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Surface and length estimation based on Crofton's formula

Catherine Aaron, Alejandro Cholaquidis and Ricardo Fraiman

July 13, 2021

Abstract

We study the problem of estimating the surface area of the boundary ∂S , of a sufficiently smooth set $S \subset \mathbb{R}^d$ when the available information is only a finite subset $\mathcal{X} \subset S$. We propose two estimators, the first one make use of the Devroye-Wise support estimator, and the second one make use of the α -convex hull of set \mathcal{X} , denoted by $C_\alpha(\mathcal{X})$. We derive results which depend on the Hausdorff distance between S and \mathcal{X} for the Devroye-Wise based estimator, and the Hausdorff distance between ∂S and $\partial C_\alpha(\mathcal{X})$ for the second one.

These results allow to obtain the rates of convergence of both estimators when \mathcal{X} is an iid sample, and also when it is the discretization of the trajectory of a reflected Brownian motion with drift.

The two proposed estimators are based on Crofton's formula, which, roughly speaking, states that the $d - 1$ dimensional surface area of a smooth enough set is the mean number of intersection of randomly chosen lines, which can be approximated by means of Monte Carlo method. For the Devroye-Wise based estimator, our proposal is not just a plug-in method. Regarding the other estimator, we prove that $d - 1$ -dimensional surface area of $C_\alpha(\mathcal{X})$, converges to the $d - 1$ surface area of ∂S , which also can be approximated by means of Monte Carlo and Crofton formula.

1 Introduction

Let $S \subset \mathbb{R}^d$ be a compact set, we aim to estimate its surface area, i.e. the $(d - 1)$ -Hausdorff measure of its boundary ∂S . Surface area estimation has been extensively considered in stereology (see for instance [7, 8], and [25]). It has also been studied as a further step in the theory of nonparametric set estimation (see [28]), and has practical applications in medical imaging (see [18]). Also surface area estimation is widely used in magnetic resonance imaging (MRI) techniques.

The three- and two-dimensional cases are addressed in [9], where the authors propose parametric estimators when the available data are the distances to S , from a sample outside the set, but at a distance smaller than a given $R > 0$.

When, as in image analysis, one can observe n data points from two distinguishable sets of random data-points (one from inside S and the other one from outside S), estimation of the surface area of the boundary has been considered, for any $d \geq 2$ in [18, 20, 26, 28] and [36]. The proposals given in [18, 28] and [20] aim to estimate the Minkowski content of ∂S . In [20] a very general convergence

result is obtained, while in [18] a convergence rate of order $n^{-1/2d}$ is obtained under some mild hypotheses, and later on, in [28] a convergence rate of order $n^{-1/(d+1)}$ is achieved, under stronger assumptions. In [26] a very nice fully data driven method, based on the Delaunay triangulation is proposed under an homogeneous point process sampling scheme. The asymptotic rate of convergence of the variance is given, but there is no global convergence rate because no result is obtained for the bias. Lastly, in [36] a parameter-free procedure, based on the Voronoi triangulation is proposed, and a rate of convergence of order $\lambda^{-1/d}$ is obtained, under a Poisson Point Process (PPP) sampling scheme (where λ is the intensity of the PPP).

We propose two surface area estimators, that make use of the Croftons formula, in any dimension, when the available data is only a finite set $\mathcal{X} \subset S$. In this setting only the two dimensional case has been yet studied. Assuming that \mathcal{X} is an iid sample the convex case has been first addressed in [10] (using the Crofton's formula). Later on, under the α -convexity assumption, [4] obtained the convergence of the α -shape's perimeter to the perimeter of the support. When the data are given by a trajectory from a reflected Brownian motion, (with or without drift) a consistency result is obtained in Theorem 4 in [12].

Although the 2-dimensional case has many significant applications, this is also the three dimensional case, since, for instance, surface area is an important biological parameter, in organs such, as the lungs. Higher dimensional study also have importance, at least for a theoretically point of view since in [32], the surface area of the boundary plays a significant role as a parameter of a probability distribution, being able to estimate it allows to apply plug-in methods.

Crofton's formula, proved by Crofton in 1868 for dimension two, and extended to arbitrary dimensions (see [35]), states that the surface of ∂S equals the integral of the number of intersections with ∂S of lines in \mathbb{R}^d (see Equations (2) and (3) for explicit Crofton's formulas for $d = 2$ and $d \geq 2$, respectively).

As previously announced we will propose two different surface area estimators, both make use of Crofton's formula. One of them use the Devroye-Wise support estimator, denoted by $\hat{S}_{\varepsilon_n}(\mathcal{X})$ (see [21]), and the other one use the α -convex hull support estimator, denoted by $C_\alpha(\mathcal{X})$ (see [33]), where n is the cardinality of \mathcal{X} and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

The Devroye-Wise based surface estimator is not just a plug-in method, because in general the number of intersections of a line with ∂S may not converge to the number of intersections of that line with $\partial \hat{S}_{\varepsilon_n}(\mathcal{X})$. We prove that this estimator attains a rate proportional to $d_H(\mathcal{X}, S)^{1/2}$ (where d_H denotes the Hausdorff distance). This rate can be improved to $d_H(\mathcal{X}, S)$ when adding a reasonable shape assumption on ∂S . These rates are known when \mathcal{X} is an iid sample, see the comments below.

To estimate the surface area when the support estimator is $C_\alpha(\mathcal{X})$, we first extend the result in [19]. More precisely we prove that, in any dimension, the surface area of the hull's boundary, i.e. $|\partial C_\alpha(\mathcal{X}_n)|_{d-1}$, converges to $|\partial S|_{d-1}$. This result is interesting in itself, but in practice to compute $|\partial C_\alpha(\mathcal{X}_n)|_{d-1}$ is difficult, especially for dimension $d > 2$. However, we will see that, by means of the Crofton's formula, it can be easily estimated, via Monte-Carlo method.

These results can be applied to many deterministic or random situations, to obtain explicit convergence rates. We focus on two random situations: the case $\mathcal{X} = \mathcal{X}_n = \{X_1, \dots, X_n\}$ of iid drawn on S (with a density bounded from

below by a positive constant), and the case of random trajectories of reflected diffusions on S . In particular, we provide convergence rates when the trajectory is the result of a reflected Brownian motion (see [12, 13]). This last setting has several applications in ecology, where the trajectory is obtained by recording the location of an animal (or several animals) living in an area S , that is called its home range (the territorial range of the animal), and X_t represent the position at time t transmitted by the instrument (see for instance [6, 12, 13], and references therein).

The rate of convergence of the surface area estimator, based on $\hat{S}_{\varepsilon_n}(\mathcal{X}_n)$, when \mathcal{X}_n is an iid sample, is of order $n^{-1/2d}$, that can be improved to $n^{-1/d}$, depending on smoothness assumptions on ∂S .

With the α -convex hull support estimation, when \mathcal{X}_n is an iid sample, we obtain a rate of order $n^{-2/(d+1)}$.

These rates are far from the rate $n^{-\frac{d+3}{2d+1}}$, attained for $d = 2$ in [5], but for the volume of S , which is conjectured to be minimax optimal. However the rates we obtain are the same (or even better) than the ones obtained in the easier inside and outside sampling scheme.

The rest of this paper is organized as follows. In Section 1, we introduce the notation and some well-known geometric restrictions. Section 2 aims to present Crofton's formula, first for dimension two and then for the general case. After that, we introduce the main geometric restrictions required in one of the main theorems. Section 3 introduces the algorithms from a mathematical standpoint, and explains the heuristics behind them. The computational aspects of the algorithms are given in Section 5 and the main results are stated in Section 6, their proofs are given in the Appendix.

Some preliminary definitions

The following notation will be used throughout the paper.

Given a set $S \subset \mathbb{R}^d$, we denote by $\overset{\circ}{S}$, \overline{S} and ∂S the interior, closure and boundary of S , respectively, with respect to the usual topology of \mathbb{R}^d . We also write $\text{diam}(S) = \sup_{(x,y) \in S \times S} \|x - y\|$. The parallel set of S of radius ε is $B(S, \varepsilon) = \{y \in \mathbb{R}^d : \inf_{x \in S} \|y - x\| \leq \varepsilon\}$.

If $A \subset \mathbb{R}^d$ is a Borel set, then $|A|_d$ denotes its d -dimensional Lebesgue measure. When $A \subset \mathbb{R}^d$ is a $(d-1)$ -dimensional manifold then $|A|_{d-1}$ denotes its $(d-1)$ -Hausdorff measure.

We denote by $\mathcal{B}(x, \varepsilon)$ the closed ball in \mathbb{R}^d , of radius ε , centred at x , and $\omega_d = |\mathcal{B}_d(x, 1)|_d$. Given two compact non-empty sets $A, C \subset \mathbb{R}^d$, the *Hausdorff distance* or *Hausdorff-Pompei distance* between A and C is defined by

$$d_H(A, C) = \inf\{\varepsilon > 0 : \text{such that } A \subset B(C, \varepsilon) \text{ and } C \subset B(A, \varepsilon)\}.$$

The $(d-1)$ -dimensional sphere in \mathbb{R}^d is denoted by \mathcal{S}^{d-1} , while the half-sphere in \mathbb{R}^d is denoted by $(\mathcal{S}^+)^{d-1}$, i.e., $(\mathcal{S}^+)^{d-1} = (\mathbb{R}^{d-1} \times \mathbb{R}^+) \cap \mathcal{S}^{d-1}$. Given M a sufficiently smooth $(d-1)$ -manifold and $x \in M$ the affine tangent space of M at x is denoted by $T_x M$. When $S \subset \mathbb{R}^d$ is a regular (i.e.: $S = \overline{S}$), compact, and have \mathcal{C}^1 regular boundary ∂S , for any $x \in \partial S$ we can define η_x the unit outward vector of $(T_x \partial S)^\perp$ such that, for $t > 0$ small enough $x + t\eta_x \in S^c$, η_x is named the outward normal unit vector at x .

Given a vector $\theta \in (\mathcal{S}^+)^{d-1}$ and a point y , $r_{\theta, y}$ denotes the line $\{y + \lambda\theta, \lambda \in \mathbb{R}\}$. If y_1 and y_2 are two points in $r_{\theta, y}$, then $y_i = y + \lambda_i\theta$; with a slight abuse of

notation, we write $y_1 < y_2$ when $\lambda_1 < \lambda_2$.

We will now recall some well-known shape restrictions in set estimation.

Definition 1.1. A set $S \subset \mathbb{R}^d$ is said to be α -convex, for $\alpha > 0$, if $S = C_\alpha(S)$, where

$$C_\alpha(S) = \bigcap_{\{\mathring{B}(x,\alpha): \mathring{B}(x,\alpha) \cap S = \emptyset\}} \left(\mathring{B}(x,\alpha)\right)^c, \quad (1)$$

is the α -convex hull of S .

When S is α -convex, a natural estimator of S from a random sample \mathcal{X}_n of points (drawn from a distribution with support S), is $C_\alpha(\mathcal{X}_n)$, see [33].

Definition 1.2. A set $S \subset \mathbb{R}^d$ is said to satisfy the outside α -rolling condition if for each boundary point $s \in \partial S$ there exists an $x \in S^c$ such that $\mathcal{B}(x, \alpha) \cap \partial S = \{s\}$. A compact set S is said to satisfy the inside α -rolling condition if $\overline{S^c}$ satisfies the outside α -rolling condition at all boundary points.

2 Crofton's formula

Crofton in 1868 proved the following result (see [16]): given γ a regular plane curve (i.e. there exists a differentiable parametrization $c: [0, 1] \rightarrow \gamma \subset \mathbb{R}^2$ such that $\|c'(t)\| > 0$ for all t), then its length $|\gamma|_1$ can be computed by

$$|\gamma|_1 = \frac{1}{2} \int_{\theta=0}^{\pi} \int_{p=-\infty}^{+\infty} n_\gamma(\theta, p) dp d\theta, \quad (2)$$

$n_\gamma(\theta, p)$ being the number of intersections of γ with the line $r_{\theta^*, \theta p}$, where $\theta^* \in (S^+)^1$ is orthogonal to θ , and $dp d\theta$ is the 2-dimensional Lebesgue measure, see Figure 1. This result has been generalized to \mathbb{R}^d for any $d > 2$, and also to Lie groups, see [35].

