



HAL
open science

Existence of $B^k_{\alpha, \beta}$ -Structures on C^k -Manifolds

Yuri Ximenes Martins, Rodney Josué Biezuner

► **To cite this version:**

Yuri Ximenes Martins, Rodney Josué Biezuner. Existence of $B^k_{\alpha, \beta}$ -Structures on C^k -Manifolds. 2020. hal-02907893

HAL Id: hal-02907893

<https://hal.science/hal-02907893>

Preprint submitted on 27 Jul 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Existence of $B_{\alpha,\beta}^k$ -Structures on C^k -Manifolds

Yuri Ximenes Martins* and Rodney Josué Biezuner†

December 16, 2019

*Departamento de Matemática, ICEx, Universidade Federal de Minas Gerais,
Av. Antônio Carlos 6627, Pampulha, CP 702, CEP 31270-901, Belo Horizonte, MG, Brazil*

Abstract

In this paper we introduce $B_{\alpha,\beta}^k$ -manifolds as generalizations of the notion of smooth manifolds with G -structure or with k -bounded geometry. These are C^k -manifolds whose transition functions $\varphi_{ji} = \varphi_j \circ \varphi_i^{-1}$ are such that $\partial^\mu \varphi_{ji} \in B_{\alpha(r)} \cap C^{k-\beta(r)}$ for every $|\mu| = r$, where $B = (B_r)_{r \in \Gamma}$ is some sequence of presheaves of Fréchet spaces endowed with further structures, $\Gamma \subset \mathbb{Z}_{\geq 0}$ is some parameter set and α, β are functions. We present embedding theorems for the presheaf category of those structural presheaves B . The existence problem of $B_{\alpha,\beta}^k$ -structures on C^k -manifolds is studied and it is proved that under certain conditions on B , α and β , the forgetful functor from C^k -manifolds to $B_{\alpha,\beta}^k$ -manifolds has adjoints.

1 Introduction

One can think of a “ n -dimensional manifold” as a topological space M in which we assign to each neighborhood U_i in M a bunch of coordinate systems $\varphi_i : U_i \rightarrow \mathbb{R}^n$. The regularity of M is determined by the space in which the transition functions $\varphi_{ji} : \varphi_i(U_{ij}) \rightarrow \mathbb{R}^n$ live, where $U_{ij} = U_i \cap U_j$ and $\varphi_{ji} = \varphi_j \circ \varphi_i^{-1}$. Indeed, if $B : \text{Open}(\mathbb{R}^n)^{op} \rightarrow \mathbf{Fre}$ is some presheaf of Fréchet spaces in \mathbb{R}^n , we can define a *n -dimensional B -manifold* as one whose regularity is governed by B , i.e, whose transition functions φ_{ji} belong to $B(\varphi_i(U_{ij}))$. For instance, given $0 \leq k \leq \infty$, a C^k -manifold (in the classical sense) is just a C^k -manifold in this new sense if we regard C^k as the presheaf of k -times continuously differentiable functions.

We could study B -manifolds when $B \subset C^k$, calling them *B^k -manifolds*. For instance, if $k \geq 1$, then any C^k -manifold can always be regarded as a C^∞ -manifold [1], showing that smooth manifolds belong to that class of B -manifolds, since $C^\infty \subset C^k$. Another example are the analytic manifolds, for which $B = C^\omega$. Recall that for $B = C^k$, with $k > 1$, we have regularity conditions not only on the transition functions φ_{ji} , but also on the derivatives $\partial^\mu \varphi_{ji}$, for each $|\mu| \leq k$. In fact, $\partial^\mu \varphi_{ji} \in C^{k-|\mu|}(\varphi_i(U_{ij}))$. We notice, however, that if $B \subset C^k$ is a subpresheaf, then $B \subset C^{k-|\mu|}$, but it is not necessarily true that $\partial^\mu \varphi_{ji} \in B(\varphi_i(U_{ij}))$. For instance, if $B = L^p \cap C^\infty$, then φ_{ji} are smooth L^p -integrable functions, but besides being smooth, the derivatives of a L^p -integrable smooth functions need not be L^p -integrable [13].

*yurixm@ufmg.br (corresponding author)

†rodneyjb@ufmg.br

The discussion above motivates us to consider n -dimensional manifolds which satisfy *a priori* regularity conditions on the transition functions and also on their derivatives. Indeed, let now $B = (B_r)_{r \in \Gamma}$ be a sequence of presheaves of Fréchet spaces with $r \in \mathbb{Z}_{\geq 0}$ and redefine a B^k -manifold as one in which $\partial^\mu \varphi_{ji} \in B_r(\varphi_i(U_{ij})) \cap C^{k-r}(\varphi_i(U_{ij}))$ for every $r \leq k$ and $|\mu| = r$. Many geometric objects can be put in this new framework. Just to exemplify we mention two of them.

Example 1.1. Given a linear group $G \subset GL(n; \mathbb{R})$, if we take $B_0 = C^k$, B_1 as the presheaf of C^1 -functions whose jacobian matrix belongs to G and $B_r = C^{k-r}$, with $r \geq 2$, then a B^k -manifold describes a C^k -manifold endowed with a G -structure in the sense of [15, 20]. This example includes many interesting situations, such as semi-riemannian manifolds, almost-complex manifolds, regularly foliated manifolds, etc.

Example 1.2. For a fixed $p \in [1, \infty]$, consider $B_r = L^p$ for every r . Then a B^k -manifold is a C^k -manifold whose transition functions and its derivatives are L^p -integrable. This means that $\varphi_{ji} \in W^{k,p}(\varphi_i(U_{ij})) \cap C^k(\varphi_i(U_{ij}))$, where $W^{k,p}(\Omega)$ are the Sobolev spaces, so that a $(L^p)^k$ -manifold is a “Sobolev manifold”. The case $p = \infty$ corresponds to the notion of k -bounded structure on a n -manifold [9, 19, 12].

This is the first of a sequence of articles which aim to begin the development of a general theory of B^k -manifolds, including the unification of certain aspects of G -structures and Sobolev structures on C^k -manifolds. Our focus is on regularity results for geometric objects on B^k -manifolds. The motivation is as follows. When studying geometry on a smooth manifold M we usually know that geometric objects τ on M (e.g, affine connections, bundles, nonlinear differential operators of a specific type, solutions of differential equations, etc.) exist with certain regularity, meaning that their local coefficients $\tau_{j_1, \dots, j_s}^{i_1, \dots, i_r}$ belong to some space, but in many situations we would like to have existence of more regular objects, meaning to have τ such that $\tau_{j_1, \dots, j_s}^{i_1, \dots, i_r}$ belong to a smaller space.

For instance, in gauge theory and when we need certain uniform bounds on the curvature, it is desirable to work with connections whose coefficients are not only smooth, but actually L^p -integrable or even uniformly bounded [22, 9]. In the study of holomorphic geometry, we consider holomorphic or hermitian connections over complex manifolds, whose coefficients are holomorphic [14, 3]. It is also interesting to consider symplectic connections, which are important in formal deformation quantization of symplectic manifolds [6, 10]. These examples immediately extend to the context of differential operators, since they can be regarded as functions depending on connections on vector bundles over M [5]. The regularity problem for solutions of differential equations is also very standard and by means of Sobolev-type embedding results and bootstrapping methods it is strongly related to the regularity problem for the underlying differential operator defining the differential equation [13].

Since the geometric structures are defined on M , the existence of some τ such that $\tau_{j_1, \dots, j_s}^{i_1, \dots, i_r}$ satisfy a specific regularity condition should depend crucially on the existence of a (non-necessarily maximal) subatlas for M whose transition functions themselves satisfy additional regularity conditions, leading us to consider B^k -manifolds in the sense introduced above. Let S be a presheaf of Fréchet spaces and let us say that τ is S -regular if there exists an open covering of M by charts $\varphi_i : U_i \rightarrow \mathbb{R}^n$ in which $\tau_{j_1, \dots, j_s}^{i_1, \dots, i_r} \in S(\varphi_i(U_{ij}))$. The geometric motivation above can then be rephrased into the following problem:

- let M be a C^k -manifold. Given S , find some presheaf $B(S)$, depending on S , such that if M has a $B(S)^k$ -structure, then there exist S -regular geometric structures on it.

In the sequence of articles we intent to work on this question for different kinds of geometric structures: affine connections, nonlinear tensors, differential operators and fiber bundles. Notice, on the other hand, that if S is a presheaf for which the previous problem has a solution for some kind of S -regular geometric structures, we have an immediate existence question:

- under which conditions does a C^k -manifold admit a $B(S)^k$ -structure?

Opening this sequence of works, in the present one we will discuss the general existence problem of B^k -structures on C^k -manifolds. Actually, we will work on more general entities, which we call $B_{\alpha,\beta}^k$ -manifolds, where α and β are functions and the transition functions satisfy $\partial^\mu \varphi_{ji} \in B_{\alpha(r)}(\varphi_i(U_{ij})) \cap C^{k-\beta(r)}(\varphi_i(U_{ij}))$. Thus, for $\alpha(r) = r = \beta(r)$ a $B_{\alpha,\beta}^k$ -manifold is the same as a B^k -manifold. We also study some aspects of the presheaf-category $\mathbf{C}_{n,\beta}^{k,\alpha}$ of the presheaves B . The present paper has the following two main results:

Theorem A. *There are full embeddings*

- (1) $\iota : \mathbf{C}_{n,\beta;l}^{k,\alpha} \hookrightarrow \mathbf{C}_{n,\beta}^{l,\alpha}$, if $l \leq k$;
- (2) $f_* : \mathbf{C}_{n,\beta}^{k,\alpha} \hookrightarrow \mathbf{C}_{r,\beta'}^{k,\alpha}$, for any continuous injective map $f : \mathbb{R}^n \rightarrow \mathbb{R}^r$;
- (3) $j : \mathbf{C}_{n,\beta;\beta'}^{k,\alpha} \hookrightarrow \mathbf{C}_{r,\beta'}^{k,\alpha}$, if $\beta' \leq \beta$.

Theorem B. *If B is ordered, fully left-absorbing (resp. fully right-absorbing) and has retractible (B, k, α, β) -diffeomorphisms, all of this in the same intersection presheaf \mathbb{X} , then the choice of a retraction \bar{r} induces a left-adjoint (resp. right-adjoint) for the forgetful functor F from $B_{\alpha,\beta}^k$ -manifolds to C^k -manifolds, which actually independes of \bar{r} . In particular, if B is fully absorbing, then F is ambidextrous adjoint.*

This article is structured as follows. In Section 2 we introduce the classes of Fréchet spaces which will be used in the next sections. These are the (Γ, ϵ) -spaces, where Γ is a set of indexes (in general $\Gamma \subset \mathbb{Z}_{\geq 0}$) and $\epsilon : \Gamma \times \Gamma \rightarrow \Gamma$ is some function. The (Γ, ϵ) -spaces are itself sequences $B = (B_i)_{i \in \Gamma}$ of nuclear Fréchet spaces. The map ϵ is used in order to consider *multiplicative structures* on B , which are given by a family of continuous linear maps $*_{ij} : B_i \otimes B_j \rightarrow B_{\epsilon(i,j)}$, where the tensor product is the projective one. Many other structures on B are required, such as additive structures, distributive structures, intersection structures and closure structures.

In Section 3 we define the $C_{\alpha,\beta}^k$ -presheaves, which are the structural presheaves for the $B_{\alpha,\beta}^k$ -manifolds, and study the presheaf category of them. We begin by constructing a sheaf-theoretic version of the concepts and results of Section 2. Thus, in this section B is not a single (Γ, ϵ) -space, but a presheaf of them on \mathbb{R}^n . The $C_{\alpha,\beta}^k$ -presheaves are those presheaves of (Γ, ϵ) -spaces which are well-related with the presheaf $C^{k-\beta}$ in a sense defined in that section. In Section 4 Theorem A is proved.

In Section 5 $B_{\alpha,\beta}^k$ -manifolds are more precisely defined and some basic properties are proven. For instance, we establish conditions on B , α and β , under which the category $\mathbf{Diff}_{\alpha,\beta}^{B,k}$ of $B_{\alpha,\beta}^k$ -manifolds and $B_{\alpha,\beta}^k$ -morphisms between them becomes well-defined. Some examples are also given. Finally, in Section 6 we introduce the notions of ordered presheaf, fully left/right-absorbing presheaf and presheaf with retractible (B, k, α, β) -diffeomorphisms, which are needed for the first theorem.

A proof of the Theorem B is given and as a consequence we get the existence of some limits and colimits in the category of $B_{\alpha,\beta}^k$ -manifolds.

2 (Γ, ϵ) -Spaces

Let **Set** be the category of all sets and let $\Gamma \in \mathbf{Set}$ be a set of indexes¹. A *nuclear Fréchet Γ -space* (or Γ -space, for short) is a family $(B_i)_{i \in \Gamma}$ of nuclear Fréchet spaces. Equivalently, it is a Γ -graded vector space $B = \bigoplus_i B_i$ whose components are nuclear Fréchet Γ -spaces. In other words, it is a functor $B : \Gamma \rightarrow \mathbf{NFre}$ from the discrete category defined by Γ to the category of nuclear Fréchet spaces and continuous linear maps. Morphisms are pairs (ξ, μ) , where $\mu : \Gamma \rightarrow \Gamma$ is a function and ξ is a family of continuous linear maps $\xi_i : B_i \rightarrow B'_{\mu(i)}$, with $i \in \Gamma$. Thus, it is an endofunctor μ together with a natural transformation $\xi : B \Rightarrow B' \circ \mu$. Composition is defined by horizontal composition of natural transformations.

Let \mathbf{Fre}_Γ be the category of Γ -spaces and notice that we have an inclusion $\iota : [\Gamma; \mathbf{NFre}] \hookrightarrow \mathbf{NFre}_\Gamma$ given by $\iota(B) = B$ and $\iota(\xi) = (\xi, id_\Gamma)$, where $[\mathbf{C}; \mathbf{D}]$ denotes the functor category. Since ι is faithful it reflects monomorphisms and epimorphisms, which means that if a morphism (ξ, μ) in \mathbf{NFre}_Γ is such that each ξ_i is a monomorphism (resp. epimorphism) in \mathbf{NFre} , then it is a monomorphism (resp. epimorphism) in \mathbf{NFre}_Γ . We have a functor $\mathcal{F} : \mathbf{Set}^{op} \rightarrow \mathbf{Cat}$ such that $\mathcal{F}(\Gamma) = \mathbf{Fre}_\Gamma$.

