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# Topological and Geometric Obstructions on Einstein-Hilbert-Palatini Theories

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## Abstract

In this article we introduce  $A$ -valued Einstein-Hilbert-Palatini functional ( $A$ -EHP) over a  $n$ -manifold  $M$ , where  $A$  is an arbitrary graded algebra, as a generalization of the functional arising in the study of the first order formulation of gravity. We show that if  $A$  is weak  $(k, s)$ -solvable, then  $A$ -EHP is non-null only if  $n < k + s + 3$ . We prove that essentially all algebras modeling classical geometries (except semi-Riemannian geometries with specific signatures) satisfy this condition for  $k = 1$  and  $s = 2$ , including Hitchin's generalized complex geometry, Pantilie's generalized quaternionic geometries and all other generalized Cayley-Dickson geometries. We also prove that if  $A$  is concrete in some sense, then a torsionless version of  $A$ -EHP is non-null only if  $M$  is Kähler of dimension  $n = 2, 4$ . We present our results as obstructions to  $M$  being an Einstein manifold relative to geometries other than semi-Riemannian.

*Keywords:* Einstein-Hilbert-Palatini functional topological obstructions, geometric Einstein-manifolds, weak  $(k, s)$ -solvable graded algebras.

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## 1. Introduction

The standard theory of gravity is General Relativity [32, 16], which is formulated in Lorentzian geometry: spacetime is regarded as a Lorentzian manifold  $(M, g)$  whose Lorentzian metric  $g$  is a critical point of the Einstein-Hilbert action functional

$$S_{EH}[g] = \int_M (R_g - 2\Lambda) \cdot \omega_g, \quad (1.1)$$

defined on the space of all possible Lorentzian metrics in  $M$ ,  $\Lambda \in \mathbb{R}$  is the cosmological constant and  $R_g$  and  $\omega_g$  are, respectively, the scalar curvature and the volume form of  $g$ . The Euler-Lagrange equations of  $S_{EH}$  are the Einstein's equations

$$\text{Ric}_g - \frac{1}{2}R_g g + \Lambda g = 0, \quad (1.2)$$

which are equivalent to saying that  $(M, g)$  is an Einstein manifold, i.e,  $\text{Ric}_g = \frac{2}{n-2}\Lambda g$ . The problem of determining which manifolds admit an Einstein structure is old and a general classification of them remains an open problem [7, 28].

The action functional  $S_{EH}$  can be considered not only in Lorentzian metrics, but also in metrics of arbitrary signature. The Euler-Lagrange equations have the same form and they are also equivalent to the Einstein manifold condition. Despite the similarity, being Einstein depends strongly on the signature (and, therefore, on the underlying geometry). For instance, if  $M$  is compact, orientable, non-parallelizable and Kähler, then it does not admit any Lorentzian metric<sup>1</sup>, while Calabi-Yau manifolds, hyperKähler and quaternion Kähler manifolds are the basic examples of Riemannian Einstein manifolds.

Due to the difficulty of classifying semi-Riemannian Einstein-manifolds, many generalizations and closely related concepts were introduced and studied, such as generalized Einstein manifolds [1], quasi-Einstein manifolds [10], generalized quasi-Einstein manifolds [11], super quasi-Einstein manifolds [24], and many others. Each of these generalizations concerns semi-Riemannian manifolds. Here we will consider another generalization: the problem of finding Einstein manifolds for geometries other than semi-Riemannian. Because semi-Riemannian Einstein manifolds are critical points of the Einstein-Hilbert functional, the idea is to define Einstein manifolds relative to other geometries as critical points of functionals which are analogues of  $S_{EH}$  in those geometries. The classical examples of geometries can be described by tensors  $t$  on  $M$  fulfilling additional integrability conditions. Considering connections on  $M$  compatible with  $t$  (in the sense that  $\nabla t = 0$ ) we can take their curvature tensor and Ricci curvature. In order to write down the analogous action we need to contract the Ricci curvature with the tensor  $t$ , getting the *scalar curvature*  $R_t$  of  $t$ . If  $M$  is oriented we can define an action functional  $S[t] = \int (R_t - 2\Lambda) \cdot \omega$  on the moduli space of those tensors. This makes sense only when  $t$  has rank two. For example, we can write a direct analogue of General Relativity in symplectic geometry, with action functional  $S[\omega] = \int_M (\pi^{ij} \text{Ric}_{ij} - 2\Lambda) \cdot \omega^n$  on a  $2n$ -dimensional manifold, where  $\pi^{ij}$  are the components of the bivector induced by  $\omega$ . But the tensor  $\text{Ric}_{ij}$  is always symmetric, so that  $\pi^{ij} \text{Ric}_{ij} = 0$  and the action therefore is trivial [8, 9]. Also, there is no obvious analogue of  $S_{EH}$  (and, consequently, of Einstein manifolds) relative to Kähler, quaternion Kähler and hyper-Kähler geometry. This fact should not be viewed as a contradiction to the above cited fact that Calabi-Yau manifolds, quaternion Kähler and hyper-Kähler manifolds are Riemannian Einstein manifolds nor to the applications of those geometries in classical (i.e., semi-Riemannian) General Relativity, specially in the canonical formalism [2, 14].

We recall, however, that Einstein's equations can be obtained as critical points of another functional, usually known as the *Einstein-Hilbert-Palatini* (EHP) *action functional*, which arises in the so-called *first order formulation of gravity* (also called *tetradic gravity*). Instead of a semi-Riemannian metric, this functional is defined on the space of reductive Cartan connections on the frame

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<sup>1</sup>In fact, Kähler condition requires  $b_i(M) = 0$  for  $i$  odd, so that  $\chi(M) = \sum_i b_{2i}(M)$ , which is positive, since  $M$  is non-parallelizable. On the other hand, compact manifolds admit Lorentzian metrics only if  $\chi(M) = 0$ .

bundle  $FM \rightarrow M$ , relative to the group reduction  $O(r, s) \hookrightarrow \mathbb{R}^{r,s} \rtimes O(r, s)$ . These can be identified with pairs of 1-forms  $e$  and  $\omega$  in  $FM$ , called *tetrad* and *spin connection*, with values in  $\mathbb{R}^{n-1,1}$  and  $\mathfrak{o}(n-1, 1)$ , respectively. It is usual to assume that  $e$  is pointwise an isomorphism. The action itself is given by

$$S_{EHP}[e, \omega] = \int_M \text{tr}(\lambda_{\times}^{n-2} e \lambda_{\times} \Omega + \frac{\Lambda}{(n-1)!} \lambda_{\times}^n e), \quad (1.3)$$

where  $\lambda_{\times}$  is a type of “wedge product<sup>2</sup>” induced by matrix multiplication in  $O(r, s)$ ,  $\lambda_{\times}^k \alpha = \alpha \lambda_{\times} \dots \lambda_{\times} \alpha$  and  $\Omega = d\omega + \omega \lambda_{\times} \omega$  is the curvature of  $\omega$ . It is a well-known fact [4, 33] that in dimension  $n = 4$  and Lorentzian signature varying (1.3) in relation to  $e$  and  $\omega$  we get

$$e \lambda_{\times} \Omega + \frac{\Lambda}{(n-1)!} e \lambda_{\times} e \lambda_{\times} e = 0 \quad \text{and} \quad e \lambda_{\times} \Theta = 0, \quad (1.4)$$

where  $\Theta = d\omega + \omega \lambda_{\times} e$  is the *torsion* of  $\omega$ . Lorentz metrics on  $M$  can be identified as pullbacks of the canonical Minkowski metric on  $\mathbb{R}^{r,s}$  via tetrad and the connections  $\omega$  are compatible with them. Furthermore, since  $e$  is pointwise an isomorphism, the second equation in (1.4) is equivalent to the torsion-free condition  $\Theta = 0$ , so that  $\omega$  is necessarily a Levi-Civita connection. Via these identifications, the first equation in (1.4) is just Einstein’s equation. Therefore, a 4-dimensional Einstein manifold can also be regarded as a critical point (relative to  $e$ ) of  $S_{EHP}$ .

