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# Graph Reconstruction, Functorial Feynman Rules and Superposition Principles

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**Abstract** In this article functorial Feynman rules are introduced as large generalizations of physicists Feynman rules, in the sense that they can be applied to arbitrary classes of hypergraphs, possibly endowed with any kind of structure on their vertices and hyperedges. We show that the reconstruction conjecture for classes of (possibly structured) hypergraphs admit a sheaf-theoretic characterization, allowing us to consider analogous conjectures. We propose an axiomatization for the notion of superposition principle and prove that the functorial Feynman rules work as a bridge between reconstruction conjectures and superposition principles, meaning that a conjecture for a class of hypergraphs is satisfied only if each functorial Feynman rule defined on it induces a superposition principle. Applications in perturbative euclidean quantum field theory and graph theory are given.

**Keywords** graph reconstruction · Feynman graphs · Feynman rules · superposition principle

**Mathematics Subject Classification (2010)** 18D10 · 81T18 · 05C60 · 81Q30

## 1 Introduction

Graphs (and their generalizations such as hypergraphs) appear in the most different areas of mathematics and physics, generally parametrizing definitions and constructions. For instance: in algebraic geometry the Deligne-Mumford moduli stack of stable curves is stratified by certain graphs [1, 2]; in symplectic topology the stable maps, which play an important role in the study of  $J$ -holomorphic curves

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and Gromov-Witten theory, are defined by making use of graphs [3, 4]; Kontsevich's formula for deformation quantization of Poisson manifolds is parametrized by graphs [5, 6]; the Ribbon graphs (fat graphs) are used to compute the weak homotopy type of the geometric realization of mapping class group of surfaces with marked points [7, 8, 9]; in perturbative quantum field theory the Feynman graphs parametrize the possible worldlines of relativistic quantum particles [10, 11]; the moduli space of marked surfaces (and, therefore, stable graphs) is also used to parametrize the worldsheet of closed strings [12, 13]; the recent amplituhedrons, which parametrize the scattering amplitudes, are certain hypergraphs [14].

In the abstract study of graph theory there are important conjectures, known as *reconstruction conjectures*, stating that in order to describe a graph (belonging to a certain fixed class) it is necessary and sufficient to describe subgraphs obtained by some deleting process. It is known that these conjectures are true for some classes (such as trees, regular graphs, maximal planar graphs) and false for others (such as digraphs and infinite graphs), but the general classification remains broadly open (see [15, 16] for a review and an exposition).

On the other hand, in a completely different perspective, there are the so-called *superposition principles* which state that physical properties of a physical system are totally determined by the corresponding properties of certain subsystems. They typically occur when the physical property in question is described by a linear partial differential equation. For example, Maxwell's and Schrödinger's equations are linear, so that we have wave superposition and wave function superposition (quantum superposition). Thinking in this way, it is natural to regard a superposition principle as some kind of reconstruction phenomenon.

When considering systems of perturbative quantum field theory we have both graphs (or even hypergraphs) and physical properties: Feynman graphs and scattering amplitudes. They are related via certain rules, known as Feynman rules [10, 11]. It then makes sense to consider the reconstruction conjecture for Feynman graphs and to ask about superposition principles for scattering amplitudes. Furthermore, it is natural to ask if Feynman rules play some role between these two types of reconstruction processes. In this article, our objective is to give a positive answer, but in a much more general setup.

More precisely, we show that the physicists Feynman rules can be axiomatized under a very general frame, being regarded as functors defined in some category of structured hypergraphs and taking values into the category of analytic expressions (which will define the scattering amplitudes) of some monoidal category (which plays the role of a context where functional analysis can be done). We call these functors *Feynman functors*. We prove that the classic Feynman rules can be extended for structured hypergraphs and for any suitable context for functional analysis, establishing the general existence of Feynman functors. We also prove that any other Feynman functor is conjugated (in a very nice way, which we call quasi essentially injective conjugation) to that obtained extending the classic Feynman rules, establishing uniqueness.

There are two fundamental steps in showing that Feynman functors behave as a bridge between reconstruction conjectures and superposition principles:

- s1) *showing that the reconstruction conjectures admit a sheaf-theoretic characterization.*  
For each finite set  $V$ , let  $\mathbf{S}_{\mathfrak{s}, V}$  denote the category of hypergraphs which have structure of type  $\mathfrak{s}$  and vertex set  $V$ . Varying  $V$  we get a prestack  $S_{\mathfrak{s}}$ . By making

use of a deleting process, say  $\mathcal{D}$ , we get a new prestack  $\mathcal{D}S_{\mathfrak{s}}$  and a morphism  $\mathcal{D}_{\mathfrak{s}} : S_{\mathfrak{s}} \Rightarrow \mathcal{D}S_{\mathfrak{s}}$ , i.e, a family of functors  $\mathcal{D}_{\mathfrak{s},V} : \mathbf{S}_{\mathfrak{s},V} \rightarrow \mathcal{D}\mathbf{S}_{\mathfrak{s},V}$ . We show that different reconstruction conjectures consist in different choices of  $\mathcal{D}$  and that the corresponding  $\mathcal{D}_{\mathfrak{s}}$  is objectwise essentially injective;

- s2) *proving that we can always consider Feynman functors which are not only arbitrary functors, but actually monoidal and essentially injective.* Monoidal property means that the analytic expression of two disjoint structured hypergraphs is the product of the corresponding analytic expressions. In turn, the essentially injectivity property means that the hypergraphs are totally described by their analytic expression.

Let  $\mathbf{A}$  be a monoidal category endowed with a structure of context for doing functional analysis, and let  $Z_V : \mathbf{S}_{\mathfrak{s},V} \rightarrow \mathbf{A}_{\tau V}$  be a Feynman functor assigning to each hypergraph with  $\mathfrak{s}$ -structure an analytic expression in  $\mathbf{A}$ . Let  $\mathcal{D}$  be a deleting process and suppose that we have a canonical morphism from  $\mathcal{D}_{\mathfrak{s},V}G$  to the disjoint union of the parts of  $G$  obtained via deleting (which generally happens). Since  $Z$  is monoidal, for each  $G$  we have a corresponding morphism

$$Z_V(G) \rightarrow \bigotimes_{\text{pieces}} Z(\text{pieces of } G).$$

Suppose now that the reconstruction conjecture induced by  $\mathcal{D}$  is satisfied. Then, because  $Z_V$  is essentially injective, it follows that two hypergraphs  $G$  and  $G'$  are isomorphic iff their analytic expressions have the same decomposition in terms of the analytic expressions of the pieces. This conclusion is precisely one example of a superposition principle (in the sense axiomatized here). Let us call it the  *$\mathcal{D}$ -superposition principle*. Thus, from s1) and s2) there follows our main result:

**Theorem 1** *The  $\mathcal{D}$ -reconstruction conjecture for a prestack  $S_{\mathfrak{s}}$  of  $\mathfrak{s}$ -structured hypergraphs is true only if for any Feynman functor the  $\mathcal{D}$ -superposition principle holds.*

This theorem can be regarded both as an obstruction to the validity of reconstruction conjectures and as a source of new superposition principles. This relation becomes more involved when we think of the role of quantum field theory. Indeed, consider  $\mathbf{S}_{\mathfrak{s}}$  as the category of hypergraphs which parametrize the worldvolume of particles, strings or branes (for instance, of Feynman graphs of QED or some other gauge theory). Suppose we find  $\mathcal{D}$  such that the  $\mathcal{D}$ -reconstruction conjecture is true. Then each Feynman functor (in particular that obtained by the Feynman rules of QED, etc.) will produce a new superposition principle for the scattering amplitudes.

On the other hand, since  $\mathbf{S}_{\mathfrak{s}}$  are the Feynman graphs of a physical theory, we can analyze whether these superposition principles for the scattering amplitudes exist or not, looking at concrete experiments of LHC, trying to find a counterexample. If found, it will be a strong indicative that the  $\mathcal{D}$ -reconstruction conjecture is false. Another approach is to notice that the existence of new superposition principles in quantum theories produce many logic implications [17, 18], so that assuming the validity of the  $\mathcal{D}$ -conjecture we could verify if the induced logic implications contradict those that are experimentally realized.

This paper is organized as follows. In Section 2 we define what is meant by a  $\mathfrak{s}$ -structured hypergraph and give many examples of objects that can be regarded as such. We also show that the category of all categories that can be embedded into

$\mathbf{S}_s$  for some  $s$  is complete. In Section 3 the sheaf-theoretic characterization of the classical reconstruction conjecture is given and analogous conjectures are defined, as needed for step s1). We also prove that if one work with hypergraphs which contain labelings as part of their structures, then many reconstruction conjectures are true. In Section 4 the notions of context for functional analysis and analytic expressions are axiomatized and many examples are given. In Section 5 Feynman functors and functorial Feynman rules are defined and the existence and uniqueness up to quasi essentially injective conjugation is established, as required for step s2). We also show how to recover the classic Feynman rules from this general approach. In Section 6 the notion of superposition principle is formalized and a formal proof of Theorem 1 is given. Finally, in Section 7 some applications of our results on hypergraph theory, manifold topology and perturbative quantum field theory are presented.

*Remark 1* Along this article, by an oplax monoidal functor we mean one where only the morphisms of the products are reverted. So,  $F : \mathbf{C} \rightarrow \mathbf{C}'$  is oplax if it becomes endowed with natural transformations  $F(X \otimes Y) \rightarrow F(X) \otimes' F(Y)$  and a morphism  $1' \rightarrow F(1)$  making the appropriated diagrams commutative.

## 2 Structured Hypergraphs

There are several ways to define a hypergraph. For us, a *hypergraph*  $G$  consists of a finite (possibly empty) set  $V$  of vertices and for each  $j > 1$  a finite (possibly empty) set  $E_j$  of  $j$ -edges and a  $j$ -adjacency function  $\psi_j : E_j \rightarrow \text{bin}(V, j)$  such that  $\psi_1 = id_V$ , where  $\text{bin}(V, j)$  denotes the set of  $j$ -subsets of  $V$ . We usually write  $E_1 = V$ , so that a 1-edge is just a vertice. If  $v \in \psi_j(e)$  we say that  $e$  is adjacent to  $v$ . For each  $v \in V$  and each  $j > 1$ , let  $d_j(v)$  denote the number of  $j$ -edges that are adjacent to  $v$ . Using these notations, we have [19]:

$$\sum_{j>1} \sum_v d_j(v) = \sum_{j>1} j|E_j|. \quad (1)$$

Let  $\mathbb{N}$  be the set of natural number regarded as a discrete category and notice that for each set  $X$  the rule  $j \mapsto \text{bin}(X, j)$  extends to a functor  $\text{bin}(X, -) : \mathbb{N} \rightarrow \mathbf{Set}$ . Thus, a hypergraph is equivalently a pair  $(E, \psi)$ , where  $E : \mathbb{N} \rightarrow \mathbf{FinSet}$  and  $\psi : E \Rightarrow \text{bin}(E(1), -)$  is a natural transformation such that  $\psi_1 = id$ . Here,  $\mathbf{FinSet}$  denotes the category of finite sets. The equivalence between both definitions is obtained via the identifications  $E(j) = E_j$ .

The rule assigning to each set  $X$  its set  $\text{bin}(X, j)$  of  $j$ -subsets also extends to a functor  $\text{bin}(-, j) : \mathbf{Set} \rightarrow \mathbf{Set}$ . We can then define a *morphism*  $f : G \rightarrow G'$  of between hypergraphs  $(E, \psi)$  and  $(E', \psi')$  as a natural transformation  $f : E \Rightarrow E'$  which commutes with adjacencies, i.e, such that  $\psi' \circ f = \text{bin}(f_1, -) \circ \psi$ . We have the category  $\mathbf{Hyp}$  of hypergraphs. Under the operation of taking disjoint unions of hypergraphs it becomes a symmetric monoidal category whose neutral object is the empty hypergraph.

Sometimes we will work with *bounded hypergraphs*. We say that  $G$  is *bounded* if there is some  $b$  such that  $E_j = \emptyset$  for each  $j \geq b$ . The smallest of these  $b$ 's is the *bounding degree* of  $G$ . For each  $b > 0$  we have monoidal subcategories  $\mathbf{Hyp}^b \subset \mathbf{Hyp}$  of *b-bounded hypergraphs* with fixed bounding degree  $b$ . For instance, if  $b = 2$  this is

the category of what is known as finite pseudographs or finite graphs, depending on the author and we will write **Grph** instead of **Hyp**<sup>2</sup>.

*Remark 2* Let  $\mathbf{C} \subset \mathbf{Hyp}$  be some category of hypergraphs. For fixed  $V$  we can consider the full subcategory  $\mathbf{C}_V \subset \mathbf{C}$  of hypergraphs in  $\mathbf{C}$  whose vertex set is  $V$  or empty. Even if  $\mathbf{C}$  is a monoidal subcategory, if  $V \neq \emptyset$  then  $\mathbf{C}_V$  is not monoidal. Indeed, if  $G, G'$  have the same vertex set  $V$ , then  $G \sqcup G'$  has vertex set  $V \sqcup V$ . This is one of the motivations for considering reconstruction conjectures in a sheaf-theoretic perspective, as will be discussed in the next section.

In the following we will work with *structured hypergraphs*, in that for any  $j \geq 1$  we have functions  $\varepsilon_j : E_j \rightarrow s_j$ . We think of  $s_j$  as a set of  $j$ -structures in the set of  $j$ -edges. Thus,  $\varepsilon_j$  assigns to each  $j$ -edge a corresponding  $j$ -structure. We form the category  $\mathbf{S} \subset \mathbf{Hyp}$  whose morphisms are hypergraph morphisms  $f : G \rightarrow G'$  preserving  $j$ -structures. In more precisely terms, for each functor  $s : \mathbb{N} \rightarrow \mathbf{Set}$ , called a *functor of structures*, we define a category  $\mathbf{S}_s$  as follows. Objects are *s-structured hypergraphs*, i.e, pairs  $(G, \varepsilon)$ , where  $G = (E, \psi)$  is a hypergraph and  $\varepsilon : E \Rightarrow s$  is a natural transformation. Morphisms  $f : (G, \varepsilon) \rightarrow (G', \varepsilon')$  are hypergraph morphisms between the underlying hypergraphs such that  $\varepsilon' \circ \text{bin}(f, -) = \varepsilon$ . If we are working with bounded hypergraphs we can consider  $\mathbf{S}^b \subset \mathbf{Hyp}^b$ , defined analogously.

Given  $s$ -structured hypergraphs  $(G, \varepsilon)$  and  $(G', \varepsilon')$ , the disjoint union  $G \sqcup G'$  can be naturally regarded as a  $s$ -structured hypergraph with the transformation  $(\varepsilon \sqcup \varepsilon')_j : E_j \sqcup E'_j \rightarrow s_j$  given by the composition below, where the second map is the codiagonal. Also, the empty hypergraph has a unique  $s$ -structure, so that  $\mathbf{S} \subset \mathbf{Hyp}$  is actually a monoidal subcategory.

$$E_j \sqcup E'_j \xrightarrow{\varepsilon_j \sqcup \varepsilon'_j} s_j \sqcup s_j \xrightarrow{(\varepsilon \sqcup \varepsilon')_j} s_j$$

*Remark 3* When  $s_j : \mathbb{N} \rightarrow \mathbf{Ab}$ , i.e, when the structural functor takes values in the category of abelian groups, the map  $(\varepsilon \sqcup \varepsilon')_j$  can be identified with  $\varepsilon_j + \varepsilon'_j$ . In these situations we say that  $s$  is an *additive structure*. In this paper, essentially all structures will take values in **FinSet**, so that we can always think of them as additive structures by replacing a finite set  $[n]$  with the abelian group  $\mathbb{Z}_n$  that it generates.

The last construction can be easily generalized by considering not only one functor of structure  $s$ , but a family of them. In fact, let  $\mathfrak{s} : \mathbb{N} \rightarrow [\mathbb{N}; \mathbf{Set}]$  be a functor that to each  $k$  assigns a structure functor  $\mathfrak{s}_k : \mathbb{N} \rightarrow \mathbf{Set}$ , so that  $\mathfrak{s}_{k,j} \in \mathbf{Set}$  (equivalently,  $\mathfrak{s}$  can be regarded as a bifunctor  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbf{Set}$ )<sup>1</sup>. Let  $\text{cst}_E : \mathbb{N} \rightarrow [\mathbb{N}; \mathbf{Set}]$  be the constant functor in  $E$ . We define a  *$\mathfrak{s}$ -structured hypergraph* as previously: it is a pair  $(G, \varepsilon)$ , but now  $\varepsilon$  is a natural transformation  $\varepsilon : \text{cst}_E \Rightarrow \mathfrak{s}$ . Thus, it is a rule that to each  $k$  assigns a natural transformation  $\varepsilon_k : E \Rightarrow \mathfrak{s}_k$ , which means that for fixed  $j \geq 1$  we have a family  $\varepsilon_{k,j} : E_j \rightarrow \mathfrak{s}_{k,j}$ . Morphisms of  $\mathfrak{s}$ -structured hypergraphs also are defined analogously, so that we have a monoidal category  $\mathbf{S}_{\mathfrak{s}}$ . If we are working with bounded hypergraphs we have the corresponding monoidal category  $\mathbf{S}_{\mathfrak{s}}^b$ .

<sup>1</sup> Here  $[\mathbf{C}; \mathbf{D}]$  denotes the functor category.

## 2.1 Embedded Subcategories

In this subsection we will discuss some category of hypergraphs that can be embedded into the category of structured hypergraphs in an essentially injective way. More precisely, we will give examples of subcategories  $\mathbf{C} \subset \mathbf{Hyp}$  such that for each fixed  $V$  the corresponding  $\mathbf{C}_V$  can be realized, up to equivalence, as a full subcategory of  $\mathbf{S}_{\mathfrak{s},V}$  for some  $\mathfrak{s}$ , meaning that there exists a fully faithful (and, therefore, essentially injective) inclusion functor  $\iota : \mathbf{C}_V \hookrightarrow \mathbf{S}_{\mathfrak{s},V}$ .

*Example 1 (colouring)* Recall that a (vertex) *colouring* for a hypergraph  $G$  consists of a finite set  $A \subset \mathbb{N}$  of colors and a function  $c : V \rightarrow S$  (assigning to each vertice its color) such that each hyperedge contains at least two vertices of distinct colors. In other words, for each  $j$  the composition  $c_j = \text{bin}(V, c) \circ \psi_j$  is non-constant. We will work with  $A = [n] = \{1, 2, \dots, n\}$ . Notice that for graphs we recover the usual notion of graph coloring. A *n-colored* hypergraph is one in which a coloring with a set of  $n$  colors was fixed, i.e, it is a pair  $(G, c)$  with  $c : V \rightarrow [n]$ . A morphism  $f : (G, c) \rightarrow (G', c')$  of *n-colored* hypergraphs is a hypergraph morphism which preserves the coloring, i.e,  $c' \circ f = c$ . Denote by  $n\mathbf{Col}$  the category of *n-colored* hypergraphs. Define  $\mathfrak{s}$  by  $\mathfrak{s}_{1,1} = [n]$  and  $\mathfrak{s}_{j,k} = 0$  if  $j \neq 1$  or  $k \neq 1$ . Let  $n\mathbf{S}_{\mathfrak{s},V}$  be the full subcategory of  $\mathbf{S}_{\mathfrak{s},V}$  whose objects are  $\mathfrak{s}$ -structured graphs  $(G, \epsilon)$  with  $\epsilon_{1,1} = c$  and  $\epsilon_{j,k} = 0$  if  $j \neq 1$  or  $k \neq 1$ . The condition  $c' \circ f = c$  means precisely that  $c' \circ f = c$ , so that  $n\mathbf{Col}_V \simeq \mathbf{S}_{\mathfrak{s},V}$ .