More generally, let us define a (Γ, ϵ) -space as a Γ -space B endowed with a map $\epsilon : \Gamma \times \Gamma \rightarrow \Gamma$. A morphism between a (Γ, ϵ) -space B and a (Γ, ϵ') -space B' is a morphism (ξ, μ) of Γ -spaces such that the first diagram below commutes. Let $\mathbf{Fre}_{\Gamma, \epsilon}$ be the category of (Γ, ϵ) -spaces for a fixed ϵ and let $\mathbf{NFre}_{\Gamma, \Sigma}$ be the category of (Γ, ϵ) -spaces for all ϵ . Notice that $\mathbf{NFre}_{\Gamma, \Sigma} \simeq \coprod_\epsilon \mathbf{NFre}_{\Gamma, \epsilon}$ and that there exists a forgetful functor $U : \mathbf{NFre}_{\Gamma, \Sigma} \rightarrow \mathbf{NFre}_\Gamma$ that forgets ϵ .

$$\begin{array}{ccc}
 \Gamma \times \Gamma & \xrightarrow{\epsilon} & \Gamma \\
 \mu \times \mu \downarrow & & \downarrow \mu \\
 \Gamma \times \Gamma & \xrightarrow{\epsilon'} & \Gamma
 \end{array}
 \quad
 \mathbf{NFre}_{\Gamma, \Sigma} \xrightarrow{\simeq} \coprod_\epsilon \mathbf{NFre}_{\Gamma, \epsilon} \xrightarrow{\coprod_\epsilon F_\epsilon} \coprod_\epsilon \mathbf{NFre}_{\Gamma \times \Gamma} \xrightarrow{\nabla} \mathbf{NFre}_{\Gamma \times \Gamma}$$

$\xrightarrow{F_\Sigma}$

For any $\epsilon : \Gamma \times \Gamma \rightarrow \Gamma$ we have a functor $F_\epsilon : \mathbf{NFre}_{\Gamma, \epsilon} \rightarrow \mathbf{NFre}_{\Gamma \times \Gamma}$ defined by $F_\epsilon(B, \epsilon) = B \circ \epsilon$ and $F_\epsilon(\xi, \mu) = (\xi \circ \epsilon, \mu \circ \epsilon)$. In the following we will write B_ϵ to denote $B \circ \epsilon$. Notice that F_ϵ is well-defined precisely because of the commutativity of the first diagram. Thus, we also have a functor $F_\Sigma : \mathbf{NFre}_{\Gamma, \Sigma} \rightarrow \mathbf{NFre}_{\Gamma \times \Gamma}$ given by the composition above, where ∇ is the codiagonal. Furthermore, recalling that the category of Fréchet spaces is symmetric monoidal with the projective tensor product \otimes [21, 8], given Γ, Γ' we have a functor

$$\otimes_{\Gamma, \Gamma'} : \mathbf{NFre}_\Gamma \times \mathbf{NFre}_{\Gamma'} \rightarrow \mathbf{NFre}_{\Gamma \times \Gamma'}$$

playing the role of an *external product*, given by $(B \otimes B')_{(i,j)} = B_i \otimes B_j$ on objects and by $(\xi, \mu) \otimes (\eta, \nu) = (\xi \otimes \eta, \mu \times \nu)$ on morphisms. By composing with the diagonal and the forgetful functor U ,

¹In all the article and in the examples considered Γ will be $\mathbb{Z}_{\geq 0}$ or some finite product of copies of $\mathbb{Z}_{\geq 0}$. However, all the results of this section can be generalized to the case in which Γ is an arbitrary (i.e, not necessarily discrete) category.

for each Γ we get the functor \otimes_Γ below.

$$\mathbf{NFre}_{\Gamma,\Sigma} \xrightarrow{U} \mathbf{NFre}_\Gamma \xrightarrow{\Delta} \mathbf{NFre}_\Gamma \times \mathbf{NFre}_\Gamma \xrightarrow{\otimes_{\Gamma,\Gamma}} \mathbf{NFre}_{\Gamma \times \Gamma}$$

\otimes_Γ

A *multiplicative structure* in a (Γ, ϵ) -space (B, ϵ) is a morphism $(*, \epsilon) : \otimes_\Gamma(B, \epsilon) \rightarrow F_\Sigma(B, \epsilon)$, i.e., a family $*_{ij} : B_i \otimes B_j \rightarrow B_{\epsilon(i,j)}$ of continuous linear maps, with $i, j \in \Gamma$. A *multiplicative Γ -space* is a (Γ, ϵ) space in which a multiplicative structure has been fixed. A *weak morphism* between two multiplicative Γ -spaces $(B, \epsilon, *)$ and $(B', \epsilon', *')$ is an arbitrary morphism in the arrow category of $\mathbf{NFre}_{\Gamma \times \Gamma}$, i.e, it is given by morphisms $(\zeta, \psi) : \otimes_\Gamma(B, \epsilon) \rightarrow \otimes_\Gamma(B', \epsilon')$ and $(\eta, \nu) : F_\Sigma(B, \epsilon) \rightarrow F_\Sigma(B', \epsilon')$ in the category $\mathbf{NFre}_{\Gamma \times \Gamma}$ such that the diagrams below commute.

$$\begin{array}{ccc} B_i \otimes B_j & \xrightarrow{*_{ij}} & B_{\epsilon(i,j)} \\ \zeta_{ij} \downarrow & & \downarrow \eta_{\epsilon(i,j)} \\ (B' \otimes B')_{\psi(i,j)} & \xrightarrow{*'_{\psi(i,j)}} & B'_{\epsilon'(\psi(i,j))} \end{array} \quad \begin{array}{ccc} \Gamma \times \Gamma & \xrightarrow{\epsilon} & \Gamma \\ \psi \downarrow & & \downarrow \nu \\ \Gamma \times \Gamma & \xrightarrow{\epsilon'} & \Gamma \end{array}$$

A *strong morphism* (or simply *morphism*) between multiplicative Γ -spaces is a weak morphism such that there exists $(\xi, \mu) : (B, \epsilon) \rightarrow (B', \epsilon')$ for which $(\zeta, \psi) = \otimes_\Gamma(\xi, \mu)$ and $(\eta, \nu) = F_\Sigma(\xi, \mu)$. In explicit terms this means that $\nu = \mu$, $\psi = \mu \times \mu$, $\zeta = \xi \otimes \xi$ and $\eta = \xi$, so that the diagrams above become the diagrams below. Notice that the second diagram only means that (ξ, μ) is a morphism of (Γ, ϵ) -spaces. We will denote by $\mathbf{NFre}_{\Gamma,*}$ the category of multiplicative Γ -spaces and strong morphisms.

$$\begin{array}{ccc} B_i \otimes B_j & \xrightarrow{*_{ij}} & B_{\epsilon(i,j)} \\ \xi_i \otimes \xi_j \downarrow & & \downarrow \xi_{\epsilon(i,j)} \\ B'_{\mu(i)} \otimes B'_{\mu(j)} & \xrightarrow{*'_{\mu(i)\mu(j)}} & B'_{\epsilon'(\mu(i)\mu(j))} \end{array} \quad \begin{array}{ccc} \Gamma \times \Gamma & \xrightarrow{\epsilon} & \Gamma \\ \mu \times \mu \downarrow & & \downarrow \mu \\ \Gamma \times \Gamma & \xrightarrow{\epsilon'} & \Gamma \end{array}$$

An *additive Γ -space* is a multiplicative Γ -space $(B, +, \epsilon)$ such that $\epsilon_{ii} = i$ and such that $+_{ii} : B_i \otimes B_i \rightarrow B_i$ is the addition $+_i$ in B_i . Let $\mathbf{NFre}_{\Gamma,+} \subset \mathbf{NFre}_{\Gamma,*}$ denote the full subcategory of additive Γ -spaces. The inclusion has a left-adjoint $F(*, \epsilon) = (+, \delta)$ given by

$$\delta(i, j) = \begin{cases} \epsilon(i, j), & i \neq j \\ i & j = i \end{cases} \quad \text{and} \quad +_{ij} = \begin{cases} *_{ij}, & i \neq j \\ +_i & j = i \end{cases}.$$

Example 2.1. Any Γ -space $B = (B_i)_{i \in \Gamma}$ admits many trivial additive structures, defined as follows. Let $\theta : \Gamma \times \Gamma \rightarrow \Gamma$ be any function and define an additive structure by

$$\delta_\theta(i, j) = \begin{cases} \theta(i, j), & i \neq j \\ i & j = i \end{cases} \quad \text{and} \quad +_{ij} = \begin{cases} 0_{\theta(i,j)}, & i \neq j \\ +_i & j = i \end{cases},$$

where $0_{\theta(i,j)}$ is the function constant in the zero vector of $B_{\theta(i,j)}$. All these trivial structures are actually strongly isomorphic. Indeed, let B_θ denote B with the trivial additive structure defined by θ . As one can see, we have a natural bijection

$$\mathbf{IMor}_{\Gamma,+}(B_\theta, B_{\theta'}) \simeq \mathbf{IMor}_\Gamma(B, B'),$$

where the right-hand side is the set of μ -injective morphisms between B and B' , i.e, morphisms (ξ, μ) of Γ -spaces such that $\mu : \Gamma \rightarrow \Gamma$ is injective. Furthermore, this bijection preserve isomorphisms, so that we also have a bijection

$$\text{IISO}_{\Gamma,+}(B_{\theta}, B'_{\theta'}) \simeq \text{IISO}_{\Gamma}(B, B').$$

Since for $B' = B$ the right-hand side contains at least the identity, it follows that $B_{\theta} \simeq B_{\theta'}$ for every θ and θ' .

Example 2.2. Suppose that Γ has a partial order \leq . Any increasing or decreasing sequence of nuclear Fréchet spaces, i.e, such that $B_i \subset B_j$ if either $i \leq j$ or $j \leq i$, has a canonical additive structure, where $+_{ij}$ is given by the sum in $B_{\max(i,j)}$ or $B_{\min(i,j)}$, respectively, so that $\delta(i, j) = \max(i, j)$ or $\delta(i, j) = \min(i, j)$. In particular, for any open set $U \subset \mathbb{R}^n$ and any order-preserving integer function $p : \Gamma \rightarrow [0, \infty]$, the sequences $B_i = L^{p(i)}(U)$ and $B'_i = C^{p(i)}(U)$ are decreasing, where in $L^{p(i)}(U)$ we take the Banach structure given by the $L^{p(i)}$ -norm and in $C^{p(i)}(U)$ we consider Fréchet family of seminorms

$$\|f\|_{r,l} = \sup_{|\mu|=r} \sup_{x \in K_l} \|\partial^{\mu} f(x)\|, \quad (1)$$

where $0 \leq r \leq p(i)$ and (K_l) is any countable sequence of compacts such that every other compact $K \subset U$ is contained in K_l for some l [21]. Thus, both sequences become an additive Γ -space in a natural way. They will be respectively denoted by $L^p(U)$ and $C^p(U)$. We will be specially interested in the function $p(i) = k - \alpha(i)$, for $i \leq k$, where α is some other function. In this case, we will write $C^{k-\alpha}(U)$ instead of $C^p(U)$. More precisely, we will consider the sequence given by $B_i = C^{k-\alpha(i)}(U)$, if $\alpha(i) \leq k$, and $B_i = 0$, if $\alpha(i) \geq k$. In this situation, $\delta(i, j) = k - \max(\alpha(i), \alpha(j))$.

Example 2.3. Similarly, if $U \subset \mathbb{R}^n$ is a nice open set such that the Sobolev embeddings are valid [13], then for any fixed integers $p, q, r > 0$ with $p < q$, we have a continuous embedding $W^{r,p}(U) \hookrightarrow W^{\ell(r),q}(U)$, where $\ell(r) = r + n \frac{(p-q)}{pq}$ and $W^{k,p}(U)$ is the Sobolev space. Thus, by defining $B_i = W^{l(i),q}(U)$, where $l(i) = \ell^i(r)$, i.e, $l(0) = r$, $l(1) = \ell(r)$ and $l(i) = \ell(\ell(\dots \ell(r)\dots))$, we get again a decreasing sequence of Banach spaces and therefore an additive $\mathbb{Z}_{\geq 0}$ -space.

Example 2.4. As in Example 2.2, suppose Γ ordered and consider an integer function $p : \Gamma \rightarrow [0, \infty]$. In any open set $U \subset \mathbb{R}^n$ pointwise multiplication of real functions give us bilinear maps $\cdot_{ij} : C^{p(i)}(U) \times C^{p(j)}(U) \rightarrow C^{\min(p(i), p(j))}(U)$, which are continuous in the Fréchet structure (1), so that with $\delta(i, j) = \min(p(i), p(j))$, the $\mathbb{Z}_{\geq 0}$ -space $C^p(U)$ is multiplicative. If $p(i) = k - \alpha(i)$, then $\delta(i, j) = k - \max(\alpha(i), \alpha(j))$.

Example 2.5. From Hölder's inequality, pointwise multiplication of real functions also defines continuous bilinear maps $\cdot_{ij} : L^i(U) \times L^j(U) \rightarrow L^{i \star j / (i+j)}(U)$ for $i, j \geq 2$ such that $i \star j = i \cdot j / (i+j)$ is an integer [13, 21]. Let \mathbb{Z}_S be the set of all integers $m \geq 2$ such that for every two given $m, m' \in \mathbb{Z}_S$ the sum $m + m'$ divides the product $m \cdot m'$. Suppose Γ ordered and choose a function $p : \Gamma \rightarrow \mathbb{Z}_S$. Then $L^p(U)$ has a multiplicative structure with $\epsilon(i, j) = p(i) \star p(j)$. Even if $i \star j$ is not an integer we get a multiplicative structure. Indeed, let $\epsilon'(i, j) = \text{int}(\epsilon(i, j))$, where $\text{int}(k)$ denotes the integer part of a real number. Then $\epsilon'(i, j) \leq \epsilon(i, j)$, so that $L^{\epsilon(i,j)}(U) \subset L^{\epsilon'(i,j)}(U)$, and we can assume \cdot_{ij} as taking values in $L^{\epsilon'(i,j)}(U)$.

Example 2.6. Given $1 \leq i, j \leq \infty$, notice that $i \star j \geq 1/2$, which is equivalent to saying the number $r(i, j) = i \cdot j / (i + j - i \cdot j)$, i.e, the solution of

$$\frac{1}{i} + \frac{1}{j} = \frac{1}{r} + 1,$$

also satisfies $1 \leq r \leq \infty$. Thus, from Young's inequality, convolution product defines a continuous bilinear map [13, 21]

$$*_{ij} : L^i(\mathbb{R}^n) \times L^j(\mathbb{R}^n) \rightarrow L^{r(i,j)}(\mathbb{R}^n),$$

so that for any function $p : \Gamma \rightarrow [1, \infty]$ the sequence $L^p(\mathbb{R}^n)$ has a multiplicative structure with $\epsilon(i, j) = r(p(i), p(j))$.

Example 2.7. A *Banach Γ -space* is a Γ -space $B = (B_i)_{i \in \Gamma}$ such that each B_i is a Banach space. Suppose that Γ is actually a monoid $(\Gamma, +, 0)$. Notice that a multiplicative structure in B with $\epsilon(i, j) = i + j$ is a family of continuous bilinear maps $*_{ij} : B_i \times B_j \rightarrow B_{i+j}$. Since the category of Banach spaces and continuous linear maps has small coproducts, the coproduct $\mathcal{B} = \bigoplus_{i \in \Gamma} B_i$ exists as a Banach space and $(\mathcal{B}, *)$ is actually a Γ -graded normed algebra.

We say that two multiplicative structures $(*, \epsilon)$ and $(+, \delta)$ on the same Γ -space B are *left-compatible* if the following diagrams are commutative. The first one makes sense precisely because the second one is commutative (σ is the map $\sigma(x, (y, z)) = ((x, y), (x, z))$). A *left-distributive Γ -space* is a Γ -space endowed with two left-compatible multiplicative structures. Morphisms are just morphisms of Γ -spaces which preserve both multiplicative structures. Let $\mathbf{Fre}_{\Gamma, *}^l$ be the category of all of them. Our main interest will be when $(+, \delta)$ is actually an additive structure, justifying the notation.

$$\begin{array}{ccc} B_i \otimes (B_j \otimes B_k) & \xrightarrow{\sigma} & (B_i \otimes B_j) \otimes (B_i \otimes B_k) \xrightarrow{*_{ij} \otimes *_{ik}} B_{\epsilon(i,j)} \otimes B_{\epsilon(i,k)} \\ \text{id} \otimes +_{jk} \downarrow & & \downarrow +_{\epsilon(i,j)\epsilon(i,k)} \\ B_i \otimes B_{\delta(j,k)} & \xrightarrow{*_{i\delta(j,k)}} & B_{\epsilon(i,\delta(j,k))} \end{array}$$

$$\begin{array}{ccc} \Gamma \times (\Gamma \times \Gamma) & \xrightarrow{\sigma} & (\Gamma \times \Gamma) \times (\Gamma \times \Gamma) \xrightarrow{\epsilon \times \epsilon} \Gamma \times \Gamma \\ \text{id} \times \delta \downarrow & & \downarrow \delta \\ \Gamma \times \Gamma & \xrightarrow{\epsilon} & \Gamma \end{array}$$

In a similar way, we say that $(*, \epsilon)$ and $(+, \delta)$ are *right-compatible* if the diagrams below commute, with $\sigma((x, y), z) = ((x, z), (y, z))$. A *right-distributive Γ -space* is a Γ -space together with a right-compatible structure. Morphisms are morphisms of Γ -spaces preserving those structures. Let $\mathbf{Fre}_{\Gamma, *}^r$ be the corresponding category. Finally, let $\mathbf{Fre}_{\geq, +, *} = \mathbf{Fre}_{\Gamma, +, *}^l \cap \mathbf{Fre}_{\Gamma, +, *}^r$ be the category of *distributive Γ -spaces*, i.e, the category of Γ -spaces endowed with two multiplicative structures which are both left-compatible and right-compatible.

$$\begin{array}{ccc} (B_i \otimes B_j) \otimes B_k & \xrightarrow{\sigma} & (B_i \otimes B_k) \otimes (B_j \otimes B_k) \xrightarrow{*_{ik} \otimes *_{jk}} B_{\epsilon(i,k)} \otimes B_{\epsilon(j,k)} \\ +_{ij} \otimes \text{id} \downarrow & & \downarrow +_{\epsilon(i,k)\epsilon(j,k)} \\ B_{\delta(i,j)} \otimes B_k & \xrightarrow{*_{\delta(i,j)k}} & B_{\epsilon(\delta(i,j),k)} \end{array}$$

$$\begin{array}{ccccc}
(\Gamma \times \Gamma) \times \Gamma & \xrightarrow{\sigma} & (\Gamma \times \Gamma) \times (\Gamma \times \Gamma) & \xrightarrow{\epsilon \times \epsilon} & \Gamma \times \Gamma \\
\delta \times id \downarrow & & & & \downarrow \delta \\
\Gamma \times \Gamma & \xrightarrow{\epsilon} & & & \Gamma
\end{array}$$

Example 2.8. The additive and multiplicative structures for C^p described in Example 2.2 and Example 2.4 define a distributive structure. Any Γ -space B , when endowed with a trivial additive structure and the induced multiplicative structure, becomes a distributive Γ -space.

