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**Surface and length estimation based on Crofton´s formula.**

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**Abstract**

We study the problem of estimating the surface area of the boundary of a sufficiently smooth set when the available information is only a set of points (random or not) that becomes dense (with respect to Hausdorff distance) in the set or the trajectory of a reflected diffusion. We obtain consistency results in this general setup, and we derive rates of convergence for the iid case or when the data corresponds to the trajectory of a reflected Brownian motion. We propose an algorithm based on Crofton's formula, which estimates the number of intersections of random lines with the boundary of the set by counting, in a suitable way (given by the proposed algorithm), the number of intersections with the boundary of two different estimators: the Devroye–Wise estimator and the $\alpha$-convex hull of the data. As a by-product, our results also cover the convex case, for any dimension.

# 1 Introduction

Surface area estimation has been extensively considered in stereology (see Baddeley, Gundersen and Cruz-Orive (1986); Baddeley and Vedel Jensen (2005), Gokhale (1990)). It has also been studied as a further step in the theory of nonparametric set estimation (see Pateiro-López and Rodríguez-Casal (2008)), and has practical applications in medical imaging (see Cuevas, Fraiman, and Rodríguez-Casal (2007)).

The aim of this work is to propose estimators (and to derive their convergence rates) of the surface area (i.e., the $(d-1)$-dimensional Hausdorff measure) of the boundary $\partial S$ of an unknown set $S \subset \mathbb{R}^d$. They are based on a finite subset of points $\mathcal{X}_n = \{X_1, \dots X_n\} \subset S$, or on the trajectory of a reflected diffusion on $S$, that becomes dense in the set with respect to the Hausdorff distance, and makes use of two well known support estimators together with Crofton's formula.

This well-known formula, proved by Crofton in 1868 for dimension two, and extended to arbitrary dimensions (see Santaló (2004)), states that the surface of $\partial S$ equals the integral of the number of intersections with $\partial S$ of lines in $\mathbb{R}^d$ (see Equations (4) and (5) for explicit Crofton formulas for $d = 2$ and $d \geq 2$, respectively).

We propose to 'estimate' the number of intersections with $\partial S$ of lines, by using two different support estimators. First we consider the Devroye–Wise estimator (see Devroye and Wise (1980)), and next the $\alpha$-convex hull estimator (see Rodríguez-Casal (2007)). By using the Devroye–Wise estimator, our Crofton-based surface estimator attains a rate proportional to $d_H(\mathcal{X}_n, S)^{1/2}$ (where $d_H$ denotes the Hausdorff distance). The estimator is not just a plug-in, because in general the number of intersections of a line with $\partial S$ is

different from the number of intersections of that line with the Devroye–Wise estimator. This result can be applied to many deterministic or random situations, to obtain explicit convergence rates. We focus on two random situations: the case of iid drawn on $S$ (with a density bounded from below), and the case of random trajectories of reflected diffusions on $S$. In particular, we provide convergence rates when the trajectories are the result of reflected Brownian motion (see Cholaquidis et al. (2016, 2020)). This last setting has several applications in ecology, such as home-range estimation, where the trajectory is obtained by recording the location of an animal (or several animals) living in an area $S$ that is called the home range (see for instance Cholaquidis et al. (2016, 2020), Baíllo and Chacón (2018) and references therein).

To use Crofton's formula when the support estimator is the $\alpha$-convex hull of a sample $\mathcal{X}_n$ (denoted by $C_\alpha(\mathcal{X}_n)$), we first prove that the surface area of the hull boundary, i.e. $|C_\alpha(\mathcal{X}_n)|_{d-1}$, converges to $|\partial S|_{d-1}$. This result is interesting in itself, but in practice to compute $|C_\alpha(\mathcal{X}_n)|_{d-1}$ is much more difficult, especially for dimension $d > 2$. However, by means of the Crofton formula, this can be done easily. The convergence of $|C_\alpha(\mathcal{X}_n)|_1$ to $|\partial S|_1$ was proved in Cuevas, Fraiman and Pateiro-López (2012), and we prove it for any dimension in Theorem 2.

When the available data is only a sample in the set, the two dimensional length estimation problem (under convexity assumptions) has been previously addressed in Bräker and Hsing (1998) using Crofton's formula.

With a different sample scheme, where there are two distinguishable sets of random data-points (one from inside $S \subset \mathbb{R}^d$ for $d \geq 2$, and the other one from $S^c$), the problem of the estimation of the surface area of the boundary has been considered in Cuevas, Fraiman, and Rodríguez-Casal (2007), Pateiro-López and Rodríguez-Casal (2008), Jiménez and Yukich (2011) and Cuevas, Fraiman and Györfi (2013).
The three- and two-dimensional cases are addressed in Berrendero et al. (2014), where the authors propose parametric estimators when the available data are the distances to $S$, from a sample outside the set, but at a distance smaller than a given $R > 0$.

