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Estimation of surface area

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Abstract

We study the problem of estimating the surface area of the boundary ∂S of a sufficiently smooth set $S \subset \mathbb{R}^d$ when the available information is only a finite subset $\mathbb{X}_n \subset S$. We propose two estimators: the first makes use of the Devroye–Wise support estimator. It is based on Crofton's formula, which, roughly speaking, states that the $d-1$ dimensional surface area of a smooth enough set is the mean number of intersections of randomly chosen lines. For that purpose we propose an estimator of the number of intersections of such lines with support based on the Devroye–Wise support estimators. The second surface area estimator makes use of the α-convex hull of $\mathbb{X}_n$, denoted by $C_\alpha(\mathbb{X}_n)$. More precisely, it is the $(d-1)$-dimensional surface area of $C_\alpha(\mathbb{X}_n)$, denoted by $|C_\alpha(\mathbb{X}_n)|_{d-1}$, which is proven to converge to the $(d-1)$-dimensional surface area of ∂S. Moreover, $|C_\alpha(\mathbb{X}_n)|_{d-1}$ can be computed using Crofton's formula.

Our results depend on the Hausdorff distance between S and $\mathbb{X}_n$ for the Devroye–Wise estimator, and the Hausdorff distance between ∂S and $\partial C_\alpha(\mathbb{X}_n)$ for the second estimator.

1 Introduction

1.1 On surface area and length estimation

The estimation of surface areas has been extensively considered in stereology (see, for instance, [8, 9] and [27]). It has also been studied as a further step in the theory of nonparametric set estimation (see [32]), and has practical applications in medical imaging (see [19]). In addition, the estimation of a surface area is widely used in magnetic resonance imagining techniques, see [28].

The three- and two-dimensional cases are addressed in [10], which proposed parametric estimators when the available data are the distances to S, from a sample outside the set, but at a distance smaller than a given $R > 0$.

The 2-dimensional case has many important applications. This is also true of the three dimensional case. For instance, surface area is an important biological parameter in organs such as the lungs, see for instance [40]. The higher

dimensional study is also important, at least from a theoretical point of view, since in [36], it is shown that the boundary surface plays an important role as a parameter of a probability distribution, which allows applying plug-in methods. Up to our knowledge, the only paper that tackles the surface area estimation problem in any dimension, when only "inside" data are available, is [22], and no convergence rates are given.

When, as in image analysis, one can observe n data points from two distinguishable sets of random data-points (one from inside S and the other from outside S), the estimation of the surface area of the boundary has been treated, for any $d \geq 2$, in [19, 21, 29, 32] and [41]. The proposals given in [19, 32] and [21] aim to estimate the Minkowski content of ∂S. In [21] a very general convergence result is obtained, and in [19] a convergence rate of order $n^{-1/2d}$ is obtained under some mild hypotheses, and later on, in [32], a convergence rate of order $(\log(n)/n)^{1/(d+1)}$ is achieved, under stronger assumptions. In [29] a very nice fully data-driven method, based on the Delaunay triangulation, is proposed under an homogeneous point process sampling scheme. The asymptotic rate of convergence of the variance is given, but there is no global convergence rate because no result is obtained for the bias. Lastly, in [41], a parameter-free procedure, based on the Voronoi triangulation, is proposed, and a rate of convergence of order $\lambda^{-1/d}$ is obtained, under a Poisson Point Process (PPP) sampling scheme (where λ is the intensity of the PPP).

1.2 Roadmap

When $S \subset \mathbb{R}^d$ is a compact set, we aim to estimate its surface area, i.e. the $(d-1)$-Hausdorff measure of its boundary ∂S.

We propose two surface area estimators, at any finite dimension, when the available data is only a finite set $\mathbb{X}_n \subset S$. In this setting the two dimensional case has been mostly studied. Assuming that $\mathbb{X}_n$ is an iid sample, the convex case was first addressed in [11] (using Crofton's formula). Later on, under the α-convexity assumption, [5] obtained the convergence of the α-shape's perimeter to the perimeter of the support and the associated convergence rates are derived. When the data are given by a trajectory from a reflected Brownian motion, (with or without drift), a consistency result is obtained in Theorem 4 in [13].

Crofton's formula, proved in 1868 for convex subsets of $\mathbb{R}^2$, and extended to arbitrary dimensions (see [39]), says that the surface area of ∂S equals the integral of the number of intersections with ∂S of lines in $\mathbb{R}^d$ (see Equations (3) and (4) for explicit versions of Crofton's formula for $d = 2$ and $d \geq 2$, respectively).

As previously announced, we will propose two different estimators for the surface area. One of them uses the Devroye–Wise support estimator

$$\hat{S}_{\varepsilon_n}(\mathbb{X}_n) = \cup_{i=1}^n \mathcal{B}(X_i, \varepsilon_n) \tag{1}$$

see [23], and the other one uses the α-convex hull support estimator

$$C_\alpha(\mathbb{X}_n) = \bigcap_{\{x : d(x, \mathbb{X}_n) \geq \alpha\}} \mathring{\mathcal{B}}(x, \alpha)^c \tag{2}$$

see [37], where n is the cardinality of $\mathbb{X}_n$, $\varepsilon_n \to 0$ as $n \to \infty$ and $\mathring{\mathcal{B}}(x, \alpha)^c$ denotes the complement of the open ball in $\mathbb{R}^d$ centred at x, of radius $\alpha > 0$.

The surface area estimator we propose based on Devroye–Wise support estimator is not just a plug-in method, because in general the number of intersections of a line with $\partial\hat{S}_{\varepsilon_n}(\mathbb{X}_n)$ may not converge to the number of intersections of that line with ∂S. We prove that this estimator converges at a rate proportional to $d_H(\mathbb{X}_n, S)^{1/2}$ (where d_H denotes the Hausdorff distance). This rate can be improved to $d_H(\mathbb{X}_n, S)$ when adding a reasonable assumption on the shape of ∂S. These rates are known when $\mathbb{X}_n$ is an iid sample, see Corollary 2.

To estimate the surface area when the support estimator is $C_\alpha(\mathbb{X}_n)$, we first extend the results in [20]. More precisely, we prove that, in any dimension, the surface area of the hull's boundary, i.e. $|\partial C_\alpha(\mathbb{X}_n)|_{d-1}$, converges to $|\partial S|_{d-1}$. This result is interesting in itself, but, in practice, it is difficult to compute $|\partial C_\alpha(\mathbb{X}_n)|_{d-1}$, especially for dimension $d > 2$. However, we will see that, by means of Crofton's formula, it can be easily estimated, via the Monte Carlo method.

These results can be applied to many deterministic or random situations, to obtain explicit convergence rates. We focus on two random situations: the case $\mathbb{X}_n = \{X_1, \ldots, X_n\}$ of iid drawn on S (with a density bounded from below by a positive constant), and the case of random trajectories of reflected diffusions on S. In particular, we provide convergence rates when the trajectory is the result of a reflected Brownian motion (see [13, 14]). This last setting has several applications in ecology, where the trajectory is obtained by recording the location of an animal (or several animals) living in an area S, which is called its home range (the territorial range of the animal), and X_t represent the position at time t transmitted by the instrument (see for instance [7, 13, 14], and references therein).

The rate of convergence of the surface area estimator, based on $\hat{S}_{\varepsilon_n}(\mathbb{X}_n)$, when $\mathbb{X}_n$ is an iid sample, is of order $n^{-1/2d}$, which can be improved to $n^{-1/d}$, depending on the assumptions on the smoothness of ∂S.

With the estimation of the support that uses the α-convex hull, when $\mathbb{X}_n$ is an iid sample, we obtain a rate of order $n^{-2/(d+1)}$.

The rest of this paper is organized as follows.

In Section 2, we introduce: 1) the notation. 2) Crofton's formula, first for dimension two and then for the general case. 3) The main geometric restrictions required in one of the main theorems. Section 3 introduces the surface area estimator based on Devroye–Wise support estimator. Main results regarding this estimator are stated in subsection 3.3. The computational aspects of this estimator are studied in subsection 3.4. The approach based on the α-convex hull is introduced in Section 4. A discussion of the rates of convergence is given in Section 5. An algorithm based on the Monte Carlo method for the estimator based on the α-hull is introduced in Section 6. Lastly, Section 7 treates the performance of our estimator in a simulation study. All proofs are deferred to the Appendix.

2 Background

2.1 Notations

Given a set $S \subset \mathbb{R}^d$, we denote by $\mathring{S}$, $\overline{S}$ and ∂S the interior, closure and boundary of S, respectively, with respect to the usual topology of $\mathbb{R}^d$. We

also write $\mathrm{diam}(S) = \sup_{(x,y)\in S\times S} ||x-y||$. The parallel set of S of radius ε is $B(S,\varepsilon) = \{y \in \mathbb{R}^d : \inf_{x\in S} \|y-x\| \leq \varepsilon\}$.

If $A \subset \mathbb{R}^d$ is a Borel set, then $|A|_d$ denotes its d-dimensional Lebesgue measure. When $A \subset \mathbb{R}^d$ is a $(d-1)$-dimensional manifold, then $|A|_{d-1}$ denotes its $(d-1)$-Hausdorff measure.

We denote by $\mathcal{B}(x,\varepsilon)$ the closed ball in $\mathbb{R}^d$, of radius ε, centred at x, and $\omega_d = |\mathcal{B}_d(x,1)|_d$. Given two compact non-empty sets $A, C \subset \mathbb{R}^d$, the Hausdorff distance between A and C is defined by

$$d_H(A,C) = \inf\{\varepsilon > 0 : \text{such that } A \subset B(C,\varepsilon) \text{ and } C \subset B(A,\varepsilon)\}.$$

The $(d-1)$-dimensional sphere in $\mathbb{R}^d$ is denoted by $\mathcal{S}^{d-1}$, while the half-sphere in $\mathbb{R}^d$ is denoted by $(\mathcal{S}^+)^{d-1}$, i.e. $(\mathcal{S}^+)^{d-1} = (\mathbb{R}^{d-1} \times \mathbb{R}^+) \cap \mathcal{S}^{d-1}$. Given M a sufficiently smooth $(d-1)$-manifold and $x \in M$, the affine tangent space of M at x is denoted by T_xM. When $S \subset \mathbb{R}^d$ is regular (i.e. compact and satisfying $S = \overline{\mathring{S}}$), and has a $\mathcal{C}^1$ regular boundary ∂S, then for any $x \in \partial S$ we can define η_x the outward normal unit vector at x, that is, the unit vector of $(T_x\partial S)^\perp$ such that, for $t > 0$ small enough, $x + t\eta_x \in S^c$.

Given a vector $\theta \in (\mathcal{S}^+)^{d-1}$ and a point y, $r_{\theta,y}$ denotes the line $\{y + \lambda\theta, \lambda \in \mathbb{R}\} = y + \mathbb{R}\theta$. If y_1 and y_2 are two points in $r_{\theta,y}$, then $y_i = y + \lambda_i\theta$; with a slight abuse of notation, we write $y_1 < y_2$ when $\lambda_1 < \lambda_2$.

2.2 Crofton's formula

In 1868, Crofton proved the following result (see [17]): given a convex set in the plane, whose boundary is denoted by γ, then its length $|\gamma|_1$ can be computed by

$$|\gamma|_1 = \frac{1}{2}\int_{\theta=0}^{\pi}\int_{p=-\infty}^{+\infty} n_\gamma(\theta,p)dpd\theta, \tag{3}$$

$n_\gamma(\theta,p)$ being the number of intersections of γ with the line $r_{\theta^*,\theta p}$, where $\theta^* \in (\mathcal{S}^+)^1$ is orthogonal to θ, and $dpd\theta$ is the 2-dimensional Lebesgue measure, see Figure 1. This result has been generalized to compact (not necessarily convex) sets in $\mathbb{R}^d$ for any $d > 2$, and also to Lie groups, see [39].

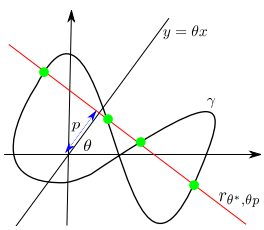

Figure 1: The function n_γ counts the number of intersections of γ with the line $r_{\theta^*,\theta p}$ determined by θ and p.

To introduce the general Crofton formula in $\mathbb{R}^d$ for a compact $(d-1)$-dimensional manifold M, let us define first the constant

$$\beta(d) = \Gamma(d/2)\Gamma((d+1)/2)^{-1}\pi^{-1/2},$$

where Γ stands for the well known Gamma function. Let $\theta \in (\mathcal{S}^+)^{d-1}$. Then, θ determines a $(d-1)$-dimensional linear space $\theta^\perp = \{v : \langle v, \theta \rangle = 0\}$. Given $y \in \theta^\perp$, let us write $n_M(\theta, y) = \#(r_{\theta,y} \cap M)$, where $\#$ is the cardinality of the set, see Figure 2.

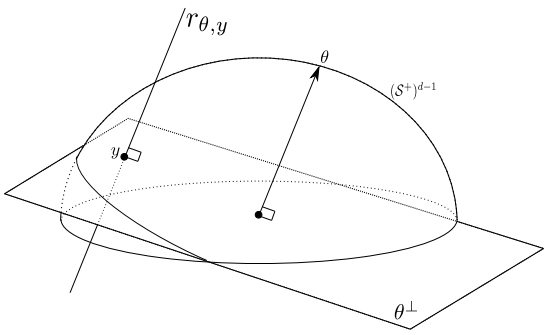

Figure 2: The line $r_{\theta,y} = y + \mathbb{R}\theta$ is shown, where $y \in \theta^\perp$ and $\theta \in (\mathcal{S}^+)^{d-1}$.

It is proved in [26] (see Theorem 3.2.26) that if M is a $(d-1)$-dimensional rectifiable set, then the integralgeometric measure of M (which will be denoted by $I_{d-1}(M)$, and is defined by the right-hand side of (4)) equals its $(d-1)$-dimensional Hausdorff measure, i.e.

$$|M|_{d-1} = I_{d-1}(M) = \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \theta^\perp} n_M(\theta, y) d\mu_{d-1}(y) d\theta. \tag{4}$$

The measure $d\theta$ is the uniform measure on $(\mathcal{S}^+)^{d-1}$ (with total mass 1) and μ_{d-1} the $d-1$ dimensional Lebesgue measure.

2.3 Restrictions on the shape

We will now recall some well-known restrictions that are put on the shape in set estimation.

Definition 2.1. For $\alpha > 0$, a set $S \subset \mathbb{R}^d$ is said to be α-convex if $S = C_\alpha(S)$, where $C_\alpha(S)$ is the α-convex hull of S, defined in (2), replacing $\mathbb{X}_n$ by S.

When S is α-convex, a natural estimator of S from a random sample $\mathbb{X}_n$ of points (drawn from a distribution with support S), is $C_\alpha(\mathbb{X}_n)$, see [37].

Definition 2.2. A set $S \subset \mathbb{R}^d$ is said to satisfy the outside α-rolling condition if for each boundary point $s \in \partial S$ there exists an $x \in S^c$ such that $\mathcal{B}(x, \alpha) \cap \partial S = \{s\}$. A compact set S is said to satisfy the inside α-rolling condition if $\overline{S^c}$ satisfies the outside α-rolling condition.

Following the notation in [25], let $\mathrm{Unp}(S)$ be the set of points $x \in \mathbb{R}^d$ with a unique projection on S.

Definition 2.3. For $x \in S$, let $reach(S, x) = \sup\{r > 0 : \mathring{\mathcal{B}}(x, r) \subset \mathrm{Unp}(S)\}$. The reach of S is defined by $reach(S) = \inf\left\{reach(S, x) : x \in S\right\}$, while S is of positive reach if $reach(S) > 0$.

Remark 1. *Throughout this paper we assume that ∂S is the boundary of a compact set $S \subset \mathbb{R}^d$ such that $S = \overline{\mathring{S}}$. We also assume that S fulfills the outside and inside α-rolling conditions, and then ∂S is rectifiable (see Theorem 1 in [42]). From this it follows that $I_{d-1}(\partial S) = |\partial S|_{d-1} < \infty$, which implies (by (4)) that, except for a set of measure zero with respect to $d\mu_{d-1}(y)d\theta$, any line $r_{\theta,y}$ meets ∂S a finite number of times: $n_{\partial S}(\theta, y) < \infty$. From Theorem 1 in [42], it also follows that ∂S is a $\mathcal{C}^1$ manifold, which allows us for each $x \in \partial S$, to define its unit outward normal vector η_x.*

For the estimator of the surface area based on the Devroye–Wise estimator we will assume that ∂S satisfies a technical hypothesis, referred to as (C, ε_0)-regularity.