Example 2.9. The additive structure in L^p given in Example 2.2 is generally *not* left/right-compatible with the multiplicative structures of Example 2.5 and Example 2.6. Indeed, as one can check, the sum $+$ in L^p is compatible with the pointwise multiplication iff the function $p : \Gamma \rightarrow \mathbb{Z}_{\geq 0}$ is such that $p(i) \star p(j) \leq p(i) \star p(k)$ whenever $p(j) \leq p(k)$, where $i, j, k \in \Gamma$. For fixed p we can clearly restrict to the subset Γ_p in which the desired condition is satisfied, so that $(L^{p(i)})_{i \in \Gamma_p}$ is distributive for every p . Similarly, the $+$ is compatible with the convolution product $*$ in $L^{p(\cdot)}$ iff $r(p(i), p(j)) \leq r(p(i), p(k))$ whenever $p(j) \leq p(k)$.

A Γ -ambient is a pair (\mathbf{X}, γ) , where \mathbf{X} is a category with pullbacks and γ is an embedding $\gamma : \mathbf{NFre}_\Gamma \hookrightarrow \mathbf{X}$. Let $X \in \mathbf{X}$ and consider the corresponding slice category $\gamma(\mathbf{NFre}_\Gamma)/X$, i.e, the category of morphisms $\gamma(B) \rightarrow X$ in \mathbf{X} , with B a Γ -space, and commutative triangles with vertex X . Let $\text{Sub}_\Gamma(X, \gamma)$ be the full subcategory whose objects are monomorphisms $\iota : \gamma(B) \hookrightarrow X$. If $(\gamma(B), \iota)$ belongs to $\text{Sub}_\Gamma(X, \gamma)$ we say that it is a Γ -subspace of (X, γ) . Finally, let $\text{Span}_\Gamma(X, \gamma)$ be the category of spans $\gamma(B) \hookrightarrow X \leftarrow \gamma(B')$ of Γ -subspaces of (X, γ) . If $(\gamma(B), \iota, \gamma(B'), \iota')$ is a span, we say that $(\mathbf{X}, \gamma, X, \iota, \iota')$ is an *intersection structure between B and B' in X* and we define the corresponding *intersection in X* as the object $B \cap_{X, \gamma} B' \in \mathbf{X}$ given by the pullback between ι and ι' ; the object X itself is called the *base object* for the intersection. We say that a span in $\text{Span}_\Gamma(X, \gamma)$ is *proper* if the corresponding pullback actually belongs to \mathbf{NFre}_Γ , i.e, if there exists some $L \in \mathbf{NFre}_\Gamma$ such that $\gamma(L) \simeq B \cap_{X, \gamma} B'$. Notice that, since γ is an embedding, when L exists it is unique up to isomorphisms. Thus, we will write $B \cap_X B'$ in order to denote any object in the isomorphism class. We will also require that the universal maps $u_\gamma : B \cap_{X, \gamma} B' \rightarrow \gamma(B)$ and $u'_\gamma : B \cap_{X, \gamma} B' \rightarrow \gamma(B')$ also induce maps $u : B \cap_X B' \rightarrow B$ and $u' : B \cap_X B' \rightarrow B'$, which clearly exist if the embedding γ is full. Let $\text{PSpan}_\Gamma(X, \gamma)$ be the full subcategory of proper spans.

$$\begin{array}{ccc}
B \cap_{X, \gamma} B' & \longrightarrow & \gamma(B') \\
\downarrow & & \downarrow \iota' \\
\gamma(B) & \xrightarrow{\iota} & X
\end{array}
\quad
\begin{array}{ccc}
B_\epsilon \cap_{X, \gamma_\Sigma} B'_{\epsilon'} & \longrightarrow & \gamma_\Sigma(B'_{\epsilon'}) \\
\downarrow & & \downarrow j' \\
\gamma_\Sigma(B_\epsilon) & \xrightarrow{j} & X
\end{array}
\tag{2}$$

Recall the functor $F_\Sigma : \mathbf{NFre}_{\Gamma, \Sigma} \rightarrow \mathbf{NFre}_{\Gamma \times \Gamma}$ assigning to each (Γ, ϵ) -space (B, ϵ) the corresponding $(\Gamma \times \Gamma)$ -space $B \circ \epsilon \equiv B_\epsilon$. Let $(\mathbf{X}, \gamma_\Sigma)$ be a $(\Gamma \times \Gamma)$ -ambient. Let $\text{Sub}_{\Gamma, \Sigma}(X, \gamma_\Sigma)$ be the category of monomorphisms $j : F_\Sigma(\gamma_\Sigma(B, \epsilon)) \hookrightarrow X$ for a fixed X , i.e, the category of (Γ, ϵ) -subspaces of X . The *intersection in X* between two of them is the second pullback above, where for simplicity we write $\gamma_\Sigma(B) \equiv F_\Sigma(\gamma_\Sigma(B, \epsilon))$. The intersection is *proper* if the resulting object belongs to $\mathbf{NFre}_{\Gamma, \Sigma}$. Let $\text{PSpan}_{\Gamma, \Sigma}(X, \gamma_\Sigma)$ be the category of those spans. Now, let $(B, *, \epsilon)$ and $(B', *', \epsilon')$ be two multiplicative Γ -spaces. An *intersection structure* between them is a tuple $\mathbb{X} = (\mathbf{X}, \gamma_\Sigma, X, j, j')$ consisting of a $(\Gamma \times \Gamma)$ -ambient $(\mathbf{X}, \gamma_\Sigma)$, and object X and a span (j, j') , as in (2). The *intersection*

object between $*$ and $*$ ' in \mathbb{X} is the pullback $\text{pb}(*, *'; X, \gamma_\Sigma)$ below. By universality we get the dotted arrow $* \cap *'$. We say that an intersection structure is *proper* if not only the span (j, j') is proper, but also the intersection object belongs to $\mathbf{NFre}_{\Gamma, \Sigma}$ (in the same previous sense). In this case, the object in $\mathbf{NFre}_{\Gamma, \Sigma}$ will be denoted by $\text{pb}(*, *'; X)$, its components by $\text{pb}_{ij}(*, *'; X)$ and we will define the *intersection number function* between $*$ and $*$ ' in \mathbb{X} as the function $\#_{*, *'; \mathbb{X}} : \Gamma \times \Gamma \rightarrow [0, \infty]$ given by $\#_{*, *'; \mathbb{X}}(i, j) = \dim_{\mathbb{R}} \text{pb}_{ij}(*, *'; X)$. We say that two multiplicative Γ -spaces $(B, *, \epsilon)$ and $(B', *, \epsilon')$ (or that $*$ and $*$ ' have *nontrivial intersection in \mathbb{X}* if $\#_{*, *'; \mathbb{X}} \geq 1$. Finally, we say that $*$ and $*$ ' are *nontrivially intersecting* if they have nontrivial intersection in some intersection structure.

$$\begin{array}{ccc}
\text{pb}(*, *'; X, \gamma_\Sigma) & \xrightarrow{\quad} & \gamma_\Sigma(B'_{\epsilon'} \otimes B_{\epsilon}) \\
\downarrow & \searrow^{* \cap *'} & \downarrow \gamma_\Sigma(*') \\
& & B_{\epsilon} \cap_{X, \gamma_\Sigma} B'_{\epsilon'} \xrightarrow{\quad} \gamma_\Sigma(B'_{\epsilon'}) \\
& & \downarrow & \downarrow j' \\
\gamma_\Sigma(B_{\epsilon} \otimes B_{\epsilon}) & \xrightarrow{\gamma_\Sigma(*)} & \gamma_\Sigma(B_{\epsilon}) \xrightarrow{j} & X
\end{array}$$

Example 2.10. Let (\mathbf{X}, γ) be any Γ -ambient such that \mathbf{X} has coproducts. Given B, B' , let $X = \gamma(B) \oplus \gamma(B')$ and notice that we have monomorphisms $\iota : \gamma(B) \rightarrow X$ and $\iota' : \gamma(B') \rightarrow X$, so that we can consider the intersection $B \cap_{X, \gamma} B'$ having the coproduct as a base object. However, this is generally trivial. For instance, if $\mathbf{X} = \mathbf{Set}_\Gamma$ is the category of Γ -sets, i.e, sequences $(X_i)_{i \in \Gamma}$ of sets, then the intersection above is the empty Γ -set, i.e, $X_i = \emptyset$ for each $i \in \Gamma$. Furthermore, if \mathbf{X} is some category with null objects which is freely generated by Γ -sets, i.e, such that there exists a forgetful functor $U : \mathbf{X} \rightarrow \mathbf{Set}_\Gamma$ admitting a left-adjoint, then the intersection $B \cap_{X, \gamma} B'$ is a null object. In particular, if $\mathbf{X} = \mathbf{Vec}_{\mathbb{R}, \Gamma}$ is the category of Γ -graded real vector spaces, with Span_Γ denoting the left-adjoint, then the intersection object $\text{Span}_\Gamma(B \cap_{X, \gamma} B')$ is the trivial Γ -graded vector space. Let us say that an intersection structure \mathbb{X} is *vectorial* if is proper and defined in a Γ -ambient $(\mathbf{Vec}_{\mathbb{R}, \Gamma}, \gamma)$ such that γ create null-objects (in other words, B is the trivial Γ -space iff $\gamma(B)$ is the trivial Γ -vector space, i.e, iff $\gamma(B)_i \simeq 0$ for each i). In this case, it then follows that for the base object $X = \gamma(B) \oplus \gamma(B')$ we have $B \cap_{X, \gamma} B' \simeq 0$ when regarded as a Γ -space.

Example 2.11. The intersection between two Γ -spaces in a vectorial intersection structure is not necessarily trivial; it actually depends on the base object X . Indeed, suppose that B and B' are such that there exists a Γ -set S for which $B \subset S$ and $B' \subset S$. Let $X = \text{Span}_\Gamma(B \cup B')$, where the union is defined componentwise. We have obvious inclusions and the corresponding intersection object is given by $B \cap_{X, \gamma} B' \simeq \text{Span}_\Gamma(B \cap B')$, where the right-hand side is the intersection as Γ -vector spaces, defined componentwise, which is nontrivial if $B \cap B'$ is nonempty as Γ -sets. For instance, $C^p(U) \cap_{X, \gamma} C^q(U)$ and $C^p(U) \cap_{X, \gamma} L^q(U)$ are nontrivial in them for every p, q . We will say that a vectorial intersection structure with base object $X = \text{Span}_\Gamma(B \cup B')$ is a *standard intersection structure*. Notice that two standard intersection structures differ from the choice of γ and from the maps ι and ι' which define the span in X .

Vectorial intersection structures have the following good feature:

Proposition 2.1. *Let $(B, \epsilon, *)$ and $(B', \epsilon', *')$ be multiplicative structures and let \mathbb{X} be a vectorial intersection structure between them. If at least B or B' is nontrivial, then the intersection space*

$\text{pb}(*, *'; X)$ is nontrivial too, independently of the base object X . In other words, nontrivial multiplicative structures have nontrivial intersection in any vectorial intersection structure.

Proof. Since γ_Σ creates null objects, it is enough to work in the category of Γ -vector spaces. On the other hand, since a Γ -vector space $V = (V_i)_{i \in \gamma}$ is nontrivial iff at least one V_i is nontrivial, it is enough to work with vector spaces. Thus, just notice that if $T : V \otimes W \rightarrow Z$ and $T' : V' \otimes W' \rightarrow Z$ are arbitrary linear maps, then the pullback between them contains a copy of $V \oplus W \oplus V' \oplus W'$. \square

Corollary 2.1. *Nontrivial multiplicative structures are always nontrivially intersecting.*

Proof. Straightforward from the last proposition and from the fact that vectorial intersection structures exist. \square

Remark 2.1. The proposition explains that the intersection space $\text{pb}(*, *'; X)$ can be nontrivial even if the intersection $B_\epsilon \cap_X B'_\epsilon$ is trivial.

Let $(B, *, +, \epsilon, \delta)$ and $(B', *', +', \epsilon', \delta')$ be two distributive Γ -spaces. A *distributive intersection structure* between them is an intersection structure $\mathbb{X}_{*, *'}$ between $*$ and $*'$ together with an intersection structure $\mathbb{X}_{+, +'}$ between $+$ and $+'$ whose underlying Γ_Σ -ambient are the same. In other words, it is a tuple $\mathbb{X}_{*, +} = (\mathbf{X}, \gamma_\Sigma, X_*, X_+, J_*, J'_*, J_+, J'_+)$ such that (X_*, J_*, J'_*) and (X_+, J_+, J'_+) are spans as in (2). A *full intersection structure* between distributive Γ -spaces is a pair $\mathbb{X} = (\mathbb{X}_0, \mathbb{X}_{*, +})$, where $\mathbb{X}_{*, +} = (\mathbb{X}_*, \mathbb{X}_+)$ is a distributive intersection structure and $\mathbb{X}_0 = (\mathbf{X}_0, \gamma, X, \iota, \iota')$ is an intersection structure between the Γ -spaces B and B' , whose underlying ambient category \mathbf{X}_0 is equal to the ambient category \mathbf{X} of \mathbb{X}_* and \mathbb{X}_+ . We say that the triple $(\mathbf{X}, \gamma, \gamma_\Sigma)$ is a *full Γ -ambient* and that (X, X_*, X_+) is the *full base object*. We also say that \mathbb{X} is *vectorial* if both \mathbb{X}_0 , \mathbb{X}_* and \mathbb{X}_+ are vectorial. The *full intersection space* of the distributive structures B and B' in the full intersection structure \mathbb{X} is the triple consisting of the intersection spaces $B \cap_{X, \gamma} B'$ in \mathbb{X}_0 , $\text{pb}(*, *'; X_*, \gamma_\Sigma)$ in \mathbb{X}_* and $\text{pb}(+, +'; X_+, \gamma_\Sigma)$ in \mathbb{X}_+ , denoted simply by $B \cap_{\mathbb{X}} B'$. It is *nontrivial* (and in this case we say that B and B' have *nontrivial intersection in \mathbb{X}*) if each of the three components are nontrivial. When \mathbb{X} is vectorial, the object representing $B \cap_{\mathbb{X}} B'$ in $\mathbf{NFre}_\Gamma \times \mathbf{NFre}_{\Gamma, \Sigma} \times \mathbf{NFre}_{\Gamma, \Sigma}$ will also be denoted by $B \cap_{\mathbb{X}} B'$, i.e,

$$B \cap_{\mathbb{X}} B' = (B \cap_X B', \text{pb}(*, *'; X_*), \text{pb}(+, +'; X_+))$$

In some situations we will need to work with a special class of ambient categories \mathbf{X} which becomes endowed with a monoidal structure that has a nice relation with some closure operator. We finish this section introducing them. Let $(\mathbf{X}, \otimes, 1)$ be a monoidal category, let $\text{Ar}(\mathbf{X})$ be the arrow category and recall that we have two functors $s, t : \text{Ar}(\mathbf{X}) \rightarrow \mathbf{X}$ which assign to each map f its source $s(f)$ and target $t(f)$. A *weak closure operator* in \mathbf{X} is a functor $\text{cl} : \text{Ar}(\mathbf{X}) \rightarrow \text{Ar}(\mathbf{X})$ such that $t(\text{cl}(f)) = t(f)$. If $s(f) = B$ and $t(f) = X$, we write $\text{cl}_X(B)$ to denote $s(\text{cl}(f))$. A *weak closure structure* is a pair (cl, μ) given by a weak closure operator endowed with a natural transformation $\mu : \text{id} \Rightarrow \text{cl}$, translating the idea of embedding a space onto its closure. The monoidal product \otimes in \mathbf{X} induces a functor $\overline{\otimes}$ in the arrow category given $f \otimes f'$ on objects and as below on morphisms.

$$\begin{array}{ccc} \begin{array}{ccc} X & \xrightarrow{h_x} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{h_y} & Y' \end{array} & \overline{\otimes} & \begin{array}{ccc} Z & \xrightarrow{l_z} & Z' \\ g \downarrow & & \downarrow g' \\ W & \xrightarrow{l_w} & W' \end{array} \\ & & = & \begin{array}{ccc} X \otimes Z & \xrightarrow{h_x \otimes l_z} & X' \otimes Z' \\ f \otimes g \downarrow & & \downarrow f' \otimes g' \\ Y \otimes W & \xrightarrow{h_y \otimes l_w} & Y' \otimes W' \end{array} \end{array}$$