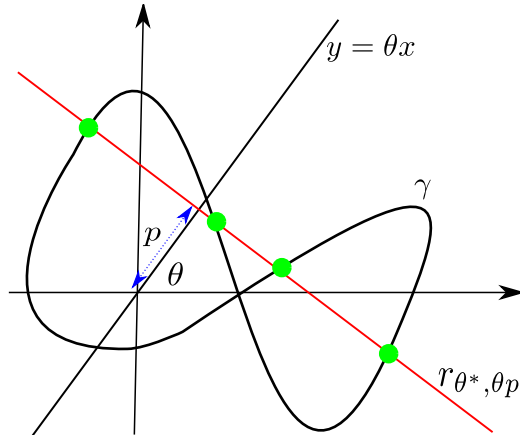


Figure 1: The function n_γ counts the number of intersections of γ with the line $r_{\theta^*, \theta p}$ determined by θ and p .

To introduce the general Crofton's formula in \mathbb{R}^d for a compact $(d-1)$ -dimensional manifold M , let us define first the constant

$$\beta(d) = \Gamma(d/2)\Gamma((d+1)/2)^{-1}\pi^{-1/2},$$

where Γ stands for the well known Gamma function. Let $\theta \in (\mathcal{S}^+)^{d-1}$, θ determine a $(d-1)$ -dimensional linear space $\theta^\perp = \{v : \langle v, \theta \rangle = 0\}$. Given $y \in \theta^\perp$, let us write $n_M(\theta, y) = \#(r_{\theta, y} \cap M)$, where $\#$ is the cardinality of the set. see Figure 2.

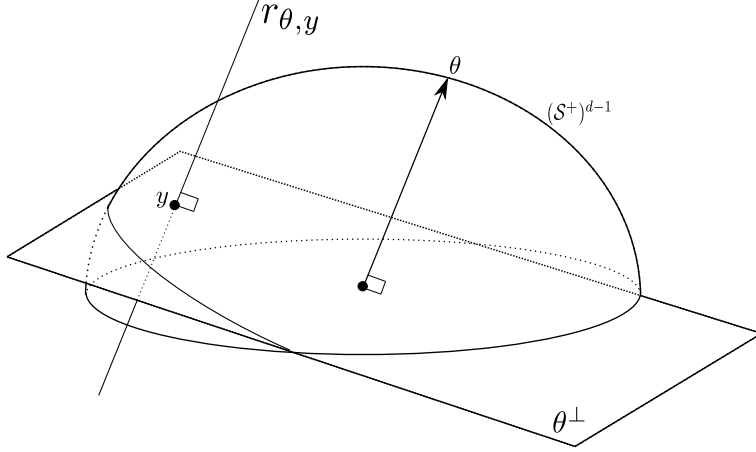


Figure 2: The line $r_{\theta, y} = y + \lambda\theta$ is shown, where $y \in \theta^\perp$ and $\theta \in (\mathcal{S}^+)^{d-1}$.

It is proved in [24] (see Theorem 3.2.26) that if M is an $(d-1)$ -dimensional rectifiable set, then the integralgeometric measure of M (which will be denote by $I_{d-1}(M)$, and is defined by the right-hand side of 3) equals its $(d-1)$ -dimensional Hausdorff measure, i.e.,

$$|M|_{d-1} = I_{d-1}(M) = \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \theta^\perp} n_M(\theta, y) d\mu_{d-1}(y) d\theta. \quad (3)$$

The measure $d\theta$ is the uniform measure on $(\mathcal{S}^+)^{d-1}$ (with total mass 1).

Remark 1. *Along this paper we assume that ∂S is the boundary of a compact set $S \subset \mathbb{R}^d$ such that $S = \bar{S}$. We also assume that S fulfills the outside and inside α -rolling condition, and then ∂S is rectifiable (see Theorem 1 in [37]). From this it follows that $I_{d-1}(\partial S) = |\partial S|_{d-1} < \infty$, which implies (by (3)) that, except for a set of measure zero with respect to $d\mu_{d-1}(y)d\theta$, any line $r_{\theta, y}$ meets ∂S a finite number of times: $n_{\partial S}(\theta, y) < \infty$. From Theorem 1 in [37], it also follows that ∂S is a C^1 manifold, which allows us to consider for all $x \in \partial S$, η_x , the unit outward normal vector.*

For the Devroye–Wise based surface area estimator we will assume that ∂S satisfies a technical hypothesis named (C, ε_0) -regularity.

Definition 2.1. Let us define $E_\theta(\partial S) = \{x \in \partial S, \langle \eta_x, \theta \rangle = 0\}$ and F_θ its orthogonal projection onto θ^\perp . Let us define, for $\varepsilon > 0$,

$$\varphi_\theta(\varepsilon) = |\theta^\perp \cap B(F_\theta, \varepsilon)|_{d-1}.$$

We will say that ∂S is (C, ε_0) -regular if for all θ and all $\varepsilon \in (0, \varepsilon_0)$, $\varphi'_\theta(\varepsilon)$ exists and $\varphi'_\theta(\varepsilon) \leq C$.

Once the rolling balls condition is imposed, we will show through some examples that the (C, ε_0) -regularity of the boundary is not a too restrictive hypothesis.

For instance, a polyhedron with ‘rounded corners’, such as in Figure 3, satisfies the (C, ε_0) -regularity of the boundary. Under regularity and geometric conditions on ∂S , the (C, ε_0) -regularity is related to the conjecture proposed in [3].

To find sets that satisfy the inside and outside α rolling ball properties but without a (C, ε_0) -regular boundary, the idea is to consider a set with some E_θ having infinitely many connected components, such as the one shown in Figure 6, whose boundary is locally around some boundary point, the hypograph of the function $x^5 \sin(1/x)$.

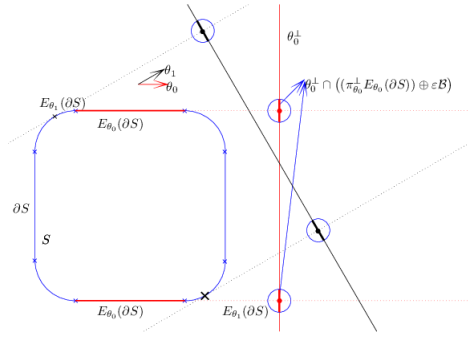


Figure 3: (a) smooth square

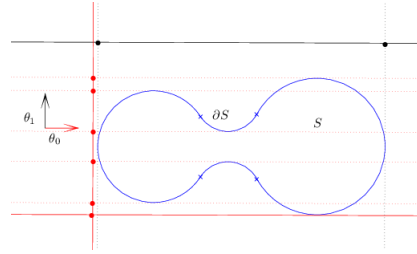


Figure 4: (b) 2D peanut

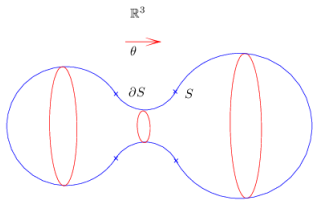


Figure 5: (c) 3D peanut

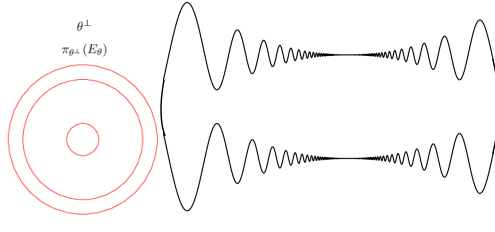


Figure 6: (d) an ‘infinite wave’ shape

- (a) The first set, presented in Figure 3, is a square with ‘round angles’, it has a 2-regular boundary.
- (b) The second set, presented in Figure 4, is a 2-dimensional ‘peanut’ made of 4 arcs of circle. It has a 6-regular boundary.
- (c) The third set, presented in Figure 5, is the surface of revolution generated by (b). The number of connected components of E_θ is bounded by 3 and the maximal length of a component is bounded by L , the length of the maximal perimeter (shown in blue in the figure). Thus, it is C -regular with $C \leq 3L$.
- (d) The rolling ball condition is not sufficient to guarantee the (C, ε_0) regularity of the boundary: this happens if, for instance, we replace in the

smooth square shown in (a) a flat piece of the boundary by the graph of the function $x^5 \sin(1/x)$. To illustrate this behaviour, Figure 6 shows a set such that the number of connected components of E_θ (with a horizontal θ) is infinite.

For the Devroye-Wise type estimator we will also show that the convergence rate can be quadratically improved if we additionally assume that the number of intersections between any line and ∂S is bounded from above (that exclude the case of a linear part in ∂S).

Definition 2.2. Given $S \subset \mathbb{R}^d$, we say that ∂S has a bounded number of linear intersections if there exists N_S such that, for all $\theta \in (\mathcal{S}^+)^{d-1}$ and $y \in \theta^\perp$, $n_{\partial S}(\theta, y) \leq N_S$.

3 Definitions of the estimators

3.1 Devroye–Wise based approach

3.1.1 A conjecture on the Devroye–Wise estimator

Given that in general the set S is unknown, we can first propose the natural plug-in idea of computing $|\partial \hat{S}|_{d-1}$ where \hat{S} is an estimator of S . There are different kinds of set estimators, depending on the geometric restrictions imposed on S and the structure of the data (see [12, 21] and references therein). One of the most studied in the literature, which is also universally consistent, is the Devroye–Wise estimator (see [21]), given by

$$\hat{S}_{\varepsilon_n}(\mathcal{X}) = \bigcup_{x \in \mathcal{X}} \mathcal{B}(x, \varepsilon_n),$$

where $\varepsilon_n \rightarrow 0$ is a sequence of positive real numbers and \mathcal{X} the “observations” set, not necessarily finite. This all-purpose estimator has the advantage that it is quite easy to compute the intersection of a line with its boundary (i.e. the points in the line at a distance of exactly ε_n from the sample).

We conjecture that the plug-in estimator $|\partial \hat{S}_{\varepsilon_n}(\mathcal{X})|_{d-1}$ has the following behaviour:

1. if $\varepsilon_n < d_H(\mathcal{X}, S)$, then $\partial \hat{S}_{\varepsilon_n}(\mathcal{X})$ does not converge to ∂S and $|\partial \hat{S}_{\varepsilon_n}(\mathcal{X})|_{d-1}$ does not converge to $|\partial S|_{d-1}$.
2. if $\varepsilon_n = d_H(\mathcal{X}, S)$ then $\partial \hat{S}_{\varepsilon_n}(\mathcal{X})$ converges to ∂S with the best possible rate but $|\partial \hat{S}_{\varepsilon_n}(\mathcal{X})|_{d-1}$ does not converge to $|\partial S|_{d-1}$.
3. if $\varepsilon_n \gg d_H(\mathcal{X}, S)$ and $\varepsilon_n \rightarrow 0$ then $\partial \hat{S}_{\varepsilon_n}(\mathcal{X})$ converges to ∂S and $|\partial \hat{S}_{\varepsilon_n}(\mathcal{X})|_{d-1}$ converges to $|\partial S|_{d-1}$. The optimal rate being obtained when $\varepsilon_n = \mathcal{O}(d_H(\mathcal{X}, S)^{2/3})$ and then $||\partial \hat{S}_{\varepsilon_n}(\mathcal{X})|_{d-1} - |\partial S|_{d-1}| = \mathcal{O}(d_H(\mathcal{X}, S)^{2/3})$.

This conjecture is supported by the following example. In a deterministic setting, let us consider for $d = 2$, $S = \mathcal{B}(0, 1)$, where the observations set \mathcal{X} is assumed to fulfil $d_H(\mathcal{X}_n, S) = \mathcal{O}(n^{-1})$ and

$$\hat{S}_{\varepsilon_n}(\mathcal{X}) = S \cup \bigcup_{k=1}^n \mathcal{B}((\cos(2k\pi/n), \sin(2k\pi/n)), \varepsilon_n).$$

See Figure 7. This is possible only if $\varepsilon_n \geq \sin(\pi/n)$. In this configuration, we obtain that $|\partial S_{\varepsilon_n}(\mathcal{X})|_1 = 2\pi\varepsilon_n + 2n\varepsilon_n \arcsin(\sin(\pi/n)/\varepsilon_n)$, so if $\varepsilon_n = \varepsilon_0/n$, with $\varepsilon_0 > 0$, $d_H(\partial S, \partial \hat{S}_{\varepsilon_n}(\mathcal{X})) \rightarrow 0$ (with the best possible rate) but $|\partial S_{\varepsilon_n}(\mathcal{X})|_1 \rightarrow c'\pi$ with $c' > 2$, and thus is a not consistent estimator of the perimeter of the support. On the contrary, if $\varepsilon_n \rightarrow 0$ such that $n\varepsilon_n \rightarrow +\infty$, it follows that, $|\partial \hat{S}_{\varepsilon_n}(\mathcal{X})|_1 - 2\pi \sim A\varepsilon_n + B((n\varepsilon_n)^{-2}) + o((n\varepsilon_n)^{-3})$ for some positive constants A and B . So the optimal rate, attained for $\varepsilon_n = \mathcal{O}(n^{-2/3})$, is $|\partial \hat{S}_{\varepsilon_n}(\mathcal{X})|_1 - 2\pi = \mathcal{O}(d_H(S, \mathcal{X})^{2/3})$. We conjecture that, with the use of the Devroye-Wise estimator we can only achieve (in general) a slow convergence rate given by $|\partial \hat{S}_{\varepsilon_n}(\mathcal{X})|_1 - 2\pi = \mathcal{O}(d_H(S, \mathcal{X})^{2/3})$.