Neither the non-iid case nor the non-convex case, when only a sample inside the set is available and $d > 2$, have been addressed in the literature. A result for this general setup has also been considered for $d = 2$ in Theorem 4 in Cholaquidis et al. (2016). To fill this gap, we propose to use consistent estimators that are based on Crofton´s formula. Explicit rates of convergence of the estimators are given for the iid case and for reflected Brownian motion, with and without drift.

The rest of this paper is organized as follows. In Section 2, we introduce the notation and also some well-known geometric restrictions. In Section 3, we introduce Crofton's formula, first for dimension two and then for the general case. After that, we introduce the main geometric restriction required in one of the main theorems. Section 4 introduces the algorithms from a mathematical standpoint, and explains the heuristics behind them. The computational aspects of the algorithms are given in Section 5 and the main results

are stated in Section 6, their proofs are given in the Appendix.

## 2 Some preliminaries

*The following notation will be used throughout the paper.*

Given a set $S \subset \mathbb{R}^d$, we will denote by $\mathring{S}$, $\overline{S}$ and $\partial S$ the interior, closure and boundary of $S$, respectively, with respect to the usual topology of $\mathbb{R}^d$. We will also write $\mathrm{diam}(S) = \sup_{(x,y)\in S\times S} ||x-y||$. The parallel set of $S$ of radius $\varepsilon$ will be denoted by $B(S,\varepsilon)$, that is, $B(S,\varepsilon) = \{y \in \mathbb{R}^d : \inf_{x\in S} \|y-x\| \leq \varepsilon\}$. If $A \subset \mathbb{R}^d$ is a Borel set, then $\mu_d(A)$ (sometimes just $\mu(A)$) or $|A|_d$) will denote its $d$-dimensional Lebesgue measure. We will denote by $\mathcal{B}(x,\varepsilon)$ the closed ball in $\mathbb{R}^d$, of radius $\varepsilon$, centred at $x$, and $\omega_d = \mu_d(\mathcal{B}_d(x,1))$. Given two compact non-empty sets $A, B \subset \mathbb{R}^d$, the *Hausdorff distance* or *Hausdorff–Pompei distance* between $A$ and $C$ is defined by

$$d_H(A,C) = \inf\{\varepsilon > 0 : \text{such that } A \subset B(C,\varepsilon) \text{ and } C \subset B(A,\varepsilon)\}. \tag{1}$$

The $(d-1)$-dimensional sphere in $\mathbb{R}^d$ is denoted by $\mathcal{S}^{d-1}$, while the half-sphere in $\mathbb{R}^d$ is denoted by $(\mathcal{S}^+)^{d-1}$, i.e, $(\mathcal{S}^+)^{d-1} = (\mathbb{R}^{d-1} \times \mathbb{R}^+) \cap \mathcal{S}^{d-1}$. Given $M$ a sufficiently smooth manifold and $x \in M$, we denote by $\eta_x$ the unit outward normal vector at $x$. The affine tangent space of $M$ at $x$ is denoted by $T_xM$.

Given a vector $\theta \in (\mathcal{S}^+)^{d-1}$ and a point $y$, $r_{\theta,y}$ denotes the line $\{y + \lambda\theta, \lambda \in \mathbb{R}\}$. If $y_1$ and $y_2$ are two points in $r_{\theta,y}$, then $y_i = y + \lambda_i\theta$; with a slight abuse of notation, we write $y_1 < y_2$ when $\lambda_1 < \lambda_2$.

We will now recall some well-known shape restrictions in set estimation.

**Definition 1.** *A set $S \subset \mathbb{R}^d$ is said to be $\alpha$-convex, for $\alpha > 0$, if $S = C_\alpha(S)$, where*

$$C_\alpha(S) = \bigcap_{\left\{\mathring{\mathcal{B}}(x,\alpha):\ \mathring{\mathcal{B}}(x,\alpha)\cap S=\emptyset\right\}} \left(\mathring{\mathcal{B}}(x,\alpha)\right)^c, \tag{2}$$

*is the $\alpha$-convex hull of $S$. When $S$ is $\alpha$-convex, a natural estimator of $S$ from a random sample $\mathcal{X}_n$ of points (drawn from a distribution with support $S$), is $C_\alpha(\mathcal{X}_n)$.*