The advantage of this new approach is that unlike  $S_{EH}$ , the EHP functional  $S_{EHP}$  (and, consequently, the notion of Einstein manifold) makes sense relative to any geometry. In fact, recall that tensor geometries on  $M$  are equivalent to reductions on the structural group of  $FM$ , while connections compatible with tensors are equivalent to Cartan connections for the corresponding group reductions. The difference from the previous problematic approach is that, while  $S_{EH}$  involves scalar curvature (which makes canonical sense only for particular tensorial geometries),  $S_{EHP}$  contains only the curvature form, which can be defined for arbitrary Cartan connections. Not only this, EHP theories can be defined in a purely algebraic sense: given an  $R$ -algebra  $A$  endowed with a vector decomposition  $A \simeq A_0 \oplus A_1$ , we can define a *reductive  $A$ -connection* in a bundle  $P \rightarrow M$  as a pair of 1-forms  $e : TP \rightarrow A_0$  and  $\omega : TP \rightarrow A_1$ . The corresponding  *$A$ -valued EHP theory* ( $A$ -EHP) is given by the functional<sup>3</sup>

$$S_{EHP}[e, \omega] = \int_M (\wedge_*^{n-2} e \wedge_* \Omega + \frac{\Lambda}{(n-1)!} \wedge_*^n e), \quad (1.5)$$

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<sup>2</sup>In this paper we will work with many different types of wedge products, satisfying very different properties. Therefore, in order to avoid confusion, we will not follow the literature, but introduce specific symbols for each of them.

<sup>3</sup>In the way it is written, it only makes sense when the algebra  $\Lambda(P; A)$  satisfies some associativity-like polynomial identity. We will see, nonetheless, that it is possible to define  $S_{EHP}$  for arbitrary algebras.

where  $\wedge_*$  is the wedge product induced by the multiplication  $*$ :  $A \otimes A \rightarrow A$  and  $\Omega = d\omega + \omega \wedge_* \omega$  is the *field strength* (or *curvature* of  $\omega$ ). An *A-valued Einstein manifold* is then defined as a manifold  $M$  endowed with an *A-connection*  $A = e + \omega$  fulfilling the Euler-Lagrange equations of (1.5). This definition can be enlarged even more to include graded geometries, where  $A$  is a graded algebra and  $P \rightarrow M$  is a graded bundle over a non-graded manifold.

Since EHP theories make sense in arbitrary geometries defined by abstract algebras, we can ask if for each of those geometries the corresponding theory is nontrivial in some sense, implying the existence of *A-valued Einstein manifolds*. The minimal requirement is, of course, that the functional must be non-null (recall that the symplectic version of  $S_{EH}$  vanishes identically, so that it is expected that a similar situation will happen in EHP). In this article we give general obstruction results for the minimal nontriviality of *A-EHP* (and, therefore, for the existence of *A-valued Einstein manifolds*) in terms of properties of the underlying algebra. We will prove two main theorems, which are different in nature. The first of them applies to concrete and abstract EHP functionals and gives obstructions on the dimension of the base manifold, while the second applies only to concrete geometries, but gives obstructions on the dimension and on the topology of the base manifold.

**Theorem A.** *Let  $P \rightarrow M$  be a graded bundle over a  $n$ -dimensional manifold and  $A \simeq \oplus_m A^m$  a graded  $R$ -algebra endowed with a  $R$ -module decomposition  $A_0 \oplus A_1$  such that each  $A_0 \cap A^m$  is a weak  $(k_m, s_m)$ -solv submodule of  $A_0$ . Let  $(k, s)$  be  $\min_m(k_m, s_m)$ . If  $n \geq k + s + 1$ , then the *A-EHP* action functional is homogeneous (i.e.,  $\Lambda = 0$ ). If  $n \geq k + s + 3$ , then it is trivial.*

**Theorem B.** *Let  $M$  be an  $n$ -dimensional Berger manifold endowed with an  $H$ -structure. If there are natural numbers  $k_1 + k_r = n$  such that  $\mathfrak{h} \subset \mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r)$  properly, then the torsionless EHP functional is nontrivial only if  $n = 2, 4$  and  $M$  is Kähler. In particular, if  $M$  is compact and  $H^2(M; \mathbb{C}) = 0$ , then it must be a K3-surface.*

This paper is organized as follows: in Section 2 we study the solvability conditions appearing in Theorem A and this theorem is stated in a more general context and proved. In Section 3 we recall some facts concerning algebra extensions and Theorem B is stated also in a more general context and proved. In Section 4 many examples are given, where by an example we mean a specific algebraic context fulfilling the solvability hypothesis of Theorem A, so that the corresponding EHP theory cannot be realized in the underlying geometry.

## 2. Theorem A

Let  $(A, *)$  be an  $R$ -algebra, where  $R$  is a commutative ring. Given a smooth manifold  $P$ , let  $\Lambda(P; A)$  denote the  $\mathbb{N}$ -graded algebra of *A-valued forms* on  $P$  with the wedge product  $\wedge_*$  induced by  $*$ . An *A-valued reductive connection* (or simply *A-connection*) on a smooth bundle  $P \rightarrow M$ , relative to a  $R$ -module decomposition  $A \simeq A_0 \oplus A_1$ , is a pair  $\nabla = e + \omega$  of 1-forms  $e : TP \rightarrow A_0$  and

$\omega : TP \rightarrow A_1$ . Therefore, the space of  $A$ -connections is given by

$$\text{Conn}_A(P) = \Lambda^1(P; A_0) \oplus \Lambda^1(P; A_1).$$

The *pure field strength* (or *curvature*) and the *torsion* of an  $A$ -connection  $\nabla$  are respectively defined as the  $A$ -valued 2-forms  $\Omega = d\omega + \omega \wedge_* \omega$  and  $\Theta = de + \omega \wedge_* e$ . If  $(A, *)$  is associative, then given  $\Lambda \in \mathbb{R}$  there is no ambiguity in considering the *Hilbert-Palatini form*

$$\alpha_{n,\Lambda} = \wedge_*^{n-2} e \wedge_* \Omega + \frac{\Lambda}{(n-1)!} \wedge_*^n.$$

In this case, fixed any FABS  $j$  and any trace transformation  $\text{tr}$  (see Appendix, p. 17) we have a natural map  $\text{tr} \circ j : \Lambda(P; A) \rightarrow \Lambda(M; \mathbb{R})$  preserving the  $\mathbb{N}$ -grading, allowing us to define the  *$A$ -valued EHP action functional* in  $P \rightarrow M$  as the functional (if  $\Lambda = 0$  we say that this is the homogeneous  $A$ -valued EHP functional)

$$S_{n,\Lambda} : \text{Conn}_A(P) \rightarrow \mathbb{R} \quad \text{such that} \quad S_{n,\Lambda}[e, \omega] = \int_M \text{tr}(j(\alpha_{n,\Lambda})).$$

