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Geometric Obstructions on Gravity

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Abstract

These are notes for a short course and some talks gave at Department of Mathematics and at Department of Physics of Federal University of Minas Gerais, based on the author’s paper [26]. Some new information and results are also presented. Unlike the original work, here we try to give a more physical emphasis. In this sense, we present obstructions to realize gravity, modeled by the tetradic Einstein-Hilbert-Palatini (EHP) action functional, in a general geometric setting. In particular, we show that if spacetime has dimension $n \geq 4$, then the cosmological constant plays no role in any “concrete geometries” other than Lorentzian. If $n \geq 6$, then the entire EHP theory is trivial, meaning that Lorentzian geometry is (essentially) the only “concrete geometry” in which gravity (i.e, the EHP action functional) makes sense. Examples of “concrete geometries” include those locally modeled by group reductions $H \hookrightarrow O(k; A)$ for some k and some algebra A , so that Riemannian geometry, Hermitian geometry, Kähler geometry and symplectic geometry, as well as Type II geometry, Hitchin’s generalized complex geometry and G_2 -geometry are included. We also study EHP theory in “abstract geometries”, such as graded geometry (and hence supergeometry), and we show how the obstruction results extend to this context. We construct two theories naturally associated to EHP, which we call the geometric/algebraic dual of EHP, and we analyze the effect of the obstructions in these new theories. Finally, we speculate (and provide evidence for) the existence of a “universal obstruction condition”.

1 Introduction

Einstein’s theory of gravity is formulated in Lorentzian geometry: spacetime is regarded as an orientable four-dimensional Lorentzian manifold (M, g) whose Lorentzian metric g is a critical point of the Einstein-Hilbert action functional

$$S_{EH}[g] = \int_M (R_g - 2\Lambda) \cdot \omega_g, \quad (1)$$

defined on the space of all possible Lorentzian metrics in M . Here, $\Lambda \in \mathbb{R}$ is a parameter (the cosmological constant), while R_g and ω_g are, respectively, the scalar curvature and the volume-form

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of g , locally written as $g^{ij} \text{Ric}_{ij}$ and $\sqrt{|\det g|} dx^1 \wedge \dots \wedge dx^4$, where Ric_{ij} are the components of the Ricci tensor. Such critical points are precisely those satisfying the vacuum Einstein's equation

$$\text{Ric}_{ij} - \frac{1}{2} R_g g_{ij} + \Lambda g_{ij} = 0. \quad (2)$$

The action (1) is about tensors, so that we can say that General Relativity is generally formulated in a “tensorial approach”. Electromagnetism (or, more generally, Yang-Mills theories) was also initially formulated in a “tensorial approach”. Indeed, the action functional for electromagnetism is given by

$$S_{YM}[A] = -\frac{1}{4} \int_M F_{ij} \tilde{F}^{ij},$$

where F and \tilde{F} are the “electromagnetic tensor” and the “dual electromagnetic tensor”, with components

$$F_{ij} = \partial_i A_j - \partial_j A_i \quad \text{and} \quad \tilde{F}^{ij} = \frac{1}{2} \epsilon^{ijkl} F_{kl} \sqrt{|\det g|},$$

respectively. Varying this action we get Maxwell's equations in their tensorial formulation:

$$\partial_k \tilde{F}^{kl} + A_k \tilde{F}^{kl} = 0. \quad (3)$$

Notice that this is actually **one half** of the actual Maxwell's equations, corresponding to the “inhomogeneous equations”. In order to get the full equations we need to take into account an external set of equations, corresponding to the “homogeneous part” of Maxwell's equations:

$$\epsilon^{ijklmn} \partial_l F_{mn} = 0. \quad (4)$$

It happens that, when we move from the tensorial language to the language of differential forms and connections on bundles, we rediscover (4) as a *geometric identity* (Bianchi identity), so that (4) actually holds *a priori* and not *a posteriori*, as was suggested using tensorial language. This is not a special feature of Yang-Mills theories. In fact, General Relativity (GR) also manifests this type of behavior: alone, Einstein's equation (2) does not completely specify a system of (GR); we also need to assume that the connection in question is precisely the Levi-Civita connection of g . But, as in the case of Bianchi identity for Yang-Mills theories, this assumption can be avoided if we use the language of differential forms.

Indeed, in the so-called *first order formulation of gravity* (also known as *tetradic gravity*) we can rewrite Einstein-Hilbert action functional as the *Einstein-Hilbert-Palatini* (EHP) *action*:

$$S_{EHP}[e, \omega] = \int_M \text{tr}(e \wedge_{\times} e \wedge_{\times} \Omega + \frac{\Lambda}{6} e \wedge_{\times} e \wedge_{\times} e \wedge_{\times} e), \quad (5)$$

where e and ω are 1-forms in the frame bundle FM with values in the Lie algebras $\mathbb{R}^{3,1}$ and $\mathfrak{o}(3,1)$, called *tetrad* and *spin connection*, respectively, and \wedge_{\times} is a type of “wedge product”¹ induced by matrix multiplication in $O(3,1)$. Because M is Lorentzian, its frame bundle is structured over $O(3,1)$ and the spin connection is actually a connection in this bundle. The tetrad e is such that

¹In this paper we will work with many different types of “wedge products”, satisfying very different properties. Therefore, in order to avoid confusion, we will not follow the literature, but introduce specific symbols for each of them.

for every $a \in FM$ the corresponding map $e_a : FM_a \rightarrow \mathbb{R}^{3,1}$ is an isomorphism, so that it has the geometrical meaning of a soldering form. Furthermore, in (5) we have a 2-form Ω with values in $\mathfrak{o}(3,1)$, representing the curvature $\Omega = d\omega + \omega \lambda_{\times} \omega$ of ω . The translational algebra and the Lorentz algebra fits into the Poincaré Lie algebra $\mathfrak{iso}(3,1) = \mathbb{R}^{3,1} \times \mathfrak{o}(3,1)$. The equations of motion for the EHP action are [42]

$$d_{\omega}e \lambda_{\times} e = 0 \quad \text{and} \quad e \lambda_{\times} \Omega + \frac{\Lambda}{6} e \lambda_{\times} e \lambda_{\times} e = 0.$$

The second of them is just Einstein's equation (2) rewritten in the language of forms, while the first, due to the fact that e is an isomorphism, is equivalent to $d_{\omega}e = 0$. As the 2-form $\Theta = d_{\omega}e$ describes precisely the torsion of the spin connection ω , the first equation of motion implies that ω is actually the only torsion-free connection compatible with the metric: the *Levi-Civita connection*.

These examples emphasize that the language of differential forms seems to be a nice way to describe physical theories. Indeed, [18, 19] started a program that attempts to unify different physical theories by using the same type of functional on forms, the so-called *Hitchin's functional*. Furthermore, in [11] it was shown that all known "gravity theories defined by forms" can really be unified in that they are particular cases of a single *topological M-theory*. Motivated by this philosophy, in this article we will consider generalizations of classical EHP theories (5).

First of all notice that if the underlying manifold is n -dimensional spacetime (instead of four-dimensional), (5) can be immediately generalized by considering $\mathfrak{iso}(n-1,1)$ -valued forms and an action functional given by

$$S_{n,\Lambda}[e, \omega] = \int_M \text{tr}(e \lambda_{\times} \dots \lambda_{\times} e \lambda_{\times} \Omega + \frac{\Lambda}{(n-1)!} e \lambda_{\times} \dots \lambda_{\times} e). \quad (6)$$

However, this is not the only generalization that can be considered. We notice that in the modern language of differential geometry, the data defined by the pairs (ω, e) above corresponds to *reductive Cartan connections* on the frame bundle FM with respect to the inclusion $O(n-1,1) \hookrightarrow \text{Iso}(n-1,1)$ of the Lorentz group into the Poincaré group. Indeed, given a G -bundle $P \rightarrow M$ and a structural group reduction $H \hookrightarrow G$, we recall that a *Cartan connection* in P for such a reduction is a G -connection ∇ in P which projects isomorphically onto $\mathfrak{g}/\mathfrak{h}$ in each point. Explicitly, it is an (horizontal and equivariant) \mathfrak{g} -valued 1-form $\nabla : TP \rightarrow \mathfrak{g}$ such that in each $a \in P$ the composition below is an isomorphism of vector spaces:

$$TP_a \begin{array}{c} \xrightarrow{\nabla_a} \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathfrak{h} \\ \underbrace{\hspace{1.5cm}}_{\simeq} \end{array}$$

Due to the decomposition (as vector spaces) $\mathfrak{g} \simeq \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{h}$, the existence of the isomorphism above allows us to write $\nabla_a = e_a + \omega_a$ in each point. If this varies smoothly we say that the Cartan connection is *reductive* (or *decomposable*).

Now, looking at (6) we see that the groups $O(n-1,1) \hookrightarrow \text{Iso}(n-1,1)$ and the bundle FM do not appear explicitly, so that we can think of considering an analogous action for other group reductions $H \hookrightarrow G$ in other bundle P . Intuitively, this means that we are trying to realize gravity (as modeled by EHP theory) in different geometries other than Lorentzian. But, *is (6) always nontrivial?* In other words, *is it possible to realize gravity in any geometry?* Notice that it is natural to expect that the algebraic properties of H and G will be related to the fundamental properties of the corresponding version of (6), so that a priori there should exist some abstract algebraic conditions

under which (6) is trivial. In this article we will search for such nontrivial conditions, which we call *geometric obstructions*. Thus, one can say that *we will do Geometric Obstruction Theory applied to EHP theory*. For instance, one of the results that we will prove is the following, which emphasizes that some notion of “solvability” is crucial:

Theorem A. Let M be an n -dimensional spacetime and $P \rightarrow M$ be a $\mathbb{R}^k \rtimes H$ -bundle, endowed with the group reduction $H \hookrightarrow \mathbb{R}^k \rtimes H$. If \mathfrak{h} is a (k, s) -solv algebra and $n \geq k + s + 1$, then the cosmological constant plays no role; if $n \geq k + s + 3$, then the entire EHP theory is trivial.

This type of obstruction theorem gives restrictions only on the dimension of spacetime. We also show that if we restrict to torsion-free connections, then there are nontrivial geometric obstructions that gives restrictions not only on the dimension, but also on the topology of spacetime. For instance,

Theorem B. Let M be an n -dimension Berger manifold endowed with an H -structure. If $\mathfrak{h} \subset \mathfrak{so}(n)$, then the torsionless EHP theory is nontrivial only if $n = 2, 4$ and M is Kähler. In particular, if M is compact and $H^2(M; \mathbb{C}) = 0$, then it must be a K3-surface.

The generalization of EHP from Lorentzian geometry to arbitrary geometry is not the final step. Indeed, looking at (6) again we see that it remains well defined if one forgets that e and ω together define a Cartan connection. In other words, all one needs is the fact they are 1-forms that take values in some algebra. This leads us to consider some kind of *algebra-valued EHP theories*, defined on certain algebra-valued differential forms, for which we show that Theorem A remains valid almost *ipsis litteris*. We also show that if the algebra in question is endowed with a grading, then the “solvability condition” in Theorem A can be weakened.

This paper is organized as follows: in Section 2 we review some facts concerning algebra-valued differential forms and we study the “solvability conditions” that will appear in the obstructions results. In Section 3 we introduce EHP theory into the “linear”/ “extended-linear” contexts and we prove many obstruction theorems, including Theorem A and Theorem B. In Section 4 EHP is internalized into the “abstract context”, where matrix algebras are replaced by arbitrary (possibly graded) algebras. We then show how the “fundamental obstruction theorem”, namely Theorem A, naturally extends to this context. We also give geometric obstructions to realize EHP into the “full graded context”, i.e, into “graded geometry” and, therefore, into “supergeometry”. This section ends with a conjecture about the existence of some kind of “universal geometric obstruction”. Finally, in Section 5 many examples are given, where by an “example” we mean an specific algebra fulfilling the hypothesis of some obstruction result, so that EHP cannot be realized in the underlying geometry.

2 Polynomial Identities in Algebra-Valued Forms

Let A be a real vector space² and P be a smooth manifold. A A -valued k -form in P is a section of the bundle $\Lambda^k TP^* \otimes A$. In other words, it is a rule α assigning to any $a \in P$ a skew-symmetric k -linear map

$$\alpha_a : TP_a \times \dots \times TP_a \rightarrow A.$$

²All definitions and almost all results of this section hold analogously for modules over any commutative ring R of characteristic zero. For some results in Subsection 2.4 we must assume that the module is free, which is guaranteed if R is a field.

The collection of such maps inherits a canonical vector space structure which we will denote by $\Lambda^k(P; A)$. Given an A -valued k -form α and a B -valued l -form β we can define an $(A \otimes B)$ -valued $(k + l)$ -form $\alpha \otimes \beta$, such that

$$(\alpha \otimes \beta)_a(v, w) = \frac{1}{(k + l)!} \sum_{\sigma} \text{sign}(\sigma) \alpha_a(v_{\sigma}) \otimes \beta_a(w_{\sigma}),$$

where $v = (v_1, \dots, v_k)$ and $w = (v_{k+1}, \dots, v_{k+l})$ and σ is a permutation of $\{1, \dots, k + l\}$. This operator is obviously bilinear, so that it extends to a bilinear map

$$\otimes : \Lambda^k(P; A) \times \Lambda^l(P; B) \rightarrow \Lambda^{k+l}(P; A \otimes B).$$

We are specially interested when A is an algebra, say with multiplication $* : A \otimes A \rightarrow A$. In this case we can compose the operation \otimes above with $*$ in order to get an exterior product \wedge_* , as shown below.

$$\Lambda^k(P; A) \times \Lambda^l(P; A) \xrightarrow{\otimes} \Lambda^{k+l}(P; A \otimes A) \xrightarrow{*} \Lambda^{k+l}(P; A) \quad (7)$$

$\xrightarrow{\wedge_*}$

Explicitly, following the same notations above, if α and β are A -valued k and l forms, we get a new A -valued $(k + l)$ -form by defining

$$(\alpha \wedge_* \beta)_a(v, w) = \frac{1}{(k + l)!} \sum_{\sigma} \text{sign}(\sigma) \alpha_a(v_{\sigma}) * \beta_a(w_{\sigma}).$$

This new product defines an \mathbb{N} -graded algebra structure on the total A -valued space

$$\Lambda(P; A) = \bigoplus_k \Lambda^k(P; A),$$

whose properties are deeply influenced by the properties of the initial product $*$. For instance, recall that many properties of the algebra $(A, *)$ can be characterized by its polynomial identities (PI's), i.e., by polynomials that vanish identically when evaluated in A . Just to mention a few, some of these properties are commutativity and its variations (as skew-commutativity), associativity and its variations (Jacobi-identity, alternativity, power-associativity), and so on [22, 12].

The following proposition shows that each such property is satisfied in A iff it is satisfied (in the graded-sense) in the corresponding algebra of A -valued forms.

Proposition 2.1. *Any PI of degree m in A lifts to a PI in the graded algebra of A -valued forms. Reciprocally, every PI in the graded algebra restricts to a PI in A .*

Proof. Let f be a polynomial of degree m . Given arbitrary A -valued forms $\alpha_1, \dots, \alpha_m$ of degrees k_1, \dots, k_m , we define a corresponding A -valued form of degree $k = k_1 + \dots + k_m$ as

$$\hat{f}(\alpha_1, \dots, \alpha_m)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma} \text{sign}(\sigma) f(\alpha_1(w_{\sigma(1)}), \alpha_2(w_{\sigma(2)}), \dots, \alpha_m(w_{\sigma(m)})),$$

where

$$w_{\sigma(1)} = (v_{\sigma(1)}, \dots, v_{\sigma(k_1)}), \quad w_2 = (v_{\sigma(k_1+1)}, \dots, v_{\sigma(k_1+k_2)}), \quad \text{and so on.}$$

Therefore, \hat{f} is a degree m polynomial in $\Lambda(P; A)$. It is clear that if f vanishes identically then \hat{f} vanishes too, so that each PI in A lifts to a PI in $\Lambda(P; A)$. On the other hand, if we start with a polynomial F in $\Lambda(P; A)$, we get a polynomial $F|_A$ in A of the same degree by restricting F to constant 1-forms. Clearly, if F is a PI, then $F|_A$ is also a PI. \square

It follows that the induced product \wedge_* is associative, alternative, commutative, and so on, iff the same properties are satisfied by $*$. Notice that, because the algebra $\Lambda(P; A)$ is graded, its induced properties must be understood in the graded sense. For instance, by commutativity and skew-commutativity of \wedge_* one means, respectively,

$$\alpha \wedge_* \beta = (-1)^{kl} \beta \wedge_* \alpha \quad \text{and} \quad \alpha \wedge_* \beta = -(-1)^{kl} \beta \wedge_* \alpha.$$

Let us analyze some useful examples.

Example 2.1 (*Cayley-Dickson forms*). There is a canonical construction, called *Cayley-Dickson construction* [2], which takes an algebra $(A, *)$ endowed with an involution $\overline{(-)} : A \rightarrow A$ and returns another algebra $\text{CD}(A)$. As a vector space it is just $A \oplus A$, while the algebra multiplication is given by

$$(x, y) * (z, w) = (x * z - \overline{w} * y, z * x + y * \overline{z}).$$

This new algebra inherits an involution $\overline{(x, y)} = (\overline{x}, -y)$, so that the construction can be iterated. It is useful to think of $\text{CD}(A) = A \oplus A$ as being composed of a “real part” and an “imaginary part”. We have a sequence of inclusions into the “real part”

$$A \longrightarrow \text{CD}(A) \longrightarrow \text{CD}^2(A) \longrightarrow \text{CD}^3(A) \longrightarrow \dots$$

For any manifold P , such a sequence then induce a corresponding sequence of inclusions into the algebra of forms

$$\Lambda(P; A) \longrightarrow \Lambda(P; \text{CD}(A)) \longrightarrow \Lambda(P; \text{CD}^2(A)) \longrightarrow \Lambda(P; \text{CD}^3(A)) \longrightarrow \dots$$

The Cayley-Dickson construction weakens any PI of the starting algebra [2, 37] and, therefore, due to Proposition 2.1, of the corresponding algebra of forms. For instance, if we start with the commutative and associative algebra (\mathbb{R}, \cdot) , endowed with the trivial involution, we see that $\text{CD}(\mathbb{R}) = \mathbb{C}$, which remains associative and commutative. But, after an iteration we obtain $\text{CD}^2(\mathbb{R}) = \text{CD}(\mathbb{C}) = \mathbb{H}$, which is associative but not commutative. Another iteration gives the octonions \mathbb{O} which is non-associative, but alternative. The next is the sedenions \mathbb{S} which is non-alternative.