*Example 2 (labeled hypergraphs)* By a *labeling* of a hypergraph  $G$  we mean a bijection  $\varphi_1 : V \rightarrow [|V|]$ , where  $[n] = \{1, \dots, n\}$ . Notice that this map induces bijections  $\text{bin}(\varphi_1, j)$  for every  $j > 1$  and, by restriction, corresponding bijections  $\varphi_j : E_j \rightarrow [|E_j|]$ . A *labeled hypergraph* is one in which a labeling has been chosen, i.e, it is a pair  $(G, \varphi_1)$ . A morphism of labeled hypergraphs  $(G, \varphi_1)$  and  $(G', \varphi'_1)$  is a triple  $(f, \alpha, \alpha')$ , where  $f : G \rightarrow G'$  is a morphism of hypergraphs and  $\alpha$  and  $\alpha'$  are families of maps such that the diagram below commutes. Notice that, since  $\varphi_j$  and  $\varphi_1$  are bijective,  $\alpha$  and  $\alpha'$  are actually determined by  $f$ ,  $\varphi$  and  $\varphi'$ .

$$\begin{array}{ccccccc}
 [|E_j|] & \xrightarrow{\varphi_j} & E_j & \xrightarrow{f_j} & E'_j & \xleftarrow{\varphi'_j} & [|E'_j|] \\
 \alpha_j \downarrow & & \psi_j \downarrow & & \psi'_j \downarrow & & \alpha'_j \downarrow \\
 \text{bin}(|V|, j) & \xrightarrow{\text{bin}(\varphi_1, j)} & \text{bin}(V, j) & \xrightarrow{\text{bin}(f_1, j)} & \text{bin}(V', j) & \xleftarrow{\text{bin}(\varphi'_1, j)} & \text{bin}(|V'|, j)
 \end{array}$$

We then have a category  $\mathbf{Hyp}^\ell$  of labeled hypergraphs. Let us show that for every fixed finite set  $V$  we have a canonical functor  $\iota : \mathbf{Hyp}_V^\ell \hookrightarrow \mathbf{S}_{\mathfrak{s},V}$  for some  $\mathfrak{s}$ . From the above will then follows that  $\iota$  is essentially injective, as desired. We define  $\mathfrak{s}_{k,j}$  as the group  $\mathbb{Z}_{|V|}$ , if  $j = k = 1$ , and the trivial group otherwise, i.e, if  $j \neq 1$  or  $k \neq 1$ . Now, for every labeled hypergraph  $(G, \varphi)$  with vertex set  $E(1) = V$ , take  $\epsilon_{k,j} : E_j \rightarrow \mathfrak{s}_{k,j}$  as the trivial map if  $k \neq 1$  or  $j \neq 1$ , and  $\epsilon_{1,1} = \varphi_1$ . For a morphism  $f$  of labeled graphs, it is clear that  $\epsilon'_{k,j} \circ \text{bin}(f, j) = \epsilon_{k,j}$ , so that it becomes a morphism of structured graphs, and thus  $\mathbf{Hyp}_V^\ell \hookrightarrow \mathbf{S}_{\mathfrak{s},V}$ .

*Remark 4* If  $(G, \varphi)$  and  $(G', \varphi')$  are labeled hypergraphs such that  $|E_j| = |E'_j|$  for some  $j$ , then it follows from the commutativity of the last diagram that any

morphism  $f : G \rightarrow G'$  has the  $j$ th component  $f_j$  completely determined by the labelings, i.e,  $f_j = \varphi_j \circ \varphi_j^{-1}$ . In particular, it is a bijection. Consequently, if there exists an isomorphism between two labeled graphs, then it is unique. Furthermore, if we are working in  $\mathbf{Hyp}_V^\ell$ , then we can always assume  $f_1 = id_V$ .

*Example 3 (Feynman graphs)* A *Feynman graph* is a graph  $G$  that has further structure satisfying additional properties. Precisely, it has a decomposition  $V = V^0 \sqcup V^1$  into *external* (or *false*) *vertices* and *internal* (or *fundamental*) *vertices*, respectively, and a function  $g : V \rightarrow \mathbb{Z}_{\geq 0}$ , called *genus map*. This data is required to satisfy:

- c1) if  $v \in V^0$ , then  $d(v) = 1$  and  $g(v) = 0$ ;
- c2) there are no edges between two external vertices.

From the conditions above we see that there exists  $E^0 \subset E$  with  $|E^0| = |V^0|$ , so that we also have a decomposition  $E = E^0 \sqcup E^1$  into *external edges* (or *tails*) and *internal edges*, respectively, where  $E^1 = E - E^0$ . A *morphism* of Feynman graphs is a hypergraph morphism preserving the additional structure, i.e, such that  $f_1$  maps  $V^i$  into  $V'^i$  and  $g' \circ f_1 = g$ . Consequently,  $f_2$  preserves  $E^0$  and, therefore,  $E^1$ . Let  $\mathbf{Fyn}$  denote the category of Feynman graphs. Notice that to give a decomposition of  $V$  is equivalent to giving a surjective function  $\pi : V \rightarrow \mathbb{Z}_2$  by  $\pi_V^{-1}(i) = V^i$ . Take

$$\mathfrak{s}_{k,1} = \begin{cases} \mathbb{Z}_2, & k = 1 \\ \mathbb{Z}_{\geq 0}, & k = 2 \\ 0, & k > 2 \end{cases}$$

with  $\mathfrak{s}_{k,j} = 0$  if  $j > 1$ . If  $(G, \pi, g)$  is a Feynman graph on  $V$ , define  $\epsilon_{1,1} = \pi$ ,  $\epsilon_{2,1} = g$  and  $\epsilon_{k,j} = 0$ , otherwise. For a given graph morphism  $f$  the condition  $\epsilon'_{k,j} \circ \text{bin}(f, j) = \epsilon_{k,j}$  means precisely that  $f$  preserves  $\pi$  and  $g$ . Thus,  $f$  is a Feynman graph morphism iff it is a morphism of  $\mathfrak{s}$ -structured graphs. This gives the desired embedding  $\mathbf{Fyn}_V \hookrightarrow \mathbf{S}_{\mathfrak{s},V}$ .

*Remark 5* If to conditions c1) and c2) above we add

- c3) if  $g(v) = 0$ , then  $d(v) \geq 3$ ;
- c4) if  $g(v) = 1$ , then  $d(v) \geq 1$ ,

then we have what is known as *stable graphs*, since this class of graphs contains those arising in the study of stable curves [1, 20]. If  $\mathbf{Stb} \subset \mathbf{Fyn}$  is the full subcategory of stable graphs, then  $\mathbf{Stb}_V$  can also be embedded in  $\mathbf{S}_{\mathfrak{s},V}$  for the same  $\mathfrak{s}$  as  $\mathbf{Fyn}_V$ .

*Example 4 (structured Feynman hypergraphs)* Feynman graphs can be generalized in two directions: allowing hyperedges and allowing additional structures. In the first case we say that we have a *Feynman hypergraph*, while in the second one we say that we have a *structured Feynman graph*. It is straightforward to verify (following the same kind of construction used in the previous examples) that these classes of graphs produce categories  $\mathbf{HypFyn}_V$  and  $\mathbf{Fyn}_{\mathfrak{s},V}$  which can be embedded in  $\mathbf{S}_{\mathfrak{s},V}$  for some  $\mathfrak{s}$ . Special examples are the so-called *generalized Feynman graphs* [21, 22] and *sectored Feynman graphs* [23, 24, 25]<sup>2</sup>. For instance, in generalized

<sup>2</sup> The terminology is not standard: some authors use generalized Feynman graphs to refer to sectored Feynman graphs. There is also a notion of *generalized Feynman amplitudes*, introduced in [26], which (to the best of the author's knowledge) it is not directly related with the other ones.



Feynman graphs we have an additional decomposition of  $V^1$  into *positive* and *negative* vertices, i.e,  $V^{1,+} \sqcup V^{1,-}$  (which means that now we need to use a projection  $\pi : V \rightarrow \mathbb{Z}_3$  onto  $\mathbb{Z}_3$  instead of onto  $\mathbb{Z}_2$ ) and the edge set  $E$  is constrained by the condition that there is no edges between vertices of the same signal.

*Example 5 (ribbon graphs)* An alternative way of defining a graph  $G$  is as being given by a set  $V$  of vertices, a set  $E$  of edges, an incidence function  $s : V \rightarrow E$  and an involution  $i : E \rightarrow E$  without fixed points. Morphisms  $f : G \rightarrow G'$  are defined as functions  $f_V : V \rightarrow V'$  and  $f_E : E \rightarrow E'$  such that  $f_E \circ s = s' \circ f_V$  and  $f_E \circ i = i' \circ f_E$ . Denote by  $\mathbf{Grph}'$  this category. So,  $\mathbf{Grph}' \simeq \mathbf{Hyp}^2$ . Of special interest is the subcategory  $\mathbf{Ribb}$  of *ribbon* (or *fat*) graphs. They become endowed with a permutation  $\sigma : E \rightarrow E$  satisfying the following property:

- r) let  $\langle \sigma \rangle$  be the cyclic group generated by  $\sigma$ . It acts on  $E$  giving a decomposition, which must coincide with that induced by the fibers  $s^{-1}(x)$  of  $s$ .

The morphisms between ribbon graphs are graph morphisms which preserve the permutations. Let  $\mathbf{Ribb}_0$  be the category of graphs with a permutation  $\sigma : E \rightarrow E$  but which does not necessarily satisfies r). It is clear that  $\mathbf{Ribb}_0 \hookrightarrow \mathbf{S}_E^1$  for some  $\mathfrak{s}$ . But  $\mathbf{Ribb}$  is a full subcategory of  $\mathbf{Ribb}_0$ , so that it can also be embedded into a category of structured graphs.

The intersection of two embedded subcategories of structured hypergraphs remains a category of structured hypergraphs. More precisely, if  $\mathbf{C} \hookrightarrow \mathbf{S}_\mathfrak{s}$  and  $\mathbf{C}' \hookrightarrow \mathbf{S}_{\mathfrak{s}'}$ , then  $\mathbf{C} \cap \mathbf{C}'$  can be embedded in both  $\mathbf{S}_\mathfrak{s}$  and  $\mathbf{S}_{\mathfrak{s}'}$ . For instance, in the last example  $\mathbf{Fyn}_\mathfrak{s} = \mathbf{Fyn} \cap \mathbf{S}_\mathfrak{s}$ . More generally, arbitrary limits of categories of structured hypergraphs remain a category of structured hypergraphs. In fact, let  $\mathfrak{S}$  be the category defined as follows. Its objects are categories  $\mathbf{C}$  such that there exists a functor of structures  $\mathfrak{s}$  and an essentially injective embedding  $\iota : \mathbf{C} \hookrightarrow \mathbf{S}_\mathfrak{s}$ , while the morphisms are functors. So  $\mathfrak{S}$  it is actually a full subcategory of  $\mathbf{Cat}$ . Since full embeddings are monadic, it follows that they reflect limits [27]. Thus:

**Proposition 1** *The category  $\mathfrak{S}$  is complete.*

We end this section with a convention which will be specially important in the construction of Feynman functors in Section 5.

*Remark 6* Let  $(G, \varphi)$  and  $(G', \varphi')$  two labeled hypergraphs with vertex set  $V$ . Regarding them as structured hypergraphs as done in Example 2 we find ambiguities when considering the induced labeling in  $G \sqcup G'$ . In order to fix this we will use the following convention: let  $[[V]]' = \{1', \dots, |V|'\}$  be a copy of  $[[V]]$  and in the disjoint union

$$[[V]] \sqcup [[V]]' = \{1, \dots, |V|, 1', \dots, |V|'\}$$

consider the ordering

$$1 \leq 2 \leq \dots \leq |V| \leq 1' \leq 2' \leq \dots \leq |V|'.$$

Define in  $V \sqcup V$  a similar ordering, i.e,  $\varphi(1) \leq \dots \leq \varphi(|V|) \leq \varphi(1') \leq \dots \leq \varphi(|V|')$ . Then define the labeling in  $G \sqcup G'$  as the unique bijection such that this ordering is preserved.

### 3 Reconstruction Conjectures

Given two sets  $V, V'$ , with  $V' \subset V$ , let  $V'^c = V - V'$  denote the complement of  $V'$  in  $V$ . We have a pair of adjoint functors

$$D_{V',V} : \mathbf{Hyp}_V \rightleftarrows \mathbf{Hyp}_{V'} : I_{V',V},$$

defined as follows. The right adjoint  $I_{V',V}$  is just the inclusion functor. More precisely, if  $G$  is a hypergraph with vertex set  $V'$ , then  $I_{V',V}(G)$  is the graph obtained by adding the elements of  $V'^c$  as isolated vertices. On the other hand,  $D_{V',V}$  is the functor that takes a hypergraph  $G$ , with vertex set  $V$ , and delete the vertices  $V'^c$  together with their adjacent hyperedges.

So, if we fix a set  $X$  and define  $\mathbb{X}$  as the category whose objects are subsets  $V \subset X$  and whose morphisms are inclusions, varying  $V, V'$  we get functors  $I : \mathbb{X} \rightarrow \mathbf{Cat}$  and  $D : \mathbb{X}^{op} \rightarrow \mathbf{Cat}$ . Notice that the process of adding (resp. deleting) a finite number of vertices is equivalent to iterating the process of adding (resp. deleting) a single vertice. This means that we have a distinguished class  $J \subset \text{Mor}(\mathbb{X})$  of morphisms, given by inclusions  $j_x : V' \hookrightarrow V$ , with  $V' = V - x$  for some  $x \in V$ . We can think of  $J$  as a rule assigning to each  $V \in \mathbb{X}$  the collection of morphisms  $J(V) = (j_x)_{x \in V}$ , which we call the *covering family of  $V$* . It is clear that if  $V' \subset V$ , then  $V' - x \subset V - x$ . Since the only morphisms in  $\mathbb{X}$  are inclusions and the only covering families are  $(j_x)$ , the previous condition implies that given a covering family  $J(V)$  and a morphism  $V' \rightarrow V$ , then for any covering family  $J(V')$  and each  $j'_{x'} \in J(V')$  there exists some  $j_x \in J(V)$  (actually  $j_{x'}$ ) such that the diagram below commutes.

$$\begin{array}{ccc} V' - x' & \xrightarrow{f - x'} & V - x' \\ j'_{x'} \downarrow & & \downarrow j_x \\ V' & \xrightarrow{f} & V \end{array}$$

In other words,  $J$  is a coverage for  $\mathbb{X}$ . Therefore, we can talk about stacks on the site  $(\mathbb{X}, J)$ . More precisely, we have a reflexive subcategory of presheaves

$$\iota : \mathbf{Stack}(\mathbb{X}, J) \rightleftarrows [\mathbb{X}^{op}; \mathbf{Cat}] : L$$

whose reflection  $L$  preserve finite limits. This reflexive subcategory is just the localization of the presheaf category at the local isomorphism system associated with  $J$ . If  $j \in J$ , then  $Y(j)$  belongs to this system, where  $Y$  denotes the Yoneda embedding. Therefore,  $LY(j)$  is an isomorphism, so that from Yoneda lemma  $LF(j)$  is an isomorphism for every  $F : \mathbb{X}^{op} \rightarrow \mathbf{Cat}$ . Particularly, it is for the deleting functor  $D$  above and for every subfunctor  $C \subset D$ . This means that *after localization, the process of deleting vertices becomes an equivalence*. Furthermore: the same holds for every subfunctor  $C \subset D$ . Such a subfunctor assigns to each  $V \subset X$  a subcategory  $\mathbf{C}_V \subset \mathbf{Hyp}_V$  which is invariant by vertex deleting, i.e, we get an induced functor  $C_{V',V} : \mathbf{C}_V \rightarrow \mathbf{C}_{V-V'}$ . So, after localizing, deleting vertices is an equivalence independently of the class of hypergraphs considered.

We should not expect the same result before localizing, since in general there are much more graphs with  $|V|$  vertices than graphs with  $|V| - 1$  vertices. But, we can ask if in a given class of graphs, i.e, for a given subfunctor  $C \subset D$ , for every  $j_x \in$

$J$  the corresponding functor  $C(j_x) : \mathbf{C}_V \rightarrow \mathbf{C}_{V-x}$  is at least essentially injective. This remains a very strong requirement, since we are asking if any information of a hypergraph  $G \in \mathbf{C}_V$  can be recovered from the information after deleting a single *fixed* vertice  $x \in V$ . Thus, we can think of taking all  $x \in V$  into account simultaneously. More precisely, we can ask if the induced composition below is essentially injective.

$$\begin{array}{ccc} & \Delta \mathbf{C}_V & \\ \mathbf{C}_V & \xrightarrow{\Delta} \prod_{x \in V} \mathbf{C}_V & \xrightarrow{\prod_{x \in V} C(j_x)} \prod_{x \in V} \mathbf{C}_{V-x} \end{array}$$

This is just a sheaf theoretically formulation of what is usually known in hypergraph theory as the Reconstruction Conjecture for the class of graphs defined by  $C$ . In fact, calling an isomorphism in the image of  $\Delta \mathbf{C}_V$  a *hypomorphism*, the assertion that  $\Delta \mathbf{C}_V$  is essentially injective is equivalent to:

*Conjecture 1 (RC-C)* Two hypergraphs in  $\mathbf{C}_V$  are isomorphic iff they are hypomorphic.

*Remark 7* Of course, if the categories  $\mathbf{C}_V$  actually belong to  $\mathbf{Grph}_V$ , i.e, if we are in the context of graphs instead of general hypergraphs, then the above discussion reproduces the same conjecture, but now in graph theory. Some consequences of this categorical description (not directly related with the sheaf structure and specially concerning obstructions to the existence of non-nilpotent graph invariants) are in a work in preparation.

### 3.1 Disjoint Reconstruction

In the last section we gave a sheaf-theoretically description of the classical graph reconstruction conjecture. Our approach, on the other hand, has a problem:

- the prestack  $D$  is morphismwise adjoint to  $I$ , so that it is natural to believe that  $I$  should appear in any fundamental construction involving  $D$ . However, it was not used in the construction of  $\Delta D_V$ .