But, if  $(A, *)$  does not satisfy associative-like polynomial identities (PIs), in order to define Hilbert-Palatini forms and, consequently,  $A$ -EHP theories, we need to work with  $A$  up to those associative-like PIs. More precisely, let us consider the associativity-like polynomials

$$(x_1 \cdot \dots \cdot (x_{s-2} \cdot (x_{s-1} \cdot x_s))), \quad (((x_1 \cdot x_2) \cdot (x_3 \cdot x_4)) \cdot \dots \cdot x_s), \quad \text{etc.},$$

Due to the structure of the Hilbert-Palatini forms, we are interested in the more specific polynomials

$$(x \cdot \dots \cdot (x \cdot (x \cdot y))), \quad \text{etc.} \quad \text{and} \quad (x \cdot \dots \cdot (x \cdot (x \cdot x))), \quad \text{etc.} \quad (2.1)$$

When evaluated at  $\Lambda(P; A)$ , the set of these polynomials generates an ideal  $\mathfrak{I}(P, A)$  and we can take the quotient  $\Lambda(P; A)/\mathfrak{I}(P, A)$ , whose quotient map we denote by  $\pi$ . Fixing a FABS invariant under the ideal  $\mathfrak{I}(P, A)$  (see Appendix, p. 17) we can now consider  $S_{n,\Lambda}$  for arbitrary algebras:

$$S_{n,\Lambda}[e, \omega] = \int_M \text{tr}(j(\pi(\alpha_{n,\Lambda}))). \quad (2.2)$$

Suppose now that (as an  $R$ -module)  $A$  is also  $\mathfrak{m}$ -graded for some abelian group  $(\mathfrak{m}, +)$ , meaning that  $A \simeq \oplus_{\mathfrak{m}} A^{\mathfrak{m}}$  with  $m \in \mathfrak{m}$ . Such a decomposition induces a decomposition in each  $A_i$ , with  $i = 0, 1$ , by  $A_i \simeq \oplus_{\mathfrak{m}} A_i^{\mathfrak{m}}$ , where  $A_i^{\mathfrak{m}} = A_i \cap A^{\mathfrak{m}}$ . Suppose that  $P \rightarrow M$  is also  $\mathfrak{m}$ -graded, in that  $TP \simeq \oplus_{\mathfrak{m}} E^{\mathfrak{m}}$  for vector bundles  $E^{\mathfrak{m}} \rightarrow M$ . In this case, the previous notions can also be generalized to the graded context. Indeed, given  $l \in \mathfrak{m}$  we define an  *$A$ -connection of degree  $l$  in  $P$*  as a pair  $\nabla = e + \omega$  of  $\mathfrak{m}$ -graded  $A$ -valued 1-forms of degree

$l$ , i.e, as a sequence of 1-forms  $e^m : E^m \rightarrow A_0^{m+l}$  and  $\omega^m : E^m \rightarrow A_1^{m+l}$ . The space of those connections is then

$$\text{Conn}_A^l(P) = \oplus_m (\Lambda^1(E^m; A_0^{m+l}) \oplus \Lambda^1(E^m; A_1^{m+l})).$$

The curvature and the torsion are defined analogously, being determined by the curvature and the torsion of each connection  $\nabla^m = e^m + \omega^m$ . As discussed in the Appendix, the notions of FABS and trace transformation can be internalized into the category of graded principal bundles (in the sense above), allowing us to define (for each graded-FABS and each graded-trace transformations) the  $\mathfrak{m}$ -graded  $A$ -EHP functional of degree  $l$  in  $P \rightarrow M$  by the same expression as (2.2), but now with domain  $\text{Conn}_A^l(P)$ .

We will restate Theorem A considering the following more general conditions on the  $\mathfrak{m}$ -graded algebra  $A$  endowed with the a  $R$ -module decomposition  $A \simeq A_0 \oplus A_1$ :

- (G1)  $A_0$  is indeed a subalgebra and each  $A_0^j = A^j \cap A_0$  is a  $(k_j, s_j)$ -weak solvable submodule of  $A_0$ .
- (G2) each  $A^j$  is a  $(k_j, s_j)$ -weak solvable submodule of  $A$ .

Before restating and proving Theorem A, let us explain what we mean by a  $(k, s)$ -weak solvable submodule of an  $R$ -algebra  $A$ . We say that an algebra  $(A, *)$  is  $(k, s)$ -nil, with  $k, s \geq 0$  if (for any smooth manifold  $P$ ) every  $A$ -valued  $k$ -form in  $P$  has nilpotency degree  $s$ , i.e, if for every  $\alpha \in \Lambda^k(P; A)$  we have  $\wedge_*^s \alpha \neq 0$ , but  $\wedge_*^{s+1} \alpha = 0$ . The  $(k, s)$ -solv algebras are those that can be decomposed as finite sums of  $(k, s)$ -nil algebras. We say that a submodule  $V \subset A$  is a  $(k, s)$ -nil submodule of  $A$  if every  $V$ -valued  $k$ -form has nilpotency degree  $s$  when regarded as a  $A$ -valued form. Similarly, we say that  $V$  is a  $(k, s)$ -solv submodule if it decomposes as a sum of  $(k, s)$ -nil submodules. When  $A = \oplus_m A^m$  is  $\mathfrak{m}$ -graded, there are other kind of submodules  $V \subset A$  that can be introduced. For instance, we say that  $V$  is *graded*  $(k, s)$ -solv if each  $V_m = V \cap A_m$  is a  $(k, s)$ -solv submodule. In any submodule  $V$  of a  $\mathfrak{m}$ -graded algebra we get a corresponding grading by  $V \simeq \oplus_m V_m$ . So, any  $V$ -valued form  $\alpha$  can be written as  $\alpha = \sum_m \alpha^m$ . We say that  $V$  is *weak*  $(k, s)$ -nilpotent if for every  $k$ -form  $\alpha$ , any polynomial  $p_{s+1}(\alpha^{m_1}, \dots, \alpha^{s+1})$  vanishes. Similarly, we say that  $V$  is *weak*  $(k, s)$ -solvable if it decomposes as a sum of weak  $(k, s)$ -nilpotent submodules.

Examples will be given in Section 4. Just to mention, if an algebra  $A$  is nilpotent (resp. solvable) of degree  $s$ , then it is  $(1, s)$ -nil (resp.  $(1, s)$ -solv). Similarly, degree  $s$  nilpotent (resp. solvable) subalgebras of any algebra are  $(k, s)$ -nil (resp.  $(k, s)$ -solv) submodules.

## Restatement and Proof

Now we can give a more precise statement for Theorem A (in the formulation below it is more general than that present in the Introduction).

**Theorem 1.** *Let  $M$  be an  $n$ -dimensional manifold,  $P \rightarrow M$  be a bundle such that  $TP$  is  $\mathfrak{m}$ -graded and  $(A, *)$  be a  $\mathfrak{m}$ -graded algebra endowed with a  $R$ -module decomposition  $A_0 \oplus A_1$ . Given  $l \geq 0$ , assume that  $A[-l]$  satisfies condition (G1) or (G2). Furthermore, let  $(k, s)$  be the minimum of  $(k_j, s_j)$ . If  $n \geq k+s+1$ , then for any compatible graded FABS and any trace transformation, the corresponding graded  $A$ -EHP of degree  $l$  equals the homogeneous ones. If  $n \geq k+s+3$ , then  $A$ -EHP is trivial.*

We will divide the proof in two cases:

1. when only the algebra  $A$  is graded;
2. when both  $A$  and  $P$  are graded and the theory may have arbitrary degree.