Definition 2.4. Let $E_\theta(\partial S) = \{x \in \partial S, \langle \eta_x, \theta \rangle = 0\}$. The image of $E_\theta(\partial S)$ by the orthogonal projection onto $\theta^\perp$ is denoted by $F_\theta = \pi_{\theta^\perp}(E_\theta(\partial S))$ (which for non-degenerate cases is a $(d-2)$-dimensional submanifold of $\theta^\perp$). We also denote by $B(F_\theta, \varepsilon)$ its parallel set of radius ε.

Define, for $\varepsilon > 0$,

$$\varphi_\theta(\varepsilon) = \left|\theta^\perp \cap B(F_\theta, \varepsilon)\right|_{d-1}.$$

- We will say that ∂S is (C, ε_0)-regular if for all θ and all $\varepsilon \in (0, \varepsilon_0)$, $\varphi'_\theta(\varepsilon)$ exists and $\varphi'_\theta(\varepsilon) \leq C$.
- If ∂S is (C, ε_0)-regular for some $\varepsilon_0 > 0$, we will say that ∂S is C-regular.

Once the rolling balls condition is imposed, we will show through some examples in Figure 3 that the (C, ε_0)-regularity of the boundary is quite mild.

(a) The first set, presented in Figure 3, is a unit square with 'round angles'. For all θ, $F_\theta = \pi_{\theta^\perp}(E_\theta(\partial S)) = \{x_1(\theta), x_2(\theta)\}$ with $||x_1(\theta) - x_2(\theta)|| \geq 1$. Thus for $\varepsilon < 1/2$, and for all θ, $\varphi_\theta(\varepsilon) = 4\varepsilon$ and is thus ∂S is $(4, 0.5)$-regular (in particular 4-regular).

(b) The second set, presented in Figure 4, is a 2-dimensional 'peanut' made of 4 circular arcs. For all θ and ε small enough, we have $\varphi_\theta(\varepsilon) = 2c_\theta\varepsilon$ where c_θ is the number of connected components of F_θ, which is less than 6, from which it follows that S has a 12-regular boundary.

(c) The third set, presented in Figure 5, is the surface of revolution generated by (b). Here we have that for all θ, E_θ is a 1-dimensional manifold with less than 3 connected components. The maximal length of a component is bounded by L, the length of the maximal perimeter (shown in blue in the figure). The reach of each E_θ is (uniformly in θ) lower bounded by $\alpha > 0$. All these assertions allow claiming that ∂S is $6L$-regular.

(d) The rolling ball condition is not sufficient to guarantee the (C, ε_0)-regularity of the boundary: this fails if, for instance, S is such that $\partial S = S_1 \cup S_2 \cup S_3 \cup S_4$ (see Figure 6) with:

$$S_1 = \{(x, 1 + (500)(x^2 - 1)^{10} x^5 \sin(1/x)), x \in [-1, 1] \setminus \{0\}\} \cup \{(0, 1)\};$$

$$S_2 = \{(x, -1 - (500)(x^2 - 1)^{10} x^5 \sin(1/x)), x \in [-1, 1] \setminus \{0\}\} \cup \{(0, -1)\};$$

$$S_3 = \{(1 + \cos(\theta), \sin(\theta)), \theta \in [-\pi/2, \pi/2]\};$$

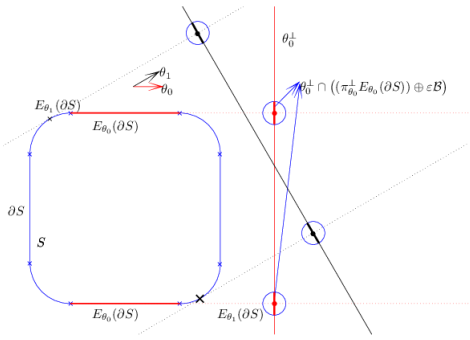

Figure 3: (a) smooth square

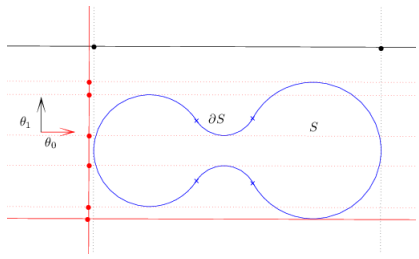

Figure 4: (b) 2D peanut

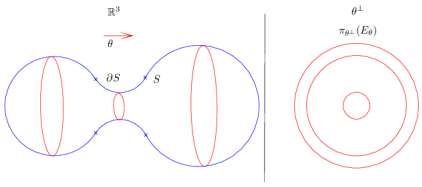

Figure 5: (c) 3D peanut

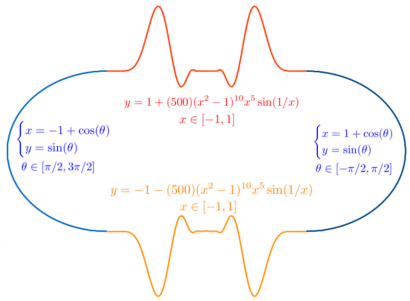

Figure 6: (d) an 'infinite wave' shape

$$S_4 = \{(-1+\cos(\theta), \sin(\theta)), \theta \in [\pi/2, 3\pi/2]\}.$$

It can easily be proved that such a set satisfies the rolling ball condition for any $r_0 \leq 1/80$ but $\varphi_0'(\varepsilon) \to +\infty$ when $\varepsilon \to 0$ which implies that ∂S is not C-regular.

For the Devroye–Wise type estimator we will also show that the convergence rate can be quadratically improved if we additionally assume that the number of intersections between any line and ∂S is bounded from above (this excludes the case of a linear part in ∂S such as in Figure 3).

Definition 2.5. Given $S \subset \mathbb{R}^d$, we say that ∂S has a bounded number of linear intersections if there exists an N_S such that for all $\theta \in (\mathbb{S}^+)^{d-1}$ and $y \in \theta^\perp$, $n_{\partial S}(\theta, y) \leq N_S$.

Remark 2. *The previous definition can be replaced with a weaker requirement, by asking that ∂S has a bounded number of linear insersections for almost all line with respect to $\mu_{d-1}(y)d\theta$, and the corresponding results remain true.*

3 Surface area estimation based on the Devroye–Wise estimator

3.1 A conjecture on the Devroye–Wise estimator

Since in general the set S is unknown, we first propose the natural plug-in idea of computing $|\partial\hat{S}|_{d-1}$ where $\hat{S}$ is an estimator of S. There are different kinds

of set estimators, depending on the geometric restrictions imposed on S and the structure of the data (see [13, 23] and references therein). One of the most studied in the literature, which is also universally consistent, is the Devroye–Wise estimator (see [23]), introduced in (1). This all-purpose estimator has the advantage that it is quite easy to compute the intersection of a line with its boundary, as follows: Given a line $r_{\theta,y}$, we can compute $\mathbb{Y}_i = \partial\mathcal{B}(X_i, \varepsilon_n) \cap r_{\theta,y}$, and then $\mathbb{Z}_i = \{y \in \mathbb{Y}_i, d(y, \mathbb{X}_n) \geq \varepsilon_n\}$, so we have that, with probability one,

$$\cup_i \mathbb{Z}_i = r_{\theta,y} \cap \partial\hat{S}_{\varepsilon_n}(\mathbb{X}_n).$$

Indeed, suppose, on the contrary, that there exists a $z \in \cup_i \mathbb{Z}_i$ and $z \in \mathring{\hat{S}}_{\varepsilon_n}$. Then we have $d(z, \mathbb{X}_n) = \varepsilon_n$ and $z \in \mathcal{H}\{X_i, d(X_i, z) = \varepsilon_n\}$ (where $\mathcal{H}(E)$ is the convex hull of E). Thus, there are at least $d+1$ observations on the same hypersphere of given radius ε_n, but this event has probability 0, see [31].

We conjecture that the plug-in estimator $|\partial\hat{S}_{\varepsilon_n}(\mathbb{X}_n)|_{d-1}$ satisfies the following:

1. If $\varepsilon_n < d_H(\mathbb{X}_n, S)$, then $\partial\hat{S}_{\varepsilon_n}(\mathbb{X}_n)$ does not converge to ∂S and $|\partial\hat{S}_{\varepsilon_n}(\mathbb{X}_n)|_{d-1}$ does not converge to $|\partial S|_{d-1}$.

2. If $\varepsilon_n = d_H(\mathbb{X}_n, S)$, then $\partial\hat{S}_{\varepsilon_n}(\mathbb{X}_n)$ converges to ∂S with the best possible rate but $|\partial\hat{S}_{\varepsilon_n}(\mathbb{X}_n)|_{d-1}$ does not converge to $|\partial S|_{d-1}$ but greatly overestimates it.

3. If $\varepsilon_n \gg d_H(\mathbb{X}_n, S)$ and $\varepsilon_n \to 0$, then $\partial\hat{S}_{\varepsilon_n}(\mathbb{X}_n)$ converges to ∂S and $|\partial\hat{S}_{\varepsilon_n}(\mathbb{X}_n)|_{d-1}$ converges to $|\partial S|_{d-1}$ but we can expect that the rate is greater than ε_n (namely $||\partial\hat{S}_{\varepsilon_n}(\mathbb{X}_n)|_{d-1} - |\partial S|_{d-1}| \geq \mathcal{O}(\varepsilon_n) \gg d_H(\mathbb{X}_n, S)$). Indeed if S fulfills the outside and inside rolling ball conditions, then for n large enough we have that $B(S, \varepsilon_n - d_H(\mathbb{X}_n, S)) \subset \hat{S}_{\varepsilon_n}(\mathbb{X}_n) \subset B(S, \varepsilon_n)$, which in turn gives that $||\partial\hat{S}_{\varepsilon_n}(\mathbb{X}_n)|_{d-1} - |\partial S|_{d-1}| \geq \mathcal{O}(\varepsilon_n) \gg d_H(\mathbb{X}_n, S)$.

3.2 A surface estimator based on the Devroye–Wise estimator

The aim of this section is to propose an estimator for the surface area based on the Devroye–Wise support estimator and Crofton's formula. It can attain a convergence rate of order $d_H(\mathbb{X}, S)$. The whole procedure is defined for any set $\mathbb{X}$, not necessarily finite, because we will apply our estimator to the case in which $\mathbb{X}$ is the trajectory of a Brownian motion. If $\mathbb{X}$ is not finite, then for a given $\varepsilon > 0$, we write $\hat{S}_\varepsilon(\mathbb{X}) = B(\mathbb{X}, \varepsilon)$. The procedure replaces $n_{\partial S}(\theta, y)$ by $\hat{n}_{\epsilon,\mathbb{X}}(\theta, y)$ introduced in Definition 3.1, and then integrates $\hat{n}_{\epsilon,\mathbb{X}}(\theta, y)$ as in Crofton's formula, see (5). We will prove that (see Remark 4), by the (C, ε_0)-regularity of the boundary, with probability one $r_{\theta,y}$ is not included in any $(d-1)$-dimensional affine tangent space (tangent to ∂S). Then $n_{\partial S}(\theta, y) = 2k_S(\theta, y)$ where $k_S(\theta, y)$ is the number of connected components of $r_{\theta,y} \cap S$.

Definition 3.1. Let ε be a positive real number, and $\mathbb{X} \subset S$ a set (not necessarily finite). Consider a line $r_{\theta,y}$. If $\hat{S}_\varepsilon(\mathbb{X}) \cap r_{\theta,y} = \emptyset$, define $\hat{n}_{\varepsilon,\mathbb{X}}(\theta, y) = 0$. If not, then:

- denote by $I_1, \ldots, I_m$ the connected components of $\hat{S}_\varepsilon(\mathbb{X}) \cap r_{\theta,y}$. Order this sequence in such a way that $I_i = (a_i, b_i)$, with $a_1 < b_1 < \cdots < a_m < b_m$.

- If for some consecutive intervals $I_i, I_{i+1}, \ldots, I_{i+\ell}$, for all $a_i < \lambda < b_{i+\ell}$ and $t = \theta + \lambda y \in r_{\theta,y}$, $d(t, \mathbb{X}) \leq 4\varepsilon$, define $A_i = (a_i, b_{i+\ell})$.
- Let j be the number of disjoint open intervals $A_1, \ldots, A_j$ that this process ended with. Then define $\hat{n}_{\varepsilon,\mathbb{X}}(\theta, y) = 2j$.

To roughly summarize this, we consider the connected components of $\hat{S}_\varepsilon \cap r_{\theta,y}$ and 'link or glue' the ones that are in the same connected component of $\hat{S}_{4\varepsilon} \cap r_{\theta,y}$. In the sequel, we will refer to this process as the gluing procedure.

To gain some insight into the relation between $\hat{n}_{\varepsilon,\mathbb{X}}(\theta, y)$ and $n_{\partial \hat{S}_\varepsilon(\mathbb{X})}(\theta, y)$, observe that $\hat{n}_{\varepsilon,\mathbb{X}}(\theta, y) \leq n_{\partial \hat{S}_\varepsilon(\mathbb{X})}(\theta, y)$. We also have that $\hat{n}_{\varepsilon,\mathbb{X}}(\theta, y) \leq n_{\partial \hat{S}_{4\varepsilon}(\mathbb{X})}(\theta, y)$. Indeed, let $C_1, \ldots, C_K$ be the connected components of $r_{\theta,y} \cap \hat{S}_{4\varepsilon}$ and note that:

1. For each j there exists an index i such that $I_j \subset C_i$.
2. If $d(C_i, \mathbb{X}) > \varepsilon$ for all j, we have that $I_j \cap C_i = \emptyset$.
3. If $d(C_i, \mathbb{X}) \leq \varepsilon$, there exists an $I_j \subset C_i$ and all the I_j such that $I_j \subset C_i$ are glued by the proposed procedure. Thus there exists a unique j' such that $A_{j'} \subset C_j$.

Our first proposed estimator is

$$\hat{I}_{d-1}(\mathbb{X}, \varepsilon) = \frac{1}{\beta(d)} \int_{\theta \in (\mathbb{S}^+)^{d-1}} \int_{y \in \theta^\perp} \hat{n}_{\varepsilon,\mathbb{X}}(\theta, y) d\mu_{d-1}(y) d\theta. \tag{5}$$

Under the assumption that ∂S has a bounded number N_S of linear intersections (see Definition 2.5), we will consider, for a given $N_0 \geq N_S$,

$$\hat{I}^{N_0}_{d-1}(\mathbb{X}, \varepsilon) = \frac{1}{\beta(d)} \int_{\theta \in (\mathbb{S}^+)^{d-1}} \int_{y \in \theta^\perp} \min(\hat{n}_{\varepsilon,\mathbb{X}}(\theta, y), N_0) d\mu_{d-1}(y) d\theta.$$

3.3 Main results on the Devroye–Wise based estimator.

Theorem 3.2. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside α-rolling conditions. Assume also that S is (C, ε_0)-regular for some positive constants C and ε_0. Let $\mathbb{X}_n = \{X_1, \ldots, X_n\} \subset S$. Let $\varepsilon_n \to 0$ be such that $d_H(\mathbb{X}_n, S) \leq \varepsilon_n$. Then*

$$\hat{I}_{d-1}(\mathbb{X}_n, \varepsilon_n) = |\partial S|_{d-1} + \mathcal{O}(\sqrt{\varepsilon_n}). \tag{6}$$

Moreover, for n large enough,

$$|\mathcal{O}(\sqrt{\varepsilon_n})| \leq \frac{4C\, diam(S)}{3\beta(d)\sqrt{\alpha}} \sqrt{\varepsilon_n}.$$

The idea of the proof of Theorem 3.2 consists on proving that for the lines that are 'far enough' (fulfilling $L(\varepsilon)$ for some $\varepsilon > 0$) from the tangent spaces, our algorithm allows a perfect estimation of $n_{\partial S}(\theta, y)$. For the rest of the lines we will prove in Corollary 5 that, under (C, ϵ_0)-regularity, the integral of $\hat{n}_{\varepsilon_n,\mathbb{X}_n}(\theta, y)$ on the set of these lines, is bounded from above by $C'\varepsilon_n^{1/2}$, C' being a positive constant. Roughly speaking, a line fulfilling condition $L(\varepsilon)$, does not not met the estimator $\partial \hat{S}_{\epsilon_n}$ too many times.