In order to fix these pathologies, notice that despite of  $I : \mathbb{X} \rightarrow \mathbf{Cat}$  not being strong monoidal (since for arbitrary  $V, V'$  we do not have an equivalence between  $\mathbf{Hyp}_{V \sqcup V'}$  and  $\mathbf{Hyp}_V \times \mathbf{Hyp}_{V'}$ ), it is lax comonoidal (because for generic  $V, V'$  there are more hypergraphs over  $V \sqcup V'$  than pairs of hypergraphs over  $V$  and  $V'$ ). Consequently, we have a natural transformation

$$\xi_{V',V}^I : \mathbf{Hyp}_V \times \mathbf{Hyp}_{V'} \rightarrow \mathbf{Hyp}_{V \sqcup V'}$$

sending pairs  $(G, G')$  into its disjoint union  $G \sqcup G'$ . We introduced the upper index to emphasize that  $\xi^I$  depends on  $I$ . So, instead of  $\Delta D_V$  we can consider the composition  $dD_V$  below.

$$\begin{array}{ccc} & \Delta D_V & \\ \mathbf{Hyp}_V & \xrightarrow{\Delta} \prod_{x \in V} \mathbf{Hyp}_V & \xrightarrow{\prod_{x \in V} D(j_x)} \prod_{x \in V} \mathbf{Hyp}_{V-x} \\ & \searrow dD_V & \downarrow \xi^I \\ & & \mathbf{Hyp}_{\prod_x (V-x)} \end{array} \quad (2)$$

This clearly fixes the initial problem for  $D$ , but this does not makes sense for arbitrary  $C \subset D$ , since  $C_{V,V'}$  may not commute with  $\xi_{V',V}^\ell$ . That is, given  $G \in \mathbf{C}_V$  and  $G' \in \mathbf{C}_{V'}$ , there is no guarantee that  $G \sqcup G' \in \mathbf{C}_{V \sqcup V'}$ . If  $C \subset D$  is a subfunctor satisfying this condition we will say that it is *proper*. In this case, for every  $V$  the map  $dC_V$  is well defined.

*Example 6 (structured presheaves)* A prestack of  $\mathfrak{s}$ -structured hypergraphs is proper. More precisely, given any functor  $\mathfrak{s} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbf{Set}$ , the presheaf  $S_\mathfrak{s}$  that to any  $V$  assigns  $\mathbf{S}_{\mathfrak{s},V}$  is proper. Indeed, given  $(G, \epsilon)$  and  $(G', \epsilon')$  in  $\mathbf{S}_{\mathfrak{s},V}$  and  $\mathbf{S}_{\mathfrak{s},V'}$ , respectively, we can always introduce structure  $\epsilon \sqcup \epsilon'$  in  $G \sqcup G'$ . In particular, the presheaf  $S^\ell$  such that  $S^\ell(X) = \mathbf{Hyp}_X^\ell$  is proper.

*Example 7* The trivial prestack  $\emptyset$  such that  $\emptyset_V = \emptyset$  for every  $V$  is trivially proper.

*Remark 8* Beware that subfunctors of proper subfunctors need not be proper. That is, if  $C \subset D$  is proper, then for arbitrary  $C' \subset C$  it is not true that  $C' \subset D$  is proper. On the other hand, the intersection  $S_\mathfrak{s}^\ell = S^\ell \cap S_\mathfrak{s}$  is proper, because  $S_\mathfrak{s}^\ell = S_{\mathfrak{s}'}^\ell$  for some  $\mathfrak{s}'$ .

Knowing that the problem is fixed by replacing  $\Delta C$  with  $\mathbf{d}C$ , we can think of a conjecture somewhat analogous to  $C$ -RC. In order to do this, let us say that the isomorphisms in the image of each  $\mathbf{d}C_V$  are *weak hypomorphisms*. So, we can then conjecture the Disjoint Reconstruction Conjecture:

*Conjecture 2 (dRC-C)* Two hypergraphs in a proper presheaf  $C \subset D$  are isomorphic iff they are weakly hypomorphic. In other words,  $\mathbf{d}C_V$  is essentially injective.

*Remark 9* From now on, if  $C$  is any prestack of hypergraphs, we will use the following simplified notations:

1.  $\sqcap \mathbf{C}_V$  instead of  $\prod_{x \in V} \mathbf{C}_{V-x}$ ;
2.  $\sqcup \mathbf{C}_V$  instead of  $\coprod_{x \in V} \mathbf{C}_{V-x}$ ;
3.  $\mathbf{C}_{dV}$  instead of  $\mathbf{C}_{\prod_{x \in V} V-x}$ .

### 3.2 Category of Reconstruction Conjectures

Let us now see that both conjectures RC and dRC can be considered in an axiomatic background. For doing this we define a *hypergraph reconstruction context* as given by the following data:

1. for each set  $X$  we have a subcategory  $\mathfrak{C} \subset [\mathbb{X}^{op}; \mathbf{Cat}]$  of *applicable prestacks*, such that if  $C \in \mathfrak{C}$ , then  $C_V \subset \mathbf{Hyp}$  for each  $V \subset X$ ;
2. a functor  $\mathcal{D} : \mathfrak{C} \rightarrow [\mathbb{X}^{op}; \mathbf{Cat}]$ , playing the role of a “deleting process” and that to any applicable prestack  $C$  it assigns the *prestack of pieces*  $\mathcal{D}C$ , such that for each  $V \subset X$  we have the *category of pieces*  $\mathcal{D}C_V$ ;
3. a natural transformation  $\gamma : \iota_{\mathfrak{C}} \Rightarrow \mathcal{D}$ , where  $\iota_{\mathfrak{C}}$  is the inclusion of applicable prestacks into the category of prestacks.

We will represent the reconstruction context simply by its deleting process  $\mathcal{D}$ , except when we need more details. We say that two hypergraphs are  *$\mathcal{D}$ -hypomorphic* if their image by  $\mathcal{D}C_V$  are isomorphic. So, given a reconstruction context  $\mathcal{D}$  and an applicable prestack  $C \in \mathfrak{C}$  we can consider the following  $\mathcal{D}$ -reconstruction conjecture for  $C$ .

*Conjecture 3 (D-RC-C)* For each  $V$ , two hypergraphs in  $C_V$  are isomorphic iff they are  $\mathcal{D}$ -hypomorphic, i.e, each functor  $\gamma_V : C_V \rightarrow DC_V$  is essentially injective.

Before giving examples, two important remarks:

*Remark 10* In the previous sections, the prestacks of hypergraphs  $C : \mathbb{X}^{op} \rightarrow \mathbf{Cat}$  were such that for each  $V \subset X$  the corresponding  $C_V$  is not only an arbitrary subcategory of  $\mathbf{Hyp}$ , but actually a subcategory of  $\mathbf{Hyp}_V$ , i.e, we worked with prestacks that assign to each  $V$  a category of hypergraphs with vertex set  $V$ . From now on, we will work with prestacks which a priori take values only in  $\mathbf{Hyp}$ . This will be specially important in proving existence and uniqueness of Feynman rules. In order to distinguish between these situations, we will say that  $C$  is a *concrete prestack* if  $C_V \subset \mathbf{Hyp}_V$ , using the bold notation  $\mathbf{C}_V$  instead of  $C_V$  (as we have used in previous sections).

*Remark 11* We say that an applicable prestack  $C \in \mathfrak{C}$  is *proper* if it becomes endowed with a natural transformation  $\xi : C_V \times C_{V'} \rightarrow C_{V \sqcup V'}$ . Recall that in the last sections we defined a proper (concrete) prestack as such that  $\xi : (G, G') \mapsto G \sqcup G'$  is well defined. This means that any proper concrete prestack (in the older context) is proper (in the newer sense). However, the reciprocal is not true, due to the last remark. In order to emphasize that this new concept is more general, we will call them *concretely proper* prestacks.

*Example 8 (RC and dRC)* In order to recover the classical reconstruction conjecture, take  $\mathfrak{C}$  as the whole category of prestacks,  $DC_V = \prod \mathbf{C}_V$  and  $\gamma_V = \Delta \mathbf{C}_V$ . For recovering the disjoint reconstruction conjecture, take  $\mathfrak{C}$  as the subcategory of proper prestacks,  $DC_V = \mathbf{C}_{dV}$  and  $\gamma_V = d\mathbf{C}_V$ .

*Example 9 (trivial context)* We also have a trivial reconstruction context  $\mathcal{I}$ , in which  $\mathfrak{C}$  is the whole category of prestacks and  $\mathcal{D}$  and  $\gamma$  are the identity functors. Notice that in it any reconstruction conjecture is satisfied.

*Example 10 (restrictions)* Let  $\mathcal{D}$  be a reconstruction context with category of applicable prestacks  $\mathfrak{C}$  and natural transformation  $\gamma : \iota_{\mathfrak{C}} \Rightarrow \mathcal{D}$ . For any subcategory  $\mathfrak{D} \subset \mathfrak{C}$  we get a context  $\mathcal{D}|_{\mathfrak{D}}$  by restricting  $\mathcal{D}$  and  $\gamma$  to  $\mathfrak{D}$ .

We will build some categories of reconstruction conjectures, allowing us to compare two of different conjectures. A *left morphism* between two reconstruction contexts  $\mathcal{D}$  and  $\mathcal{D}'$  is given by a functor  $F : \mathfrak{C} \rightarrow \mathfrak{C}'$  between the categories of applicable prestacks, together with a natural transformation  $\xi : \mathcal{D} \Rightarrow \mathcal{D}' \circ F$  between the deleting process, such that the first diagram below commutes, i.e, if for every  $C \in \mathfrak{C}$  we have  $\xi_C \circ \gamma_C = \gamma'_{F(C)}$ . This gives us a category  $LRC$ . By inverting the direction of  $\xi$  we define *right morphisms*, which produce a dual category  $RRC$ , characterized by the second diagram below.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \iota_{\mathfrak{C}} & \xrightarrow{\gamma} & \mathcal{D} \\
 \Downarrow \iota_F & & \Downarrow \xi \\
 \iota_{\mathfrak{C}' \circ F} & \xrightarrow{\gamma' \circ F} & \mathcal{D}' \circ F
 \end{array} & & \begin{array}{ccc}
 \iota_{\mathfrak{C}} & \xrightarrow{\gamma} & \mathcal{D} \\
 \Downarrow \iota_F & & \Downarrow \xi \\
 \iota_{\mathfrak{C}' \circ F} & \xrightarrow{\gamma' \circ F} & \mathcal{D}' \circ F
 \end{array}
 \end{array} \tag{3}$$

*Example 11 (canonical morphisms)* For any reconstruction context  $\mathcal{D}$  there is a canonical right morphism  $I_R : \mathcal{D} \rightarrow \mathcal{I}$ , defined as follows. In applicable prestacks it is the inclusion functor, i.e.,  $F = \iota_{\mathcal{C}}$ . Among deleting processes it is just the transformation  $\xi_C = \gamma_C$  of  $\mathcal{D}$ . The existence of left morphisms  $I_L : \mathcal{D} \rightarrow I$  is more restrictive: if we keep the canonical choice  $F = \iota_{\mathcal{C}}$ , the condition  $\xi_C \circ \gamma_C = id_C$  implies that  $\xi$  is a retraction for  $\gamma$ . In the general case, the commutativity condition is  $\xi \circ \gamma = id \circ F$ , which is also some kind of retraction requirement (let us say that  $\gamma$  has  $\xi$  as a  $F$ -retraction). So, we have the following proposition:

**Proposition 2** *Given a prestack  $F : \mathcal{C} \rightarrow [\mathbb{X}^{op}; \mathbf{Cat}]$ , there is a left morphism  $D \rightarrow I$  of reconstruction contexts coinciding with  $F$  in applicable prestacks iff  $\gamma$  has a  $F$ -retraction.*

We could think of getting morphisms in the opposite direction, i.e., from the trivial context  $I$  to a given context  $D$ . This is an even more strong requirement. This is essentially because a priori we have no canonical functor  $F : [\mathbb{X}^{op}; \mathbf{Cat}] \rightarrow \mathcal{C}$ . The commutativity conditions implies that if it exists, then it must be a retraction for the inclusion of  $\mathcal{C}$  in  $[\mathbb{X}^{op}; \mathbf{Cat}]$ . In the case of left morphisms this is enough. For right morphisms  $I \rightarrow D$  we also need  $\gamma$  to have a  $F$ -retraction.

*Example 12 (from RC to dRC)* Let  $\mathcal{C}$  be the category of proper stacks. Let  $\Delta$  and  $d$  be the contexts of the RC and dRC, as defined in Example 8. We have a right morphism  $F : d \rightarrow \Delta$ , defined as follows. In applicable prestacks we define  $F : \mathcal{C} \rightarrow [\mathbb{X}^{op}; \mathbf{Cat}]$  as the inclusion of proper prestacks and  $\xi : \Delta \Rightarrow d$  as the composition  $\xi^I \circ u$  in diagram (2). Notice that the commutativity of (3) follows directly from the commutativity of (2).

**Proposition 3** *Let  $F : \mathcal{D} \rightarrow \mathcal{D}'$  be a morphism. For a given prestack  $C \in \mathcal{C}$ :*

1. *if  $F$  is left and  $\mathcal{D}'$ -RC- $F(C)$  holds, then  $\mathcal{D}$ -RC- $C$  holds;*
2. *if  $F$  is right and  $\mathcal{D}$ -RC- $C$  holds, then  $\mathcal{D}'$ -RC- $F(C)$  holds.*

*Proof* For the first case, notice that the commutativity of (3) gives us  $\xi_C \circ \gamma_C = \gamma'_{F(C)}$  and recall that essentially injective functors behave as monomorphisms, so that if  $\gamma'_{F(C)}$  is essentially injective, then  $\gamma_C$  is too. But the validity of  $\mathcal{D}'$ -RC- $F(C)$  is, by definition, the guarantee that  $\gamma'_{F(C)}$  is essentially injective. For the second case, (3) gives  $\gamma_C = \xi_C \circ \gamma'_{F(C)}$ . Now use the same argument of the first case.  $\square$

In some cases we have a left morphism, we know that  $\mathcal{D}$ -RC- $C$  holds and we would like to conclude that  $\mathcal{D}'$ -RC- $F(C)$  holds. In other words, we would like to have conditions under which the hypothesis of the first part of the last proposition implies the conclusion of the second part, and vice-versa. This can be easily ensured if we work with a special class of morphisms. We say that a left morphism  $F : \mathcal{D} \rightarrow \mathcal{D}'$  is a left  $C$ -implication if the transformation  $\xi : \mathcal{D}' \circ F \Rightarrow \mathcal{D}$  is objectwise essentially injective. Right  $C$ -implications are defined analogously.

**Proposition 4** *Let  $F : \mathcal{D} \rightarrow \mathcal{D}'$  be a morphism. For a given prestack  $C \in \mathcal{C}$ :*

1. *if  $F$  is left  $C$ -implication and  $\mathcal{D}$ -RC- $C$  holds, then  $\mathcal{D}'$ -RC- $F(C)$  holds;*
2. *if  $F$  is right  $C$ -implication and  $\mathcal{D}'$ -RC- $F(C)$  holds, then  $\mathcal{D}$ -RC- $C$  holds.*

*Proof* As in the last proposition, for the first case (3) gives  $\xi_C \circ \gamma_C = \gamma'_{F(C)}$ . Since composition of essentially injective functors remains essentially injective, it is done. The second case is analogous.  $\square$

**Corollary 1** *For any  $\mathcal{D}$  and any  $C \in \mathfrak{C}$ , the conjecture  $\mathcal{D}$ -RC-C holds iff the canonical morphism  $I_R : \mathcal{D} \rightarrow \mathcal{I}$  is a right C-implication.*

*Proof* Straightforward.  $\square$

As a final result, let us show that reconstruction conjectures are invariant by a certain base-change.

**Proposition 5** *Let  $\mathcal{D}$  be a reconstruction context and suppose that  $\mathcal{D}$ -RC-C holds for some applicable prestack  $C \in \mathfrak{C}$ . In this case, if  $f : A \rightarrow C$  is some objectwise essentially injective morphism in  $\mathfrak{C}$ , then  $\mathcal{D}$ -RC-A holds.*

*Proof* Since  $\gamma : \mathfrak{v} \Rightarrow D$  is a natural transformation, we have  $Df \circ \gamma_A = \gamma_C \circ f$ . By hypothesis  $f$  and  $\gamma_C$  are essentially injective, so that  $Df \circ \gamma_A$ , and therefore  $\gamma_A$ , is also.  $\square$

### 3.3 Reconstruction of Labeled Structured Hypergraphs

In this subsection we will show that the RC is true for any prestack  $S_s^\ell$  of labeled  $\mathfrak{s}$ -structured hypergraphs. As a consequence, since from Example 2 and Example 6 this prestack is proper, it will follow from Proposition 3 and Example 12 that dRC- $S_s^\ell$  holds.

**Theorem 2** *For any functor of structures  $\mathfrak{s}$ , the  $S_s^\ell$ -RC holds. In other words,  $\Delta S_{\mathfrak{s},V}^\ell$  is essentially injective for every  $V$ .*

*Proof* Notice that  $\mathbf{S}_{\mathfrak{s},V}^\ell = \mathbf{S}_{\mathfrak{s},V} \cap \mathbf{Hyp}_V^\ell$ . Therefore,  $\mathbf{S}_{\mathfrak{s},V}^\ell$  can be embedded in an essentially injective way in  $\mathbf{Hyp}_V^\ell$ . This can also be verified explicitly by a counting process:

*Addendum.* Regard  $\mathfrak{s}$  as a bifunctor  $\mathfrak{s} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbf{Set}$  and let  $N : \mathbb{X}^{op} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbf{Cat} \times \mathbf{Ab}$  be the functor given by  $N(V, i, j) = \emptyset \times \mathbb{N} \simeq \mathbb{N}$  for every  $V, i, j$ . We have a natural transformation  $k : S_s^\ell \times \mathfrak{s} \Rightarrow N$  such that for every  $V, i, j$  the map  $k_{i,j}^V : \mathbf{S}_{\mathfrak{s},V}^\ell \times s_{i,j} \rightarrow \mathbb{N}$  is the rule that to any labeled  $\mathfrak{s}$ -structured hypergraph  $(G, \Phi, \epsilon)$  and to any element  $a_{ij} \in s_{i,j}$  it assigns the cardinality of  $\epsilon_{i,j}^{-1}(a_{ij})$ . We have

$$(n-1)k_{i,j}^V(G, a_{ij}) = \sum_{x \in V} k_{i,j}^{V-x}(G-x, a_{ij}), \quad (4)$$

where  $n = |V|$ . Suppose now that  $G, G' \in \mathbf{S}_{\mathfrak{s},V}^\ell$  are isomorphic only as labeled hypergraphs. From Remark 4 there is a unique isomorphism, determined by the labelings. This implies that  $G$  and  $G'$  will have the same counting (4). In particular,  $k_{i,j}^V(G', a_{ij}) = k_{i,j}^V(G, a_{ij})$ , allowing us to conclude that  $\epsilon'_{i,j} = \epsilon_{i,j}$  for every  $i, j$ , and therefore  $\epsilon' = \epsilon$ . Consequently,  $G'$  and  $G$  are also isomorphic as  $\mathfrak{s}$ -structured hypergraphs.