Example 2.2 (*Lie algebra valued forms*). Another interesting situation occurs when $(A, *)$ is a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$. In this case, we will write $\alpha[\wedge]_{\mathfrak{g}}\beta$ or simply $\alpha[\wedge]\beta$ instead of³ $\alpha \wedge_{[\cdot, \cdot]}\beta$. Lie algebras are not associative and in general are not commutative. This means that the corresponding algebra $\Lambda(P; \mathfrak{g})$ is not associative and not commutative. On the other hand, any Lie algebra is skew-commutative and satisfies the Jacobi identity, so that these properties lift to $\Lambda(P; \mathfrak{g})$, i.e, we have

$$\alpha[\wedge]\beta = (-1)^{kl+1}\beta[\wedge]\alpha$$

and

$$(-1)^{km}\alpha[\wedge](\beta[\wedge]\gamma) + (-1)^{kl}\beta[\wedge](\gamma[\wedge]\alpha) + (-1)^{lm}\gamma[\wedge](\alpha[\wedge]\beta) = 0$$

for any three arbitrarily given \mathfrak{g} -valued differential forms α, β, γ of respective degrees k, l, m .

³Some authors prefer the somewhat ambiguous notations $[\alpha \wedge \beta]$, $[\alpha, \beta]$ or $[\alpha; \beta]$. The reader should be very careful, because for some authors these notations are also used for the product $[\wedge]_{\mathfrak{g}}$ without the normalizing factor $(k+l)!$. Here we are adopting the notation introduced in [25].

We end with an important remark.

Remark 2.1. In the study of classical differential forms we know that if α is an **odd**-degree form, then $\alpha \wedge \alpha = 0$. This follows directly from the fact that the algebra (\mathbb{R}, \cdot) is commutative. Indeed, in this case $\Lambda(P; \mathbb{R})$ is graded-commutative and so, for α of **odd**-degree k , we have

$$\alpha \wedge \alpha = (-1)^{k^2} \alpha \wedge \alpha = -\alpha \wedge \alpha,$$

implying the condition $\alpha \wedge \alpha = 0$. Notice that the same argument holds for any commutative algebra $(A, *)$. Dually, analogous arguments show that if $(A, *)$ is skew-commutative, then $\alpha \wedge_* \alpha = 0$ for any **even**-degree A -valued form. So, for instance, $\alpha[\wedge]_{\mathfrak{g}} \alpha = 0$ for any given Lie algebra \mathfrak{g} .

2.1 Matrix Algebras

In Example 2.2 above, we considered the algebra $\Lambda(P; \mathfrak{g})$ for an arbitrary Lie algebra \mathfrak{g} . We saw that, because a Lie algebra is always skew-commutative, the corresponding product $[\wedge]$ is also skew-commutative, but now in the graded sense, i.e.,

$$\alpha[\wedge]\beta = (-1)^{kl+1}\beta[\wedge]\alpha$$

for every \mathfrak{g} -valued forms α, β . As a consequence (explored in Remark 2.1) we get $\alpha[\wedge]\alpha = 0$ for even-degree forms.

From now on, let us assume that the Lie algebra \mathfrak{g} is not arbitrary, but a subalgebra of $\mathfrak{gl}(k; \mathbb{R})$, for some k . In other words, we will work with Lie algebras of $k \times k$ real matrices. We note that in this situation $\Lambda(P; \mathfrak{g})$ can be endowed with an algebra structure other than $[\wedge]$. In fact, for each k there exists an isomorphism

$$\mu : \Lambda^l(P; \mathfrak{gl}(k; \mathbb{R})) \simeq \text{Mat}_{k \times k}(P; \Lambda^l(P; \mathbb{R}))$$

given by $[\mu(\alpha)]_{ij}(a) = [\alpha(a)]_{ij}$, allowing us to think of every \mathfrak{g} -valued l -form α as a $k \times k$ matrix $\mu(\alpha)$ of classical forms. It happens that matrix multiplication gives an algebra structure on

$$\text{Mat}_{k \times k}(P; \Lambda(P; \mathbb{R})) = \bigoplus_l \text{Mat}_{k \times k}(P; \Lambda^l(P; \mathbb{R})),$$

which can be pulled-back by making use of the isomorphism μ , giving a new product on the graded vector space $\Lambda(P; \mathfrak{g})$, which we will denote by the symbol “ λ ”.

Because \mathfrak{g} is a matrix Lie algebra, its Lie bracket is the commutator of matrices, so that we have an identity relating both products. From Proposition 2.1 it follows that we have an analogous identity between the corresponding “wedge products” λ and $[\wedge]$:

$$\alpha[\wedge]\beta = \alpha \lambda \beta - (-1)^{kl} \beta \lambda \alpha. \tag{8}$$

This relation clarifies that, while the product $[\wedge]$ is skew-commutative for arbitrary Lie algebras, we **cannot** conclude the same for the product λ , i.e., it is not always true that $\alpha \lambda \beta = (-1)^{kl+1} \beta \lambda \alpha$. Consequently, *it is **not** true that $\alpha \lambda \alpha = 0$ for every even-degree \mathfrak{g} -valued form*. It is easy to understand why: recalling that λ is induced by matrix multiplication, the correspondence between PI's on the algebra $(\mathfrak{g}, *)$ and on the algebra of \mathfrak{g} -forms teaches us that *λ is skew-commutative exactly when the matrix multiplication is skew-commutative*, in other words, iff \mathfrak{g} is a Lie algebra of skew-commutative matrices.

Useful examples are given in the following lemma.

Lemma 2.1. *The condition $\alpha \wedge \alpha = 0$ is satisfied for even-degree forms with values into subalgebras*

$$\mathfrak{g} \subset \mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r).$$

Proof. By definition, $\mathfrak{so}(k)$ is the algebra of skew-symmetric matrices, which anti-commute, thus the result holds for $\mathfrak{g} = \mathfrak{so}(k)$ for every k . It is obvious that if $\bar{\mathfrak{g}}$ is some algebra of skew-commuting matrices, then every subalgebra $\mathfrak{g} \subset \bar{\mathfrak{g}}$ is also of skew-commuting matrices. Therefore, the result holds for subalgebras $\mathfrak{g} \subset \mathfrak{so}(k)$. But it is also clear that the direct sum of algebras of anti-commuting matrices remains an algebra of anti-commuting matrices, so that if \mathfrak{g}_1 and \mathfrak{g}_2 fulfill the lemma, then $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ fulfills too. In particular, the lemma holds for subalgebras $\mathfrak{g} \subset \mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2)$. Finite induction ends the proof. \square

Remark 2.2. We could have given a proof of the last lemma without using its invariance under direct sums. In fact, assuming that it holds for subalgebras of $\mathfrak{so}(k)$, notice that for every decomposition $k = k_1 + \dots + k_r$ we have a canonical inclusion

$$\mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r) \hookrightarrow \mathfrak{so}(k).$$

While the last lemma is a useful source of examples in which the discussion above applies, let us now give a typical non-example.

Example 2.3. Let H be a linear group of $k \times k$ real matrices. We have a canonical action of H on the additive abelian group \mathbb{R}^k , allowing us to consider the semi-direct product $G = \mathbb{R}^k \rtimes H$, of which $H \hookrightarrow G$ can be regarded as subgroup. The same holds in the level of Lie algebras, so that $\mathfrak{g} \simeq \mathbb{R}^k \rtimes \mathfrak{h}$. As a vector space we have $\mathfrak{g}/\mathfrak{h} \simeq \mathbb{R}^k$. Now, because $\mathfrak{g}/\mathfrak{h}$ is abelian, it follows that $\alpha[\wedge]\beta = 0$ for every two $\mathfrak{g}/\mathfrak{h}$ -valued forms and from (8) we then conclude that $\alpha \wedge \alpha = 0$ for every odd-degree form α . In particular, if $A = \omega + e$ is a reductive Cartan connection for the reduction $H \hookrightarrow \mathbb{R}^k \rtimes H$, then e is a $\mathfrak{g}/\mathfrak{h}$ -valued 1-form, so that $e \wedge e = 0$. Consequently, the action (5) is trivial. But, for $H = O(3,1)$ this is precisely General Relativity, which is clearly nontrivial. Therefore, at some moment we made a mistake! The problem is that we applied the relation (8) to \mathbb{R}^n -valued forms, but the product \wedge is **not** defined for such forms. Indeed, it can be defined only for Lie algebras arising from matricial algebras.

The example above teaches us two things:

1. The way it was defined, the product \wedge makes sense only for matrix Lie algebras. Particularly, it **does not** makes sense for a semi-direct sum of a matrix algebra with other algebra, so that \wedge is **not** the product appearing in the classical EHP action (5). Therefore, if we need to abstract the structures underlying EHP theory we need to show how to extend \wedge to a new product \wedge_{\rtimes} defined on forms taking values in semi-direct sums. This will be done in Subsection 2.2;
2. Once \wedge_{\rtimes} is defined, the corresponding Einstein-Hilbert-Palatini action functional can be trivial. In fact, looking at (6) we see that we have terms like $e \wedge_{\rtimes} e \wedge_{\rtimes} \dots \wedge_{\rtimes} e$, where the number of e 's depends on the spacetime dimension. So, if we are in an algebraic context in which $\alpha \wedge_{\rtimes} \alpha = 0$ for **odd**-degree forms, then $e \wedge_{\rtimes} e = 0$ and, consequently, the action will vanishes in arbitrary spacetime dimensions! On the other hand, if $\alpha \wedge_{\rtimes} \alpha = 0$ for **even**-degree forms, then $(e \wedge_{\rtimes} e) \wedge_{\rtimes} (e \wedge_{\rtimes} e) = 0$ and the theory is trivial when $n \geq 6$. *This simple idea is the*

core of almost all obstruction results that will be presented here. Notice that what we need is $\lambda_{\times}^k \alpha = 0$ for some k , which is a nilpotency condition. The correct nilpotency conditions that will be used in our abstract obstruction theorems will be discussed in Subsections 2.3 and 2.4.

2.2 Splitting Extensions

In this section we will see how to define the product λ_{\times} abstractly. We start by recalling the notion of “extension of an algebraic object”. Let \mathbf{Alg} be a category of algebraic objects, meaning that we have a null object $0 \in \mathbf{Alg}$ (understood as the trivial algebraic entity) and such that every morphism $f : X \rightarrow Y$ has a kernel and a cokernel, computed as the pullback/pushout below.

$$\begin{array}{ccc} \ker(f) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} \text{coker}(f) & \longleftarrow & 0 \\ \uparrow & & \uparrow \\ Y & \xleftarrow{f} & X \end{array}$$

Example 2.4. The category \mathbf{Grp} of groups and the category $\mathbf{Vec}_{\mathbb{R}}$ of real vector space are examples of models for \mathbf{Alg} . A non-example is \mathbf{LieAlg} , the category of Lie algebras, since a Lie algebra morphism $f : \mathfrak{h} \rightarrow \mathfrak{g}$ may not have a cokernel, which means that the vector space $\mathfrak{g}/f(\mathfrak{h})$ may not have a canonical Lie algebra structure.

If a morphism has a kernel and a cokernel we can then define its *image* as the kernel of its cokernel. Consequently, internal to \mathbf{Alg} we can talk of an “exact sequence”: this is just an increasing sequence of morphisms $f_i : X_i \rightarrow X_{i+1}$ such that for every i the kernel of f_{i+1} coincides with the image of f_i . An *extension* is just a short exact sequence, i.e, an exact sequence condensed in three consecutive objects. More precisely, in a short exact sequence as the one below we say that the middle term A was obtained as an *extension of H by E* .

$$0 \longrightarrow E \xrightarrow{i} A \xrightarrow{j} H \longrightarrow 0 \tag{9}$$

Remark 2.3. From the last example we conclude that we cannot talk of extensions internal to the category of Lie algebras. This does **not** mean that we cannot define a “Lie algebra extension” following some other approach. In fact, notice that we have a forgetful functor $U : \mathbf{LieAlg} \rightarrow \mathbf{Vec}_{\mathbb{R}}$, so that we can define a *Lie algebra extension* as a sequence of maps in \mathbf{LieAlg} which is exact in $\mathbf{Vec}_{\mathbb{R}}$. The same strategy allows us to enlarge the notion of extension in order to include categories that, a priori, are not models for \mathbf{Alg} .

Now, a typical example of extensions.

Example 2.5. The Poincaré group is just an extension of the Lorentz group by the translational group. More generally, given a Lie group H endowed with an action $H \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ we can form the semi-direct product $\mathbb{R}^n \rtimes H$, which fits into the canonical exact sequence below, where the first map is an inclusion and the second is obtained restricting to pairs $(0, h)$.

$$0 \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}^n \rtimes H \longrightarrow H \longrightarrow 0 \tag{10}$$

The extensions in the last example are special: in them we know how to include the initial object H into its extension $\mathbb{R}^n \rtimes H$. An extension with this property is called *splitting*. More precisely, we

say that an abstract extension as (9) is *splitting* if there exists a morphism $s : H \rightarrow A$ such that $j \circ s = id_H$.

$$0 \longrightarrow E \xrightarrow{i} A \begin{array}{c} \xrightarrow{j} \\ \xleftarrow{s} \end{array} H \longrightarrow 0 \quad (11)$$

The name comes from the fact that in some good situations, the category \mathbf{Alg} have a notion of “product”, say $\#$, such that a sequence (9) is splitting iff A “splits” as $A \simeq E\#H$. For instance, if \mathbf{Alg} is an abelian category, then the Splitting Lemma shows that such a product $\#$ is just the coproduct \oplus , i.e, the “direct sum”. For general groups or Lie algebras, $\#$ is the corresponding notion of semi-direct product/sum.

Now, assume that the ambient category \mathbf{Alg} is actually monoidal, meaning that we have a fixed bifunctor $\otimes : \mathbf{Alg} \times \mathbf{Alg} \rightarrow \mathbf{Alg}$ and an object $1 \in \mathbf{Alg}$ such that \otimes is associative (up to natural isomorphisms) and has 1 as a neutral object (also up to natural isomorphisms). We can then talk of *monoids in \mathbf{Alg}* . These are objects $X \in \mathbf{Alg}$ endowed with morphisms $*$: $X \otimes X \rightarrow X$ and $e : 1 \rightarrow X$ which satisfy the associativity-type and neutral element-type diagrams.

The main point is that we can use sections morphisms to transfer (to pullback) monoid structures. Indeed, if $f : X \rightarrow Y$ is a morphism with section $s : Y \rightarrow X$, then for any monoid structure $(*, e)$ in Y we get a corresponding monoid structure $(*', e')$ in X with $*$ ' as shown below and $e' = s \circ e$. In particular, if in an splitting extension (11) H is a monoid, then we can use the section $s : H \rightarrow A$ to get a monoid structure in A .

$$X \otimes X \xrightarrow{f \otimes f} Y \otimes Y \xrightarrow{*} Y \xrightarrow{s} X \quad (12)$$

$\xleftarrow{\quad *' \quad}$

In the case when \mathbf{Alg} admits a product $\#$ characterizing splitting extensions it is natural to write $*_{\#}$ and $e_{\#}$ (instead of $*$ ' and e') to denote the monoid structure induced on an extension.

Example 2.6. Let us take the category $\mathbf{Vec}_{\mathbb{R}}$ endowed with the monoidal structure given by the tensor product $\otimes_{\mathbb{R}}$. Its monoid objects are just real algebras. From the last paragraphs, if A is a splitting extension of a vector space H by another vector space E and H is an algebra, say with product $*$, then A is automatically an algebra with product $*_{\oplus}$. Therefore, for a given manifold P we will have not only a wedge product \wedge_* between H -valued forms, but also a product $\wedge_{*_{\oplus}}$. In the very particular case when H is a matrix algebra, recall that \wedge_* is denoted by \wedge , which motivate us to denote the corresponding $\wedge_{*_{\oplus}}$ by \wedge_{\oplus} . It is exactly this kind of multiplication that appears in (5).

2.3 (k, s) -Nil Algebras

In the last subsection we showed how to build the products that will enter in the abstract definition of EHP theory. Here we will discuss the nilpotency conditions that will be imposed into these products in order to get obstruction theorems for the corresponding EHP theory.

We start by recalling that an element $v \in A$ in an algebra $(A, *)$ has *nilpotency degree s* if $v^s \neq 0$, but $v^{s+1} = 0$, where $v^i = v * \dots * v$. In turn, the nilpotency degree of A is the minimum over the nilpotency degree of its elements. An algebra with non-zero nilpotency degree is called a *nil algebra*. This “nil” property can also be characterized as a PI: A is nil iff for some $s \neq 0$ the polynomial $p_1(x) = x^s$ does not vanish identically, but x^{s+1} does.

The first nontrivial examples of nil algebras are the anti-commutative algebras, which include Lie algebras, for which the nilpotency degree is $s = 1$. For such kind of objects we usually consider the more restrictive notion of *nilpotent algebra*. Indeed, we say that A is *nilpotent* of degree s if not only $p_1(x) = x^{s+1}$ vanishes, but also

$$p_{s+1}(x_1, \dots, x_{s+1}) = x_1 \cdot \dots \cdot (x_{s-1} \cdot (x_s \cdot x_{s+1}))$$

and any other polynomial obtained from p_{s+1} by changing the ordering of the parenthesis. In general, being nilpotent is much stronger than being nil. For associative algebras, on the other hand, such concepts coincide [41, 36]. A fact more easy to digest is that an anti-commutative and associative algebra A is nilpotent iff it is an *Engel's algebra*, in the sense that the PI's

$$p_{s,1}(x, y) = p_{s+1}(x, \dots, x, y) = 0$$

are satisfied for any “ p_{s+1} -type” polynomial. Due to Engel's theorem, other important examples of Engel's algebras are the Lie algebras.

From Proposition 2.1, if an algebra A is nil or nilpotent, then the corresponding graded-algebra of A -valued exterior forms, with the product \wedge_* , have the same properties, but in the **graded** sense. Notice that $x^{s+1} = 0$ iff $x^{s+1} = -x^{s+1}$, so that the graded-nil condition becomes $x^{s+1} = (-1)^{k^{s+1}+1}x^{s+1}$, where $k = \deg x$, which is nontrivial only when k is even. The conclusion is the following: *if $(A, *)$ is nil with nilpotency degree $s > 0$, then $\Lambda^{\text{even}}(P; A)$ is graded-nil with the same degree.*

The next example sets Lemma 2.1 in terms of this new language.