A *lax closure operator* in \mathbf{X} is a weak closure operator which is lax monoidal relative to $\bar{\otimes}$. This means that for any pair of arrows (f, f') we have a corresponding arrow morphism $\phi_{f,f'} : \text{cl}(f) \bar{\otimes} \text{cl}(f') \rightarrow \text{cl}(f \bar{\otimes} f')$ which is natural in f, f' . In particular, if $s(f) = B, s(f') = B', t(f) = X$ and $t(f') = X'$, we have a morphism $\phi : \text{cl}_X(B) \bar{\otimes} \text{cl}_{X'}(B') \rightarrow \text{cl}_{X \bar{\otimes} X'}(B \bar{\otimes} B')$. A *lax closure structure* is a weak closure structure (cl, μ) whose weak closure operator is actually a lax closure operator (cl, ϕ) such that the first diagram below commutes, meaning that ϕ and μ are compatible. If f and f' have source/target as above, we have in particular the second diagram.

$$\begin{array}{ccc}
& \text{cl}(f) \bar{\otimes} \text{cl}(f') & \\
\mu_f \bar{\otimes} \mu_{f'} \nearrow & \downarrow \phi_{f,f'} & \\
f \bar{\otimes} f' \xrightarrow{\mu_{f \bar{\otimes} f'}} & \text{cl}(f \bar{\otimes} f') &
\end{array}
\qquad
\begin{array}{ccc}
& \text{cl}_X(B) \bar{\otimes} \text{cl}_{X'}(B') & \\
\mu_B \bar{\otimes} \mu_{B'} \nearrow & \downarrow \phi_{B,B'} & \\
B \bar{\otimes} B' \xrightarrow{\mu_{B \bar{\otimes} B'}} & \text{cl}_{X \bar{\otimes} X'}(B \bar{\otimes} B') &
\end{array}
\tag{3}$$

A *monoidal Γ -ambient* is a Γ -ambient (\mathbf{X}, γ) whose ambient category \mathbf{X} is a monoidal category $(\mathbf{X}, \bar{\otimes}, 1)$ such that $\gamma : \mathbf{NFre}_\Gamma \rightarrow \mathbf{X}$ is a strong monoidal functor in the sense of [16, 2], i.e, lax and oplax monoidal in a compatible way. Thus, for any two Γ -spaces B, B' we have an isomorphism $\psi_{B,B'} : \gamma(B) \bar{\otimes} \gamma(B') \simeq \gamma(B \bar{\otimes} B')$. Let \mathbf{X} be a monoidal Γ -ambient $(\mathbf{X}, \gamma, \bar{\otimes})$. A *closure structure* for a Γ -space B in \mathbf{X} is a lax closure structure (cl, μ, ϕ) in \mathbf{X} such that any morphism $f : \gamma(B) \rightarrow X$ (not necessarily a subobject) admits an extension $\hat{\mu}_f$ relative to $\mu_f : \gamma(B) \rightarrow \text{cl}_X(\gamma(B))$ as in the diagram below. A *Γ -space with closure in \mathbf{X}* is Γ -space B endowed with a closure structure $\mathbf{c} = (\text{cl}, \mu, \phi)$ in \mathbf{X} . Notice that the diagrams above make perfect sense in the category $\mathbf{NFre}_{\Gamma, \Sigma}$ of (Γ, ϵ) -spaces, so that we can also define (Γ, ϵ) -spaces with closure in a monoidal Γ -ambient $(\mathbf{X}, \gamma, \bar{\otimes})$. Let $\mathbf{NFre}_{\Gamma, \Sigma, \mathbf{c}}(\mathbf{X}) \subset \mathbf{NFre}_{\Gamma, \Sigma}$ be the full subcategory of those (Γ, ϵ) -spaces.

$$\begin{array}{ccc}
& & X \\
& \nearrow f & \uparrow \hat{\mu}_f \\
\gamma(B) \xrightarrow{\mu_f} & \text{cl}_X(\gamma(B)) &
\end{array}
\tag{4}$$

3 $C_{n,\beta}^{k,\alpha}$ -Presheaves

- Let Γ be a set such that $\Gamma \cap \mathbb{Z}_{\geq 0} \neq \emptyset$ and let $\Gamma_{\geq 0} = \Gamma \cap \mathbb{Z}_{\geq 0}$. In the following we will consider only $\Gamma_{\geq 0}$ -spaces.

We begin by introducing a presheaf version of the previous concepts. A *presheaf of $\Gamma_{\geq 0}$ -spaces in \mathbb{R}^n* is just a presheaf $B : \text{Open}(\mathbb{R}^n)^{op} \rightarrow \mathbf{NFre}_{\Gamma_{\geq 0}}$ of $\Gamma_{\geq 0}$ -spaces. Let \mathbf{B}_n be the presheaf category of them. Given a $\Gamma_{\geq 0}$ -ambient (\mathbf{X}, γ) , let $X : \text{Open}(\mathbb{R}^n)^{op} \rightarrow \mathbf{X}$ be a presheaf, i.e, let $X \in \text{Psh}(\mathbb{R}^n; \mathbf{X})$. We say that $B \in \mathbf{B}_n^k$ is a *subobject of (X, γ)* if it becomes endowed with a natural transformation $\iota : \gamma \circ B \hookrightarrow X$ which is objectwise a monomorphism. As in the previous section, let $\text{Span}_\Gamma(X, \gamma)$ be the category of spans of those subobjects. We have functors

$$\text{Base} : \text{Span}_\Gamma(X, \gamma) \rightarrow \text{Psh}(\mathbb{R}^n; \mathbf{X}) \quad \text{and} \quad \text{Pb} : \text{Span}_\Gamma(X, \gamma) \rightarrow \text{Psh}(\mathbb{R}^n; \mathbf{X})$$

which to each span (ι, ι') assigns the *base presheaf* $\text{Base}(\iota, \iota') = X$ and which evaluate the pullback of (ι, ι') , i.e, $\text{Pb}(\iota, \iota')(U) = B(U) \cap_{X(U), \gamma} B'(U)$. In the following we will write $\text{Pb}(\iota, \iota') \equiv B \cap_{X, \gamma} B'$

whenever there is no risk of confusion. We say that a span (ι, ι') is *proper* if it is objectwise proper, i.e, if there exists $L \in \mathbf{B}_n$ such that $\gamma \circ L \simeq B \cap_{X, \gamma} B'$. If exists, then L is unique up to natural isomorphisms and will be denoted by $B \cap_X B'$. Furthermore, we also demand that there exists $u : B \cap_X B' \Rightarrow B$ and $u' : B \cap_X B' \Rightarrow B'$ such that $\gamma \circ u \simeq u_\gamma$ and $\gamma \circ u' \simeq u'_\gamma$. We call $\mathbb{X} = (\mathbf{X}, \gamma, X, \iota, \iota')$ and $B \cap_{X, \gamma} B'$ the *intersection structure presheaf* (ISP) and the *intersection presheaf* between B and B' in (X, γ) , respectively. We say that B and B' have *nontrivial intersection in \mathbb{X}* if objectwise they have nontrivial intersection, i.e, if $(B \cap_X B')(U)$ have positive real dimension for every U .

If (B, ϵ) and (B', ϵ') are now presheaves of $(\Gamma_{\geq 0}, \epsilon)$ -spaces in \mathbb{R}^n , i.e, if they take values in $\mathbf{NFre}_{\Gamma_{\geq 0}, \Sigma}$ instead of in $\mathbf{NFre}_{\Gamma_{\geq 0}}$, let $\mathbf{B}_{n, \Sigma}$ be the associated presheaf category. We can apply the same strategy in order to define the category $\text{Span}_{\Gamma, \Sigma}(X, \gamma_\Sigma)$ of spans of subobjects of (X, γ_Σ) . If $j : \gamma_\Sigma \circ B_\epsilon \Rightarrow X$ and $j' : \gamma_\Sigma \circ B'_{\epsilon'} \Rightarrow X$ are two of those spans, we define an *intersection structure presheaf* between (B, ϵ) and (B', ϵ') as the tuple $\mathbb{X} = (\mathbf{X}, \gamma_\Sigma, X, j, j')$. Similarly, pullback and projection onto the base presheaf define functors

$$\text{Base}_\Sigma : \text{Span}_{\Gamma, \Sigma}(X, \gamma_\Sigma) \rightarrow \text{Psh}(\mathbb{R}^n; \mathbf{X}) \quad \text{and} \quad \text{Pb}_\Sigma : \text{Span}_{\Gamma, \Sigma}(X, \gamma_\Sigma) \rightarrow \text{Psh}(\mathbb{R}^n; \mathbf{X}).$$

We will write $\text{Pb}_\Sigma(j, j') \equiv B_\epsilon \cap_{X, \gamma_\Sigma} B'_{\epsilon'}$. If the span is proper, the representing object in $\mathbf{B}_{n, \Sigma}$ is denoted simply by $B_\epsilon \cap_X B'_{\epsilon'}$, so that $\gamma_\Sigma \circ (B_\epsilon \cap_X B'_{\epsilon'}) \simeq B_\epsilon \cap_{X, \gamma_\Sigma} B'_{\epsilon'}$. When $(B, \epsilon, *)$ and $(B', \epsilon', *')$ are presheaves of multiplicative $\Gamma_{\geq 0}$ -spaces, meaning that they take values in $\mathbf{NFre}_{\Gamma_{\geq 0}, *}$, we denote their presheaf category $\mathbf{B}_{n, *}$ and define the *intersection space presheaf* for an intersection structure presheaf $\mathbb{X}_{*, *'} = (\mathbf{X}, \gamma_\Sigma, X, j, j')$ between $*$ and $*'$ as the presheaf which objectwise is the pullback below in the category $\text{Psh}(\mathbb{R}^n; \mathbf{X})$. In other words, it is $\text{Pb}_\Sigma((j \circ \gamma_\Sigma \circ *, j' \circ \gamma_\Sigma \circ *'))$. If these spans are proper we denote the representing object in $\mathbf{B}_{n, \Sigma}$ by $\text{pb}(*, *'; X)$ and we say that $*$ and $*'$ are *nontrivially intersecting* in \mathbb{X} if that representing object has objectwise positive dimension.

$$\begin{array}{ccc} \text{pb}(*, *'; X, \gamma_\Sigma) & \xrightarrow{\quad\quad\quad} & \gamma_\Sigma \circ (B'_{\epsilon'} \otimes B_{\epsilon'}) \\ \downarrow & & \downarrow \gamma_\Sigma \circ *' \\ & B_\epsilon \cap_{X, \gamma_\Sigma} B'_{\epsilon'} \xrightarrow{\quad\quad\quad} \gamma_\Sigma \circ B'_{\epsilon'} & \\ & \downarrow & \downarrow j' \\ \gamma_\Sigma \circ (B_\epsilon \otimes B_\epsilon) & \xrightarrow{\gamma_\Sigma \circ *} & \gamma_\Sigma \circ B_\epsilon \xrightarrow{j} X \end{array}$$

Let us now consider presheaves of distributive $\Gamma_{\geq 0}$ -spaces in \mathbb{R}^n , i.e, which assigns to each open subset $U \subset \mathbb{R}^n$ a corresponding distributive $\Gamma_{\geq 0}$ -space. Let $\mathbf{B}_{n, *, +}$ be the presheaf category of them. Define a *full intersection structure presheaf* (full ISP) as a tuple

$$\mathbb{X} = (\mathbf{X}, \gamma, \gamma_\Sigma, X, X_*, X_+, \iota, \iota', j_*, j'_*, j_+, j'_+),$$

where $(\mathbf{X}, \gamma, \gamma_\Sigma)$ is a full Γ -ambient, $X, X_*, X_+ \in \text{Psh}(\mathbb{R}^n; \mathbf{X})$ are presheaves and (ι, ι') , (j_*, j'_*) and (j_+, j'_+) are spans of subobjects of X, X_* and X_+ , respectively. Given $B, B' \in \mathbf{B}_{n, *, +}$ we say that a full ISP \mathbb{X} is *between B and B'* if B is on the domain of ι, j_* and j_+ , while B' is on the domain of ι', j'_* and j'_+ . Thus, e.g, $\iota : \gamma \circ B \Rightarrow X$, $j_* : \gamma_\Sigma \circ B_\epsilon \Rightarrow X_*$ and $j'_+ : \gamma_\Sigma \circ B'_{\epsilon'} \Rightarrow X_+$. We can also write \mathbb{X} as $\mathbb{X} = (\mathbb{X}_0, \mathbb{X}_*, \mathbb{X}_+)$, where

$$\mathbb{X}_0 = (\mathbf{X}, \gamma, X, \iota, \iota'), \quad \mathbb{X}_* = (\mathbf{X}, \gamma_\Sigma, X_*, j_*, j'_*) \quad \text{and} \quad \mathbb{X}_+ = (\mathbf{X}, \gamma_\Sigma, X_+, j_+, j'_+)$$

are ISP between B and B' , between $*$ and $*'$, and between $+$ and $+'$, respectively. The *full intersection presheaf* between B and B' in a full ISP \mathbb{X} is the triple consisting of the intersection presheaves between B and B' in \mathbb{X}_0 , between $*$ and $*'$ in \mathbb{X}_* and between $+$ and $+'$ in \mathbb{X}_+ . When \mathbb{X} is proper, the corresponding full intersection presheaf has a representing object $B \cap_{\mathbb{X}} B'$ in $\mathbf{B}_n \times \mathbf{B}_{n,\Sigma} \times \mathbf{B}_{n,\Sigma}$, given by

$$B \cap_{\mathbb{X}} B' = (B \cap_{\mathbb{X}_0} B', \text{pb}(*, *'; X_*), \text{pb}(+, +' ; X_+)).$$

We say that B and B' have *nontrivial intersection* in a proper full ISP \mathbb{X} if each of the three presheaves in $B \cap_{\mathbb{X}} B'$ are nontrivial, i.e, have objectwise positive real dimension.

Example 3.1. By means of varying U and restricting to $\Gamma_{\geq 0}$, for every fixed $n \geq 0$, each multiplicative Γ -space in Examples 2.2-2.6 defines a presheaf of multiplicative $\Gamma_{\geq 0}$ -spaces in \mathbb{R}^n . Furthermore, by the same process, from Example 2.8 and Example 2.9 we get presheaves of distributive $\Gamma_{\geq 0}$ -structures in \mathbb{R}^n . Vectorial ISP can be build by following Example 2.10 and Example 2.11.

Let us introduce $C_{n,\beta}^{k,\alpha}$ -presheaves, which will be the structural presheaves appearing in the definition of $B_{\alpha,\beta}^k$ -manifold. Let $K', B' \in \mathbf{B}_n$. An *action* of K' in B' is just a morphism $\kappa' : K' \otimes B' \Rightarrow B'$, where the tensor product is defined objectwise. A *morphism* between actions $\kappa : K \otimes B \Rightarrow B$ and $\kappa' : K' \otimes B' \Rightarrow B'$ is just a morphism (f, g) in the arrow category such that $f = \ell \otimes g$. More precisely, it is a pair (ℓ, g) , where $\ell : K \Rightarrow K'$ and $g : B \Rightarrow B'$ are morphisms in \mathbf{B}_n such that the diagram below commutes. Let \mathbf{Act}_n be the category of actions and morphisms between them.

$$\begin{array}{ccc} K' \otimes B' & \xrightarrow{\kappa'} & B' \\ \ell \otimes g \uparrow \parallel & & \uparrow \parallel g \\ K \otimes B & \xrightarrow{\kappa} & B \end{array}$$

Given $B, B' \in \mathbf{B}_n$, an action $\kappa' : K' \otimes B' \Rightarrow B'$ and a proper ISP \mathbb{X}_0 between B and B' , we say that B is *compatible* with κ' in \mathbb{X}_0 if there exists an action $\kappa : A \otimes B \cap_{\mathbb{X}_0} B' \Rightarrow B \cap_{\mathbb{X}_0} B'$ in the intersection presheaf $B \cap_{\mathbb{X}_0} B'$ and a morphism $\ell : A \Rightarrow K'$ such that (ℓ, u') is a morphism of actions, where $u' : B \cap_{\mathbb{X}_0} B' \Rightarrow B'$ is the morphism such that $\gamma \circ u' \simeq u'_\gamma$, existing due to the properness hypothesis. If it is possible to choose (A, ℓ) such that is objectwise a monomorphism, B is called *injectively compatible* with κ' in \mathbb{X}_0 . Given $B, B' \in \mathbf{B}_{n,*,+}$ we say that B is *compatible with B'* in a proper full ISP $\mathbb{X} = (\mathbb{X}_0, \mathbb{X}_*, \mathbb{X}_+)$ if:

1. the proper full ISP \mathbb{X} is between them;
2. they have nontrivial intersection in \mathbb{X} ;
3. there exists an action $\kappa' : K' \otimes B' \Rightarrow B'$ in B' such that B is injectively compatible with κ' in \mathbb{X}_0 .