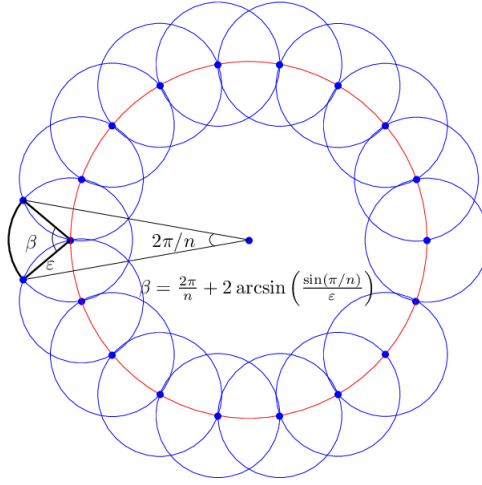


Figure 7: In this figure the sample and ε_n are chosen to illustrate our conjecture.

3.1.2 The Devroye-Wise based surface area estimator

The Devroye-Wise based surface area estimator estimate the number of intersection between ∂S and a given line, with the intersections between that line and $\hat{S}_{\varepsilon_n}(\mathcal{X})$, and “glue” some close intersections. This algorithm, detailed in Definition 3.1 attains a convergence rate of order $d_H(\mathcal{X}, S)$ under reasonable hypotheses (see Theorem 4.2), and thus is better than the direct plug-in estimator $|\partial \hat{S}_{\varepsilon_n}(\mathcal{X})|_{d-1}$ which was conjecture to be of order $\mathcal{O}(d_H(S, \mathcal{X})^{2/3})$

To estimate $n_{\partial S}(\theta, y)$, note that when $r_{\theta, y}$ is not included in a $(d-1)$ -dimensional affine tangent space (tangent to ∂S), then $n_{\partial S}(\theta, y) = 2k_S(\theta, y)$ where $k_S(\theta, y)$ is the number of connected components of $r_{\theta, y} \cap S$.

Definition 3.1. Let ε_n a sequence of positive real numbers, such that $\varepsilon_n \rightarrow 0$ and $\mathcal{X} \subset S$ a set not necessarily finite. Consider a line $r_{\theta, y}$. If $\hat{S}_{\varepsilon_n}(\mathcal{X}) \cap r_{\theta, y} = \emptyset$, define $\hat{n}_{\varepsilon_n}(\theta, y) = 0$, otherwise:

- denote by I_1, \dots, I_m the connected components of $\hat{S}_{\varepsilon_n}(\mathcal{X}) \cap r_{\theta, y}$. Order this sequence in such a way that $I_i = (a_i, b_i)$, with $a_1 < b_1 < a_2 < b_2 < \dots < a_m < b_m$.

- If for some consecutive intervals $I_i, I_{i+1}, \dots, I_{i+l}$, for all $a_i < t < b_{i+l}$ and $t \in r_{\theta, y}$, $d(t, \mathcal{X}_n) \leq 4\varepsilon_n$, define $A_i = (a_i, b_{i+l})$.
- Let j be the number of disjoint open intervals A_1, \dots, A_j that this process ended with. Then define $\hat{n}_{\varepsilon_n}(\theta, y) = 2j$.

Our first proposed estimator is

$$\hat{I}_{d-1}(\partial S) = \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \theta^\perp} \hat{n}_{\varepsilon_n}(\theta, y) d\mu_{d-1}(y) d\theta.$$

Under the assumption that ∂S has a bounded number N_S of linear intersections (see Definition 2.2) we will consider, for a given $N_0 \geq N_S$,

$$\hat{I}_{d-1}^{N_0}(\partial S) = \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \theta^\perp} \min(\hat{n}_{\varepsilon_n}(\theta, y), N_0) d\mu_{d-1}(y) d\theta.$$

3.2 α -convex hull based approach

The α -convex hull of a finite set of n points \mathcal{X}_n (defined by (1) with $S = \mathcal{X}_n$), which is also a consistent estimator of S under some regularity conditions (see for instance [33]), has the advantage that the $(d-1)$ -dimensional Lebesgue measure of its boundary converges to the $(d-1)$ -dimensional Lebesgue measure of ∂S (see Theorem 4.3 below). Unfortunately, the computation of $|\partial C_\alpha(\mathcal{X}_n)|_{d-1}$ is very difficult. This, together with the fact that $\partial C_\alpha(\mathcal{X}_n)$ is a rectifiable set (see the comment before Remark 1), suggests using Crofton's formula to estimate $|\partial C_\alpha(\mathcal{X}_n)|_{d-1}$ with a Monte-Carlo method. Then our second proposed estimator is

$$\check{I}_{d-1}(\partial S) = \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \theta^\perp} \check{n}_\alpha(\theta, y) d\mu_{d-1}(y) d\theta, \quad \text{where}$$

$$\check{n}_\alpha(\theta, y) = n_{\partial C_\alpha(\mathcal{X}_n)}(\theta, y).$$

In this case, the computation of the intersection of a line with $\partial C_\alpha(\mathcal{X}_n)$ is not as direct as in the Devroye–Wise estimator. However, weaker regularity restrictions on ∂S will be required (see Corollary 4) to get the consistency of $\check{I}_{d-1}(\partial S)$ with a better convergence rate.

4 Main results

In this section we will state our main results. All proofs are given in the Appendix.

4.1 Convergence rates for the Devroye–Wise based estimator under α -rolling condition and (C, ε_0) -regularity.

Theorem 4.1. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside α -rolling conditions. Assume also that S is (C, ε_0) -regular for some positive*

constants C and ε_0 . Let $\mathcal{X}_n = \{X_1, \dots, X_n\} \subset S$. Let $\varepsilon_n \rightarrow 0$ such that $d_H(\mathcal{X}_n, S) \leq \varepsilon_n$. Then

$$\hat{I}_{d-1}(\partial S) = |\partial S|_{d-1} + \mathcal{O}(\sqrt{\varepsilon_n}). \quad (4)$$

Moreover, for n large enough,

$$|\mathcal{O}(\sqrt{\varepsilon_n})| \leq \frac{4C \operatorname{diam}(S)}{3\beta(d)\sqrt{\alpha}} \sqrt{\varepsilon_n},$$

C being the constant of the (C, ε_0) -regularity of S .

From Theorem 4.1 and Theorem 4 in [17], we can obtain the rate of convergence for the iid case:

Corollary 1. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the inside and outside α -rolling conditions. Assume also that S is (C, ε_0) -regular for some positive constants C and ε_0 . Let X_1, \dots, X_n be an iid sample of X with distribution P_X supported on S . Assume that P_X has density f (w.r.t. μ_d) bounded from below by some $c > 0$. Let $\varepsilon_n = C'(\ln(n)/n)^{1/d}$ and $C' > (6/(c\omega_d))^{1/d}$. Then with probability one, for n large enough,*

$$\hat{I}_{d-1}(\partial S) = |\partial S|_{d-1} + \mathcal{O}\left(\left(\frac{\ln(n)}{n}\right)^{\frac{1}{2d}}\right).$$

Notice that, as it mentioned in Section 5.2 in [17], if $\varepsilon_n = 2 \max_i \min_{j \neq i} \|X_i - X_j\|$, then with probability one, for n large enough $\varepsilon_n \leq 2d_H(\mathcal{X}_n, S)$, which together with Corollary 1, entails that, with the aforementioned choice for ε_n , our proposal is fully data driven, for the iid case.

4.2 Convergence rates for the Devroye-Wise based estimator under α -rolling condition, (C, ε_0) -regularity and bounded number of linear intersections.

If the number of linear intersection of ∂S is assumed to be bounded by a constant N_S , the use of $\min(\hat{n}_{\varepsilon_n}, N_0)$ (for any $N_0 \geq N_S$) allows us to obtain better convergence rates.

Theorem 4.2. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside α -rolling conditions. Assume also that S is (C, ε_0) -regular for some positive constants C and ε_0 and that ∂S has a number of linear intersection bounded by N_S . Let $\mathcal{X}_n = \{X_1, \dots, X_n\} \subset S$. Let $\varepsilon_n \rightarrow 0$ such that $d_H(\mathcal{X}_n, S) \leq \varepsilon_n$ and $N_0 \geq N_S$. Then*

$$\hat{I}_{d-1}^{N_0}(\partial S) = |\partial S|_{d-1} + \mathcal{O}(\varepsilon_n).$$

Moreover, for n large enough,

$$|\mathcal{O}(\varepsilon_n)| \leq \frac{5}{\beta(d)} C N_0 \varepsilon_n,$$

C being the constant of the (C, ε_0) -regularity of S .

As before we give the convergence rate associated to the iid setting and the reflected Brownian motion hypothesis as two corollaries of Theorem 4.2.

Corollary 2. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the inside and outside α -rolling conditions. Assume also that S is (C, ε_0) -regular for some positive constants C and ε_0 and that ∂S has a bounded number of linear intersections. Let X_1, \dots, X_n be iid random vectors with distribution P_X , supported on S . Assume that P_X has density f (w.r.t. μ_d) bounded from below by some $c > 0$. Let $\varepsilon_n = C'(\ln(n)/n)^{1/d}$ and $C' > (6/(c\omega_d))^{1/d}$. Then with probability one, for n large enough,*

$$\hat{I}_{d-1}^{N_0}(\partial S) = |\partial S|_{d-1} + \mathcal{O}\left(\left(\frac{\ln(n)}{n}\right)^{\frac{1}{d}}\right).$$

It is worth to be mentioned that the estimator $\hat{I}_{d-1}^{N_0}(\partial S)$ attains a better convergence rate than the one conjectured in Section 4.1 for $|\partial \hat{S}_{\varepsilon_n}(\mathcal{X}_n)|_{d-1}$, that is, when just we compute the $(d-1)$ -dimensional surface area of the Devroye Wise estimator. Indeed, we obtained a rate of order $d_H(S, \mathcal{X}_n)$ while the plug-in Devroye-Wise estimator $|\partial \hat{S}_{\varepsilon_n}(\mathcal{X}_n)|_{d-1}$, was conjectured to be of order $d_H(S, \mathcal{X}_n)^{2/3}$. Here again the choice of $\varepsilon_n = 2 \max_i \min_j \|X_i - X_j\|$ is suitable, but now, the price to pay is the selection of the parameter N_0 .

4.2.1 The reflected Brownian motion

In a more general setting, the conclusion of Theorem 4.2 holds when the set of points \mathcal{X}_n is replaced by the trajectory \mathcal{X}_T , of any stochastic process $\{X_t\}_{t>0}$ included in S , observed in $[0, T]$, such that $d_H(\mathcal{X}_T, S) \rightarrow 0$ as $T \rightarrow \infty$. Observe that the estimator $\hat{I}_{d-1}^{N_0}(\partial S)$ is well defined even when \mathcal{X}_T is not a finite set, see Definition 3.1. We will assume that S is bounded with connected interior and ∂S is \mathcal{C}^2 . This is the case (for example) of some reflected diffusions and in particular the reflected Brownian motion (RBM). This has been recently proven in Corollary 1 in [12], for RBM without drift (see also [13] and [14] for the RBM with drift). RBM with drift is defined as follows: Given a d -dimensional Brownian motion $\{B_t\}_{t \geq 0}$, departing from $B_0 = 0$ and defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_x)$, the RBM with drift is the (unique) solution to the following stochastic differential equation on S :

$$X_t = X_0 + B_t - \frac{1}{2} \int_0^t \nabla_f(X_s) ds - \int_0^t \eta_{X_s} \xi(ds), \quad \text{where } X_t \in \bar{D}, \quad \forall t \geq 0,$$

where the drift, $\nabla_f(x)$, is given by the gradient of a function f , and is assumed to be Lipschitz, $\{\xi_t\}_{t \geq 0}$ is the corresponding local time: i.e., a one-dimensional continuous non-decreasing process with $\xi_0 = 0$ that satisfies $\xi_t = \int_0^t \mathbb{I}_{\{X_s \in \partial S\}} d\xi_s$. Since the drift function is given by the gradient of a function f and the set S is compact, we have that the stationary distribution π has a density g bounded from below by a constant $c > 0$.