Example 2.7. If A is anti-commutative, then it is nil with degree $s = 1$, which implies that even A -valued forms are also nil with degree $s = 1$, i.e. $\alpha \wedge_* \alpha = 0$ for even forms. If A and B are nil of respective degrees r and s , then the direct sum $A \oplus B$ is also nil, with degree given by $\min\{r, s\}$. Subalgebras of nil algebras are also nil algebras. Consequently,

$$A \hookrightarrow A_1 \oplus \dots \oplus A_r$$

is nil when each A_i is nil. This is exactly a generalization of Lemma 2.1.

The same kind of discussion applies to the nilpotent property. The analysis is easier when A is skew-commutative and satisfies an associativity or Jacobi identity, because we can work with $p_{s,1}$ instead of p_{s+1} . Noting that $p_{s,1}$ is a PI iff $x^s \cdot y = -x^s \cdot y$ we conclude that the graded-nilpotent property is described by

$$x^s \cdot y = (-1)^{k^{sl}+1}(x^s \cdot y),$$

where $k = \deg x$ and $l = \deg y$. This condition is nontrivial iff $k^{sl} + 1$ is odd, i.e. iff k^{sl} is even, which implies that k and l have the same parity. Summarizing: *if A is nilpotent of degree s , then $\Lambda^{\text{odd}}(P; A)$ or $\Lambda^{\text{even}}(P; A)$ is graded-nilpotent with the same degree.*

Notice that the nil property gives nontrivial conditions only on even-degree forms. On the other hand, while the nilpotency property can be used to give nontrivial conditions on even-degree or odd-degree forms, it is too strong for most purposes. This leads us to define an intermediary concept of (k, s) -nil algebra containing both nil and nilpotent algebras as particular examples. Indeed, given integers $k, s > 0$, we will say that an algebra $(A, *)$ is (k, s) -nil if in the corresponding graded-algebra $\Lambda(P; A)$ every A -valued k -form has nilpotency degree s .

2.4 (k, s) -Solv Algebras

In this subsection we will explore more examples of (k, s) -nil algebras. In the study of Lie algebras (as well as of other kinds of algebras), there is a concept of *solvable algebra* which is closely related to the concept of *nilpotent algebra*. Indeed, to any algebra A we can associate two decreasing sequences A^n and $A^{(n)}$ of ideals, respectively called the *lower central series* and the *derived series*, inductively defined as follows:

$$A^0 = A, \quad A^k = \bigoplus_{i+j=k} A^i * A^j \quad \text{and} \quad A^{(0)} = A, \quad A^{(k)} = A^{(k-1)} * A^{(k-1)},$$

where, for given subsets $X, Y \subset A$, by $X * Y$ we mean the ideal generated by all products $x * y$, with $x \in X$ and $y \in Y$. Clearly, the polynomials like p_{s+1} are PI's for A iff $A^{s+1} = 0$. So, A is nilpotent of degree s iff its lower central series stabilizes in zero after $s + 1$ steps. Analogously, we say that A is *solvable of solvability degree s* if its derived series stabilizes in zero after $s + 1$ steps.

We notice that $A^{(k+1)}$ can be regarded not only as an ideal of A , but indeed as an ideal of $A^{(k)}$. Furthermore, the quotient $A^{(k)}/A^{(k+1)}$ subalgebra is always commutative [36]. Starting with $A^{(1)}$ we get the first exact sequence (the first line) below. Because we are working over fields, the quotient is a free module and then the sequence splits, allowing us to write $A \simeq A^{(1)} \oplus A/A^{(1)}$. Inductively we then get $A \simeq A'_s \oplus \dots \oplus A'_0$, where $A'_i := A^{(i)}/A^{(i+1)}$. Summarizing: *as a vector space, a solvable algebra can be decomposed into a finite sum of spaces each of them endowed with a structure of commutative algebra.*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^{(s)} & \longrightarrow & A^{(s-1)} & \longrightarrow & A/A^{(1)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \\ 0 & \longrightarrow & A^{(2)} & \longrightarrow & A^{(1)} & \longrightarrow & A^{(1)}/A^{(2)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A^{(1)} & \longrightarrow & A & \longrightarrow & A/A^{(1)} \longrightarrow 0 \end{array}$$

With the remarks above in our minds, let us prove that any solvable algebra is “almost” a (k, s) -nil algebra.

Proposition 2.2. *Let $(A, *)$ be a solvable algebra and let α be an A -valued k -form in a smooth manifold P . If k is odd and α is pointwise injective, then $\alpha \wedge_* \alpha = 0$.*

Proof. Let $A \simeq A'_s \oplus \dots \oplus A'_0$ be the decomposition above. Given a k -form α , assume that α_a is injective for every $a \in P$. Then, by the isomorphism theorem, α_a induces an isomorphism from its domain on its image. Notice that the subspace $\text{img}(\alpha) \subset A$ can be decomposed as $\overline{A}_s \oplus \dots \oplus \overline{A}_0$, where $\overline{A}_i = \text{img}(\alpha) \cap A'_i$. Let V_i be the preimage $\alpha_a^{-1}(\overline{A}_i)$, so that the domain of α_a decomposes as $V'_s \oplus \dots \oplus V'_1$, allowing us to write $\alpha_a = \alpha_a^s + \dots + \alpha_a^0$. We assert that $\alpha \wedge_* \alpha = 0$. From the bilinearity of \wedge_* it is enough to verify that $\alpha_a^i \wedge_* \alpha_a^j = 0$ for every i, j and $a \in P$. If $i \neq j$ this is immediate because α_a^i and α_a^j are nonzero in different subspaces. So, let us assume $i = j$. In this

case, since A is solvable, the algebras A'_i are commutative, so that by the discussion in Section 2 we have $\beta \wedge_* \beta = 0$ for every A'_i -valued odd-degree form and, in particular, $\alpha_a^i \wedge_* \alpha_a^i = 0$. \square

Up to the injectivity hypothesis, the last proposition is telling us that solvable algebras are $(k, 1)$ -nil algebras for every k odd. So, we can say that solvable algebras are $(k, 1)$ -nil “on the class of injective forms”. This motivates us to define the following: given a class $\mathcal{C}^k \subset \Lambda^k(P; A)$ of A -valued k -forms, we say that A is a (\mathcal{C}^k, s) -nil algebra if any A -valued form belonging to \mathcal{C}^k has nilpotency degree s .

We are now in position of generalizing the last proposition. Indeed, the structure of its proof is very instructive in the sense that it can be easily abstracted by noticing that the “solvable” hypothesis over A was used only to get a decomposition of $A \simeq A'_s \oplus \dots \oplus A'_0$ in terms of commutative algebras. The commutativity, in turn, was important only to conclude $\alpha \wedge_* \alpha = 0$. Therefore, using the same kind of proof we immediately obtain an analogous result if we consider algebras A endowed with a vector space decomposition $A \simeq A_s \oplus \dots \oplus A_0$ where each A_i is a (k, s) -nil algebra. We will call this kind of algebras (k, s) -solv algebras, because they generalize solvable algebras in the same sense as (k, s) -nil algebras generalize nil and nilpotent algebras.

Summarizing, in this new terminology we have the following result.

Proposition 2.3. *Every (k, s) -solv algebra is (\mathcal{C}^k, s) -nil over the class of pointwise injective forms.*

Remark 2.4. Exactly as nilpotent algebras are always solvable, (k, s) -nil algebras are (k, s) -solv. The last proposition shows that the reciprocal is almost true. On the other hand, we could have analogously defined (\mathcal{C}^k, s) -solv algebras and, in this case, if \mathcal{C}^k is contained in the class of pointwise injective forms, the last proposition would be rephrased as stating an equivalence between the concepts of (\mathcal{C}^k, s) -solv algebras and (\mathcal{C}^k, s) -nil algebras.

Remark 2.5. In Subsection 4.2 we will work with special vector subspaces of a given algebra. Let us seize the opportunity to introduce them. Given an arbitrary algebra A , we say that a vector subspace $V \subset A$ is a (k, s) -nil subspace of A if every V -valued k -form has nilpotency degree s when regarded as a A -valued form. Similarly, we say that V is a (k, s) -solv subspace if it decomposes as a sum of (k, s) -nil subspaces. When $A = \bigoplus_m A^m$ is \mathfrak{m} -graded, there are other kind of subspaces $V \subset A$ that can be introduced. For instance, we say that V is graded (k, s) -solv if each $V_m = V \cap A_m$ is a (k, s) -solv subspace. In any subspace V of a \mathfrak{m} -graded algebra we get a corresponding grading by $V \simeq \bigoplus_m V_m$. So, any V -valued form α can be written as $\alpha = \sum_m \alpha^m$. We say that V is weak (k, s) -nilpotent if for every k -form α , any polynomial $p_{s+1}(\alpha^{m_1}, \dots, \alpha^{s+1})$ vanishes. Similarly, we say that V is weak (k, s) -solvable if it decomposes as a sum of weak (k, s) -nilpotent subspaces.

2.5 Functorial Algebra Bundle System

Now, let us discuss the last ingredient before applying Geometric Obstruction Theory to EHP theories. In previous subsections, P was an arbitrary smooth manifold. Let us now assume that it is the total space of a G -bundle $\pi : P \rightarrow M$. EHP theories (which are our aim) are not about forms on the total space P , but about forms on the base manifold M . So, we need some process allowing us to replace A -valued forms in P by forms in M with coefficients in some other bundle, say E_A . More precisely, we are interested in rules assigning to every pair (P, A) a corresponding algebra bundle E_A , whose typical fiber is A , in such a way that there exists a graded-subalgebra $S(P, A) \subset \Lambda(P; A)$ and a canonical morphism $j : S(P, A) \rightarrow \Lambda(M; E_A)$.

It is more convenient to think of this in categorical terms. Let $\mathbf{Alg}_{\mathbb{R}}$ be the category of real finite-dimensional⁴ algebras, $\mathbb{Z}\mathbf{Alg}_{\mathbb{R}}$ be the category of \mathbb{Z} -graded real algebras and, given a manifold M , let \mathbf{Bun}_M and $\mathbf{Alg}_{\mathbb{R}}\mathbf{Bun}_M$ denote the categories of bundles and of \mathbb{R} -algebra bundles over M , respectively. As in the first diagram below, we have two canonical functors, which assign to each pair (A, P) the corresponding superalgebra of A -valued forms in P , and to each algebra bundle E over M the graded algebra of E -valued forms in M . We also have the projection $(P, A) \mapsto A$. So, our problem of passing from forms in P to forms in M could be solved by searching for a functor F making commutative the second diagram below.

$$\begin{array}{ccc}
 & \mathbf{Alg}_{\mathbb{R}}\mathbf{Bun}_M & \mathbf{Alg}_{\mathbb{R}} \overset{E_-}{\dashrightarrow} \mathbf{Alg}_{\mathbb{R}}\mathbf{Bun}_M \\
 & \downarrow \Lambda(M; -) & \downarrow \Lambda(M; -) \\
 \mathbf{Bun}_M \times \mathbf{Alg}_{\mathbb{R}} & \xrightarrow{\Lambda(-; -)} \mathbb{Z}\mathbf{Alg}_{\mathbb{R}} & \mathbf{Bun}_M \times \mathbf{Alg}_{\mathbb{R}} \xrightarrow{\Lambda(-; -)} \mathbb{Z}\mathbf{Alg}_{\mathbb{R}} \\
 & & \uparrow \pi_1
 \end{array}$$

But this would be a stronger requirement; for instance, it would imply the equality of two functors, a fact that can always be weakened by making use of natural transformations. Therefore, we could search for functors E_- endowed with natural transformations j , as in the diagram below. This condition would remain stronger than we need: it requires that for **any** bundle $P \rightarrow M$ and for **any** algebra A we have a canonical algebra morphism $\Lambda(P; A) \rightarrow \Lambda(M; E_A)$. We would like to include rules that are defined only for certain classes of bundles and algebras. This leads us to work in subalgebras \mathbf{C} of $\mathbf{Bun}_M \times \mathbf{Alg}_{\mathbb{R}}$ as in the second diagram below.

$$\begin{array}{ccc}
 \mathbf{Alg}_{\mathbb{R}} \overset{E_-}{\dashrightarrow} \mathbf{Alg}_{\mathbb{R}}\mathbf{Bun}_M & & \mathbf{Alg}_{\mathbb{R}} \overset{E_-}{\dashrightarrow} \mathbf{Alg}_{\mathbb{R}}\mathbf{Bun}_M \\
 \uparrow \pi_1 & \searrow j & \uparrow \pi_1 \\
 \mathbf{Bun}_M \times \mathbf{Alg}_{\mathbb{R}} & \xrightarrow{\Lambda(-; -)} \mathbb{Z}\mathbf{Alg}_{\mathbb{R}} & \mathbf{C} \xrightarrow{\Lambda(-; -)} \mathbb{Z}\mathbf{Alg}_{\mathbb{R}} \\
 & & \downarrow \Lambda(M; -)
 \end{array}$$

But, once again, we should consider weaker conditions. Indeed, the above situation requires that for $(P, A) \in \mathbf{C}$ the algebra morphism $\Lambda(P; A) \rightarrow \Lambda(M; E_A)$ is “canonical” in the entire algebra $\Lambda(P; A)$. It may happen that Λ be “canonical” only in some subalgebra $S(P; A) \subset \Lambda(P; A)$, meaning that it is first defined in $S(P; A)$ and then trivially extended to $\Lambda(P; A)$. Therefore, the correct approach seems to be to replace Λ by another functor S endowed with an objectwise injective transformation $\iota : S \Rightarrow \Lambda$, as in the following diagram.

$$\begin{array}{ccc}
 \mathbf{Alg}_{\mathbb{R}} \overset{E_-}{\dashrightarrow} \mathbf{Alg}_{\mathbb{R}}\mathbf{Bun}_M & & \\
 \uparrow \pi_1 & \searrow j & \downarrow \Lambda(M; -) \\
 \mathbf{C} & \xrightarrow{S(-, -)} \mathbb{Z}\mathbf{Alg}_{\mathbb{R}} & \\
 & \searrow \iota & \\
 & \Lambda(-; -) &
 \end{array}$$

In sum, given a manifold M , the last diagram describes, in categorical terms, the transition between algebra-valued forms in bundles over M and algebra bundle-valued forms in M . The input

⁴Actually, we can work in the infinite-dimensional setting, but in this case we have to take many details into account. For instance, we would need to work with topological algebras, bounded linear maps, etc.

needed to do this transition corresponds to the dotted arrows in the last diagram. We will say that they define a *functorial algebra bundle system* (FABS) for the manifold M . Concretely, a FABS for M consists of

1. a category \mathbf{C} of pairs (P, A) , where $P \rightarrow M$ is a bundle and A is a real algebra;
2. a functor E_- assigning to any algebra $A \in \mathbf{C}$ a corresponding algebra bundle $E_A \rightarrow M$ whose typical fiber is A ;
3. a functor $S(-; -)$ that associates an algebra to each pair $(P, A) \in \mathbf{C}$;
4. natural transformations $\iota : S(-; -) \Rightarrow \Lambda(-; -)$ and $\xi : S(-; -) \Rightarrow \Lambda(M; -)$ such that ι is objectwise injective.

Such systems always exists, as showed by the next examples. The fundamental properties and constructions involving FABS will appear in a work under preparation [27].

Example 2.8 (*trivial case*). Starting with any subcategory \mathbf{C} of pairs (P, A) that contains only the trivial algebra, up to natural isomorphisms there exists a single E_- : the constant functor at the trivial algebra bundle $M \times 0 \rightarrow M$. Noting that $\Lambda(M; M \times 0) \simeq 0$ since the trivial algebra is a terminal object in $\mathbf{Alg}_{\mathbb{R}}$, independently of the choice of S and $\iota : S \Rightarrow \Lambda$ there is only one $j : S \Rightarrow \Lambda(M; -)$: the trivial one.

Example 2.9 (*almost trivial case*). The last situation has the defect that it can be applied only for subcategories \mathbf{C} whose algebraic part is trivial. It is easy, on the other hand, to build FABS for arbitrarily given algebras. Indeed, let \mathbf{C} be defined by pairs (P, A) , where A is an arbitrary algebra, but P is the trivial principal bundle $M \times GL(A) \rightarrow M$. Putting E_A as the trivial algebra bundle $M \times A \rightarrow A$, we have a 1-1 correspondence between A -valued forms in P and forms in M with values in E_A . Therefore, we actually have a map $j : \Lambda(P; A) \rightarrow \Lambda(M; E_A)$ leading us to take $S = \Lambda$ and $\iota = j$.

Example 2.10 (*standard case*). In the first example we considered \mathbf{C} containing arbitrary bundles, but we paid the price of working only with trivial algebras. In the second example we were faced with a dual situation. A middle term can be obtained by working with pairs (P, A) , where $P \rightarrow M$ is a G -bundle whose group G becomes endowed with a representation $\rho : G \rightarrow GL(A)$. In that case we define E_- as the rule assigning to each A the corresponding associated bundle $P \times_{\rho} A$. The functor S is such that $S(P, A)$ is the algebra $\Lambda_{\rho}(P; A)$ of ρ -equivariant A -valued forms α in P , i.e, of those satisfy the equation $R_g^* \alpha = \rho(g^{-1}) \cdot \alpha$, where here $R : G \times P \rightarrow P$ is the canonical free action characterizing P as a principal G -bundle. This algebra of ρ -equivariant forms naturally embeds into $\Lambda(P; A)$, giving ι . Finally, it is a standard fact [24] that each ρ -equivariant A -valued form on P induces an $P \times_{\rho} A$ -valued form on M , defining the transformation ξ . This is the standard approach used in the literature, so that we will refer to it as the *standard FABS*.

Example 2.11 (*canonical case*). There is an even more canonical situation: that obtained from the latter by considering only associative algebras underlying matrix Lie algebras. Indeed, in this case we have also a distinguished representation $\rho : G \rightarrow GL(\mathfrak{g})$, given by the adjoint representation. The corresponding bundles $E_{\mathfrak{g}}$ correspond to what is known as the *adjoint bundles*, leading us to say that this is the *adjoint FABS*.

3 Concrete Obstructions

After the algebraic prolegomena developed in the previous section, we are ready to introduce and study EHP theories in the general setting. We will start in Subsection 3.1 by considering a very simple framework, modeled by inclusions of linear groups, where our first fundamental obstruction theorem is obtained. At each following subsection the theory will be redefined (in the direction of the full abstract context) in such a way that the essence of the obstruction theorem will remain the same.