From Theorem 3.2 and Theorem 4 in [18], we can obtain the rate of convergence for the iid case:

Corollary 1. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the inside and outside α-rolling conditions. Assume also that S is (C, ε_0)-regular for some positive constants C and ε_0. Let $\mathbb{X}_n = \{X_1, \ldots, X_n\}$ be the set of observations of an iid sample of X with distribution P_X supported on S. Assume that P_X has density f (w.r.t. μ_d) bounded from below by some $c > 0$. Let $\varepsilon_n = C'(\ln(n)/n)^{1/d}$ and $C' > (6/(c\omega_d))^{1/d}$. Then with probability one, for n large enough,*

$$\hat{I}_{d-1}(\mathbb{X}_n, \varepsilon_n) = |\partial S|_{d-1} + \mathcal{O}\left(\left(\frac{\ln n}{n}\right)^{\frac{1}{2d}}\right).$$

As mentioned in Section 5.2 in [18], if $\varepsilon_n = 2\max_i \min_{j \neq i} ||X_i - X_j||$, then with probability one, for n large enough, $\varepsilon_n \leq 2d_H(\mathbb{X}_n, S)$, which together with Corollary 1, entails that, with the aforementioned choice for ε_n, our proposal is fully data driven, for the iid case.

If the number of linear intersections of ∂S is assumed to be bounded by a constant N_S, the use of $\min(\hat{n}_{\varepsilon_n}, N_0)$ (for any $N_0 \geq N_S$) allows us to obtain better convergence rates.

Theorem 3.3. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside α-rolling conditions. Assume also that S is (C, ε_0)-regular for some positive constants C and ε_0, and that the number of linear intersections of ∂S is bounded by N_S. Let $\mathbb{X}_n = \{X_1, \ldots, X_n\} \subset S$. Let $\varepsilon_n \to 0$ be such that $d_H(\mathbb{X}_n, S) \leq \varepsilon_n$ and $N_0 \geq N_S$. Then*

$$\hat{I}^{N_0}_{d-1}(\mathbb{X}_n, \varepsilon_n) = |\partial S|_{d-1} + \mathcal{O}(\varepsilon_n).$$

Moreover, for n large enough, $|\mathcal{O}(\varepsilon_n)| \leq 4C(N_0 + N_S)\varepsilon_n/\beta(d)$.

As before, we give the convergence rate associated to the iid setting and the reflected Brownian motion hypothesis as two corollaries of Theorem 3.3.

Corollary 2. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the inside and outside α-rolling conditions. Assume also that S is (C, ε_0)-regular for some positive constants C and ε_0 and that ∂S has a bounded number of linear intersections. Let $\mathbb{X}_n = \{X_1, \ldots, X_n\}$ be the set of observations of an iid sample with distribution P_X, supported on S. Assume that P_X has density f (w.r.t. μ_d) bounded from below by some $c > 0$. Let $\varepsilon_n = C'(\ln(n)/n)^{1/d}$ and $C' > (6/(c\omega_d))^{1/d}$. Then with probability one, for n large enough,*

$$\hat{I}^{N_0}_{d-1}(\mathbb{X}_n, \varepsilon_n) = |\partial S|_{d-1} + \mathcal{O}\left(\left(\frac{\ln(n)}{n}\right)^{\frac{1}{d}}\right).$$

Here again the choice of $\varepsilon_n = 2\max_i \min_j ||X_i - X_j||$ is suitable, but now, the price to pay is the selection of the parameter N_0.

In a more general setting, the conclusion of Theorem 3.3 holds when the set of points $\mathbb{X}_n$ is replaced by the trajectory $\mathbb{X}_T$ of any stochastic process $\{X_t\}_{t>0}$ included in S, observed in $[0, T]$, such that $d_H(\mathbb{X}_T, S) \to 0$ as $T \to \infty$. Observe that the estimator $\hat{I}^{N_0}_{d-1}(\mathbb{X}, \varepsilon)$ is well defined even when $\mathbb{X}_T$ is not a finite set, see Definition 3.1. We will assume that S is bounded with connected interior

and ∂S is $\mathcal{C}^2$. This is the case (for example) of some reflected diffusions and in particular the reflected Brownian motion (RBM). This has been recently proven in Corollary 1 in [13], for RBM without drift (see also [14] and [15] for the RBM with drift). The definition of an RBM with drift is as follows: Given a d-dimensional Brownian motion $\{B_t\}_{t\geq 0}$ departing from $B_0 = 0$ and defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}_x)$, an RBM with drift is the (unique) solution to the following stochastic differential equation on S:

$$X_t = X_0 + B_t - \frac{1}{2}\int_0^t \nabla_f(X_s)ds - \int_0^t \eta_{X_s}\xi(ds), \qquad \text{where } X_t \in \overline{D},\ \forall t \geq 0,$$

where the drift, $\nabla_f(x)$, is given by the gradient of a function f, and is assumed to be Lipschitz, $\{\xi_t\}_{t\geq 0}$ is the corresponding local time, i.e. a one-dimensional continuous non-decreasing process with $\xi_0 = 0$ that satisfies $\xi_t = \int_0^t \mathbb{I}_{\{X_s \in \partial S\}} d\xi_s$. Since the drift is given by the gradient of a function and S is compact, we have that its stationary distribution has a density bounded from below by a constant.

Corollary 3. *Let $S \subset \mathbb{R}^d$ be a non-empty compact set with connected interior such that $S = \overline{\mathring{S}}$, and suppose that S fulfills the outside and inside α-rolling conditions. Assume also that S is (C, ε_0)-regular for some positive constants C and ε_0 and that the number of linear intersections of ∂S is bounded by N_S. Let $\mathbb{X}_T \subset S$ be as before. Then, with probability one, for T large enough,*

$$\hat{I}^{N_0}_{d-1}(\mathbb{X}_T, \varepsilon_T) = |\partial S|_{d-1} + o\left(\left(\frac{\ln(T)^2}{T}\right)^{\frac{1}{d}}\right),$$

where $\varepsilon_T = o((\ln(T)^2/T)^{1/d})$.

3.4 The algorithm

We will now describe an algorithm to compute $\hat{n}_{\varepsilon,\mathbb{X}_n}(\theta, y)$ for a given (θ, y), when the input is a finite set of n elements and $\varepsilon > 0$. For a reflected diffusion, we take $\mathbb{X}_n \subset \mathbb{X}_T$ to be a dense enough subset of n points. Observe that this is not restrictive since $\mathbb{X}_T$ is stored as a finite set of points in a computer.

1. For each i, compute $d_i := d(r_{\theta,y}, X_i) = \sqrt{||X_i - y||^2 - \langle X_i - y, \theta\rangle^2}$.

2. Compute the connected components I_i of $r_{\theta,y} \cap \hat{S}_\varepsilon(\mathbb{X}_n)$ according to the following steps: Initialize the list of the extremes of these intervals by listz= $\emptyset$ and listl= $\emptyset$. Then, for $i = 1$ to n:

   - If $d_i = \varepsilon$, then $N_i = 1$, $\ell_1 = \langle X_i - y, \theta\rangle$ and $z_1 = \mathcal{B}(X_i, \varepsilon) \cap r_{\theta,y} = y + \ell_1\theta$
   - If $d_i < \varepsilon$, then $N_i = 2$ and compute $\ell_1 = \langle X_i - y, \theta\rangle - \sqrt{\varepsilon^2 - d_i^2}$ and $\ell_2 = \langle X_i - y, \theta\rangle + \sqrt{\varepsilon^2 - d_i^2}$. Then $z_1 = y + \ell_1\theta$ and $z_2 = y + \ell_2\theta$ such that $\{z_1, z_2\} = \mathcal{B}(X_i, \varepsilon_n) \cap r_{\theta,y}$.
   - For $j = 1$ to N_i: if $d(z_j, \mathbb{X}_n) \geq \varepsilon$, do listz=listz$\cup\{z_j\}$ and listl=listl$\cup\{\ell_j\}$.

     From the comments at the beginning of subsection 3.1, we know that, with probability one, listz equals $r_{\theta,y} \cap \partial\hat{S}_\varepsilon$.

- Sort listl. With probability one, listl has an even number, $2m$, of elements (see the comments at the beginning of subsection 3.2), and define a_i and b_i such that $\ell_{2(i-1)+1} = a_i$, $\ell_{2i} = b_i$ (which correspond to the a_i and b_i in Definition 3.1 i.e. (a_i, b_i) are the connected components of $r_{\theta,y} \cap \hat{S}_\varepsilon(\mathbb{X})$).

3 Obtain the a_i' and b_i' such that $I_i' = (a_i', b_i')$ are the connected components of $\hat{S}_{4\varepsilon}(\mathbb{X}_n) \cap r_{\theta,y}$ by using the same procedure.

4. Lastly, compute $\hat{n}_{\varepsilon,\mathbb{X}}(\theta, y)$, as follows:
initialization $\hat{n}_{\varepsilon,\mathbb{X}_n}(\theta, y) = m$.

   For $i = 1$ to $m - 1$:

   - If there exists k such that $(b_i, a_{i+1}) \subset I_k'$, then:

$$\hat{n}_{\varepsilon,\mathbb{X}_n}(\theta, y) = \hat{n}_{\varepsilon,\mathbb{X}_n}(\theta, y) - 1.$$

5. $\hat{n}_{\varepsilon,\mathbb{X}_n}(\theta, y) = 2\hat{n}_{\varepsilon,\mathbb{X}_n}(\theta, y)$.

4 The approach based on the α-convex hull

4.1 The estimator based on the α'-hull assuming the α-rolling ball condition

In [5] it was proved that, in dimension two, under some regularity assumptions, the length of the boundary of the α-shape of an iid sample converges to the length of the boundary of the set. The α-shape has the very good property that its boundary is very easy to compute, and hence so is its surface measure. Unfortunately we are not sure that the results can be extended to higher dimensions. Nevertheless, considering the α-convex hull (which is quite close to the α-shape) allows extending the results on the surface measure to any dimension. The following deterministic theorem states that, for all $0 < \alpha' < \alpha$, the surface measure of the boundary of the α'-convex hull $\mathbb{X}_n \subset S$ converges to $|\partial S|_{d-1}$ with a rate that depends on $d_H(\partial C_{\alpha'}(\mathbb{X}_n), \partial S)$.

Theorem 4.1. *Let $S \subset \mathbb{R}^d$ be a compact set such that ∂S is a $(d-1)$-dimensional $\mathcal{C}^2$ manifold with reach $\alpha > 0$. Let $\alpha' < \alpha$ be a positive constant and let $\mathbb{X}_n \subset S$ be a finite set such that $d_H(\mathbb{X}_n, S) < \frac{1}{2}\frac{\alpha\alpha'}{\alpha+\alpha'}$ and $d_H(\partial C_{\alpha'}(\mathbb{X}_n), \partial S) \leq \varepsilon_n$ with*

$$\varepsilon_n \leq \min\left\{\frac{\alpha\alpha'}{16(\alpha+\alpha')}, \frac{1}{(d-1)\alpha}\right\}.$$

Then

1. *$\pi_{\partial S} : \partial C_{\alpha'}(\mathbb{X}_n) \to \partial S$ (where $\pi_{\partial S}(x)$ denotes the projection onto ∂S) is one to one, and*

2. $\left||\partial S|_{d-1} - |\partial C_{\alpha'}(\mathbb{X}_n)|_{d-1}\right| \leq (d-1)\left(\frac{3}{2}\alpha + 32\frac{\alpha+\alpha'}{\alpha\alpha'}\right)\varepsilon_n(1 + o(1))$.

As previously, we can deduce from the deterministic theorem and results in [5] the convergence rates under the iid assumption.

Corollary 4. *Let $S \subset \mathbb{R}^d$ be a compact set such that ∂S is a $(d-1)$-dimensional $\mathcal{C}^2$ manifold with reach $\alpha > 0$. Let $\mathbb{X}_n = \{X_1, \dots, X_n\}$ be an iid sample of X with distribution P_X supported on S. Assume that P_X has density f (w.r.t. μ_d) bounded from below by some $c > 0$. Suppose $\alpha' < \alpha$. Then with probability one, for n large enough,*

$$\left| |\partial S|_{d-1} - |\partial C_{\alpha'}(\mathbb{X}_n)|_{d-1} \right| = \mathcal{O}\left(\left(\frac{\ln(n)}{n} \right)^{\frac{2}{d+1}} \right).$$

In this case we do not need the additional hypothesis of (C, ε_0)-regularity; the convergence rate is far better than the one given in Theorem 3.2, where the price to pay is the computational cost when d increases. Indeed, as detailed in next section the computation of the α-convex hull requires to start by the computation of the Delaunay complex . With regard to the parameter selection α', a fully data driven (but computationally expensive) method is proposed in [38].

4.2 Computation with the use of Crofton's formula

Unfortunately, the explicit computation of $|\partial C_\alpha(\mathbb{X}_n)|_{d-1}$ is very difficult. However, from the results in Lemma 8.7 we derive that we can make use Crofton's formulae and the Monte Carlo method to estimate $|\partial C_\alpha(\mathbb{X}_n)|_{d-1}$. This, as we will see, is based on the fact that the computation of $\check{n}_\alpha(\theta, y) := n_{\partial C_\alpha(\mathbb{X}_n)}(\theta, y)$ is feasible. It requires first the computation of the α-convex hull, as well as the convex hull, of $\mathbb{X}_n$. Recall that the convex hull $\mathcal{H}(\mathbb{X}_n)$ of $\mathbb{X}_n$ is equal to the intersection of a finite number of half-spaces $\mathcal{H}(\mathbb{X}_n) = \bigcap_{i=1}^K H_i$ with $H_i = \{x \in \mathbb{R}^d, \langle x - y_i, u_i \rangle \leq 0\}$ for some $\{y_1, \dots, y_K\} \subset \mathbb{R}^d$ and $\{u_1, \dots, u_K\} \subset \mathcal{S}^{d-1}$.

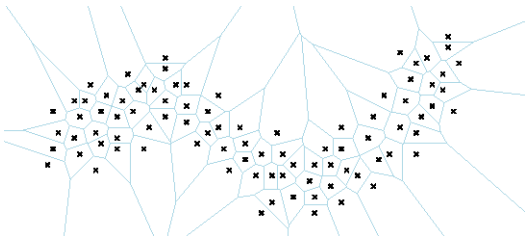

Figure 7: Points in $\mathbb{R}^2$

In [24] it is proved, for dimension 2, but mentioned that the generalization is not difficult, that $C_\alpha(\mathbb{X}_n)^c$ is the union of a finite number of balls and the aforementioned half-spaces. The centres O_i of these balls, and their radii r_i, are obtained by computing the Delaunay complex. The computational cost of the Delaunay complex being the main part of the computational cost of our algorithm, which is defined as follows:

1. Compute all the Delaunay simplices $\sigma_i = \mathcal{H}(\{X_{i_1}, \dots, X_{i_{d+1}}\})$, i.e. those such that $\mathring{\mathcal{B}}(O_i, r_i) \cap \mathbb{X}_n = \emptyset$ and $\partial\mathcal{B}(O_i, r_i)$ is the sphere circumscribed to $X_{i_1}, \dots, X_{i_{d+1}}$.

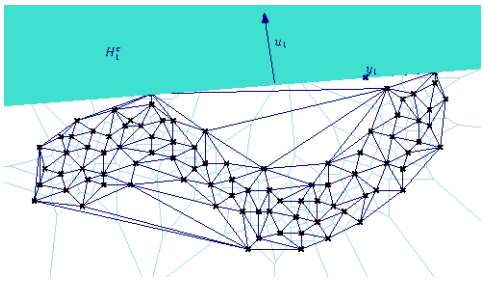

Figure 8: Points in $\mathbb{R}^2$, The associated Delaunay complex and an half space H_1^c

2. Sort the indices so that r_i are decreasing, and define $K' = \#\{r_i, r_i \geq \alpha'\}$.
3. Define $B_i^+ = \mathring{\mathcal{B}}(O_i, r_i)$ for $i \in \{1, \ldots, K'\}$. Clearly $r_i \geq \alpha'$ for all $i = 1, \ldots, K'$.

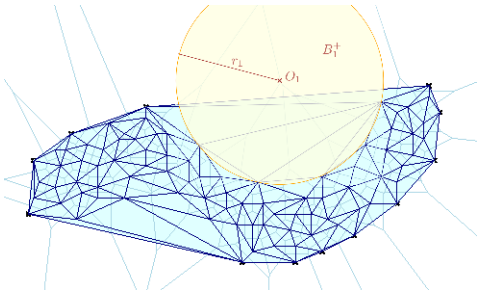

Figure 9: The convex hull of the points (blue) and a Ball B_1^+

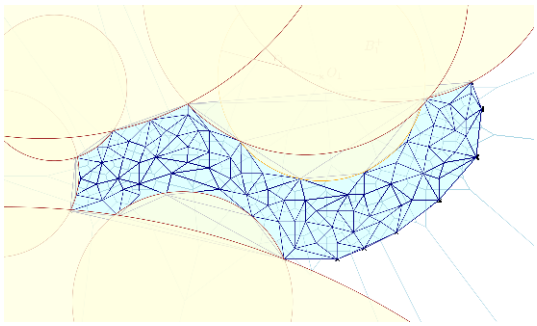

Figure 10: The convex hull of the points (blue) and all the B_i^+

4. Compute the faces of the boundary of the α'-shape (see [24]), which are the $f_i = \mathcal{H}(\{X_{i_1}, \ldots, X_{i_d}\})$ such that there exists a unique $j \geq K' + 1$ such that $f_i \subset \sigma_j$.