Corollary 3. *Let $S \subset \mathbb{R}^d$ be a non-empty compact set with connected interior such that $S = \bar{S}$, and suppose that S fulfils the outside and inside α -rolling conditions. Assume also that S is (C, ε_0) -regular for some positive constants C and ε_0 and that ∂S has a number of linear intersection bounded by N_S . Let $\mathcal{X}_T \subset S$ as before. Then, with probability one, for T large enough,*

$$\hat{I}_{d-1}^{N_0}(\partial S) = |\partial S|_{d-1} + o\left(\left(\frac{\ln(T)}{T}\right)^{\frac{1}{d}}\right).$$

4.3 α' -hull based estimator under α -rolling ball condition

In [4] it has been proved that, in dimension two, under some regularity assumptions, the length of the boundary of the α -shape of an iid sample converges to the length of the boundary of the set. The α -shape has the very good property that its boundary is very easy to compute, and so its surface measure. Unfortunately we are not sure that the results can be extended to higher dimension. Nevertheless considering the α -convex hull (which is quite close to the α -shape) allows to extend the results on the surface measure for any dimension. The price to pay is the difficulty to obtain an explicit formula for the surface measure of the α -convex hull. We so propose to skip this problem by a Monte-Carlo estimation based on Crofton's formula. The following theorem states that the surface measure of the boundary of the α -convex hull $\mathcal{X}_n \subset S$ converges toward $|\partial S|_{d-1}$ with a rate that depends on $d_H(\partial C_\alpha(\mathcal{X}), \partial S)$.

Theorem 4.3. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the inside and outside α -rolling conditions. Let $\alpha' < \alpha$ be a positive constant and let $\mathcal{X}_n \subset S$ be such that $d_H(\partial C_{\alpha'}(\mathcal{X}_n), \partial S) \leq \varepsilon_n$ with $\varepsilon_n \leq \min((\alpha\alpha')/(16(\alpha + \alpha')), 1/((d-1)\alpha))$. Let us assume that $C_{\alpha'}(\mathcal{X}_n)$ is a finite union of compact \mathcal{C}^2 , $(d-1)$ -dimensional manifold. Then*

1. $\pi_{\partial S} : \partial C_{\alpha'}(\mathcal{X}_n) \rightarrow \partial S$ (where $\pi_{\partial S}(x)$ denotes the projection onto ∂S) is one to one
2. $||\partial S|_{d-1} - |\partial C_{\alpha'}(\mathcal{X}_n)|_{d-1}| = \left(3\alpha + 64 \frac{\alpha+\alpha'}{\alpha\alpha'}\right) \varepsilon_n$,
3. as a consequence $\check{I}_{d-1}(\partial S) = \left(3\alpha + 64 \frac{\alpha+\alpha'}{\alpha\alpha'}\right) \varepsilon_n$.

As previously, one can deduce from the deterministic theorem, the convergence rates under the iid assumption. Now we have, as a corollary of Theorem 4.3 and [4]:

Corollary 4. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the inside and outside α -rolling conditions. Let $\mathcal{X}_n = \{X_1, \dots, X_n\}$ be an iid sample of X with distribution P_X supported on S . Assume that P_X has density f (w.r.t. μ_d) bounded from below by some $c > 0$. Suppose $\alpha' \leq \alpha$. Then with probability one, for n large enough,*

1. $||\partial S|_{d-1} - |\partial C_{\alpha'}(\mathcal{X}_n)|_{d-1}| = \mathcal{O}((\ln(n)/n)^{2/(d+1)})$,
2. as a consequence

$$\check{I}_{d-1}(\partial S) = |\partial S|_{d-1} + \mathcal{O}\left(\left(\frac{\ln(n)}{n}\right)^{\frac{2}{d+1}}\right).$$

In this case we do not need the additional hypothesis of (C, ε_0) -regularity, the convergence rate is far better than the one given in Theorem 4.1, where the price to pay is the computational cost when d increases. With regard to the parameter selection α' , a fully data driven (but computationally expensive) method is proposed in [34].

4.4 On the rates of convergence

Observe that we obtained in Corollary 4 the same convergence rate as the one provided in [4] for $d = 2$, where is also conjectured as suboptimal with regard to the result obtained in [31] (see Chapter 8). Indeed, as mentioned in [4], if the measure of the symmetric difference between S and an estimator \hat{S}_n is bounded by ε_n , we can only expect that plug-in methods allow to estimate $|\partial S|_{d-1}$ with a convergence rate ε_n . Thus, in the iid setting, the estimator defined by (6) (respectively (7) to (9)) can be seen as “optimal” relatively to the use of the Devroye–Wise support estimator (respectively the α -convex hull support estimator), since they achieve the best possible convergence rate for those estimators. This is nevertheless far from being optimal. Indeed, the minimax rate is conjecture to be $n^{-\frac{d+3}{2d+2}}$, because it is the minimax rate for the volume estimation problem (see [5]) and, in [30] it is proved that the minimax rate is the same for the volume estimation problem and the surface area estimation problem (at least in the image setting, that usually extends to the iid setting). Unfortunately attaining this optimal rate for the surface area estimation problem is much more involved, even in the easier setting with data uniformly drawn in S and S^c with perfect identification, no estimator attaining this rate has been proposed yet.

5 Computational and practical aspects of the algorithms

The algorithms to compute $\hat{n}_{\varepsilon_n}(\theta, y)$ and $\check{n}_\alpha(\theta, y)$ work for any finite set \mathcal{X}_n (not necessarily random). The general case for stochastic processes indexed by $T \in \mathbb{R}^+$ is obtained by replacing the set \mathcal{X}_n in the algorithm by a discretization of a trajectory of the process observed in $[0, T]$ (which is not restrictive since, the trajectories are always stocked as a finite number of points in a computer).

Let us first describe the algorithms that allows to compute the estimations of $n_{\partial S}(\theta, y)$ for a given (θ, y) .

5.1 Devroye–Wise based approach

To compute $\hat{n}_{\varepsilon_n}(\theta, y)$ for a given (θ, y) we proceed as follows.

1. Identify the centers $\mathcal{Y}_{n'} = \{Y_1, \dots, Y_{n'}\}$ of the boundary balls of $\hat{S}_{\varepsilon_n}(\mathcal{X}_n)$ (see [2]), i.e., the points $X_i \in \mathcal{X}_n$ such that

$$\max\{\|x - X_i\| : x \in \text{Vor}(X_i)\} \geq \varepsilon_n,$$

where $\text{Vor}(X_i) = \{x \in \mathbb{R}^d \text{ s.t. for all } j : \|x - X_i\| \leq \|x - X_j\|\}$ denotes the Voronoi cell of X_i .

2. Compute $d_i = d(r_{\theta, y}, Y_i)$.
3. Compute the connected components I_i , of $r_{\theta, y} \cap \hat{S}_{\varepsilon_n}(\mathcal{X}_n)$, according to the following steps: Initialize the list of the extremes of these intervals by $\text{list} = \emptyset$, and then, for $i = 1$ to n' :
 - If $d_i \leq \varepsilon_n$ then compute $\{z_1, z_2\} = \mathcal{B}(Y_i, \varepsilon_n) \cap r_{\theta, y}$.

– For $j = 1$ to 2 : if $d(z_j, \mathcal{X}_n) \geq \varepsilon_n$ do list=list $\cup\{z_j\}$.

The a_i and b_i (and so the I_i) introduced in Definition 3.1 are obtained by a sorting procedure applied to the points z_j .

4 Obtain the a'_i and b'_i such that $I'_i = (a'_i, b'_i)$ are the connected components of $\hat{S}_{4\varepsilon_n}(\mathcal{X}_n) \cap r_{\theta,y}$ by using the same procedure.

5. Lastly, compute $\hat{n}_{\varepsilon_n}(\theta, y)$, as follows:

initialization $\hat{n}_{\varepsilon_n}(\theta, y) = m$. For $i = 1$ to $m - 1$

- If there exists k such that $(b_i, a_{i+1}) \subset I'_k$ then: $\hat{n}_{\varepsilon_n}(\theta, y) = \hat{n}_{\varepsilon_n}(\theta, y) - 1$

5.2 α -convex hull based approach

It is much more involved to compute $\tilde{n}_\alpha(\theta, y)$: it requires the computation of the α -convex hull, as well as the convex hull, of the set \mathcal{X}_n . Recall that the convex hull of a sample is equal to the intersection of a finite number of half-spaces. In [22] it is proved, for dimension 2, but mentioned that the generalization is not difficult, that $C_\alpha(\mathcal{X}_n)^c$ is the union of a finite number of balls and the aforementioned half-spaces. The centers O_i of these balls, and their radii r_i , are obtained by computing the Delaunay complex of the points. Let us write $C_\alpha(\mathcal{X}_n)^c = \bigcup_i E_i$, where E_i is either a half-space or a ball. Observe that if the line $r_{\theta,y}$ is chosen at random (w.r.t. $d\mu_{d-1}d\theta$), $r_{\theta,y} \cap E_i$ contains fewer than 3 points.

Initialize list= \emptyset . Then:
for all i ,

- compute $r_{\theta,y} \cap \partial E_i$
- For all $z \in r_{\theta,y} \cap \partial E_i$
 1. If for all $j \ z \notin \mathring{E}_j$ do list=list $\cup\{z\}$

then $\tilde{n}(\theta, y) = \#\text{list}$.

5.3 Integralgeometric estimations via a Monte Carlo method

Once we have estimated $n_{\partial S}(\theta, y)$ by $\hat{n}_{\varepsilon_n}(\theta, y)$ for any given (θ, y) , $\hat{I}_{d-1}(\partial S)$ can be calculated via the Monte-Carlo method, as follows. Generate a random sample $\theta_1, \dots, \theta_k$ uniformly distributed on $(\mathcal{S}^+)^{d-1}$. For each $i = 1, \dots, k$, build a random sample $\aleph_i = \{y_1^i, \dots, y_\ell^i\}$ uniformly distributed on the $(d-1)$ -dimensional hyper-cube $[-L, L]^{d-1} \subset \theta_i^\perp$, where $L = \max_{j=1, \dots, n} \|X_j\|$, and

independent of $\theta_1, \dots, \theta_k$. Then, the estimators are given by

$$\hat{I}_{d-1}^{(\ell,k)}(\partial S) = \frac{(2L)^{d-1}}{\beta(d)} \frac{1}{\ell k} \sum_{i=1}^k \sum_{j=1}^{\ell} \hat{n}_{\varepsilon_n}(\theta_i, y_j^i) \quad (5)$$

$$\hat{I}_{d-1}^{(\ell,k,N_0)}(\partial S) = \frac{(2L)^{d-1}}{\beta(d)} \frac{1}{\ell k} \sum_{i=1}^k \sum_{j=1}^{\ell} \min(\hat{n}_{\varepsilon_n}(\theta_i, y_j^i), N_0) \quad (6)$$

$$\check{I}_{d-1}^{(\ell,k)}(\partial S) = \frac{(2L)^{d-1}}{\beta(d)} \frac{1}{\ell k} \sum_{i=1}^k \sum_{j=1}^{\ell} \check{n}_r(\theta_i, y_j^i). \quad (7)$$

6 Appendix

6.1 Proof of Theorems 4.1 and 4.2

Sketch of the proof of Theorems 4.1 and 4.2

The idea is to consider separately two subsets of the set of lines that intersect $\partial \hat{S}_{\varepsilon_n}(\mathcal{X}_n)$:

1. If a line $r_{\theta,y} = y + \lambda\theta$ is ‘far enough’ (fulfilling condition $L(\varepsilon)$ for some $\varepsilon > 0$, see Definition 6.1) from the tangent spaces, then our algorithm allows a perfect estimation of $n_{\partial S}(\theta, y)$, see Lemma 6.5.
2. Considering the set of lines that are not ‘far enough’ from the tangent spaces (denoted by $\mathcal{A}_{\varepsilon_n}(\theta)$), see Definition 6.1), Corollary 5 states that, under (C, ε_0) -regularity, the integral of $\hat{n}_{\varepsilon_n}(\theta, y)$ on $\mathcal{A}_{\varepsilon_n}(\theta)$ is bounded from above by $C'\varepsilon_n^{1/2}$, C' being a positive constant. Theorem 4.2 states that the previous bound can be improved to $C'\varepsilon_n$, under (C, ε_0) -regularity, if ∂S has a bounded number of linear intersections.