Example 3.2. Suppose that C is a presheaf of increasing $\Gamma_{\geq 0}$ -Fréchet algebras, i.e, such that $C(U)$ is a $\Gamma_{\geq 0}$ -space for which each $C(U)_i$ is a Fréchet algebra such that there exists a continuous embedding of topological algebras $C(U)_i \subset C(U)_j$ if $i \leq j$. Let $p : \Gamma_{\geq 0} \rightarrow \Gamma_{\geq 0}$ be any function such that $i \leq p(i)$. For $K' = C$ and $B' = C_p$ we have an action $\kappa' : K' \otimes B' \Rightarrow B'$ such that $\kappa'_{U,i} : C(U)_i \otimes C(U)_{p(i)} \rightarrow C(U)_{p(i)}$ is obtained by embedding $C(U)_i$ in $C(U)_{p(i)}$ and then using

the Fréchet algebra multiplication of $C(U)_{p(i)}$. In other words, under the hypothesis C is actually a presheaf of multiplicative $\Gamma_{\geq 0}$ -spaces, with $\epsilon(i, j) = \max(i, j)$, so that $\kappa'_{U,i} = *_{i,p(i)}$. An analogous discussion holds for decreasing presheaves.

As a particular case of the last example, we see that pointwise multiplication induces an action of the presheaf C^k , such that $C^k(U)_i = C^k(U)$ for every $U \subset \mathbb{R}^n$ and $i \in \Gamma_{\geq 0}$, on the presheaf $C^{k-\beta}$, such that $C^{k-\beta}(U)_i = C^{k-\beta(i)}(U)$, where $\beta : \Gamma_{\geq 0} \rightarrow [0, k]$ is some fixed integer function. Given $\alpha : \Gamma_{\geq 0} \rightarrow \Gamma_{\geq 0}$ and $\beta : \Gamma_{\geq 0} \rightarrow [0, k]$, we define a $C_{n,\beta}^{k,\alpha}$ -presheaf in a proper full ISP \mathbb{X} as a presheaf $B \in \mathbf{B}_{n,*,+}$ which such that B_α is compatible with $C^{k-\beta}$ in \mathbb{X} . Thus, B is a $C_{n,\beta}^{k,\alpha}$ -presheaf in \mathbb{X} if for every $U \subset \mathbb{R}^n$:

1. *the intersection spaces $B_\alpha(U) \cap_{X(U)} C^{k-\beta}(U)$, etc., have positive real dimension.* This means that at least in some sense (i.e, internal to X) the abstract spaces defined by B have a concrete interpretation in terms of differentiable functions satisfying some further properties/regularity. Furthermore, under this interpretation, the sum and the multiplication in B are the pointwise sum and multiplication of differentiable functions added of properties/regularity;
2. *there exists a subspace $A(U) \subset C^k(U)$ and for every $i \in \Gamma_{\geq 0}$ a morphism $*_{U,i}$ making commutative the diagram below.* This means that if we regard the abstract multiplication of B as pointwise multiplication of functions with additional properties, then that multiplication becomes closed under a certain set of smooth functions.

$$\begin{array}{ccc}
C^k(U) \otimes C^{k-\beta(i)}(U) & \xrightarrow{\cdot_{U,i}} & C^{k-\beta(i)}(U) \\
\uparrow & & \uparrow \\
A(U) \otimes B_{\alpha(i)}(U) \cap_{X(U)} C^{k-\beta(i)}(U) & \xrightarrow{*_{U,i}} & B_{\alpha(i)}(U) \cap_{X(U)} C^{k-\beta(i)}(U)
\end{array} \tag{5}$$

If, in addition, the intersection structure \mathbb{X}_0 between B_α and $C^{k-\beta}(U)$ is such that the image of each $u' : B_{\alpha(i)} \cap_{X(U)} C^{k-\beta(i)}(U) \rightarrow C^{k-\beta(i)}(U)$ is closed, we say that B is a *strong $C_{n,\beta}^{k,\alpha}$ -presheaf* in \mathbb{X} . Finally we say that B is a *nice $C_{n,\beta}^{k,\alpha}$ -presheaf* in \mathbb{X} if it is possible to choose A such that $\dim_{\mathbb{R}} A(U) \cap C_b^\infty(U) \geq 1$, where $C_b^\infty(U)$ is the space of bump functions. Notice that being nice does not depends on the ISP \mathbb{X} .

Example 3.3. From Example 3.2 and Proposition 2.1 we see that the presheaf of distributive structures C^{k-} is a $C_{n,\beta}^{k,\alpha}$ -presheaf in the full ISP which is objectwise the standard vectorial intersection structure of Example 2.11, with $A = C^\infty$. Since $C_b^\infty(U) \subset C^\infty(U)$ for every U , we conclude that C^{k-} is actually a nice $C_{n,\beta}^{k,\alpha}$ -presheaf.

Example 3.4. The same arguments of the previous example can be used to show that the presheaf L of distributive $\Gamma_{\geq 0}$ -spaces $L(U)_i = L^i(U)$, endowed with the distributive structure given by pointwise addition and multiplication, is a nice $C_{n,\beta}^{k,\alpha}$ -presheaf in the standard ISP, for $A(U) = C_b^\infty(U)$ or $A(U) = \mathcal{S}(U)$, where $\mathcal{S}(U)$ is the Schwarz space [13]. An analogous conclusion is valid if we replace pointwise multiplication with convolution product.

Let $\mathbf{C}_{n,\beta}^{k,\alpha}$ be the category whose objects are pairs (B, \mathbb{X}) , where \mathbb{X} is a proper full ISP and $B \in \mathbf{B}_{n,*,+}$ is a presheaf of distributive $\Gamma_{\geq 0}$ -spaces which is a $C_{n,\beta}^{k,\alpha}$ -presheaf in \mathbb{X} . Morphisms $\xi :$

$(B, \mathbb{X}) \Rightarrow (B', \mathbb{X}')$ are just morphisms $\xi : B \Rightarrow B'$ in $\mathbf{B}_{n,*,+}$. We have a projection $\pi_n : \mathbf{C}_{n,\beta}^{k,\alpha} \rightarrow \mathbf{Cat}$ assigning to each pair (B, \mathbb{X}) the ambient category \mathbf{X} in the full Γ -ambient $(\mathbf{X}, \gamma, \gamma_\Sigma)$ of \mathbb{X} , where \mathbf{Cat} is the category of all categories. Notice that such projection really depends only on n (and not on α, β and k). Let \mathbf{Cat}_n be the image of π_n and for each $\mathbf{X} \in \mathbf{Cat}_n$ let $\mathbf{C}_{n,\beta}^{k,\alpha}(\mathbf{X})$ be the preimage $\pi_n^{-1}(\mathbf{X})$. Closing the section, let us study the dependence of the fiber $\pi_n^{-1}(\mathbf{X})$ on the variables k, n and β . We need some background.

Let \mathbb{X} and \mathbb{Y} be two (not necessarily proper) ISP in a Γ -ambient (\mathbf{X}, γ) . A *connection* between them is a transformation $\xi : X \Rightarrow Y$ between the underlying base presheaves such that for every two presheaves B, B' of Γ -subspaces for which both \mathbb{X} and \mathbb{Y} are between B and B' , the diagram below commutes. Thus, by universality we get the dotted arrow.

$$\begin{array}{ccc}
 B \cap_{X,\gamma} \gamma \circ B' & \xrightarrow{\quad} & \gamma \circ B' \\
 \downarrow & \dashrightarrow^{\bar{\xi}} & \downarrow j' \\
 & B \cap_{Y,\gamma} B' & \\
 \downarrow & \swarrow & \downarrow j' \\
 \gamma \circ B & \xrightarrow{j} & X \\
 \downarrow & \searrow & \downarrow \\
 & & Y \\
 & \xrightarrow{\ell} &
 \end{array}
 \quad \ell'
 \tag{6}$$

Suppose now that \mathbf{X} is a monoidal Γ -ambient and that B and B' are presheaves of (Γ, \mathbf{c}) -spaces in \mathbf{X} , i.e, which objectwise belong to $\mathbf{NFre}_{\Gamma,\mathbf{c}}(\mathbf{X})$ (recall the notation in the end of Section 2). Notice that for any connection $\xi : X \Rightarrow Y$ we have the first commutative diagram below, where $\mathbf{c} = (\text{cl}, \mu)$ is the closure structure. We say that a presheaf of Γ -spaces Z is *central* for $(B, B', \mathbb{X}, \mathbb{Y}, \mathbf{c})$ if it becomes endowed with an embedding $\iota : B \cap_{X,\gamma} B' \Rightarrow Z$ and a morphism η such that the second diagram below commutes.

$$\begin{array}{ccc}
 B \cap_{X,\gamma} B' & \xrightarrow{\mu \circ \iota_X} & \text{cl}_X(B \cap_{X,\gamma} B') \\
 \downarrow \bar{\xi} & & \downarrow \text{cl} \circ \bar{\xi} \\
 B \cap_{Y,\gamma} B' & \xrightarrow{\mu \circ \iota_Y} & \text{cl}_Y(B \cap_{Y,\gamma} B')
 \end{array}
 \quad
 \begin{array}{ccc}
 B \cap_{X,\gamma} B' & \xrightarrow{\mu \circ \iota_X} & \text{cl}_X(B \cap_{X,\gamma} B') \\
 \downarrow \bar{\xi} & \searrow \iota & \downarrow \text{cl} \circ \bar{\xi} \\
 & Z & \dashrightarrow^{\eta} \\
 B \cap_{Y,\gamma} B' & \xrightarrow{\mu \circ \iota_Y} & \text{cl}_Y(B \cap_{Y,\gamma} B'_{\epsilon'})
 \end{array}$$

Remark 3.1. If \mathbf{X} is complete/cocomplete then η always exists, at least up to universal natural transformations. Indeed, notice that η is actually the extension of $\mu_Y \circ \bar{\xi}$ by ι (equivalently, the extension of $\text{cl} \circ \bar{\xi} \circ \mu_X$ by ι), so that due to completeness/cocompleteness we can take the left/right Kan extension [16, 7].

Remark 3.2. In an analogous way we can define connections between presheaves of (Γ, ϵ) -spaces and central (Γ, ϵ) -presheaves relative to some closure structure. These can also be regarded as Kan extensions, so that if \mathbf{X} is complete/cocomplete they will also exist up to universal natural transformations.

4 Theorem A

We say that a ISP $\mathbb{X} = (\mathbf{X}, \gamma, X, \iota, \iota')$ is *monoidal* if the underlying Γ -ambient (\mathbf{X}, γ) is monoidal. A full ISP $\mathbb{X} = (\mathbb{X}_0, \mathbb{X}_*, \mathbb{X}_+)$ is *partially monoidal* if the ISP \mathbb{X}_0 is monoidal. Fixed monoidal Γ -ambient $(\mathbf{X}, \gamma, \otimes)$ and given an integer function $p : \Gamma_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ such that $p(i) \leq k - \beta(i)$ for each $i \in \Gamma_{\geq 0}$, define a p -structure on a $C_{n,\beta}^{k,\alpha}$ -presheaf B in a partially monoidal full proper ISP $\mathbb{X} = (\mathbb{X}_0, \mathbb{X}_*, \mathbb{X}_+)$, as another non-necessarily proper monoidal ISP $\mathbb{Y}_0 = (\mathbf{X}, \gamma, Y)$, together with a closure structure $\mathbf{c} = (\text{cl}, \mu, \phi)$ and a connection $\xi : Y_0 \Rightarrow X$ such that $Z = B_\alpha \cap_{X,\gamma} C^p$ is central for $(B_\alpha, C^{k-\beta}, \mathbb{X}_0, \mathbb{Y}_0, \mathbf{c})$ relative to the canonical embedding $\iota : B_\alpha \cap_{X,\gamma} C^{k-\beta} \hookrightarrow Z$ induced by universality of pullbacks applied to $C^{k-\beta} \hookrightarrow C^p$, which exists due to the condition $p \leq k - \beta$. Define a $C_{n,\beta;p}^{k,\alpha}$ -presheaf in \mathbb{X} as a $C_{n,\beta}^{k,\alpha}$ -presheaf in B in \mathbb{X} endowed with a p -structure and let $\mathbf{C}_{n,\beta;p}^{k,\alpha}$ be the full subcategory of them. Furthermore, let $\mathbf{C}_{n,\beta;p}^{k,\alpha}(\mathbf{X}_{\text{full}}) \subset \mathbf{C}_{n,\beta}^{k,\alpha}(\mathbf{X})$ the corresponding full subcategories of pairs (B, \mathbb{X}) for which $\pi_n(B, \mathbb{X}) = \mathbf{X}$ and γ is a full functor.

We can now prove Theorem A. It will be a consequence of the following more general theorem:

Theorem 4.1. *For any category with pullbacks \mathbf{X} , there are full embeddings*

- (1) $\iota : \mathbf{C}_{n,\beta;p}^{k,\alpha}(\mathbf{X}_{\text{full}}) \hookrightarrow \mathbf{C}_{n,p}^{k,\alpha}(\mathbf{X})$, if $p - \beta \leq k$;
- (2) $f_* : \mathbf{C}_{n,\beta}^{k,\alpha}(\mathbf{X}) \hookrightarrow \mathbf{C}_{r,\beta}^{k,\alpha}(\mathbf{X})$, for any continuous injective map $f : \mathbb{R}^n \rightarrow \mathbb{R}^r$.

Proof. We begin by proving (1). Notice that the fiber $\pi_n^{-1}(\mathbf{X})$ is a nonempty category only if \mathbf{X} for every k, α, β is part of a partially monoidal proper full ISP, with γ full, in which at least one $C_{n,\beta}^{k,\alpha}$ -presheaf is defined. Since for empty fibers the result is obvious, we assume the above condition. Thus, given $(B, \mathbb{X}) \in \mathbf{C}_{n,\beta;p}^{k,\alpha}(\mathbf{X}_{\text{full}})$ (which exists by the assumption on \mathbf{X}), we will show that it actually belongs to $\mathbf{C}_{n,p}^{k,\alpha}(\mathbf{X})$. Since we are working with full subcategories this will be enough for (1). We assert that B_α and C^p have nontrivial intersection in \mathbb{X} . Since the functor γ creates null-objects, it is enough to prove that $B_\alpha \cap_{X,\gamma} C^p$, $(B_\alpha)_\epsilon \cap_{X_*,\gamma_\Sigma} (C^p)_{\epsilon'}$ and $(B_\alpha)_\delta \cap_{X_*,\gamma_\Sigma} (C^p)_{\delta'}$. But, since $\delta' = \epsilon' = \min$ and since $p \leq \beta - k$, from universality and stability of pullbacks under monomorphisms we get monomorphisms $B_\alpha \cap_{X,\gamma} C^{k-\beta} \hookrightarrow B_\alpha \cap_{X,\gamma} C^p$, etc. Now, being B a $C_{n,\beta}^{k,\alpha}$ -presheaf in \mathbb{X} , the left-hand sides $B_\alpha \cap_{X,\gamma} C^{k-\beta}$, etc., are nontrivial, which implies B_α and C^p have nontrivial intersection in \mathbb{X} . By the same arguments we get the commutative diagram below, where A is the presheaf arising from the $C_{n,\beta}^{k,\alpha}$ -structure of B . Our task is to extend \star_β by replacing $k - \beta$ with p . If $p(i) - \beta(i) = k$ there is nothing to do for such i . Thus, suppose $p(i) - \beta(i) < k$ for all i . More precisely, our task is to get the dotted arrow in the second diagram below, where the long vertical arrows arise from the universality of pullbacks, as above. We use simple arrows instead of double arrows in order to simplify the notation.

$$\begin{array}{ccc}
C^\infty \otimes C^p & \longrightarrow & C^p \\
\uparrow & & \uparrow \\
C^\infty \otimes C^{k-\beta} & \longrightarrow & C^{k-\beta} \\
\uparrow & & \uparrow \\
A \otimes (B_\alpha \cap_X C^{k-\beta}) & \xrightarrow{\star_\beta} & B_\alpha \cap_X C^{k-\beta}
\end{array}
\quad
\begin{array}{ccc}
A \otimes (B_\alpha \cap_X C^p) & \xrightarrow{\star_p} & B_\alpha \cap_X C^p \\
\downarrow & & \downarrow \\
C^\infty \otimes C^p & \longrightarrow & C^p \\
\downarrow & & \downarrow \\
C^\infty \otimes C^{k-\beta} & \longrightarrow & C^{k-\beta} \\
\downarrow & & \downarrow \\
A \otimes (B_\alpha \cap_X C^{k-\beta}) & \xrightarrow{\star_\beta} & B_\alpha \cap_X C^{k-\beta}
\end{array}$$