**Definition 2.** *A set $S \subset \mathbb{R}^d$ is said to be standard with respect to a Borel measure $\nu$ at a point $x$ if there are $\lambda > 0$ and $\delta > 0$ such that*

$$\nu(\mathcal{B}(x,\varepsilon) \cap S) \geq \delta\mu_d(\mathcal{B}(x,\varepsilon)), \quad 0 < \varepsilon \leq \lambda. \tag{3}$$

*A set $S \subset \mathbb{R}^d$ is said to be standard if (3) holds for all $x \in S$.*

**Definition 3.** *A set $S \subset \mathbb{R}^d$ is said to satisfy the outside $\alpha$-rolling condition if for each boundary point $s \in \partial S$ there exists an $x \in S^c$ such that $\mathcal{B}(x,\alpha) \cap \partial S = \{s\}$. A compact set $S$ is said to satisfy the inside $\alpha$-rolling condition if $\overline{S^c}$ satisfies the outside $\alpha$-rolling condition at all boundary points.*

Proposition 1 in Aaron, Cholaquidis and Cuevas (2017) proves that a Borel set that satisfies the inside rolling condition is standard w.r.t. any measure $\nu$ with density $f$ (w.r.t. $\mu_d$) bounded from below by some $f_0 > 0$.

## 3 Crofton's formula

Crofton in 1868 proved the following result (see Crofton (1868)): given $\gamma$ a regular plane curve (i.e. there exists a differentiable parametrization $c : [0,1] \to \gamma \subset \mathbb{R}^2$ such that $||c'(t)|| > 0$ for all $t$), then its length $I_1(\gamma)$ can be computed by

$$I_1(\gamma) = \frac{1}{2} \int_{\theta=0}^{\pi} \int_{p=-\infty}^{+\infty} n_\gamma(\theta, p) dp d\theta, \tag{4}$$

$n_\gamma(\theta, p)$ being the number of intersections of $\gamma$ with the line $r_{\theta^*,\theta p}$, where $\theta^* \in (\mathcal{S}^+)^{d-1}$ is orthogonal to $\theta$, and $dpd\theta$ is 2-dimensional Lebesgue measure, see Figure 1. This result has been generalized to $\mathbb{R}^d$ for any $d > 2$, and also to Lie groups, see Santaló (2004).

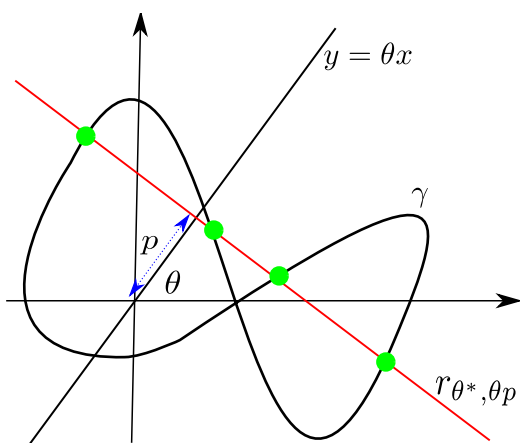


Figure 1: The function $n_\gamma$ counts the number of intersections of $\gamma$ with the line $r_{\theta^*,\theta p}$ determined by $\theta$ and $p$ with the curve.

To introduce the general Crofton formula in $\mathbb{R}^d$, which measures the 'volume' of an hyper-surface $M$ (i.e., a $(d-1)$-dimensional manifold), let us define first the constant

$$\beta(d) = \Gamma(d/2)\Gamma((d+1)/2)^{-1}\pi^{-1/2},$$

where $\Gamma$ stands for the well known Gamma function. Let $B \subset \mathbb{R}^d$ be a Borel set and $\theta \in (\mathcal{S}^+)^{d-1}$, $\theta$ determine a $d-1$ dimensional linear space $\theta^\perp = \{v : \langle v, \theta \rangle = 0\}$. Given $y \in \theta^\perp$, let us write $n_M(\theta, y) = \#(r_{\theta,y} \cap M)$, see Figure 2.