Thus, in Subsection 3.2 we define a version of EHP for reductions $H \hookrightarrow G$ such that G is not a linear group, but a splitting extension of H . We then show that, as desired, the fundamental obstruction theorem remains valid, after minor modifications.

Independently, if one is interested in matrix EHP theories or EHP theories arising from splitting extensions, we also work with reductive Cartan connections. These are 1-forms $\nabla : TP \rightarrow \mathfrak{g}$ that decompose as $\nabla = e + \omega$, where $e : TP \rightarrow \mathfrak{g}/\mathfrak{h}$ is a pointwise isomorphism and $\omega : TP \rightarrow \mathfrak{h}$ is an H -connection. What happens if we forget the pointwise isomorphism hypothesis on e ? In Subsection 3.3 we discuss the motivations to do this and we show that the fundamental obstruction theorem not only holds, but becomes stronger.

Finally, in Subsection 3.5 we show that if we restrict our attention to connections $\nabla = e + \omega$ such that ω is torsion-free, then we get, as a consequence of Berger's classification theorem, a new obstruction result which is independent of the fundamental obstruction theorem. In particular, this new result implies topological obstructions to the spacetime in which the EHP theory is being described.

3.1 Matrix Gravity

Let us start by considering a smooth G -bundle $P \rightarrow M$ over an orientable smooth manifold M , with G a linear group, and fix a group structure reduction $H \hookrightarrow G$ of P . Given an integer $k > 0$ and $\Lambda \in \mathbb{R}$, we define the *homogeneous* and the *inhomogeneous linear* (or *matrix*) *Hilbert-Palatini form* of degree k of a reductive Cartan connection $\nabla = e + \omega$ in P , relative to the group structure reduction $H \hookrightarrow G$, as

$$\alpha_k = e \wedge \dots \wedge e \wedge \Omega \quad \text{and} \quad \alpha_{k,\Lambda} = \alpha_k + \frac{\Lambda}{(k-1)!} e \wedge \dots \wedge e, \quad (13)$$

respectively. In α_k the term e appears $(k-2)$ -times, while in right-hand part of $\alpha_{k,\Lambda}$ it appears k -times. Because we are working with matrix Lie algebras we have the adjoint FABS, so that these \mathfrak{g} -valued forms correspond to k -forms in M with values in the adjoint bundle $P \times_{\text{ad}} \mathfrak{g}$, respectively denoted by $j(\alpha_k)$ and $j(\alpha_{k,\Lambda})$.

We define the *homogeneous* and the *inhomogeneous linear EHP theories* in P with respect to $H \hookrightarrow G$ as the classical field theories whose spaces of configurations are the spaces of reductive Cartan connections $\xi = e + \omega$ and whose action functionals are respectively given by

$$S_n[e, \omega] = \int_M \text{tr}(j(\alpha_n)) \quad \text{and} \quad S_{\Lambda,n}[e, \omega] = \int_M \text{tr}(j(\alpha_{n,\Lambda})). \quad (14)$$

Warning: The expression ‘‘linear EHP’’ is used here to express the fact that the EHP action is realized in a geometric background defined by linear groups. It **does not** mean that the equations of motion were linearized or something like this.

Remark 3.1. A priori, H is an arbitrary subgroup of G , which implies that \mathfrak{h} is a Lie subalgebra of \mathfrak{g} . It would be interesting to think of \mathfrak{g} as the Lie algebra extension of some \mathfrak{r} by \mathfrak{h} . This makes sense only when $\mathfrak{h} \subset \mathfrak{g}$ is not just a subalgebra, but an ideal, which is true iff $H \subset G$ is a **normal** subgroup.

With the remark above in mind, we can state our first obstruction theorems. In them we will consider theories for reductions $H \hookrightarrow G$ fulfilling one of the following conditions:

- (C1) the subgroup $H \subset G$ is normal and, as a matrix algebra, \mathfrak{g} is an splitting extension by \mathfrak{h} of a (k, s) -nil algebra \mathfrak{r} .
- (C2) as a matrix algebra, \mathfrak{g} is a (k, s) -nil algebra.

Theorem 3.1. *Let M be an n -dimensional spacetime and $P \rightarrow M$ be a G -bundle, endowed with a group reduction $H \hookrightarrow G$ fulfilling (C1) or (C2). If $n \geq k + s + 1$, then the linear inhomogeneous EHP theory equals the homogeneous ones. If $n \geq k + s + 3$, then both are trivial.*

Proof. Assume (C1). As vector spaces we can write $\mathfrak{g} \simeq \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{h}$ and, because \mathfrak{g} is a Lie algebra extension of \mathfrak{r} by \mathfrak{h} , it follows that $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{r}$. How the extension splitting, \mathfrak{r} is a subalgebra of \mathfrak{g} and $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{r}$ is indeed a Lie algebra isomorphism, So, $\mathfrak{g}/\mathfrak{h}$ can be regarded as a (k, s) -nil algebra and, therefore, $\lambda^{s+1}\alpha = 0$ for every $\mathfrak{g}/\mathfrak{h}$ -valued k -form in P . If $\nabla = e + \omega$ is a reductive Cartan connection in P relative to $H \hookrightarrow G$, then e is a $\mathfrak{g}/\mathfrak{h}$ -valued 1-form in P and, because $\mathfrak{g}/\mathfrak{h}$ is a subalgebra of \mathfrak{g} , $\lambda^k e$ is a $\mathfrak{g}/\mathfrak{h}$ -valued k -form. Consequently,

$$(\lambda^{k+s+1}e) \lrcorner \alpha = 0 \tag{15}$$

for any \mathfrak{g} -valued form α . In particular, for

$$\alpha = \frac{\Lambda}{(n-1)!} \lambda^{n-(k+s+1)} e$$

(15) is precisely the inhomogeneous part of $\alpha_{\Lambda, n}$. Therefore, for every $n \geq k + s + 1$ we have $\alpha_{\Lambda, n} = \alpha_n$ and, consequently, $j(\alpha_k) = j(\alpha_{\Lambda, n})$, implying $S_n[e, \omega] = S_{n, \Lambda}[e, \omega]$ for any Cartan connection $\nabla = e + \omega$, and then $S_n = S_{n, \Lambda}$. On the other hand, for

$$\alpha = (\lambda^{n-(k+s+1)+2}e) \lrcorner \Omega$$

we see that (15) becomes exactly α_n . Therefore, when $n \geq k + s + 3$ we have $\alpha_n = 0$, implying (because j is linear) $j(\alpha_n) = 0$ and then $S_n = 0$. But $k + s + 3 > k + s + 1$, so that $S_{n, \Lambda} = 0$ too. This ends the proof when the reduction $H \hookrightarrow G$ fulfills condition (C1). If we assume (C2) instead of (C1), we can follow exactly the same arguments, but now thinking of e as a \mathfrak{g} -valued 1-form. \square

Recall that, as remarked in Section 2, examples of (k, s) -nil algebras \mathfrak{g} include nil and nilpotent algebras. In particular, from Lemma 2.1 we see that subalgebras of $\mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{s}(k_r)$ are (k, s) -nil for every (k, s) such that k is even and $s \geq 1$. This leads us to the following corollary:

Corollary 3.1. *For subalgebras as above, in a spacetime of dimension $n \geq 4$, the inhomogeneous and the homogeneous linear EHP are equal. If $n \geq 6$ both are trivial.*

3.2 Gravity Arising From Extensions

Now, let us move on to a slightly more abstract situation: when G is not a linear group, but a splitting extension of a given linear group H , i.e, we will deal with group reductions $H \hookrightarrow \mathbb{R}^k \rtimes H$. First of all, notice that if $P \rightarrow M$ is a bundle structured over $H \subset GL(k; \mathbb{R})$, then we can always extend its group structure to $\mathbb{R}^k \rtimes H$. Indeed, up to isomorphisms this bundle is classified by a map $f : M \rightarrow BH$ and getting a group extension is equivalent to lifting f as shown below. It happens that this lifting actually exists, since we are working with splitting extensions (second diagram). It is in this context that we will now internalize EHP theories.

$$\begin{array}{ccc}
 & \mathbb{R}^k \rtimes H & \\
 & \nearrow & \downarrow j \\
 M & \xrightarrow{f} & H
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \mathbb{R}^k \rtimes H & \\
 & \nearrow^{sof} & \downarrow j \Big) s \\
 M & \xrightarrow{f} & H
 \end{array}$$

Thus, given a linear group H , consider bundles $P \rightarrow M$ endowed with group reductions $H \hookrightarrow \mathbb{R}^k \rtimes H$. As previously, fixed $k > 0$ and $\Lambda \in \mathbb{R}$ we define *homogeneous* and *inhomogeneous extended-linear Hilbert-Palatini form* of reductive Cartan connections $\nabla = e + \omega$ in P as

$$\alpha_k = e \wedge_{\times} \dots \wedge_{\times} e \wedge_{\times} \Omega \quad \text{and} \quad \alpha_{k,\Lambda} = \alpha_k + \frac{\Lambda}{(k-1)!} e \wedge_{\times} \dots \wedge_{\times} e. \quad (16)$$

Furthermore, fixed a FABS, the corresponding *extended-linear EHP theories* are defined by (14). At first sight, the only difference between the “extended-linear” and the “linear” theories is that we now consider the induced product \wedge_{\times} instead of \wedge . However, in the context of Geometric Obstruction Theory, this replacement makes a fundamental difference. For instance, the condition (C1) used to get Theorem 3.1 no longer makes sense, because $H \hookrightarrow \mathbb{R}^k \rtimes H$ generally is **not** a normal subgroup. Immediate substitutes for (C1) and (C2) are

(C1’) as an associative algebra $(\mathfrak{h}, \wedge_{\times})$ is (k, s) -nil;

(C2’) as an associative algebra, $(\mathbb{R}^k \rtimes \mathfrak{h}, \wedge_{\times})$ is (k, s) -nil.

But, differently from (C1) and (C2), these new conditions are intrinsically related. It is obvious that (C2)’ implies (C1)’, because \mathfrak{h} is a subalgebra of $\mathbb{R}^k \rtimes \mathfrak{h}$. The reciprocal is also valid, as shown in the next lemma.

Lemma 3.1. *Let $(A, *)$ be an algebra, B a vector space and $f : A \rightarrow B$ a linear map that admits a section $s : B \rightarrow A$. In this case, if $(A, *)$ is (k, s) -nil, then $(B, *')$ is too, where $*'$ is the pulled-back multiplication (12).*

Proof. Recall that any algebra X induces a corresponding graded-algebra structure in $\Lambda(P; X)$ according to (7). Due to the functoriality of $\Lambda(P; -)$, we then get the commutative diagram below, where the horizontal rows are just (7) for the algebras $(A, *)$ and $(B, *')$, composed with the diagonal map. The commutativity of this diagram says just that $\wedge_{*'}^2 \alpha = \wedge_*^2 f(\alpha)$ for every $\alpha \in \Lambda(P; B)$. From the same construction we get, for each given $s \geq 2$, a commutative diagram that implies

$\wedge_*^s \alpha = \wedge_*^s f(\alpha)$. Therefore, if $(A, *)$ is (k, s) -nil it immediately follows that $(B, *')$ is (k, s) -nil too.

$$\begin{array}{ccccccc}
\Lambda(P; B) & \xrightarrow{\Delta} & \Lambda(P; B) \otimes \Lambda(P; B) & \longrightarrow & \Lambda(P; B \otimes B) & \longrightarrow & \Lambda(P; B) \\
\Lambda s \uparrow \downarrow \Lambda f & & (\Lambda s) \otimes (\Lambda s) \uparrow \downarrow (\Lambda f) \otimes (\Lambda f) & & \Lambda(s \otimes s) \uparrow \downarrow \Lambda(f \otimes f) & & \Lambda s \uparrow \downarrow \Lambda f \\
\Lambda(P; A) & \xrightarrow{\Delta} & \Lambda(P; A) \otimes \Lambda(P; A) & \longrightarrow & \Lambda(P; A \otimes A) & \longrightarrow & \Lambda(P; A)
\end{array}$$

□

As a consequence of the lemma above, the version of Theorem (3.1) for extended-linear EHP is the following:

Theorem 3.2. *Let M be an n -dimensional spacetime and $P \rightarrow M$ be a H -bundle, where H is a linear group such that (\mathfrak{h}, \wedge) is (k, s) -nil. If $n \geq k + s + 1$, then the inhomogeneous extended-linear EHP theory equals the homogeneous ones. If $n \geq k + s + 3$, then both are trivial.*

Proof. From the last lemma we can assume that $(\mathfrak{g}, \wedge_\times)$ with $\mathfrak{g} = \mathbb{R}^l \rtimes \mathfrak{h}$ is (k, s) -nil. An argument similar to that of the (C2)-case in Theorem 3.1 gives the result. □

Corollary 3.2. *In a spacetime of dimension $n \geq 4$, the cosmological constant plays no role in any extended-linear EHP theory with $\mathfrak{h} \subset \mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r)$. If $n \geq 6$ the full theory is trivial.*

Remark 3.2. This simple result will be our primary source of examples of geometric obstructions. This is due to the fact that essentially all “classical” geometry satisfies the hypothesis of the last corollary, implying that EHP cannot be realized in them in higher dimensions.

Remark 3.3. For future reference, notice that any linear EHP theory induces, in a canonical way, a corresponding extended linear EHP theory. Indeed, if $G \subset GL(k; \mathbb{R})$ is a linear Lie group, then any subgroup $H \subset G$ inherits a canonical action on \mathbb{R}^k , allowing us to consider the semidirect product $\mathbb{R}^k \rtimes H$ and then H as a subgroup of it. So, if we start with a linear theory for the inclusion $H \hookrightarrow G$ we can always move to a extended-linear theory for the inclusion $H \hookrightarrow \mathbb{R}^k \rtimes H$. We will refer to this induced theory as the *canonical extended-linear theory associated* to a given linear theory.

3.3 Gauged Gravity

In the last subsection we defined EHP theories in linear geometries $H \hookrightarrow G$ and in extended-linear geometries $H \hookrightarrow \mathbb{R}^k \rtimes H$ as direct analogues of the EHP action functional. This means that we considered theories on (reductive) Cartan connections for the given group reduction, which are pairs $\nabla = e + \omega$, where ω is an usual H -connection and e is a pointwise isomorphism.

In order to regard EHP theories as genuine (reductive) gauge theories, it is necessary to work with arbitrary (i.e, not necessarily Cartan) reductive connections. In practice, this can be obtained by just forgetting the hypothesis of pointwise isomorphism in e . This leads us to define *homogeneous* and *inhomogeneous gauge linear/extended-linear Hilbert-Palatini forms* of a reductive (not necessarily Cartan) connection $\nabla = e + \omega$ as in (13). Similarly, we can then define *homogeneous* and *inhomogeneous gauge linear/extended-linear EHP theories* as in (14), whose configuration space is this new space of arbitrary reductive connections.

A linear map may not be an isomorphism when it is not injective or/and when it is not surjective. We would like that gauge EHP theories have a nice physical interpretation, leading us to ask: *what is the physical motivation for e be injective or/and surjective?*

At least for geometries described by G -structures (which in our context means extend-linear geometries), the quotient $\mathfrak{g}/\mathfrak{h}$ can be identified with some \mathbb{R}^l endowed with a tensor t such that the pair (\mathbb{R}^l, t) is the “canonical model” for the underlying geometry. In such cases, the injectivity hypothesis on $e_a : TP_a \rightarrow \mathfrak{g}/\mathfrak{h}$ is important to ensure that we have a good way to pull-back this “canonical geometric model” to each fiber of TP . For instance, when $G = \mathbb{R}^l \rtimes O(n-1, 1)$ and $H = O(n-1, 1)$, the quotient $\mathfrak{g}/\mathfrak{h}$ is just Minkowski space endowed with its standard metric η , and the injectivity of e implies that $g = e^*\eta$ is also a Lorentzian metric TP . But, even if we forget injectivity, the tensor $g = e^*\eta$ **remains** well-defined, but now it is no longer non-degenerate. Furthermore, for some models of Quantum Gravity, this non-degeneracy is welcome (say to ensure the possibility of topology change [20, 1]), which means that *physically it is really interesting to consider reductive connections $\xi = \omega + e$ such that e is pointwise non-injective*. With this in mind, we will do gauge EHP theories precisely for this kind of reductive connections.

On the other hand, in the context of Geometric Obstruction Theory, it is natural to expect that working with gauge EHP theories such that e is non-injective will impact the theory more than usual. But, how deep will be this impact? Notice that the proofs of Theorem 3.1 and Theorem 3.2 were totally based on the fact that in each point e take values in some (k, s) -nil algebra. In general, as discussed in Subsection 2.4, being (k, s) -solv is weaker than being (k, s) -nil. But, if we are in the class of injective forms, Proposition 2.3 shows that both concepts agree. Therefore, when working with Cartan connections, no new results can be obtained if we replace “ (k, s) -nil” with “ (k, s) -solv”. However, forgetting injectivity we may build stronger obstructions.

In order to get these stronger obstructions, let us start by recalling that until this moment we worked with the class of **reductive** connections. These are \mathfrak{g} -valued 1-forms ξ which, respectively to the vector space decomposition $\mathfrak{g} \simeq \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{h}$, can be globally written as $\xi = e + \omega$. This notion of “reducibility” extends naturally to k -forms with values in A , endowed with some vector space decomposition $A \simeq A_s \oplus \dots \oplus A_0$. Indeed, we say that an A -valued k -form α in a smooth manifold P is *reductive* (or *decomposable*) respectively to the given vector space decomposition if it can be globally written as $\alpha = \alpha_s + \dots + \alpha_0$. We can immediately see that when P is parallelizable every A -valued k -form is reductive respectively to an arbitrarily given decomposition of A . It then follows that *in an arbitrary manifold algebra-valued k -forms are locally reductive*.

After this digression, we state and proof the obstruction result for gauge linear/extended-linear EHP theories. Here, once fixed a group reduction $H \hookrightarrow G$, instead of (C1) and (C2) (which are nilpotency conditions) we will consider the following analogous solvability conditions:

- (S0) the group G is isomorphic to $\mathbb{R}^l \rtimes H$ and (\mathfrak{h}, λ) is (k, s) -solv;
- (S1) the subgroup $H \subset G$ is normal and, as a matrix algebra, (\mathfrak{g}, λ) is a splitting extension by \mathfrak{h} of a (k, s) -solv algebra \mathfrak{r} ;
- (S2) the full algebra (\mathfrak{g}, λ) is a (k, s) -solv algebra.

