$ from $S_{2,\delta} + S_{3,\kappa}$ is equivalent to a specific new emergence of $S_{1,\varepsilon(\ell)}$ from $S_{3,\kappa}$.

Proof. Assume first that $S_{1,\varepsilon(\ell)}$ is additive. From the second hypothesis we conclude that $\Psi_{1,\varepsilon(\ell)} = \Psi_{2,F(\varepsilon(\ell))}$, $\Psi_{1,\varepsilon(\ell)} = \Psi_{3,G(\varepsilon)}$ and $\Psi_{2,\delta} = \Psi_{3,H(\delta)}$ for certain functions F, G, H . Summing the first two of these conditions and using the third one with $\delta = F(\varepsilon(\ell))$ we find

$$2\Psi_{1,\varepsilon(\ell)} = \Psi_{3,H(F(\varepsilon(\ell)))} + \Psi_{3,G(\varepsilon(\ell))}. \quad (4)$$

On the other hand, since $S_{1,\varepsilon(\ell)}$ is additive, we get

$$\Psi_{1,\varepsilon(\ell)} = \Psi_{3,H(F(\varepsilon(\ell)/2))} + \Psi_{3,G(\varepsilon(\ell)/2)}, \quad (5)$$

so that $S_1[\varphi; \varepsilon(\ell)] = (S_2 + S_3)[\varphi; K(\varepsilon(\ell))]$ with $K(\varepsilon(\ell)) = (G(\varepsilon(\ell)/2), H(F(\varepsilon(\ell)/2)))$, finishing the first part of the proof in the additive case. For the second part, assume that $S_{3,\kappa}$ is additive on parameters. In this case, the right-hand side of the expression above is equivalent to $\Psi_{3,H(F(\varepsilon(\ell)/2)+G(\varepsilon(\ell)/2))}$, meaning that $S_1[\varphi; \varepsilon(\ell)] = S_3[\varphi; L(\varepsilon(\ell))]$ with $L(\varepsilon(\ell)) = H(F(\varepsilon(\ell)/2)) + G(\varepsilon(\ell)/2)$. Now, observe that nowhere in the proof have we used that the parameter space contains the null vector or opposite vectors. This means that the same arguments work equally well in the partially additive setting. \square

Remark 4.1. One can similarly show that if $S_{1,\varepsilon(\ell)}$ is additive or partially additive which emerges from $S_{2,\delta}$, then it emerges from $c \cdot S_{2,\delta}$ for every $c \neq 0$. Indeed, if $\Psi_{1,\varepsilon(\ell)} = \Psi_{2,F(\varepsilon(\ell))}$ and if $\Psi_{2,F(\varepsilon(\ell))} = \frac{c}{c} \Psi_{2,F(\varepsilon(\ell))} = c \Psi_{2,F(\varepsilon(\ell))/c}$, then $\Psi_{1,\varepsilon} = c \Psi_{2,F(\varepsilon(\ell))/c}$. Reciprocally, if $S_{1,\varepsilon(\ell)}$ emerges from $c \cdot S_{2,\delta}$, with $c \neq 0$, then it also emerges from $S_{2,\delta}$.

Given $\ell \geq 0$, we say that a nice background \mathcal{NB} has *degree ℓ square roots* if it has a degree ℓ nowhere vanishing algebra of parameters or degree ℓ algebra of parameters and if for every $\varepsilon(\ell)$ there exists some $\varepsilon(\ell)^{1/2}$ such that $(\varepsilon(\ell)^{1/2})^2 \equiv \varepsilon(\ell)^{1/2} * \varepsilon(\ell)^{1/2} = \varepsilon(\ell)$. We write \mathcal{NB}^ℓ to denote this fact.

Example 4.3. From Remark 2.4, if \mathcal{NB} has a degree ℓ nowhere vanishing algebra of parameters or degree ℓ algebra of parameters, then it also has for every $\ell' = n\ell$, with $n \geq 1$. Relatively to this componentwise structure, one can show that if \mathcal{NB} has degree ℓ square roots, then it also has for every $\ell' = n\ell$, i.e, if \mathcal{NB} is of type \mathcal{NB}^ℓ , then it also is of type $\mathcal{NB}^{n\ell}$.

We can now work on the third step.

Lemma 4.2. *Let $S_{1,\varepsilon(\ell)}$, $S_{2,\delta}$ and $S_{3,\kappa}$ be three NPT with parameterized operators $\Psi_{1,\varepsilon(\ell)}$, $\Psi_{2,\delta}$ and $\Psi_{3,\kappa}$, as above. Assume that:*

1. *the theory $S_{1,\varepsilon(\ell)}$ is multiplicative or partially multiplicative on parameters;*
2. *the theory $S_{1,\varepsilon(\ell)}$ emerges from both $S_{2,\delta}$ and $S_{3,\kappa}$, while theory $S_{2,\delta}$ emerges from $S_{3,\kappa}$;*
3. *the underlying nice background has degree ℓ square roots.*

Then $S_{1,\varepsilon(\ell)}$ emerges from the composition $S_{2,\delta} \circ S_{3,\kappa}$. If in addition, $S_{3,\kappa}$ is also multiplicative or partially multiplicative on parameters, then the emergence of $S_{1,\varepsilon(\ell)}$ from $S_{2,\delta} \circ S_{3,\kappa}$ is equivalent to a specific new emergence of $S_{1,\varepsilon(\ell)}$ from $S_{3,\kappa}$.