Since $(B, \mathbb{X}) \in \mathbf{C}_{n,\beta;p}^{k,\alpha}(\mathbf{X}_{\text{full}})$, there exists a ISP \mathbb{Y}_0 , a connection $\xi : Y \Rightarrow X$ and a closure structure $\mathbf{c} = (\text{cl}, \mu, \phi)$ such that $B_\alpha \cap_{X,\gamma} C^p$ is central for $(B_\alpha, C^{k-\beta}, \mathbb{X}_0, \mathbb{Y}_0, \mathbf{c})$. Since any morphism extends to the closure and recalling that γ is strong monoidal, we have the diagram below. It commutes due to the commutativity of (3) and (4) and due to the naturality of ψ . We are also using that $\gamma \circ B \cap_X B' \simeq B \cap_{X,\gamma} B'$. Furthermore, μ_A is $\mu_{\gamma(\iota_A)}$, where $\iota_A : A \hookrightarrow C^\infty$ is the embedding. Again we use simple arrows instead of double arrows.

$$\begin{array}{ccc}
\text{cl}_{\gamma(C^\infty)}(\gamma(A)) \otimes \text{cl}_X(B_\alpha \cap_{X,\gamma} C^{k-\beta}) & \xleftarrow{id \otimes (\text{cl} \circ \bar{\xi})} & \text{cl}_{\gamma(C^\infty)}(\gamma(A)) \otimes \text{cl}_Y(B_\alpha \cap_{Y,\gamma} C^{k-\beta}) \\
\downarrow \phi & & \downarrow \phi \\
\text{cl}_{\gamma(C^\infty) \otimes X}(\gamma(A) \otimes (B_\alpha \cap_{X,\gamma} C^{k-\beta})) & \xrightarrow{\hat{\mu}_{\gamma(\star_\beta) \circ \psi}} & B_\alpha \cap_{X,\gamma} C^{k-\beta} \\
\downarrow \mu_{A \otimes X} & \nearrow \gamma(\star_\beta) & \downarrow \mu_{A \otimes Y} \\
\gamma(A \otimes (B_\alpha \cap_X C^{k-\beta})) & \xrightarrow{\gamma(id \otimes \iota)} & \gamma(A \otimes (B_\alpha \cap_X C^p)) \\
\downarrow \psi \downarrow \psi^{-1} & & \downarrow \psi \downarrow \psi^{-1} \\
\gamma(A) \otimes (B_\alpha \cap_{X,\gamma} C^{k-\beta}) & \xrightarrow{\gamma(id) \otimes \gamma(\iota)} & \gamma(A) \otimes (B_\alpha \cap_{X,\gamma} C^p)
\end{array}$$

Let us consider the composition morphism

$$f_\beta^p = \hat{\mu}_{\gamma(\star_\beta) \circ \psi} \circ \phi \circ (id \otimes (\text{cl} \circ \bar{\xi})) \circ (\mu_A \otimes \eta) \circ \psi^{-1} : \gamma(A \otimes B_\alpha \cap_X C^p) \rightarrow \gamma(B_\alpha \cap_X C^{k-\beta}).$$

On the other hand, we have $\gamma(\iota_{b,p}) : \gamma(B_\alpha \cap_X C^{k-\beta}) \hookrightarrow \gamma(B_\alpha \cap_X C^p)$ arising from the embedding $C^{k-\beta} \hookrightarrow C^p$. Composing them we get a morphism

$$\gamma(\iota_{b,p}) \circ f_\beta^p : \gamma(A \otimes B_\alpha \cap_X C^p) \rightarrow \gamma(B_\alpha \cap_X C^p). \quad (7)$$

Since γ is fully-faithful, there exists exactly one $\star_p : A \otimes B_\alpha \cap_X C^p \rightarrow B_\alpha \cap_X C^p$ such that $\gamma(\star_p) = \gamma(\iota_{b,p}) \circ f_\beta^p$, which is our desired map. That it really extends \star_β follows from the commutativity of all diagrams involved in the definition of \star_p . For (2), recall that any continuous map $f : \mathbb{R}^n \rightarrow \mathbb{R}^r$ induces a pushforward functor $f^{-1} : \mathbf{Psh}(\mathbb{R}^n) \rightarrow \mathbf{Psh}(\mathbb{R}^r)$ between the corresponding presheaf categories of presheaves of sets, which becomes an embedding when f is injective [17]. Since $(f^{-1}F)(U) = F(f^{-1}(U))$, it is straightforward to verify that if (B, \mathbb{X}, A) belongs to $\mathbf{C}_{n,\beta}^{k,\alpha}(\mathbf{X})$, then $(f^{-1}B, f^{-1}A, f^{-1}\mathbb{X}) \in \mathbf{C}_{r,\beta}^{k,\alpha}(\mathbf{X})$, where $f^{-1}\mathbb{X}$ is defined componentwise. Thus, we have an injective

on objects functor $f^{-1} : \mathbf{C}_{n,\beta}^{k,\alpha}(\mathbf{X}) \rightarrow \mathbf{C}_{r,\beta}^{k,\alpha}(\mathbf{X})$. Because we are working with full subcategories, it follows that f^{-1} is full and therefore a full embedding. \square

Corollary 4.1. *Let $\mathbf{C}_{n,\beta;l}^{k,\alpha}(\mathbf{X}_{\text{full}})$ and $\mathbf{C}_{n,\beta;\beta'}^{k,\alpha}(\mathbf{X}_{\text{full}})$ be the category $\mathbf{C}_{n,\beta;p}^{k,\alpha}(\mathbf{X}_{\text{full}})$ for $p_l(i) = l - \beta(i)$ and $p_{\beta'}(i) = k - \beta'(i)$, respectively. Then there are full embeddings*

- $j : \mathbf{C}_{n,\beta;\beta'}^{k,\alpha}(\mathbf{X}_{\text{full}}) \hookrightarrow \mathbf{C}_{n,\beta}^{k,\alpha}(\mathbf{X}_{\text{full}})$, if $\beta' \leq \beta$;
- $\iota : \mathbf{C}_{n,\beta;l}^{k,\alpha}(\mathbf{X}_{\text{full}}) \hookrightarrow \mathbf{C}_{n,\beta}^{l,\alpha}(\mathbf{X}_{\text{full}})$, if $l \leq k$.

Proof. Just notice that if $\beta' \leq \beta$ implies $p_{\beta'} - \beta \leq k$ and that $l \leq k$ implies $p_l - \beta \leq k$, and then uses (1) of Theorem 4.1. \square

Theorem A. *There are full embeddings*

- (1) $\iota : \mathbf{C}_{n,\beta;l}^{k,\alpha} \hookrightarrow \mathbf{C}_{n,\beta}^{l,\alpha}$, if $l \leq k$;
- (2) $f_* : \mathbf{C}_{n,\beta}^{k,\alpha} \hookrightarrow \mathbf{C}_{r,\beta}^{k,\alpha}$, for any continuous injective map $f : \mathbb{R}^n \rightarrow \mathbb{R}^r$;
- (3) $j : \mathbf{C}_{n,\beta;\beta'}^{k,\alpha} \hookrightarrow \mathbf{C}_{r,\beta'}^{k,\alpha}$, if $\beta' \leq \beta$.

Proof. Straightforward from Corollary 4.1 and condition (2) of Theorem 4.1. \square

Remark 4.1. The requirement of γ being full is a bit strong. Indeed, our main examples of Γ -ambients are the vectorial ones. But requiring a full embedding $\gamma : \mathbf{NFre}_{\Gamma} \rightarrow \mathbf{Vec}_{\mathbb{R},\Gamma}$ is a really strong condition. When looking at the proof of Theorem 4.1 we see that the only time when the full hypothesis on γ was needed is to ensure that the morphism (7) in $\mathbf{Psh}(\mathbb{R}^n; \mathbf{X})$ is induced by a morphism in \mathbf{B}_n . Thus, the hypothesis of being full can be clearly weakened. Actually, the hypothesis on γ being an embedding can also be weakened. In the end, the only hypothesis needed in order to develop the previous results is that γ creates null-objects and some class of monomorphisms. Since from now on we will not focus on the theory of $\mathbf{C}_{n,\beta}^{k,\alpha}$ -presheaves itself, we will not modify our hypothesis. On the other hand, in futures works concerning the study of the categories $\mathbf{C}_{n,\beta}^{k,\alpha}$ this refinement on the hypothesis will be very welcome.

5 $B_{\alpha,\beta}^k$ -Manifolds

Let \mathbb{X} be a proper full intersection structure and let $B \in \mathbf{C}_{n,\beta}^{k,\alpha}$ be a $\mathbf{C}_{n,\beta}^{k,\alpha}$ -presheaf in \mathbb{X} be a intersection structure. A C^k -function $f : U \rightarrow \mathbb{R}^m$ is called a (B, k, α, β) -function in (\mathbb{X}, m) if $\partial^{\mu} f_j \in B_{\alpha(i)}(U) \cap_{X(U)} C^{k-\beta(i)}(U)$ for all $|\mu| = i$ and $i \in \Gamma_{\geq 0}$. Due to the compatibility between the operations of B_{α} and $C^{k-\beta}$ at the intersection, it follows that the collection $B_{\alpha,\beta}^k(U; \mathbb{X}, m)$ of all (B, k, α, β) -functions in (\mathbb{X}, m) is a real vector space. This will become more clear in the next proposition. First, notice that by varying $U \subset \mathbb{R}^n$ we get a presheaf (at least of sets) $B_{\alpha,\beta}^k(-; \mathbb{X}, m)$. Recall that a *strong* $\mathbf{C}_{n,\beta}^{k,\alpha}$ -presheaf is one in which $B_{\alpha} \cap_X C^{k-\beta} \hookrightarrow C^{k-\beta}$ is objectwise closed.

Proposition 5.1. *For every $B \in \mathbf{C}_{n,\beta}^{k,\alpha}$, every \mathbb{X} and every $m \geq 0$, the presheaf of (B, k, α, β) -functions in (\mathbb{X}, m) is a presheaf of real vector spaces. If B is strong, then it is actually a presheaf of nuclear Fréchet spaces.*

Proof. Consider the following spaces:

$$\mathbb{C}_{\alpha,\beta}^k(U, m) = \prod_{\kappa(m)} C^{k-\beta(i)}(U) \quad \text{and} \quad \mathbb{B}\mathbb{C}_{\alpha,\beta}^k(U; \mathbb{X}, m) = \prod_{\kappa(m)} B_{\alpha(i)}(U) \cap_{X(U)} C^{k-\beta(i)}(U),$$

where $\prod_{\kappa(m)} = \prod_{j=1}^m \prod_i \prod_{|\alpha|=i} \prod_{m^i}$. Since they are countable products of nuclear Fréchet spaces, they have a natural nuclear Fréchet structure. Consider the map $j^k : C^k(U; \mathbb{R}^m) \rightarrow \mathbb{C}^{k,\alpha}(U, m)$, given by $j^k f = (\partial^\mu f_j)_{\mu,j}$, with $|\mu| = i$, and notice that $B_{\alpha,\beta}^k(U; \mathbb{X}, m)$ is the preimage of j^k by $\mathbb{B}\mathbb{C}_{\alpha,\beta}^k(U; \mathbb{X}, m)$. The map j^k is linear, so that any preimage has a linear structure, implying that the space of (B, k, α, β) -functions is linear. But j^k is also continuous in those topologies, so that if $\mathbb{B}\mathbb{C}_{\alpha,\beta}^k(U; \mathbb{X}, m)$ is a closed subset in $\mathbb{C}_{\alpha,\beta}^k(U, m)$, then $B_{\alpha,\beta}^k(U; \mathbb{X}, m)$ is a closed subset of a nuclear Fréchet spaces and therefore it is also nuclear Fréchet. This is ensured precisely by the strong hypothesis on B . \square

Example 5.1. Even if B is not strong, the space of (B, k, α, β) -functions may have a good structure. Indeed, let $p \in [1, \infty]$ be fixed, let $\alpha(i) = p$, $\beta(i) = i$ and let L be the presheaf $L(U)_i = L^i(U)$, regarded as a $C_{n,\beta}^{k,\alpha}$ -presheaf in the standard ISP. Thus, $L_\alpha(U) \cap_{X(U)} C^{k-\beta}(U)$ is the $\Gamma_{\geq 0}$ -space of components $L^p(U) \cap C^{k-i}(U)$. A (B, k, α, β) -function is then a C^k -map such that $\partial^\mu f_j \in L^p(U) \cap C^{k-i}(U)$ for every $|\mu| = i$. Therefore, the space of all of them is the strong (in the sense of classical derivatives) Sobolev space $W^{k,p}(U; \mathbb{R}^m)$, which is a Banach space [13]. But $L^p(U) \cap C^{k-i}(U)$ is clearly not closed in $C^{k-i}(U)$, since sequences of L^p -integrable C^k -functions do not necessarily converge to L^p -integrable maps [13].

Remark 5.1. In order to simplify the notation, if $m = n$ the space of (B, k, α, β) -functions will be denoted by $B_{\alpha,\beta}^k(U; \mathbb{X})$ instead of $B_{\alpha,\beta}^k(U; \mathbb{X}, n)$.

Let M be a Hausdorff paracompact topological space. A n -dimensional C^k -structure in M is a C^k -atlas in the classical sense, i.e, a family \mathcal{A} of coordinate systems $\varphi_i : U_i \rightarrow \mathbb{R}^n$ whose domains cover M and whose transition functions $\varphi_j \circ \varphi_i^{-1} : \varphi(U_{ij}) \rightarrow \mathbb{R}^n$ are C^k , where $U_{ij} = U_i \cap U_j$. A n -dimensional C^k -manifold is a pair (M, \mathcal{A}) , where \mathcal{A} is a maximal C^k -structure. Given a $C_{n,\beta}^{k,\alpha}$ -presheaf B in a proper full ISP \mathbb{X} , define a $(B_{\alpha,\beta}^k, \mathbb{X})$ -structure on a C^k -manifold (M, \mathcal{A}) as a subatlas $\mathcal{B}_{\alpha,\beta}^k(\mathbb{X}) \subset \mathcal{A}$ such that $\varphi_j \circ \varphi_i^{-1} \in B_{\alpha,\beta}^{k,n}(\varphi_i(U_{ij}); \mathbb{X}(\varphi_i(U_{ij})))$. A $(B_{\alpha,\beta}^k, \mathbb{X})$ -manifold is one in which a $(B_{\alpha,\beta}^k, \mathbb{X})$ -structure has been fixed. A $(B_{\alpha,\beta}^k, \mathbb{X})$ -morphism between two C^k -manifolds is a C^k -function $f : M \rightarrow M'$ such that $\phi f \varphi^{-1} \in B_{\alpha,\beta}^{k,n}(\varphi(U); \mathbb{X}(\varphi(U)))$, for every $\varphi \in \mathcal{B}_{\alpha,\beta}^k(\mathbb{X})$ and $\phi \in \mathcal{B}'_{\alpha,\beta}(\mathbb{X})$. The following can be easily verified:

1. A C^k -manifold (M, \mathcal{A}) admits a $(B_{\alpha,\beta}^k, \mathbb{X})$ -structure iff there exists a subatlas $\mathcal{B}_{\alpha,\beta}^k(\mathbb{X})$ for which the identity $id : M \rightarrow M$ is a (B, k, α, β) -morphism in \mathbb{X} .
2. If M_j is a $(B_{\alpha_j, \beta_j}^k, \mathbb{X})$ -manifold, with $j = 1, 2$, then there exists a $(B_{\alpha,\beta}^k, \mathbb{X})$ -morphism between them only if $\beta(i) \geq \max_j \{\beta_j(i)\}$ for every i . In particular, if $\beta_j(i) = i$, then such a morphism exists only if $i \leq \beta(i)$.

Example 5.2. In the standard vectorial ISP \mathbb{X} the Example 1.1 and Example 1.2 contains the basic examples of $(B_{\alpha,\beta}^k, \mathbb{X})$ -manifolds.

We would like to consider the category $\mathbf{Diff}_{\alpha,\beta}^{B,k}(\mathbb{X})$ of $(B_{\alpha,\beta}^k, \mathbb{X})$ -manifolds with $(B_{\alpha,\beta}^k, \mathbb{X})$ -morphisms between them. The next lemma reveals, however, that the composition of those morphisms is not well-defined.