   Also compute Ω_i (resp. ρ_i), which is the center (resp. radius) of the sphere

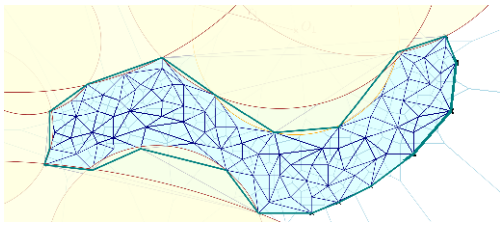

Figure 11: The convex hull of the points, all the B_i^+ and the boundary faces (green)

circumscribed to $X_{i_1}, \ldots, X_{i_d}$ in the plane spanned by $X_{i_1}, \ldots, X_{i_d}$.

Now we have two different cases.

(a) f_i is a face of $\partial\mathcal{H}(\mathbb{X}_n)$, i.e. there exists j' such that $f_i \subset H_{j'}$. Then define $w_i = u_{j'}$.

(b) f_i is not a face of $\partial\mathcal{H}(\mathbb{X}_n)$, thus there exists $j' \leq K'$ such that $f_i \subset \sigma_{j'}$. Then define $w_i = (O_j - O_{j'})/||O_j - O_{j'}||$, with $j \geq K+1$ such that $f_i \subset \sigma_j$.

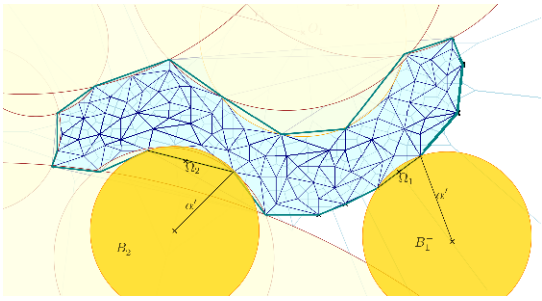

Figure 12: The convex hull of the points, all the B_i^+, the boundary faces (green) and two B^-. B_1^- correspond to case (a) and B_2^- corresponds to case (b).

Define $B_i^- = \mathring{\mathcal{B}}(\Omega_i + \sqrt{\alpha'^2 - \rho_i^2} w_i, \alpha')$. Then

$$C_{\alpha'}(\mathbb{X}_n)^c = \left(\bigcup_i H_i^c\right) \cup \left(\bigcup_i B_i^-\right) \cup \left(\bigcup_i B_i^+\right). \tag{7}$$

To simplify notation, we write $C_\alpha(\mathbb{X}_n)^c = \bigcup_i B_i$. Observe that if the line $r_{\theta,y}$ is chosen at random (w.r.t. $d\mu_{d-1}d\theta$), with probability one we have $r_{\theta,y} \cap \partial B_i$ contains less than three points.

Initialize list=$\emptyset$. Then:
for each i,

- compute $r_{\theta,y} \cap \partial B_i$.

- For all $z \in r_{\theta,y} \cap \partial B_i$, if for all j $z \notin \mathring{B}_j$, then do list=list$\cup\{z\}$.

then $\check{n}(\theta, y) = \#$list.

5 Discussion of the rates of convergence

In Corollary 4 we obtained the same convergence rate as the one provided in [5] for $d = 2$, conjectured as suboptimal. As mentioned in [5], if the measure of the symmetric difference between S and an estimator $\hat{S}_n$ is bounded by ε_n, we can only expect that plug-in methods allow estimating $|\partial S|_{d-1}$ with a convergence rate ε_n. Thus, in the iid setting, the estimator defined by (6) (respectively (7) to (9)) can be seen as 'optimal' relative to the use of the Devroye–Wise support estimator (respectively the α-convex hull support estimator), since they achieve the best possible convergence rates for those estimators. This is nevertheless far from being optimal: the minimax rate is conjectured to be $n^{-\frac{d+3}{2d+2}}$, which is the minimax rate for the volume estimation problem (see [6]), and in [34] it is proved that the minimax rate is the same for the volume estimation problem and the surface area estimation problem (at least in the image setting, which usually extends to the iid setting). Unfortunately, attaining this optimal rate for the surface area estimation problem is much more involved, even in the easier setting with data uniformly drawn in S and S^c with perfect identification: no estimator attaining this rate has been proposed yet.

6 Integralgeometric estimations via a Monte Carlo method and numerical experiments

To estimate the surface area with a Monte Carlo method, we propose the following classical procedure. Generate a random sample $\theta_1, \dots, \theta_k$ uniformly distributed on $(\mathcal{S}^+)^{d-1}$. For each $i = 1, \dots, k$, draw a random sample $\aleph_i = \{y_1^i, \dots, y_\ell^i\}$ uniformly distributed on $[-L, L]^{d-1} \subset \theta_i^\perp$, independent of $\theta_1, \dots, \theta_k$, where $L = \max_{j=1,\dots,n} ||X_j||$. Then, the estimators are given by

$$\hat{\hat{I}}_{d-1}^{(\ell,k)}(\partial S) = \frac{(2L)^{d-1}}{\beta(d)} \frac{1}{\ell k} \sum_{i=1}^{k} \sum_{j=1}^{\ell} \hat{n}_{\varepsilon_n, \mathbb{X}_n}(\theta_i, y_j^i) \tag{8}$$

$$\hat{\hat{I}}_{d-1}^{(\ell,k,N_0)}(\partial S) = \frac{(2L)^{d-1}}{\beta(d)} \frac{1}{\ell k} \sum_{i=1}^{k} \sum_{j=1}^{\ell} \min(\hat{n}_{\varepsilon_n, \mathbb{X}_n}(\theta_i, y_j^i), N_0) \tag{9}$$

$$\check{\hat{I}}_{d-1}^{(\ell,k)}(\partial S) = \frac{(2L)^{d-1}}{\beta(d)} \frac{1}{\ell k} \sum_{i=1}^{k} \sum_{j=1}^{\ell} \check{n}_r(\theta_i, y_j^i). \tag{10}$$

7 Simulation study

The performance of (8) and (10) is shown through a simulation study. We consider the sets $S_d = \mathcal{B}_d(O, 1) \setminus \mathring{\mathcal{B}}_d(O, r)$ for $d = 2, 3$ and $r = 0.5, 0.6, 0.7, 0.8$ and 0.9. On each set we draw n iid random vectors supported on S_d, whose common distribution is $X = RZ$, R being a real valued random variable uniform

on $[0, r]$ and Z a random vector (independent of R), supported on the $d-1$ dimensional sphere.

We choose $k = 4000$ and $\ell = 1$, at equation (8) and the same for (10). The random lines were generated as before. For both estimators the samples sizes were $n = 50, 100, 200, 500, 1000, 2000$ and 4000, and in all cases we considered uniformly distributed random vectors on S_d. For (8) we computed the parameter ε_n as follows: for each sample point we computed the distance to it's closest point in the sample. Then we choose the third quantile of these n distances. For (10) we estimated the parameter α with the data-driven estimator proposed in [38].

Let us define

$$E_i^{DW} = \frac{\hat{\hat{I}}_{d-1}^{(1,1000)}(\partial S) - |S_d|_{d-1}}{|S_d|_{d-1}} \tag{11}$$

and

$$E_i^{CH} = \frac{\check{\check{I}}_{d-1}^{(1,1000)}(\partial S) - |S_d|_{d-1}}{|S_d|_{d-1}} \tag{12}$$

In Figure 13 we show, for each d and r, 10 black curves corresponding to 10 replications of (11)

It is also shown in red 10 replications of (12). For the first three values of r_0 the estimators behave similarly, while for the last two ones the estimator (8) outperforms (10).

8 Appendix

8.1 Proofs of Theorems 3.2 and 3.3

Sketch of the proofs of Theorems 3.2 and 3.3

The idea is to consider separately two subsets of the set of lines that intersect $\partial \hat{S}_{\varepsilon_n}(\mathbb{X}_n)$:

1. If a line $r_{\theta,y} = y + \mathbb{R}\theta$ is 'far enough' (fulfilling condition $L(\varepsilon)$ for some $\varepsilon > 0$, see Definition 8.1) from the tangent spaces, then our algorithm allows a perfect estimation of $n_{\partial S}(\theta, y)$, see Lemma 8.5.

2. Considering the set of lines that are not 'far enough' from the tangent spaces (denoted by $\mathcal{A}_{\varepsilon_n}(\theta)$), see Definition 8.1), Corollary 5 states that, under (C, ϵ_0)-regularity, the integral of $\hat{n}_{\varepsilon_n,\mathbb{X}_n}(\theta, y)$ on $\mathcal{A}_{\varepsilon_n}(\theta)$ is bounded from above by $C'\varepsilon_n^{1/2}$, C' being a positive constant. Theorem 3.3 states that the previous bound can be improved to $C'\varepsilon_n$, under (C, ϵ_0)-regularity, if ∂S has a bounded number of linear intersections.

8.1.1 Condition $L(\varepsilon)$

We now define the two sets of lines to be treated separately: The lines that are 'far' from an affine tangent space, and the lines that are 'close to being tangent' to ∂S. More precisely, recall that under the rolling ball hypothesis, the unit outer normal vector η_x at x is well defined (see Remark 1). Now we define

$$\mathcal{T}_S = \{x + (\eta_x)^{\perp} : \ x \in \partial S\},$$

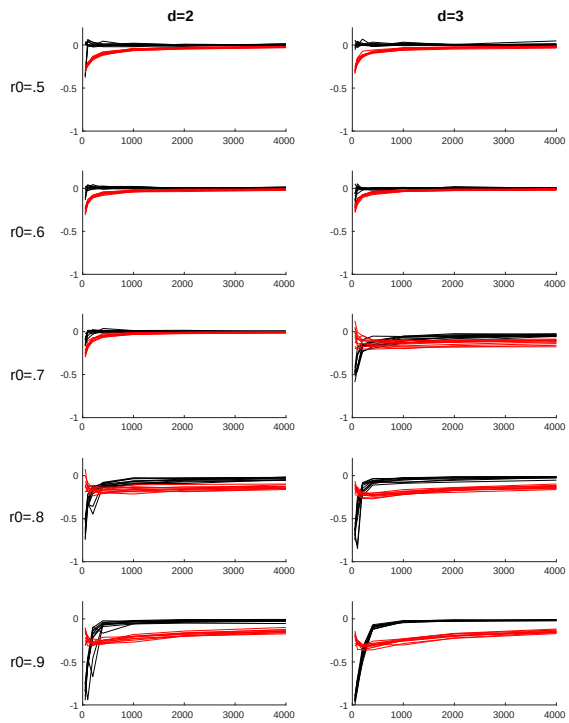

Figure 13: We show in each pannel 10 replications of (11) (black) and (12) (red), for different values of d and r. Varying $n \in \{50, 100, 200, 500, 1000, 2000, 4000\}$.

the collection of all the affine $(d-1)$-dimensional tangent spaces.

Definition 8.1. Let $\varepsilon \geq 0$. A line $r_{\theta,y} = y + \mathbb{R}\theta$ fulfills **condition** $L(\varepsilon)$ if y is at a distance larger than 4ε from all the affine hyper-planes $w + \eta^{\perp} \in \mathcal{T}_S$ satisfying $\langle \eta, \theta \rangle = 0$, i.e. for all $x \in \partial S$ such that $\langle \eta_x, \theta \rangle = 0$ we have that $d(y, x + \eta_x^{\perp}) > 4\varepsilon$

For a given θ, we define

$$\mathcal{A}_\varepsilon(\theta) = \Big\{ y \in \theta^{\perp} : ||y|| \leq diam(S) \text{ and } r_{\theta,y} \text{ does not satisfy } L(\varepsilon) \Big\}.$$

Remark 3. *Notice that $\ell_\theta(y) = \min_{x \in E_\theta(\partial S)} d(y, x + \eta_x^{\perp})$ is well defined because E_θ is compact and $x \mapsto \eta_x$ is a continuous function, due to the regularity of ∂S. Moreover, if $y \in \theta^{\perp}$, then $\ell_\theta(y) = d(y, F_\theta)$ and, as a consequence, for all $t \in (0, d(y, F_\theta)/4)$, $r_{\theta,y}$ satisfies the condition $L(d(y, F_\theta)/4 - t)$.*

8.1.2 Some useful lemmas

Lemma 8.2. *Let S be a compact set fulfilling the outside and inside α-rolling conditions. Let $r_{\theta,y}$ be a line that fulfills condition $L(0)$ and $r_{\theta,y} \cap \partial S \neq \emptyset$. Then $r_{\theta,y}$ intersects ∂S in a finite number of points.*

Proof. Because S fulfills the outside and inside α-rolling conditions, Theorem 1 in [42] implies that for any $x \in \partial S$, the affine $(d-1)$-dimensional tangent space $T_x\partial S$ exists. If $r_{\theta,y}$ fulfills $L(0)$, then $r_{\theta,y}$ is not included in any hyper-plane tangent to S. Suppose that $\partial S \cap r_{\theta,y}$ is not finite. Then, by compactness, one can extract a subsequence $t'_n \in \partial S \cap r_{\theta,y}$ that converges to $y' \in \partial S \cap r_{\theta,y}$.

1. Because t'_n and y' are in $r_{\theta,y}$, we have that, for all n, $(t'_n - y')/||t'_n - y'|| = \pm\theta$.

2. Because t'_n and y' are in ∂S, which is a $(d-1)$-dimensional $\mathcal{C}^1$ manifold (see Theorem 1 in [42]), and $t'_n \to y'$, we have $\lim_{n\to+\infty}(t'_n - y')/||t'_n - y'|| \in T_{y'}\partial S$, (see Definition 4.3 in [25]).

These two facts imply that $\theta \in T_{y'}\partial S$, which contradicts the assumption that $r_{\theta,y}$ is not included in any hyper-plane tangent to S. □

Lemma 8.3. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside α-rolling conditions. Let $\varepsilon > 0$ be such that $\varepsilon < \alpha/4$ and $\nu = 2[2\varepsilon(\alpha - 2\varepsilon)]^{1/2}$. For any line $r_{\theta,y}$ fulfilling condition $L(\varepsilon)$ and $r_{\theta,y} \cap \partial S \neq \emptyset$, we have that $r_{\theta,y}$ meets ∂S at a finite number of points $t_1, \dots, t_k$, where $t_{i+1} - t_i \geq 2\nu$ for all $i = 1, \dots, k-1$. Consequently, if $\varepsilon < \alpha/4$, then $k = \#(r_{\theta,y} \cap \partial S) \leq diam(S)\varepsilon^{-1/2}/(4\sqrt{\alpha})$.*

Proof. If a line fulfills condition $L(\varepsilon)$, then it fulfills condition $L(0)$, consequently, the fact that $r_{\theta,y}$ intersects ∂S in a finite number of points follows from Lemma 8.2. Let us denote by $t_1 < \cdots < t_k$ the intersection of $r_{\theta,y}$ with ∂S.

Let us denote by η_{t_i} and $\eta_{t_{i+1}}$ the outer normal vectors at t_i and t_{i+1}, respectively. We have two cases: the open interval $(t_i, t_{i+1}) \subset S^c$ or $(t_i, t_{i+1}) \subset \mathring{S}$. Let us consider the first case (the proof for the second one is similar).

Because $(t_i, t_{i+1}) \subset \overline{S^c}$ and S fulfills the inside α-rolling condition on t_i, there exists a $z \in S$ such that $t_i \in \partial\mathcal{B}(z, \alpha)$ and $\mathcal{B}(z, \alpha) \subset S$. In particular, $\mathcal{B}(z, \alpha) \cap (t_i, t_{i+1}) = \emptyset$, which implies $\langle \eta_{t_i}, \theta \rangle \geq 0$.

Reasoning in the same way but with t_{i+1}, we get $\langle \eta_{t_{i+1}} \theta \rangle \leq 0$. Given that $r_{\theta,y}$ is not included in any tangent hyperplane, we have that $\langle \eta_{t_i}, \theta \rangle > 0$ and $\langle \eta_{t_{i+1}}, \theta \rangle < 0$.