Returning to the proof, from Proposition 5 we only need to prove RC for labeled hypergraphs, i.e. that  $\Delta D_V^\ell : \mathbf{Hyp}_V^\ell \rightarrow \square \mathbf{Hyp}_{V-x}^\ell$  is essentially injective for every  $V$ . So, let  $(G, \varphi)$  and  $(G', \varphi')$  be two labeled hypergraphs and suppose that there exists  $f_x : (G - x, \varphi_x) \simeq (G' - x, \varphi'_x)$  for every  $x$ . Since we are working with isomorphic labeled hypergraphs defined over the same vertex set, Remark 4 allows us to assume that the isomorphism on the vertices is given by the identity map, i.e.  $(f_x)_1 = id$  for every  $x$ . Furthermore, the bijection  $(f_x)_j : (E_j)_x \rightarrow (E'_j)_x$  on the  $j$ -edges must be given by  $(\varphi_j)_x^{-1} \circ (\varphi'_j)_x^{-1}$ , so that it can be clearly extended to a bijection between  $E_j$  and  $E'_j$  preserving the labelings, which gives  $G \simeq G'$ .  $\square$

**Corollary 2** *If  $C \subset S_s^\ell$  is concretely proper and objectwise essentially injective, then both conjectures RC-C and dRC-C holds.*

*Proof* It follows directly from Proposition 5.  $\square$

#### 4 Contexts for Functional Analysis

When working with functional analysis we are dealing with certain classes of spaces, each one with an associated “dual space”, and for which we know how to take tensor products. This leads us to define a *context for functional analysis* (or simply a *context*) as a monoidal category  $(\mathbf{A}, \otimes, 1)$  endowed with a functor  $\cdot^\vee : \mathbf{A}^{op} \rightarrow \mathbf{A}$ , assigning to each object  $U \in \mathbf{A}$  its dual  $U^\vee$ , which are compatible in the sense that we have a *compatibility transformation*  $\cdot^\vee \otimes \cdot^\vee \Rightarrow (\cdot \otimes \cdot)^\vee$  and a distinguished morphism  $i : 1 \rightarrow 1^\vee$ .

We define a *morphism* between two contexts  $(\mathbf{A}, \otimes, \cdot^\vee)$  and  $(\mathbf{B}, \otimes, \cdot^*)$  as a functor  $F : \mathbf{A} \rightarrow \mathbf{B}$  which weakly preserves tensor products and duals. In other words, it is an oplax monoidal functor<sup>3</sup> together with a transformation  $F(\cdot^\vee) \Rightarrow F(\cdot)^*$  such that the diagram below commutes. We then have the category  $\mathbf{Cnxt}$  of contexts and morphisms between them.

$$\begin{array}{ccccc}
 F(U^\vee \otimes V^\vee) & \xrightarrow{\quad\quad\quad} & F((U \otimes V)^\vee) & & \\
 \downarrow & & \downarrow & & \\
 F(U^\vee) \otimes F(V^\vee) & & F(U \otimes V)^* & & F(1_\otimes) \xrightarrow{\quad\quad\quad} F(1_{\otimes}^\vee) \\
 \downarrow & & \downarrow & & \uparrow & \uparrow \\
 F(U)^* \otimes F(V)^* & \xrightarrow{\quad\quad\quad} & (F(U) \otimes F(V))^* & & 1_\otimes \xrightarrow{\quad\quad\quad} 1_{\otimes}^*
 \end{array}$$

*Remark 12* In some cases, the compatibility transformation  $\cdot^\vee \otimes \cdot^\vee \Rightarrow (\cdot \otimes \cdot)^\vee$  and the distinguished map  $1 \rightarrow 1^\vee$  are isomorphisms. In such cases we say that we have *strong contexts*. With the same notion of morphisms they define a full subcategory  $\mathbf{SCnxt} \subset \mathbf{Cnxt}$ .

<sup>3</sup> Recall our convention that in an oplax monoidal functor the arrow between the neutral objects remains in the correct direction.



*Example 13 (linear algebra)* Given a field  $\mathbb{K}$ , the category  $\mathbf{Vec}_{\mathbb{K}}$  of  $\mathbb{K}$ -vector spaces defines a context for functional analysis when endowed with the tensor product monoidal structure  $(\otimes_{\mathbb{K}}, \mathbb{K})$ , with  $U^{\vee}$  being the linear dual  $U^*$ . This is *not* a strong context, since for arbitrary vector spaces the canonical transformation  $U^* \otimes_{\mathbb{K}} V^* \rightarrow (U \otimes_{\mathbb{K}} V)^*$  is not an isomorphism. On the other hand, the full subcategory  $\mathbf{FinVec}_{\mathbb{K}}$  of finite-dimensional  $\mathbb{K}$ -vector spaces is a strong context. More generally, the transformation  $U^* \otimes_{\mathbb{K}} V^* \rightarrow (U \otimes_{\mathbb{K}} V)^*$  exists but may not be an isomorphism for arbitrary  $R$ -modules; but they are if we consider finitely generated projective  $R$ -modules [28]. So, with analogous structure,  $\mathbf{Mod}_R$  is a context and the category of finitely generated projective  $R$ -modules is a strong context.

*Example 14 (locally convex)* Now we can take  $\mathbf{A}$  as some category of topological real vector spaces and continuous linear maps and think of defining  $U \otimes V$  and  $U^{\vee}$  as  $U \otimes_{\mathbb{R}} V$  and  $B(U; \mathbb{R}) \subset U^*$ , respectively, endowed with some topology<sup>4</sup>. There are many possible choices of topology, of course. For locally convex spaces (lcs), there are at least three canonical ways to topologize  $U \otimes V$ : the projective, the injective and the inductive topologies. The choice of each of them produce a symmetric monoidal category  $(\mathbf{LCS}, \otimes, \mathbb{R})$  [29]. Also, there are many topologies in  $B(U; \mathbb{R})$ , such as the topology of pointwise convergence and uniform convergence in bounded sets. With each of them, we get a pseudocontext structure in  $\mathbf{LCS}$ .

*Example 15 (nuclear Fréchet)* When the spaces are nuclear, the injective and the projective topologies coincide [29, 30, 20], so that we have a more canonical pseudocontext structure. In general, none of the topologies in  $U^{\vee} = B(U; \mathbb{R})$  will induce isomorphisms  $\cdot^{\vee} \otimes \cdot^{\vee} \simeq (\cdot \otimes \cdot)^{\vee}$ , but they exist if we restrict to the full subcategory  $\mathbf{NucFrec}$  of nuclear Fréchet spaces, showing that  $(\mathbf{Frec}, \otimes_p, \cdot^{\vee})$  is a strong context [20, 30]. Also, for Fréchet spaces the projective and the inductive topologies coincide [31], meaning that in  $\mathbf{NucFrec}$  we have a canonical symmetric monoidal structure.

*Example 16 (subcontext)* Let  $(\mathbf{A}, \otimes, \cdot^{\vee})$  be a context. We define a *subcontext* as a full subcategory  $\mathbf{C} \subset \mathbf{A}$  such that  $1 \in \mathbf{C}$  and which is closed under  $\otimes$  and  $\cdot^{\vee}$ , meaning that  $U \otimes V$  and  $U^{\vee}$  belongs to  $\mathbf{C}$  when  $U, V \in \mathbf{C}$ . It then follows that  $(\mathbf{C}, \otimes, \cdot^{\vee})$  is a context and that the inclusion functor  $\iota : \mathbf{C} \hookrightarrow \mathbf{A}$  is a morphism of contexts. Furthermore, if  $\mathbf{A}$  is strong, then  $\mathbf{C}$  is also (the reciprocal is false as the last example shows). In particular, the full subcategories  $\mathbf{Ban}$  of Banach spaces and  $\mathbf{Hilb}$  of Hilbert spaces are subcontexts of  $\mathbf{Frec}$  and therefore define themselves strong contexts for functional analysis.

*Example 17 (categories with duals)* There are many flavors of monoidal categories whose objects or morphisms have duals (in the monoidal sense), e.g, autonomous categories, pivotal categories, spherical categories, spacial categories, compact closed categories and dagger monoidal categories (see [32] for a survey). All of them define a version of strong contexts whose  $\cdot^{\vee}$  is actually an ana-functor. Via Tannaka duality, they can be characterized as the representation category of certain monoid objects [33, 34].

We can consider monoidal categories  $\mathbf{A}$  endowed with a functor  $\cdot^{\vee} : \mathbf{A}^{op} \rightarrow \mathbf{A}$  without any compatibility condition. In this case we will say that we have a *pseudocontext*. A *morphism* between two of them is just a lax monoidal functor  $F : \mathbf{A} \rightarrow \mathbf{B}$

<sup>4</sup> Here,  $B(U; V)$  denotes the space of bounded linear maps.

together with a transformation  $F(\cdot^\vee) \Rightarrow F(\cdot)^*$ , giving us a category  $\mathbf{PCntx}$  which contains  $\mathbf{Cntx}$ .

#### 4.1 Analytic Expressions

From now on we will assume that the pseudocontexts  $(\mathbf{A}, \otimes, \cdot^\vee)$  are such that:

1. the category  $\mathbf{A}$  have countable limits, coproducts and cokernels;
2. the tensor product  $\otimes$  and the functor of duals  $\cdot^\vee$  preserve limits.

For physical interpretation we will also require that they become endowed with an additional functor  $K : \mathbf{A} \rightarrow \mathbf{Set}$  and for every  $k \in \mathbb{Z}_{\geq 0}$  a transformation  $S_k \times K(U^{\otimes k}) \rightarrow K(U^{\otimes k})$  which is objectwise a group action<sup>5</sup>. We require that the quotient  $K(U^{\otimes k})/S_k$  belongs to the image of  $K$  and have a single pre-image, so that it uniquely defines an object in  $\mathbf{A}$ . The obvious notation for this object should be  $\text{Sym}^k(U)$ , but here we will use  $P^k U$  to denote it. For  $k > 1$  we will say that this is the *object of  $k$ -propagators* in  $U$  (2-propagators are called propagators for short).

Given  $X \in \mathbf{A}$  we define an *object of formal power series* as the countable product of copies of  $X$ , i.e, as the product  $\prod_i X_i$ , with  $X_i \simeq X$ . We usually use the powers of a formal parameter  $h$  to indicate the order of the products, writing  $\prod_i X_i h^i$  or  $X[[h]]$  for short. We can consider power series not only with a single parameter  $h$ , but with a family  $\mathfrak{h} = (h_n)_{n \in \mathbb{Z}_{\geq 0}}$  of them. These will be given by  $X[[\mathfrak{h}]] = \prod_n \prod_i X_i^n h_n^i$  with  $X_i^n \simeq X$ . In physical contexts  $\mathfrak{h}$  will represent the family of fundamental parameters over which we will do perturbation theory.

Write  $PU = \prod_k P^k U$  and  $\mathcal{O}U = PU^\vee$ . Given a family of parameters  $\mathfrak{h}$  we define an *interacting term* in  $U$  relative to  $\mathfrak{h}$  as a morphism  $I : 1 \rightarrow \mathcal{O}U[[\mathfrak{h}]]$ . A *morphism* of interacting terms is a morphism in the under category  $1/\mathbf{A}$ , giving us a full subcategory  $\mathbf{Int}_{U, \mathfrak{h}}$  of  $1/\mathbf{A}$ .

Feynman rules will take structured hypergraphs and assigns to each of them interacting terms and propagators which together will fit into analytic expressions. Let  $A^k U$  denote  $(U^\vee)^{\otimes k} \otimes U^{\otimes k}$  and let  $AU = \prod_k A^k U$  (since the tensor product  $\otimes$  preserves countable products, if  $\mathbf{A}$  is a *strong context*, we can also write  $AU = PU \otimes PU^\vee$ ). An *analytic expression* in  $U$  is a morphism  $a : 1 \rightarrow AU$ . A *morphism* is a morphism in  $1/\mathbf{A}$ , so that we have a full subcategory  $\mathbf{A}_U$ . We say that an analytic expression has *order  $k$*  if it takes values in  $A^k U$  instead of  $AU$ , defining a category  $\mathbf{A}_U^k$ . We have  $\mathbf{A}_U \simeq \prod_k \mathbf{A}_U^k$ .

**Proposition 6** *For every  $U \in \mathbf{A}$ , the category  $\mathbf{A}_U$  acquires a canonical monoidal structure, induced from the monoidal structure in  $\mathbf{A}$ .*

*Proof* Recall that if  $(\mathbf{C}, \otimes, 1)$  is any monoidal category and then for any comonoid object  $X \in \mathbf{C}$ , say with coproduct  $\mu : X \rightarrow X \otimes X$  and counit  $\eta : X \rightarrow 1$ , then the under category  $X/\mathbf{C}$  can be endowed with a monoidal product  $X/\otimes$ , defined by tensoring with  $\otimes$  and precomposing with  $\mu$ , whose neutral object is  $\eta$ . Particularly, for  $X = 1$  we get that  $1/\mathbf{C}$  is monoidal. Let  $(\mathbf{A}, \otimes, \cdot^\vee)$  be a context. Let  $\mathbf{A}U \subset \mathbf{A}$  be the full subcategory whose single object is  $AU$ . Since  $\otimes$  and  $\cdot^\vee$  preserve countable products, we see that  $AU$  is actually a monoid object, so that  $\mathbf{A}U$  is a monoidal

<sup>5</sup> Here  $S_k$  is the permutation group and  $U^{\otimes k} = ((U \otimes U) \otimes U) \dots$ ,  $k$  times.

subcategory. On the other hand,  $\mathbf{A}_U \simeq 1/\mathbf{A}U$ , from where we get the desired monoidal structure on analytic expressions.  $\square$

**Proposition 7** *Every pseudocontext morphism  $F : \mathbf{A} \rightarrow \mathbf{B}$  induces, for each  $U \in \mathbf{A}$ , a functor  $F_U : \mathbf{A}_U \rightarrow \mathbf{B}_{F(U)}$  between the corresponding categories of analytic expressions.*

*Proof* Given an analytic expression  $a : 1 \rightarrow AU$  in  $\mathbf{A}$ , define  $F_U(a)$  as the following composition.

$$1_{\otimes} \longrightarrow F(1) \xrightarrow{F(a)} F(AU) \xrightarrow{u} \prod_k F(A^k U) \longrightarrow \prod_k B^k F(U) = BF(U)$$

The first and the last arrows appear because  $F$  is oplax monoidal and because a pseudocontext morphism becomes endowed with a transformation  $F(\cdot^\vee) \rightarrow F(\cdot)^*$ . Furthermore,  $u$  is from the universality of products in  $\mathbf{B}$ . From the definition of  $F_U(a)$  it immediately follows that  $F_U$  is functorial.  $\square$

*Remark 13* Unless the pseudocontext morphism  $F : \mathbf{A} \rightarrow \mathbf{B}$  is a bilax monoidal functor (instead of only oplax monoidal), the induced functor  $F_U : \mathbf{A}_U \rightarrow \mathbf{B}_{F(U)}$  generally will not be monoidal. Indeed, recall that the monoidal structure of  $\mathbf{A}_U$  is essentially the monoid object structure of  $AU$  in  $\mathbf{A}$ . Therefore, saying that  $F_U$  is monoidal we are saying that  $F$  maps the monoid  $AU$  into the monoid  $BF(U)$ , which is not a typical property of oplax functors, but which is clearly satisfied when  $F$  is bilax or strong monoidal.

*Remark 14* We could think of replacing the products  $\prod_k$  by coproducts  $\coprod_k$  in the above definitions and constructions. This would make life easier. But, in order to do this, we should assume that  $\mathbf{A}$  has countable colimits (instead of countable limits) and that  $\otimes$  and  $\cdot^\vee$  preserve colimits (instead of limits). For our purposes, these assumptions are restrictive: if  $\otimes$  preserves colimits, then it has right adjoint, meaning that  $(\mathbf{A}, \otimes, 1)$  is a closed monoidal category and automatically excluding  $(\mathbf{Frec}, \otimes_p, \cdot^\vee)$  as a possible context.

## 5 Feynman Functors

We can finally introduce and prove existence and uniqueness of functorial Feynman functors. These assign to each presheaf of hypergraphs an analytic expression in some pseudocontext. If the pseudocontext is sufficiently well behaved, then it will be able to evaluate these analytic expressions, giving us some kind of “amplitude of probability” assigned to each hypergraph.

Given a prestack  $C : \mathbb{X}^{op} \rightarrow \mathbf{Cat}$  of hypergraphs (generally regarded as an applicable prestack), a pseudocontext  $(\mathbf{A}, \otimes, \cdot^\vee)$  and a functor  $\tau : \mathbf{FinSet} \rightarrow \mathbf{A}$ , we define a *functorial Feynman functor* (or *Feynman functor*) for  $C$  with values in  $(\mathbf{A}, \tau)$ , denoted by  $Z : C \rightarrow (\mathbf{A}, \tau)$ , as a function that to each  $V \in \mathbf{FinSet}$  assigns a functor  $Z_V : C_V \rightarrow 1/\mathbf{A}$  such that for every  $G \in C_V$  we have  $Z_V(G) \in \mathbf{A}_{\tau(VG)}$ , where  $VG$  is the vertex set of  $G$ . We say that a Feynman functor is *complete* when each  $Z_V$  is essentially injective, meaning that the structured hypergraphs can be totally described by their associated analytic expressions.

We are also interested in *monoidal Feynman functors*. In order to define them we need to work with  $C$  proper, which means that it becomes endowed with a transformation  $\xi_{V,V'} : C_V \times C_{V'} \rightarrow C_{V \sqcup V'}$ , as discussed in Remark 11. From it we obtain the following transformation, which takes into account three finite sets instead of only two. Notice that  $\xi_{(V,V'),V''} \simeq \xi_{V,(V',V'')}$ , where the isomorphism means that any three hypergraphs have isomorphic image under these maps.

$$\begin{array}{ccc} (C_V \times C_{V'}) \times C_{V''} & \xrightarrow{\xi_{V,V'} \times id_{V''}} & C_{V \sqcup V'} \times C_{V''} \\ & \searrow \xi_{(V,V'),V''} & \downarrow \xi_{(V \sqcup V'),V''} \\ & & C_{(V \sqcup V') \sqcup V''} \end{array} \quad (5)$$

We say that  $Z$  is *oplax monoidal* if for every  $V, V' \in \mathbf{FinSet}$ , every  $G \in C_V$  and every  $G' \in C_{V'}$  we have morphisms

$$\mu_{V,V'} : Z_{V \sqcup V'}(\xi_{V,V'}(G, G')) \rightarrow Z_V(G) \otimes Z_{V'}(G') \quad \text{and} \quad \nu : Z_{\emptyset}(\emptyset) \rightarrow id_1 \quad (6)$$

in  $1/\mathbf{A}$  satisfying the usual comonoid-like diagrams (e.g, the associativity diagram is that presented below<sup>6</sup>). Similarly, we define the situations when  $Z$  is *lax monoidal* and *strong monoidal* (or simply *monoidal*) by reverting the arrow or by requiring that they are isomorphisms, respectively.

$$\begin{array}{ccc} Z_{V,(V',V'')}(\xi_{V,(V',V'')}(G, G', G'')) & \xrightarrow{\simeq} & Z_{(V,V'),V''}(\xi_{(V,V'),V''}(G, G', G'')) \\ \mu_{V,(V',V'')} \downarrow & & \downarrow \mu_{(V,V'),V''} \\ Z_V(G) \otimes Z_{V',V''}(\xi_{V',V''}(G', G'')) & & Z_{V,V'}(\xi_{V,V'}(G, G')) \otimes Z_{V''}(G'') \\ id \times \mu_{V',V''} \downarrow & & \downarrow \mu_{V,V'} \times id \\ Z_V(G) \otimes (Z_{V'}(G') \otimes Z_{V''}(G'')) & \xrightarrow{\simeq} & (Z_V(G) \otimes Z_{V'}(G')) \otimes Z_{V''}(G'') \end{array} \quad (7)$$

*Remark 15* When  $C$  is concretely proper,  $C_V \subset \mathbf{Hyp}_V$  for every  $V$ , so that

$$Z_{V \sqcup V'}(\xi(G, G')) \in \mathbf{A}_{\tau(V \sqcup V')} \quad \text{and} \quad Z_V(G) \otimes Z_{V'}(G') \in \mathbf{A}_{\tau V} \otimes \mathbf{A}_{\tau V'}.$$

Therefore, in order to formalize the notion of lax (resp. oplax) monoidal Feynman functors, instead of requiring the morphisms (6) to belong to  $1/\mathbf{A}$ , in these cases we could require that  $\tau : (\mathbf{FinSet}, \sqcup, \emptyset) \rightarrow (\mathbf{A}, \otimes, 1)$  is lax (resp. oplax) monoidal. However, this would produce a much more rigid concept, e.g,  $\tau$  could not be chosen as the constant functor in some  $U \in \mathbf{A}$ , with  $U \neq 1$ . And, as we will see, it is precisely when  $\tau = \text{cst}$  that the explicit connection with perturbative quantum field theory is made.