6.1.1 Condition $L(\varepsilon)$

Let us define the two sets of lines to be treated separately: The lines that are ‘far’ from an affine tangent space, and the lines that are ‘close to being tangent’ to ∂S . More precisely, assume that ∂S is smooth enough so that for all $x \in \partial S$, the unit outer normal vector η_x at x is well defined. Now we define

$$\mathcal{T}_S = \{x + (\eta_x)^\perp : x \in \partial S\},$$

the collection of all the affine $(d-1)$ -dimensional tangent spaces.

Definition 6.1. Let $\varepsilon \geq 0$. A line $r_{\theta,y} = y + \lambda\theta$ fulfills **condition** $L(\varepsilon)$ if y is at a distance larger than 4ε from all the affine hyper-planes $w + \eta^\perp \in \mathcal{T}_S$ satisfying $\langle \eta, \theta \rangle = 0$.

For a given θ , we define

$$\mathcal{A}_\varepsilon(\theta) = \left\{ y \in \theta^\perp : \|y\| \leq \text{diam}(S) \text{ and } r_{\theta,y} \text{ does not satisfy } L(\varepsilon) \right\}.$$

6.1.2 Some useful lemmas

Lemma 6.2. *Let S be a compact set fulfilling the outside and inside α -rolling conditions. Let $r_{\theta,y}$ be a line that fulfills condition $L(0)$ and $r_{\theta,y} \cap \partial S \neq \emptyset$. Then $r_{\theta,y}$ intersects ∂S in a finite number of points.*

Proof. Because S fulfills the outside and inside α -rolling conditions, Theorem 1 in [37] implies that for any $x \in \partial S$, the affine $(d-1)$ -dimensional tangent space $T_x \partial S$ exists. If $r_{\theta,y}$ fulfills $L(0)$, then $r_{\theta,y}$ is not included in any hyper-plane tangent to S . Suppose that $\partial S \cap r_{\theta,y}$ is not finite. Then, by compactness, one can extract a subsequence t'_n that converges to $y' \in \partial S$. Note that for all $(n,p) \in \mathbb{N}^2$ $(t'_n - t'_{n+p}) / \|t'_n - t'_{n+p}\| = \pm\theta$, which implies that $(t'_n - y') / \|t'_n - y'\| = \pm\theta$. Lastly, if $n \rightarrow \infty$, then $\theta \in T_{y'} \partial S$. Considering y' , we have $y' \in \partial S$, $\theta \in T_{y'} \partial S$ and $y' \in r_{\theta,y}$, which contradicts the assumption that $r_{\theta,y}$ is not included in any hyper-plane tangent to S . \square

Lemma 6.3. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside α -rolling conditions. Let $\varepsilon > 0$ such that $\varepsilon < \alpha/4$ and $\nu = 2\sqrt{2\varepsilon(\alpha - 2\varepsilon)}$. For any line $r_{\theta,y}$ fulfilling condition $L(\varepsilon)$ and $r_{\theta,y} \cap \partial S \neq \emptyset$, we have that $r_{\theta,y}$ meets ∂S at a finite number of points t_1, \dots, t_k , where $t_{i+1} - t_i \geq 2\nu$ for all $i = 1, \dots, k-1$. Consequently, if $\varepsilon < \alpha/4$, then $k \leq \text{diam}(S)\varepsilon^{-1/2}/(4\sqrt{\alpha})$.*

Proof. Note that if a line fulfills condition $L(\varepsilon)$, then it fulfills condition $L(0)$. Consequently, the fact that $r_{\theta,y}$ intersects ∂S in a finite number of points follows from Lemma 6.2. Let us denote by $t_1 < \dots < t_k$ the intersection of $r_{\theta,y}$ with ∂S . Proceeding by contradiction, assume that for some i , $t_{i+1} - t_i < 2\nu$. Let us denote by η_{t_i} and $\eta_{t_{i+1}}$ the outer normal vectors at t_i and t_{i+1} , respectively. We have two cases: the open interval $(t_i, t_{i+1}) \subset S^c$ or $(t_i, t_{i+1}) \subset \dot{S}$. Let us consider the first case (the proof for the second one is similar).

Because $(t_i, t_{i+1}) \subset S^c$ and S fulfills the inside α -rolling condition on t_i , there exists $z \in S$ such that $t_i \in \partial \mathcal{B}(z, \alpha)$ and $\mathcal{B}(z, \alpha) \subset S$. In particular, $\mathcal{B}(z, \alpha) \cap (t_i, t_{i+1}) = \emptyset$, which implies $\langle \eta_{t_i}, \theta \rangle \geq 0$.

Reasoning in the same way but with t_{i+1} , we get $\langle \eta_{t_{i+1}}, \theta \rangle \leq 0$. Given that $r_{\theta,y}$ is not included in any tangent hyperplane, we have that $\langle \eta_{t_i}, \theta \rangle > 0$ and $\langle \eta_{t_{i+1}}, \theta \rangle < 0$. Because S fulfills the inside and outside α -rolling conditions, ∂S is a $(d-1)$ -dimensional \mathcal{C}^1 manifold whose normal vector is Lipschitz (see Theorem 1 in [37]). By Theorem 3.8 in [15], there exists a curve $\gamma : [0, 1] \rightarrow \partial S$ such that $\gamma(0) = t_i$, $\gamma(1) = t_{i+1}$ and $d(\gamma(t), r_{\theta,y}) \leq 4\varepsilon$ for all t . From $\langle \eta_{t_i}, \theta \rangle > 0$ and $\langle \eta_{t_{i+1}}, \theta \rangle < 0$, it follows that there exists an $s_0 \in (0, 1)$ such that $\langle \eta_{\gamma(s_0)}, \theta \rangle = 0$, which contradicts the hypothesis that y is at a distance larger than 4ε from all the $(d-1)$ -dimensional hyperplanes tangent to S . This proves that $t_{i+1} - t_i \geq 2\nu$ for all $i = 1, \dots, k-1$. \square

Lemma 6.4. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside α -rolling ball conditions and with a (C, ε_0) -regular boundary. Then for all $\varepsilon \leq \min\{\varepsilon_0, \alpha/4\}$,*

$$\int_{\theta \in (S^+)^{d-1}} \int_{y \in \mathcal{A}_\varepsilon(\theta)} n_{\partial S}(\theta, y) d\mu_{d-1}(y) d\theta \leq C \frac{\text{diam}(S)}{\sqrt{\alpha}} \sqrt{\varepsilon}.$$

Moreover if ∂S has bounded number of linear intersections then

$$\int_{\theta \in (S^+)^{d-1}} \int_{y \in \mathcal{A}_\varepsilon(\theta)} n_{\partial S}(\theta, y) d\mu_{d-1}(y) d\theta \leq 4CN_S \varepsilon. \quad (8)$$

Proof. From the proof of the previous lemma, it follows that for any $y \in E_\theta$ with $d(y, F_\theta) = l$ and $l < 4\varepsilon$, $n_{\partial S}(\theta, y) \leq \text{diam}(S)l^{-1/2}/(4\sqrt{\alpha})$. Hence,

$$\begin{aligned}
& \int_{\theta \in (S^+)^{d-1}} \int_{y \in \mathcal{A}_\varepsilon(\theta)} n_{\partial S}(\theta, y) d\mu_{d-1}(y) d\theta \\
&= \int_{\theta \in (S^+)^{d-1}} \int_{l=0}^{4\varepsilon} \int_{\{y \in \theta^\perp : d(y, F_\theta) = l\}} n_{\partial S}(\theta, y) d\mu_{d-2}(y) dl d\theta \\
&\leq \int_{\theta \in (S^+)^{d-1}} \int_{l=0}^{4\varepsilon} \int_{\{y \in \theta^\perp : d(y, F_\theta) = l\}} \frac{1}{4} \text{diam}(S)(\alpha l)^{-1/2} d\mu_{d-2}(y) dl d\theta \\
&\leq \int_{\theta \in (S^+)^{d-1}} \int_{l=0}^{4\varepsilon} \frac{1}{4} \text{diam}(S)(\alpha l)^{-1/2} \int_{\{y \in \theta^\perp : d(y, F_\theta) = l\}} d\mu_{d-2}(y) dl d\theta \\
&\leq \int_{\theta \in (S^+)^{d-1}} \int_{l=0}^{4\varepsilon} \frac{1}{4} \text{diam}(S)(\alpha l)^{-1/2} |\{y \in \theta^\perp : d(y, F_\theta) = l\}|_{d-2} dl d\theta.
\end{aligned}$$

By the definition of φ_θ ,

$$\left| \{y \in \theta^\perp : l \leq d(y, F_\theta) \leq l + dl\} \right|_{d-1} = \varphi_\theta(l + dl) - \varphi_\theta(l).$$

From the (C, ε_0) -regularity of ∂S and the mean value theorem we obtain

$$\left| \{y \in \theta^\perp : d(y, F_\theta) = l\} \right|_{d-2} \leq \sup_{\varepsilon \in (0, \varepsilon_0)} \varphi'_\theta(\varepsilon) \leq C,$$

which implies

$$\begin{aligned}
& \int_{\theta \in (S^+)^{d-1}} \int_{y \in \mathcal{A}_\varepsilon(\theta)} n_{\partial S}(\theta, y) d\mu_{d-1}(y) d\theta \leq \\
& \int_{\theta \in (S^+)^{d-1}} \int_{l=0}^{4\varepsilon} C \frac{1}{4} \text{diam}(S)(\alpha l)^{-1/2} dl d\theta \leq C \frac{\text{diam}(S)}{\sqrt{\alpha}} \sqrt{\varepsilon}.
\end{aligned}$$

Applying exactly the same calculus, under the hypothesis of bounded number of linear intersections for ∂S , we get

$$\int_{\theta \in (S^+)^{d-1}} \int_{y \in \mathcal{A}_\varepsilon(\theta)} n_{\partial S}(\theta, y) d\mu_{d-1}(y) d\theta \leq \int_{\theta \in (S^+)^{d-1}} \int_{l=0}^{4\varepsilon} CN_S dl d\theta \leq 4CN_S \varepsilon.$$

□

Lemma 6.5. *Let S be a compact set fulfilling the outside and inside α -rolling conditions. Let $\mathcal{X}_n = \{X_1, \dots, X_n\} \subset S$. Let $\varepsilon_n \rightarrow 0$ be such that $d_H(\mathcal{X}_n, S) \leq \varepsilon_n$. Let $r_{\theta, y} = y + \lambda\theta$ be any line fulfilling condition $L(\varepsilon_n)$. Then, for n large enough such that $4\varepsilon_n < \alpha$, $n_{\partial S}(\theta, y) = \hat{n}_{\varepsilon_n}(\theta, y)$.*

Proof. Note that the choice of ε_n ensures that $S \subset \hat{S}_{\varepsilon_n}(\mathcal{X}_n)$, thus

$$r_{\theta, y} \cap S \subset r_{\theta, y} \cap \hat{S}_{\varepsilon_n}(\mathcal{X}_n). \tag{9}$$

First, we will prove that

$$\hat{n}_{\varepsilon_n}(\theta, y) \geq n_{\partial S}(\theta, y). \tag{10}$$

Because $\hat{n}_{\varepsilon_n}(\theta, y)$ is not the number of connected components of $r_{\theta, y} \cap \hat{S}_{\varepsilon_n}(\mathcal{X}_n)$, (10) does not follow directly from (9). If $r_{\theta, y} \cap \partial S = \emptyset$ inequality (10) holds. Assume $r_{\theta, y} \cap \partial S \neq \emptyset$. Let $t_1 < \dots < t_k$ be the intersection of $r_{\theta, y}$ with ∂S (this set is finite due to Lemma 6.2). Let us prove that

$$\text{if } (t_i, t_{i+1}) \subset S^c, \text{ then: } \exists s \in (t_i, t_{i+1}) \text{ such that } d(s, S) > 4\varepsilon_n. \quad (11)$$