If, for some i, $t_{i+1} - t_i < 2\nu$, then, by Theorem 3.8 in [16], there exists a curve $\gamma : [0, 1] \to \partial S$ such that $\gamma(0) = t_i$, $\gamma(1) = t_{i+1}$ and $d(\gamma(t), r_{\theta,y}) \leq 4\varepsilon$ for all t. We also have the inside and outside α-rolling conditions, something which implies that $x \mapsto \eta_x$ is Lipschitz (see Theorem 1 in [42]). From $\langle \eta_{t_i}, \theta \rangle > 0$ and $\langle \eta_{t_{i+1}}, \theta \rangle < 0$, it follows that there exists an $s_0 \in (0, 1)$ such that $\langle \eta_{\gamma(s_0)}, \theta \rangle = 0$, which contradicts the hypothesis that y is at a distance larger than 4ε from all the $(d-1)$-dimensional hyperplanes tangent to S. This proves that $t_{i+1} - t_i \geq 2\nu$ for all $i = 1, \dots, k-1$. □

Lemma 8.4. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside α-rolling conditions, with a (C, ε_0)-regular boundary. Then for all $\varepsilon \leq \min\{\varepsilon_0, \alpha\}/4$,*

$$\int_{\theta \in (\mathbb{S}^+)^{d-1}} \int_{y \in \mathcal{A}_\varepsilon(\theta)} n_{\partial S}(\theta, y) d\mu_{d-1}(y) d\theta \leq 2C \frac{diam(S)}{\sqrt{\alpha}} \sqrt{\varepsilon}.$$

Moreover, if ∂S has bounded number of linear intersections, then

$$\int_{\theta \in (\mathbb{S}^+)^{d-1}} \int_{y \in \mathcal{A}_\varepsilon(\theta)} n_{\partial S}(\theta, y) d\mu_{d-1}(y) d\theta \leq 4CN_S\varepsilon. \tag{13}$$

Proof. Observe that

$$\int_{\theta\in(\mathbb{S}^+)^{d-1}}\int_{y\in\mathcal{A}_\varepsilon(\theta)} n_{\partial S}(\theta,y)d\mu_{d-1}(y)d\theta$$
$$=\int_{\theta\in(\mathbb{S}^+)^{d-1}}\int_{\ell=0}^{4\varepsilon}\int_{\{y\in\theta^\perp:d(y,F_\theta)=\ell\}} n_{\partial S}(\theta,y)d\mu_{d-2}(y)d\ell d\theta.$$

According to Remark 3 if $y\in\theta^\perp : d(y,F_\theta)=\ell$, then, for all $t\in(0,\ell)$, $r_{\theta,y}$ fulfills $L(\ell/4-t)$. From the proof of the previous lemma, it follows that for any $y\in\theta^\perp$ with $d(y,F_\theta)=\ell$ and $\ell<4\varepsilon$, and any $t\in(0,\ell/4)$

$$n_{\partial S}(\theta,y)\le \operatorname{diam}(S)(\ell/4-t)^{-1/2}/(4\sqrt{\alpha}).$$

Hence, with $t\to 0$ we obtain $n_{\partial S}(\theta,y)\le \operatorname{diam}(S)(\ell)^{-1/2}/(2\sqrt{\alpha})$, from which:

$$\int_{\theta\in(\mathbb{S}^+)^{d-1}}\int_{\ell=0}^{4\varepsilon}\int_{\{y\in\theta^\perp:d(y,F_\theta)=\ell\}} n_{\partial S}(\theta,y)d\mu_{d-2}(y)d\ell d\theta =$$
$$\le \int_{\theta\in(\mathbb{S}^+)^{d-1}}\int_{\ell=0}^{4\varepsilon}\int_{\{y\in\theta^\perp:d(y,F_\theta)=\ell\}} \frac{1}{2}\operatorname{diam}(S)(\alpha\ell)^{-1/2}d\mu_{d-2}(y)d\ell d\theta$$
$$\le \int_{\theta\in(\mathbb{S}^+)^{d-1}}\int_{\ell=0}^{4\varepsilon} \frac{1}{2}\operatorname{diam}(S)(\alpha\ell)^{-1/2}\int_{\{y\in\theta^\perp d(y,F_\theta)=\ell\}} d\mu_{d-2}(y)d\ell d\theta$$
$$\le \int_{\theta\in(\mathbb{S}^+)^{d-1}}\int_{\ell=0}^{4\varepsilon} \frac{1}{2}\operatorname{diam}(S)(\alpha\ell)^{-1/2}|\{y\in\theta^\perp : d(y,F_\theta)=\ell\}|_{d-2}d\ell d\theta.$$

By the definition of φ_θ,

$$\left|\{y\in\theta^\perp:\ell\le d(y,F_\theta)\le\ell+d\ell\}\right|_{d-1}=\varphi_\theta(\ell+d\ell)-\varphi_\theta(\ell).$$

From the (C,ε_0)-regularity of ∂S and the mean value theorem we obtain

$$\left|\{y\in\theta^\perp : d(y,F_\theta)=\ell\}\right|_{d-2}\le \sup_{\varepsilon\in(0,4\varepsilon_0)}\varphi'_\theta(\varepsilon)\le C,$$

which implies

$$\int_{\theta\in(\mathbb{S}^+)^{d-1}}\int_{y\in\mathcal{A}_\varepsilon(\theta)} n_{\partial S}(\theta,y)d\mu_{d-1}(y)d\theta\le$$
$$\int_{\theta\in(\mathbb{S}^+)^{d-1}}\int_{\ell=0}^{4\varepsilon} C\frac{1}{2}\operatorname{diam}(S)(\alpha\ell)^{-1/2}d\ell d\theta\le 2C\frac{\operatorname{diam}(S)}{\sqrt{\alpha}}\sqrt{\varepsilon}.$$

Applying exactly the same reasoning, under the hypothesis of the boundedness of the number of linear intersections for ∂S, we get

$$\int_{\theta\in(\mathbb{S}^+)^{d-1}}\int_{y\in\mathcal{A}_\varepsilon(\theta)} n_{\partial S}(\theta,y)d\mu_{d-1}(y)d\theta\le\int_{\theta\in(\mathbb{S}^+)^{d-1}}\int_{\ell=0}^{4\varepsilon} CN_S d\ell d\theta\le 4CN_S\varepsilon.$$

□

Remark 4. *If in the proof of Lemma 8.4 we take $\ell = 0$, we obtain that the measure of the set of lines belonging to some half-space tangent to ∂S is 0.*

Lemma 8.5. *Let S be a compact set fulfilling the outside and inside α-rolling conditions. Let $\mathbb{X}_n = \{X_1, \dots, X_n\} \subset S$. Let $\varepsilon_n \to 0$ be such that $d_H(\mathbb{X}_n, S) \leq \varepsilon_n$. Let $r_{\theta,y} = y + \mathbb{R}\theta$ be any line fulfilling condition $L(\varepsilon_n)$. Then, for n large enough so that $4\varepsilon_n < \alpha$, $n_{\partial S}(\theta, y) = \hat{n}_{\varepsilon_n, \mathbb{X}_n}(\theta, y)$.*

Proof. Throughout this proof we will use the following notation when $r_{\theta,y} \cap \partial S \neq \emptyset$. Let $t_1 < \ldots < t_{2k}$ be the intersection of $r_{\theta,y}$ with ∂S. This set is finite due to Lemma 8.2 and is an even number because condition $L(\varepsilon_n)$ is fulfilled. Also $[t_{2(i-1)+1}, t_{2i}] \subset S$ for all $i = 1, \dots, k$ and $(t_{2i}, t_{2i+1}) \subset S^c$ for all $i = 1, \dots, k-1$.

First, we will prove that

$$\hat{n}_{\varepsilon_n, \mathbb{X}_n}(\theta, y) \geq n_{\partial S}(\theta, y). \tag{14}$$

If $r_{\theta,y} \cap \partial S = \emptyset$, then inequality (14) holds. Assume $r_{\theta,y} \cap \partial S \neq \emptyset$. We will now prove that

$$\text{if } (t_i, t_{i+1}) \subset S^c, \text{ then: } \exists s \in (t_i, t_{i+1}) \text{ such that } d(s, S) > 4\varepsilon_n. \tag{15}$$

Because S fulfills the inside α-rolling condition on t_i, there exists a $z_i \in S$ such that $t_i \in \partial\mathcal{B}(z_i, \alpha)$ and $\mathcal{B}(z_i, \alpha) \subset S$. Since $\mathcal{B}(z_i, \alpha) \cap (t_i, t_{i+1}) = \emptyset$, it follows that $\langle \eta_{t_i}, \theta \rangle \geq 0$ (recall that $\eta_{t_i} = (t_i - z_i)/\alpha$ and $t_{i+1} - t_i = ||t_{i+1} - t_i||\theta$). Reasoning in the same way but with t_{i+1}, $\langle \eta_{t_{i+1}}, \theta \rangle \leq 0$. By condition $L(\varepsilon_n)$, we obtain

$$\langle \eta_{t_i}, \theta \rangle > 0 \text{ and } \langle \eta_{t_{i+1}}, \theta \rangle < 0. \tag{16}$$

Suppose that for all $t \in (t_i, t_{i+1})$ we have $d(t, \partial S) \leq 4\varepsilon_n$. Take n large enough so that $4\varepsilon_n < \alpha$. Because S fulfills the outside and inside α-rolling conditions, by Lemma 2.3 in [33], ∂S has positive reach greater than α. Then, by Theorem 4.8 in [25], $\gamma = \{\gamma(t) = \pi_{\partial S}(t), t \in (t_i, t_i + 1)\}$, the orthogonal projection onto ∂S of the interval (t_i, t_{i+1}) is well defined and is a continuous curve in ∂S. By Theorem 1 in [42], the map from ∂S to $\mathbb{R}^d$ $x \mapsto \eta_x$ is Lipschitz. Thus, $t \mapsto \langle \eta_{\gamma(t)}, \theta \rangle$ is a continuous function of t for all $t \in (t_i, t_{i+1})$, which, together with (16), ensures the existence of an $s \in (t_i, t_{i+1})$ such that $d(s, \gamma(s)) \leq 4\varepsilon_n$ and $\theta \in \eta^{\perp}_{\gamma(s)}$, which contradicts the assumption that $r_{\theta,y}$ fulfills condition $L(\varepsilon_n)$. This proves (15).

From (15) we easily obtain (since $s \in S^c$ and $\mathbb{X}_n \subset S$) that

$$\text{if } (t_i, t_{i+1}) \subset S^c, \text{ then: } \exists s \in (t_i, t_{i+1}) \text{ such that } d(s, \mathbb{X}_n) > 4\varepsilon_n. \tag{17}$$

To conclude (14) let us write, for $i = 1, \dots, k$, $I'_i = [t_{2(i-1)+1}, t_{2i}]$ for the connected components of $S \cap r_{\theta,y}$. Since $d_H(\mathbb{X}_n, S) < \varepsilon_n$, $S \subset \hat{S}_{\varepsilon_n}(\mathbb{X}_n)$. Then, for $i = 1, \dots, k$, there exists a j such that $I'_i \subset I_j$, I_j being a connected component of $\hat{S}_{\varepsilon_n} \cap r_{\theta,y}$. Note now that (17) ensures that, for all $i \neq i'$, if $I'_i \subset I_j$ and $I'_{i'} \subset I_{j'}$ then $I_{j'}$ and I_j are not in the same connected component of $\hat{S}_{4\varepsilon_n}(\mathbb{X}_n)$ thus they are not glued, and then $\hat{n}_{\varepsilon_n, \mathbb{X}_n}(\theta, y) \geq n_{\partial S}(\theta, y)$.
Next, we will prove the opposite inequality,

$$\hat{n}_{\varepsilon_n, \mathbb{X}_n}(\theta, y) \leq n_{\partial S}(\theta, y). \tag{18}$$

Assume first $r_{\theta,y} \cap \partial S \neq \emptyset$. Consider $t^* \in (t_i, t_{i+1}) \subset S^c$ and $t^* \in \hat{S}_{\varepsilon_n}(\mathbb{X}_n)$. Equation (18) will be derived from the fact that $(t^*, t_{i+1}] \subset \hat{S}_{4\varepsilon_n}(\mathbb{X}_n) \cap r_{\theta,y}$ or

$[t_i, t^*) \subset \hat{S}_{4\varepsilon_n}(\mathbb{X}_n) \cap r_{\theta,y}$ and thus the connected component of $\hat{S}_{\varepsilon_n}(\mathbb{X}_n) \cap r_{\theta,y}$ that contained t^* is glued with the one that contains $[t_{i-1}, t_i]$ or with the one that contains $[t_i, t_{i+1}]$.

Introduce $\psi(t) : (t_i, t_{i+1}) \to \mathbb{R}$ defined by $\psi(t) = d(t, \partial S)$. Consider the points $t \in (t_i, t_{i+1})$ such that $d(t, \partial S) < \alpha$, and let $p_t \in \partial S$ be such that $||p_t - t|| = d(t, \partial S)$. By item (3) in Theorem 4.8 in [25], $\psi'(t) = \langle \eta_{p_t}, \theta \rangle$.

Let X_j be the closest observation to t^* (recall that because $t^* \in \hat{S}_{\varepsilon_n}(\mathbb{X}_n)$, we have $||X_j - t^*|| \leq \varepsilon_n$). Now, because there exists a point $p^* \in [t^*, X_j] \cap \partial S$, we obtain that $\psi(t^*) \leq \varepsilon_n$ and, because $r_{\theta,y}$ fulfils $L(\varepsilon_n)$, $\langle \eta_{p_{t^*}}, \theta \rangle \neq 0$.

Assume that, for instance, $\langle \eta_{p_{t^*}}, \theta \rangle < 0$. Then $\psi(t^*) \leq \varepsilon_n$ and $\psi'(t^*) < 0$. Suppose that there exists a $t' \in (t^*, t_{i+1})$ such that $\psi(t') \geq \varepsilon_n$ and consider $t'' = \inf\{t > t^*, \psi(t') \geq \varepsilon_n\}$. Then for all $t \in (t^*, t'')$, we have $\psi(t) \leq \varepsilon_n < \alpha$, and thus ψ is differentiable on this interval (using again item (3) of Theorem 4.8 in [25]). From the fact that $\psi(t'') \geq \psi(t^*)$ and $\psi'(t^*) < 0$ we deduce that there exists a $\tilde{t} \in (t^*, t'')$ such that $\psi'(\tilde{t}) = 0$, which contradicts $L(\varepsilon_n)$ because $\psi(\tilde{t}) \leq \varepsilon_n$. To summarize, we have shown that if $\langle \eta_{p_{t^*}}, \theta \rangle < 0$, then for all $t \in (t^*, t_{i+1})$ we have that $d(t, \partial S) \leq \varepsilon_n$, and thus $(t^*, t_{i+1}) \subset \hat{S}_{2\varepsilon_n}(\mathbb{X}_n) \subset \hat{S}_{4\varepsilon_n}(\mathbb{X}_n)$.

Symmetrically, if $\langle \eta_{p_{t^*}}, \theta \rangle > 0$, then $(t_i, t^*) \subset \hat{S}_{2\varepsilon_n}(\mathbb{X}_n) \subset \hat{S}_{4\varepsilon_n}(\mathbb{X}_n)$.

Thus we now have that if $r_{\theta,y} \cap \partial S \neq \emptyset$, then $\hat{n}_{\varepsilon_n,\mathbb{X}_n}(\theta, y) \leq n_{\partial S}(\theta, y)$.

Now we are going to prove that for a line $r_{\theta,y}$ fulfilling condition $L(\varepsilon_n)$ we can not have $r_{\theta,y} \cap \partial S = \emptyset$ and $\hat{n}_{\varepsilon_n,\mathbb{X}_n}(\theta, y) > 0$. Reasoning by contradiction, upon assuming that $r_{\theta,y} \cap \partial S = \emptyset$ and $\hat{n}_{\varepsilon_n,\mathbb{X}_n}(\theta, y) > 0$, we have that $0 < \min\{||x - y||, x \in r_{\theta,y}, y \in S\} \leq \varepsilon_n$. Now the regularity condition also gives that if this minimum is realized for x^* and y^* we have $y^* \in \partial S$ and $\theta \in T_{y^*}\partial S$, which contradicts condition $L(\varepsilon_n)$. □

Lemma 8.6. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside α-rolling conditions. Let $\mathbb{X}_n \subset S$ and suppose $\varepsilon_n \to 0$ is a sequence such that $d_H(\mathbb{X}_n, S) \leq \varepsilon_n$, while $r_{\theta,y} = y + \mathbb{R}\theta$ and $A_1, \ldots, A_k$ are the sets in Definition 3.1, $A_i = (a_i, b_i)$ for $i = 1, \ldots, k$. Now suppose that the sets are indexed in such a way that $a_1 < b_1 < a_2 < \ldots < b_k$. Then for all $i = 2, \ldots, k$, we have that $||a_i - b_{i-1}|| > 3\sqrt{\varepsilon_n \alpha}$ and for all $i = 1, \ldots, k$, $||b_i - a_i|| > 3\sqrt{\varepsilon_n \alpha}$, for n large enough so that $3\sqrt{\alpha \varepsilon_n} < \alpha/2$, which implies*

$$\hat{n}_{\varepsilon_n,\mathbb{X}_n}(\theta, y) \leq \frac{diam(S)}{3\sqrt{\alpha}} \varepsilon_n^{-1/2}.$$

Proof. Assume by contradiction that for some i, $||a_i - b_{i-1}|| \leq 3\sqrt{\varepsilon_n \alpha}$. By construction, $[b_{i-1}, a_i] \subset \hat{S}_{\varepsilon_n}(\mathbb{X}_n)^c \subset S^c$. Because a_i and b_i are on $\partial \hat{S}_{\varepsilon_n}(\mathbb{X}_n)$, we have $d(a_i, \mathbb{X}_n) = d(b_{i-1}, \mathbb{X}_n) = \varepsilon_n$.