*Proof of first case.* The proofs considering (G1) or (G2) are very similar, so we will only explain the (G1) case. Since everything in the EHP action is linear, grading-preserving and invariant under the action of  $\mathfrak{J}(P; A)$ , it is enough to prove that  $\alpha_{n,\Lambda} = 0$  for some representative Einstein-Hilbert form in the quotient  $\Lambda(P; A)/\mathfrak{J}(P; A)$ . We choose

$$\alpha_{n,\Lambda} = \wedge_*^{n-2} e \wedge_* \Omega + \frac{\Lambda}{(n-1)!} \wedge_*^n e.$$

Under the hypothesis, we can locally write  $e = \sum_m e^m$  with  $e^m : TP \rightarrow A_0^m$ . From condition (G1) each  $A_0^m$  is a weak  $(k_m, s_m)$ -solvable subspace, so that it writes as a sum  $A_0^m = \oplus_i V_i^m$  of weak  $(k_m, s_m)$ -nilpotent subspaces, which means that we can write  $e^m = \sum_i e_i^m$  locally and, therefore,  $e = \sum_m \sum_i e_i^m$ . Consequently, for every  $l$  we have  $\wedge_*^l e = p_l(e_{i_1}^{m_1}, \dots, e_{i_l}^{m_l})$  for some polynomial of degree  $l$ . If we now consider the minimum  $(k, s)$  (over  $m$ ) of  $(k_m, s_m)$ , the fact that each  $A_0^m$  is weakly  $(k_m, s_m)$ -nilpotent then implies  $\wedge_*^{k+s+1} e = 0$ . Consequently,

$$(\wedge_*^{k+s+1} e) \wedge_* \alpha = 0 \tag{2.3}$$

for any  $A$ -valued form  $\alpha$ . In particular, for

$$\alpha = \frac{\Lambda}{(n-1)!} \wedge_*^{n-(k+s+1)} e$$

(2.3) is precisely the inhomogeneous part of  $\alpha_{\Lambda, n}$ . Therefore, for every  $n \geq k+s+1$  we have  $\alpha_{n,\Lambda} = \alpha_n$  and consequently  $\pi(\alpha_n) = \pi(\alpha_{n,\Lambda})$ , implying  $S_n[e, \omega] = S_{n,\Lambda}[e, \omega]$  for any  $A$ -valued connection, and thus  $S_n = S_{n,\Lambda}$ . On the other hand, for

$$\alpha = (\wedge_*^{n-(k+s+1)+2} e) \wedge_* \Omega$$

we see that (2.3) becomes exactly  $\alpha_n$ . Therefore, when  $n \geq k+s+3$  we have  $\alpha_n = 0$ , implying (because  $j$  is linear)  $j(\alpha_n) = 0$  and thus  $S_n = 0$ . But  $k+s+3 > k+s+1$ , so that  $S_{n,\Lambda} = 0$  too, ending this first case.

*Proof of second case.* Notice that to give a morphism  $f : A' \rightarrow A$  of degree  $l$  between two graded algebras is the same as giving a zero degree morphism



$f : A' \rightarrow A[-l]$ , where  $A[-l]$  is the graded algebra obtained by shifting  $A$ . Consequently, the obstructions of a degree  $l$  graded  $A$ -EHP are just the obstructions of degree zero graded  $A[-l]$ -EHP, so that it is enough to analyze theories of degree zero. If we assume that  $TP$  is a graded bundle, so that  $TP \simeq \oplus_m E^m$ , and if  $e$  has degree zero, then the only change in comparison to the previous “partially graded” context is that instead of decomposing  $e$  as a sum  $\sum_m e^m$ , we can now write it as a genuine direct sum  $e = \oplus_m e^m$ , with  $e^m : E^m \rightarrow A_0 \cap A^m$ , meaning that the same argumentation of the case case applies here, ending the proof.  $\square$

### 3. Theorem B

Until now we worked in full generality and we showed that action functionals defined on spaces of algebra valued forms suffer minimal obstructions (in terms of the properties of the algebra) which affect the possible dimensions of the base manifold. Here we will show that if we restrict ourselves to certain concrete situations the base manifold suffers not only those minimal obstructions, but also strong topological obstructions.

Recall that given algebras  $E$ ,  $A$  and  $H$ , we say that  $E$  is an *extension of  $A$  by  $H$*  when they fit into a short exact sequence (as  $R$ -modules):

$$0 \longrightarrow H \xrightarrow{i} E \xrightarrow{\pi} A \longrightarrow 0$$

Furthermore, the extension is called *separable* if  $\pi$  has a section, i.e, a  $R$ -linear map  $s : A \rightarrow E$  such that  $\pi \circ s = id_A$ . In this case we can use  $s$  to pushforward the product of  $A$  to  $E$ , as below:

$$E \otimes E \xrightarrow{\pi \otimes \pi} A \otimes A \xrightarrow{*} A \xrightarrow{s} E \quad (3.1)$$

$\underbrace{\hspace{10em}}_{*'}$

Concerning this new product, all that we need is the following result:

**Lemma 2.** *The algebra  $(A, *)$  is  $(k, s)$ -nil iff  $(E, *')$  is  $(k, s)$ -nil.*

*Proof.* Relative to the new product  $A$  is clearly a subalgebra of  $E$ . Therefore,  $A$  is  $(k, s)$ -nil only if  $(E, *')$  is. In order to get the reciprocal, recall that any algebra  $A$  induces a corresponding graded-algebra structure in  $\Lambda(P; A)$ . Due to the functoriality of  $\Lambda(P; -)$ , we then get the commutative diagram below, where the horizontal rows are just (3.1) for the algebras  $(E, *)$  and  $(A, *')$ , composed with the diagonal map. The commutativity of this diagram says just that  $\wedge_*^2 \alpha = \wedge_*^2 f(\alpha)$  for every  $\alpha \in \Lambda(P; A)$ . From the same construction we get, for each given  $s \geq 2$ , a commutative diagram that implies  $\wedge_*^s \alpha = \wedge_*^s f(\alpha)$ . Therefore, if  $(A, *)$  is  $(k, s)$ -nil it immediately follows that  $(E, *')$  is  $(k, s)$ -nil



horizontability and equivariance. So, for any  $\nabla = e + \omega$  the corresponding 1-form  $(\iota \circ \kappa)^*\omega \equiv \omega|_N$  is an  $H$ -connection in  $FN$  whose holonomy is contained in  $H$ . Since  $H$  is the Lie integration of a  $k$ -algebra we have

$$H \subset SO(k_1) \times \dots \times SO(k_r) \subset SO(k).$$

In particular, the holonomy of  $\omega|_N$  is contained in  $SO(k)$ , implying that  $\omega|_N$  is compatible with some Riemannian metric  $g$  in  $N$ . But, we are working with torsion-free connections, so that  $\omega|_N$  is actually the Levi-Civita connection of  $g$  and, because  $N$  is irreducible, de Rham decomposition theorem implies that there exists  $i \in 1, \dots, r$  such that  $k_i = k$  and  $k_j = 0$  for  $j \neq i$ . Without loss of generality we can take  $i = 1$ . Because  $N$  is a Berger manifold, Berger's theorem [6] applies, implying that the holonomy of  $\omega|_N$  is classified, giving conditions (B1) and (B2).  $\square$