Proof. We will work only in the multiplicative case. The partially multiplicative one will follow from the same argument used in Lemma 4.1. The first part of the proof also follows the same lines of Lemma 4.1, the only difference being that equation (4) is now replaced with

$$\Psi_{1,\varepsilon(\ell)} \circ \Psi_{1,\varepsilon(\ell)} = \Psi_{3,H(F(\varepsilon(\ell)))} \circ \Psi_{3,G(\varepsilon(\ell))} = \Psi_{3,G(\varepsilon(\ell))} \circ \Psi_{3,H(F(\varepsilon(\ell)))}.$$

Since $S_{1,\varepsilon(\ell)}$ is multiplicative and since the background \mathcal{NB}^ℓ has degree ℓ square roots we can write an analogue for (5):

$$\Psi_{1,\varepsilon(\ell)} = \Psi_{3,H(F(\varepsilon(\ell)^{1/2}))} \circ \Psi_{3,G(\varepsilon(\ell)^{1/2})} = \Psi_{3,G(\varepsilon(\ell)^{1/2})} \circ \Psi_{3,H(F(\varepsilon(\ell)^{1/2}))},$$

so that $S_1[\varphi, \varepsilon(\ell)] = (S_2 \circ S_3)[\varphi; K(\varepsilon(\ell))]$ with $K(\varepsilon(\ell)) = (G(\varepsilon(\ell)^{1/2}), H(F(\varepsilon(\ell)^{1/2})))$, finishing this first part. For the second part, assuming $S_{3,\kappa}$ multiplicative on parameters, just notice that the right-hand side of the above expression becomes

$$\Psi_{3,G(\varepsilon(\ell)^{1/2}) * H(F(\varepsilon(\ell)^{1/2}))} = \Psi_{3,H(F(\varepsilon(\ell)^{1/2})) * G(\varepsilon(\ell)^{1/2})},$$

finishing the proof. \square

Corollary 4.1. *Let $S_{\varepsilon(\ell)}$ be a multiplicative or partially multiplicative NPT defined in \mathcal{NB}^ℓ and with operator $\Psi_{\varepsilon(\ell)}$. Then, for every $l, m \geq 1$, the theories $S_{\varepsilon(\ell)}^l$ and $S_{\varepsilon(\ell)}^m$ emerges each one from the other, where $S_{\varepsilon(\ell)}^i = S_{\varepsilon(\ell)} \circ \dots \circ S_{\varepsilon(\ell)}$.*

Proof. Fixed $m = 1$, use induction on l . The base of induction is the fact that a theory always emerges from itself. For the induction step, use previous lemma. This implies that $S_{\varepsilon(\ell)}$ emerges from $S_{\varepsilon(\ell)}^l$ for every $l \geq 1$. Consequently, $S_{\varepsilon(\ell)}^{m-1} \circ S_{\varepsilon(\ell)}$ emerges from $S_{\varepsilon(\ell)}^{m-1} \circ S_{\varepsilon(\ell)}^l$ for every $m, l \geq 1$, where, by definition $S_{\varepsilon(\ell)}^0$ is the theory whose operator is the identity. Since $S_{\varepsilon(\ell)}^i \circ S_{\varepsilon(\ell)}^j = S_{\varepsilon(\ell)}^{i+j}$, we conclude that $S_{\varepsilon(\ell)}^m$ emerges from $S_{\varepsilon(\ell)}^{m-1+l} = S_{\varepsilon(\ell)}^{l(m)}$, with $m \leq l(m)$. On the other hand, since the identity is an invertible map, we see that $S_{\varepsilon(\ell)}^{l(m)}$ also emerges from $S_{\varepsilon(\ell)}^m$. \square

Recall that if a nice background \mathcal{NB} is actioned by degree ℓ' parameters, we write $\mathcal{NB}_{\ell'}$ to denote this fact. Furthermore, by the above, if \mathcal{NB} has degree ℓ square roots, we write \mathcal{NB}^ℓ . Thus, from now on, if \mathcal{NB} has both properties we will write $\mathcal{NB}_{\ell'}^\ell$.

5 Fourth Step

As a consequence of the previous lemmas we can prove the fourth step. We say that a functional calculus in a nice background $\mathcal{NB}_{\ell''}$ is *unital* if its domain $C_{\ell''}(P; \mathbb{K})$ contains the constant function $f \equiv 1$.

Lemma 5.1. *Let $S_{\varepsilon(\ell)}$ be a homomorphic or partially homomorphic on parameters NPT, defined in a nice background $\mathcal{NB}_{\ell''}^\ell$ and whose parameterized operator is $\Psi_{\varepsilon(\ell)}$. Let $C_{\ell''}(P; \mathbb{K})$ be the domain of a unital functional calculus in $\mathcal{NB}_{\ell''}^\ell$. Then $S_{\varepsilon(\ell)}$ emerges from every theory whose operator is of the form*

$$p_{\ell''}^l[\Psi] = \sum_{i=1}^l m_{\delta_i(\ell'')}^i[\Psi] = \sum_{i=1}^l f_i(\delta_i(\ell''))\Psi^i,$$

where $\ell'' = l\ell'$, $f_i \in C_{\ell''}^\ell(P; \mathbb{K})$ for $i = 1, \dots, l$ and such that Ψ is right-invertible³.

Proof. For each $j = 1, \dots, l$, let $\Gamma_j = \sum_{i=j}^l f_i(\delta_i(\ell''))\Psi^{i-1}$ and notice that

$$\sum_{i=1}^l \Psi_{i, \delta_i(\ell'')} = \left(\sum_{i=1}^l f_i(\delta_i(\ell''))\Psi^{i-1} \right) \circ \Psi = \Gamma_1(\delta_1(\ell'')) \circ \Psi.$$

³The condition on f_i makes sense due Remark 2.4.