*Example 18 (trivial Feynman functor)* For any  $C$  and any  $(\mathbf{A}, \tau)$  there always exists a trivial monoidal Feynman functor  $\text{cst}_1 : C \rightarrow (\mathbf{A}, \tau)$  assigning to each  $V$  the functor  $\text{cst}_1 : C_V \rightarrow 1/\mathbf{A}$  constant in the neutral object  $id_1$  of the monoidal category  $1/\mathbf{A}$ . In the next section we will show that nontrivial Feynman functors also exist.

<sup>6</sup> In order to simplify the notation we wrote  $Z_{V,V'}$  instead of  $Z_{V \sqcup V'}$ . Furthermore, notice that it makes sense due to the commutativity of (5).

*Example 19 (Feynman subfunctor)* Suppose given a Feynman functor  $Z : C \rightarrow (\mathbf{A}, \tau)$ . Precomposition defines a Feynman functor  $Z' : C' \rightarrow (\mathbf{A}, \tau)$  for any subfunctor  $C' \subset C$ , and:

1. if  $Z$  is lax monoidal and  $C' \subset C$  is proper, then  $Z'$  is lax monoidal. The same holds for oplax or strong monoidal;
2. if  $Z$  is complete and  $C' \subset C$  is essentially injective, then  $Z'$  is complete.

Our aim in this section is to show that when fixed a decomposition system in a prestack  $C$ , we can obtain Feynman functors  $F : C \rightarrow (\mathbf{A}, \tau)$  constructively by means of following certain rules, called *Feynman rules*. So, let us begin by defining what we mean by a decomposition system.

Let  $\mathbf{C} \subset \mathbf{Hyp}$  be a category of hypergraphs. A *decomposition system of order  $n$*  in  $\mathbf{C}$  is a rule  $\mathfrak{d}$  assigning to each  $G \in \mathbf{C}$  decompositions  $EG_j = EG_j^0 \sqcup EG_j^1 \sqcup \dots \sqcup EG_j^n$ , for each  $j \geq 1$ . In other words, we have a decomposition of the vertex set  $VG = EG_1$  and of each set  $EG_j$  of  $j$ -edges into  $n$  parts. We do not require conditions on the adjacency functions  $\psi_j$ . We say that  $v^0 \in EG_1^0$  and  $e_j^0 \in EG_{j>1}^0$  are the *external vertices* and *external  $j$ -edges*, respectively. If  $i > 0$ , the elements  $v^i$  of  $EG_1^i$  and  $e_j^i$  of  $EG_j^i$  are called *internal vertices* and *internal  $j$ -edges of type  $i$* .

We say that  $\mathfrak{d}$  is *functorial* if each hypergraph morphism  $f : G \rightarrow G'$  preserve the decomposition. It is straightforward to check that the choice of a functorial decomposition system induces an embedding  $C_V \hookrightarrow \mathbf{S}_s$  for certain  $s$ . We define a functorial decomposition system  $\mathfrak{d}$  for a prestack  $C$  as a rule that to any  $V \subset X$  it assigns a functorial decomposition system  $\mathfrak{d}_V$  for  $C_V$ .

*Example 20 (trivial)* Each  $\mathbf{C} \subset \mathbf{Hyp}$  possesses a trivial functorial decomposition system of arbitrary order: just define  $VG^0 = VG$ ,  $EG_{j>1}^0 = EG_{j>1}$  and the remaining pieces given by the empty set, i.e.,  $EG_j^i = \emptyset$  if  $i > 0$ .

*Example 21 (incidence)* On the other hand, each  $\mathbf{C}$  also possesses a nontrivial functorial decomposition of order 2. In fact, recall that to each hypergraph  $G$  we assign a bipartite graph  $IG$ : its *incidence graph*, whose set of vertices is  $V_{IG} = VG \sqcup EG$ , where  $EG = \sqcup_j EG_j$ . There exists a 2-edge between  $x, x' \in V_{IG}$  iff  $x \in VG$  and if  $x' \in EG$  is some hyperedge adjacent to  $x$ . The rule  $G \mapsto IG$  is functorial and actually an equivalence  $\mathbf{Hyp} \simeq \mathbf{BipGrph}$  between hypergraphs and bipartite graphs [19]. In particular, each  $\mathbf{C} \subset \mathbf{Hyp}$  is equivalent to some category of bipartite graphs. But bipartite graphs have, by definition, a decomposition of order 2. So, any  $\mathbf{C}$  has an induced decomposition, which we denote by  $\mathfrak{d}_i$ . Furthermore, each prestack  $C$  can be endowed with this decomposition. Particularly, each  $C$  can be embedded into a structured prestack  $S_s$  for certain  $s$ .

*Example 22 (Feynman)* For any  $V$ , the category  $\mathbf{Fyn}_V$  (and, therefore the prestack) of Feynman graphs have a canonical functorial decomposition system of order 2, as explained in Example 3. Analogously for Feynman hypergraphs and generalized Feynman hypergraphs introduced in Example 4.

*Example 23 (structured)* Any prestack  $S_s$  of additively structured hypergraphs (e.g, those which are finitely structured - see Remark 3) can be endowed with a nontrivial functorial decomposition system of order 2. Indeed, define  $VG^0$  as the subset of  $VG$  consisting of all vertices  $v$  such that

- s1)  $d_2(v) = 1$  and  $d_{j>2}(v) = 0$ ;

- s2) for every  $k$ ,  $\mathfrak{s}_{k,1}(v) = 0$ ;  
s3) if exists  $j$ -edge between  $v$  and  $v'$ , then  $v'$  is not in  $VG^0$ .

In other words,  $VG^0$  is the collection of vertices that have trivial structure and that belong to a single 2-edge. Then take  $VG^1 = VG - VG^0$ . Conditions s1) and s2) establish a subset  $EG_2^0 \subset EG_2$  and conditions s2) and s3) give us  $|EG_2^0| = |VG^0|$ . We then have a decomposition  $EG_2 = EG_2^0 \sqcup EG_2^1$ . Finally, for  $j > 2$  define  $EG_j^1 = EG_j$  and  $EG_j^{i \neq 1} = \emptyset$ . We will denote this decomposition system by  $\mathfrak{d}_s$ .

*Remark 16* For future reference, we define a *standard decomposition*  $\mathfrak{d}$  for a prestack as being one codifying the fundamental property of Examples 22 and 23

- st1) the number of external vertices is equal to the number of external edges. In other words,  $|VG^0| = \sum_j |EG_j^0|$ ;  
st2) each  $j$ -edge in  $EG_j^0$  is adjacent to at least one vertex in  $VG^0$ .

*Remark 17* In our construction we will need to work with decompositions such that  $|EG_j^0| \leq 1$ , which cannot be satisfied by standard decompositions of order 2 with  $|VG^0| > 1$ . This leads us to consider decompositions such that  $|EG_j^0| \leq 1$  and  $|VG^0| \leq 1$ . They will be called *normal*. If  $\mathfrak{d}$  is a functorial normal decomposition of a prestack  $C$  we say that the pair  $(C, \mathfrak{d})$  is *normal* and that  $\mathfrak{d}$  is a *normal structure* for  $C$ . Starting with any pair  $(C, \mathfrak{d})$ , where  $\mathfrak{d}$  is of order  $n$ , we can always build a normal pair  $(\mathcal{N}C, \mathcal{N}\mathfrak{d})$ , with  $\mathcal{N}\mathfrak{d}$  also of order  $n$ , called the *normalization* of  $(C, \mathfrak{d})$  and defined as follows:  $\mathcal{N}C_V$  is the category of hypergraphs  $\mathcal{N}_\mathfrak{d}G$  obtained from hypergraphs  $G \in C_V$  by deleting  $VG^0$  and  $EG_j^0$ , and consider the decomposition  $\mathcal{N}\mathfrak{d}_V$  given by

$$V\mathcal{N}_\mathfrak{d}G^0 = \emptyset = E\mathcal{N}_\mathfrak{d}G_j^0, \quad V\mathcal{N}_\mathfrak{d}G^{i>0} = VG^{i>0}, \quad \text{and} \quad E\mathcal{N}_\mathfrak{d}G_j^{i>0} = EG_j^{i>0}.$$

Furthermore, any prestack  $C$  admits a normal structure  $\mathfrak{d}_0$  of order 2: just take  $VG^0 = \varphi = EG_j^0$  and  $VG^1 = VG$  and  $EG_j^1 = EG_j$ .

*Remark 18* From Example 21 any prestack  $C$  admits a decomposition  $\mathfrak{d}_i$ , implying  $C \subset S_s$ . By Example 23  $C$  can then be endowed with a new decomposition  $\mathfrak{d}_s$ , which is standard by the last remark. In other words, each  $C$  admits a nontrivial standard decomposition.

Let  $(C, \mathfrak{d})$  be a prestack of hypergraphs endowed with a (non necessarily functorial) decomposition system  $\mathfrak{d}$ , say of order  $n$ , let  $(\mathbf{A}, \otimes, \cdot^\vee)$  be a pseudocontext and let  $\tau : \mathbf{FinSet} \rightarrow \mathbf{A}$  be a functor. A *Feynman rule* in  $(C, \mathfrak{d})$  with coefficients in  $(\mathbf{A}, \tau)$  is a rule  $FR$  that to each  $V \in \mathbf{FinSet}$  and each hypergraph  $G \in C_V$  it assigns:

1. functions  $r^i : VG^i \rightarrow \mathbb{Z}_{\geq 0}$  and  $r_j^i : EG_j^i \rightarrow \mathbb{Z}_{\geq 0}$ , with  $0 \leq i \leq n$  and  $j \geq 1$ , called the *degrees* or *weights* of the decomposition  $\mathfrak{d}$ ;
2. functors assigning to each external vertice, etc., a tensor in  $\tau VG \in \mathbf{A}$  or  $(\tau VG)^\vee \in \mathbf{A}$  of corresponding degree. Precisely, functors

$$FR^i : VG^i \rightarrow 1/(\tau(VG)^\vee)^{\otimes_{r^i}} \quad \text{and} \quad FR_j^i : EG_j^i \rightarrow 1/(\tau VG)^{\otimes_{r_j^i}},$$

regarding  $VG^i$  and  $EG_j^i$  as discrete categories, while

$$1/(\tau(VG)^\vee)^{\otimes_{r^i}} \quad \text{and} \quad 1/(\tau VG)^{\otimes_{r_j^i}}$$

denotes the full subcategories of  $1/\mathbf{A}$  consisting of all morphisms

$$1 \rightarrow 1/(\tau(VG)^\vee)^{\otimes r^i(v^i)} \quad \text{and} \quad 1/(\tau(VG))^{\otimes r_j^i(e_j^i)},$$

for every  $v^i \in VG^i$  and  $e_j^i \in EG_j^i$ , respectively. Recall that  $1/U^0$  is the category with a single object  $id_1$ .

*Example 24 (classic Feynman rules)* Let  $C$  be any prestack endowed with a functorial decomposition  $\mathfrak{d}$ . Let  $(\mathbf{A}, \otimes, \cdot^\vee)$  be any pseudocontext endowed with any functor  $\tau : \mathbf{FinSet} \rightarrow \mathbf{A}$ . An usual setup for Feynman rules  $FR : (C, \mathfrak{d}) \rightarrow (\mathbf{A}, \tau)$  is to take for each  $G \in C_V$  the degree functions as  $r^0 \equiv 0$ ,  $r_j^0 \equiv 1$ ,  $r_j^{i \geq 0} \equiv j$  and  $r^{i \geq 0}(v) = d(v) = \sum_j d_j(v)$ , so that

$$FR^0 : VG^0 \rightarrow 1/(\tau(VG)^\vee)^0$$

must be the constant functor in  $id_1$  (i.e, it must assign the trivial tensor in  $\tau(VG)$  to each external vertice). Furthermore, to each hypergraph  $G$  consider two formal parameters  $t, o$  such that if  $G$  is  $b$ -bounded and  $\mathfrak{d}$  has order  $n$ , then  $t^j = 0$  for  $j > b$  and  $o^i = 0$  for  $i > n$ . Take the object  $\tau(VG)[[t, o]]$  of formal power series in  $t$ , which decomposes as a product  $\prod_{j,i} X_{i,j} o^i t^j$ , with  $X_{i,j} \simeq \tau(VG)$ . Then define  $FR_j^0$  as any rule assigning a generalized element  $FR_j^0(e)$  of degree  $(0, j)$  in  $\tau(VG)$ , i.e, as a morphism  $1 \rightarrow \tau(VG)[[t, o]]$  which actually take values in  $X_{0,j}$ . Moreover, define  $FR_j^i$  as any map assigning  $j$ -propagators in  $\tau(VG)$  to internal  $j$ -edges. More precisely, recall that  $P\tau(VG)[[t, o]] = \prod_{k,i,j} P_{i,j}^k o^i t^j$ , with  $P_{i,j}^k \simeq \text{Sym}^k(\tau(VG))$ , and take  $FR_j^i(e) \in 1/P_{i,j}^j$ . Finally, take  $FR^{i \geq 0}$  as any rule assigning to each internal vertex  $v^i$  a corresponding interacting term in  $\tau(VG)^\vee$ , relative to  $t$ , of degree  $d(v)$ . In other words, for each  $v$  we have a morphism  $FR^i(v) : 1 \rightarrow \mathcal{O}\tau(VG)[[t]]$ , where  $\mathcal{O}\tau(VG)[[t, o]] = \prod_{k,i,j} I_{i,j}^k o^i t^j$ , with  $I_{i,j}^k \simeq \text{Sym}^k(\tau(VG)^\vee)$ , which actually take values in  $I_{i,1}^{d(v)}$ .

*Example 25 (structured Feynman rules)* We can particularize and consider  $C$  as a prestack  $S_{\mathfrak{s}}$  of  $\mathfrak{s}$ -structured hypergraphs. Furthermore, we can take  $\tau : \mathbf{FinSet} \rightarrow \mathbf{A}$  as constant in some object  $U$ , called *background object*, so that all data assigned by  $FR$  are tensors on  $U$ . We can also use  $\mathfrak{s}$  to require more properties on  $FR^{i \geq 0}$ , as follows. To each  $s_{k,j}$  we consider a new formal parameter  $h_{k,j}$ , so that

$$\mathcal{O}\tau(VG)[[h]] = \prod_{k,i,j,l_{1,1},l_{1,2},\dots} I_{i,j,l_{1,1},l_{2,1},\dots}^k t^j o^i h_{1,1}^{l_{1,1}} h_{1,2}^{l_{1,2}} \dots$$

Then, for a given  $(G, \epsilon)$  a  $\mathfrak{s}$ -structure graph, take  $FR^i(v)$  as a generalized element in  $I_{i,1,\epsilon_{1,1}(v),\epsilon_{2,1}(v),\dots}^{d(v)}$ .

*Example 26 (physicists Feynman rules)* Let us now particularize even more and consider  $C$  as the concrete prestack  $C_V = \mathbf{Fyn}_V$  of Feynman graphs of Example 3 endowed with the functorial decomposition of Example 22, so that  $b = 2$  and  $n = 1$ . Also, we have two nontrivial structures  $s_{1,1} = \mathbb{Z}_2$  and  $s_{2,1} = \mathbb{Z}_{\geq 0}$ , corresponding to the decomposition and the genus map, so that we have two parameters  $h_{1,1}$  and  $h_{2,1}$ . In this particular setting, let us denote  $h_{2,1} \equiv \hbar$ . Furthermore, let us consider the pseudocontext of Example 14, allowing us to work with **LCS** instead

of with  $\mathbb{R}/\mathbf{LCS}$ , as explained in Example 27. Thus, according to the previous example, fixed a background object  $U$ , we get Feynman rules by giving functions  $FR_2^0 : EG^0 \rightarrow U$ ,  $FR_2^1 : EG^1 \rightarrow P^2U$  and  $FR^1 : VG^1 \rightarrow \mathcal{O}U[\hbar]$  such that  $FR^1(v) \in I_{g(v)}^{d(v)}$ . In turn, these functions are exactly the data defining the Feynman rule of a perturbative QFT as will be more detailed explored in Section 7.

Suppose now that the decomposition system  $\mathfrak{d}$  in  $C$  is functorial. In this case, for any Feynman rule  $FR$  in  $(C, \mathfrak{d})$ , given a morphism  $f : G \rightarrow G'$  we get the induced diagrams below (without the segmented arrow). We say  $FR$  is *functorial* when for each  $f$  there exists the dotted arrows  $r^i f$  and  $r_j^i f$  completing these diagrams (of functors between discrete categories) in a commutative way. As we will see, if we assume Axiom of Choice, any Feynman rule becomes functorial.

$$\begin{array}{ccc}
 VG^i & \xrightarrow{FR^i} & 1/(\tau VG^V)^{\otimes_{r^i}} \\
 \downarrow f^i & & \downarrow r^i f \\
 VG'^i & \xrightarrow{FR'^i} & 1/(\tau VG'^V)^{\otimes_{r'^i}}
 \end{array}
 \qquad
 \begin{array}{ccc}
 EG_j^i & \xrightarrow{FR_j^i} & 1/(\tau VG)^{\otimes_{r_j^i}} \\
 \downarrow f_j^i & & \downarrow r_j^i f \\
 EG_j'^i & \xrightarrow{FR_j'^i} & 1/(\tau VG')^{\otimes_{r_j'^i}}
 \end{array}
 \quad (8)$$

Let  $\mathbf{A}$  be a category with a distinguished object  $1 \in \mathbf{A}$ . Let  $\mathbb{D}$  be a collection of categories. We say that  $(\mathbf{A}, 1)$  has the *Hahn-Banach property* (or simply that it is Hahn-Banach) *relative to*  $\mathbb{D}$  when for any  $\mathbf{D}, \mathbf{D}' \in \mathbb{D}$ , any full subcategories  $\mathbf{C}, \mathbf{C}' \subset 1/\mathbf{A}$  and any functors  $F : \mathbf{D} \rightarrow \mathbf{D}'$ ,  $G : \mathbf{D}' \rightarrow \mathbf{C}'$  and  $H : \mathbf{D} \rightarrow \mathbf{C}$ , the corresponding  $e$  admit an extension in  $\mathbf{Cat}$  relatively to any functor  $H$ , as in diagram (9). There,  $e \circ m$  denotes the mono-epi decomposition of  $G \circ F$  obtained by taking its image in  $\mathbf{Cat}$ . We say that  $(\mathbf{A}, 1)$  is *Hahn-Banach relative to a functor*  $\sigma : \mathbf{FinSet} \rightarrow \mathbf{Cat}$  if it is Hahn-Banach relative to  $\mathbb{D} = \text{img}(\sigma)$ .

$$\begin{array}{ccc}
 \mathbf{D} & \xrightarrow{H} & \mathbf{C} \\
 \downarrow F & \searrow e & \downarrow \hat{e} \\
 & & \text{img}(G \circ F) \\
 \mathbf{D}' & \xrightarrow{G} & \mathbf{C}' \\
 & & \downarrow m
 \end{array}
 \quad (9)$$

*Example 27 (locally convex spaces)* Let  $\mathbf{LCS}$  be the category of lcs endowed with the real line  $\mathbb{R}$  as a distinguished object. We assert that it is Hahn-Banach relative to the subcategory  $\mathbf{FinVec}_{\mathbb{R}}$  of finite-dimensional vector spaces. With the inductive topology in  $\otimes$ ,  $(\mathbf{LCS}, \otimes, \mathbb{R})$  is closed monoidal, whose internal hom  $[U; U']$  between  $U$  and  $U'$  is given by  $B(U; U')$  endowed with the topology of pointwise convergence. Therefore,  $[\mathbb{R}; U] \simeq U$  and we can work directly with  $\mathbf{LCS}$  instead of  $\mathbb{R}/\mathbf{LCS}$ . Since a continuous linear map taking values in a finite-dimensional vector space is the same thing as a finite combination of linear functionals, the Hahn-Banach condition above is a direct consequence of Hahn-Banach theorem for lcs [30, 31]. The inclusion  $\mathbf{Vec}_{\mathbb{R}} \hookrightarrow \mathbf{Set}$  is reflective. Let  $\mathbb{R}[\cdot] : \mathbf{FinSet} \rightarrow \mathbf{FinVec}_{\mathbb{R}}$  be the



restriction of the the left adjoint to the category of finite sets and let  $\iota : \mathbf{FinVec}_{\mathbb{R}} \hookrightarrow \mathbf{LCS}$  be the inclusion. We can verify that  $\mathbf{LCS}$  is Hahn-Banach relative to  $\iota \circ \mathbb{R}[\cdot]$ .