Theorem 3.3. *Let M be an n -dimensional spacetime and $P \rightarrow M$ be a G -bundle, endowed with a group reduction $H \hookrightarrow G$ fulfilling (S0), (S1) or (S2). If $n \geq k + s + 1$, then the inhomogeneous gauge EHP theory equals the homogeneous ones. If $n \geq k + s + 3$, then both are trivial.*

Proof. By definition, a (k, s) -solv algebra A comes endowed with a decomposition $A \simeq A_s \oplus \dots \oplus A_0$ by (k, s) -nil algebras A_i . Therefore, under condition (S1) if, e is an $\mathfrak{g}/\mathfrak{h}$ -valued 1-form, where $\mathfrak{g}/\mathfrak{h}$ is

(k, s) -solv, the digression above tells us that it can be locally written as $e = e_s + \dots + e_0$. Furthermore, we have

$$\lambda^r e = \lambda^r e_s + \dots + \lambda^r e_0,$$

because for $i \neq j$ the forms e_i and e_j are non-zero in different spaces. Since each e_i take values in a (k, s) -nil algebra, it then follows that $\lambda^{k+s+1} e = 0$ locally, which implies $\lambda^{k+s+1} e \lambda \alpha = 0$ for every \mathfrak{g} -valued form α . The remaining part of the proof is identical to that of Theorem 3.1, starting after equation (15). Case (S2) is analogous to (S1) and (after using Lemma 3.1) case (S0) is analogous to (S2). \square

3.4 Dual Gravity

Here we will see that there are two “dual theories” associated to any EHP theory and we will see how the geometric obstructions of the actual EHP theory relate to the obstructions affecting these new theories.

We start by recalling that the fields in a gauge EHP theory (being it linear or extended-linear) are reductive connections for a given reduction $H \hookrightarrow G$. As a vector space we always have $\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ and these connections are given by 1-forms $e : TP \rightarrow \mathfrak{g}/\mathfrak{h}$ and $\omega : TP \rightarrow \mathfrak{h}$ such that ω is a connection. We can then think of a EHP theory as being determined by four variables: the fields e and ω together with the spaces $\mathfrak{g}/\mathfrak{h}$ and \mathfrak{h} . This allows us to define two types of “dual EHP theories” by interchanging such variables.

More precisely, we define the *geometric dual* of a given EHP theory as that theory having the same space of configurations, but whose action functional ${}^*S_{n,\Lambda}$ is ${}^*S_{n,\Lambda}[e, \omega] = S_{n,\Lambda}[\omega, e]$. Explicitly, for any fixed FABS we have

$${}^*S_{n,\Lambda}[e, \omega] = \int_M \text{tr}(J({}^*\alpha_{n,\Lambda})), \quad \text{with} \quad {}^*\alpha_{n,\Lambda} = \lambda_*^{n-2} \omega \lambda_* E + \frac{\Lambda}{(n-1)!} \lambda_*^n \omega, \quad (17)$$

where $E = de + e \lambda_* e$ and λ_* must be interpreted as λ or λ_\times depending if the starting theory is linear or extended-linear.

On the other hand, we define the *algebraic dual* of a EHP theory as that theory having an action functional with the same shape, but now defined in a “dual configuration space”. This is the theory whose action function $S_{n,\Lambda}^*$ is given by

$$S_{n,\Lambda}^*[e, \omega] = \int_M \text{tr}(J(\alpha_{n,\Lambda}^*)), \quad \text{with} \quad \alpha_{n,\Lambda}^* = \lambda_*^{n-2} e \lambda_* \Omega + \frac{\Lambda}{(n-1)!} \lambda_*^n e, \quad (18)$$

where e and ω take value in \mathfrak{h} and $\mathfrak{g}/\mathfrak{h}$, respectively. Furthermore, as above, λ_* must be interpreted as λ or λ_\times , depending of the case.

In very few words we can say that the *geometric dual* of a given theory makes a change at the *dynamical level*, in the sense that only the *action functional* is changed. Dually, the *algebraic dual* of a given theory produce changes only at the *kinematic level*, meaning that the dualization refers to the *configuration space*. This can be summarized in the table below.

theories / variables	e	ω	$\mathfrak{g}/\mathfrak{h}$	\mathfrak{h}
EHP	e	ω	$\mathfrak{g}/\mathfrak{h}$	\mathfrak{h}
* EHP	ω	e	$\mathfrak{g}/\mathfrak{h}$	\mathfrak{h}
EHP*	e	ω	\mathfrak{h}	$\mathfrak{g}/\mathfrak{h}$

Table 1: Relation between EHP theories and its geometric and algebraic dualizations.

Once such dual theories are introduced, let us now see how the obstruction results for EHP theories apply to them. Let us start by focusing on the geometric dual. Our main result until now is Theorem 3.3. It is essentially determined by two facts. First, e take values in an algebra fulfilling some “solvability condition” (i.e, any one of hypotheses (S0), (S1) or (S2)); and second, the EHP action contains powers of e .

When we look at the action (17) of the geometric dual theory we see that it contains powers of ω (instead of e). Therefore, assuming that ω take values in some “solvable” algebra we will get a result analogous to Theorem 3.3. Looking at Table 1 we identify that in the geometric dual theory ω takes values in \mathfrak{h} which is a subalgebra of \mathfrak{g} , where the product \wedge makes sense, so that the “solvability condition” should be on \mathfrak{h} . Dually, the action (18) also contains powers, now of e . Table 1 shows that here e take values in \mathfrak{h} , so that *if (\mathfrak{h}, \wedge) is “solvable”, then not only the geometric dual theory is trivial, but also the algebraic dual one!* Formally, we have the following result, whose proof follows that of Theorem 3.3

Theorem 3.4. *Let M be an n -dimensional spacetime and $P \rightarrow M$ be a G -bundle, endowed with a group reduction $H \hookrightarrow G$. Assume that (\mathfrak{h}, \wedge) is a (k, s) -solv algebra. If $n \geq k + s + 3$, then both the geometric dual and the algebraic dual of gauge EHP theory are trivial.*

Corollary 3.3. *If condition (S0) is satisfied and $n \geq k + s + 3$, then gauge EHP theory and all its duals are trivial.*

Remark 3.4. In the **linear** context, where G is a linear Lie group and $H \subset G$ is a subgroup, Theorem 3.4 above implies that if (\mathfrak{h}, \wedge) is (k, s) -solv, then the duals of the actual EHP theory are trivial. This **does not** mean that the actual EHP is trivial. In fact, Corollary 3.3 needs condition (S0), which subsumes that we are working in the **extended-linear** context.

3.5 The Role of Torsion

Until this moment we have given conditions under which gauge EHP theories and their duals are trivial. One of this conditions is (S0), which makes sense in the extended-linear context and states that the subalgebra (\mathfrak{h}, \wedge) is (k, s) -solv. Indeed, from Corollary 3.3, if this condition holds then EHP and their duals are all trivial for $n \geq k + s + 3$. Due to Lemma 2.1, the standard examples of $(1, 2)$ -nil algebras are subalgebras of $\mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r)$, so that *for such types of geometry EHP and their duals are trivial if $n \geq 6$* . Here we will see that if we work with a special class of connections, then there are even more obstructions, due essentially to Berger’s classification theorem on Riemannian holonomy [4, 29].

We start by recalling that if ω is a connection on a G -principal bundle $\pi : P \rightarrow M$, each point $a \in P$ determines a Lie subgroup $\text{Hol}(\omega, a) \subset G$, called the *holonomy group of ω in a* , which measures the failure of a loop in M , based in $\pi(a)$, remaining a loop after horizontal lifting [24]. In typical situations, P is the frame bundle FM of a n -manifold M , regarded as a G -structure

under some reduction $G \hookrightarrow GL(n; \mathbb{R})$. Such a G -structure, in turn, generally is determined as the space of frame transformations which preserve some additional tensor field t in M . In this context, the condition $\text{Hol}(\omega, a) \subset G$ is equivalent to saying that there exists one such tensor field which is parallel relative to ω , i.e, such that $\nabla_\omega t = 0$, where ∇_ω is the covariant derivative on tensors induced by ω . Particularly, $SO(n)$ -reductions of FM correspond to Riemannian metrics on an oriented manifold, so that if ω is a connection on FM such that $\text{Hol}(\omega, a) \subset SO(n)$, then $\nabla_\omega g = 0$ for some Riemannian metric g . Consequently, *a torsion-free connection ω in FM whose holonomy is contained in $SO(n)$ is the Levi-Civita connection of some metric.*

Recall that, as pointed in Remark 2.2, for every $k_1 + \dots + k_r = n$ we have a canonical inclusion

$$SO(k_1) \times \dots \times SO(k_r) \hookrightarrow SO(n) \tag{19}$$

as diagonal block matrices. Given a connection ω with special Riemannian holonomy (meaning that it is contained in $SO(n)$) we can ask: *when is it indeed contained in the product subgroup above?* From de Rham decomposition theorem [33, 43] we see that if ω is torsion-free (and, therefore, the Levi-Civita connection of a metric g) and M is simply connected, this happens iff (M, g) is locally isometric⁵ to a product of Riemannian k_i -manifolds (M_i, g_i) , with $i = 1, \dots, r$, and $\text{Hol}(\omega)$ is actually the product of $\text{Hol}(\omega_i) \subset SO(k_i)$, where ω_i is the Levi-Civita connection of g_i .

Assuming (M, g) simply connected and locally irreducible in the above sense, the holonomy reduction (19) does not exist. In this case, it is natural to ask for which proper subgroups $G \hookrightarrow SO(n)$ the holonomy of the Levi-Civita connection of g can be reduced. When (M, g) is not locally isometric to a symmetric space, we have a complete classification of such proper subgroups, given by *Berger's classification theorem*, as in Table 2.

$G \subset SO(n)$	$\dim(M)$	nomenclature
$U(n)$	$2n$	Kähler
$SU(n)$	$2n$	Calabi-Yau
$Sp(n) \cdot Sp(1)$	$4n$	Quaternionic-Kähler
$Sp(n)$	$4n$	Hyperkähler
G_2	7	G_2
$\text{Spin}(7)$	8	$\text{Spin}(7)$

Table 2: Berger's classification theorem for Riemannian signature.

Let us see how we can use Berger's classification theorem to get geometric obstructions similar to those given in Theorem 3.4 and in Corollary 3.3. We will need some definitions. We say that a linear group H is a k -group if there are nonnegative integers k_1, \dots, k_r , with $k_1 + \dots + k_r = k$, such that (\mathfrak{h}, λ) is a subalgebra of $\mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r)$. In other words, k -groups are fundamental examples of $(1, 2)$ -nil algebras. Furthermore, we say that a manifold N is a *Berger k -manifold* if it has dimension k and it is simply connected, locally irreducible and locally non-symmetric. A bundle $P \rightarrow M$ is called *k -proper* if there exists an immersed Berger k -manifold $N \hookrightarrow M$ whose frame bundle is a subbundle of P , i.e, such that $FN \subset i^*P$. The motivating (trivial) examples are the following:

Example 3.1. The frame bundle of a Berger n -manifold is, of course, n -proper.

⁵If (M, g) is geodesically complete, the decomposition is global.

Example 3.2. If $f : N \hookrightarrow M$ an immersion with N a Berger k -manifold, then FM is automatically k -proper, because $FN \subset f^*FM$.

Despite the notions introduced above, in the next theorem we will work with *torsionless (extended-linear) EHP theories*, i.e, extended-linear EHP theories restricted to reductive connections $\nabla = e + \omega$ such that $\Theta_\omega = d_\omega e = 0$. For the case of linear EHP theories, recall that as discussed in Remark 3.3, to any of them we have an associated canonical extended-linear theory.

Theorem 3.5. *Let $P \rightarrow M$ be a k -proper H -bundle over a n -manifold M . If H is a k -group, then $k_1 = k$ and $k_{i>1} = 0$. Furthermore, for any FABS, a torsionless extended-linear EHP theory based on H is nontrivial only if one of the following conditions is satisfied*

(B1) $k = 2, 4$ and M contains a Kähler Berger k -manifold;

(B2) $k = 4$ and M contains a quaternionic-Kähler Berger k -manifold.

Proof. From Lemma 2.1, the hypothesis on \mathfrak{h} implies that it is $(1, 2)$ -nil and, therefore, because we are working with extended-linear theories, condition (S0) is satisfied. Consequently, by Theorem 3.3 the actual (and, in particular, the torsionless) EHP theory is trivial if $n \geq 6$, so that we may assume $n < 6$. Since $P \rightarrow M$ is k -proper, M contains at least one immersed Berger k -manifold $\iota : N \hookrightarrow M$ such that $FN \subset \iota^*P$. Let κ denote the inclusion of FN into ι^*P . On the other hand, we also have an immersion $\iota^*P \hookrightarrow P$, which we denote by ι too. Lie algebra-valued forms can be pulled-back and the pullback preserves horizontability and equivariance. So, for any $\nabla = e + \omega$ the corresponding 1-form $(\iota \circ \kappa)^*\omega \equiv \omega|_N$ is an H -connection in FN and its holonomy is contained in H . By hypothesis H is an k -group so that, via Lie integration,

$$H \subset SO(k_1) \times \dots \times SO(k_r) \subset SO(k).$$

In particular, the holonomy of $\omega|_N$ is contained in $SO(k)$, implying that $\omega|_N$ is compatible with some Riemannian metric g in N . But, we are working with torsion-free connections, so that $\omega|_N$ is actually the Levi-Civita connection of g and, because N is irreducible, de Rham decomposition theorem implies that there exists $i \in 1, \dots, r$ such that $k_i = k$ and $k_j = 0$ for $j \neq i$. Without loss of generality we can take $i = 1$. Because N is a Berger manifold, Berger's theorem applies, implying that the holonomy of $\omega|_N$ is classified by Table 2, giving conditions (B1) and (B2). \square

Corollary 3.4. *Let M be a Berger n -manifold with an H -structure, where H is a n -group. In this case, for any FABS, a torsionless extended-linear EHP based on H is nontrivial only if M has dimension $n = 2, 4$ and admits a Kähler structure.*

Proof. The result follows from the last theorem by considering the bundle $P \rightarrow M$ as the frame bundle $FM \rightarrow M$ and from the fact that every orientable four-dimensional smooth manifold admits a quaternionic-Kähler structure [5, 34]. \square

Remark 3.5. This corollary shows how topologically restrictive it is to internalize *torsionless* extended-linear EHP in geometries other than Lorentzian. Indeed, if the spacetime M is compact and 2-dimensional, then it must be \mathbb{S}^2 . On the other hand, in dimension $n = 4$ compact Kähler structures exist iff the Betti numbers $b_1(M)$ and $b_3(M)$ are zero, so that $\chi(M) = b_2(M) + 2$. As a consequence, if we add the (mild) condition $H^2(M; \mathbb{R}) \simeq 0$ on the hypothesis of Corollary 3.4 we conclude that M must be a K3-surface!

The last theorem was obtained as a consequence of Theorem 3.3 and of Berger’s classification theorem. So, this is a geometric obstruction result for *extended-linear* EHP theories. However, due to Theorem 3.4 and Corollary 3.3 we can get exactly the same result for the geometric and algebraic *duals* of *linear* EHP. The same result does not hold for the *actual linear* EHP theories, because there is no analogue of Theorem 3.3 or Corollary 3.3 for them, as emphasized in Remark 3.4.

On the other hand, Berger’s theorem remains valid, allowing us to get an obstruction result for linear EHP theories independently of the previous ones. Indeed, recall that (as pointed in Remark 3.3) we can always associate a canonical (extended-linear) theory to a given linear one. Let us define a *torsionless linear EHP theory* as that obtained by restricting a linear theory to the space of connections such that the corresponding canonical theory is a torsionless extended-linear EHP theory in the sense introduced above. We then have the following:

Theorem 3.6. *Let $P \rightarrow M$ be a k -proper G -bundle over a n -manifold M endowed with a group structure reduction $H \hookrightarrow G$. If H is a k -group, then $k_1 = k$ and $k_{i>1} = 0$. Furthermore, a torsionless linear EHP theory for this reduction is nontrivial only if one of the following conditions is satisfied*

- (B1’) k is even and M contains a Kähler Berger k -manifold;
- (B2’) k is divisible by 4 and M contains a quaternionic-Kähler Berger k -manifold;
- (B3’) $k = 7$ and M contains a Berger k -manifold with a G_2 -structure.

Proof. The proof follows the same lines of the last theorem, except by the first argument, which makes explicit use of Theorem 3.3. □

The above results are about different versions of torsionless *gauge* EHP theories. By this we mean that no requirement was made on e . We close this section remarking that if we work not on the *gauge* context, but on the *Cartan* context (in the sense that e is a pointwise isomorphism), then there is a physical appeal for working with torsionless connections.

Indeed, recall that by the very abstract definition, a *classical theory* is given by an *action functional* $S : \text{Conf} \rightarrow \mathbb{R}$ defined on some *space of configurations*. The interesting classical theories are those in that Conf has some kind of “smooth structure” relative to which S can be regarded as a “smooth function” and, as such, has a “derivative”. In such cases, there exists a distinguished subspace $\text{Cut}(S) \subset \text{Conf}$ constituted by the “critical points” of S . This is the *phase space* of the underlying classical theory, which contains all configurations which a priori can be observed in nature.

If S is now the action of EHP theory, we find that the phase space is determined by the pairs (e, ω) which satisfy the equations

$$\lambda_*^{n-2} e \lambda_* \Omega + \frac{\Lambda}{(n-1)!} \lambda_*^n e = 0 \quad \text{and} \quad \lambda_*^{n-2} e \lambda \Theta_\omega = 0$$

simultaneously, where (once again) λ_* must be interpreted as λ or λ_\times depending if we are in the linear or in the extended-linear context. The first of these equations is just an analogue of Einstein’s equations. The second, in turn, if we are working with *Cartan connections*, reduces to $\Theta_\omega = 0$, i.e., to the “torsionless” condition previously used.

4 Abstract Obstructions

Until this moment we considered EHP theory for group structure reductions $H \hookrightarrow G$ other than $O(n-1, 1) \hookrightarrow \text{Iso}(n-1, 1)$, where G could be an arbitrary linear group or some semidirect product $\mathbb{R}^k \rtimes H$. In other words, we realized gravity, as modeled by EHP theories, in other geometries than Lorentzian geometry. We then showed that for certain classes of geometries the corresponding theory is actually trivial, meaning that we have geometric obstructions.