The projection $\pi_S : [b_{i-1}, a_i] \to \partial S$ is uniquely defined because ∂S has reach at least α and $d(t, \partial S) \leq d(t, a_i) + d(a_i, \partial S) \leq ||a_i - b_{i-1}|| + d(a_i, \mathbb{X}_n)$ for all $t \in (b_{i-1}, a_i)$, $||a_i - b_{i-1}|| \leq 3\sqrt{\varepsilon_n \alpha} < \alpha/2$ and $d(a_i, \partial S) \leq \varepsilon_n \leq \alpha/2$. Moreover, π_S is a continuous function.

Hence $\max_{x \in [b_{i-1}, a_i]} ||x - \pi_S(x)|| \geq \varepsilon_n - d_H(S, \mathbb{X}_n)$, and the maximum is attained at some $x_0 \in [b_{i-1}, a_i]$. First, we show that $||x_0 - \pi_S(x_0)|| \geq 3\varepsilon_n$. Indeed, suppose by contradiction that for all $t \in (b_{i-1}, a_i)$, $d(t, \partial S) \leq 3\varepsilon_n$. Then $d(t, \mathbb{X}_n) \leq 4\varepsilon_n$, which contradicts the definition of the points a_i and b_i. The fact that $||x_0 - \pi_S(x_0)|| \geq 3\varepsilon_n > d(a_i, S) = d(b_{i-1}, S)$ guarantees that

$x_0 \in (b_{i-1}, a_i)$ and that η_0, the outward unit normal vector to ∂S at $\pi_S(x_0)$, is normal to θ.

Let $z_0 = \pi_S(x_0) + \eta_0\alpha$. Observe that $d(a_i, S) \leq \varepsilon_n$ and $d(b_{i-1}, S) \leq \varepsilon_n$. From the outside α-rolling condition at $\pi_S(x_0)$, $||x_0 - \pi_S(x_0)|| \leq \alpha$ and using the fact that η_0 is normal to θ, we have (see Figure 14)

$$r_{\theta,y} \cap \mathcal{B}(z_0, \alpha - \varepsilon_n) \subset [b_{i-1}, a_i],$$

which implies, see Figure 14, that $||a_i - b_{i-1}|| \geq 2\sqrt{(\alpha - \varepsilon_n)^2 - (\alpha - \ell)^2}$, where $\ell = d(x_0, \pi_S(x_0))$. Therefore,

$$||a_i - b_{i-1}|| \geq 2\sqrt{(\ell - \varepsilon_n)(2\alpha - \ell - \varepsilon_n)}. \tag{19}$$

If we bound $\ell \geq 3\varepsilon_n$ and use the fact that $\ell = o(1)$, which follows from $\ell \leq ||b_{i-1} - a_i|| + \varepsilon_n \leq 3\sqrt{\varepsilon_n\alpha} + \varepsilon_n$, then we get, from (19),

$$||a_i - b_{i-1}|| \geq 2\sqrt{2\varepsilon_n(2\alpha - \ell - \varepsilon_n)} = 2\sqrt{4\varepsilon_n\alpha(1 + o(1)))} = 4\sqrt{\alpha\varepsilon_n}(1 + o(1)),$$

and for n large enough this contradicts $||a_i - b_{i-1}|| \leq 3\sqrt{\alpha\varepsilon_n}$. Then, the number

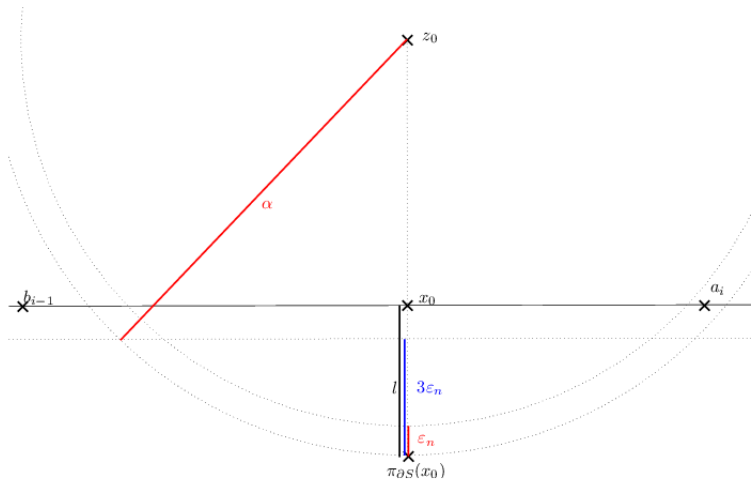

Figure 14: $||a_i - b_{i-1}|| \geq 2\sqrt{(\alpha - \varepsilon_n)^2 - (\alpha - \ell)^2}$, where $\ell = d(x_0, \pi_S(x_0))$.

of disjoint intervals A_i is bounded from above by $\text{diam}(S)/(3\sqrt{\varepsilon_n\alpha})$. The proof that for all $i = 1, \ldots, k$, $||b_i - a_i|| > 3\sqrt{\varepsilon_n\alpha}$ follows the same ideas, we will give a sketch of the proof. Let $b_i > a_i$ (recall that we ordered the points $a_1 < b_1 < \ldots < a_k < b_k$) be such that $||a_i - b_i|| \leq 3\sqrt{\varepsilon_n\alpha}$. Proceeding as before, $\max_{x \in [a_i, b_i]} ||x - \pi_S(x)|| \geq 3\varepsilon_n$ and it is attained at some $x_0 \in (a_i, b_i)$. Let $z_0 = \pi_S(x_0) - \eta_0\alpha$, with η_0 being the outward unit normal vector to ∂S at $\pi_S(x_0)$. Then $r_{\theta,y} \cap \mathcal{B}(z_0, \alpha) \subset [a_i, b_i]$ since $[a_i, b_i] \notin \mathcal{B}(z_0, \alpha)$ and $\mathcal{B}(z_0, \alpha) \subset S$. From $r_{\theta,y} \cap \mathcal{B}(z_0, \alpha) \subset [a_i, b_i]$ it follows as before that $||a_i - b_i|| \geq 4\sqrt{\varepsilon_n\alpha}$, which is a contradiction.

Lastly, $\hat{n}_{\varepsilon_n, \mathbb{X}_n}(\theta, y) \leq \text{diam}(S)/(3\sqrt{\varepsilon_n\alpha})$. □

Corollary 5. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside α-rolling conditions and with a (C, ε_0)-regular boundary. For n large enough so that $3\sqrt{\alpha\varepsilon_n} < \min(\alpha/2, \varepsilon_0)$, we have*

$$\int_\theta \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} \hat{n}_{\varepsilon_n, \mathbb{X}_n}(\theta, y) d\mu_{d-1}(y) d\theta \leq C \frac{diam(S)}{3\sqrt{\alpha}} \sqrt{\varepsilon_n}.$$

8.1.3 Proof of Theorem 3.2

Without loss of generality, we can assume that $0 \in S$. Recall that for $\theta \in (\mathcal{S}^+)^{d-1}$, $\mathcal{A}_{\varepsilon_n}(\theta)$ is the set of all $y \in \theta^\perp$ such that $||y|| \leq \text{diam}(S)$ and $r_{\theta,y}$ does not fulfill $L(\varepsilon_n)$. First, from Lemma 8.5, we have that $||\partial S|_{d-1} - \hat{I}_{d-1}(\mathbb{X}, \varepsilon)|$ is bounded from above by

$$\frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} |\hat{n}_{\varepsilon_n, \mathbb{X}_n}(\theta, y) - n_{\partial S}(\theta, y)| d\mu_{d-1}(y) d\theta,$$

which is bounded from above by

$$\frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} \hat{n}_{\varepsilon_n, \mathbb{X}_n}(\theta, y) d\mu_{d-1}(y) d\theta + \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} n_{\partial S}(\theta, y) d\mu_{d-1}(y) d\theta.$$

Now, by Corollary 5 and Lemma 8.4, we get that

$$||\partial S|_{d-1} - \hat{I}_{d-1}(\mathbb{X}, \varepsilon)| \leq \frac{7C\text{diam}(S)}{3\beta(d)\sqrt{\alpha}} \sqrt{\varepsilon_n},$$

for n large enough.

8.1.4 Proof of Theorem 3.3

The proof of Theorem 3.3 is basically the same as the previous one. Since $N_0 \geq N_S$ Lemma 8.5 ensures that, for all $r_{y,\theta}$ not in $\mathcal{A}_{\varepsilon_n}(\theta)$, $\min(\hat{n}(\theta, y), N_0) = n_{\partial S}(\theta, y)$, for n large enough that $4\varepsilon_n < \alpha$. Thus we still have, for n large enough, $||\partial S|_{d-1} - \hat{I}^{N_0}_{d-1}(\partial S)|$ is bounded from above

$$\frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} n_{\partial S}(\theta, y) d\mu_{d-1}(y) d\theta + \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} N_0 d\mu_{d-1}(y) d\theta.$$

Now, by applying (13) for the first part and a similar calculation for the second part, we get that

$$||\partial S|_{d-1} - \hat{I}^{N_0}_{d-1}(\partial S)| \leq \frac{4C(N_S + N_0)}{\beta(d)} \varepsilon_n, \tag{20}$$

for n large enough.

8.1.5 Proof of Corollary 3

By Corollary 1 in [15], we know that, with probability one, for T large enough, $d_H(\mathbb{X}_T, S) \leq \varepsilon_T \to 0$, where $\varepsilon_T = o((\ln(T)^2/T)^{1/d})$. Let $\mathbb{X}_n = \{X_{t_1}, \ldots, X_{t_n}\}$ be a discretization of $\mathbb{X}_T$ such that $t_i - t_{i-1} = T/n$ and $t_n = T$. Put $\varepsilon_n = d_H(\mathbb{X}_n, S)$; then $\varepsilon_n \geq \varepsilon_T$. It is clear that, for a fixed T, ε_n decreases to ε_T as $n \to \infty$. To emphasize the dependence on the set, we will write $\hat{I}_{d-1}^{N_0}(\partial S, \mathbb{X}_n)$ for the estimator based on $\mathbb{X}_n$, and $\hat{I}_{d-1}^{N_0}(\partial S, \mathbb{X}_T)$ for the estimator based on $\mathbb{X}_T$ (both defined using Definition 3.1). Then by (20), to prove Corollary 3 it is enough to prove $\hat{I}_{d-1}^{N_0}(\partial S, \mathbb{X}_n) \to \hat{I}_{d-1}^{N_0}(\partial S, \mathbb{X}_T)$ as $n \to \infty$, for arbitrary fixed T. Fix θ and y. It is clear that $\hat{n}(\theta, y)(\partial S, \mathbb{X}_n) \to \hat{n}(\theta, y)(\partial S, \mathbb{X}_T)$ as $n \to \infty$, and so Corollary 3 follows by the dominated convergence theorem, using the fact that $\min\{\hat{n}(\theta, y), N_0\} \leq N_0$.

8.2 Proofs for the estimator based on the α-hull

Theorem 4.1 will be easily obtained from the two following geometric lemmas and Theorem 3 in [37].

Here we need to introduce some new notation. If f is a function, then $\nabla_f(x)$ denotes its gradient and $\mathcal{H}_f$ its Hessian matrix. Given two sets $C, D \subset \mathbb{R}^d$, we write $C \approx D$ if there exists an homeomorphism between C and D. In what follows, $M \subset \mathbb{R}^d$ will be a compact set, and $\mathcal{C}^2$ a $(d-1)$-dimensional manifold (with or without boundary). Then for all x in M, there exists an $r_x > 0$ such that either

i) for all $r \leq r_x$, $\mathring{\mathcal{B}}(x, r) \cap M \approx \mathring{\mathcal{B}}_{d-1}(0, 1)$, or

ii) for all $r \leq r_x$, $\mathring{\mathcal{B}}(x, r) \cap M \approx \mathring{\mathcal{B}}_{d-1}(0, 1) \cap \{(x_1, \ldots, x_{d-1}) : x_1 \geq 0\}$.

The set of points satisfying condition *i)* constitute $\text{int}(M)$, while the set of points satisfying *ii)* constitute ∂M. We have that ∂M is a $(d-2)$-dimensional manifold without boundary and, as a consequence, $|\partial M|_{d-1} = 0$.

Given a point $x \in M$, $N_x M = \{v \in \mathbb{R}^d : \langle v, u \rangle = 0, \forall u \in T_x M\}$ is the 1-dimensional orthogonal subspace. If M is a manifold as before, and $\partial M = \emptyset$, we define for any compact set $E \subset M$ (E is not necessarily a manifold) its interior $\text{int}(E) = \{x \in E : \exists r_x \text{ such that for all } r \leq r_x, \mathring{\mathcal{B}}(x, r) \cap E \approx \mathring{\mathcal{B}}_{d-1}(0, 1)\}$. We have $\text{int}(E)$ is a manifold (without boundary and, when is not empty $\text{int}(E)$ has the same dimension as M).

Lemma 8.7. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the inside and outside α-rolling conditions. Let $\alpha' < \alpha$ be a positive constant. Let $\mathbb{X}_n = \{X_1, \ldots, X_n\} \subset S$ be such that :*

i. $d_H(\partial C_{\alpha'}(\mathbb{X}_n), \partial S) \leq \varepsilon_n$ *with* $\varepsilon_n < \frac{\alpha\alpha'}{2(\alpha+\alpha')}$ *(notice that we then have* $\varepsilon_n \leq \alpha'/2$ *and* $\varepsilon_n \leq \alpha/4$*).*

ii. $d_H(\mathbb{X}_n, S) < \frac{1}{3}\frac{\alpha\alpha'}{\alpha+\alpha'}$ *note that* $\frac{1}{3}\frac{\alpha\alpha'}{\alpha+\alpha'} \leq \frac{\alpha'}{3}$

Then,

1. there exist $C_1(\mathbb{X}_n), \ldots, C_K(\mathbb{X}_n)$ *such that:*

(a) $\bigcup_{i=1}^K C_i(\mathbb{X}_n) \subset \partial C_{\alpha'}(\mathbb{X}_n)$

(b) $|\partial C_{\alpha'}(\mathbb{X}_n) \setminus (\bigcup_{i=1}^K C_i(\mathbb{X}_n))|_{d-1} = 0$

(c) $C_i(\mathbb{X}_n)$ *is a* $\mathcal{C}^2$ $(d-1)$*-dimensional manifold*

(d) $C_i(\mathbb{X}_n) \cap C_j(\mathbb{X}_n) = \emptyset$ *when* $i \neq j$

for all $x \in \bigcup_{i=1}^K C_i(\mathbb{X}_n)$*, there exists a* $\hat{\eta}_x$*, the unit normal (to* $\partial C_{\alpha'}(\mathbb{X}_n)$*), a vector pointing outward (with respect to* $C_{\alpha'}(\mathbb{X}_n)$*) from* x *that satisfies*

$$\langle \hat{\eta}_x, \eta_{\pi_{\partial S}(x)} \rangle \geq 1 - \frac{2(\alpha+\alpha')}{\alpha\alpha'}\varepsilon_n.$$

2. $\pi_{\partial S} : \partial C_{\alpha'}(\mathbb{X}_n) \to \partial S$ *the orthogonal projection onto* ∂S *is one to one.*

3. $\partial C_{\alpha'}(\mathbb{X}_n) \approx \partial S$

Proof. Let us prove first that there are no isolated points in $\partial C_{\alpha'}(\mathbb{X}_n)$. Indeed, suppose by contradiction that there exists x is an isolated point of $\partial C_{\alpha'}(\mathbb{X}_n)$. That is, there exists $r > 0$ such that $B(x,r) \cap \partial C_{\alpha'}(\mathbb{X}_n) = \{x\}$. By connectedness of $B(x,r) \setminus \{x\}$ we have or $B(x,r) \setminus \{x\} \subset C_{\alpha'}(\mathbb{X}_n)^c$ or $B(x,r) \setminus \{x\} \subset C_{\alpha'}(\mathbb{X}_n)$. The second case contradicts $x \in \partial C_{\alpha'}(\mathbb{X}_n)$ since $C_{\alpha'}(\mathbb{X}_n)$ is a close set. Thus we have $B(x,r) \setminus \{x\} \subset C_{\alpha'}(\mathbb{X}_n)^c$. Let us introduce $x^* = \pi_{\partial S}(x)$, then $||x - x^*|| \leq \varepsilon_n$. Let us denote $\eta^* = \eta_{x^*}$. By definition of x^*, $x = x^* - ||x - x^*||\eta^*$. Let us introduce $O = x^* - \alpha\eta^*$. From the inner rolling ball property, $B(O,\alpha) \subset S$. Let us define $y = x - \min(r,\varepsilon_n)\eta^*$. From $y \in C_{\alpha'}(\mathbb{X}_n)^c$ it follows that there exists O_y such that $||O_y - y|| < \alpha'$ and $B(O_y,\alpha') \cap \mathbb{X}_n = \emptyset$. From $d_H(\mathbb{X}_n, S) < \alpha'$ we have $||O_y - O|| > \alpha$, and thus $[O, O_y] \cap \partial B(O,\alpha) \neq \emptyset$. Let us define $z = [O, O_y] \cap \partial B(O,\alpha)$, then $z \in S$ and $B(z, (\alpha' + \alpha - ||O_y - O||)) \cap \mathbb{X}_n = \emptyset$. We will prove that $\alpha' + \alpha - ||O_y - O|| \geq d_H(\mathbb{X}_n, S)$, which is a contradiction.