**Corollary 4.** *Let  $M$  be a Berger  $n$ -manifold with an  $H$ -structure, where  $H$  is such that  $\mathfrak{h}$  is a  $n$ -algebra. In this case, for any FABS, the EHP theory with values into  $\mathbb{R}^n \rtimes H$  is nontrivial only if  $M$  has dimension  $n = 2, 4$  and admits a Kähler structure.*

*Proof.* The result follows from the last theorem by considering the bundle  $P \rightarrow M$  as the frame bundle  $FM \rightarrow M$  and from the fact that every orientable four-dimensional smooth manifold admits a quaternionic-Kähler structure [7, 27].  $\square$

This corollary shows how topologically restrictive it is to internalize *torsionless* EHP in geometries other than Lorentzian. Indeed, if the manifold  $M$  is compact and 2-dimensional, then it must be  $\mathbb{S}^2$ . On the other hand, in dimension  $n = 4$  compact Kähler structures exist iff the Betti numbers  $b_1(M)$  and  $b_3(M)$  are zero, so that  $\chi(M) = b_2(M) + 2$ . As a consequence, if we add the (mild) condition  $H^2(M; \mathbb{R}) \simeq 0$  on the hypothesis of Corollary 4 we conclude that  $M$  must be a K3-surface!

## 4. Examples

In this section we will give realizations of the obstruction theorems studied previously. Due to the closeness with Lorentzian EHP (which is where EHP theories are usually formulated), our focus is on concrete situations, meaning that we will be focused in  $A$ -EHP functionals for  $A$  a  $k$ -algebra or an splitting extension of a  $k$ -algebra. Even so, some abstract examples will also be given.

### Concrete Examples

Let us start by considering  $A$ -EHP for  $A$  an  $\mathbb{R}$ -algebra endowed with a distinguished subalgebra  $\overline{A} \subset A$ . We then have two vector space decompositions  $A \simeq A_0 \oplus A_1$ : one for which  $A_0 = \overline{A}$  and  $A_1 = A/\overline{A}$  and another one for the opposite. From Theorem 1 and the fact that  $k$ -algebras are  $(1, 2)$ -nil we have the following conclusions for  $n \geq 6$ :

- (E1) if  $\bar{A}$  is an ideal and  $A/\bar{A}$  is a  $k$ -algebra, then the EHP functional relative to the opposite decomposition is trivial;
- (E2) if  $\bar{A}$  is a  $k$ -subalgebra, then the EHP functional relative to the first decomposition is trivial;
- (E3) if  $A$  is  $k$ -algebra, then EHP is trivial in both decompositions.

Conditions (E2) and (E3) are immediately satisfied for  $k$ -groups, i.e., Lie subgroups of  $SO(k_1) \times \dots \times SO(k_r)$ , or, equivalently, Lie subalgebras of  $\mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_r)$ . Indeed, if  $G_0$  is a such subgroup, then the EHP theory for any associative algebra algebra  $A$  extending  $\mathfrak{g}_0$  obviously satisfies (E2), so that if  $n \geq 6$  the dual theories are trivial. On the other hand, for any subalgebra  $A_0 \subset \mathfrak{g}_0$  the EHP theory with values into  $\mathfrak{h}$  clearly satisfies (E3) so that in dimension  $n \geq 6$  both the dual and the actual EHP theory are trivial. Some obvious examples of subgroups of  $k$ -groups  $G_0 \subset SO(k)$  are given in the table below. Except for  $Spin(4) \hookrightarrow O(8)$ , which arises from the exceptional isomorphism  $Spin(4) \simeq SU(2) \times SU(2)$ , all the other inclusions appear in Berger's classification [6].

Let us focus on (E2). We can think of each element of Table 1 as included in some  $GL(k; \mathbb{R})$ , i.e., as a  $G$ -structure on a manifold and then as a geometry. We could get more examples by taking finite products of arbitrary elements in the table. In terms of geometry, this can be interpreted as follows. Recall that a distribution of dimension  $k$  on an  $n$ -manifold can be regarded as a  $G$ -structure for  $G = GL(k) \times GL(n - k)$ . Therefore, we can think of a product  $O(k) \times O(n - k)$  as a distribution of Riemannian leaves on a Riemannian manifold,  $U(k) \times U(n - k)$  as a hermitan distribution, and so on.

$(n, k)$	$(1, k)$	$(2, k)$	$(4, k)$	$(1, 7)$	$(1, 8)$
$H \subset O(n \cdot k)$	$SO(k)$	$U(k)$	$Sp(k)$	$G_2$	$Spin(4)$
	$U(k)$	$SU(k)$	$Sp(k) \cdot Sp(1)$		$Spin(7)$

Table 1: First examples of concrete geometric obstructions

Other special cases where condition (E2) applies are in the table below. In the first line,  $O(k, k)$  is the so-called *Narain group* [23], i.e., the orthogonal group of a metric with signature  $(k, k)$ , whose maximal compact subgroup is  $O(k) \times O(k)$ . Second line follows from an inclusion similar to  $U(k) \hookrightarrow O(2k)$ , first studied by Hitchin and Gualtieri [17, 15, 18], while the remaining lines are particular cases of the previous ones. The underlying flavors of geometry arose from the study of Type II gravity and Type II string theory [12, 20]. That condition (E2) applies for the second and third lines follows from the fact that by complexifying  $U(k, k) \hookrightarrow O(2k, 2k)$  we obtain  $U(k, k) \hookrightarrow O(4k; \mathbb{C})$ .

$G$	$G_0$	geometry
$O(k, k)$	$O(k) \times O(k)$	Type II
$O(2k, 2k)$	$U(k, k)$	Generalized Complex
$O(2k, 2k)$	$SU(k, k)$	Generalized Calabi-Yau
$O(2k, 2k)$	$U(k) \times U(k)$	Generalized Kähler
$O(2k, 2k)$	$SU(k) \times SU(k)$	Generalized Calabi

Table 2: More examples of concrete geometric obstructions

About these two tables, some remarks:

1. Table 1 contains any classical flavors of geometry, except symplectic and complex. The reason is that  $\mathrm{Sp}(k; \mathbb{R})$  and  $GL(k; \mathbb{C})$  are not contained in some  $O(r)$ . But this does not mean that condition (E2) is not satisfied by those geometries. Indeed, generalized complex geometry contains both of them [15], so that they actually fulfill (E2).
2. We could create a third table with “exotic  $k$ -groups”, meaning that a priori they are not related to any classical geometry, so that they describe some kind of “exotic geometry”. For instance, in [25] all Lie subgroups  $G_0 \subset O(k)$  satisfying

$$\frac{(k-3)(k-4)}{2} + 6 < \dim H < \frac{(k-1)(k-2)}{2}$$

were classified and in arbitrary dimension  $k$  there are fifteen families of them. Other exotic (rather canonical, in some sense) subgroups that we could add are maximal tori. Indeed, both  $O(2k)$  and  $U(k)$  have maximal tori, say denoted by  $T_O$  and  $T_U$ , so that the reductions  $T_O \hookrightarrow O(2k)$  and  $T_U \hookrightarrow U(k)$  will satisfy condition (E2). More examples of exotic subgroups to be added are the *point groups*, i.e.,  $H \subset \mathrm{Iso}(\mathbb{R}^k)$  fixing at least one point. Without loss of generality we can assume that this point is the origin, so that  $H \subset O(k)$ . Here we have the symmetric group of any spherically symmetric object in  $\mathbb{R}^k$ , such as regular polyhedrons and graphs embedded on  $\mathbb{S}^{k-1}$ .