In Subsection 4.1 we will generalize EHP theories even more by replacing the algebras \mathfrak{h} and \mathfrak{g} , which are induced by the Lie groups G and H , by general (not necessarily satisfying PI's) algebras. We call these theories *algebra-valued EHP theories*⁶. One motivation to consider this new generalization is the following. In the previous extended-linear context, a group reduction $H \hookrightarrow \mathbb{R}^n \rtimes H$ on a manifold M is (as discussed in the beginning of Subsection 3.5) a geometry modeled by some kind of tensor t in M . A *global symmetry* of (M, t) is an automorphism in the category of H -structures, i.e, a difeo such that $f^*t = t$. If we apply this to the canonical geometric model (\mathbb{R}^n, t) we see that its group of global symmetries $\text{Aut}(\mathbb{R}^n, t)$ is precisely $\mathbb{R}^n \rtimes H$. Therefore, the algebra $(\mathbb{R}^n \rtimes \mathfrak{h}, \wedge_{\rtimes})$ takes the role of the associative algebra of *infinitesimal global symmetries of the canonical geometric model*, which is where the reductive connections for the reduction $H \hookrightarrow \mathbb{R}^n \rtimes H$ take values.

It happens that in many physical situations we have more hidden/internal/worldsheet symmetries, so that the *full* algebra of infinitesimal symmetries has a more abstract structure than just an associative algebra, leading us to consider “reductive connections with values in arbitrary algebras”. Once the notion of algebra-valued EHP theories is introduced, we show that the previous obstruction results (Theorem 3.3), hold *ipsis litteris* in this new abstract context.

A typical situation in that the full algebra of infinitesimal symmetries is not just an associative algebra is when the underlying “canonical geometric model” is not a cartesian space \mathbb{R}^n endowed with a tensor, but actually a supercartesian space $\mathbb{R}^{n|m}$ endowed with a supertensor. Indeed, in this case the algebra of infinitesimal symmetries inherits a \mathbb{Z}_2 -graded algebra structure. This motivates us to analyze the effects of gradings in the obstruction results, which is done in Subsection 4.2.

Notice that if the pair (\mathbb{R}^n, t_n) describes geometry, the pair $(\mathbb{R}^{n|m}, t_{n|m})$ describes *supergeometry*. Therefore, it would be natural to consider not only “EHP with values in superalgebras”, but actually “super EHP theories”. Closing the section, in Subsection 4.3 we present an approach to the notion of “reductive graded-connection”, allowing us to define “graded EHP theories”, and we show how to extend the obstruction results to this context.

4.1 Algebra-Valued Gravity

We start by recalling that a reductive connection in P for $H \hookrightarrow G$ is a \mathfrak{g} -valued 1-form ∇ which decomposes as $\omega + e$, where ω is a \mathfrak{h} -valued 1-form and e is a $\mathfrak{g}/\mathfrak{h}$ -valued 1-form (not necessarily an isomorphism, due to previous discussion). Notice that it is completely determined by the pair (ω, e) and by the vector space decomposition $\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$, a fact that was already used in Subsection 3.4. In typical cases \mathfrak{g} is actually a splitting extension (the previous extended-linear context) or $H \subset G$ is a normal subgroup (previous linear context), so that both \mathfrak{h} and $\mathfrak{g}/\mathfrak{h}$ acquire algebra structures.

⁶We could call such theories *algebraic EHP theories*, but this nomenclature would suggest that exactly as EHP theories are about connections, “algebraic EHP” should be about “algebraic connections”. The problem is that the notion of “algebraic connection” actually exists in the literature [9, 23, 13], being applied in a more general context.

This leads us to the following generalization: given an \mathbb{R} -algebra $(A, *)$ endowed with a vector space decomposition $A \simeq A_0 \oplus A_1$, where A_0 and A_1 have algebra structures $*_0$ and $*_1$ (not necessarily subalgebras of A), we define an A -connection in a manifold P as an A -valued 1-form ∇ in P which is reductive respective to $A \simeq A_1 \oplus A_0$. In }

{kmore concrete terms, it is an A -valued 1-form ∇ which is written as $\nabla = \omega + e$, where ω and e take values in A_1 and A_0 , respectively. The *curvature* of ω is the 2-form $\Omega = d\omega + \omega \wedge_1 \omega$ in A_1 , where \wedge_1 is the product induced by $*_1$. Similarly, the *torsion* of ω is the A -valued 2-form $\Theta_\omega = de + \omega \wedge_* e$. Some related concepts are considered in [6, 7].

With these structures in hand we can generalize gauge EHP theories (about \mathfrak{g} -valued connections) to *algebra-valued EHP theories* (about A -connections in the above sense). Indeed, given a spacetime manifold M , this generalization is obtained following the following steps:

1. consider some analogue of the Hilbert-Palatini forms (13);
2. show that this Hilbert-Palatini form induces a corresponding form in M with values in some bundle;
3. turn this bundle-valued form into a real-valued form;
4. define the action functional as the integral over M of this real-valued form.

In order to do the second step, the immediate idea is to select a FABS, say defined on a subcategory $\mathbf{C} \subset \mathbf{Bun}_M \times \mathbf{Alg}_{\mathbb{R}}$, as in Section 2.5. However, in order to realize the first step we need to work with FABS fulfilling certain “invariance property”: we say that a FABS on \mathbf{C} is *invariant* by a functor $I : \mathbf{C} \rightarrow \mathbb{Z}\mathbf{Alg}_{\mathbb{R}}$ if

- (a) for all $(P, A) \in \mathbf{C}$ the corresponding $I(P, A)$ is an ideal of $S(P; A)$, so that we can take the quotient functor S/I and we have a natural transformation $\pi : S \Rightarrow S/I$;
- (b) there exists another functor $J : \mathbf{C} \rightarrow \mathbb{Z}\mathbf{Alg}_{\mathbb{R}}$ such that $J(P, A)$ is an ideal of $\Lambda(M; E_A)$ and whose projection we denote by π' ;
- (c) there exists a natural transformation $j' : S/I \Rightarrow \Lambda(M; -)/J$ such that $j' \circ \pi = \pi' \circ j$, i.e, the diagram below commutes for every (P, A) .

$$\begin{array}{ccc}
 S(P, A) & \xrightarrow{J(P, A)} & \Lambda(M; E_A) \\
 \pi_{(P, A)} \downarrow & & \downarrow \pi'_{(P, A)} \\
 S(P, A)/I(P, A) & \xrightarrow{j'_{(P, A)}} & \Lambda(M; E_A)/J(P, A)
 \end{array} \tag{20}$$

Returning to the first step, notice that a priori the algebras A and A_1 may not be associative, so that due to ambiguities we cannot simply replace “ \wedge ” by “ \wedge_* ” and “ \wedge_0 ” in (13) to get an A -valued version of α_{HP} ; instead, we should take into account all possibilities simultaneously. This can be done as follows. Recall the “associativity-like” polynomials

$$(x_1 \cdot \dots \cdot (x_{s-2} \cdot (x_{s-1} \cdot x_s))), \quad (((x_1 \cdot x_2) \cdot (x_3 \cdot x_4)) \cdot \dots \cdot x_s), \quad \text{etc.},$$

which were introduced in Subsection 2.3 in order to describe nilpotency and nil properties as PIs. Due to the structure of the Hilbert-Palatini forms, we are interested in the more specific polynomials

$$(x \cdot \dots \cdot (x \cdot (x \cdot y))), \quad \text{etc.} \quad \text{and} \quad (x \cdot \dots \cdot (x \cdot (x \cdot x))), \quad \text{etc.} \quad (21)$$

When evaluated at $\Lambda(P; A)$, the set of these polynomials generates an ideal $\mathfrak{I}(P, A)$ and the idea is to consider FABS which are invariant by it.

More precisely, given a bundle $P \rightarrow M$ over a manifold M and an algebra A endowed with vector space decomposition $A \simeq A_0 \oplus A_1$, we say that a FABS over M , defined in \mathbf{C} , is *compatible* with the given data if

- (a) the pair (P, A) belongs to \mathbf{C} ;
- (b) each possibility of defining $\alpha_{n,\Lambda}$ (i.e, the evaluation of each polynomial (21) in each A -connection) belongs to $S(P, A)$;
- (c) the FABS in question is invariant by \mathfrak{I} , as defined above.

Considering FABS satisfying (a)-(c) we realize the first two steps needed to define ‘‘algebra-valued EHP theory’’. In order to realize the third step, we need something like a ‘‘trace’’. A *trace transformation* for a FABS on \mathbf{C} invariant by I is a natural transformation tr between $\Lambda(M; E_-)/J$ and the constant functor in $\Lambda(M; \mathbb{R})$. In other words, it is a rule that assigns (in a natural way) a map of graded-algebras $\text{tr}_{(P,A)} : \Lambda(M; E_A) \rightarrow \Lambda(M; \mathbb{R})$ to every pair $(P, A) \in \mathbf{C}$.

Now, we can finally define what is an EHP theory in the general algebraic setting. Given a principal bundle $P \rightarrow M$, an algebra A and a vector space decomposition $A \simeq A_0 \oplus A_1$, choose a compatible FABS endowed with a trace transformation. The corresponding *A -valued inhomogeneous⁷ EHP theory* in P is the classical theory whose configuration space is the space of all reductive A -connections in P and whose action functional is given by (we omit the subscripts in the maps π , j' and tr in order to simplify the notation)

$$S_{n,\Lambda}[e, \omega] = \int_M \text{tr}(j'(\pi(\alpha_{HP}))),$$

where $\alpha_{HP} \in S(P, A)$ is any ‘‘Hilbert-Palatini’’-type form, say

$$\alpha_{HP} = (e \wedge_* \dots (\wedge_*(e \wedge_* (e \wedge_* \Omega))) + \frac{\Lambda}{(n-1)!} (e \wedge_* \dots (\wedge_*(e \wedge_* (e \wedge_* \Omega)))).$$

Notice that we recover the gauge linear EHP theories discussed previously by taking A as a matrix Lie algebra \mathfrak{g} and considering the adjoint FABS endowed with the classical trace map. Therefore, at least in this restricted domain we have obstruction Theorems 3.1, 3.3 and 3.4. These results remain valid if:

1. *we replace the adjoint FABS by any other compatible FABS* (indeed, due to the linearity of the FABS, once one shows that $\alpha_{HP} = 0$ it immediately follows that the bundle-valued form $j'(\pi(\alpha_{HP}))$ is also trivial independently of the FABS chosen);
2. *we replace the classical trace by any other trace transformation* (also due to the linearity of trace transformations);

⁷The corresponding A -valued *homogeneous* EHP theory is defined analogously.

3. we replace the algebra \mathfrak{g} by any other algebra A endowed with a vector space decomposition $A \simeq A_0 \oplus A_1$ (this follows from the structure of the proofs of Theorems 3.1, 3.3 and 3.4, which in essence depends only on general algebraic hypotheses on the decomposition $\mathfrak{g} \simeq \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{h}$).

Summarizing: the obstruction theorems hold not only in the domain of matrix algebras, adjoint FABS and classical trace, but also for arbitrary algebras, arbitrary compatible FABS and arbitrary trace transformations. Concretely, we have the following general obstruction theorem whose proof is in essence Remarks 1-3 above.

Theorem 4.1. *Let M be an n -dimensional spacetime and $P \rightarrow M$ be a bundle and $(A, *)$ an algebra endowed with a vector space decomposition $A_0 \oplus A_1$ fulfilling one of the following conditions*

(A1) *the algebra $(A_0, *_0)$ is a (k, s) -solv subalgebra⁸ $(A, *)$;*

(A2) *$(A, *)$ is itself (k, s) -solv.*

If $n \geq k + s + 1$, then for any compatible FABS and any trace transformation, the corresponding inhomogeneous A -valued EHP theory equals the homogeneous ones. If $n \geq k + s + 3$, then both theories are trivial.

4.2 Graded-Valued Gravity

Here we shall indicate how the previous discussion can be extended to the case when the background algebra A is itself graded (for more details see [27]). Given a monoid \mathfrak{m} , let $P \rightarrow M$ be a principal bundle and A be a \mathfrak{m} -graded real algebra. For any vector space decomposition $A \simeq A_0 \oplus A_1$ where A_i are algebras, we define a *connection in P with values in the graded algebra A* exactly as previously: as a A -valued 1-form ∇ in P which writes as $\nabla = e + \omega$ for $e : TP \rightarrow A_0$ and $\omega : TP \rightarrow A_1$. The only difference is that now the \mathfrak{m} -grading of A induces a corresponding grading in each A_i by $\oplus_m A_i^m$, where $A_i^m = A_i \cap A^m$, which means that locally we can write e and ω as

$$e = \sum_m e^m \quad \text{and} \quad \omega = \sum_m \omega^m,$$

where $e^m : TP \rightarrow A_0^m$ and $\omega^m : TP \rightarrow A_1^m$ are just the projections of e and ω onto the corresponding A_i^m .

In order to extend FABS to this graded context we notice that if A is \mathfrak{m} -graded, then the algebra of A -valued exterior forms $\Lambda(P; A)$ is $(\mathbb{Z} \times \mathfrak{m})$ -graded. On the other hand, an algebra bundle E_A whose typical fiber is A is not necessarily \mathfrak{m} -graded, because the pointwise decomposition may not vary continuously to allow us to globally decompose E_A as $\oplus_m E_{A_m}$. Therefore, in order to incorporate graded-algebras it is enough to work with “graded FABS” characterized by the following diagram:

⁸We can forget the subalgebra hypothesis by modifying a little bit the ideal $\mathcal{J}(P, A)$ used to define the notion of “compatible FABS”.

$$\begin{array}{ccc}
\mathfrak{m}\mathbf{Alg} & \xrightarrow{E_-} & \mathbf{AlgBun}_M \\
\uparrow \pi_1 & & \downarrow \Lambda(M; -) \\
& & \mathbb{Z}\mathbf{Alg} \\
& \nearrow j & \uparrow \\
\mathbf{C} & \xrightarrow{S(-, -)} & (\mathbb{Z} \times \mathfrak{m})\mathbf{Alg} \\
& \downarrow \iota & \downarrow \Lambda(-; -) \\
& & \mathbf{C}
\end{array}$$

For a chosen bundle $P \rightarrow M$ and a \mathfrak{m} -graded algebra $A \simeq A_0 \oplus A_1$, we define a *compatible graded FABS* exactly as in the last subsection. Notice that now the natural transformation $\iota : S \Rightarrow \Lambda$ is $(\mathbb{Z} \times \mathfrak{m})$ -graded, while $j : S \Rightarrow \Lambda(M; E_-)$ remains \mathbb{Z} -graded (due precisely to the fact that E_A may not be \mathfrak{m} -graded). In particular, the diagram (20) remains in $\mathbb{Z}\mathbf{AlgBun}_M$. This means that the trace transformation that we need to consider is also only \mathbb{Z} -graded, as previously. For every such compatible graded FABS endowed with a trace transformation tr we define the corresponding *A-valued inhomogeneous EHP theory* as the classical theory whose configuration space is the collection of all A -valued connections in P , and whose action functional is given by

$$S_{n, \Lambda}[e, \omega] = \int_M \text{tr}(j'(\pi(\alpha_{HP}))). \quad (22)$$

Thus, up to minor modifications everything works as in the last subsection. The crux of these “minor modifications” is that we can now locally decompose e and ω . This allows us to get a stronger version of Theorem 4.1 following the same strategy used to get Theorem 3.3 from Theorem 3.1. Indeed, consider the following conditions about the vector space decomposition $A \simeq A_0 \oplus A_1$:

- (G1) $(A_0, *_0)$ is a subalgebra and each $A_0^j = A^j \cap A_0$ is a (k_j, s_j) -weak solvable subspace of A_0 (recall definition in Remark 2.5).
- (G2) restricted to each A^j the algebra A is (k_j, s_j) -weak solvable.

We can now prove

Theorem 4.2. *Let M be an n -dimensional spacetime, $P \rightarrow M$ be a bundle and $(A, *)$ be a \mathfrak{m} -graded algebra endowed with a vector space decomposition $A_0 \oplus A_1$ fulfilling conditions (G1) or (G2) above. Furthermore, let (k, s) be the minimum of (k_j, s_j) . If $n \geq k + s + 1$, then for any compatible FABS and any trace transformation, the corresponding inhomogeneous A -valued EHP theory equals the homogeneous ones. If $n \geq k + s + 3$, then both theories are trivial.*

Proof. The proofs for (G1) and (G2) are very similar, so that we will only explain the (G1) case. Once again, since everything is linear and grading-preserving, it is enough to prove that $\alpha_{HP} = 0$ for some representative Einstein-Hilbert form. Particularly, we can prove for

$$\alpha_{HP} = \wedge_*^{n-2} e \wedge_* \Omega + \frac{\Lambda}{(n-1)!} \wedge_*^n e.$$

Under the hypothesis, we can locally write $e = \sum_m e^m$ with $e^m : TP \rightarrow A_0^m$. From condition (G1) each A_0^m is a weak (k_m, s_m) -solvable subspace, so that it writes as a sum $A_0^m = \oplus_i V_i^m$ of weak

(k_m, s_m) -nilpotent subspaces, which means, in particular, that we can write $e^m = \sum_i e_i^m$ locally and, therefore, $e = \sum_m \sum_i e_i^m$. Consequently, for every l we have $\wedge_*^l e = p_l(e_{i_1}^{m_1}, \dots, e_{i_l}^{m_l})$ for some polynomial of degree l . If we now consider the minimum (k, s) (over m) of (k_m, s_m) , the fact that each A_0^m is weakly (k_m, s_m) -nilpotent then implies $\wedge_*^{k+s+1} e = 0$. The remaining steps in the proof are identical to every other given in the previous theorems. \square

4.3 Graded Gravity

The last section was about “gauge theories with values in graded algebras”. We would like to work not only with graded algebras but in the “full graded context”, i.e, with genuine graded gauge theories (particularly, with graded EHP theories). A reductive A -valued gauge theory is about A -valued connections, i.e, 1-forms $\nabla : TP \rightarrow A$ which are reductive respective to some decomposition $A \simeq A_0 \oplus A_1$. So, “graded gauge theories” should be about “ A -valued graded connections”. There are many approaches to formalize the notion of “graded connection”. For instance, we have:

1. Quillen superconnections [32], which are defined as operators on graded vector bundles over a non-graded manifold;
2. connections on graded manifolds in the spirit of Kostant-Berezin-Leites, which are defined for graded principal bundles $P \rightarrow M$ over graded manifolds [40];
3. connections on principal ∞ -bundles, which can be applied in the domain of any cohesive ∞ -topos (particularly in the ∞ -topos of formal-super-smooth manifolds), where the notion of differential cohomology can be axiomatized and a ∞ -bundle with connection is defined as a cocycle of such cohomology [14, 35, 38].