Because $x \in C_{\alpha'}(\mathbb{X}_n)$ we have $||O_y - x|| \geq \alpha'$. Let us write $O_y = y + a\eta^* + bw$ with $||w|| = 1$ and $w \in (\eta^*)^\perp$, by $||O_y - x|| \geq \alpha'$ and $||O_y - y|| < \alpha'$ it quickly comes that $a^2 + b^2 \leq (\alpha')^2$ and $a \leq \frac{\min(r,\varepsilon_n)}{2}$.

Now

$$\begin{aligned}
||O_y - O||^2 &= (\alpha - \min(r,\varepsilon_n) - ||x - x^*|| + a)^2 + b^2 \\
&\leq (\alpha - \min(r,\varepsilon_n) - ||x - x^*||)^2 + 2a(\alpha - \min(r,\varepsilon_n) - ||x - x^*||) + (\alpha')^2 \\
&\leq (\alpha + \alpha' - \min(r,\varepsilon_n) - ||x - x^*||)^2 - 2(\alpha' - a)(\alpha - \min(r,\varepsilon_n) - ||x - x^*||)
\end{aligned}$$

Thus $||O_y - O||$ is bounded from above by

$$\begin{aligned}
&\leq (\alpha + \alpha' - \min(r,\varepsilon_n) - ||x - x^*||)\sqrt{1 - 2\frac{(\alpha' - a)(\alpha - \min(r,\varepsilon_n) - ||x - x^*||)}{(\alpha + \alpha' - \min(r,\varepsilon_n) - ||x - x^*||)^2}} \\
&\leq \alpha + \alpha' - \min(r,\varepsilon_n) - ||x - x^*|| - \frac{(\alpha' - a)(\alpha - \min(r,\varepsilon_n) - ||x - x^*||)}{(\alpha + \alpha' - \min(r,\varepsilon_n) - ||x - x^*||)}
\end{aligned}$$

and

$$\alpha + \alpha' - ||O_y - O|| \geq (\alpha' - a)\frac{\alpha - \min(r,\varepsilon_n) - ||x - x^*||}{\alpha + \alpha' - \min(r,\varepsilon_n) - ||x - x^*||}$$

$$\alpha + \alpha' - ||O_y - O|| \geq \frac{(\alpha' - \varepsilon_n/2)(\alpha - 2\varepsilon_n)}{\alpha + \alpha'} \geq \frac{\alpha\alpha'}{\alpha + \alpha'}\frac{3}{8} > \frac{1}{3}\frac{\alpha\alpha'}{\alpha + \alpha'} > d_H(\mathbb{X}_n, S)$$

Which, as announced lead to a contradiction.

From (7) it follows that, for some N,

$$\partial C_{\alpha'}(\mathbb{X}_n) = \bigcup_{i=1}^{N} \Big(\partial B_i \setminus \bigcup_{j=1}^{N} B_j\Big).$$

Here, the B_i are balls of radius r_i larger than α' or half-spaces (by abuse of notation, if B_i is an half-space we will put $r_i = +\infty$).

Our first step consists in proving that:

1. If $x \in \partial C_{\alpha'}(\mathbb{X}_n) \setminus \mathbb{X}_n$, then for all i such that $x \in \partial B_i \setminus \bigcup_{j=1}^{N} B_j$, we have $r_i = \alpha'$.

2. If $x \in \partial C_{\alpha'}(\mathbb{X}_n) \cap \mathbb{X}_n$, then there exists an i such that $x \in \partial B_i \setminus \bigcup_{j=1}^{N} B_j$ $r_i = \alpha'$.

We define $S_i = \partial B_i \setminus \bigcup_{j=1}^{N} B_j$.

Suppose that $x \in \partial C_{\alpha'}(\mathbb{X}_n) \setminus \mathbb{X}_n$. Consider first the case $x \in \partial B_i = \mathcal{S}(O_i, r_i)$ with $r_i \geq \alpha'$. If $r_i > \alpha'$, then, introducing $\Omega_i = x + (\alpha'/r_i)(O_i - x) = O_i + (r_i - \alpha')(x - O_i)/r_i$, we have that $\mathcal{B}(\Omega_i, \alpha') \cap \mathbb{X}_n \subset (\mathring{\mathcal{B}}(O_i, r_i) \cup \{x\}) \cap \mathbb{X}_n = \emptyset$. Hence, $d(\Omega_i, \mathbb{X}_n) > \alpha'$, and by continuity, there exists a $t > 0$ so small that $d(\Omega_i + (t/r_i)(x - O_i), \mathbb{X}_n) > \alpha'$, that is, $\mathcal{B}(\Omega_i + (t/r_i)(x - O_i), \alpha') \subset C_{\alpha'}(\mathbb{X}_n)^c$ and so $x \in C_{\alpha'}(\mathring{\mathbb{X}}_n)^c$. This is impossible: to conclude this first step, if $x \in \partial C_{\alpha'}(\mathbb{X}_n) \setminus \mathbb{X}_n$ with $x \in \partial B_i = \mathcal{S}(O_i, r_i)$, then $r_i = \alpha'$.

Second, consider the case $x \in B_i$ with $B_i = \{z, \langle z - y_i, u_i\rangle > 0\}$ where u_i is a unit vector. We can conclude, similarly, on introducing $\Omega_i = x + \alpha' u_i$, that $\mathcal{B}(\Omega_i, \alpha') \cap \mathbb{X}_n = \emptyset$ and $\mathcal{B}(\Omega_i - t u_i, \alpha') \subset C_{\alpha'}(\mathbb{X}_n)^c$ (for some positive but small enough t) and so $x \in C_{\alpha'}(\mathring{\mathbb{X}}_n)^c$.

If $x \in \partial C_{\alpha'}(\mathbb{X}_n) \cap \mathbb{X}_n$, then by the preliminary result, there exists a sequence (x_k) in $\partial C_{\alpha'}(\mathbb{X}_n) \setminus \{x\}$ with $x_k \to x$. Because $\mathbb{X}_n$ is finite, it follows that for k large enough, $x_k \in \partial C_{\alpha'}(\mathbb{X}_n) \cap \mathbb{X}_n^c$. Because the number of possible S_i is finite, we can extract from (x_k) a sequence (x'_k) such that there exists an $S_i = \partial B_i$ such that for all k, $x'_k \in S_i$ making $k \to +\infty$, and then we have $x \in S_i$.

Our second step consists in proving that if there exists an $x \in \partial B_i \setminus (\bigcup_j B_j)$, then

$$\langle \hat{\eta}_{x,i}, \eta_{\pi_{\partial S}(x)} \rangle \geq 1 - \frac{2(\alpha + \alpha')}{\alpha\alpha'} \varepsilon_n, \tag{21}$$

where $\hat{\eta}_{x,i} = \frac{O_i - x}{\alpha'}$ and $x^* = \pi_{\partial S}(x)$. Observe that from the first step we know that $B_i = \mathring{\mathcal{B}}(O_i, \alpha')$. Write η_{x^*} for the outward (from S) unit normal vector of ∂S at x^* and $O^* = x^* - \alpha \eta_{x^*}$.

Note first that

$$\mathring{\mathcal{B}}(O_i, \alpha') \subset C_{\alpha'}(\mathbb{X}_n)^c \text{ and } \mathcal{B}(O^*, \alpha) \subset S. \tag{22}$$

Introduce $y^* = [O^*, O_i] \cap \partial\mathcal{B}(O_i, \alpha')$ and $y = [O^*, O_i] \cap \partial\mathcal{B}(O^*, \alpha)$ (see Figure 15). Then, from the second inclusion in (22), we get $y \in S$, and from the first inclusion in (22) we get $d(y, C_{\alpha'}(\mathbb{X}_n)) \geq ||y - y^*||$. Then $||y - y^*|| \leq \varepsilon_n$, which in turn implies

$$\alpha + \alpha' - ||O_i - O^*|| \leq \varepsilon_n. \tag{23}$$

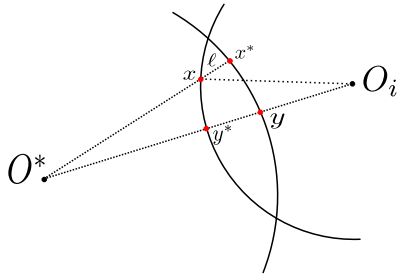

Figure 15: $x \in \partial C_{\alpha'}(\mathbb{X}_n)$, $x^* = \pi_{\partial S}(x)$, $O_i = x + \alpha' \hat{\eta}_{x,i}$ and $O^* = x^* - \alpha \eta_{x^*}$. Observe that $\mathring{B}(O_i, \alpha') \cap B(O^*, \alpha) \neq \emptyset$ by (24)
.

From $x^* = \pi_{\partial S}(x)$ we get that $x^* = x + \ell \eta_{x^*}$ where $\ell = ||x - x^*|| \leq \varepsilon_n$. Then $O_i = O^* + (\alpha - \ell)\eta_{x^*} + \alpha' \hat{\eta}_{x,i}$ and

$$\begin{aligned}
\alpha + \alpha' - ||O_i - O^*|| &= \alpha + \alpha' - \sqrt{(\alpha')^2 + (\alpha - \ell)^2 + 2\alpha'(\alpha - \ell)\langle \hat{\eta}_{x,i}, \eta_{x^*} \rangle} \\
&= \alpha + \alpha' - \sqrt{(\alpha' + \alpha - \ell)^2 - 2\alpha'(\alpha - \ell)(1 - \langle \hat{\eta}_{x,i}, \eta_{x^*} \rangle)} \\
&\geq \ell + \frac{\alpha'(\alpha - \ell)(1 - \langle \hat{\eta}_{x,i}, \eta_{x^*} \rangle)}{\alpha + \alpha' - \ell} \geq \frac{\alpha' \alpha (1 - \langle \hat{\eta}_{x,i}, \eta_{x^*} \rangle)}{2(\alpha + \alpha')}, \qquad (24)
\end{aligned}$$

where in the first inequality of the last line we bounded $A\sqrt{1 - 2B/A^2} \leq A(1 - B/A^2) = A - B/A$, and in the last inequality $\alpha - \ell \geq \alpha/2$, thus, combined with Equation (23), we can conclude the proof of Equation (21).

We will now, as the third step, conclude the proof of assertion 1. Note that if B_i is a ball (and not an half-space), then $\partial B_i \cap B_j^c = \partial B_i \cap P_{i,j}$ where $P_{i,j}$ the following closed half space.

$$P_{i,j} = \begin{cases} B_j^c & \text{if } B_j \text{ is an open half space} \\ \{x : ||x - O_i||^2 - r_i^2 \leq ||x - O_j||^2 - r_j^2\} & \text{if } B_j = \mathring{B}(O_j, r_j). \end{cases}$$

Thus $S_i = \partial B_i \bigcap \left(\bigcup_j B_j \right)^c = \partial B_i \bigcap \left(\bigcap_{j \neq i} B_j^c \right) = \partial B_i \cap H_i$, where H_i is a convex polygon.

Put $C_i(\mathbb{X}_n) = \partial B_i \cap \mathring{H}_i \setminus \mathbb{X}_n$. We are going to prove that the $C_i(\mathbb{X}_n)$ satisfy conditions (a), (b), (c) and (d) of assertion 1. First note that (a) is obvious by construction.

Suppose $x \in \partial C_{\alpha'}(\mathbb{X}_n) \setminus \mathbb{X}_n$. By the first step, we know that there exists a B_{i_0} which is a ball of radius α' such that $x \in S_{i_0}$ and thus we are in the situation where $x \in \partial B_{i_0} \cap H_{i_0}$ with H_{i_0} a convex polygon. If now $x \in \partial C_{\alpha'}(\mathbb{X}_n) \setminus \mathbb{X}_n$ but $x \notin \cup C_i(\mathbb{X}_n)$, we must have $x \in \partial B_{i_0} \cap \partial H_{i_0}$. This gives

$$\partial C_{\alpha'}(\mathbb{X}_n) \setminus \big(\bigcup_i C_i(\mathbb{X}_n) \big) \subset \mathbb{X}_n \bigcup \big(\bigcup_{i, r_i = \alpha'} \partial B_i \cap \partial H_i \big)$$

and thus $|\partial C_{\alpha'}(\mathbb{X}_n) \setminus \big(\bigcup_i C_i(\mathbb{X}_n) \big)|_{d-1} = 0$, which proves (b).

We will now prove that if $i \neq j$ and B_i and B_j are two balls, then $(\partial B_i \cap \mathring{H}_i) \cap (\partial B_j \cap H_j) = \emptyset$. Suppose by contradiction that $(S_i \cap \mathring{H}_i) \cap (S_j \cap H_j) \neq \emptyset$. Then $||x - O_i||^2 - r_i^2 < ||x - O_j||^2 - r_j^2$ and $||x - O_i||^2 - r_i^2 \geq ||x - O_j||^2 - r_j^2$, which is a contradiction. Thus, if $C_i(\mathbb{X}_n)$ and $C_j(\mathbb{X}_n)$ are both non-empty, we have that B_i and B_j are two balls, and if $i \neq j$, $C_i(\mathbb{X}_n) \cap C_j(\mathbb{X}_n) = \emptyset$, which proves (d).

This also proves that if $x \in C_i(\mathbb{X}_n)$, then there exists an $r_x > 0$ small enough so that $\partial C_{\alpha'}(\mathbb{X}_n) \cap \mathcal{B}(x, r_x) = \partial B_i \cap \mathcal{B}(x, r_x)$. Thus $\partial C_{\alpha'}(\mathbb{X}_n) \cap \mathcal{B}(x, r_x)$ is a $\mathcal{C}^2$, $d-1$ dimensional manifold. Moreover, the tangent space at x is given by $(x - O_i)^\perp$. Also, the unit normal (to $\partial C_{\alpha'}(\mathbb{X}_n)$) vector $(O_i - x)/||x - O_i||$ is well defined, and points outwards to $C_{\alpha'}(\mathbb{X}_n)$. This concludes the proof of (c) and also the proof of 1).

The proof of 2) follows the same ideas used to prove Theorem 3 in [1]. We are going to give the main steps of the proof (adapted to our case).

We first prove the surjectivity. For any $x^* \in \partial S$, we introduce $O^* = x^* - \alpha \eta_{x^*}$ and $x = x^* - 2\varepsilon_n \eta_{x^*}$. From the inside and outside α-rolling conditions it follows that S has reach $\alpha > 0$, and so $\pi_{\partial S}([x, x^*]) = x^*$, where we used that $2\varepsilon_n < \alpha$. To prove that $x \in C_{\alpha'}(\mathbb{X}_n)$ we proceed by contradiction. If $x \notin C_{\alpha'}(\mathbb{X}_n)$, then there exists an O with $||O - x|| \leq \alpha'$ and $\mathring{\mathcal{B}}(O, \alpha') \subset C_{\alpha'}(\mathbb{X}_n)^c$.