Since Ψ is right-invertible and since the nice background $\mathcal{NB}_{\ell'}^{\ell}$ has unital functional calculus, from Lemma 3.1 it follows that $S_{\varepsilon(\ell)}$ emerges from $1 \cdot \Psi$. On the other hand, again from Lemma 3.1 we see that the theory defined by the operator Γ_1 also emerges from that defined by $1 \cdot \Psi$. Thus, if $S_{\varepsilon(\ell)}$ itself emerges from Γ_1 one can use Lemma 4.2 to conclude that it actually emerges from $\Gamma_1 \circ \Psi$. In turn, notice that $\Gamma_1 = f_1 \cdot I + \Gamma_2 \circ \Psi = \Gamma_2 \circ \Psi + f_1 \cdot I$. But, since I is right-invertible and since $f_1 \in C_{\ell'}^{\ell'}(P; \mathbb{K})$, from Remark 2.4 and from Lemma 3.1 we get that $S_{\varepsilon(\ell)}$ emerges from the theory defined by $f_1 \cdot I$, while by the same argument we see that $\Gamma_2 \circ \Psi$ emerges from $f_1 \cdot I$. Therefore, if $S_{\varepsilon(\ell)}$ emerges from $\Gamma_2 \circ \Psi$ we will be able to use Lemma 4.1 to conclude that it emerges from Γ_1 , finishing the proof. It happens that, as done for $\Gamma_1 \circ \Psi$, we see that Γ_2 emerges from Ψ and we already know that $S_{\varepsilon(\ell)}$ emerges from Ψ . Thus, our problem is to prove that $S_{\varepsilon(\ell)}$ emerges from Γ_2 instead of from Γ_1 . A finite induction argument proves that if $S_{\varepsilon(\ell)}$ emerges from Γ_l , then it emerges from Γ_j , for each $j = 1, \dots, l$. Recall that $\Gamma_l = f_l \cdot \Psi^{l-1}$. Since $f_l \in C_{\ell'}^{\ell'}(P; \mathbb{K})$ we can use Lemma 3.1 to see that $S_{\varepsilon(\ell)}$ really emerges from Γ_l . \square

Let $\text{Map}(\text{Par}(P)^{\ell'}; \mathbb{K})$ be the algebra of functions $f : \text{Par}(P)^{\ell'} \rightarrow \mathbb{K}$ and let $\text{Map}(\text{Par}(P)^{\ell'}; \mathbb{K})[x]$ be the corresponding polynomial algebra. Recall that any polynomial ring has a grading by the degree, so that we can write

$$\text{Map}(\text{Par}(P)^{\ell'}; \mathbb{K})[x] \simeq \bigoplus_{l \geq 0} \text{Map}_l(\text{Par}(P)^{\ell'}; \mathbb{K})[x],$$

where the right-hand side consists of the sum over the \mathbb{K} -vector spaces of polynomials with fixed degree l . By extension of scalars $\text{Nice}(E)$ can be regarded as an algebra over $\text{Map}(\text{Par}(P)^{\ell'}; \mathbb{K})$, denoted by $\text{Nice}_{\mathbb{K}}^{\ell'}(E)$, so that for every $l \geq 1$ we have an evaluation map

$$ev_{\ell', l} : \text{Map}_l(\text{Par}(P)^{\ell'}; \mathbb{K})[x] \times \text{Nice}(E) \rightarrow \text{Nice}_{\mathbb{K}}^{\ell'}(E) \quad (6)$$

which takes a polynomial $p_{\ell'}^l[x] = \sum_{i=0}^l f_i \cdot x^i$ and an operator Ψ and produces $ev_l(p_{\ell'}^l[x], \Psi) = p_{\ell'}^l[\Psi] = \sum_{i=0}^l f_i \cdot \Psi^i$, where $\Psi^i = \Psi \circ \dots \circ \Psi$. Let $\text{Poly}_{\mathbb{K}, l}^{\ell'}(E)$ denote the image of $ev_{\ell', l}$. If one fixes an operator Ψ such that $\Psi^0, \Psi^1, \Psi^2, \dots$ are linearly independent as objects of $\text{Nice}(E)$ the map $ev_{\ell', l}$ becomes injective. More precisely, we have the following result:

Proposition 5.1. *Let Ψ be an operator such that Ψ^i is linearly independent of Ψ^j for every $1 \leq i, j \leq l$, for some $l \geq 1$. Then the evaluation morphism at Ψ*

$$ev_{\ell', \leq l}^{\Psi} : \text{Map}_{\leq l}(\text{Par}(P)^{\ell'}; \mathbb{K})[x] \rightarrow \text{Nice}_{\mathbb{K}}^{\ell'}(E) \quad (7)$$

is injective, where the left-hand side is the subspace of polynomials of degree $d \leq l$.

Proof. The kernel of $ev_{\ell', \leq l}^{\Psi}$ consists of those $p_{\ell'}^d[x] = \sum_{i \leq d} f_i \cdot x^i$ such that $ev_{\ell', \leq l}^{\Psi}(p_{\ell'}^d[x]) = 0 = \sum_{i \leq d} f_i \cdot \Psi^i$. Due to the linear independence hypothesis, this is the case iff $f_i = 0$, i.e., $p_{\ell'}^d[x] = 0$. \square

Observe that we have another evaluation

$$ev_{\ell', l}^l : \text{Poly}_{\mathbb{K}, l}^{\ell'}(E) \times [\text{Par}(P)^{\ell'}]^l \rightarrow \text{Nice}(E)$$

such that $ev_{\ell',l}^l(p_{\ell'}^l[\Psi], \delta_I(\ell')) = p_{\ell'}^l[\Psi](\delta_I(\ell')) = \sum_{i=0}^l f_i(\delta_i(\ell'))\Psi^i$, where $\delta_I(\ell') = (\delta_1(\ell'), \dots, \delta_l(\ell'))$. Therefore, for every $l \geq 1$ we have a complete evaluation

$$ev_{\ell'}^l : \text{Map}_l(\text{Par}(P)^{\ell'}; \mathbb{K})[x] \times \text{Nice}(E) \times [\text{Par}(P)^{\ell'}]^l \rightarrow \text{Nice}(E)$$

given by $ev_{\ell'}^l(p_{\ell'}^l[x], \Psi, \delta_I(\ell')) = ev_{\ell',l}^l(ev_{\ell',l}(p_{\ell'}^l[x], \Psi), \delta_I(\ell'))$. Let $\underline{\text{Nice}}(E)$ be the subset of operators $\Psi \in \text{Nice}(E)$ whose extension $\tilde{\Psi} \in B(\mathcal{H}(E))$ is right-invertible, let $C_{\ell'}(P; \mathbb{K}) \subset \text{Map}(\text{Par}(P)^{\ell'}; \mathbb{K})$ be a subset and let $C_{\ell;l}(P; \mathbb{K})$ the set of polynomials $p_{\ell'}^l[x] = \sum f_i \cdot x^i$ of degree l whose coefficients belongs to $C_{\ell'}(P; \mathbb{K})$. Restricting (6), we get a map

$$\underline{ev}_{\ell';l}^C : C_{\ell;l}(P; \mathbb{K}) \times \underline{\text{Nice}}(E) \rightarrow \text{Nice}_{\mathbb{K}}^{\ell'}(E).$$