Here we will not work with any of the models above. Instead, given \mathfrak{m} -graded algebra A , we assume that the bundle TP is also \mathfrak{m} -graded (in the sense that it decomposes as a sum of vector bundles $TP \simeq \oplus_m E^m$) and we define an A -valued graded connection in P of degree l as a smooth 1-form $\nabla : TP \rightarrow A$ which has degree l , meaning that it decomposes as a sequence of usual vector-valued 1-forms $\nabla^m : E^m \rightarrow A^{m+l}$. Particularly, for us the *reductive graded connections* are those that can be written as $\nabla = e + \omega$, where each e and ω can themselves be decomposed as maps of degree l

$$e^m, \omega^m : E^m \rightarrow A^{m+l}, \quad \text{with} \quad e^m + \omega^m = \nabla^m.$$

With the notion of “reductive graded connections of degree l ”, we can define “graded EHP theories of degree l ” in a natural way. Indeed, chosen a \mathfrak{m} -graded FABS, the corresponding \mathfrak{m} -graded A -valued EHP theory of degree l is the classical theory given by the action functional (22) restricted to the class of graded connections of degree l .

Remark 4.1. If we think of A as a graded-algebra describing the infinitesimal symmetries of the theory and if we interpret the graded structure of the bundle TP as induced by the different flavors of fundamental objects of the theory, then it is more natural to consider the connections of degree zero, because they will map each piece E^m into each corresponding subspace of infinitesimal symmetries A^m . On the other hand, in some situations (say in the BV-BRST formalism, where we have ghosts, anti-ghosts and anti-fields) we need to work with “shifted symmetries”, meaning that degree $l > 0$ graded connections should also have some physical meaning.

Now, let us focus on the geometric obstructions of the A -valued EHP theory that appear in the full graded context. First of all, notice that to give a morphism $f : A' \rightarrow A$ of degree l between two graded algebras is the same as giving a zero degree morphism $f : A' \rightarrow A[-l]$, where $A[-l]$ is the graded algebra obtained shifting A . Consequently, the obstructions of a degree l graded A -valued EHP theory are just the obstructions of degree zero graded $A[-l]$ -valued EHP theory, so that it is enough to analyze theories of degree zero.

The core idea of the proof of Theorem 4.2 was to use that A is graded and the hypothesis (G1) or (G2) in order to conclude that $\wedge_*^{k+s+1}e = 0$. More precisely, the grading of A was needed in order to write e as $e \simeq \sum_m e^m$, with $e^m : TP \rightarrow A_0^m$, while the ‘‘solvability hypothesis’’ (G1) or (G2) allowed us to decompose each e^m as $e^m = \oplus_i e_i^m$, so that $e = \oplus_{i,m} e_i^m$. The hypothesis (G1) or (G2) was then used once again to yield $\wedge_*^{k+s+1}e = 0$.

If we assume that TP is a graded bundle, so that $TP \simeq \oplus_m E^m$, and if e has degree zero, then the only change in comparison to the previous ‘‘partially graded’’ context is that instead of decomposing e as a sum $\sum_m e^m$, we can now write it as a genuine direct sum $e = \oplus_m e^m$, with $e^m : E^m \rightarrow A_0 \cap A^m$. Therefore, under the same hypothesis (G1) or (G2), for theories of **degree zero** we get exactly the same obstruction results. Due to the argument of the last paragraph we then have the following general obstruction result, which applies for graded theories of arbitrary degree.

Theorem 4.3. *Let M be an n -dimensional spacetime, $P \rightarrow M$ be a bundle such that TP is \mathfrak{m} -graded and $(A, *)$ be a \mathfrak{m} -graded algebra endowed with a vector space decomposition $A_0 \oplus A_1$. Given $l \geq 0$, assume that $A[-l]$ satisfies condition (G1) or (G2). Furthermore, let (k, s) be the minimum of (k_j, s_j) . If $n \geq k + s + 1$, then for any compatible graded FABS and any trace transformation, the corresponding inhomogeneous A -valued graded EHP theory of degree l equals the homogeneous ones. If $n \geq k + s + 3$, then both theories are trivial.*

Assume that the \mathfrak{m} -graded algebra A is bounded from below, meaning that \mathfrak{m} has a partial order \leq and that there is $m_o \in \mathfrak{m}$ such that $A^m \simeq 0$ if $m < m_o$. Consider a \mathfrak{m} -graded theory of degree l taking values in that kind of algebra. In order to apply the last theorem we need to verify one of conditions (G1) or (G2). Such conditions are about $A[-l]$. This algebra has grading $A[-l]^m = A^{m+l}$. Because A is bounded, the first solvability condition falls on A^{m_o+l} , i.e, *no condition is needed between m_o and $m_o + l$* . This is, in essence, the new phenomenon obtained when we work with ‘‘full graded theories’’ in the sense introduced above. As a corollary:

Corollary 4.1. *In the same notation of the last theorem, assume that A is bounded from below and from above, respectively in degrees m_o and m_1 . If l does not divide $m_1 - m_o$, then (G1) and (G2) are automatically satisfied and, therefore, for any graded bundle P the corresponding graded EHP theory of degree l is trivial. Otherwise, i.e, if $m_1 - m_o = k.l$ for some k , then conditions (G1) and (G2) need to be fulfilled exactly for k terms.*

A particular consequence is the following:

Corollary 4.2. *Let $A \simeq A^0 \oplus A^1$ be a \mathbb{Z}_2 -graded algebra, endowed with a vector space decomposition $A \simeq A_0 \oplus A_1$, where A_i is not necessarily A^i and A_0 is a subalgebra of A . If $A_0 \cap A^1$ is a weak (k, s) -solvable subspace of A_0 , then any A -valued \mathbb{Z}_2 -graded EHP theory of degree one over a n -dimensional spacetime M is trivial if $n \geq k + s + 3$.*

Examples will be explored in next section.

4.4 Some Speculation

When we look at the previous obstruction theorems, all of them (except Theorem 3.5) were conceived as abstractions of a single “fundamental obstruction theorem”, namely Theorem 3.1, introduced in the most concrete situation: the *linear/matrix context*. Despite the fact that “derived obstruction theorems” hold in more abstract contexts, they are very closely related to the first one, in that they require the same kind of hypothesis on the underlying algebra: a “solvability condition”. Furthermore, the more abstract the context is, the weaker the required “solvability condition” is. Indeed, Theorem 3.1 (for linear EHP theories) requires “ (k, s) -nil condition”, while Theorem 3.3 (for gauge EHP theories) requires “ (k, s) -solv condition”. Furthermore, Theorem 4.1 (for arbitrary A -valued EHP theories) requires “partially (k, s) -solv condition”, in the sense that $A \simeq A_0 \oplus A_1$, with A_0 (k, s) -solv, and Theorem 4.2 (for arbitrary graded-valued EHP theories) requires “locally (k, s) -solv condition”, meaning that each A_0^m is weak (k, s) -solvable.

On the other hand, the same strategy used here can a priori be applied to get geometric obstructions for any classical theory (Conf, S) whose space of configurations Conf is some class of algebra-valued smooth forms. Let us call such theories *smooth forms theories*. However, recall that our strategy here was based in working first in the “linear context”, where we identify a “fundamental algebra condition”. Then, abstracting the context we could consider weaker conditions than the “fundamental” one. Therefore, in order to use this strategy in other results we need to find a corresponding “fundamental algebra condition”. This leads us to speculate:

Conjecture (roughly). *Any smooth forms theory admits a fundamental algebraic condition.*

For instance, in the way that it is stated, it is easy to verify that this conjecture is true for “polynomial smooth forms theories”, i.e, for theories whose configuration space is $\text{Conf} = \Lambda(P; A)$, for some algebra A endowed with a vector space decomposition $A \simeq A_1 \oplus \dots \oplus A_k$, and whose action functional is

$$S[\alpha_1, \dots, \alpha_k] = \int_M J(p_s(\alpha_1, \dots, \alpha_k)),$$

where J is some FABS and p_s is a polynomial of degree s in $\Lambda(P; A)$. Indeed, the desired condition is that p_s be a PI of A . Now, notice that EHP theories are “polynomial smooth forms theories of degree” in the above sense. Therefore, they have a “fundamental algebra condition” given by the vanishing of p_s . This is a *nilpotent condition*, which is much stronger than the (k, s) -solv condition previously established. This teaches us that the same smooth forms theory may admit two fundamental algebraic conditions, leading us to search for the optimal one:

Conjecture (roughly). *Any smooth forms theory admits an optimal fundamental algebraic condition.*

We can go further and ask if there is some kind of “universal algebraic condition”. More precisely, suppose a collection \mathcal{C} of smooth forms theories satisfying last conjecture is given. Given two of those theories, if the optimal algebraic condition of one is contained in the optimal algebraic condition of the other, then both can be simultaneously trivialized. If this is not the case, the union of those optimal conditions will clearly trivialize them simultaneously. But, the union of optimal conditions is not necessarily the optimal one. This leads us to define the *optimal fundamental algebraic condition* of \mathcal{C} as the weaker algebraic condition under which each classical theory in \mathcal{C} becomes trivial, and to speculate its existence:

Conjecture (roughly). *Any collection \mathcal{C} of smooth forms theories admits an optimal fundamental algebraic condition.*

5 Examples

In the present section we will give realizations of the obstruction theorems studied previously. Due to the closeness with the genuine Lorentzian EHP theory, our focus is on the “concrete context”, meaning that we will give many examples of “concrete geometries” which realize the obstruction theorems of Section 3. Even so, some examples for the “abstract context” of Section 4 will also be given.

5.1 Linear Examples

We start by considering the “fully linear” context of Subsection 3.1, i.e, EHP theories defined on a G -principal bundle $P \rightarrow M$ respective to a group reduction $H \hookrightarrow G$, where G is a linear group and M is a smooth n -manifold. The objects of interest are the classical reductive Cartan connections on P , i.e, pairs (e, ω) of 1-forms such that e takes values in $\mathfrak{g}/\mathfrak{h}$ and ω takes values in \mathfrak{h} . It follows from Corollary 3.1 and Theorem 3.4 that if $n \geq 6$ and

- (E1) $H \subset G$ is normal, with $\mathfrak{g}/\mathfrak{h} \subset \mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r)$, then the corresponding gauge EHP theory is trivial;
- (E2) $\mathfrak{h} \subset \mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r)$, then the geometric/algebraic dual EHP theories are trivial;
- (E3) $\mathfrak{g} \subset \mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r)$, then both EHP and the dual theories are trivial.

Conditions (E2) and (E3) are immediately satisfied if we identify k -groups, i.e, Lie subgroups of $SO(k_1) \times \dots \times SO(k_r)$, or, equivalently, Lie subalgebras of $\mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r)$. Indeed, if A is a such subgroup, then the EHP theory for $A \hookrightarrow G_A$, where G_A is any matrix extension of A , obviously satisfies (E2), so that if $n \geq 6$ the dual theories are trivial. On the other hand, if we consider EHP theories for reductions $H \hookrightarrow A$, then (E3) is clearly satisfied and in dimension $n \geq 6$ both the dual and the actual EHP theory are trivial.

Some obvious examples of subgroups of $A \subset SO(k)$, are given in the table below. Except for $Spin(4) \hookrightarrow O(8)$, which arises from the exceptional isomorphism $Spin(4) \simeq SU(2) \times SU(2)$, all the other inclusions already appeared in Subsection 3.5. Let us focus on condition (E2). In this case we can think of each element of Table 3 as included in some $GL(k; \mathbb{R})$, i.e, as a G -structure on a manifold and then as a geometry (recall Table 2).

(n, k)	$(1, k)$	$(2, k)$	$(4, k)$	$(1, 7)$	$(1, 8)$
$H \subset O(n, k)$	$SO(k)$	$U(k)$	$Sp(k)$	G_2	$Spin(4)$
	$U(k)$	$SU(k)$	$Sp(k) \cdot Sp(1)$		$Spin(7)$

Table 3: First classical examples of geometric obstructions

We could get more examples by taking finite products of arbitrary elements in the table. In terms of geometry, this can be interpreted as follows. Recall that a regular distribution of dimension k on an n -manifold can be regarded as a G -structure for $G = GL(k) \times GL(n - k)$. Therefore, we can

think of a product $O(k) \times O(n-k)$ as a regular distribution of Riemannian leaves on a Riemannian manifold, $U(k) \times U(n-k)$ as a hermitan distribution, and so on.

Other special cases where condition (E2) applies are in table below. In the first line, $O(k, k)$ is the so-called *Narain group* [28], i.e, the orthogonal group of a metric with signature (k, k) , whose maximal compact subgroup is $O(k) \times O(k)$. Second line follows from an inclusion similar as $U(k) \hookrightarrow O(2k)$, first studied by Hitchin and Gualtieri [16, 15, 17], while the remaining lines are particular cases of the previous ones. The underlying flavors of geometry arose from the study of Type II gravity and Type II string theory [8, 21]. That condition (E2) applies for the second and third lines follows from the fact that complexifying $U(k, k) \hookrightarrow O(2k, 2k)$ we obtain $U(k, k) \hookrightarrow O(4k; \mathbb{C})$, as will be discussed in the next section.

G	H	geometry
$O(k, k)$	$O(k) \times O(k)$	Type II
$O(2k, 2k)$	$U(k, k)$	Generalized Complex
$O(2k, 2k)$	$SU(k, k)$	Generalized Calabi-Yau
$O(2k, 2k)$	$U(k) \times U(k)$	Generalized Kähler
$O(2k, 2k)$	$SU(k) \times SU(k)$	Generalized Calabi

Table 4: More examples of classical geometric obstructions

About these two tables, some remarks:

1. Table 3 contains any “classical” flavors of geometry, except symplectic geometry. The reason is that the symplectic group $\text{Sp}(k; \mathbb{R})$ is not contained in some $O(r)$. But this does not mean that condition (E2) cannot be satisfied by symplectic geometry. Indeed, generalized complex geometry contains symplectic geometry [15], so that Table 4 implies that symplectic geometry fulfill condition (E2).
2. We could create a third table with “exotic k -groups”, meaning that a priori they are not related to any “classical geometry”, so that they describe some kind of “exotic geometry”. For instance, in [30] all Lie subgroups $H \subset O(k)$ satisfying

$$\frac{(k-3)(k-4)}{2} + 6 < \dim H < \frac{(k-1)(k-2)}{2}$$

were classified and in arbitrary dimension k there are fifteen families of them. Other exotic (rather canonical, in some sense) subgroups that we could add are maximal tori. Indeed, both $O(2k)$ and $U(k)$ have maximal tori, say denoted by T_O and T_U , so that the reductions $T_O \hookrightarrow O(2k)$ and $T_U \hookrightarrow U(k)$ will satisfy condition E2. More examples of exotic subgroups to be added are the *point groups*, i.e, $H \subset \text{Iso}(\mathbb{R}^k)$ fixing at least one point. Without loss of generality we can assume that this point is the origin, so that $H \subset O(k)$. Here we have the symmetric group of any spherically symmetric object in \mathbb{R}^k , such as regular polyhedrons and graphs embedded on \mathbb{S}^{k-1} .

3. If we are interested only in condition (E2), then the tables above can be enlarged by including embeddings of $O(k_1) \times \dots \times O(k_r)$ into some other larger group \tilde{G} . Indeed, in this condition it only matters that H is a k -group. Particularly, $O(k)$ admits some exceptional embeddings (which arise from the classification of simple Lie algebras), as in the table below [44].

k	3	9	10	12	16
$O(k) \hookrightarrow$	G_2	F_4	E_6	E_7	E_8

Table 5: Exceptional embeddings of orthogonal groups.

Due to the above discussion, Tables 3 and 4, as well as the possible “table of exotic k -groups”, are not only a source of examples for condition (E2), but also for condition (E3). Indeed, we can just consider G in condition (E3) as some “ H ” in the tables and take an arbitrary $H \subset G$. This produces a long list of examples, because G/H is a priori an arbitrary homogeneous space subject only to the condition that G is a k -group. In geometric terms, if a geometry fulfills condition (E2), then any induced “homogeneous geometry” fulfills (E3). On the other hand, differently from what happens with condition (E2), Table 5 *cannot* be used to get more examples of condition (E3). Indeed, if $H \hookrightarrow G$ is a reduction fulfilling (E3) and $G \hookrightarrow \tilde{G}$ is an embedding, then $H \hookrightarrow \tilde{G}$ fulfills (E3) iff it is satisfied by $G \hookrightarrow \tilde{G}$ (see diagram below).

$$\begin{array}{ccccccc}
 H & \hookrightarrow & G & \hookrightarrow & O(k_1) \times \dots \times O(k_r) & \hookrightarrow & \tilde{G} \\
 & & & & \searrow & & \nearrow \\
 & & & & & &
 \end{array}$$

Let us now analyze condition (E1). First of all we notice that it is very restrictive, because we need to work with reductions $H \hookrightarrow G$ in which $H \subset G$ is normal. For instance, in the typical situations above, H is not normal. Even so, there are two dual conditions under which $H \hookrightarrow G$ fulfills (E1): when G is a k -group with $G/H \subset G$ and, dually, when H is k -group with $G/H \subset H$. In the first situation, we are just in condition (E3) for $G/H \hookrightarrow G$, while in the second we are in condition (E3) for $G/H \hookrightarrow H$. Therefore, there are not many new examples here.

Conclusion. *Assume $n \geq 6$. So, for any compatible FABS:*

1. *geometric/algebraic dual EHP is trivial in each geometry modeled by tables above;*
2. *the actual gauge EHP is trivial in each homogeneous geometry associated to the first two tables above.*

5.2 Extended-Linear Examples

In the last subsection we constructed explicit examples of geometric obstructions for linear EHP theories and their duals. Let us now consider extended-linear theories. This means that we will work with bundles $P \rightarrow M$ structured over $\mathbb{R}^k \rtimes H$, where H is a linear group and M is a n -dimensional smooth manifold. Furthermore, we will take into account reductive connections for the group reduction $H \hookrightarrow \mathbb{R}^k \rtimes H$. Our obstruction theorem is now Corollary 3.3, which implies that if $n \geq 6$ and

- $\mathfrak{h} \subset \mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r)$, then EHP action and its duals are trivial.