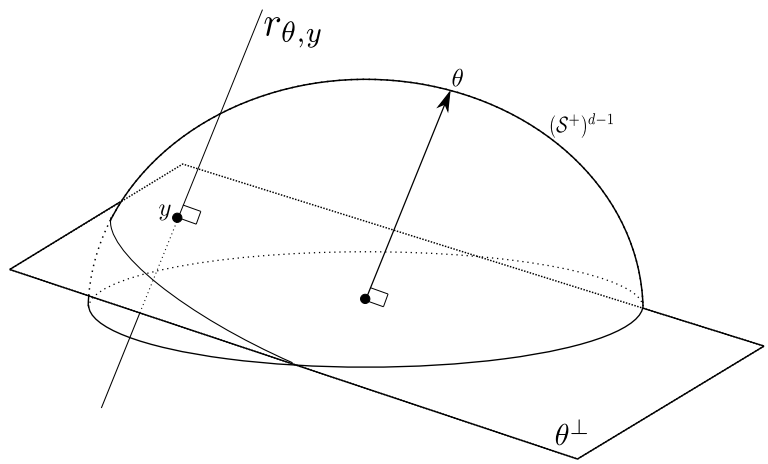


Figure 2: The line $r_{\theta,y} = y + \lambda\theta$ is shown, where $y \in \theta^\perp$ and $\theta \in (\mathcal{S}^+)^{d-1}$.

Let $I_{d-1}(\partial B)$ be the $d-1$ integralgeometric measure of $\partial B$, i.e.,

$$I_{d-1}(M) = \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \theta^\perp} n_M(\theta, y) d\mu_{d-1}(y) d\theta. \tag{5}$$

The measure $d\theta$ is the uniform measure on $(\mathcal{S}^+)^{d-1}$ (with total mass 1).

It is proved in Federer (1969) (see Theorem 3.2.26) that if $M$ is an $(d-1)$-dimensional rectifiable set, then the integralgeometric measure of $M$ equals its $(d-1)$-dimensional Hausdorff measure: $I_{d-1}(M) = H^{d-1}(M)$.

**Remark 1.** *Throughout this paper we will assume that $\partial S$ is the boundary of a compact set $S \subset \mathbb{R}^d$ such that $S = \overline{int(S)}$. We will also assume that $S$ fulfills the outside and inside $\alpha$-rolling condition and then $\partial S$ is rectifiable (see Theorem 1 in Walther, G. (1999)). From this it follows that $I_{d-1}(\partial S) = H^{d-1}(\partial S) < \infty$, which implies (by (5)) that, except for a set of measure zero with respect to $d\mu_{d-1}(y) \times d\theta$, any line $r_{\theta,y}$ meets $\partial S$ a finite number of times: $n_{\partial S}(\theta, y) < \infty$. From Theorem 1 in Walther, G. (1999), it also follows that $\partial S$ is a $\mathcal{C}^1$ manifold, which allows us to consider for all $x \in \partial S$, $\eta_x$, the unit outward normal vector.*

For a given $\theta$ we will separate the integral with respect to $d\mu_{d-1}(y)$ in (5), as a sum of two integrals. In the first one, we will consider the lines (defined by $y \in \theta^\perp$) that are *far* (properly defined later as condition $L(\varepsilon)$ in Definition 4) from all of the tangent spaces to $\partial S$, while in the second integral we will consider those lines that are *close* to some tangent space. To control the measure of these last lines, we need to introduce the following shape restriction.

**Definition 4.** *Let us define* $E_\theta(\partial S) = \{x \in \partial S, \langle \eta_x, \theta\rangle = 0\}$ *and* $F_\theta$ *its normal projection onto* $\theta^\perp$*. Let us define, for* $\varepsilon > 0$*,*

$$\varphi_\theta(\varepsilon) = \left|\theta^\perp \cap B(F_\theta, \varepsilon)\right|_{d-1}.$$

*We will say that* $\partial S$ *is* $(C, \varepsilon_0)$*-regular if for all* $\theta$ *and all* $\varepsilon \in (0, \varepsilon_0)$ $\varphi'_\theta(\varepsilon)$ *exists and* $\varphi'_\theta(\varepsilon) \leq C$*.*

When we use the Devroye–Wise estimator we will assume the $(C, \varepsilon_0)$-regular boundary condition. Once the rolling balls condition is imposed, we will show through some examples that the $(C, \varepsilon_0)$-regularity of the boundary is not a too restrictive hypothesis.

For instance, a polyhedron with 'rounded corners', such as in Figure 3, satisfies the $(C, \varepsilon_0)$-regularity of the boundary. Under regularity and geometric conditions on $\partial S$, the $(C, \varepsilon_0)$-regularity is related to the conjecture proposed in Alesker (2018).

To find sets that satisfy the inside and outside $\alpha$ rolling ball properties but without a $(C, \varepsilon_0)$-regular boundary, the only case that we were able to construct is a set with some $E_\theta$ having infinitely many connected components, such as the one shown in Figure 6, whose boundary is locally around some boundary point, which is the hypograph of the function $x^5 \sin(1/x)$.