3. If we are interested only in condition (E2), then the tables above can be enlarged by including embeddings of  $O(k_1) \times \dots \times O(k_r)$  into some other larger group  $\tilde{G}$ . In this condition it only matters that  $H$  is a  $k$ -group. Particularly,  $O(k)$  admits some exceptional embeddings (which arise from the classification of simple Lie algebras), as in the table below.

$k$	3	9	10	12	16
$O(k) \hookrightarrow$	$G_2$	$F_4$	$E_6$	$E_7$	$E_8$

Table 3: Exceptional embeddings of orthogonal groups.

4. Despite (E2) making sense for arbitrary real algebras  $A$  with distinguished  $k$ -subalgebra  $A_0 \subset A$ , the examples above were all for matrix

algebras underlying Lie algebras of matrix Lie groups. But, as pointed out previously, in order to get (E2) we only need that  $A_0$  be a  $k$ -algebra, meaning that **any** extension (not necessarily a matrix extension) will satisfy (E2). In particular, for every  $G_0$  in the above tables  $\mathbb{R}^k \rtimes \mathfrak{g}_0$  satisfies this condition. Recall that EHP theories with values in those algebras are immediate generalizations of Lorentzian EHP theory. Therefore, what we showed is that the EHP functional cannot be realized (in dimension  $n \geq 6$ ) in essentially all flavors of geometry other than semi-Riemannian with specific signature.

5. Condition (E2) is on the Lie algebra level. A Lie group and its universal covering have the same Lie algebra. Therefore, we can double the size of our current tables by adding the universal cover (when it exists) of each group. For instance, in Table 1 the spin groups  $\text{Spin}(4)$  and  $\text{Spin}(7)$  were added due to exceptional isomorphisms on the level of Lie groups. Now, noticing that for  $k > 2$  (in particular for  $k = 4, 7$ )  $\text{Spin}(k)$  is the universal cover of  $SO(k)$ , we can automatically add all spin groups to our table. We can also add the universal covering of the symplectic group  $Sp(k; \mathbb{R})$ , the metaplectic group.
6. Due to the above discussion, we can use Tables 1 and 2, as well as the “table of exotic  $k$ -groups”, as a source of examples for condition (E3). Indeed, we can just consider the algebra  $A$  in condition (E3) as the algebra of some  $H$  in the mentioned tables and take a fixed subalgebra  $A_0 \subset \mathfrak{g}$ . This produces a long list of examples, because  $G/G_0$  is a priori an arbitrary homogeneous space subjected only to the condition of  $G$  being a  $k$ -group (here  $G_0$  is the integrated Lie group of  $A_0$ ). In geometric terms, if a geometry fulfills condition (E2), then any induced “homogeneous geometry” fulfills (E3). On the other hand, differently from what happens with condition (E2), Table 3 *cannot* be used to get more examples of condition (E3). Indeed, if  $G_0 \hookrightarrow G$  is a reduction fulfilling (E3) and  $G \hookrightarrow \tilde{G}$  is an embedding, then  $G_0 \hookrightarrow \tilde{G}$  fulfills (E3) iff it is fulfilled by  $G \hookrightarrow \tilde{G}$  (see diagram below).

$$\begin{array}{ccccccc}
 H & \hookrightarrow & G & \hookrightarrow & O(k_1) \times \dots \times O(k_r) & \hookrightarrow & \tilde{G} \\
 & & & & \searrow & & \nearrow \\
 & & & & & & 
 \end{array}$$

Let us now analyze condition (E1). First of all we notice that it is very restrictive, because we need to work with subalgebras  $A_0 \subset A$  of  $k$ -algebras which are ideals. In terms of Lie groups this means that we need to restrict to normal subgroups  $G_0 \subset G$  such that the quotient  $G/G_0$  is a  $k$ -group. For instance, in the typical situations above,  $G_0$  is not normal. Even so, there are two dual conditions under which  $G_0 \subset G$  fulfills (E1): when  $G$  is a  $k$ -group with  $G/G_0 \subset G$  and, dually, when  $G_0$  is  $k$ -group with  $G/G_0 \subset G_0$ . However, in both situations we are also in (E3), so that there is nothing much new here.

### Cayley-Dickson Examples

Here we will show that the list of concrete examples can be extended indefinitely, allowing us to conclude that EHP theories are trivial in an infinite

number of geometries. This will be done by making use of the Cayley-Dickson construction. Let  $(A, *)$  be an arbitrary  $\mathbb{R}$ -algebra with involution  $(\overline{-}) : A \rightarrow A$ . For each fixed  $l > 0$  we get a sesquilinear map  $s : A^l \times A^l \rightarrow A$  such that  $s(x, y) = \sum_i \overline{x_i} * y_i$ . Consider the subspace of all  $l \times l$  matrices  $M \in \text{Mat}_{l \times l}(A)$  with coefficients in  $A$  which preserve  $s$ , in the sense that  $s(Mx, My) = s(x, y)$ . We say that these are the *unitary matrices in  $A$* , and we denote this set by  $U(k; A)$ . If the involution is trivial (i.e, the identity map), then call them the *orthogonal matrices in  $A$* , writing  $O(k; A)$  to denote the corresponding space.

Recall that Cayley-Dickson takes an involutive algebra  $A$  and gives another involutive algebra  $\text{CD}(A)$  with weakened PIs [29]. As an  $\mathbb{R}$ -module, the newer algebra is given by a sum  $A \oplus A$  of real and imaginary parts. We have an inclusion  $A \hookrightarrow \text{CD}(A)$ , obtained by regarding  $A$  as the real part, which induces an inclusion into the corresponding unitary groups  $U(k; A) \hookrightarrow U(k; \text{CD}(A))$ . This inclusion can be extended in the following way

$$U(k; A) \hookrightarrow U(k; \text{CD}(A)) \hookrightarrow U(2k; A),$$

defined by setting the real part and the imaginary part as diagonal block matrices. Iterating we see that for every  $k$  and every  $l$  there exists an inclusion

$$U(k; \text{CD}^l(A)) \hookrightarrow U(2^l k; A). \quad (4.1)$$

We assert that, at least when the starting algebra  $A$  is finite-dimensional, each  $U(k; \text{CD}^l(A))$  is a Lie group. Taking  $A = \mathbb{R}$  in the inclusion above we will then get a corresponding sequence of matrix Lie algebras (and, therefore, a sequence of underlying associative algebras). Consequently, each of such algebras will satisfy condition (E2), meaning that, for every  $k > 0$  and  $l \geq 0$ , the EHP theories with values into extensions of  $\mathfrak{u}(k; \text{CD}^l(A))$  are trivial.

$$U(k; \text{CD}^l(\mathbb{R})) \longrightarrow \dots \longrightarrow U(2^{l-2}k; \mathbb{H}) \longrightarrow U(2^{l-1}k; \mathbb{C}) \longrightarrow O(2^l k) .$$

The idea for proving the assertion is to reproduce the proof that the standard unitary/orthogonal groups  $U(k)$  and  $O(k)$  are Lie groups. The starting point is to notice that the involution of  $A$  induces an involution  $(-)^{\dagger}$  in  $\text{Mat}_{k \times k}(A)$ , defined by composition with the transposition map. The next step is to see that  $k \times k$  matrices preserve the involution  $s$  iff they are invertible with  $M^{-1} = M^{\dagger}$ , i.e, iff  $M \cdot M^{\dagger} = 1_k = M^{\dagger} \cdot M$ . Since  $(\overline{-}) : A \rightarrow A$  is an algebra morphism, we have the usual property  $(M \cdot N)^{\dagger} = M^{\dagger} \cdot N^{\dagger}$ , allowing us to characterize the unitary matrices as those satisfying  $M \cdot M^{\dagger} = 1_k$ . Therefore, defining the map

$$f : \text{Mat}_{k \times k}(A) \rightarrow \text{Mat}_{k \times k}(A) \quad \text{by} \quad f(M) = M \cdot M^{\dagger},$$

in order to proof that  $U(k; A)$  is a Lie group it is enough to verify that the above map is, in some sense, a submersion (which will give the smooth structure) and that the multiplication and inversion maps are smooth. When  $A$  is finite-dimensional this is immediate (although this requirement is not necessary).