Notice that this is exactly the same condition as (E2). This means that *if we are working in the extended-linear context, in dimension $n \geq 6$ EHP theory cannot be realized in any geometry of the last subsection.*

Remark 5.1. Recall that it is in the extended-linear context, with $P = FM$, that “abstract EHP theory” becomes very close to the concrete one, in that the geometries of the last subsection really describe geometries (in the most classical sense, in terms of tensors) on the manifold M . Therefore, the fact that EHP is trivial in a lot of situations is a strong manifestation that the geometry of gravity is very rigid, especially in higher dimension.

5.3 Cayley-Dickson Examples

In the last two sections we presented obstructions to the realization of linear/extended-linear (dual) EHP theories in commonly studied geometries, as well as in some “exotic” geometries. Here we would like to show that there are also obstructions for less studied (and some never studied in detail) kinds of geometry. So, for now, let $(A, *)$ be an arbitrary \mathbb{R} -algebra and let $\text{Mat}_{k \times l}(A)$ be the \mathbb{R} -module of $k \times l$ matrices with coefficients in A . The multiplication of A induces a corresponding multiplication

$$\cdot : \text{Mat}_{k \times l}(A) \times \text{Mat}_{l \times m}(A) \rightarrow \text{Mat}_{k \times l}(A).$$

In particular, for $k = 0 = m$ we see that for every l we have a bilinear map

$$b : A^l \times A^l \rightarrow A, \quad \text{given by } b(x, y) = x_1 * y_1 + \dots + x_l * y_l.$$

where we used the identifications

$$\text{Mat}_{0 \times l}(A) \simeq A^l \simeq \text{Mat}_{l \times 0}(A) \quad \text{and} \quad \text{Mat}_{0 \times 0}(A) \simeq A.$$

Remark 5.2. The bilinear map above is symmetric iff the algebra A is commutative. Furthermore, its non-degeneracy depends on whether A has zero divisors or not, and a priori it is not possible to ask about its positive definiteness, because it takes values in A and not in \mathbb{R} . Thus, it is **not** an inner product in $\text{Mat}_{0 \times l}(A)$. But, if we choose $A = \mathbb{R}$, then it is the standard inner product of \mathbb{R}^l . Now, assume that A is endowed with an involution $\overline{(-)} : A \rightarrow A$. In this case, it can be combined with b to get a sesquilinear map s in A^l , as in the diagram below.

$$\begin{array}{ccc} A^l \times A^l & \xrightarrow{\overline{(-)}^l \times id} & A^l \times A^l \xrightarrow{b} A \\ & \searrow s & \nearrow \end{array}$$

Explicitly,

$$s(x, y) = \overline{x}_1 * y_1 + \dots + \overline{x}_l * y_l.$$

We can now consider the subspace of all $l \times l$ matrices $M \in \text{Mat}_{l \times l}(A)$ with coefficients in A which preserve the sesquilinear form s , in the sense that $s(Mx, My) = s(x, y)$. We say that these are the *unitary matrices in A , relative to the involution s induced by the involution $\overline{(-)}$* , and we denote this set by $U(k; A)$. If the involution is trivial (i.e, the identity map), then call them the *orthogonal matrices in A* , writing $O(k; A)$ to denote the corresponding space.

Example 5.1. If we consider $A = \mathbb{R}, \mathbb{C}$ and the trivial involution we get, respectively, the real and the complex orthogonal groups $O(k)$ and $O(k; \mathbb{C})$. If we consider the canonical involution of \mathbb{C} , we get the unitary group $U(k)$, while if we consider \mathbb{H} with its canonical involution we get the quaternionic unitary group $U(k; \mathbb{H})$, which already have appeared in Tables 2 and 3 with the more used notation $Sp(k)$.

which imply that (in spacetime dimension $n \geq 6$) EHP theory and its duals are trivial for each of these geometries (because they satisfy (E2)). In sum, Table 3 can be augmented to include Cayley-Dickson geometries in \mathbb{R} of arbitrary order, as shown below.

G	H	geometry
$\mathbb{R}^k \rtimes H$	$O(k)$	Riemannian
$\mathbb{R}^{2k} \rtimes H$	$U(k; \mathbb{C})$	hermitean
$\mathbb{R}^{4k} \rtimes H$	$U(k; \mathbb{H})$	quaternionic
$\mathbb{R}^{8k} \rtimes H$	$U(k; \mathbb{O})$	octonionic
$\mathbb{R}^{16k} \rtimes H$	$U(k; \mathbb{S})$	sedenionic
\vdots	\vdots	\vdots
$\mathbb{R}^{2^l k} \rtimes H$	$U(k; \text{CD}^l(\mathbb{R}))$	real CD of order l
\vdots	\vdots	\vdots

Table 6: Extended examples of concrete geometric obstructions

In Subsections 5.1 and 5.2 we showed that EHP theory cannot be realized in the most classical flavors of geometry. These, however, constitute a *finite* amount. As a corollary of the construction above, we now know that EHP theory is actually trivial in an *infinite* number of geometries:

Corollary 5.1. *In dimension $n \geq 6$, the EHP theory and its duals cannot be realized in an infinite number of geometries.*

Table 6 can, in turn, be extended in three different directions:

1. by adding distinguished subgroups. For every involutive algebra A we can define the subgroup $SU(k; A) \subset U(k; A)$ of those matrices whose determinant equals the identity $1 \in A$. When A is finite-dimensional, it will be a Lie subgroup, allowing us to include $SU(k; \text{CD}^l(\mathbb{R}))$ for every $l \geq 0$ in Table 6. This means that if a geometry belongs to the “table of obstructions”, then its “oriented version” belongs too;
2. by replacing the base field. Notice that condition $\mathfrak{h} \subset \mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r)$ was used above only in order to have $\alpha \wedge \alpha = 0$ for every even-degree \mathfrak{h} -valued form, i.e, to ensure that \mathfrak{h} is $(k, 1)$ -nil for each k even. In turn, the condition $\alpha \wedge \alpha = 0$ is satisfied exactly because $\mathfrak{so}(n)$ is an algebra of skew-symmetric matrices. But this remains valid for the Lie algebra $\mathfrak{so}(n; A)$ of $SO(n; A)$, independently of the involutive algebra. This means that we can replace \mathbb{R} by any algebra A in Table 6. New interesting examples that arise from this fact are the following. For given $p, q > 0$, there is no k such that $O(p, q) \subset O(k)$, so that a priori we cannot add $O(p, q)$ to Table 6. On the other hand, after complexification, i.e, after replacing \mathbb{R} by \mathbb{C} we have $O(p, q) \otimes_{\mathbb{R}} \mathbb{C} \simeq O(p+q) \otimes_{\mathbb{R}} \mathbb{C}$ for every $p, q \geq 0$, so that they immediately enter in the “obstruction table”. This means that we can add to Table 6 “complex semi-Riemannian geometry” and, similarly, “quaternionic semi-Hermitian geometry”, and so on.
3. by making use of Lie theory. The condition that we need is on the **Lie algebra** level. It happens that in general there are many groups with the same algebra. Therefore, once we find a Lie group whose algebra fulfills what we need, we can automatically add to our “table

of obstructions” every other Lie group that induces the same algebra. In particular, we can double the size of our current table by adding the universal cover (when it exists) of each group. For instance, in Tables 3 and 6 the spin groups $\text{Spin}(4)$ and $\text{Spin}(7)$ were added due to exceptional isomorphisms on the level of **Lie groups**. Now, noticing that for $k > 2$ (in particular for $k = 4, 7$) $\text{Spin}(k)$ is the universal cover of $SO(k)$, we can automatically add all these spin groups to our table, meaning that the dual EHP cannot be realized in “spin geometry”. Similarly, we can add the universal coverings of the (connected component at the identity of) $U(k; \text{CD}^l(\mathbb{R}))$. We can also add the universal cover of the symplectic group $Sp(k; \mathbb{R})$, usually known as the *metaplectic group*. Therefore, dual EHP cannot be realized in “metaplectic geometry” too.

Up to this point we gave examples which extend Table 3. We notice, however, that Table 4 can also be extended. Indeed, for every A we have the inclusion $U(k, k; \text{CD}(A)) \subset O(2k; 2k, A)$, so that by iteration we get

$$U(k, k; \text{CD}^l(A)) \subset O(2^l k; 2^l k, A). \quad (24)$$

For $A = \mathbb{R}$ and $l = 1$ this model generalized complex geometry, leading us to say that the inclusion above models “*generalized Cayley-Dickson geometry in A of degree l* ”. For instance, if we take $A = \mathbb{R}$ and $l = 2$ this becomes generalized quaternionic geometry, which is a poorly studied theory, started with the works [31, 10]. For $l = 3, 4, \dots$ it should be “generalized octonionic geometry”, “generalized sedenionic geometry”, and so on. The authors are unaware of the existence of substantial works on these theories.

When tensoring inclusion (24) with $\text{CD}^l(A)$ we get

$$U(k, k; \text{CD}^l(A)) \subset O(2^l k; 2^l k, \text{CD}^l(A)) \simeq O(2^l k + 2^l k; \text{CD}^l(A)),$$

so that condition 2 above implies that, independently of the present development of abstract generalized Cayley-Dickson geometries, EHP theory (and its duals) is trivial in each of them.

Conclusion: *In dimension $n \geq 6$ and for any FABS, extendend-linear EHP theory and its duals cannot be realized in a lot of geometries, which include:*

1. *real Cayley-Dickson geometries of all orders, their “connected components of the identity” and their universal covering geometries;*
2. *generalized Cayley-Dickson geometries of all orders;*
3. *the exceptional geometry G_2 .*

5.4 Partially Abstract Examples

The last examples were considered “concrete” because the algebras underlying them are matrix algebras over the real numbers. So, in order to give non-concrete examples it is enough to work with algebras which are not matrix algebras “and/or” which have coefficient rings other than \mathbb{R} . Let us call the “or” examples *partially abstract examples*.

The interesting part of the partially abstract situations is that we can give abstract examples of obstructions without using the abstract theorems of Section 4. For instance, some obstructions for non-real matrix algebras were actually given in last subsection in “by replacing the base-field”.

Indeed, there we considered situations in which the new field \mathbb{K} arises as an extension of \mathbb{R} and the matrix algebra $A(k; \mathbb{K})$ in consideration is indeed a scalar extension $A(k; \mathbb{K}) = A(k; \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{K}$. But, since \mathbb{R} is characteristic zero, it follows that every extension $\mathbb{K} \supset \mathbb{R}$ is also characteristic zero, allowing us to ask: *can we give examples when \mathbb{K} has prime characteristic?*

In [3] necessary and sufficient conditions were given under which arbitrary orthogonal groups $O(q, \mathbb{K})$, where $q : V \rightarrow \mathbb{K}$ is a positive-definite quadratic form on a finite-dimension \mathbb{K} -space q and $\text{ch}(\mathbb{K}) \neq 2$, admit an embedded maximal torus $\mathbf{T}(q; \mathbb{K})$. Independently of the quadratic space (V, q) , the corresponding orthogonal group is a Lie group and, as in the real case, the algebra $\mathfrak{o}(q; \mathbb{K})$ is $(2, 1)$ -nil. Therefore, under the conditions of [3], *EHP theory cannot be realized in any of these “toroidal geometries”* $\mathbf{T}(q; \mathbb{K}) \hookrightarrow O(q; \mathbb{K})$. Notice that for $\mathbb{K} = \mathbb{R}$ the same result appeared in 5.1 as a “exotic linear example”.

5.5 Fully Abstract Examples

Finally, we give “abstract examples” of geometries in which EHP theory cannot be realized. First we deal with algebra-valued geometry, with graded geometry considered in the sequence. This means that we work with a bundle $P \rightarrow M$ endowed with A -valued connections $\nabla : TP \rightarrow A$, where A is a graded \mathfrak{m} -graded algebra $A \simeq \oplus_{\mathfrak{m}} A_{\mathfrak{m}}$. The obstruction theorems are now Theorems 4.1, 4.2 and 4.3. In summary, if

- (F1) *the algebra A admits a vector space decomposition $A \simeq A_0 \oplus A_1$, where A_0 is a subalgebra such that each $A_0^{\mathfrak{m}} = A_0 \cap A^{\mathfrak{m}}$ is a weak (k_m, s_m) -solvable subspace, then EHP theory is trivial in dimension $n \geq k + s + 1$, where $(k, s) = \min(k_m, s_m)$.*

The most basic examples are those for $\mathfrak{m} = 0$ and $A_1 = 0$, i.e, the non-graded setting with A itself (k, s) -solv. As discussed in Subsections 2.3 and 2.4, there are many natural examples of (k, s) -solv algebras, e.g, any Lie algebra is $(k, 1)$ for any even k . As a consequence, *in dimension $n \geq 4$, any Lie algebra valued EHP theory is trivial.* In particular, EHP theories with values in the Poincaré group $\mathfrak{iso}(n - 1, 1)$ are trivial. But, $\mathfrak{iso}(n - 1, 1)$ -valued EHP theory is just classical EHP theory, thus we conclude that General Relativity does not makes sense in dimension $n \geq 4$, which is absurd. We made a similar mistake in Example 2.3: while the classical EHP theory and the $\mathfrak{iso}(n - 1, 1)$ -valued EHP theory take values in the same **vector space**, at the same time that their action functionals have the same shape, the **algebra** (and, therefore, its properties) used to define the corresponding wedge product is *totally different*. Indeed, in the discussed cases, the exterior products λ_{\times} and $[\wedge]$, respectively.

If we now allow \mathfrak{m} to be nontrivial, but with $A_1 = 0$, then condition (5.5) is satisfied if each $A^{\mathfrak{m}}$ is a weak (k_m, s_m) -solvable subspace. In particular, it remains satisfied if A is itself (k, s) -solv. It happens that not only Lie algebras are solv $(2, 1)$ -nil, but also a class of graded Lie algebras. Consequently, EHP theories are also trivial in the domain of graded Lie algebras. One can generalize even more thinking in EHP theories with values in Lie superalgebras and in graded Lie superalgebras. Indeed, a \mathfrak{m} -graded Lie superalgebra \mathfrak{g} is just a \mathfrak{m} -graded Lie algebra whose underlying PI’s (i.e, skew-commutativity and Jacobi identity) hold in the graded sense. Particularly, this means that the \mathbb{Z}_2 -grading writes $\mathfrak{g} \simeq \mathfrak{g}^0 \oplus \mathfrak{g}^1$, with \mathfrak{g}^0 a \mathfrak{m} -graded Lie algebra and, therefore, $(2, 1)$ -nil. It then follows that \mathfrak{g} satisfies condition F1. Summarizing, *EHP theories cannot be realized in any “Lie algebraic” context.*

Graded Lie Superalgebras are the first examples of algebras satisfying (F1) with $A_1 \neq 0$, but they are far from being the only one. Indeed, when we look at a decomposition $A \simeq A_0 \oplus A_1$, where

A_0 is a subalgebra, it is inevitable to think of A as an extension of A_1 by A_0 , meaning that we have an exact sequence as shown below. If A_0 is (k, s) -solv, then (F1) holds. This can be interpreted as follows: *suppose that we encountered an algebra A_1 such that EHP is not trivial there. So, EHP theory will be trivial in any (splitting) extension of A_1 by a (k, s) -solv algebra.*

$$0 \longrightarrow A_0 \longrightarrow A \longrightarrow A_1 \longrightarrow 0$$

In particular, because $(\mathbb{R}^k, +)$ is abelian and, therefore, $(1, 1)$ -nil, any algebra extension by \mathbb{R}^k will produce a context in which EHP theory is trivial. For instance, recall the extended-linear context, which was obtained taking splitting extensions of matrix algebras by \mathbb{R}^k . Thus, EHP theory is trivial in every extended-linear context, so that classical EHP is trivial. This implies tha GR is trivial, showing that *we made another mistake*. Once again, *the mistake resides on the underlying “wedge products”*: *the wedge product induced on an algebra extension by the abelian group $(\mathbb{R}^k, +)$ is **not** the wedge product \wedge_{\times} studied in Subsection 2.2.* Indeed, \wedge_{\times} does **not** take the abelian structure of \mathbb{R}^k into account.

Another example about extensions is as follows: the paragraph above shows that abelian extensions of nontrivial EHP theories are trivial, but what about “super extensions”? They remain trivial. Indeed, being a Lie superalgebra, the translational superalgebra $\mathbb{R}^{k|l}$ (the cartesian super-space, regarded as a Lie superalgebra) is $(2, 1)$ -nil, so that any algebra extension by it is trivial.

Finally, let us say that we can also consider “fully exotic abstract examples” meaning geometries modeled by algebras fulfilling (F1), but that have no physical meaning. Just to mention, in [39] the authors build “mathematically exotic” examples of nilpotent algebras, which fulfill (F1). If, for any reason, one tries to model gravity as EHP with values in those algebras, one will find a trivial theory.

6 Conclusion

In this notes, based on [26], we considered EHP action functional in different contexts and we gave obstructions to realize gravity (modeled by these EHP actions) in several geometries. In particular, we showed that EHP cannot be realized in almost all “classical geometries”, including Riemannian geometry, hermitean geometry, Kähler geometry, generalized complex geometry and many extensions of them, as well as some exceptional geometries, such as G_2 -geometry. We also introduced the notions of geometric/algebraic duals of an EHP theory and we have show that many obstructions also affect them. A physical understanding of these “dual theories” is desirable.

In this process of finding geometric obstructions for EHP theories, we identified a “general obstruction”, corresponding to a “solvability condition” on the underlying algebra, which led us to speculate on the existence of a “general obstruction” for each gauge theory and of a “universal obstruction”, unifying all such “general obstructions”.

The speculations and the obstruction theorems developed here are independent of the choice of a *Functional Algebra Bundle System*, a concept conceived in the present work, which codifies the data necessary to pass from A -valued forms on a bundle $P \rightarrow M$ to bundle-valued forms on M . A specific study of the category of these objects is desirable (we have some work in developing stage [27]). For instance, which kind of limits/colimits exist in such category? It has at least initial objects (corresponding to universal FABS)?

We believe that the main contribution of the present work is to clarify that *when working with classical theories defined by algebra-valued differential forms, one needs to be very careful about the*

“wedge products” used in the action functional, given that the properties of the underlying algebra deeply affect the properties of the corresponding wedge product.

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