Let $u = (O - x)/||O - x||$, $y = [O^*, O] \cap \partial \mathcal{B}(O^*, \alpha)$ and $y^* = [O^*, O] \cap \partial \mathcal{B}(O, \alpha')$, and therefore $||y - y^*|| \leq \varepsilon_n$ which implies

$$\alpha + \alpha' - ||O - O^*|| \leq \varepsilon_n. \tag{25}$$

But now

$$\begin{aligned}
\alpha + \alpha' - ||O - O^*|| &= \alpha + \alpha' + ||(\alpha - 2\varepsilon_n)\eta_{x^*} + u||O - x||||\\
&= \alpha + \alpha' - \sqrt{||O - x||^2 + (\alpha - 2\varepsilon_n)^2 + 2||O - x||(\alpha - 2\varepsilon_n)\langle u, \eta_{x^*}\rangle}\\
&= \alpha + \alpha' - \sqrt{(||O - x|| + \alpha - 2\varepsilon_n)^2 - 2||O - x||(\alpha - 2\varepsilon_n)(1 - \langle u, \eta_{x^*}\rangle)}\\
&= \alpha + \alpha' - (||O - x|| + \alpha - 2\varepsilon_n)\sqrt{1 - \frac{2||O - x||(\alpha - 2\varepsilon_n)(1 - \langle u, \eta_{x^*}\rangle)}{(||O - x|| + \alpha - 2\varepsilon_n)^2}}\\
&\geq \alpha + \alpha' - (||O - x|| + \alpha - 2\varepsilon_n)\left(1 - \frac{||O - x||(\alpha - 2\varepsilon_n)(1 - \langle u, \eta_{x^*}\rangle)}{(||O - x|| + \alpha - 2\varepsilon_n)^2}\right)\\
&\geq 2\varepsilon_n + \alpha' - ||O - x|| + \frac{||O - x||(\alpha - 2\varepsilon_n)(1 - \langle u, \eta_{x^*}\rangle)}{(||O - x|| + \alpha - 2\varepsilon_n)} \geq 2\varepsilon_n,
\end{aligned}$$

where the last inequality follows from $||O - x|| \leq \alpha'$ and $2\varepsilon_n < \alpha$. This contradicts Equation (25). Thus $x \in C_{\alpha'}(\mathbb{X}_n)$. From the outside and inside α rolling condition (which implies α-convexity, see [20]) it follows that,

$$C_{\alpha'}(\mathbb{X}_n) \subset C_\alpha(\mathbb{X}_n) \subset C_\alpha(S) = S.$$

Then, if $x^* \in \partial S$ then $x^* \in \partial C_{\alpha'}(\mathbb{X}_n)$ or $x^* \in C_{\alpha'}(\mathbb{X}_n)^c$. In both cases there exists a $z \in (x, x^*)$ and $z \in \partial C_{\alpha'}(\mathbb{X}_n)$, such that $\pi_{\partial S}(z) = x^*$.

We now prove the injectivity. Suppose by contradiction that there are $x_1, x_2 \in \partial C_{\alpha'}(\mathbb{X}_n)$ such that $\pi_{\partial S}(x_1) = \pi_{\partial_S}(x_2) = y$. Write $\ell_i = d(x_i, \partial S)$, for $i = 1, 2$. Because $C_{\alpha'}(\mathbb{X}_n) \subset S$, we have $x_i + \ell_i \eta_y = y$ and thus $x_1 = x_2 + (\ell_2 - \ell_1)\eta_y$ and $|\ell_2 - \ell_1| \leq \varepsilon_n$ (because for $i \in \{1, 2\}$, $\ell_i \geq 0$ and $\ell_i \leq \varepsilon_n$ due to $d_H(\partial S, \partial C_{\alpha'}(\mathbb{X}_n)) \leq \varepsilon_n$). Suppose that $\ell_2 \geq \ell_1$. From the first step together

with Equation (21), we know that there exists an O_i such that $\mathring{\mathcal{B}}(O_i, \alpha') \subset C_{\alpha'}(\mathbb{X}_n)^c$, $||x_2 - O_i|| = \alpha'$ and $\langle u, \eta_y \rangle \geq 1 - 2(\alpha + \alpha')/(\alpha\alpha')\varepsilon_n$ with $u = (O_i - x_2)/\alpha'$. Then

$$\begin{aligned}
||x_1 - O_i||^2 &= (\ell_2 - \ell_1)^2 + \alpha'^2 - 2\alpha'(\ell_2 - \ell_1)\langle u, \eta_y \rangle \\
&\leq (\ell_2 - \ell_1)^2 + \alpha'^2 - 2\alpha'(\ell_2 - \ell_1) + \frac{4(\alpha + \alpha')}{\alpha}(\ell_2 - \ell_1)\varepsilon_n \\
&\leq \alpha'^2 - (\ell_2 - \ell_1)\left(2\alpha' - \frac{4(\alpha + \alpha')}{\alpha}\varepsilon_n - (\ell_2 - \ell_1)\right) \\
&\leq \alpha'^2 - (\ell_2 - \ell_1)\left(2\alpha' - \left(\frac{4(\alpha + \alpha')}{\alpha} + 1\right)\varepsilon_n\right).
\end{aligned}$$

The condition $\varepsilon_n \leq \frac{\alpha\alpha'}{4(\alpha+\alpha')}$ guarantees $2\alpha' - (4(\alpha + \alpha')/\alpha + 1)\varepsilon_n > 0$ thus, if $\ell_2 > \ell_1$, then $x_1 \in \mathring{\mathcal{B}}(O_i, \alpha')$, which is impossible (recall that $\mathring{B}(O_i, \alpha') \subset C^c_{\alpha'}$ and that $x_1 \in C_{\alpha'}(\mathbb{X}_n)$). Thus, by contradiction, $\ell_1 = \ell_2$ and $x_1 = x_2$, which concludes the proof of injectivity.

Lastly, we prove 3. Since $reach(\partial S) \geq \alpha$ and $d_H(\partial C_{\alpha'}(\mathbb{X}_n), \partial S) \leq \varepsilon_n < \alpha$, $\pi_{\partial S}$, restricted to $\partial C_{\alpha'}(\mathbb{X}_n)$, is continuous (see [25]). The continuity of $\pi^{-1}_{\partial S} : \partial S \to \partial C_{\alpha'}(\mathbb{X}_n)$ follows from the same ideas used to prove the injectivity of $\pi_{\partial S}$: we provide a sketch of the proof. It follows from $reach(\partial S) \geq \alpha$ that $\pi^{-1}_{\partial S}(x) = x - \ell(x)\eta_x$ with $\ell(x) \geq 0$. Also, $x \mapsto \eta_x$ is a continuous function (see Theorem 1 in [42]). It remains to be proved that ℓ is a continuous function. If this is not the case, we can find sequences $(y_n) \subset \partial S$ and $(y'_n) \subset \partial S$, both converging to some $y \in \partial S$), such that $\ell(y_n) \to \ell_1$ and $\ell(y'_n) \to \ell_2$. We can conclude exactly as in the proof of injectivity that we can take $x_{1,n} = y_n - \ell(y_n)\eta_{y_n}$ and $x_{2,n} = y'_n - \ell(y'_n)\eta_{y'_n}$ making $n \to +\infty$. We thus have $\partial S \approx \partial C_{\alpha'}(\mathbb{X}_n)$, which proves assertion 3, and thus concludes the proof of the lemma.

□

Lemma 8.8. *Suppose that M is a $\mathcal{C}^2$, bounded $(d-1)$-dimensional manifold with positive reach α. Let π_M denote the projection onto M and $\hat{M}$ be a $\mathcal{C}^2$, $(d-1)$-dimensional manifold such that*

1. *π_M is one to one from $\hat{M}$ to M,*
2. *for all $x \in \hat{M}$ we have $||x - \pi_M(x)|| \leq \varepsilon_1$ and $\langle \hat{\eta}_x, \eta_{\pi_M(x)} \rangle \geq 1 - \varepsilon_2$.*

Then, if $\varepsilon_1(d-1)\alpha \leq 1$ and $\varepsilon_2 \leq 1/8$, we have

$$(1 - 3\varepsilon_1\alpha - 32\varepsilon_2)^{\frac{d-1}{2}} \leq \frac{|\hat{M}|_{d-1}}{|M|_{d-1}} \leq (1 + 3\varepsilon_1\alpha + 32\varepsilon_2)^{\frac{d-1}{2}}. \tag{26}$$

Proof. Let $p \in M$ and denote by $(e_1, \ldots, e_{d-1})$ an orthonormal basis of T_pM and complete it with e_d a unit vector of N_pM. A neighborhod of p in M can be parametrized by $\varphi(x) = x + f(x)e_d = \sum_1^{d-1} x_ie_i + f(x_1, \ldots, x_{d-1})e_d$ where $x = \sum_1^{d-1} x_ie_i$ belongs to a neighborhood of p and $\nabla_f(p) = 0$, see for instance Proposition 3, point 1, in [2].

Consider now the surface element (of M) $ds(p) = dx_1 \dots dx_{d-1}$. Its image by π_M^{-1} on the surface element (of $\hat{M}$) is given by

$$d\hat{s}(p) = \sqrt{\det(J_{\pi_M^{-1}}(p)' J_{\pi_M^{-1}}(p))} dx_1 \dots dx_{d-1}.$$

The rest of the the proof consist in giving bounds for $\det(J_{\pi_M^{-1}}(p)' J_{\pi_M^{-1}}(p))$. We have that $\pi_M^{-1}(\varphi(x)) = x + \ell(x) n(x)$ where $n(x) = (-\partial f/\partial x_1, \dots, -\partial f/\partial x_{d-1}, 1) \in N_x M$, which gives that

$$J_{\pi_M^{-1}}(p) = \begin{pmatrix} I_{d-1} - \ell(p)\mathcal{H}_f(p) \\ \nabla_\ell(p) \end{pmatrix}.$$

The reach condition gives that $||\mathcal{H}_f(p)||_{op} \leq \alpha$ (see Proposition 6.1 in [30]) and $\ell(p) = ||\pi_M^{-1}(p) - p|| \leq \varepsilon_1$ so that we just have to bound $||\nabla_\ell(p)||$. Note that, for $j = 1, \dots, d-1$, we have

$$t_j = e_j + \frac{\partial \ell}{\partial x_j}(p) e_d - \ell(p) \left(\sum_1^{d-1} \frac{\partial^2 f}{\partial x_i \partial x_j} e_i \right) \in T_{\pi_M^{-1}(p)} \hat{M}.$$

Note that $\eta_p = \pm e_d$ and introduce $\hat{\eta}_{\pi_M^{-1}(p)}$. Since $t_1, \dots, t_{d-1}, \hat{\eta}_{\pi_M^{-1}(p)}$ is an orthogonal basis of $\mathbb{R}^d$, we have that

$$e_d = \sum_{i=1}^{d-1} \langle e_d, \frac{t_i}{||t_i||} \rangle \frac{t_i}{||t_i||} + \langle e_d, \hat{\eta}_{\pi_M^{-1}(p)} \rangle \hat{\eta}_{\pi_M^{-1}(p)},$$

which implies

$$1 = \sum_{i=1}^{d-1} \langle \eta_p, \frac{t_i}{||t_i||} \rangle^2 + \langle \eta_p, \hat{\eta}_{\pi_M^{-1}(p)} \rangle^2.$$

Thus, by condition 2, we have $|\langle t_j, e_d \rangle| = |\langle t_j, \eta_p \rangle| \leq \sqrt{2\varepsilon_2} ||t_j||$, which implies

$$\left| \frac{\partial \ell}{\partial x_j}(p) \right| \leq \sqrt{2\varepsilon_2} ||t_j|| \leq \sqrt{2\varepsilon_2} \left(1 + \left| \frac{\partial \ell}{\partial x_j}(p) \right| + \varepsilon_1 (d-1) \alpha \right).$$

From this, we get

$$\left| \frac{\partial \ell}{\partial x_j}(p) \right| \leq \frac{\sqrt{2\varepsilon_2}(1 + \varepsilon_1(d-1)\alpha)}{1 - \sqrt{2\varepsilon_2}}.$$

So, $J_{\pi_M^{-1}}(p)' J_{\pi_M^{-1}}(p) = I_{d-1} + E$ with E a symmetric matrix with

$$||E||_{op} \leq 2\varepsilon_1 \alpha + \varepsilon_1^2 \alpha^2 + \left(\frac{\sqrt{2\varepsilon_2}(1 + \varepsilon_1(d-1)\alpha)}{1 - \sqrt{2\varepsilon_2}} \right)^2,$$

thus we finally obtain the inequality

$$(1 - 3\varepsilon_1 \alpha - 32\varepsilon_2)^{d-1} \leq \det \left(J_{\pi_M^{-1}}(p)' J_{\pi_M^{-1}}(p) \right) \leq (1 + 3\varepsilon_1 \alpha + 32\varepsilon_2)^{d-1},$$

which concludes the proof. □

8.2.1 Proof of Theorem 4.1

Theorem 4.1 follows now from the previous lemmas.

Let $\varepsilon_n = d_H(\partial C_{\alpha'}(\mathbb{X}_n), \partial S)$ and $S_i = \pi_{\partial S}(C_i(\mathbb{X}_n))$ where the $C_i(\mathbb{X}_n)$ are the sets introduced in Lemma 8.7, we have that, for all i: $d_H(S_i, C_i(\mathbb{X}_n)) \leq \varepsilon_n$. Due to Lemma 8.7 we also have

1. $|\partial S|_{d-1} = \sum_i |S_i|_{d-1}$ and $|\partial C_{\alpha'}(\mathbb{X}_n)|_{d-1} = \sum_i |C_i(\mathbb{X}_n)|_{d-1}$.
2. for every i and all $x \in C_i(\mathbb{X}_n)$, $\langle \hat{\eta}_x, \eta_{\pi_{\partial S}(x)} \rangle \geq 1 - \frac{2(\alpha+\alpha')}{\alpha\alpha'}\varepsilon_n$.

Thus, by Lemma 8.8 we also have, for all i:

$$\left(1 - 3\alpha\varepsilon_n - \frac{64(\alpha+\alpha')}{\alpha\alpha'}\varepsilon_n\right)^{\frac{d-1}{2}} \leq \frac{|C_i(\mathbb{X}_n)|_{d-1}}{|S_i|_{d-1}} \leq \left(1 + 3\alpha\varepsilon_n + \frac{64(\alpha+\alpha')}{\alpha\alpha'}\varepsilon_n\right)^{\frac{d-1}{2}}.$$

Introduce then $A = 3\alpha + \frac{64(\alpha+\alpha')}{\alpha\alpha'}$, summing all the terms in the inequalities

$$(1 - A\varepsilon_n)^{\frac{d-1}{2}} |S_i|_{d-1} \leq |C_i(\mathbb{X}_n)|_{d-1} \leq (1 + A\varepsilon_n)^{\frac{d-1}{2}} |S_i|_{d-1},$$

gives

$$(1 - A\varepsilon_n)^{\frac{d-1}{2}} |\partial S|_{d-1} \leq |\partial C_{\alpha'}(\mathbb{X}_n)|_{d-1} \leq (1 + A\varepsilon_n)^{\frac{d-1}{2}} |\partial S|_{d-1}.$$

which concludes the proof.

8.2.2 Proof of Corollary 4

We only need to check that the conditions of Theorem 4.1 are fulfilled, with probability one, for n large enough. In [18] it is proved that $d_H(\mathbb{X}_n, S) \leq O((\ln n/n)^{1/d}$ e.a.s. so, with probability one for n large enough it is upper bounded by $\frac{1}{3}\frac{\alpha\alpha'}{\alpha+\alpha'}$. In [37] it is proven that, with probability one for n large enough, $d_H(\partial C_{\alpha'}(\mathbb{X}_n), \partial S) \leq \varepsilon_n \leq c(\ln n/n)^{2/(d+1)}$ for some given explicit constant c. Since $C_{\alpha'}(\mathbb{X}_n)^c$ is a finite union of balls and affine half-spaces, that is, $C_{\alpha'}(\mathbb{X}_n)^c = \bigcup_{i=1}^{N_1} E_i$ with $E_i = \mathring{\mathcal{B}}(O_i, r_i)$ or $E_i = \{z \in \mathbb{R}^d, \langle u_j, z\rangle > a_i\}$, it follows that

$$\partial C_{\alpha'}(\mathbb{X}_n) = \bigcup_i \left(\partial E_i \bigcap \left(\bigcup_{j\neq i} E_j\right)^c\right).$$

Now define the F_j as the connected components of the sets $\partial E_i \bigcap (\bigcup_{j\neq i} E_j)^c$. Then the F_j are closed manifolds of dimension $d_j \leq (d-1)$, and are compact since $F_j \subset C_{\alpha'}(\mathbb{X}_n)$, which is compact. Lastly, because $\partial C_{\alpha'}(\mathbb{X}_n)$ is a $(d-1)$-dimensional manifold, we must have $\partial C_{\alpha'}(\mathbb{X}_n) = \cup_{j,d_j=d-1} F_j$ (that is, the lower dimensional F_k are included in $\cup_{j,d_j=d-1} F_j$). This concludes the proof of Corollary 4.

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