Definition 5.1. Let \mathcal{NB}^{ℓ} be a nice background with degree ℓ square roots. Let us say that the set of polynomials $C_{\ell;l}(P; \mathbb{K})$ is coherent in \mathcal{NB}^{ℓ} if

1. \mathcal{NB}^{ℓ} is actioned by degree $\ell'' = \ell'$ parameters (i.e, it is of the form $\mathcal{NB}_{\ell''}^{\ell}$);
2. there exists a set $C_{\ell''}(P; \mathbb{K}) \subset \text{Map}(\text{Par}(P)^{\ell''}; \mathbb{K})$ which is the domain of unital functional calculus in $\mathcal{NB}_{\ell''}^{\ell}$ and such that $C_{\ell;l}(P; \mathbb{K}) = C_{\ell''}^{\ell'}(P; \mathbb{K})$.

After this discussion we see that Lemma 5.1 (i.e, the first version of the fourth step) can be rewritten as following:

Lemma 5.2 (Lemma 5.1 revisited). Let $S_{\varepsilon(\ell)}$ be a homomorphic or partially homomorphic on parameters NPT, defined in nice background $\mathcal{NB}_{\ell''}^{\ell}$. Then it emerges from any theory in the image of $\underline{ev}_{\ell';l}^C$, where $\ell'' = \ell'$ and $C_{\ell;l}(P; \mathbb{K})$ is coherent in $\mathcal{NB}_{\ell''}^{\ell}$.

Remark 5.1. From Remark 2.4 it follows that if $C_{\ell;l}(P; \mathbb{K})$ is coherent in \mathcal{NB}^{ℓ} , then $C_{\ell;l'}(P; \mathbb{K})$ is also coherent for every $l' \leq l$.

6 Fifth Step

The final step before proving the main result is a recurrence lemma which is obtained as a consequence of Lemma 4.1-4.2. Recall that $\text{Nice}(E)$ is an algebra and, as in any algebra A we can ask if a given element $a \in A$ divides from the left (resp. from the right) another element $b \in A$, meaning that there exists $q \in A$, called the *quotient between b and a*, such that $b = a * q$ (resp $b = q * a$). In $\text{Nice}(E)$, given two operators Ψ_2 and Ψ_3 , this means that there exists a third $Q \in \text{Nice}(E)$ such that $\Psi_2 = \Psi_3 * Q$ (resp. $\Psi_2 = Q * \Psi_3$). Given two NPT $S_{2,\delta(\ell')}$ and $S_{3,\kappa(\ell'')}$, with respective parameterized operators $\Psi_{2,\delta(\ell')}$ and $\Psi_{3,\kappa(\ell'')}$, let us say that $S_{2,\delta(\ell')}$ is *divisible from the left (resp. from the right)* by $S_{3,\kappa(\ell'')}$ if for every $\delta(\ell')$ there exists $\kappa(\ell'')$ such that $\Psi_{3,\kappa(\ell'')}$ divides $\Psi_{2,\delta(\ell')}$ from the left (resp. from the right), so that $\Psi_{2,\delta(\ell')} = \Psi_{3,\kappa(\ell'')} \circ Q_{\delta(\ell'),\kappa(\ell'')}$ (resp. $\Psi_{2,\delta(\ell')} = Q_{\delta(\ell'),\kappa(\ell'')} \circ \Psi_{3,\kappa(\ell'')}$).

Lemma 6.1. Let $S_{1,\varepsilon(\ell)}$ be a homomorphic or partially homomorphic NPT defined in a nice background $\mathcal{NB}_{\ell''}^{\ell}$ and whose parameterized operator is $\Psi_{1,\varepsilon(\ell)}$. Let Given $l \geq 1$, let $S_{j,\delta_j(\ell')}$ and $S_{k,\kappa_k(\ell'')}$, with $1 \leq j, k \leq l$ be two families of NPT, also defined in $\mathcal{NB}_{\ell''}^{\ell}$. Let $C_{\ell''}(P; \mathbb{K})$ be the domain of a functional calculus in $\mathcal{NB}_{\ell''}^{\ell}$. Assume that:

1. $S_{1,\varepsilon(\ell)}$ emerges from $S_{2_j,\delta_j(\ell')}$ and from $S_{3_k,\kappa_k(\ell')}$ for every j, k ;
2. $S_{2_j,\delta_j(\ell')}$ emerges from $S_{3_k,\kappa_k(\ell')}$ is $k = j$;
3. for every $2 \leq k \leq l$ the theory $S_{3_k,\kappa_k(\ell')}$ is divisible from the right by a monomial $m_{\kappa_k(\ell')}^{d(k)} = g_k(\kappa_k(\ell'))\Psi^{d(k)}$, where Ψ is right-invertible and $g_k \in C_{\ell'}^{\ell'}(P; \mathbb{K})$ so that $\Psi_{3_k,\kappa_k(\ell')} = Q_k \circ m_{\kappa_k(\ell')}^{d(k)}$;
4. for every $1 \leq m \leq l - 1$ the theories $S_{\delta_J,\kappa_J}^m = \sum_{j=1}^m (S_{2_j,\delta_j} \circ S_{3_j,\kappa_j})$ and $S_{2_{m+1},\delta_{m+1}}$ emerges from Q_{m+1} ;
5. for every $1 \leq m \leq l - 1$ the theory S_{δ_J,κ_J}^m emerges from $S_{2_{m+1},\delta_{m+1}}$.