Lemma 5.1. *Let B be a $C_{n,\beta}^{k,\alpha}$ -presheaf in a proper full ISP \mathbb{X} , let $\Pi(i)$ be the set of partitions of $[i] = \{1, \dots, i\}$. For each $\mu[i] \in \Pi(i)$, write $\mu[i; r]$ to denote its blocks, i.e., $\mu[i] = \sqcup_r \mu[i; r]$. Let \leq be a function assigning to each $i \in \Gamma_{\geq 0}$ orderings \leq_i in $\Pi(i)$ and $\leq_{\mu[i]}$ in the set of blocks of $\mu[i]$, as follows:*

$$\begin{aligned} \mu[i]_{\min} &\leq_i \mu[i]_{\min+1} \leq_i \dots \leq_i \mu[i]_{\max-1} \leq_i \mu[i]_{\max} \\ \mu[i; \min] &\leq_{\mu[i]} \mu[i; \min+1] \leq_{\mu[i]} \dots \leq_{\mu[i]} \mu[i; \max-1] \leq_{\mu[i]} \mu[i; \max]. \end{aligned}$$

Then, for any given open sets $U, V, W \subset \mathbb{R}^n$, composition induces a map²

$$\bar{\circ} : B_{\alpha,\beta}^{k,n}(U, V; \mathbb{X}) \times B_{\alpha,\beta}^{k,n}(V, W; \mathbb{X}) \rightarrow B_{\alpha_{\leq}, \beta_{\leq}}^{k,n}(U, W; \mathbb{X}),$$

where $B_{\alpha,\beta}^{k,n}(U, V; \mathbb{X})$ is the subspace of (B, k, α, β) -functions $f : U \rightarrow \mathbb{R}^n$ such that $f(U) \subset V$, and

$$\begin{aligned} \alpha_{\leq}(i) &= \delta(\bar{\epsilon}_{\mu[i]_{\max}}, \delta(\bar{\epsilon}_{\mu[i]_{\max-1}}, \delta(\bar{\epsilon}_{\mu[i]_{\max-2}}, \dots, \delta(\bar{\epsilon}_{\mu[i]_{\min+1}}, \bar{\epsilon}_{\mu[i]_{\min}}) \dots))) \\ \beta_{\leq}(i) &= \max_{\mu[i]} \max_{\mu[i; r]} \{\beta(|\mu[i]|), \beta(|\mu[i; r]|)\}. \end{aligned}$$

Here, if $\mu[i]$ is some partition of $[i]$, then

$$\bar{\epsilon}_{\mu[i]} = \epsilon(\alpha(|\mu[i]|), \epsilon(\mu_{\alpha}[i])),$$

where for any block $\mu[i; r]$ of $\mu[i]$ we denote $|\mu_{\alpha}[i; r]| \equiv \alpha(|\mu[i; r]|)$ and

$$\epsilon(\mu_{\alpha}[i]) \equiv \epsilon(|\mu_{\alpha}[i; \max]|, \epsilon(|\mu_{\alpha}[i; \max-1]|, |\mu_{\alpha}[i; \max-2]|, \dots, \epsilon(|\mu_{\alpha}[i; \min+1]|, |\mu_{\alpha}[i; \min]|) \dots)).$$

Proof. The proof follows from Faà di Bruno's formula giving a chain rule for higher order derivatives [13] and from the compatibility between the multiplicative/additive structures of B_{α} and $C^{k-\beta}$. First of all, notice that if $C^k(U; V)$ is the set of all C^k -functions $f : U \rightarrow \mathbb{R}^n$ such that $f(U) \subset V$, then for any $g \in C^k(V; W)$ the composition $g \circ f$ is well-defined. Thus, our task is to show that there exists the dotted arrow making commutative the diagram below.

$$\begin{array}{ccc} C^k(U; V) \times C^k(V; W) & \xrightarrow{\circ} & C^k(U; W) \\ \uparrow & & \uparrow \\ B_{\alpha,\beta}^k(U, V; \mathbb{X}) \times B_{\alpha,\beta}^k(V, W; \mathbb{X}) & \xrightarrow{\bar{\circ}} & B_{\alpha,\beta}^k(U, W; \mathbb{X}) \end{array}$$

Now, let $f \in C^k(U; V)$ and $g \in C^k(V; W)$, so that from Faà di Bruno's formula, for any $0 \leq i \leq k$ and any multi-index μ such that $|\mu| = i$, we have

$$\partial^{\mu}(g \circ f)_j = \sum_j \sum_{\mu[i]} \partial^{\mu[i]} g_j \prod_{\mu[i; r] \in \mu[i]} \partial^{\mu[i; r]} f_j \equiv \sum_j \sum_{\mu[i]} \partial^{\mu[i]} g_j f_j^{\mu[i]}.$$

²In the subspace topology this is actually a continuous map, but we will not need that here.

Consequently, if f and g are actually (B, k, α, β) -functions in \mathbb{X} , then

$$\partial^{\mu[i]} g_j \in B_{\alpha(|\mu[i]|)}(V) \cap_{X(V)} C^{k-\beta(|\mu[i]|)}(V) \quad \text{and} \quad \partial^{\mu[i;r]} f_j \in B_{\alpha(|\mu[i;r]|)}(U) \cap_{X(U)} C^{k-\beta(|\mu[i;r]|)}(U).$$

Under the choice of ordering functions \leq , from the compatibility of multiplicative structures we see that

$$\begin{aligned} f_j^{\mu[i]} &\in B_{\epsilon(\mu_\alpha[i])}(U) \cap_{X(U)} C^{k-\max\{\beta(|\mu[i;r]|)\}}(U) \\ \partial^{\mu[i]} g_j f_j^{\mu[i]} &\in B_{\epsilon_{\mu[i]}}(U) \cap_{X(U)} C^{k-\max\{\beta(|\mu[i;r]|), \beta(|\mu[i]|)\}}(U). \end{aligned}$$

Finally, compatibility of additive structures shows that $\partial^\mu(g \circ f)_j \in B_{\alpha_\leq(i)}(U) \cap_{X(U)} C^{k-\beta_\leq(i)}(U)$, so that by varying i we conclude that $g \circ f$ is a $(B, k, \alpha_\leq, \beta_\leq)$ -function in \mathbb{X} . \square

Remark 5.2. Unlike $B_{\alpha, \beta}^k(-; \mathbb{X})$, the rule $U \mapsto B_{\alpha, \beta}^k(U, U; \mathbb{X})$ is generally not a presehaf, since the restriction of a map $f : U \rightarrow U$ to an open set $V \subset U$ needs not take values in V .

The problem with the composition can be avoided by imposing conditions on B . Indeed, given an ordering function \leq as above, let us say that a $C_{\alpha, \beta}^{k, n}$ -presheaf B *preserves* \leq (or that it is *ordered*) in \mathbb{X} if there exists an embedding of presheaves $B_{\alpha_\leq} \cap_X C^{k-\beta_\leq} \hookrightarrow B_\alpha \cap_X C^{k-\beta}$.

Example 5.3. We say that B is *increasing* (resp. *decreasing*) if for any U we have embeddings $B(U)_i \hookrightarrow B(U)_j$ (resp. $B(U)_j \hookrightarrow B(U)_i$) whenever $i \leq j$, where the order is the canonical order in $\Gamma_{\geq 0} \subset \mathbb{Z}_{\geq 0}$. Suppose that $\beta_\leq(i) \leq \beta(i)$, that B is increasing (resp. decreasing) and that $\alpha_\leq(i) \leq \alpha(i)$ (resp. $\alpha_\leq(i) \geq \alpha(i)$). Thus, for any U and any i we have embeddings $B(U)_{\alpha_\leq(i)} \hookrightarrow B(U)_{\alpha(i)}$ and $B(U)_{\beta_\leq(i)} \hookrightarrow B(U)_{\beta(i)}$, so that by the universality of pullbacks and stability of monomorphisms we see that any intersection presheaf makes B ordered.

Corollary 5.1. *In the same notations and hypotheses of the previous lemma, if B is ordered relative to some intersection presheaf \mathbb{X} , then the composition induces a map*

$$\bar{\circ} : B_{\alpha, \beta}^k(U, V; \mathbb{X}) \times B_{\alpha, \beta}^k(V, W; \mathbb{X}) \rightarrow B_{\alpha, \beta}^k(U, W; \mathbb{X}).$$

Proof. Straightforward. \square

On the other hand, we also have problems with the identities: the identity map $id : U \rightarrow U$ is not necessarily a (B, k, α, β) -function in an arbitrary intersection structure \mathbb{X} for an arbitrary B . We say that B is *unital* in \mathbb{X} if $id_U \in B_{\alpha, \beta}^k(U; \mathbb{X})$ for every open set $U \subset \mathbb{R}^n$. Thus, with this discussion we have proved:

Proposition 5.2. *If B is as $C_{n, \beta}^{k, \alpha}$ -presheaf which is ordered and unital in some intersection presheaf \mathbb{X} , then the category $\mathbf{Diff}_{\alpha, \beta}^{B, k}(\mathbb{X})$ is well-defined.*

6 Theorem B

In this section we will finally prove an existence theorem of $(B_{\alpha, \beta}^k, \mathbb{X})$ -structures on C^k -manifolds under certain conditions on B , meaning absorption and retraction conditions, culminating on Theorem B. Given a $C_{n, \beta}^{k, \alpha}$ -presheaf B in \mathbb{X} and open sets $U, V \subset \mathbb{R}^n$, let $\mathbf{Diff}_{\alpha, \beta}^k(U, V; \mathbb{X})$ be the set

of (B, k, α, β) -diffeomorphisms from U to V in \mathbb{X} , i.e, the largest subset for which there exists the dotted arrow below.

$$\begin{array}{ccc} B_{\alpha,\beta}^k(U, V; \mathbb{X}) & \hookrightarrow & C^k(U; V) \\ \uparrow & & \uparrow \\ \text{Diff}_{\alpha,\beta}^k(U, V; \mathbb{X}) & \dashrightarrow & \text{Diff}^k(U; V) \end{array}$$

We say that B is *left-absorbing* (resp. *right-absorbing*) in \mathbb{X} if for every U, V, W there also exists the dotted arrow in the lower (resp. upper) square below, i.e, $g \circ f$ remains a (B, k, α, β) -diffeomorphism whenever f (resp. g) is a (B, k, α, β) -diffeomorphism and g (resp. f) is a C^k -diffeomorphism. If B is both left-absorbing and right-absorbing, we say simply that it is *absorbing* in \mathbb{X} .

$$\begin{array}{ccc} \text{Diff}^k(U; V) \times \text{Diff}_{\alpha,\beta}^k(V, W; \mathbb{X}) & \xrightarrow{\circ_r} & \text{Diff}_{\alpha,\beta}^k(U, W; \mathbb{X}) \\ \downarrow \text{id} \times \iota & & \downarrow \\ \text{Diff}^k(U; V) \times \text{Diff}^k(V; W) & \xrightarrow{\circ} & \text{Diff}^k(U; W) \\ \uparrow \iota \times \text{id} & & \uparrow \\ \text{Diff}_{\alpha,\beta}^k(U, V; \mathbb{X}) \times \text{Diff}^k(V; W) & \xrightarrow{\circ_l} & \text{Diff}_{\alpha,\beta}^k(U, W; \mathbb{X}) \end{array}$$

A more abstract description of these absorbing properties is as follows. Let \mathbf{C} be an arbitrary category and let $\text{Iso}(\mathbf{C})$ the set of isomorphisms in \mathbf{C} , i.e,

$$\text{Iso}(\mathbf{C}) = \coprod_{X, Y \in \text{Ob}(\mathbf{C})} \text{Iso}_{\mathbf{C}}(X; Y).$$

Let $\text{Com}(\mathbf{C}) \subset \text{Iso}(\mathbf{C}) \times \text{Iso}(\mathbf{C})$ be the pullback between the source and target maps $s, t : \text{Iso}(\mathbf{C}) \rightarrow \text{Ob}(\mathbf{C})$. Composition gives a function $\circ : \text{Com}(\mathbf{C}) \rightarrow \text{Iso}(\mathbf{C})$. If \mathbf{C} has a distinguished object $*$, we can extend \circ to the whole $\text{Iso}(\mathbf{C}) \times \text{Iso}(\mathbf{C})$ by defining \circ_* , such that $g \circ_* f = g \circ f$ when $(f, g) \in \text{Com}(\mathbf{C})$ and $g \circ_* f = \text{id}_*$, otherwise. Thus, $(\text{Iso}(\mathbf{C}), \circ_*)$ is a magma. Define a *left *-ideal* (resp. *right *-ideal*) in \mathbf{C} as a map I assigning to each pair of objects $X, Y \in \mathbf{C}$ a subset $I(X; Y) \subset \text{Iso}_{\mathbf{C}}(X; Y)$ such that the corresponding subset

$$I(\mathbf{C}) = \coprod_{X, Y \in \text{Ob}(\mathbf{C})} I(X; Y)$$

of $\text{Iso}(\mathbf{C})$ is actually a left ideal (resp. right ideal) for the magma structure induced by \circ_* . A *bilateral *-ideal* (or **-ideal*) in \mathbf{C} is a map I which is both left and right *-ideal. When $*$ is an initial object, we say simply that I is a left-ideal, right-ideal or ideal in \mathbf{C} .

Proposition 6.1. *A $C_{n,\beta}^{k,\alpha}$ -presheaf B is left-absorbing (resp. right-absorbing or absorbing) in a proper full ISP \mathbb{X} iff the induced rule*

$$U, V \mapsto \text{Diff}_{\alpha,\beta}^{k,n}(U, V; \mathbb{X})$$

is a left ideal (resp. right ideal or ideal) in the full subcategory of \mathbf{Diff}^k , consisting of open sets of \mathbb{R}^n and C^k -maps between them.

Proof. Immediate from the definitions above. \square

Notice that in the context of vector spaces, since these are free abelian objects, the short exact sequence below always split, so that from the splitting lemma we conclude the existence of a retraction r_U , such that $r_U \circ \iota = id$, for every U [11].

$$0 \longrightarrow B_{\alpha,\beta}^k(U; \mathbb{X}) \xrightarrow{\iota} C^k(U; \mathbb{R}^n) \xrightarrow{\pi} C^k(U; \mathbb{R}^n)/B_{\alpha,\beta}^k(U; \mathbb{X}) \longrightarrow 0$$

$\xleftarrow{r_U}$ (curved arrow from $C^k(U; \mathbb{R}^n)$ to $B_{\alpha,\beta}^k(U; \mathbb{X})$)

By restriction, for each V we have an induced retraction $r_{U,V}$, as in the first diagram below. On the other hand, the dotted arrow does not necessarily exists. In other words, $r_{U,V}$ need not preserve diffeomorphisms. We say that B has *retractible (B, k, α, β) -diffeomorphisms in \mathbb{X}* if for every U, V there exists $\bar{r}_{U,V}$ in the second diagram, not necessarily making the first diagram commutative. A *retraction presheaf in \mathbb{X}* for B is a rule \bar{r} , assigning to each U, V a retraction $\bar{r}_{U,V}$.

$$\begin{array}{ccc} B_{\alpha,\beta}^k(U, V; \mathbb{X}) & \xrightarrow{\iota} & C^k(U; V) \\ \uparrow & \xleftarrow{r_{U,V}} & \uparrow \\ \text{Diff}_{\alpha,\beta}^k(U, V; \mathbb{X}) & \xrightarrow{\quad} & \text{Diff}^k(U; V) \end{array} \quad \begin{array}{ccc} \text{Diff}_{\alpha,\beta}^k(U, V; \mathbb{X}) & \xrightarrow{\iota_{U,V}} & \text{Diff}^k(U; V) \\ \xleftarrow{\bar{r}_{U,V}} & & \end{array} \quad (8)$$

Given a C^k -manifold (M, \mathcal{A}) and a $C_{n,\beta}^{k,\alpha}$ -presheaf B in \mathbb{X} , let $C^k(\mathcal{A})$ and $B_{\alpha,\beta}^{k,n}(\mathcal{A}; \mathbb{X})$ denote the collection of not necessarily maximal C^k -structures $\mathcal{A}' \subset \mathcal{A}$ and $(B_{\alpha,\beta}^k, \mathbb{X})$ -structures $\mathcal{B}_{\alpha,\beta}^k(\mathbb{X}) \subset \mathcal{A}$, respectively. Observe that there is an inclusion $\iota_{\mathcal{A}} : B_{\alpha,\beta}^k(\mathcal{A}; \mathbb{X}) \hookrightarrow C^k(\mathcal{A})$, which take a $(B_{\alpha,\beta}^k, \mathbb{X})$ -structure and regard it as a C^k -structure. We can now finally prove that for certain classes of B the set $B_{\alpha,\beta}^{k,n}(\mathcal{A}; \mathbb{X})$ is non-empty.