*Example 28 (full subcategories)* If  $(\mathbf{A}, 1)$  is Hahn-Banach, then  $(\mathbf{C}, 1)$  is also for every full subcategory  $\mathbf{C} \subset \mathbf{A}$ . So, in particular, the pseudocontexts from Example 15 and Example 16 define Hahn-Banach pairs.

Now, the fundamental fact is that, as a consequence of the axiom of choice, any pair  $(\mathbf{A}, 1)$  is Hahn-Banach relative to the functor regarding finite sets as discrete categories. Indeed, since the domain category is discrete we can only work with objects, so that the extension problem (9) is equivalent to a problem in  $\mathbf{Set}$  which has a solution, since  $e$  is an epimorphism and we are assuming Axiom of Choice.

Thus, if  $FR : (C, \mathfrak{d}) \rightarrow (\mathbf{A}, \tau)$  is any Feynman rule, the following dotted arrows exist, so that  $FR$  is a functorial Feynman rule for  $r^i f = m^i \circ \hat{e}^i$  and  $r_j^i f = m_j^i \circ \hat{e}_j^i$ .

$$\begin{array}{ccc}
 VG^i & \xrightarrow{FR^i} & 1/(\tau(VG)^\vee)^{\otimes r^i} \\
 \downarrow f^i & \searrow e^i & \downarrow \hat{e}^i \\
 & & \text{img}(FR'^i \circ f^i) \\
 & & \downarrow m^i \\
 VG'^i & \xrightarrow{FR'^i} & 1/(\tau(VG')^\vee)^{\otimes r^i}
 \end{array}
 \qquad
 \begin{array}{ccc}
 EG_j^i & \xrightarrow{FR_j^i} & 1/(\tau(VG)^\vee)^{\otimes r_j^i} \\
 \downarrow f_j^i & \searrow e_j^i & \downarrow \hat{e}_j^i \\
 & & \text{img}(FR_j'^i \circ f_j^i) \\
 & & \downarrow m_j^i \\
 EG_j'^i & \xrightarrow{FR_j'^i} & 1/(\tau(VG')^\vee)^{\otimes r_j^i}
 \end{array}$$

### 5.1 Existence

We can now prove that functorial Feynman rules induce monoidal Feynman functors. Actually, we will need to work with Feynman rules  $FR : (C, \mathfrak{d}) \rightarrow (\mathbf{A}, \tau)$  which are *coherent* in the sense that the degrees  $r^i$  and  $r_j^i$  of  $\mathfrak{d}$  satisfy the following coherence equation:

$$\sum_{i \geq 0} \sum_{j > 2} \sum_{e_j^i} r_j^i(e_j^i) = \sum_{i \geq 0} \sum_{v_i} r^i(v_i). \quad (10)$$

*Example 29 (standard Feynman rules)* We say that a Feynman rule is *standard* if the decomposition system  $\mathfrak{d}$  is standard (in the sense of Remark 16) and if the degrees  $r^i$  and  $r_j^i$  are given by  $r^0 \equiv 1$ ,  $r^{i>0}(v^i) = d_i(v^i)$ ,  $r_j^0 \equiv 1$  and  $r_j^{i>0}(e_j^i) = j$ . For instance, the classic Feynman rules are standard. We assert if  $FR$  is standard then it is coherent. Due to the structure of the degrees, for any hypergraph  $G \in C_V$ , the left-hand side and the right-hand side (10) writes, respectively

$$\sum_{j > 2} |EG_j^0| + \sum_{i > 0} \sum_{j > 2} j |EG_j^i| \quad \text{and} \quad |VG^0| + \sum_{i > 0} \sum_{v_i} r^i(v_i).$$

Since  $\mathfrak{d}$  is standard, the first term of both sides coincide. Let  $G_{i>0}$  be the hypergraph obtained by deleting  $VG^0$ . The formula (1) applied to it shows that the second terms also coincide.

**Theorem 3** Let  $(\mathbf{A}, \otimes, \cdot^\vee)$  be a pseudocontext endowed with a functor  $\tau : \mathbf{FinSet} \rightarrow \mathbf{A}$  and let  $C^b$  be a prestack of  $b$ -bounded hypergraphs. Suppose one of the following conditions:

- c1)  $C^b \subset S^{\ell, b}$ , i.e., our hypergraphs are labeled;
- c2) the monoidal category  $(\mathbf{A}, \otimes, 1)$  is strict.

Then any coherent Feynman rule  $FR : (C^b, \mathfrak{d}) \rightarrow (\mathbf{A}, \tau)$  defines a Feynman functor  $Z : C^b \rightarrow (\mathbf{A}, \tau)$ .

*Proof* Assuming condition c1), let  $(G, \varphi)$  be a labeled  $b$ -bounded hypergraph in  $C_V^b \subset S^{\ell, b}$ . Let  $n$  be the order of  $\mathfrak{d}$ . The decompositions  $VG = \sqcup_i V^i G$  and  $E_{j>1} G = \sqcup_i E_j^i G$  with  $0 \leq i \leq n$  defined by  $\mathfrak{d}$ , induce decompositions for the bijections  $\varphi_j$ . Without loss of generality we can assume that now we have  $\varphi_1^i : [|V^i G|] \rightarrow V^i G$  and  $\varphi_{j>1}^i$ . Since morphisms of labeled hypergraphs preserve the labelings, they will preserve these decomposed labelings. With this in mind, if  $FR : (C^b, \mathfrak{d}) \rightarrow (\mathbf{A}, \tau)$  is a Feynman rule, for each set  $V \subset X$  and each labeled hypergraph  $(G, \varphi) \in C_V$ , define  $Z_V^i(G)$  as the following tensor product (in  $1/\mathbf{A}$ )

$$Z_V^i(G) = (\dots((FR^i(\varphi^i(1)) \otimes FR^i(\varphi^i(2))) \otimes \dots) \otimes FR^i(\varphi^i(|V^i G|))). \quad (11)$$

In analogous way, define  $Z_{j,V}^i(G)$  for every  $1 < j \leq b$ , i.e., by taking the tensor product in  $1/\mathbf{A}$  of the images of  $FR_j^i$ , following the order given by the labelings  $\varphi_j^i$ . Furthermore, define

$$Z_V^I(G) = (\dots((Z_V^0(G) \otimes Z_V^1(G)) \otimes Z_V^2(G)) \dots) \otimes Z_V^n(G) \quad (12)$$

and similarly  $Z_{j,V}^i(G)$  for each  $0 < i \leq n$ . Then take

$$Z_V(G) = (\dots((Z_V^I(G) \otimes Z_{j,V}^1(G)) \otimes Z_{j,V}^2(G)) \dots) \otimes Z_{j,V}^n(G). \quad (13)$$

From the coherence equation (10) we see that  $Z_V(G) \in \mathbf{A}_{\tau VG}$ . Therefore, we have a rule  $G \mapsto Z_V(G)$  assigning to each hypergraph with vertex set  $VG$  an analytic expression in  $\tau VG \in \mathbf{A}$ . Its functoriality follows from the functoriality of the Feynman rules  $FR$ . Indeed, if  $f : (G, \varphi) \rightarrow (G', \varphi')$  is a morphism in  $C_V^b$ , we have the morphisms  $r^i f$  and  $r_j^i$  in (8). Since  $f$  preserves the decomposed labelings,  $r^i f$  induce morphisms  $Z_V^i(f) : Z_V^i(G) \rightarrow Z_V^i(G')$  and analogously for  $r_j^i$ . Define  $Z_V^I(f)$  by replacing the symbol “ $G$ ” with the symbol “ $f$ ” in (12) and define  $Z_{j,V}^i(f)$  similarly. Finally, replacing “ $G$ ” with “ $f$ ” in (13) we get the desired morphism  $Z_V(f) : Z_V(G) \rightarrow Z_V(G')$ . It is straightforward to check that composition and identity morphisms are preserved, so that we have a Feynman functor  $Z : C \rightarrow (\mathbf{A}, \tau)$ . Finally, notice that condition c1) was used only to fix an ordering in the tensor products (11). Suppose now that c2) is satisfied. Then  $\mathbf{A}$  is strict, so that we can remove the parenthesis of tensor products, meaning that we do not need to care about ordering. So, the same construction works even without labelings.  $\square$

**Corollary 3 (of the proof)** A Feynman functor induced by a Feynman rule is never trivial unless the prestack  $C^b$  is trivial.

*Proof* Just look at the defining expressions (11), (12) and (13) and use the coherence equation.  $\square$

We say that two Feynman rules  $Z : C \rightarrow (\mathbf{A}, \tau)$  and  $Z' : C' \rightarrow (\mathbf{A}', \tau')$  differ by a change of coefficients if they are defined in the same prestack, i.e, if  $C' = C$ , and if there exists an isomorphism  $F : (\mathbf{A}, \otimes, \cdot^{\vee}) \rightarrow (\mathbf{A}', \otimes', \cdot^{\vee'})$  of pseudocontexts and a natural isomorphism  $\zeta : \tau \simeq \tau'$  such that for every hypergraph  $G \in C_V$  we have  $F_\zeta \circ Z = Z'$ , where  $F_\zeta : \mathbf{A}_{\tau VG} \simeq \mathbf{A}'_{\tau' VG}$  is the equivalence induced by  $F$  and  $\zeta$  (this equivalence exists, since by Proposition 7 morphisms of pseudocontexts induce functors between the categories of analytic expressions).

**Proposition 8** *Up to change of coefficients, there exists a nontrivial Feynman functor  $Z : C \rightarrow (\mathbf{A}, \tau)$  defined in any nontrivial proper prestack  $C$  (not necessarily bounded) and taking values in any pseudocontext  $(\mathbf{A}, \otimes, \cdot^{\vee})$  endowed with any functor  $\tau : \mathbf{FinSet} \rightarrow \mathbf{A}$ .*

*Proof* From Example 21 any  $C$  is subprestack of  $S_{\mathfrak{s}}^2$  for some  $\mathfrak{s}$ . By Example 23 and by Remark 16,  $S_{\mathfrak{s}}^2$  (and therefore  $C$ ) can be endowed with a standard decomposition system  $\mathfrak{d}$ , and by Example 24 we have a Feynman rule (the classical one)  $FR : (S_{\mathfrak{s}}^2, \mathfrak{d}) \rightarrow (\mathbf{A}, \tau)$  with coefficients in any  $(\mathbf{A}, \tau)$ , which induces a Feynman rule in  $(C, \mathfrak{d})$ . From Example 29 this Feynman rule is coherent, so that if  $\mathbf{A}$  is strict, then by Theorem 3  $FR$  induces a Feynman functor  $Z : C \rightarrow (\mathbf{A}, \tau)$ . But, since we are working up to change of coefficients, Mac Lane's strictification theorem for monoidal categories allows us to forget the strictness hypothesis on  $(\mathbf{A}, \otimes, 1)$ . Nontriviality of  $Z$  is due Corollary 3.  $\square$

It would be interesting to find conditions on Feynman rules under which the induced Feynman functors are monoidal and/or complete. We start by analyzing completeness. Let  $(\mathbf{C}, \otimes, 1)$  be a monoidal category and let  $F : \mathbf{I} \rightarrow \mathbf{C}$  be some functor. We say that  $F$  is *decomposable* if for any isomorphism  $f : I \rightarrow I'$  in  $\mathbf{I}$  and any decompositions  $F(I) \simeq X \otimes Y$  and  $F(I') \simeq X' \otimes Y'$  there are isomorphisms  $\alpha : X \simeq X'$  and  $\beta : Y \simeq Y'$  in  $\mathbf{C}$  such that  $F(f) = \alpha \otimes \beta$ , i.e, such that the diagram below commutes. If  $(\mathbf{C}, \otimes, \vee)$  is actually a pseudocontext, we require decompositions not only for  $F(I)$ , but also for each  $F(I)^{\otimes r}$  and  $(F(I)^{\vee})^{\otimes r}$ . We say that  $F$  is *1/decomposable* if the induced functor  $1/F : \mathbf{I} \rightarrow 1/\mathbf{C}$  is decomposable.

$$\begin{array}{ccc} F(I) & \xrightarrow{\simeq} & X \otimes Y \\ F(f) \downarrow & & \downarrow \alpha \otimes \beta \\ F(I') & \xrightarrow{\simeq} & X' \otimes Y' \end{array}$$

We say that a Feynman rule  $Z : (C, \mathfrak{d}) \rightarrow (\mathbf{A}, \tau)$  is *complete* if for each hypergraph  $G$  the maps  $FR_j^i$  are equivalences over their images. This means that different internal vertices and different external and internal hyperedges have different tensorial representations. If, in addition,  $FR^i$  are also equivalences over their images, we say that  $Z$  is *strongly complete*.

**Proposition 9** *In the same notations of Theorem 3, let  $FR : (C^b, \mathfrak{d}) \rightarrow (\mathbf{A}, \tau)$  be a coherent Feynman rule such that  $\tau$  is 1/decomposable and suppose one of the following conditions:*

- c1)  $C^b \subset S^{b, \ell}$  is concrete and  $FR$  is complete;
- c2)  $C^b \subset S^{b, \ell}$  is  $FR$  is strongly complete;

c3)  $(\mathbf{A}, \otimes, 1)$  is strict and  $FR$  is strongly complete.

Then  $FR$  induces a Feynman functor which is also complete.

*Proof* It is clear that the induced monoidal Feynman functor exists, since conditions c1), c2) and c3) above contains the conditions c1) and c2) of Theorem 3. Let us show that the induced Feynman functor is complete. We will work first with c1). Fixed  $V$ , given a morphism  $f : G \rightarrow G'$  in  $C_V^b$ , suppose that  $Z_V(f) : Z_V(G) \rightarrow Z_V(G')$  is an isomorphism. From the definition of  $Z_V(f)$  and the fact that  $\tau$  is 1/decomposable we find that  $r^i f$  and  $r_j^i$ , with  $i = 0, 1, \dots, n$  and  $1 < j \leq b$ , are all isomorphisms. Because  $FR$  is complete, each  $FR_{j>1}^i$  is an equivalence over its image. From the commutativity of diagrams (8) we then see that  $f_j^i$  are bijections. Since the prestack is concrete, we are working with hypergraphs over the same vertex set. Furthermore, since the hypergraphs are labeled, from Remark 4 we can assume  $f_V : V \rightarrow V$  equal to the identity  $id_V$ . Then each  $f^i$  is also a bijection, so that  $f$  is a hypergraph isomorphism. Assume c2) or c3). Then now  $FR$  is strongly complete, so that not only  $FR_{j>1}^i$ , but also  $FR^i$ , are equivalences over their images, implying directly that  $f_j^i$  and  $f^i$  are bijections.  $\square$

**Corollary 4** Let  $(\mathbf{A}, \otimes, \cdot^V)$  be a pseudocontext and let  $\tau$  1/decomposable. In this case:

- c1) if  $C \subset S_s^{\ell, b}$  is bounded, labeled, structured, then for any normal structure  $\mathfrak{d}$  in  $C$  there exists a nontrivial complete Feynman functor  $Z : C \rightarrow (\mathbf{A}, \tau)$ ;
- c2) if  $C$  is arbitrary, then for any normal structure  $\mathfrak{d}$  in  $C$  such a Feynman functor also exists, but only up to a change of coefficients.

*Proof* From Example 24, if  $C \subset S_s^b$ , then there exist classic Feynman rules in  $C$ . Notice that whenever classic rules of Feynman exist, we can modify them in order to assume that  $FR_{j>0}^i$  and  $FR^{i>0}$  are equivalences over their images. If  $(C, \mathfrak{d})$  is normal, the domains of  $FR^0$  and  $FR_j^0$  are the empty category or the point category, so that both functors are automatically equivalences over their images. Therefore,  $FR$  becomes strongly complete. With this in mind, c1) and c2) follow, respectively, from conditions c2) and c3) of Proposition 9.  $\square$

Recall from Example 17 that we have a rule  $\mathcal{N}$  assigning to each pair  $(C, \mathfrak{d})$  its normalization  $(\mathcal{N}_\mathfrak{d}C, \mathcal{N}_\mathfrak{d})$ . We say that some assertion about prestacks with decompositions holds *up to normalization* if it holds for every  $\mathcal{N}_\mathfrak{d}C$ .

**Corollary 5** Up to normalization and change of coefficients, there exists a nontrivial complete Feynman functor defined in any pair  $(C, \mathfrak{d})$  and taking values in any pair  $(\mathbf{A}, \tau)$  with  $\tau$  1/decomposable.

*Proof* Straightforward from condition c2) of Corollary 4.  $\square$

*Remark 19* The existence result for complete Feynman functors is much less general than that for Feynman functors: for the complete case we need to work with decomposable functors  $\tau : \mathbf{FinSet} \rightarrow \mathbf{A}$ . This hypothesis on  $\tau$  cannot be avoided because we have no analogue of Mac Lane's strictification theorem establishing that any  $(\mathbf{A}, \tau)$  is equivalent to other  $(\mathbf{A}', \tau')$  with a  $\tau'$  1/decomposable.