Then $S_{1,\varepsilon(\ell)}$ emerges from S_{δ_J,κ_J}^l .

Proof. We proceed by induction in l . First of all, notice that from the first two hypotheses and from Lemma 4.2 we see that $S_{1,\varepsilon(\ell)}$ emerges from the composition $S_{2_j,\delta_j} \circ S_{3_j,\kappa_j}$ for every $j = 1, \dots, l$. In particular, it emerges from $S_{\delta_1,\kappa_1}^1 = S_{2_1,\delta_1} \circ S_{3_1,\kappa_1}$, which is the base of induction. For every $m = 1, \dots, l - 1$ it also emerges from $S_{2_{m+1},\delta_{m+1}} \circ S_{3_{m+1},\kappa_{m+1}}$. For the induction step, suppose that $S_{1,\varepsilon(\ell)}$ emerges from $S_{\delta_J,\kappa_J}^m = \sum_{j=1}^m S_{2_j,\delta_j} \circ S_{3_j,\kappa_j}$ for every $1 \leq m \leq l - 1$ and let us show that it emerges from $S_{\delta_J,\kappa_J}^{m+1}$. Notice that

$$\begin{aligned} S_{\delta_J,\kappa_J}^{m+1} &= \sum_{j=1}^{m+1} S_{2_j,\delta_j} \circ S_{3_j,\kappa_j} = \sum_{j=1}^m (S_{2_j,\delta_j} \circ S_{3_j,\kappa_j}) + S_{2_{m+1},\delta_{m+1}} \circ S_{3_{m+1},\kappa_{m+1}} \\ &= S_{\delta_J,\kappa_J}^m + S_{2_{m+1},\delta_{m+1}} \circ S_{3_{m+1},\kappa_{m+1}}. \end{aligned}$$

From the induction hypothesis $S_{1,\varepsilon(\ell)}$ emerges from S_{δ_J,κ_J}^m , while by the above it also emerges from $S_{2_{m+1},\delta_{m+1}} \circ S_{3_{m+1},\kappa_{m+1}}$. Thus, if S_{δ_J,κ_J}^m emerges from $S_{2_{m+1},\delta_{m+1}} \circ S_{3_{m+1},\kappa_{m+1}}$ we can use Lemma 4.1 to conclude that $S_{1,\varepsilon(\ell)}$ emerges from $S_{\delta_J,\kappa_J}^{m+1}$. From Lemma 3.1 we know that Q_{m+1} , S_{δ_J,κ_J}^m and $S_{2_{m+1},\delta_{m+1}}$ emerge from $m_{\kappa_{m+1}(\ell')}^{d(m+1)}$. Due to the fourth hypothesis, by Lemma 4.2 we see that $S_{2_{m+1},\delta_{m+1}}$ and S_{δ_J,κ_J}^m emerge from the composition $S_{3_{m+1},\kappa_{m+1}(\ell')} = Q_{m+1} \circ m_{\kappa_{m+1}(\ell')}^{d(m+1)}$. Therefore, if we prove that S_{δ_J,κ_J}^m also emerges from $S_{2_{m+1},\delta_{m+1}}$, then Lemma 4.2 will imply that it emerges from $S_{2_{m+1},\delta_{m+1}} \circ S_{3_{m+1},\kappa_{m+1}}$, as desired. But this remaining condition is precisely the fifth hypothesis. \square

7 The Theorem

Let $r \geq 1$ be a positive integer and let

$$\text{Map}(\text{Par}(P)^{\ell'}; \mathbb{K})[x_1, \dots, x_r]$$

be the polynomial ring in variables x_1, \dots, x_r . In analogy to (6) we have an evaluation map

$$ev_{\ell';l;r} : \text{Map}_l(\text{Par}(P)^{\ell'}; \mathbb{K})[x_1, \dots, x_r] \times \text{Nice}(E)^r \rightarrow \text{Nice}_{\mathbb{K}}^{\ell'}(E). \quad (8)$$

- We ask: can we find subsets $X_{\ell';l}^r(P, E) = C_{\ell';l;r}(P) \times \underline{N}^r(E)$ of multivariate polynomials and operators such that Lemma 5.2 holds if we replace the image of $\underline{ev}_{\ell';l}^C$ with the image of $ev_{\ell';l;r}$ by $X_{\ell';l}^r(P, E)$?

Since we added a new integer index “ r ” and since the desired property holds in the case $r = 1$, it is natural to try to use induction arguments. In an induction argument it is highly desirable that the induction step can be set in connection with the base of induction. Recall that for any commutative ring R , we have an isomorphism of graded algebras $R[x, y] \simeq R[x][y]$, so that $R[x_1, \dots, x_r] \simeq R[x_1, \dots, x_{r-1}][x_r]$, which allows us to work in the nice setting for induction described above. The obvious idea is to try to take $\underline{N}^r(E) = \underline{\text{Nice}}(E)^r$, since for $r = 1$ we recover the set of operators used in Lemma 5.2. Furthermore, given $C_{\ell'}(P; \mathbb{K})$ as previously, we can consider the $C_{\ell'; l; r}(P)$ as the subset $C_{\ell'; l; r}(P) \subset \text{Map}_l(\text{Par}(P)^{\ell'}; \mathbb{K})[x_1, \dots, x_r]$ of multivariate polynomials whose coefficients are in $C_{\ell'; l; r}(P)$. The restriction of $ev_{l; r}$ to those subsets will be denoted by

$$\underline{ev}_{\ell'; l; r}^C : C_{\ell'; l; r}(P) \times \underline{N}^r(E) \rightarrow \text{Nice}_{\mathbb{K}}^{\ell'}(E).$$

We can now restate and prove our main theorem.