Because S fulfills the inside α -rolling condition on t_i , there exists a $z_i \in S$ such that $t_i \in \partial \mathcal{B}(z_i, \alpha)$ and $\mathcal{B}(z_i, \alpha) \subset S$. Since $\mathcal{B}(z_i, \alpha) \cap (t_i, t_{i+1}) = \emptyset$, it follows that $\langle \eta_{t_i}, \theta \rangle \geq 0$ (recall that $\eta_{t_i} = (t_i - z_i)/\alpha$ and $t_{i+1} - t_i = \|t_{i+1} - t_i\|\theta$). Reasoning in the same way but with t_{i+1} , $\langle \eta_{t_{i+1}}, \theta \rangle \leq 0$. By condition $L(\varepsilon_n)$ we obtain

$$\langle \eta_{t_i}, \theta \rangle > 0 \text{ and } \langle \eta_{t_{i+1}}, \theta \rangle < 0. \quad (12)$$

Suppose that for all $t \in (t_i, t_{i+1})$ we have $d(t, \partial S) \leq 4\varepsilon_n$. Take n large enough such that $4\varepsilon_n < \alpha$. Because ∂S fulfills the outside and inside α -rolling conditions, by Lemma 2.3 in [29] it has positive reach. Then, by Theorem 4.8 in [23], $\gamma = \{\gamma(t) = \pi_{\partial S}(t), t \in (t_i, t_{i+1})\}$, the orthogonal projection onto ∂S of the interval (t_i, t_{i+1}) is well defined and is a continuous curve in ∂S . By Theorem 1 in [37], the map from ∂S to \mathbb{R}^d that sends $x \in \partial S$ to $\eta_x \in \partial \mathcal{B}(0, 1)$ is Lipschitz. Thus, $t \rightarrow \langle \eta_{\gamma(t)}, \theta \rangle$ is a continuous function of t for all $t \in (t_i, t_{i+1})$, which, together with (12), ensures the existence of an $s \in (t_i, t_{i+1})$ such that $d(s, \gamma(s)) \leq 4\varepsilon_n$ and $\theta \in \eta_{\gamma(s)}^\perp$, which contradicts the assumption that $r_{\theta, y}$ fulfills condition $L(\varepsilon_n)$. This proves (11), which implies that

$$\text{if } (t_i, t_{i+1}) \subset S^c, \text{ then: } \exists s \in (t_i, t_{i+1}) \text{ such that } d(s, \mathcal{X}_n) > 4\varepsilon_n$$

and now (10) follows from (9).

Next we will prove the opposite inequality,

$$\hat{n}_{\varepsilon_n}(\theta, y) \leq n_{\partial S}(\theta, y). \quad (13)$$

Assume first $r_{\theta, y} \cap \partial S \neq \emptyset$. Let $\{t_1, \dots, t_k\}$ be the intersection of $r_{\theta, y}$ with ∂S (this set is finite due to Lemma 6.2).

Consider $t^* \in (t_i, t_{i+1}) \subset S^c$ and $t^* \in \hat{S}_{\varepsilon_n}(\mathcal{X}_n)$. Equation (13) will be derived from the fact that $(t^*, t_{i+1}) \subset \hat{S}_{\varepsilon_n}(\mathcal{X}_n) \cap r_{\theta, y}$ or $[t_i, t^*) \subset \hat{S}_{\varepsilon_n}(\mathcal{X}_n) \cap r_{\theta, y}$.

Introduce $\psi(t) : (t_i, t_{i+1}) \rightarrow \mathbb{R}$ defined by $\psi(t) = d(t, \partial S)$. Let us consider the points $t \in (t_i, t_{i+1})$, such that $d(t, \partial S) < \alpha$, and let $p_t \in \partial S$ such that $\|p_t - t\| = d(t, \partial S)$. By item (3) in Theorem 4.8 in [23], $\psi'(t) = \langle \eta_{p_t}, \theta \rangle$.

Let X_j be the closest observation to t^* (recall that because $t^* \in \hat{S}_{\varepsilon_n}(\mathcal{X}_n)$, we have $\|X_j - t^*\| \leq \varepsilon_n$). Now, because there exists a point $p^* \in [t^*, X_j] \cap \partial S$, we obtain that $\psi(t^*) \leq \varepsilon_n$ and, because $r_{\theta, y}$ fulfills $L(\varepsilon_n)$, $\langle \eta_{p_{t^*}}, \theta \rangle \neq 0$.

Assume that, for instance, $\langle \eta_{p_{t^*}}, \theta \rangle < 0$. Then $\psi(t^*) \leq \varepsilon_n$ and $\psi'(t^*) < 0$. Suppose that there exists a $t' \in (t^*, t_{i+1})$ such that $\psi(t') \geq \varepsilon_n$ and consider $t'' = \inf\{t > t^*, \psi(t) \geq \varepsilon_n\}$. Then for all $t \in (t^*, t'')$ we have $\psi(t) \leq \varepsilon_n < \alpha$, and thus ψ is differentiable on this interval (using again item (3) of Theorem 4.8 in [23]). From the fact that $\psi(t'') \geq \psi(t^*)$ and $\psi'(t^*) < 0$ we deduce that there exists a $\tilde{t} \in (t^*, t'')$ such that $\psi'(\tilde{t}) = 0$, which contradicts $L(\varepsilon_n)$ because $\psi(\tilde{t}) \leq \varepsilon_n$. To summarize, we have shown that if $\langle \eta_{p_{t^*}}, \theta \rangle < 0$, then $(t^*, t_{i+1}) \subset \hat{S}_{\varepsilon_n}(\mathcal{X}_n)$. Symmetrically, if $\langle \eta_{p_{t^*}}, \theta \rangle > 0$, then $(t_i, t^*) \subset \hat{S}_{\varepsilon_n}(\mathcal{X}_n)$,

which concludes the proof.

Reasoning in the same way, if $r_{\theta,y} \cap \partial S = \emptyset$ and $\hat{n}_{\varepsilon_n}(\theta, y) > 0$, a contradiction with condition $L(\varepsilon_n)$ is obtained. \square

Lemma 6.6. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside α -rolling conditions. Let $\varepsilon_n \rightarrow 0$ be a sequence such that $d_H(\mathcal{X}_n, S) \leq \varepsilon_n$, while $r_{\theta,y} = y + \lambda\theta$ and A_1, \dots, A_k are the sets in Definition 3.1, $A_i = (a_i, b_i)$ for $i = 1, \dots, k$. Now suppose that the sets are indexed in such a way that $a_1 < b_1 < a_2 < \dots < b_k$. Then for all $i = 2, \dots, k$, we have that $\|a_i - b_{i-1}\| > 3\sqrt{\varepsilon_n\alpha}$ and for all $i = 1, \dots, k$, $\|b_i - a_i\| > 3\sqrt{\varepsilon_n\alpha}$, for n large enough such that $3\sqrt{\alpha\varepsilon_n} < \alpha/2$, which implies*

$$\hat{n}_{\varepsilon_n}(\theta, y) \leq \frac{\text{diam}(S)}{3\sqrt{\alpha}} \varepsilon_n^{-1/2}.$$

Proof. Assume by contradiction that for some i , $\|a_i - b_{i-1}\| \leq 3\sqrt{\varepsilon_n\alpha}$. By construction, $[b_{i-1}, a_i] \subset \hat{S}_{\varepsilon_n}(\mathcal{X}_n)^c \subset S^c$. Because a_i and b_i are on $\partial \hat{S}_{\varepsilon_n}(\mathcal{X}_n)$, we have $d(a_i, \mathcal{X}_n) = d(b_{i-1}, \mathcal{X}_n) = \varepsilon_n$.

The projection $\pi_S : [b_{i-1}, a_i] \rightarrow \partial S$ is uniquely defined because ∂S has reach at least α and $d(t, \partial S) \leq d(t, a_i) + d(a_i, \partial S) \leq \|a_i - b_{i-1}\| + d(a_i, \mathcal{X}_n)$ for all $t \in (b_{i-1}, a_i)$, $\|a_i - b_{i-1}\| \leq 3\sqrt{\varepsilon_n\alpha} < \alpha/2$ and $d(a_i, \partial S) \leq \varepsilon_n \leq \alpha/2$. Moreover, π is a continuous function.

Hence $\max_{x \in [b_{i-1}, a_i]} \|x - \pi_S(x)\| \geq \varepsilon_n$, and the maximum is attained at some $x_0 \in [b_{i-1}, a_i]$. We will prove that $\|x_0 - \pi_S(x_0)\| \geq 3\varepsilon_n$, which guarantees that $x_0 \in (b_{i-1}, a_i)$ and that η_0 , the outward unit normal vector to ∂S at $\pi_S(x_0)$, is normal to θ . Indeed, suppose by contradiction that for all $t \in (b_{i-1}, a_i)$, $d(t, \partial S) \leq 3\varepsilon_n$. Then $d(t, \mathcal{X}_n) \leq 4\varepsilon_n$, which contradicts the definition of the points a_i and b_i . Let $z_0 = \pi_S(x_0) + \eta_0\alpha$. Observe that $d(a_i, S) \leq \varepsilon_n$ and $d(b_{i-1}, S) \leq \varepsilon_n$. From the outside α -rolling condition at $\pi_S(x_0)$, and using the fact that η_0 is normal to θ , we have (see Figure 8)

$$r_{\theta,y} \cap \mathcal{B}(z_0, \alpha - \varepsilon_n) \subset [b_{i-1}, a_i],$$

which implies, see Figure 8, that $\|a_i - b_{i-1}\| \geq 2\sqrt{(\alpha - \varepsilon_n)^2 - (\alpha - l)^2}$, where $l = d(x_0, \pi_S(x_0))$. Therefore,

$$\|a_i - b_{i-1}\| \geq 2\sqrt{(l - \varepsilon_n)(2\alpha - l - \varepsilon_n)}. \quad (14)$$

If we bound $l \geq 3\varepsilon_n$ and use the fact that $l = o(1)$, which follows from $l \leq \|b_{i-1} - a_i\| + \varepsilon_n \leq 3\sqrt{\varepsilon_n\alpha} + \varepsilon_n$, then we get, from (14),

$$\|a_i - b_{i-1}\| \geq 2\sqrt{2\varepsilon_n(2\alpha - l - \varepsilon_n)} = 2\sqrt{4\varepsilon_n\alpha(1 + o(1))} = 4\sqrt{\alpha\varepsilon_n}(1 + o(1)),$$

and for n large enough this contradicts $\|a_i - b_{i-1}\| \leq 3\sqrt{\alpha\varepsilon_n}$. Then, the number of disjoint intervals A_i is bounded from above by $\text{diam}(S)/(3\sqrt{\varepsilon_n\alpha})$. The proof that for all $i = 1, \dots, k$, $\|b_i - a_i\| > 3\sqrt{\varepsilon_n\alpha}$ follows the same ideas, and it is left to the reader. Lastly, $\hat{n}_{\varepsilon_n}(\theta, y) \leq \text{diam}(S)/(3\sqrt{\varepsilon_n\alpha})$. \square

Corollary 5. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside α -rolling conditions and with a (C, ε_0) -regular boundary. For n large enough such that $3\sqrt{\alpha\varepsilon_n} < \min(\alpha/2, \varepsilon_0)$, we have*

$$\int_{\theta} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} \hat{n}_{\varepsilon_n}(\theta, y) d\mu_{d-1}(y) d\theta \leq C \frac{\text{diam}(S)}{3\sqrt{\alpha}} \sqrt{\varepsilon_n}.$$

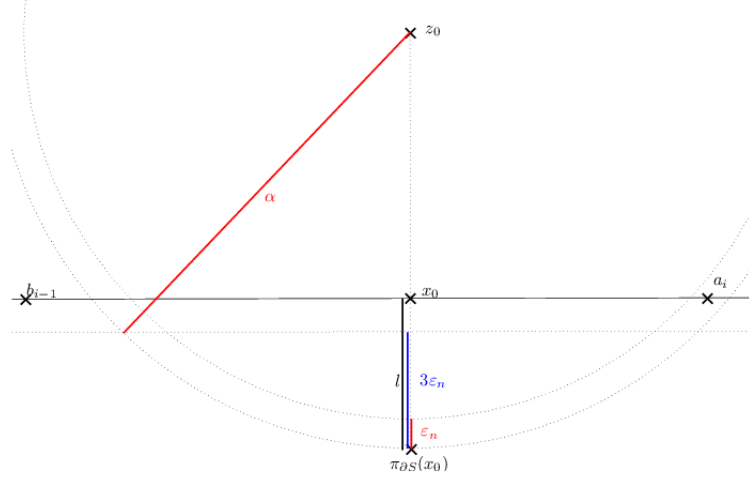


Figure 8: $\|a_i - b_{i-1}\| \geq 2\sqrt{(\alpha - \varepsilon_n)^2 - (\alpha - l)^2}$, where $l = d(x_0, \pi_S(x_0))$.