We will now discuss the monoidal property. Let  $(C, \xi)$  be a proper prestack endowed with a functorial decomposition  $\mathfrak{d}$ . We say that a Feynman rule  $FR : (C, \mathfrak{d}) \rightarrow (\mathbf{A}, \tau)$  is *oplax monoidal* if for any every  $V, V' \subset X$  and for every  $G \in C_V$  and  $G' \in C_{V'}$ , we have functors

$$FR^i(G, G') : V\xi(G, G')^i \rightarrow FR^i(VG^i) \otimes FR^i(VG'^i) \quad (14)$$

$$FR_j^i(G, G') : E\xi(G, G')_j^i \rightarrow FR_j^i(EG_j^i) \otimes FR_j^i(EG'_j{}^i), \quad (15)$$

and also  $FR^i(\emptyset) \rightarrow id_1$  and  $FR_j^i(\emptyset) \rightarrow id_1$ , fulfilling comonoid-like diagrams analogous to (7). Lax monoidal and strong monoidal Feynman rules are defined in a similar way.

*Example 30 (concrete proper)* Let  $(C, \xi)$  be a concrete proper prestack. We say that a functorial decomposition  $\mathfrak{d}$  in  $C$  is *compatible* with  $\xi$  if

$$V\xi(G, G')^i = V\xi(G, \emptyset)^i \sqcup V\xi(\emptyset, G')^i \text{ and } E\xi(G, G')_j^i = E\xi(G, \emptyset)_j^i \sqcup E\xi(\emptyset, G')_j^i \quad (16)$$

for  $0 \leq i \leq n$  and  $j > 1$ , where  $n$  is the order of  $\mathfrak{d}$ . For instance, if  $C$  is concretely proper, which means that  $\xi(G, G') = G \sqcup G'$ , then the condition above becomes

$$V(G \sqcup G')^i = VG^i \sqcup VG'^i \text{ and } E(G \sqcup G')_j^i = EG_j^i \sqcup EG'_j{}^i.$$

We notice that any Feynman rule  $FR : (C, \mathfrak{d}) \rightarrow (\mathbf{A}, \tau)$ , defined in a concrete proper prestack endowed with a compatible decomposition, is oplax monoidal in a unique way. Indeed, notice that since we have (16), the maps (14) and (15) we are looking for will be defined in a coproduct. But maps defined in a coproduct are uniquely determined by its components. So,  $FR^i(G, G')$  exists and it is totally determined by  $FR^i(G)$  and  $FR^i(G')$ .

**Proposition 10** *Under the same notations and hypotheses of Theorem 3, if  $C^b$  is proper and the coherent Feynman rule is oplax (resp. lax or strong) monoidal, then the induced Feynman functor is oplax (resp. lax or strong) monoidal.*

*Proof* We will prove only the oplax case assuming condition c2) in Theorem 3. The other cases (lax, strong and condition c1) are analogous. So, let  $FR : (C^b, \mathfrak{d}) \rightarrow (\mathbf{A}, \tau)$  be a coherent Feynman rule with  $\mathbf{A}$  strict and  $FR$  oplax, and let  $Z$  be the induced Feynman rule. Notice that, by definition  $Z_{V, V'}(\xi(G, G'))$  is the tensor product between  $Z^i(\xi(G, G'))$  and  $Z_j^i(\xi(G, G'))$ , for  $0 \leq i \leq n$ , where here we omit the subindices  $V, V'$  in order to simplify the notation (see expressions (13) and (12)). In turn,  $Z^i(\xi(G, G'))$  is the tensor product between  $FR^i(v)$ , with  $v \in V\xi(G, G')^i$ , while  $Z_j^i(\xi(G, G'))$  is the tensor product of  $FR_j^i(e)$ , for  $e \in E\xi(G, G')_j^i$ . Since  $FR$  is oplax, we have the maps (14), (15) and also  $\nu : FR(\emptyset) \rightarrow id_1$ . Therefore, tensoring  $FR^i(v)$  for every  $v$  and  $FR_j^i(e)$  for every  $e$  we get maps  $Z^i(\xi(G, G')) \rightarrow Z^i(G) \otimes Z^i(G')$  and  $Z_j^i(\xi(G, G')) \rightarrow Z_j^i(G) \otimes Z_j^i(G')$ . Tensoring them and varying  $i$  and  $j$ , we obtain a map

$$\mu_{V, V'} : Z_{V, V'}(\xi(G, G')) \rightarrow Z_V(G) \otimes Z_{V'}(G').$$

Furthermore, by definition of  $Z$  we see that  $\nu$  induces another  $\nu : Z_\emptyset(\emptyset) \rightarrow id_1$ . These maps will satisfy the comonoid-like diagrams precisely because they are finite tensor products of maps that satisfy the diagrams.  $\square$

**Corollary 6** *Up to change of coordinates, there exists a nontrivial oplax monoidal Feynman functor defined in any concrete proper prestack  $C$  and taking values in any pseudocontext  $\mathbf{A}$  endowed with any functor  $\tau : \mathbf{FinSet} \rightarrow \mathbf{A}$ .*

*Proof* Direct consequence of Example 25, Example 30 and Proposition 10.  $\square$

*Remark 20* We could think of getting a general existence result for Feynman functors which are simultaneously complete and oplax monoidal. Corollary 4 tell us that complete Feynman functors exist with the hypothesis that  $C$  is endowed with a normal structure. This condition was avoided in Corollary 5 by making use of a normalization process. Similarly, Corollary 6 states that oplax monoidal Feynman functors exist if  $C$  is concrete proper. So, we could try to build some kind of “concretization” and “propertification” processes, allowing us to say that up to them oplax Feynman functors always exist. The fundamental fact is that, even if these processes are built, we **cannot** use them to say that *up to normalization, “propertification” and “concretification” complete oplax Feynman functor always exist*. This happens because the normalization of an arbitrary nontrivial prestack cannot be concrete. Indeed, recall that being concrete means that  $C_V \in \mathbf{Hyp}_V$  for any  $V$ . However, the normalization  $\mathcal{N}C$  is such that  $V\mathcal{N}_\partial G^0 = \emptyset$ , so that  $\mathcal{N}C_V \notin \mathbf{Hyp}_V$  unless the initial decomposition  $\partial$  coincide with  $\mathcal{N}\partial$ , i.e, unless  $(C, \partial)$  is normal.

## 5.2 Uniqueness

Closing our discussion on Feynman functors, let us focus on the uniqueness problem. We will show that two complete Feynman functors are always conjugated in a suitable way.

We say that a functor  $\alpha : \mathbf{C} \rightarrow \mathbf{C}'$  is *quasi essentially injective* (qei) if it is constant in a subcategory  $c \subset \mathbf{C}$  and essentially injective in the remaining  $\mathbf{C} - c$ . We say that  $F : \mathbf{C} \rightarrow \mathbf{D}$  is *quasi essentially injectively conjugated* (qeic) to another functor  $F' : \mathbf{C}' \rightarrow \mathbf{D}'$  if there are qei functors  $\alpha$  and  $\beta$  making the following square commutative up to natural isomorphisms:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ \alpha \downarrow & \simeq \nearrow & \downarrow \beta \\ \mathbf{C}' & \xrightarrow{G} & \mathbf{D}' \end{array} \quad (17)$$

**Lemma 1** *Let  $\mathbf{D}$  a category with null objects. If a functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  is essentially injective and faithful, then it is qeic to any essentially injective functor  $G : \mathbf{C} \rightarrow \mathbf{D}$ .*

*Proof* Set  $\alpha = id_{\mathbf{C}}$  and for every  $Y \in \mathbf{D}$  define

$$\beta(Y) = \begin{cases} G(F^{-1}(Y)), & Y \in \text{img } F \\ 0, & \text{otherwise.} \end{cases}$$

Since  $F$  is essentially injective, this is well defined up to natural isomorphisms using the axiom of choice. Let  $f : Y \rightarrow Z$  be a morphism in  $\mathbf{D}$ . If  $Y$  do not belong to the image of  $F$ , define  $\beta(f) : \beta(Y) \rightarrow \beta(Z)$  as the unique map  $0 \rightarrow \beta(Z)$ . In a similar

way define  $\beta(f)$  when  $Z$  (or both  $Y$  and  $Z$ ) do not belong to  $F(\mathbf{C})$ . Finally, notice that since  $F$  is essentially injective and faithful, it is an equivalence over its image, so that if both  $Y, Z$  belong to  $F(\mathbf{C})$ , then to any  $f : Y \rightarrow Z$  corresponds a unique  $F^{-1}(f)$ . In this case, define  $\beta(f) = G(F^{-1}(f))$ . It is straightforward to check the functorial properties of  $G$ . By definition,  $\beta$  is essentially injective when restricted to  $F(\mathbf{C})$  and constant (equal to the null object) in the remaining part. Thus, both  $\alpha$  and  $\beta$  are qei. Furthermore, by construction the diagram (17) commutes up to isomorphisms, giving the desired conjugation and completing the proof.  $\square$

We say that a monoidal category  $(\mathbf{A}, \otimes, 1)$  is  $\tau$ -faithful, where  $\tau : \mathbf{FinSet} \rightarrow \mathbf{A}$  is a given functor, if the induced monoidal product in  $1/\mathbf{A}$  is faithful in both variables when restricted to the image of  $1/\tau$ , i.e, if  $f \otimes g = f' \otimes g'$  implies  $f = f'$  and  $g = g'$  for every  $f, g, f', g' \in 1/\tau(\mathbf{FinSet})$ .

**Proposition 11** *Under the same notations of Proposition 9, suppose that  $(\mathbf{A}, \otimes, 1)$  is  $\tau$ -faithful and  $\tau$  is 1/decomposable. In this case, if  $Z : C^b \rightarrow (\mathbf{A}, \tau)$  is the Feynman rule induced by a strongly complete Feynman rule  $FR$  fulfilling condition c2) or condition c3), then it is qeic to any other complete Feynman functor  $Z' : C^b \rightarrow (\mathbf{A}, \tau)$ .*

*Proof* Since  $Z$  is a complete Feynman functor,  $Z_V$  is essentially injective for every  $V$ . Therefore, by the previous lemma it is enough to prove that Feynman functors induced by strongly complete Feynman rules taking values into a  $\tau$ -faithful are also faithful, i.e, are such that if  $Z_V(f) = Z_V(g)$ , then  $f = g$ . In order to do this we need to work case by case of Proposition 9. We will give the proof only for case c3). Case c2) is analogous, needing only some care with the parentheses. By definition,  $Z_V(f)$  is a tensor product between  $Z_V^i(f)$  and  $Z_{V,j}^i(f)$ , with  $1 < j \leq b$  and  $0 \leq i \leq n$ . In turn,  $Z_V^i(f)$  is the map between  $\otimes_v FR^i(v)$  and  $\otimes_w FR^i(w)$ , with  $v \in VG^i$  and  $w \in VG^i$ , given by the restriction of  $\otimes_v r^i f$ . Furthermore,  $Z_{V,j}^i(f)$  is obtained in a similar way as a restriction of  $\otimes_e r_j^i f$  with  $e \in EG_j^i$ . Therefore,  $Z_V(f)$  is the restriction of  $\otimes_{i,j} \otimes_{v,e} r^i f \otimes r_j^i f$  to an object depending only of  $G$  (and not of  $f$ ). So, if  $Z_V(f) = Z_V(g)$ , then

$$\otimes_{i,j} \otimes_{v,e} r^i f \otimes r_j^i f = \otimes_{i,j} \otimes_{v,e} r^i g \otimes r_j^i g.$$

Since both sides are morphisms between  $1/\tau VG$  and  $1/\tau VG'$ , the fact that  $\mathbf{A}$  is  $\tau$ -faithful implies  $r^i f = r^i g$  and  $r_j^i f = r_j^i g$ . Finally, because  $FR$  is strongly proper, each  $FR^i$  and each  $FR_j^i$  is an equivalence over their images. Thus, commutativity of diagrams (8) gives us  $f^i = g^i$  and  $f_j^i = g_j^i$ , i.e,  $f = g$ .  $\square$

Putting together all parts of our construction we have the following general existence and uniqueness theorem.

**Theorem 4 (existence and uniqueness)** *Let  $C$  be a proper,  $(\mathbf{A}, \otimes, \cdot^\vee)$  be a  $\tau$ -faithful pseudocontext with null objects, where  $\tau : \mathbf{FinSet} \rightarrow \mathbf{A}$  is a 1/decomposable functor. In this case:*

1. *up to normalization, change of coefficients and qeic, there exists a unique complete Feynman functor  $Z : C \rightarrow (\mathbf{A}, \tau)$ ;*
2. *if  $C$  is concrete proper and becomes endowed with a normal structure  $\mathfrak{d}$ , then up to change of coefficients and qeic, there exists a unique monoidal and complete Feynman functor  $Z : C \rightarrow (\mathbf{A}, \tau)$ .*

## 6 Superposition

Let  $(\mathbf{C}, \otimes, 1)$  be a monoidal category. We say that an object  $X \in \mathbf{C}$  is the *superposition* of a family of objects  $A_i \in \mathbf{C}$ , with  $1 \leq i \leq n$ , if there exists an isomorphism

$$(\dots((A_1 \otimes A_2) \otimes A_3)\dots) \otimes A_n \simeq X. \quad (18)$$

The number  $n$  is called the *order* of the superposition.

*Example 31 (trivial superposition)* Each object  $X$  admits a superposition of arbitrary order. Indeed, for given  $n$ , just take  $A_1 = X$  and  $A_{i>1} = 1$ .

Let  $\mathcal{C} \subset \text{Ob}(\mathbf{C})$  be a collection of objects in  $\mathbf{C}$ . A *superposition principle* in  $\mathcal{C}$  is given by

1. a bounded from above function  $n : \mathcal{C} \rightarrow \mathbb{N}$ , which is equivalent to saying that  $n(\mathcal{C}) \subset \mathbb{N}$  is finite<sup>7</sup>. The maximum of  $n(\mathcal{C})$  will be denoted by  $N$ ;
2. a function  $\mathcal{S} : \mathcal{C} \rightarrow \text{Ob}(\mathbf{C})$  assigning to each object  $X$  in  $\mathcal{C}$  a family of objects  $SX_i$  in  $\mathbf{C}$ , with  $1 \leq i \leq N$ , such that
  - (a) for  $i > n(X)$  we have  $SX_i \simeq 1$ ;
  - (b) after taking the tensor product between the  $SX_i$ , in the same order that in (18) we get a superposition for  $X$ .

We say that a superposition principle is *functorial* if  $\mathcal{C}$  is actually a (non necessarily monoidal) subcategory of  $\mathbf{C}$  such that the function  $\mathcal{S} : \mathcal{C} \rightarrow \text{Ob}(\mathbf{C})$  extends to a functor  $\mathcal{S} : \mathcal{C} \rightarrow \mathbf{C}$ .

*Example 32 (trivial superposition principle)* In any set  $\mathcal{C}$  we have a trivial superposition principle, which assigns to each  $X \in \mathcal{C}$  its corresponding trivial superposition.

In the following we will show that nontrivial Feynman functors behave as a bridge between reconstruction conjectures and nontrivial superposition principles. In order to do this, let  $\mathcal{D}$  be a deleting process, which assigns to each applicable prestack  $C \in \mathfrak{C}$  its prestack  $DC$  of pieces, and let us define a *structure of disjoint pieces* in some  $\mathcal{C} \in \mathfrak{C}$  as:

1. a rule that for each  $V$  associates a decomposition  $V = \coprod_i \overline{V}_i$ ;
2. a subprestack  $K \subset C$  (not necessarily applicable);
3. for each  $V$  a functor  $\kappa_V : DC_V \rightarrow \prod_i K_{\overline{V}_i}$ .

*Example 33 (representable reconstruction)* If each  $DC_V$  is representable, then it maps colimits into limits, so that for any decomposition  $V = \coprod_i \overline{V}_i$  we have  $DC_{\coprod_i \overline{V}_i} \simeq \prod_i DC_{\overline{V}_i}$ . So, if  $DC_V \subset C_V$ , we can take  $K_{\overline{V}_i} = DC_{\overline{V}_i}$  and  $\kappa_V$  as the previous isomorphism.

*Example 34 (disjoint reconstruction)* The disjoint context  $DC_V = \sqcap C_V$  has a canonical structure of disjoint pieces with  $\overline{V}_i = V - v_i$ ,  $K = DC$  and  $\kappa = id$ .

<sup>7</sup> Indeed, due to the well-ordering principle a subset  $S \subset \mathbb{N}$  is always bounded from below, hence it is bounded from above iff it is bounded. Furthermore, any infinite subset of  $\mathbb{N}$  is in bijection with  $\mathbb{N}$ , so that it is unbounded. Finally, every finite subset of  $\mathbb{N}$  is bounded.