Theorem 7.1. *Let $S_{1, \varepsilon(\ell)}$ be a homomorphic or partially homomorphic on parameters NPT, defined in nice background $\mathcal{NB}_{\ell'}^{\ell}$. Let $C_{\ell''}(P; \mathbb{K})$ be the domain of a unital functional calculus in $\mathcal{NB}_{\ell''}^{\ell}$. Then $S_{1, \varepsilon(\ell)}$ emerges from any NPT in the image of $\underline{ev}_{\ell'; l; r}^C$, where $\ell'' = r\ell'$.*

Proof. After all these steps and the discussion above, the proof is almost straightforward. It is done by induction in r . The case $r = 1$ is just Lemma 5.2. Suppose that the theorem holds for each $r = 1, \dots, q$ and let us show that it holds for $r = q + 1$. Let $p_{\ell'; q+1}^l[x_1, \dots, x_{q+1}] = \sum_{|\alpha| \leq l} f_{\alpha} \cdot x^{\alpha}$ be a multivariate polynomial whose coefficients f_{α} belong to $C_{\ell''}^{\ell'}(P; \mathbb{K})$. From the isomorphism $R[x_1, \dots, x_{q+1}] \simeq R[x_1, \dots, x_q][x_{q+1}]$ one can regard $p_{\ell'; q+1}^l[x_1, \dots, x_{q+1}]$ as a univariate polynomial $p_{\ell'; q+1}^{l_{q+1}}[x_{q+1}]$ for some $1 \leq l_{q+1} \leq l$. Thus, $p_{\ell'; q+1}^l[x_1, \dots, x_{q+1}] = \sum_i g_{i; r}[x_1, \dots, x_q] \cdot x_{q+1}^i$, where $g_{i; r} \in C_{\ell'; l_i; r}(P; \mathbb{K})$ and $1 \leq l_i \leq l$ such that $\sum_{i=1}^{q+1} l_i = l$. Let $\Psi_1, \dots, \Psi_{q+1} \in \underline{\text{Nice}}(E)$ be right-invertible operators. Notice that

$$\underline{ev}_{\ell'; l; q+1}^C(p_{\ell'; q+1}^l[x_1, \dots, x_{q+1}], \Psi_1, \dots, \Psi_{q+1}) = \sum_{i=1}^{l_{q+1}} \underline{ev}_{\ell'; l_i; q}^C(g_{i; q}[x_1, \dots, x_q], \Psi_1, \dots, \Psi_q) \circ m_i[\Psi_{q+1}], \quad (9)$$

where $m_i[x]$ is the monomial $m_i[x] = 1 \cdot x^i$. Let us define S_{3_k, κ_k} as the theory with operators $\Psi_{3_k, \kappa_k} = Q_k \circ m_k[\Psi_{r+1}]$, where $Q_k = I$ for every $i = 1, \dots, l_{r+1}$. Furthermore, let S_{2_j, δ_j} be the theory defined by the operators $g_{j; q}[\Psi_1, \dots, \Psi_q]$. By the induction hypothesis, $S_{1, \varepsilon(\ell)}$ emerges from S_{2_j, δ_j} for every j . On the other hand, since $C_{\ell''}(P; \mathbb{K})$ is the domain of a unital functional calculus, from Lemma 3.1 we see that $S_{1, \varepsilon(\ell)}$ and S_{2_j, δ_j} emerges from Ψ_{3_k, κ_k} for every k . Thus, the first two conditions of Lemma 6.1 are satisfied. Since $\Psi_{3_k, \kappa_k} = Q_k \circ m_k[\Psi_{r+1}]$ it is also clear that the third hypothesis is also satisfied. The fourth one follows from Lemma 3.1. Therefore, if the fifth one holds, then we can apply Lemma 6.1 to conclude that $S_{1, \varepsilon(\ell)}$ emerges from (9), concluding the proof. Notice that the fifth condition, applied to this context, means that

$$\Xi_{q+1} = \sum_{i=1}^{l_{q+1}-1} g_{i; q}[\Psi_1, \dots, \Psi_q] \circ m_i[\Psi_{q+1}]$$

emerges from $\Xi_q = g_{l_{q+1}; q}[\Psi_1, \dots, \Psi_q]$. Notice that $\Xi_q = g_{l_{q+1}; q}[\Psi_1, \dots, \Psi_q] = \sum_{j=1}^{l(q)} \Xi_{j; q-1} \circ m_j[\Psi_q]$, where $\Xi_{j; q-1} = h_{j; q-1}[\Psi_1, \dots, \Psi_{q-1}]$, so that due to Lemma 3.1 Ξ_q and $\Xi_{j; q-1}$ emerges from $m_j[\Psi_q]$.

Thus, from Lemma 4.2 if Ξ_{q+1} emerges from $\Xi_{j;q-1}$, then Ξ_{q+1} emerges from $\Xi_{j;q-1} \circ m_j[\Psi_q]$. If, in addition Ξ_q emerges from $\Xi_{l(q);q-1}$, then we can use Lemma 6.1 to conclude that Ξ_{q+1} emerges from Ξ_q , which by the above will finish the proof. Thus, the real problem is to prove that Ξ_{q+1} emerges from $\Xi_{j;q-1}$ and that Ξ_q emerges from $\Xi_{l(q);q-1}$. We can repeat the argument to show that all we actually need to prove is that Ξ_q emerges from $\Xi_{j;q-2}$ and that Ξ_{q-1} emerges from $\Xi_{l(q-1);q-2}$, where

$$\Xi_{q-1} = t_{l(q);q-1}[\Psi_1, \dots, \Psi_{q-1}] = \sum_{j=1}^{l(q-1)} \Xi_{j;q-2} \circ m_j[\Psi_{q-1}].$$

Thus, by means of using a reverse induction, now in q , we conclude that it is enough to ensure that Ξ_3 emerges from $\Xi_{j;1}$ and that Ξ_2 emerges from $\Xi_{l(1);1}$. But $\Xi_{j;1}$ and $\Xi_{l(1);1}$ are first order univariate polynomials evaluated in a right-invertible operator. Thus, the result follows from Lemma 5.2. \square

8 Some Examples

Although this paper is focused on finding general conditions for the existence of emergence phenomena, in this section we present some concrete examples aiming to make more clear the real range of our results. We begin with the basic class of examples, which in the sequence will be generalized in many directions.