6.1.3 Proof of Theorem 4.1

Without loss of generality, we can assume that $0 \in S$. Recall that for $\theta \in (\mathcal{S}^+)^{d-1}$, $\mathcal{A}_{\varepsilon_n}(\theta)$ is the set of all $y \in \theta^\perp$ such that $\|y\| \leq \text{diam}(S)$ and $r_{\theta, y}$ does not fulfill $L(\varepsilon_n)$. First, from Lemma 6.5, we have

$$|I_{d-1}(\partial S) - \hat{I}_{d-1}(\partial S)| \leq \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} |\hat{n}_{\varepsilon_n}(\theta, y) - n_{\partial S}(\theta, y)| d\mu_{d-1}(y) d\theta.$$

So, by the triangle inequality we can bound the difference between the integral-geometric and its estimation by

$$\begin{aligned} & \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} \hat{n}_{\varepsilon_n}(\theta, y) d\mu_{d-1}(y) d\theta + \\ & \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} n_{\partial S}(\theta, y) d\mu_{d-1}(y) d\theta. \end{aligned}$$

Now, by applying Corollary 5 and Lemma 6.4, we get that

$$|I_{d-1}(\partial S) - \hat{I}_{d-1}(\partial S)| \leq \frac{4C \text{diam}(S)}{3\beta(d)\sqrt{\alpha}} \sqrt{\varepsilon_n},$$

for n large enough.

6.1.4 Proof of Theorem 4.2

The proof of Theorem 4.2 is basically the same than the previous one. Since $N_0 \geq N_S$ Lemma 6.5 ensures that, for all $r_{y, \theta}$ not in $\mathcal{A}_{\varepsilon_n}(\theta)$, $\min(\hat{n}(\theta, y), N_0) = n_{\partial S}(\theta, y)$, for n large enough such that $4\varepsilon_n < \alpha$ thus we still have, for n large enough,

$$|I_{d-1}(\partial S) - \hat{I}_{d-1}^{N_0}(\partial S)| \leq \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} |\hat{n}_{\varepsilon_n}(\theta, y) - n_{\partial S}(\theta, y)| d\mu_{d-1}(y) d\theta.$$

So, by the triangle inequality we can bound the difference between the integral-geometric and its estimation by

$$\begin{aligned} & \frac{1}{\beta(d)} \int_{\theta \in (S^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} n_{\partial S}(\theta, y) d\mu_{d-1}(y) d\theta + \\ & \frac{1}{\beta(d)} \int_{\theta \in (S^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} \hat{n}_{\varepsilon_n}(\theta, y) d\mu_{d-1}(y) d\theta. \end{aligned}$$

Now, by applying (8) for the first part and a similar calculus for the second part we get that

$$|I_{d-1}(\partial S) - \hat{I}_{d-1}^{N_0}(\partial S)| \leq \frac{C(4N_S + N_0)}{\beta(d)} \varepsilon_n \quad (15)$$

for n large enough.

6.1.5 Proof of Corollary 3

By Corollary 1 in [14], we know that, with probability one, for T large enough, $d_H(\mathcal{X}_T, S) \leq \varepsilon_T \rightarrow 0$, where $\varepsilon_T = o((\ln(T)^2/T)^{1/d})$. Let $\mathcal{X}_n = \{X_{t_1}, \dots, X_{t_n}\}$ be a discretization of \mathcal{X}_T such that $t_i - t_{i-1} = T/n$ and $t_n = T$. Let us define $\varepsilon_n = d_H(\mathcal{X}_n, S)$, then $\varepsilon_n \geq \varepsilon_T$. It is clear that, fixed T , ε_n decreases to ε_T as $n \rightarrow \infty$. To emphasize the dependence on the set, let us denote $\hat{I}_{d-1}^{N_0}(\partial S, \mathcal{X}_n)$ the estimator based on \mathcal{X}_n , and $\hat{I}_{d-1}^{N_0}(\partial S, \mathcal{X}_T)$ the estimator based on \mathcal{X}_T (both defined using Definition 3.1). Then by (15), to prove Corollary 3 it is enough to prove $\hat{I}_{d-1}^{N_0}(\partial S, \mathcal{X}_n) \rightarrow \hat{I}_{d-1}^{N_0}(\partial S, \mathcal{X}_T)$ as $n \rightarrow \infty$, for all T fixed. Fixed θ and y , it is clear that $\hat{n}(\theta, y)(\partial S, \mathcal{X}_n) \rightarrow \hat{n}(\theta, y)(\partial S, \mathcal{X}_T)$ as $n \rightarrow \infty$, then Corollary 3 follows by dominated convergence theorem, using that $\min\{\hat{n}(\theta, y), N_0\} \leq N_0$.

6.2 Proofs on the α -hull based estimator

Theorem 4.3 will be obtained from the two following lemmas. The first one states that eventually almost surely, the boundary of the α' -convex hull of an iid sample drawn on a α -convex support has some good geometrical properties.

The second one, which is purely geometric, bounds the difference between the measures of two sets, the first one having a positive reach α (as ∂S) and the second one having the same good geometrical properties as the boundary of $C_\alpha(\mathcal{X}_n)$.

We will introduce some notation. Let A and B be two sub-spaces of \mathbb{R}^d . We denote by $\angle(A, B)$ the operator norm of the difference between the orthogonal projection onto A , π_A , and the projection onto B , π_B , i.e., $\angle(A, B) = \|\pi_A - \pi_B\|_{op}$. If f is a function, then $\nabla_f(x)$ is its gradient and \mathcal{H}_f is its Hessian matrix. Given a point x in a $(d-1)$ -dimensional manifold E , $N_x E = \{v \in \mathbb{R}^d : \langle v, u \rangle = 0, \forall u \in T_x E\}$ is the 1-dimensional orthogonal subspace. If M is a compact $(d-1)$ -dimensional manifold we will denote by $\text{int}(M)$ its interior (in a manifold sense) i.e. $\text{int}(M) = \{x \in M, \exists \varepsilon_0 > 0, \text{ for all } \varepsilon \leq \varepsilon_0 \text{ we have } \hat{\mathcal{B}}(x, \varepsilon) \approx \mathcal{B}_{d-1}\}$ where \approx denote ‘‘is homeomorphic to’’ and \mathcal{B}_{d-1} denote the closed unit sphere in \mathbb{R}^{d-1} . Let us also introduce $\partial M = M \setminus \text{int}(M)$ that is a $(d-2)$ -dimensional manifold with no boundary (with no guarantee on its regularity).

Theorem 4.3 is a direct consequence of the two following lemmas.

Lemma 6.7. *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the inside and outside α -rolling conditions. Let $\alpha' < \alpha$ be a positive constant. Let $\mathcal{X}_n \subset S$ be such that $d_H(\partial C_{\alpha'}(\mathcal{X}_n), \partial S) \leq \varepsilon_n$ with $\varepsilon_n < (\alpha\alpha')/(4(\alpha + \alpha')) \leq \alpha/2$. Assume that there exists m such that $C_{\alpha'}(\mathcal{X}_n) = \cup_{i=1}^m F_i$ with F_i is a \mathcal{C}^2 , $(d-1)$ -dimensional manifold. Then*

1. *For all $i = 1, \dots, m$ and all $x \in \text{int}(F_i)$, we can uniquely define $\hat{\eta}_x$ the unit normal (to $\partial C_{\alpha'}(\mathcal{X}_n)$) outward (to $C_{\alpha'}(\mathcal{X}_n)$) pointing vector that satisfies*

$$\langle \hat{\eta}_x, \eta_{\pi_{\partial S}(x)} \rangle \geq 1 - \frac{2(\alpha + \alpha')}{\alpha\alpha'} \varepsilon_n.$$

2. $\pi_{\partial S} : \partial C_{\alpha'}(\mathcal{X}_n) \rightarrow \partial S$ the orthogonal projection onto ∂S is one to one.

Proof.

1. Let $x \in \partial C_{\alpha'}(\mathcal{X}_n)$. Because ∂S as a positive reach α and $\varepsilon_n \leq \alpha$ we have $x^* = \pi_{\partial S}(x)$ that is well define. We also have that there exists O such that $\mathcal{B}(O, \alpha') \cap C_{\alpha'}(\mathcal{X}_n) = \{x\}$ Now if we add the condition that $x \in \text{int}(F_i)$ for some i and so $\hat{\eta}_x$ is well defined it follows that $\hat{\eta}_x = (O - x)/\alpha'$.

Also introduce η_{x^*} the outward (from S) unit normal vector of ∂S at x^* and $O^* = x^* - \alpha\eta_{x^*}$.

Notice first that we have

$$\overset{\circ}{\mathcal{B}}(O, \alpha') \subset C_{\alpha'}(\mathcal{X}_n)^c \text{ and } \mathcal{B}(O^*, \alpha) \subset S. \quad (16)$$

Introduce $y^* = [O^*, O] \cap \partial \mathcal{B}(O^*, \alpha)$ and $y = [O^*, O] \cap \partial \mathcal{B}(O, \alpha')$ (See Figure 9). Then, from the second inclusion in (16), we get $y \in S$, and from the first inclusion in (16) we get $d(y, C_{\alpha'}(\mathcal{X}_n)) \geq \|y - y^*\|$. Then $\|y - y^*\| \leq \varepsilon_n$, which in turn implies

$$\alpha + \alpha' - \|O - O^*\| \leq \varepsilon_n \quad (17)$$

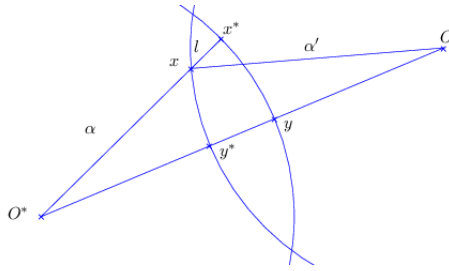


Figure 9: $x \in \partial C_{\alpha'}(\mathcal{X}_n)$, $x^* = \pi_{\partial S}(x)$, $O = x + \alpha'\hat{\eta}_x$ and $O^* = x^* - \alpha\eta_{x^*}$

From $x^* = \pi_{\partial S}(x)$ we get that $x^* = x + \ell\eta_{x^*}$ where $\ell = \|x - x^*\| \leq \varepsilon_n$.

Then $O = O^* + (\alpha - \ell)\eta_{x^*} + \alpha'\hat{\eta}_x$ and

$$\begin{aligned} \alpha + \alpha' - \|O - O^*\| &= \alpha + \alpha' - \sqrt{(\alpha')^2 + (\alpha - \ell)^2 + 2\alpha'(\alpha - \ell)\langle \hat{\eta}_x, \eta_{x^*} \rangle} \\ &= \alpha + \alpha' - \sqrt{(\alpha' + \alpha - \ell)^2 - 2\alpha'(\alpha - \ell)(1 - \langle \hat{\eta}_x, \eta_{x^*} \rangle)} \\ &= \alpha + \alpha' - (\alpha' + \alpha - \ell) \sqrt{1 - \frac{2\alpha'(\alpha - \ell)(1 - \langle \hat{\eta}_x, \eta_{x^*} \rangle)}{(\alpha' + \alpha - \ell)^2}} \\ &\geq \ell + \frac{\alpha'(\alpha - \ell)(1 - \langle \hat{\eta}_x, \eta_{x^*} \rangle)}{\alpha + \alpha' - \ell} \geq \frac{\alpha'\alpha(1 - \langle \hat{\eta}_x, \eta_{x^*} \rangle)}{2(\alpha + \alpha')}, \end{aligned}$$

where in the first inequality of the last line we bounded $A\sqrt{1 - 2B/A^2} \leq A(1 - B/A^2) = A - B/A$, and in the last inequality $\alpha - \ell \geq \alpha/2$, thus, combined to Equation 17 we can conclude the proof of the first point.