**Theorem 5** *Let  $(C, \xi)$  be a proper prestack of hypergraphs which is an applicable prestack of certain reconstruction context  $\mathcal{D}$ . Suppose given a structure of disjoint pieces in  $C$ . Then each nontrivial strong monoidal Feynman functor  $Z : C \rightarrow (\mathbf{A}, \tau)$  induces a nontrivial superposition principle in a set  $\mathcal{A}_\tau$  of analytic expressions in  $\mathbf{A}$ , which becomes endowed with canonical map  $\lambda_Z : \mathcal{G} \rightarrow \mathcal{A}_\tau$ , where  $\mathcal{G}$  is the set of all isomorphism classes of hypergraphs  $G \in C_V$  for every  $V$ .*

*Proof* We will give the proof for the case in which  $(\mathbf{A}, \otimes, 1)$  is strict. The general case is analogous, only needing some care with the parentheses. For each  $V$ , we have the functor  $\kappa_V : \mathcal{DC}_V \rightarrow \prod_i K_{\overline{V}_i}$ . On the other hand, since  $C$  is proper for any  $W, W'$  we have  $\xi : C_W \times C_{W'} \rightarrow C_{W \sqcup W'}$ , which clearly extends to a functor  $\xi : \prod_i C_{W_i} \rightarrow C_{\prod_i W_i}$ , where  $W_i$  is any finite family. Because  $K \subset C$ , such functors can be restricted to  $K$ , giving  $\xi : \prod_i K_{W_i} \rightarrow K_{\prod_i W_i}$ . Taking  $W_i = \overline{V}_i$ , let us consider  $\xi$  restricted to the image of  $\kappa_V$ . Since  $Z$  is strong monoidal, for every  $\mathcal{DG} \in \mathcal{DC}_V$  we have an isomorphism

$$Z_{\prod_i \overline{V}_i = V}(\xi(\mathcal{DG})) \simeq \otimes_i Z_{\overline{V}_i}(\mathcal{DG}_i), \quad (19)$$

where  $\mathcal{DG}_i \in K_{\overline{V}_i}$  are the components of  $\kappa_V(\mathcal{DG})$ . We can regard the isomorphism above as superposition principle for  $Z(\xi(\mathcal{DG}))$ . Recall that for every  $V$  and every  $\mathcal{DG} \in \mathcal{DC}_V$ ,  $Z_V(\mathcal{DG}) \in \mathbf{A}_{\tau V \mathcal{DG}}$ . Therefore, if  $\mathcal{A}_\tau \subset \text{Ob}(1/\mathbf{A})$  is the set of all analytic expressions in  $\tau V$ , for every  $V$ , then  $Z_V(\mathcal{DG}) \in \mathcal{A}_\tau$  for each  $V$  and each  $\mathcal{DG}$ . This means that when varying  $V$  and  $\mathcal{DG}$  in (19) we get a superposition principle in  $\mathcal{A}_\tau$ . Let  $\mathcal{DG}$  be the set of all isomorphism classes of  $\mathcal{DG} \in \mathcal{DC}_V$  for every  $V$ . So we have an obvious function  $\Lambda_Z : \mathcal{DG} \rightarrow \mathcal{A}_\tau$  assigning  $Z_V(\mathcal{DG})$  to each  $\mathcal{DG}$ . Finally, recall that in a reconstruction context there exists a transformation  $\gamma_V : C_V \rightarrow \mathcal{DC}_V$ , which produces  $\gamma : \mathcal{G} \rightarrow \mathcal{DG}$ . By composing with  $\Lambda_Z$  we get the desired  $\lambda_Z : \mathcal{G} \rightarrow \mathcal{A}_\tau$ .  $\square$

**Corollary 7** *Let  $C \in \mathfrak{C}$  be an applicable prestack of a reconstruction context  $\mathcal{D}$ . Suppose that  $C$  is proper and endowed with a structure of disjoint pieces. Then  $\mathcal{D}$ -RC- $C$  holds only if for every nontrivial complete monoidal Feynman rule  $Z : C \rightarrow (\mathbf{A}, \tau)$  the corresponding map  $\lambda_Z$  is injective.*

*Proof* Assume  $\mathcal{D}$ -RC- $C$  holds, which means that  $\gamma_V : C_V \rightarrow \mathcal{DC}_V$  is essentially injective for every  $V$ , and let  $Z : C \rightarrow (\mathbf{A}, \tau)$  be a complete monoidal Feynman rule. Recall that  $\lambda_Z : \mathcal{G} \rightarrow \mathcal{A}_\tau$  is the composition between  $\gamma : \mathcal{G} \rightarrow \mathcal{DG}$  and  $\Lambda_Z : \mathcal{DG} \rightarrow \mathcal{A}_\tau$ . Completeness of  $Z$  implies  $\Lambda_Z$  injective and essential injectivity of  $\gamma_V$  makes  $\gamma$  injective, so that  $\lambda_Z$  is also injective. Reciprocally, suppose that  $\lambda_Z$  is injective for some complete monoidal Feynman rule. By definition,  $\lambda_Z = \Lambda_Z \circ \gamma$ . Since injective functions are monomorphisms,  $\gamma_Z$  injective implies  $\gamma$  injective, which is equivalent to saying that  $\gamma_V$  is essentially injective for every  $V$ , so that  $\mathcal{D}$ -RC- $C$  holds.  $\square$

*Remark 21* As a consequence of Proposition 3 and Proposition 4, the previous corollary can be improve by replacing the hypothesis of  $\mathcal{D}$ -RC- $C$  holding with the existence of left/right morphisms  $F : \mathcal{D} \rightarrow \mathcal{D}'$  or left/right  $C$ -implications to some other reconstruction context.

**Corollary 8** *Let  $C \subset S_s^{b, \ell}$  be some concretely proper prestack of labeled structured bounded hypergraphs, regarded as an applicable prestack for the standard reconstruction*

context  $\mathcal{D} = d$  and endowed with the canonical structure of disjoint pieces. Then for every complete monoidal Feynman functor  $Z : C \rightarrow (\mathbf{A}, \tau)$  the induced map  $\lambda_Z$  is injective. Explicitly,

$$Z_V(G) \simeq Z_V(G') \quad \text{iff} \quad \bigotimes_{v \in V} Z(G - v) \simeq \bigotimes_{v \in V} Z(G' - v).$$

*Proof* Immediate from Corollary 2, Corollary 7 and from the definition of  $\lambda_Z$ .  $\square$

## 7 Applications

As the first application we present an existence result for representations of hypergraph categories in monoidal categories.

**Proposition 12** *Let  $C^b$  be  $b$ -bounded prestack of hypergraphs. Let  $(\mathbf{A}, \otimes, \cdot^\vee)$  be a pseudocontext which is  $\tau$ -faithful relative to a 1/decomposable functor  $\tau : \mathbf{FinSet} \rightarrow \mathbf{A}$ . Suppose one of the following conditions:*

- c1)  $C^b \subset S_{\mathfrak{s}}^{b,\ell}$  for some functor of structures  $\mathfrak{s}$ ;
- c2)  $(\mathbf{A}, \otimes, 1)$  is a strict monoidal category.

*In this case, the choice of a normal structure  $\mathfrak{d}$  for  $C^b$  induces an equivalence between the category  $C_V^b$ , for each  $V$ , and some subcategory of  $1/\mathbf{A}$ .*

*Proof* Notice that we are in the hypothesis of Proposition 11 and from its proof we see that if there exists some strongly complete Feynman rule  $FR$  in  $(C^b, \mathfrak{d})$ , then it induces a Feynman functor  $Z : C^b \rightarrow (\mathbf{A}, \tau)$  such that each  $Z_V : C_V^b \rightarrow 1/\mathbf{A}$  is essentially injective and faithful, so that they are equivalences over their images. Since  $\mathfrak{d}$  is chosen normal, it follows from Corollary 4 that these Feynman rules really exist.  $\square$

### 7.1 Mapping Class Group and Ribbon Graphs

One of the main problems of manifold topology is to determine the mapping class group  $\text{MCG}(M)$  of a given manifold  $M$ , i.e, the quotient of the diffeomorphism group  $\text{Diff}(M)$  by the path-component at the identity. It is a remarkable fact that for marked surfaces that object can be described by the category of ribbon graphs. More precisely, if  $\mathbf{Ribb}_3^{\text{con}} \subset \mathbf{Ribb}$  denotes the category of connected ribbon graphs whose vertices are at least trivalent and with morphisms given by ribbon graphs isomorphisms, then there exists a homotopy equivalence

$$|\mathbf{Ribb}_3^{\text{con}}| \simeq \coprod_{[\Sigma] \in \text{Iso}(\mathbf{Diff}_2^*)} B \text{MCG}(\Sigma),$$

where  $|\mathbf{C}|$  and  $BG$  denotes, respectively, the geometric realization of a category and the classifying space of a topological group. Furthermore, the coproduct is taken over the diffeomorphic classes of all marked surfaces, except for two exceptional cases: the sphere  $\mathbb{S}^2$  with one and with two marked points.

This result was proven using different methods in [7, 8, 9, 35, 36]. As a second application of the existence of complete Feynman rules we give an independent

proof of a related result. Indeed, we will show that for any fixed vertex set  $V$ , the category  $\mathbf{Ribb}_V$  can be regarded as a subcategory  $\mathbf{A}_V$  of

$$\coprod_{[\Sigma] \in \text{Iso}(\sqcup \mathbf{Diff}_2^*)} \mathbf{B} \text{MCG}(\Sigma),$$

where  $\mathbf{B}G$  is the delooping groupoid of a group and  $\sqcup \mathbf{Diff}_2^*$  means that the coproduct is take over arbitrary finite coproducts of marked surfaces.

Let  $\mathbf{Diff}^*$  be the category of marked manifolds, i.e, pairs  $(M, S)$ , where  $S \subset M$  is some finite subset of mutually distinct points, and morphisms given by smooth maps  $f : M \rightarrow M'$  such that  $f(S) \subset S'$ . The category has coproducts given by  $\coprod_i (M_i, S_i) = (\coprod_i M_i, \coprod_i S_i)$ . Two marked manifolds are isomorphic only if  $S \simeq S'$ , which means that in the isomorphism classes only the number of marked points matter. We will write  $M^s$  instead of  $(M, S)$ , where  $s = |S|$ . Let  $\mathbf{Sing}$  be the proper class of singletons and fix an injective class function  $\omega : \text{Iso}(\mathbf{Diff}^*) \rightarrow \mathbf{Sing}$ . For instance, we could take  $\omega([M]) = \{[M]\}$ . Define a category  $\text{MCG}_{\mathbf{Diff}^*}^\omega$  as follows. Objects are given by the image of  $\omega$ , i.e, we have an object for each isomorphism class of marked manifolds and this object which is a singleton  $\omega([M^s]) = *_{[M^s]}$ . Furthermore, there are morphisms between  $*_{[M^s]}$  and  $*_{[N^r]}$  iff  $[M^s] = [N^r]$  and in this case  $\text{Mor}(*_{[M^s]}, *_{[M^s]}) = \text{MCG}(M^s)$ . In particular,  $(\text{MCG}_{\mathbf{Diff}^*}^\omega)^{op} = \text{MCG}_{\mathbf{Diff}^*}^\omega$ . Furthermore, since

$$\text{MCG}_{\mathbf{Diff}^*}^\omega \simeq \coprod_{[M^s] \in \text{Iso}(\mathbf{Diff}^*)} \mathbf{B} \text{MCG}(M^s),$$

the left-hand side does not depends of  $\omega$ . Coproducts pass to  $\text{MCG}_{\mathbf{Diff}^*}^\omega$  by taking  $*_{[M^s]} \sqcup *_{[N^r]} = *_{[M^s \sqcup N^r]}$ . Furthermore, if  $f \in \text{MCG}(M^s)$  and  $g \in \text{MCG}(N^r)$ , define  $f \sqcup g : *_{[M \sqcup N]} \rightarrow *_{[M \sqcup N]}$  as the coproduct in  $\mathbf{Diff}^*$ .

Consider a full subcategory  $\mathbf{A} \subset \text{MCG}_{\mathbf{Diff}^*}^\omega$  closed by finite coproducts, so that we can take the cocartesian monoidal structure and since  $\mathbf{A}^{op} = \mathbf{A}$ , we get a pseudocontext structure  $(\mathbf{A}, \sqcup, \cdot^\vee)$  with  $\cdot^\vee : \mathbf{A} \rightarrow \mathbf{A}$  some endofunctor of  $\mathbf{A}$ . These functors are in 1-1 correspondence with rules assigning manifolds  $M^s$  such that  $\omega([M^s]) \in \mathbf{A}$  to manifolds  $N$  such that  $\omega([N^r]) \in \mathbf{A}$ , together with a group homomorphism  $\text{MCG}(M^s) \rightarrow \text{MCG}(N^r)$ . For instance, if in  $\mathbf{A}$  there exists some manifold  $X^o$  whose mapping class group is completely understood, we can take  $\cdot^\vee$  as some functor constant in such a manifold, which is equivalent to giving a representation of every  $\text{MCG}(M^s)$  in  $\text{MCG}(X^o)$ . But we can also simply take  $\cdot^\vee = id_{\mathbf{A}}$ . Let  $S \neq \emptyset$  be a connected manifold such that  $\omega([S^s]) \in \mathbf{A}$  for every  $s$  and define  $\tau_S : \mathbf{FinSet} \rightarrow \mathbf{A}$  as follows. For each finite set  $X$  we take  $\tau_S(X) = \omega([S^{|X|}])$ . Furthermore, if  $f : X \rightarrow Y$  is a map between finite sets, then  $\tau_S(f)$  is nontrivial iff  $X \simeq Y$  and in this case  $\tau_S(f) = id_{S^{|X|}}$ . objects and

Now, recall that each construction in this article was made in  $1/\mathbf{A}$ . The only reason for doing this was to have a frame closer to physics interpretation. Indeed, in this way we can talk about analytic expressions itself, while when working in  $\mathbf{A}$  we can only talk about the object of all analytic expressions. Even so, all definitions, statements and demonstrations work *ipsi literis* in  $\mathbf{A}$ . With this in mind we can search for Feynman functors taking values in  $(\mathbf{A}, \tau_S)$  for the pseudocontext  $(\mathbf{A}, \sqcup, \cdot^\vee)$  defined above.

**Proposition 13** *For each connected non-empty manifold  $S$  such that  $\omega(S^s) \in \mathbf{A}$  for every  $s$ , the pseudocontext  $\mathbf{A} \subset \text{MCG}_{\mathbf{Diff}^*}^\omega$  is  $\tau_S$ -faithful and  $\tau_S$  is decomposable.*

*Proof* Notice that  $\tau_S(\mathbf{FinSet})$  is the category whose objects are  $\omega([S^s])$ , with  $s \geq 1$ , and whose only morphisms are the identities  $id_{S^s}$ , for  $s \geq 1$ . Therefore,  $(\mathbf{A}, \sqcup, \emptyset)$  is clearly  $\tau_S$ -faithful. In order to see that  $\tau_S$  is decomposable, given finite sets  $X$  and  $X'$ , suppose we have decompositions

$$\tau_S(X) \simeq \omega([M^s]) \sqcup \omega([N^r]) \quad \text{and} \quad \tau_S(X') \simeq \omega([M'^{s'}]) \sqcup \omega([N'^{r'}]).$$

This implies  $S^{|X|} \simeq M^s \sqcup N^r$  and  $S^{|X'|} \simeq M'^{s'} \sqcup N'^{r'}$ . Since  $S$  is connected, either we have the following configuration or we have one of the other seven permutations:

- $M^s \simeq S^{|X|}$  and  $N^r = \emptyset$ , together with  $M'^{s'} \simeq S^{|X'|}$  and  $N'^{r'} = \emptyset$ .

We should proceed case by case, but since everything is analogous we will work only with the above configuration. Let  $f : X \rightarrow X'$  be an isomorphism. Then  $|X| = |X'|$ , implying  $M^s = M'^{s'} \simeq S^{|X|}$ . Since  $\tau_S(f) = id_{S^{|X|}}$ , we just take  $\alpha = id_{M^s}$ . Furthermore, since  $N^r = \emptyset = N'^{r'}$ , there exists a unique  $\beta : N^r \simeq N'^{r'}$  and we clearly have  $\tau_S(f) = \alpha \sqcup \beta$ .  $\square$

**Corollary 9** *Let  $C^b$  be a  $b$ -bounded prestack of hypergraphs. For each normal structure  $\mathfrak{d}$  and each non-empty connected manifold  $S$  there exists an equivalence between  $C_V^b$ , for every  $V$ , and a subcategory  $\mathbf{A}_V$  of any pseudocontext  $\mathbf{A} \subset \text{MCG}_{\mathbf{Diff}^*}^\omega$  such that  $\omega(S^s) \in \mathbf{A}$  for every  $s$ .*

*Proof* Straightforward from Proposition 13 and Proposition 12.  $\square$

Let  $\text{MCG}_{\sqcup \mathbf{Diff}_2^*}^\omega \subset \text{MCG}_{\mathbf{Diff}^*}^\omega$  denote the subcategory generated by compact orientable marked surfaces and finite disjoint unions of them.

**Corollary 10** *The choice of a normal decomposition  $\mathfrak{d}$  induces an equivalence between the category  $\mathbf{Ribb}_V$  and some subcategory of  $\text{MCG}_{\sqcup \mathbf{Diff}_2^*}^\omega$ .*

*Proof* Just recall that any prestack of graphs is 2-bounded and then apply the last corollary.  $\square$

*Remark 22* Our construction, however, does not give much information about the homotopy type of the classifying spaces  $B \text{MCG}(M)$ . Indeed, since we included the empty manifold in  $\text{MCG}_{\mathbf{Diff}^*}^\omega$  and the empty graph in each  $C_V^b$ , both admit initial objects, so that their geometric realizations are automatically contractible.

## 7.2 Perturbative QFT

As a final application, let us show that the validity of a reconstruction conjecture induces a new superposition principle in perturbative QFT. A *classical field theory*  $S$  (following [20]) is given by the following data:

1. a vector bundle  $\pi : E \rightarrow M$  over a compact riemannian manifold  $M$ ;
2. a positive generalized laplacian  $\mathcal{D}$  in  $E^! = E^V \otimes \text{Dens}_M^{s,+} \simeq \text{Hom}(E; \text{Dens}_M^{s,+})$ , from which we build the free functional  $S_0[s] = \int_M \mathcal{D}s$ . Typical examples occur when  $\mathcal{D}s = \langle s, Ds \rangle$ , where  $D$  is a generalized laplacian in  $E$  and  $\langle \cdot, \cdot \rangle : E \otimes E \rightarrow \text{Dens}_M^{s,+}$  is a symmetric bundle map;
3. a differential operator  $\mathfrak{D}$  between  $E^!$  and  $E$ , which is formally symmetric, i.e.,  $\mathfrak{D}^! = \mathfrak{D}$  and such that  $\mathcal{D} \circ \mathfrak{D}^! = \mathfrak{D} \circ \mathcal{D}^!$ ;

4. an element  $I \in \Gamma(E)[[\hbar]]$ , giving the full action functional  $S = S_0 + I$ .

Since  $\Gamma(E)$  is a nuclear Fréchet space, it belongs to context  $(\mathbf{NucFrec}, \otimes_p, \cdot^\vee)$  of Example 15. Let us take  $\tau : \mathbf{FinSet} \rightarrow \mathbf{NucFrec}$  as constant in  $\Gamma(E)$ . From any classical field theory (in the above sense) we can extract a full subcategory  $\mathbf{Feyn}(\mathcal{S}) \subset \mathbf{Feyn}$  simply by doing the standard Feynman graph expansion of an action functional [20]. Varying  $V$  on  $\mathbf{Feyn}(\mathcal{S})_V$  we get a prestack  $C_{\mathcal{S}}$ . Let us call the pair  $(\mathcal{S}, C_{\mathcal{S}})$  the *perturbative QFT* of  $\mathcal{S}$ . Notice, on the other hand, that from data 1-3 above we can extract a tensor  $P \in \mathrm{Sym}^2 \Gamma(E)$ , obtained as follows. Since  $M$  is compact,  $\mathcal{D}$  has a smooth heat kernel  $K_t \in \Gamma(E^1) \otimes \Gamma(E) \otimes C^\infty(\mathbb{R}_{\geq 0})$ . Composing with  $\mathfrak{D}$  we get an element of  $\mathcal{K}_t \in \Gamma(E) \otimes \Gamma(E) \otimes C^\infty(\mathbb{R}_{\geq 0})$ . Because  $\mathfrak{D}$  is symmetric,  $\mathcal{K}_t$  is symmetric too. By means of integrating we get  $P \in \mathrm{Sym}^2(\Gamma(E))$ . We can then get a Feynman rule  $FR : (C_{\mathcal{S}}, \mathfrak{d}) \rightarrow (\mathbf{NucFrec}, \tau)$  from Example 26 by fixing  $FR_2^1$  as constant in  $P$  and  $FR^1$  as determined by  $I$ . We also take  $FR^0$  constant due to the indistinguishability of quantum particles. Since  $C_{\mathcal{S}}$  is concretely proper, it follows from Example 30 and Proposition 10 that the induced Feynman functor  $Z_{\mathcal{S}} : C_{\mathcal{S}} \rightarrow (\mathbf{NucFrec}, \tau)$  is oplax monoidal. Let us define a *superposition principle* for  $(\mathcal{S}, C_{\mathcal{S}})$  as a superposition principle in the image of  $Z_{\mathcal{S}}$ .

**Proposition 14** *Let  $(\mathcal{S}, C_{\mathcal{S}})$  be a perturbative QFT whose Feynman functor  $Z_{\mathcal{S}}$  is strong monoidal. If  $C_{\mathcal{S}}$  is an applicable prestack for a reconstruction context  $\mathcal{D}$  we have a nontrivial superposition principle for  $(\mathcal{S}, C_{\mathcal{S}})$ .*

*Proof* Just apply Theorem 5.  $\square$

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