Since classical geometries (meaning geometries in the sense of Weil) are described by inclusions of Lie groups, this leads us to think of (4.1) as modeling

some flavor of geometry, say the *Cayley-Dickson geometry in  $A$ , of order  $l$* . With this nomenclature we have concluded that EHP theories are trivial in real Cayley-Dickson geometries of arbitrary order, generalizing the examples in Table 1. In the following we will show that Table 2 can be generalized as well. In fact we will show that there is a notion of *generalized Cayley-Dickson geometry in  $A$*  which, for  $A = \mathbb{R}$ , contains Hitchin's generalized complex geometry and in which EHP theories are trivial. This follows directly from the fact that for every  $A$  we have the inclusion  $U(k, k; \text{CD}(A)) \subset O(2k; 2k, A)$ , so that by iteration we get

$$U(k, k; \text{CD}^l(A)) \subset O(2^l k; 2^l k, A). \quad (4.2)$$

For  $A = \mathbb{R}$  and  $l = 1$  this model generalized complex geometry, leading us to say that the inclusion above models *generalized Cayley-Dickson geometry in  $A$  of degree  $l$* . For instance, if we take  $A = \mathbb{R}$  and  $l = 2$  this becomes generalized quaternionic geometry, which is a poorly studied theory, beginning with the works [26, 13]. For  $l = 3, 4, \dots$  it should be generalized octonionic geometry, generalized sedenionic geometry, and so on. The authors are unaware of the existence of substantial works on these theories.

When tensoring inclusion (4.2) with  $\text{CD}^l(A)$  we get

$$U(k, k; \text{CD}^l(A)) \subset O(2^l k; 2^l k, \text{CD}^l(A)) \simeq O(2^l k + 2^l k; \text{CD}^l(A)),$$

so that condition (E2) above implies that, independently of the present development of generalized Cayley-Dickson geometries, EHP theory is trivial in each of them.

### Abstract Examples

Here we give abstract examples of algebras in which EHP cannot be realized. By abstract we mean that they are not necessarily  $k$ -algebras. We will restrict to the cases in which only  $A$  is  $\mathfrak{m}$ -graded. From Theorem 1 we have that:

- (F1) when  $A$  admits a vector space decomposition  $A \simeq A_0 \oplus A_1$ , where  $A_0$  is a subalgebra such that each  $A_0^m = A_0 \cap A^m$  is a weak  $(k_m, s_m)$ -solvable subspace, then  $A$ -EHP is trivial in dimension  $n \geq k + s + 3$ , where  $(k, s)$  is the minimum over  $(k_m, s_m)$ .

The most basic examples are those for  $\mathfrak{m} = 0$  and  $A_1 = 0$ , i.e, the non-graded setting with  $A$  itself  $(k, s)$ -solv. For instance, any Lie algebra is  $(k, 1)$ -nil for any even  $k$ . As a consequence, *in dimension  $n \geq 4$ , any Lie algebra valued EHP functional is trivial*. In particular, EHP with values in the Poincaré group  $\mathfrak{iso}(n-1, 1)$  is trivial. We emphasize that this **does not** mean that the standard Lorentzian EHP theory is trivial (which would be absurd). Indeed, while Lorentzian EHP theory and  $\mathfrak{iso}(n-1, 1)$ -valued EHP theory take values in the same **vector space**, and their action functionals have the same shape, the **algebras** (and, therefore, their properties) used to define the corresponding wedge product are *totally different*.



If we now allow  $\mathfrak{m}$  to be nontrivial, but with  $A_1 = 0$ , then condition (F1) is satisfied if each  $A^m$  is a weak  $(k_m, s_m)$ -solvable subspace. In particular, it remains satisfied if  $A$  is itself  $(k, s)$ -solv. It happens that not only Lie algebras are solv  $(2, 1)$ -nil, but also graded Lie algebras. Consequently, EHP theories are also trivial in the domain of graded Lie algebras. One can generalize even more thinking of EHP theories with values in Lie superalgebras and in graded Lie superalgebras. Indeed, recall that a  $\mathfrak{m}$ -graded Lie superalgebra  $\mathfrak{g}$  is just a  $\mathfrak{m}$ -graded Lie algebra whose underlying PIs (i.e, skew-commutativity and Jacobi identity) hold in the graded sense. Particularly, this means that the  $\mathbb{Z}_2$ -grading writes  $\mathfrak{g} \simeq \mathfrak{g}^0 \oplus \mathfrak{g}^1$ , with  $\mathfrak{g}^0$  a  $\mathfrak{m}$ -graded Lie algebra and, therefore,  $(2, 1)$ -nil. It then follows that  $\mathfrak{g}$  satisfies condition (F1). Summarizing, in dimension  $n \geq 6$ , *the EHP functional cannot be realized in any Lie algebraic context.*

Graded Lie superalgebras are the first examples of algebras satisfying (F1) with  $A_1 \neq 0$ , but they are far from being the only one. Indeed, when we look at a decomposition  $A \simeq A_0 \oplus A_1$ , where  $A_0$  is a subalgebra, it is inevitable to think of  $A$  as an extension of  $A_1$  by  $A_0$ , meaning that we have an exact sequence as shown below. If  $A_0$  is  $(k, s)$ -solv, then (F1) holds. This can be interpreted as follows: suppose that we encountered an algebra  $A_1$  such that EHP is not trivial there. So, EHP will be trivial in any (splitting) extension of  $A_1$  by a  $(k, s)$ -solv algebra.

$$0 \longrightarrow A_0 \longrightarrow A \longrightarrow A_1 \longrightarrow 0$$

In particular, because  $(\mathbb{R}^k, +)$  is abelian and, therefore,  $(1, 1)$ -nil, any algebra extension by  $\mathbb{R}^k$  will produce a context in which EHP is trivial. Recall the previous concrete examples, in which we considered splitting extensions of matrix algebras by  $\mathbb{R}^k$ . This would seem to suggest that every concrete EHP theory is trivial, which is not true. Once again the problem is in the wedge products: we have the same vector space structures, but in the concrete contexts and here the way of multiplying forms with values in those spaces are different. The main difference is that the product used there does not take the abelian structure of  $\mathbb{R}^k$  into account.