2. The proof of point 2. follows the same ideas used to prove Theorem 3 in [1]. We are going to give the main steps of the proof (adapted to our case).

We prove first the surjectivity: for any $x^* \in \partial S$, let us introduce $O^* = x^* - \alpha\eta_{x^*}$ and $x = x^* - 2\varepsilon_n\eta_x$. We start proving that $x \in C_{\alpha'}(\mathcal{X}_n)$. We proceed by contradiction. If $x \notin C_{\alpha'}(\mathcal{X}_n)$ then there exist O with $\|O - x\| \leq \alpha'$ and $\mathring{B}(O, \alpha') \subset C_{\alpha'}(\mathcal{X}_n)$. Let $u = (O - x)/\|O - x\|$, $y^* = [O^*, O] \cap \partial\mathcal{B}(O^*, \alpha)$ and $y = [O^*, O] \cap \partial\mathcal{B}(O, \alpha')$, and therefore $\|y - y^*\| \leq \varepsilon_n$ which entails

$$\alpha + \alpha' - \|O - O^*\| \leq \varepsilon_n, \quad (18)$$

but now

$$\begin{aligned} \alpha + \alpha' - \|O - O^*\| &= \alpha + \alpha' - \sqrt{\|O - x\|^2 + (\alpha - 2\varepsilon_n)^2 + 2\|O - x\|(\alpha - 2\varepsilon_n)\langle u, \eta_{x^*} \rangle} \\ &= \alpha + \alpha' - \sqrt{(\|O - x\| + \alpha - 2\varepsilon_n)^2 - 2\|O - x\|(\alpha - 2\varepsilon_n)(1 - \langle u, \eta_{x^*} \rangle)} \\ &= \alpha + \alpha' - (\|O - x\| + \alpha - 2\varepsilon_n) \sqrt{1 - \frac{2\|O - x\|(\alpha - 2\varepsilon_n)(1 - \langle u, \eta_{x^*} \rangle)}{(\|O - x\| + \alpha - 2\varepsilon_n)^2}} \\ &\geq 2\varepsilon_n + \alpha' - \|O - x\| + \frac{\alpha'(\alpha - \varepsilon_n)(1 - \langle u, \eta_{x^*} \rangle)}{\alpha + \alpha' - \varepsilon_n} \geq 2\varepsilon_n, \end{aligned}$$

that contradicts Equation (18). Thus $x \in C_{\alpha'}(\mathcal{X}_n)$ and so, there exists $z \in (x, x^*)$ and $z \in \partial C_{\alpha'}(\mathcal{X}_n)$, such that $\pi_{\partial S}(z) = x^*$.

Now let us prove the injectivity. Let $x \in \partial S$ and consider $y = \arg \max\{\|x - z\| : z \in \partial C_{\alpha'}(\mathcal{X}_n), \pi_{\partial S}(z) = x\}$.

First, if there exists i such that $y \in \text{int}(F_i)$, by previous considerations there exist u a unique unitary vector such that $\mathring{B}(y + \alpha'u, \alpha') \subset (C_{\alpha'}(\mathcal{X}_n))^c$ with $\langle u, \eta_x \rangle > 1/2$, which guarantee that the segment (y, x) is included in $\mathring{B}(y + \alpha'u, \alpha')$ thus in $(C_{\alpha'}(\mathcal{X}_n))^c$. Consider now $y' \in \partial C_{\alpha'}(\mathcal{X}_n)$ such that $\pi_{\partial S}(y') = x$ then we must have $y' = x - t\eta_x$, and because $C_{\alpha'}(\mathcal{X}_n) \subset S$ we also must have $t \geq 0$ and then, due to definition of y , $t \leq \|y - x\|$ which together with $y' \in C_{\alpha'}(\mathcal{X}_n)$ implies $t = \|y - x\|$ thus $y' = y$.

Second, if y do not belong to any $\text{int}(F_i)$, it belongs to a ∂F_i and there exists a sequence $y_k \rightarrow y$ such that $y_k \in \text{int}(F_i)$. Let us proceed as

previously. Let $u_k = (O_i - y_k)/\alpha'$ be the unit normal, outward pointing, vector of F_i at y_k and $x_k = \pi_{\partial S}(y_k)$. Then $\dot{B}(y_k + \alpha' u_k, \alpha') \subset (C_{\alpha'}(\mathcal{X}_n))^c$ with $\langle u_k, \eta_{x_k} \rangle > 1/2$. Then if we take limit $k \rightarrow +\infty$, it follows that $\dot{B}(y + \alpha' u, \alpha') \subset (C_{\alpha'}(\mathcal{X}_n))^c$ with $u = (O_i - y)/\alpha'$. From Theorem 1 point (v) in [37] $\|\eta_{x_k} - \eta_x\| \leq \|x - x_k\|/\alpha$. From Theorem 4.8 point (8) in [23] and using that $d_H(\partial C_{\alpha'}(\mathcal{X}_n), \partial S) < \alpha/2$, it follows that $\|x - x_k\| \leq 2\|y - y_k\|$. Then, using that $\|u - u_k\| \leq (\|y - y_k\|)/\alpha'$, it follows that $\langle u, \eta_x \rangle > 1/2$, and we can conclude the proof as in the previous case. \square

Lemma 6.8. *Suppose that M is a C^2 , bounded $(d-1)$ -dimensional manifold with positive reach α , let π_M denotes the projection onto M and let \hat{M} be a C^2 , $(d-1)$ -dimensional manifold such that*

1. π_M is one to one from \hat{M} to M ,
2. for all $x \in \hat{M}$ we have $\|x - \pi_M(x)\| \leq \varepsilon_1$ and $\langle \hat{\eta}_x, \eta_{\pi_M(x)} \rangle \geq 1 - \varepsilon_2$,

then, if $\varepsilon_1(d-1)\alpha \leq 1$ and $\varepsilon_2 \leq 1/8$ we have

$$(1 - 3\varepsilon_1\alpha - 32\varepsilon_2)^{\frac{d-1}{2}} \leq \frac{|\hat{M}|_{d-1}}{|M|_{d-1}} \leq (1 + 3\varepsilon_1\alpha + 32\varepsilon_2)^{\frac{d-1}{2}}. \quad (19)$$

Proof. Let $p \in M$ and denote by (e_1, \dots, e_{d-1}) an orthonormal basis of $T_p M$ and complete it with e_d a unit vector of $N_p M$. A neighborhood of p in M can be parametrized as $\varphi(x) = x + f(x)e_d = \sum_{i=1}^{d-1} x_i e_i + f(x_1, \dots, x_{d-1})e_d$ where $x = \sum_{i=1}^{d-1} x_i e_i$ belongs to a neighborhood of p and with $\nabla_f(p) = 0$.

Consider now the surface element (of M) $ds(p) = dx_1 \dots dx_{d-1}$ it's image, by π_M^{-1} on the surface element (of \hat{M}) is given by $d\hat{s}(p) = \sqrt{\det(J_{\pi_M^{-1}}(p)' J_{\pi_M^{-1}}(p))} dx_1 \dots dx_{d-1}$.

The rest of the the proof consist in giving bounds for $\det(J_{\pi_M^{-1}}(p)' J_{\pi_M^{-1}}(p))$. We have that $\pi_M^{-1}(\varphi(x)) = x + \ell(x)n(x)$ where $n(x) = (-\partial f/\partial x_1, \dots, -\partial f/\partial x_{d-1}, 1) \in N_x M$ which gives that

$$J_{\pi_M^{-1}}(p) = \begin{pmatrix} I_{d-1} - \ell(p)\mathcal{H}_f(p) \\ \nabla_\ell(p) \end{pmatrix}$$

The reach condition gives that $\|\mathcal{H}_f(p)\|_{op} \leq \alpha$ and $\ell(p) = \|\pi_M^{-1}(p) - p\| \leq \varepsilon_1$ so that we just have to bound $\|\nabla_\ell(p)\|$. Notice that, for $j = 1, \dots, d-1$ we have

$$t_j = e_j + \frac{\partial \ell}{\partial x_j}(p)e_d - \ell(p) \left(\sum_{i=1}^{d-1} \frac{\partial^2 f}{\partial x_i \partial x_j} e_i \right) \in T_{\pi_M^{-1}(p)} \hat{M}.$$

Notice that $\eta_p = \pm e_d$ and introduce $\hat{\eta}_{\pi_M^{-1}(p)}$. Since $t_1, \dots, t_{d-1}, \hat{\eta}_{\pi_M^{-1}(p)}$ is an orthogonal basis of \mathbb{R}^d we have that

$$e_d = \sum_{i=1}^{d-1} \langle e_d, \frac{t_i}{\|t_i\|} \rangle \frac{t_i}{\|t_i\|} + \langle e_d, \hat{\eta}_{\pi_M^{-1}(p)} \rangle \hat{\eta}_{\pi_M^{-1}(p)},$$

which entails

$$1 = \sum_{i=1}^{d-1} \langle \eta_p, \frac{t_i}{\|t_i\|} \rangle^2 + \langle \eta_p, \hat{\eta}_{\pi_M^{-1}(p)} \rangle^2.$$

Thus, by condition 2. we have $|\langle t_j, e_d \rangle| = |\langle t_j, \eta_p \rangle| \leq \sqrt{2\varepsilon_2} \|t_j\|$ which implies

$$\left| \frac{\partial \ell}{\partial x_j}(p) \right| \leq \sqrt{2\varepsilon_2} \|t_j\| \leq \sqrt{2\varepsilon_2} \left(1 + \left| \frac{\partial \ell}{\partial x_j}(p) \right| + \varepsilon_1(d-1)\alpha \right).$$

From which we get

$$\left| \frac{\partial \ell}{\partial x_j}(p) \right| \leq \frac{\sqrt{2\varepsilon_2}(1 + \varepsilon_1(d-1)\alpha)}{1 - \sqrt{2\varepsilon_2}}.$$

So, $J_{\pi_M^{-1}}(p)' J_{\pi_M^{-1}}(p) = I_{d-1} + E$ with E a symmetric matrix with

$$\|E\|_{op} \leq 2\varepsilon_1\alpha + \varepsilon_1^2\alpha^2 + \left(\frac{\sqrt{2\varepsilon_2}(1 + \varepsilon_1(d-1)\alpha)}{1 - \sqrt{2\varepsilon_2}} \right)^2,$$

thus we finally obtain the inequality

$$(1 - 3\varepsilon_1\alpha - 32\varepsilon_2)^{d-1} \leq \det \left(J_{\pi_M^{-1}}(p)' J_{\pi_M^{-1}}(p) \right) \leq (1 + 3\varepsilon_1\alpha + 32\varepsilon_2)^{d-1},$$

that concludes the proof. \square

6.2.1 Proof of Corollary 4

We only need to check that the conditions of Theorem 4.3 are fulfilled, with probability one for n large enough. In [33] it is proven that, with probability one for n large enough, $d_H(\partial C_{\alpha'}(\mathcal{X}_n), \partial S) \leq \varepsilon_n \leq c(\ln n/n)^{2/(d+1)}$ for some given explicit constant c . Since $C_{\alpha'}(\mathcal{X}_n)^c$ is a finite union of balls and affine half spaces, that is $C_{\alpha'}(\mathcal{X}_n)^c = \bigcup_{i=1}^{N_1} E_i$ with $E_i = \mathring{B}(O_i, r_i)$ or $E_i = \{z \in \mathbb{R}^d, \langle u_j, z \rangle > a_i\}$. Then

$$\partial C_{\alpha'}(\mathcal{X}_n) = \bigcup_i \left(\partial E_i \cap \left(\bigcup_{j \neq i} E_j \right)^c \right)$$

Let us define the F_j as the connected components of the sets $\partial E_i \cap \left(\bigcup_{j \neq i} E_j \right)^c$. Then the sets F_j are closed manifolds of dimension $d_j \leq (d-1)$, and are compact since $F_j \subset C_{\alpha'}(\mathcal{X})$, which is compact. Lastly, because $\partial C_{\alpha'}(\mathcal{X})$ is a $(d-1)$ -dimensional manifold we must have $\partial C_{\alpha'}(\mathcal{X}) = \bigcup_{j, d_j=d-1} F_j$ (that is, the lower dimensional F_k are included in $\bigcup_{j, d_j=d-1} F_j$). That conclude the proof of Corollary 4.

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