Example 8.1. Consider the nice background defined by:

1. *some bounded open set* $U \subset \mathbb{R}^n$ with the canonical Riemannian metric (regarded as the spacetime);
2. *the trivial bundle* $U \times \mathbb{K}$ (regarded as the field bundle) with the global sections $C^\infty(U; \mathbb{K})$ (regarded as the fields $\varphi : U \rightarrow \mathbb{K}$);
3. *a number* $p \in [1, \infty]$, corresponding to the integrability degree;
4. *the graded Hilbert space*⁴ $W^p(U) = \bigoplus_{k \geq 0} W^{k,p}(U; \mathbb{K})$, playing the role of $\mathcal{H}(E)$;
5. *again the trivial bundle* $U \times \mathbb{K}$, now regarded as the parameter bundle;
6. *the subspace* $cst \subset C^\infty(U; \mathbb{K})$ *of constant functions* (viewed as the parameter set), so that $\text{Par}(U \times \mathbb{K}) \simeq \mathbb{K}$, endowed with the canonical algebra structure.

In this nice background, take a NPT defined by:

1. *degree* $\ell = 1$. Thus, $\varepsilon(\ell)$ belongs to $\text{Par}(U \times \mathbb{K})^\ell \simeq \mathbb{K}$, which obviously has degree $\ell = 1$ square roots. In the following, for simplicity we will write just $\varepsilon \in \mathbb{K}$ instead of $\varepsilon(1) \in \text{Par}(U \times \mathbb{K})^1$;
2. *a differential operator* $D_0 : C^k(U; \mathbb{K}) \rightarrow C^{k-d}(U; \mathbb{K})$ *in* U , where $0 \leq d \leq k$ is its degree. This has its canonical extension as a bounded operator between Sobolev spaces $\hat{D}_0 : W^{k,p}(U; \mathbb{K}) \rightarrow W^{k-d,p}(U; \mathbb{K})$, which in turn extends to a bounded operator $\Psi_0 : W^p(U) \rightarrow W^p(U)$. Indeed, let $0_{k;d} : W^{k-d,p}(U; \mathbb{K}) \rightarrow W^{k,p}(U; \mathbb{K})$ and $0_l : W^{l,p}(U; \mathbb{K}) \rightarrow W^{l,p}(U; \mathbb{K})$ be the null operators. Then $\bigoplus_{l \neq k-d} (\hat{D}_0 \oplus 0_{k;d} \oplus 0_l)$ is the desired extension;

⁴Recall that the category of Hilbert spaces is closed under direct sums (coproducts).

3. the parameterized operator $\Psi_{1,\varepsilon} = \varepsilon\Psi_0$. Notice that the rule $\varepsilon \mapsto \Psi_{1,\varepsilon}$ is additive and multiplicative, so that the corresponding NPT theory is homomorphic.

Consider, in addition:

- a list $D_{2,1}, \dots, D_{2,r} : C^\infty(U; \mathbb{K}) \rightarrow C^\infty(U; \mathbb{K})$ of smooth linear different operators in U with constant coefficients. This means that they have fundamental solutions, i.e, Green functions, and assume that these are defined in the whole U (this usually implies constrains on U [25, 12]). By the above, the operator $D_{2,i}$, with $i = 1, \dots, r$, extend to bounded operators $\Psi_{2,i} : W^p(U) \rightarrow W^p(U)$, which have right-inverses (as pseudo-differential operators) determined by the Green functions. Let $p^l[\Psi_{2,1}, \dots, \Psi_{2,r}; \delta] = \sum_{|\alpha| \leq l} f_\alpha(\delta) \Psi_2^\alpha$, where $\Psi_2^\alpha = \Psi_{2,1}^{\alpha_1} \circ \dots \circ \Psi_{2,r}^{\alpha_r}$ and $\alpha_1 + \dots + \alpha_r = |\alpha|$, be some multivariate polynomial of degree l whose coefficients are nowhere vanishing functions $f_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ depending on δ . Denote $D_2^\alpha = D_{2,1}^{\alpha_1} \circ \dots \circ D_{2,r}^{\alpha_r}$.

From Theorem 7.1 it then follows that the parameterized theory with Lagrangian density $\mathcal{L}_1(\varphi; \varepsilon) = \varepsilon\varphi^*\Psi_0\varphi$ emerges from the theory with Lagrangian density

$$\mathcal{L}_2(\varphi; \delta) = \varphi^* p^l[\Psi_{2,1}, \dots, \Psi_{2,r}; \delta] \varphi = \sum_{|\alpha| \leq l} f_\alpha(\delta) \varphi^* \Psi_2^\alpha \varphi.$$

Furthermore, the theory $\mathcal{L}_1(\varphi; \varepsilon) = \varepsilon\varphi^*D_0\varphi$ also emerges from $\mathcal{L}_2(\varphi; \delta) = \sum_{|\alpha| \leq l} f_\alpha(\delta) \varphi^* D_2^\alpha \varphi$. Here, $*$: $\mathbb{K} \rightarrow \mathbb{K}$ is the obvious involution given by $z^* = z$ if $z \in \mathbb{R}$ and $z^* = \bar{z}$ if $z \in \mathbb{C}$.

The main conclusion of the above example is the following:

Conclusion 1. In a nice open *Euclidean background* the typical real/complex kinetic scalar theories with scalar parameter emerge from any multivariate real/complex polynomial scalar theory with scalar parameter and *constant coefficients*.