Theorem 6.1. *Let B be a $C_{n,\beta}^{k,\alpha}$ -presheaf which is ordered, left-absorbing or right-absorbing, and which has retractible (B, k, α, β) -diffeomorphisms, all of this in the same proper full ISP \mathbb{X} . In this case, for any C^k -manifold (M, \mathcal{A}) , the choice of a retraction presheaf \bar{r} induces a function $\kappa_{\bar{r}} : C^k(\mathcal{A}) \rightarrow B_{\alpha,\beta}^{k,n}(\mathcal{A}; \mathbb{X})$ which is actually a retraction for $\iota_{\mathcal{A}}$. In particular, under this hypothesis every C^k -manifold has a $(B_{\alpha,\beta}^k, \mathbb{X})$ -structure.*

Proof. Let $\mathcal{A}' \subset \mathcal{A}$ be some not necessarily maximal C^k -structure and let $\varphi_i : U_i \rightarrow \mathbb{R}^n$ be its charts. The transition functions are given by $\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_{ij}) \rightarrow \varphi_j(U_{ij})$. Notice that the restricted chart $\varphi_j|_{U_{ij}}$ can be recovered by $\varphi_j|_{U_{ij}} = \varphi_{ji} \circ \varphi_i|_{U_{ij}}$ for each i such that $U_{ij} \neq \emptyset$. This motivate us to define new functions $\bar{\varphi}_{j;i} : U_{ij} \rightarrow \mathbb{R}^n$ by $\bar{\varphi}_{j;i} = \bar{r}(\varphi_{ji}) \circ \varphi_i|_{U_{ij}}$, where \bar{r} is a fixed restriction presheaf. They are homeomorphisms onto their images because they are composites of them. We assert that when varying i and j , the maps $\bar{\varphi}_{j;i}$ generate a $(B_{\alpha,\beta}^k, \mathbb{X})$ -structure, which we denote by $\bar{r}(\mathcal{A}')$. Indeed, for each i, j, k, l , then transition functions $\bar{\varphi}_{k;l} \circ \bar{\varphi}_{j;i}^{-1} : \bar{\varphi}_{j;i}(U_{ijkl}) \rightarrow \bar{\varphi}_{k;l}(U_{ijkl})$ are given by

$$\begin{aligned} \bar{\varphi}_{k;l} \circ \bar{\varphi}_{j;i}^{-1} &= [\bar{r}(\varphi_{kl}) \circ \varphi_l|_{U_{ijkl}}] \circ [\varphi_i^{-1}|_{U_{ijkl}} \circ \bar{r}(\varphi_{ji})^{-1}] \\ &= \bar{r}(\varphi_{kl}) \circ (\varphi_{li})|_{U_{ijkl}} \circ \bar{r}(\varphi_{ji})^{-1}, \end{aligned}$$

which are (B, k, α, β) -functions in \mathbb{X} , due to the absorbing properties of B in \mathbb{X} . More precisely, if B is left-absorbing, then $\bar{\tau}(\varphi_{kl}) \circ (\varphi_{li})|_{U_{ijkl}}$ is a (B, k, α, β) -function in \mathbb{X} . But $\bar{\tau}(\varphi_{ji})^{-1}$ is also a (B, k, α, β) -function in \mathbb{X} and by the ordering hypothesis on B the composite remains a (B, k, α, β) -function in \mathbb{X} . In this case, define $\kappa_{\bar{\tau}}(\mathcal{A}') = \bar{\tau}(\mathcal{A}')$. If B is right-absorbing, similar argument holds. That $\kappa_{\bar{\tau}}$ is a retraction for $\iota_{\mathcal{A}}$, i.e. that $\bar{\tau}(\iota_{\mathcal{A}}(\mathcal{B}_{\alpha, \beta}^k(\mathbb{X}))) = \mathcal{B}_{\alpha, \beta}^k(\mathbb{X})$ follows from the fact that $\bar{\tau}$ is a retraction presheaf. \square

Observe that if B is left-absorbing or right-absorbing, then it is automatically unital, so that from Proposition 5.2 under the hypotheses of the last theorem the category $\mathbf{Diff}_{\alpha, \beta}^{B, k}(\mathbb{X})$ is well-defined. We have an obvious forgetful functor $F_{\alpha, \beta}^{B, k} : \mathbf{Diff}_{\alpha, \beta}^{B, k}(\mathbb{X}) \rightarrow \mathbf{Diff}^k$ which takes a $(B_{\alpha, \beta}^k \mathbb{X})$ -manifold $(M, \mathcal{A}, \mathcal{B}_{\alpha, \beta}^k(\mathbb{X}))$ and forgets $\mathcal{B}_{\alpha, \beta}^k(\mathbb{X})$ (this is essentially an extension of the inclusion $\iota_{\mathcal{A}}$). Our task is to show that this functor has adjoints. We begin by proving existence of adjoints on the core.

We recall that if \mathbf{C} is a category, then its *core* is the subcategory $\mathbf{C}(\mathbf{C}) \subset \mathbf{C}$ obtained by forgetting all morphisms which are not isomorphisms. Every functor $F : \mathbf{C} \rightarrow \mathbf{D}$ factors through the core, so that we have an induced functor $\mathbf{C}(F) : \mathbf{C}(\mathbf{C}) \rightarrow \mathbf{C}(\mathbf{D})$. Actually, the core construction provides a functor $\mathbf{C} : \mathbf{Cat} \rightarrow \mathbf{Cat}$, where \mathbf{Cat} denotes the category of all categories [16, 7].

Theorem 6.2. *In the same notations and hypotheses of Theorem 6.1, for every restriction presheaf $\bar{\tau}$ the rule $\kappa_{\bar{\tau}}$ induces a functor $K_{\bar{\tau}} : \mathbf{C}(\mathbf{Diff}^k) \rightarrow \mathbf{C}(\mathbf{Diff}_{\alpha, \beta}^{B, k}(\mathbb{X}))$. If B is left-absorbing (resp. right-absorbing), then $K_{\bar{\tau}}$ is a left (resp. right) adjoint for the core of the forgetful functor. In particular, if B is absorbing, then $\mathbf{C}(F_{\alpha, \beta}^{B, k})$ is ambidextrous adjoint.*

Proof. Define $K_{\bar{\tau}}$ by $K_{\bar{\tau}}(M, \mathcal{A}) = (M, \mathcal{A}, \bar{\tau}(\mathcal{A}))$ on objects and by $K_{\bar{\tau}}(f) = f$ on morphisms. On objects it is clearly well-defined. On morphisms it is too, because for any $\bar{\varphi}_{j;i}, \bar{\varphi}_{k;l} \in \bar{\tau}(\mathcal{A})$ we have

$$\begin{aligned} \bar{\varphi}_{k;l} \circ f \circ \bar{\varphi}_{j;i}^{-1} &= [\bar{\tau}(\varphi_{kl}) \circ \varphi_l|_{U_{ijkl}}] \circ f \circ [\varphi_i^{-1}|_{U_{ijkl}} \circ \bar{\tau}(\varphi_{ji})^{-1}] \\ &= \bar{\tau}(\varphi_{kl}) \circ (\varphi_l f \varphi_i)|_{U_{ijkl}} \circ \bar{\tau}(\varphi_{ji})^{-1}. \end{aligned} \quad (9)$$

Since we are in the core, $(\varphi_l f \varphi_i)$ is a C^k -diffeomorphism, so that we can use the same arguments of that used in Theorem 6.1 to conclude that (9) is a (B, k, α, β) -function in \mathbb{X} . Preservation of compositions and identities is clear, so that $K_{\bar{\tau}}$ really defines a functor. Suppose that B is left-absorbing. Given a C^k -manifold (M, \mathcal{A}) and a $(B_{\alpha, \beta}^k \mathbb{X})$ -manifold $(M', \mathcal{A}', \mathcal{B}'_{\alpha, \beta}^k(\mathbb{X}))$ we assert that there is a bijection

$$\mathbf{Diff}_{\alpha, \beta}^{B, k}(K_{\bar{\tau}}(M, \mathcal{A}); (M', \mathcal{A}', \mathcal{B}'_{\alpha, \beta}^k(\mathbb{X})) \begin{array}{c} \xleftarrow{\xi_{M, M'}} \\ \xrightarrow{\iota_{M, M'}} \end{array} \mathbf{Diff}^k((M, \mathcal{A}); (M', \mathcal{A}')) \quad (10)$$

which is natural in both manifolds. Define $\iota_{M, M'}(f) = f$ and notice that this is well-defined, since locally it is given by the inclusions $\iota_{U, V}$ in (8). Define $\xi_{M, M'}(f) = f$. In order to show that this is also well-defined, let $f : (M, \mathcal{A}) \rightarrow (M', \mathcal{A}')$ a C^k -diffeomorphism and let $\bar{\varphi}_{j;i} \in \bar{\tau}(\mathcal{A})$ and $\phi \in \mathcal{B}'_{\alpha, \beta}^k(\mathbb{X})$ charts. Thus,

$$\phi \circ f \circ \bar{\varphi}_{j;i}^{-1} = (\phi \circ f \circ \varphi_i^{-1}) \circ \bar{\tau}(\varphi_{ji})^{-1}. \quad (11)$$

Due to the inclusion $\iota_{U, V}$, the chart ϕ is a C^k -chart, so that $\phi \circ f \circ \varphi_i^{-1}$ is a C^k -diffeomorphism (since f is a C^k -diffeomorphism). The left-absorption property then implies that (11) is a (B, k, α, β) -function in \mathbb{X} , meaning that $\xi_{M, M'}$ is well-defined. That (10) holds is clear; naturality follows from

the fact that the maps $\iota_{M,M'}$ and $\xi_{M,M'}$ do not depend on the manifolds. The case in which B is right-absorbing is completely analogous. \square

Corollary 6.1. *In the same notations and hypotheses of Theorem 6.1, the function $\kappa_{\bar{\tau}}$ depends on $\bar{\tau}$. More precisely, if $\bar{\tau}$ and $\bar{\tau}'$ are two retraction presheaves, then there exists a natural isomorphism $K_{\bar{\tau}} \simeq K_{\bar{\tau}'}$, so that for every C^k -manifold (M, \mathcal{A}) we have a corresponding (B, k, α, β) -diffeomorphism $(M, \bar{\tau}(\mathcal{A})) \simeq (M, \bar{\tau}'(\mathcal{A}))$.*

Proof. Straightforward from the uniqueness of the left and right adjoints [16, 7]. \square

We would like to extend Theorem 6.2 to the whole category $\mathbf{Diff}_{\alpha,\beta}^{B,k}(\mathbb{X})$. In order to do this, notice that when proving Theorem 6.2 the hypothesis that we are working on the core was used only to conclude that the local expressions $\phi f \varphi$ are C^k -diffeomorphisms, leading us to use the diffeomorphism-absorption properties. But, if instead absorbing only diffeomorphisms we can absorb every C^k -map, we will then be able to absorb $\phi f \varphi$ for every f , meaning that the same proof will still work in $\mathbf{Diff}_{\alpha,\beta}^{B,k}(\mathbb{X})$.

We say that a $C_{n,\beta}^{k,\alpha}$ -presheaf B is *fully left-absorbing* (resp. *fully right-absorbing*) in \mathbb{X} if for every U, V, W there exists the dotted arrow in the lower (resp. upper) square below. If B is both fully left-absorbing and fully right-absorbing, we say simply that it is *fully absorbing in \mathbb{X}* . There is also an abstract characterization in terms of left/right/bilateral ideals, but now considered in the magma $\text{Mor}(\mathbf{C})$ of all morphisms instead of on the magma $\text{Iso}(\mathbf{C})$ of isomorphisms.

$$\begin{array}{ccc}
C^k(U; V) \times \text{Diff}_{\alpha,\beta}^{k,n}(V, W; \mathbb{X}) - \overset{\circ_r}{\triangleright} & \text{Diff}_{\alpha,\beta}^{k,n}(U, W; \mathbb{X}) \\
\text{id} \times \iota \downarrow & \downarrow \\
C^k(U; V) \times C^k(V; W) \xrightarrow{\circ} & C^k(U; W) \\
\iota \times \text{id} \uparrow & \uparrow \\
\text{Diff}_{\alpha,\beta}^{k,n}(U, V; \mathbb{X}) \times C^k(V; W) - \overset{\circ_l}{\triangleright} & \text{Diff}_{\alpha,\beta}^{k,n}(U, W; \mathbb{X})
\end{array}$$

Theorem B. *If B is ordered, fully left-absorbing (resp. fully right-absorbing) and has retractible (B, k, α, β) -diffeomorphisms, all of this in the same intersection presheaf \mathbb{X} , then the choice of a retraction $\bar{\tau}$ induces a left-adjoint (resp. right-adjoint) for the forgetful functor F from $B_{\alpha,\beta}^k$ -manifolds to C^k -manifolds, which actually depends on $\bar{\tau}$. In particular, if B is fully absorbing, then F is ambidextrous adjoint.*

Proof. Immediate from the results and discussions above. \square

Corollary 6.2. *In the same notations of the last theorem, if B is fully left-absorbing (resp. fully right-absorbing), then $\mathbf{Diff}_{\alpha,\beta}^{B,k}(\mathbb{X})$ has all small colimits (resp. small limits) that exist in \mathbf{Diff}^k . If B is fully absorbing, then the same applies for limits and colimits simultaneously. In particular, in this last case $\mathbf{Diff}_{\alpha,\beta}^{B,k}(\mathbb{X})$ has finite products and coproducts.*

Proof. Just apply Theorem B together with the preservation of small colimits/limits by left/right-adjoint functors [16, 7] and recall that the category of C^k -manifolds has finite products and coproducts [18, 4]. \square

Acknowledgments

The first author was supported by CAPES (grant number 88887.187703/2018-00).

References

- [1] Adachi, M., *Embeddings and Immersions*, AMS, 2012.
- [2] Aguiar, M., Mahajan, S., *Monoidal Functors, Species and Hopf Algebras*, AMS 2010.
- [3] Atiyah, M., *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. 85 (1957), 181–207.
- [4] Baez, J. C., Hoffnung, A. E., *Convenient Categories of Smooth Spaces*, Trans. Amer. Math. Soc. Vol. 363, No. 11 (2011), pp. 5789–5825.
- [5] Berline, N., Getzler, E., Vergne, M., *Heat Kernels and Dirac Operators*, Springer, 2004.
- [6] Bieliavsky, P., Cahen, M., Gutt, S., Rawnsley, J., Schwachhofer, L., *Symplectic connections*, [arXiv:math/0511194](https://arxiv.org/abs/math/0511194).
- [7] Borceux, F., *Handbook of Categorical Algebra I*, Cambridge University Press, 2008.
- [8] Costello, K., *Renormalization and Effective Field Theory*, AMS 2011.
- [9] Eichhorn, J., *The Boundedness of Connection Coefficients and their Derivatives*, Math. Nachr. 152 (1991) 145–158.
- [10] Fedosov, B., *A simple geometrical construction of deformation quantization*, Journal of Differential Geometry, 40(2):213–238, 1994.
- [11] Gelfand, S. I., Manin, Y. I., *Methods of Homological Algebra*, Springer, 2003.
- [12] Greene, R. E., *Complete metrics of bounded curvature on noncompact manifolds*, Arch. Math (1978) 31: 89.
- [13] Hörmander, L., *The analysis of linear partial differential operators I*, Springer, 1983.
- [14] Kobayashi, S., *Differential Geometry of Complex Vector Bundle*, Princeton University Press, 2014.
- [15] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry, Vol.1*, Wiley-Interscience, 1996.
- [16] MacLane, S., *Categories for the Working Mathematician*, 2nd edition, Springer, 1978.
- [17] MacLane, S., Moerdijk, I., *Sheaves in Geometry and Logic*, Springer, 1994.
- [18] Moerdijk, I., Reyes, G. E., *Models for Smooth Infinitesimal Analysis*, Springer, 1991.
- [19] Müller, O., Nardmann, M., *Every conformal class contains a metric of bounded geometry*, Mathematische Annalen 363 (2015), 143–174.

- [20] Sternberg, S., *Lectures on Differential Geometry*, AMS, 1999.
- [21] Treves, F., *Topological Vector Spaces, Distributions and Kernels*, Academic Press, 1967.
- [22] Uhlenbeck, K. K., *Connections with L^p bounds on curvature*, Comm. Math. Phys. Volume 83, Number 1 (1982), 31-42.