Another example about extensions is as follows: the paragraph above shows that abelian extensions of nontrivial EHP are trivial, but what about super extensions? They remain trivial. Indeed, being a Lie superalgebra, the translational superalgebra  $\mathbb{R}^{k|l}$  (the cartesian superspace, regarded as a Lie superalgebra) is  $(2, 1)$ -nil, so that any algebra extension by it is trivial. So, if we think on supergravity as a realization of gravity not in the domain of Cartan connections but in the domain of super Cartan connections on the super Poincaré Lie algebra  $\mathfrak{siso}(n-1, 1)$  we find that EHP functional is not a good model, because it will be trivial if  $n \geq 6$ .

Finally, there are also purely abstract examples. Just to mention, in [30] the authors build “mathematically exotic” examples of nilpotent algebras, which fulfill (F1). In [5] necessary and sufficient conditions were given under which arbitrary orthogonal groups  $O(q, \mathbb{K})$ , where  $q : V \rightarrow \mathbb{K}$  is a positive-definite quadratic form on a finite-dimension  $\mathbb{K}$ -space  $q$  and  $\text{ch}(\mathbb{K}) \neq 2$ , admit an em-

bedded maximal torus  $\mathbf{T}(q; \mathbb{K})$ . Independently of the quadratic space  $(V, q)$ , the corresponding orthogonal group is a Lie group and, as in the real case, the algebra  $\mathfrak{o}(q; \mathbb{K})$  is  $(2, 1)$ -nil. Therefore, under the conditions of [5], the EHP functional will be trivial in those toroidal geometries.

## 5. Appendix

Here we will introduce a general tool that allows us to pushforward algebra-valued forms on the total space  $P$  of a bundle  $P \rightarrow M$  to the base manifold  $M$ . The first step is to build some process allowing us to replace  $A$ -valued forms in  $P$  by forms in  $M$  with coefficients in some other bundle, say  $E_A$ . It is more convenient to think of this in categorical terms. Let  $\mathbf{Alg}_R$  be the category of  $R$ -algebras,  $\mathbb{Z}\mathbf{Alg}_R$  be the category of  $\mathbb{Z}$ -graded  $R$ -algebras and, given a manifold  $M$ , let  $\mathbf{Bun}_M$  and  $\mathbf{Alg}_R\mathbf{Bun}_M$  denote the categories of bundles and of  $R$ -algebra bundles over  $M$ , respectively. A *functorial algebra bundle system* (FABS) consists of

1. a subcategory  $\mathbf{C} \subset \mathbf{Bun}_M \times \mathbb{Z}\mathbf{Alg}_R$ ;
2. a functor  $E_-$  assigning to any algebra  $A \in \mathbf{C}$  a corresponding algebra bundle  $E_A \rightarrow M$  whose typical fiber is  $A$ ;
3. a functor  $S(-; -)$  that associates an algebra to each pair  $(P, A) \in \mathbf{C}$ ;
4. natural transformations  $\iota : S(-; -) \Rightarrow \Lambda(-; -)$  and  $\xi : S(-; -) \Rightarrow \Lambda(M; -)$  such that  $\iota$  is objectwise injective.

Now that we know how to replace  $A$ -valued forms in  $P$  with  $E_A$ -valued forms in  $M$ , let us see how to transfer the latter to classical  $\mathbb{R}$ -valued forms in  $M$ . This is done by taking some “trace”. A *trace* compatible with a FABS is given by a functor  $\text{tr} : \mathbb{Z}\mathbf{Alg}_R \rightarrow \mathbb{Z}\mathbf{Alg}_{\mathbb{R}}$  together with a natural transformation  $\tau$  between  $\text{tr} \circ \Lambda(M; E_-)$  and the constant functor at  $\Lambda(M; \mathbb{R})$ . All this data fits in the following diagram:

$$\begin{array}{ccc}
 \mathbf{Alg}_{\mathbb{R}} & \overset{E_-}{\dashrightarrow} & \mathbf{Alg}_R\mathbf{Bun}_M \\
 \uparrow \pi_1 & \begin{array}{c} \dashrightarrow \\ \dashrightarrow \\ \dashrightarrow \end{array} S(-, -) & \begin{array}{c} \dashrightarrow \\ \dashrightarrow \\ \dashrightarrow \end{array} \xi \\
 \mathbf{C} & \dashrightarrow & \mathbb{Z}\mathbf{Alg}_R \\
 & \searrow \Lambda(-; -) & \downarrow \Lambda(M; -)
 \end{array}$$

Such systems exist, as shown in the next example. The fundamental properties and constructions involving FABS will appear in a work under preparation [22].

**Example 5 (standard case).** Let  $\mathbf{C}$  be composed by pairs  $(P, A)$ , where  $P \rightarrow M$  is a  $G$ -bundle whose group  $G$  becomes endowed with a representation  $\rho : G \rightarrow GL(A)$ . In that case we define  $E_-$  as the rule assigning to each  $A$  the corresponding associated bundle  $P \times_{\rho} A$ . The functor  $S$  is such that  $S(P, A)$  is the algebra  $\Lambda_{\rho}(P; A)$  of  $\rho$ -equivariant  $A$ -valued forms  $\alpha$  in  $P$ , i.e, of those

which satisfy the equation  $R_g^* \alpha = \rho(g^{-1}) \cdot \alpha$ , where  $R : G \times P \rightarrow P$  denotes the canonical free action characterizing  $P$  as a principal  $G$ -bundle. This algebra of  $\rho$ -equivariant forms naturally embeds into  $\Lambda(P; A)$ , giving  $\iota$ . Finally, it is a standard fact [21] that each  $\rho$ -equivariant  $A$ -valued form on  $P$  induces an  $P \times_\rho A$ -valued form on  $M$ , defining the transformation  $\xi$ . This is the standard approach used in the literature, so that we will refer to it as the *standard FABS*.

In order to define tgh e EHP action functional we need to consider invariant FABS. Let us introduce them. Indeed, we say that a FABS on  $\mathbf{C}$  compatible with a trace  $(\text{tr}, \tau)$  is *invariant* under a functor  $I : \mathbf{C} \rightarrow \mathbb{Z}\mathbf{Alg}_R$  if

- (a) for all  $(P, A) \in \mathbf{C}$  the corresponding  $I(P, A)$  is an ideal of  $S(P; A)$ , so that we can take the quotient functor  $S/I$  and we have a natural transformation  $\pi : S \Rightarrow S/I$ ;
- (b) there exists another  $J : \mathbf{C} \rightarrow \mathbb{Z}\mathbf{Alg}_R$  such that  $J(P, A)$  is an ideal of  $\Lambda(M; E_A)$  and whose projection we denote by  $\pi'$ ;
- (c) there exists a natural transformation  $j' : S/I \Rightarrow \Lambda(M; -)/J$  such that  $j' \circ \pi = \pi' \circ j$ , i.e, the diagram below commutes for every  $(P, A)$ ;

$$\begin{array}{ccc}
 S(P, A) & \xrightarrow{J(P, A)} & \Lambda(M; E_A) \\
 \pi_{(P, A)} \downarrow & & \downarrow \pi'_{(P, A)} \\
 S(P, A)/I(P, A) & \xrightarrow{J'(P, A)} & \Lambda(M; E_A)/J(P, A)
 \end{array} \tag{5.1}$$

- (d) the transformation  $\tau$  passes to the quotient by  $\tau \circ I$ .

Finally, notice that all these definitions make perfect sense in the more general category  $\mathbf{mAlg}_R$  of  $\mathbf{m}$ -graded  $R$ -algebras.

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