The previous class of examples is the synthesis of how we can use our abstract and general theorem in order to get concrete information which is closer to Physics. We note, however, that it can be generalized in many directions:

1. *we do not need to assume that the differential operators $D_{2,i} : C^\infty(U; \mathbb{K}) \rightarrow C^\infty(U; \mathbb{K})$ are smooth.* Indeed, we made this assumption only in order to simplify the notation. In the general situation we could consider functions $\kappa : [1, r] \rightarrow (0, \infty)$ and $\Delta : [1, r] \rightarrow [0, \infty)$, with $\Delta(i) \leq \kappa(i)$, and then work on operators $D_{2,i} : C^{\kappa(i)}(U; \mathbb{K}) \rightarrow C^{\kappa(i) - \Delta(i)}(U; \mathbb{K})$;
2. *we can work on a more general nice background.* Notice that the constructions in Example 8.1 are about building Sobolev spaces and considering extensions of differential operators to them. For the last one it is implicit the fact that the closure of smooth functions is equivalent to the Sobolev space. All of this is true in any compact Riemannian manifold (M, g) and for differential operators between vector bundles over M [20, 19, 29]. This is true even in the case of noncompact manifolds with smooth boundary, but now under some geometric assumption: if (M, g) has k -bounded geometry, then we have denseness of smooth functions on $W^{r,p}(M; \mathbb{R})$ for $1 \leq r \leq k + 2$ [20, 19, 29] - see also Footnote 2. Physically, this means that Conclusion 1 remains true in the case of *Euclidean* backgrounds with nonzero curvature and of vector or tensor fields instead of scalar ones;

3. *we can consider other kind of parameters.* In Example 8.1 we considered $\ell = 1$ and $\text{Par}(P) \simeq \mathbb{R}$. We could considered, more generally, $\text{Par}(P)$ as any algebra with square roots continuously acting on $\Gamma(E)$, where E is the field bundle (some vector bundle due the last remark). Indeed, in this case, due to the denseness of $\Gamma(E)$ on the Sobolev space $W^p(E; \mathbb{K})$ we have an induced action $*\text{Par}(P) \times W^p(E; \mathbb{K}) \rightarrow W^p(E; \mathbb{K})$, which itself induces an action $*_B$ on $B(W^p(E; \mathbb{K}))$ given by $(\varepsilon *_B \Psi)(\varphi) = \varepsilon *(\Psi\varphi)$. Furthermore, for every fixed Ψ_0 , the rule $\varepsilon \mapsto \Psi_{1,\varepsilon} = \varepsilon *_B \Psi_0$ is homomorphic, which is precisely what we need. This includes, for instance, the case of the nowhere vanishing algebra of positive self-adjoint real/complex matrices, which can be realized as a nowhere vanishing subalgebra of $\Gamma(P)$, where P is the algebra bundle $P \times B(\mathbb{K}^m)$ and $m = \text{rank}(E)$. Physically, this means that in Conclusion 1 we can replace scalar parameter with matrix parameter or more general and abstract things;
4. *we can consider other kinds of parameterized operators.* In the above remark, we consider only parameterized operators given by $\Psi_{1,\varepsilon} = \varepsilon *_B \Psi_0$, i.e, with a uniform scaling dependence on ε . More generally, we can take any representation $\rho_0 : \text{Par}(P) \rightarrow B(B(W^p(E; \mathbb{K})))$, so that for every bounded operator Ψ_0 in $W^p(E; \mathbb{K})$ we get an induced action $*_B$ of $\text{Par}(P)$ on $W^p(E; \mathbb{K})$ given by $\varepsilon *_B \varphi = [\rho_0(\varepsilon)(\Psi_0)](\varphi)$. Thus, the rule $\varepsilon \mapsto \Psi_{1,\varepsilon} = \rho_0(\varepsilon)(\Psi_0)$ is homomorphic and defines a nice parameterized operator, as desired;
5. *we can consider NPT of higher degrees.* In all previous points we worked with $\ell = 1$, i.e, with NPT of degree one. But everything remains true for higher degrees, in virtue of Remark 4.3. Physically, we can consider theories which have not a single (scalar, matrix and so on) fundamental parameter, but many of them, all of same nature (i.e, all scalar or all matrix, etc.).

If basically everything in Example 8.1 can be generalized, what are the main difficulties in generalizing this work? Here are two of them:

1. *Euclidean background.* Although our results do not require *explicitly* an Euclidean background, the examples connected with physics (as those discussed above) depend on such structure. Indeed, this appear in the construction of the Hilbert space $\mathcal{H}(E)$, which typically is built from metric structures on the spacetime M and on the field bundle E . Furthermore, we also need to regard differential operators in E as bounded operators on $\mathcal{H}(E)$, which is done (in the Euclidean case) by building Sobolev spaces. This is closely related to the problem of canonical/algebra quantization, where the classical observables (differential operators) are represented by bounded operators in a Hilbert space [??,??]. Looking at this point of view, the difficulties of avoiding the Euclidean signature are clear. Even so, if we insist in working on Lorentzian spacetimes (M, g) , one can take induced Riemannian metrics $g_X = g + 2\omega \otimes \omega$, where ω is the 1-form corresponding to a unitary timelike vector field X in M [30], and consider our results on (M, g_X) ;
2. *existence of right-inverses.* This is directly related to the first task. Indeed, recall that in Example 8.1 we assumed that the differential operators $D_{2,i}$ are of constant coefficients. Added to a global definition of their Green functions, this ensured that they are right-invertible. More generally, we could assume ellipticity conditions which are the typical approach to ensure invertibility [38, 39, 2, 26, 3]. In the *Euclidean* setting, typical operators arising in Physics are elliptic and additional conditions on their coefficients allows right-invertibility.

On the other hand, in the *Lorentzian* setting, the same operators become hyperbolic and we cannot use the standard techniques to ensure right-invertibility.

Thus, the final conclusion is the following:

Conclusion 2. In a compact (or noncompact with bounded geometry) *Euclidean background* the typical real/complex kinetic field theories with scalar/matrix/etc parameters emerge from any multivariate real/complex *elliptic* polynomial field theory with scalar/matrix/etc parameters. Furthermore, our results cannot be used to directly extend this conclusion to Lorentzian and non-elliptic settings⁵.

Acknowledgments

The first author was supported by CAPES (grant number 88887.187703/2018-00). Both authors would like to thank Fábio Dadam for reading a preliminary version of the text. The first author would like to thank Luiz Cleber Tavares de Brito for introduced him to the subject of the paper many years ago.

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⁵Same kind of difficulties was found in [15].

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