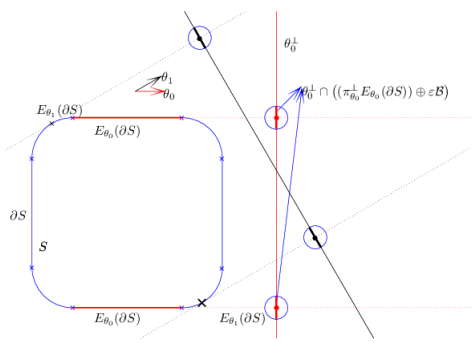


Figure 3: $(a)$ smooth square

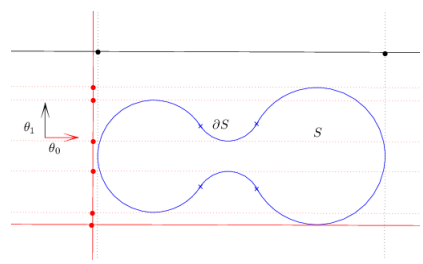


Figure 4: $(b)$ $2D$ peanut

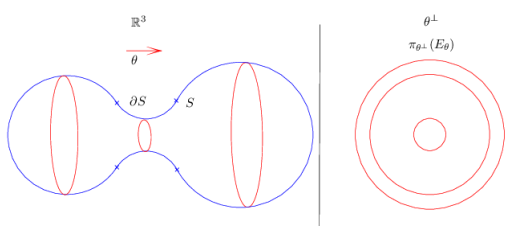


Figure 5: $(c)$ $3D$ peanut

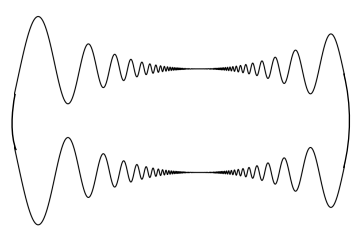

Figure 6: $(d)$ an 'infinite wave' shape

(a) The first set, presented in Figure 3, is a square with 'round angles', it has a 2-regular boundary.

(b) The second set, presented in Figure 4, is a 2-dimensional 'peanut' made of 4 arcs of circle. It has a 6-Regular boundary.

(c) The third set, presented in Figure 5, is the surface of revolution generated by (b). The number of connected components of $E_\theta$ is bounded by 3 and the maximal length of a component is bounded by $L$, the length of the maximal perimeter (shown in blue in the figure). Thus, it is $C$-regular with $C \leq 3L$

(d) The rolling ball condition is not sufficient to guarantee the $(C, \varepsilon_0)$ regularity of the boundary: this happens if, for instance, we replace in the smooth square shown in (a) a flat piece of the boundary by the graph of the function $x^5 \sin(1/x)$. To illustrate this behaviour, Figure 6 shows a set such that the number of connected components of $E_\theta$ (with a horizontal $\theta$) is infinite.

# 4 The mathematical aspects of the algorithms

## 4.1 Devroye–Wise based approach

To estimate $n_{\partial S}(\theta, y)$, note that when $r_{\theta,y}$ is not included in a $(d-1)$-dimensional affine tangent space (tangent to $\partial S$), then $n_{\partial S}(\theta, y) = 2k_S(\theta, y)$ where $k_S(\theta, y)$ is the number of connected components of $r_{\theta,y} \cap S$.

Given that in general the set $S$ is unknown, the natural idea is to plug into $n_{\partial S}$ an estimator of $S$. There are different kinds of set estimators, depending on the geometric restrictions imposed on $S$ and the structure of the data (see Devroye and Wise (1980), Cholaquidis et al. (2016)) and references therein). One of the most studied in the literature, which is also universally consistent, is the Devroye–Wise estimator (see Devroye and Wise (1980)), given by

$$\hat{S}_n(\varepsilon_n) = \bigcup_{i=1}^{n} \mathcal{B}(X_i, \varepsilon_n),$$

where $\varepsilon_n \to 0$ is a sequence of positive real numbers. This all-purposes estimator has the advantage that it is quite easy to compute the intersection of a line with its boundary (i.e. the points in the line at a distance of exactly $\varepsilon_n$ from the sample). Unfortunately, plugging in this estimator into $n_{\partial S}$ does not provide consistency (i.e. $n_{\partial \hat{S}_n}(\varepsilon_n)$ does not converges in general to $n_{\partial S}$). It needs a small adjustment, as we will explain in the following definition.

**Definition 5.** *Consider a line $r_{\theta,y}$. If $\hat{S}_n(\varepsilon_n) \cap r_{\theta,y} = \emptyset$, define $\hat{n}_{\varepsilon_n}(\theta, y) = 0$, otherwise:*

- *Denote by $I_1, \ldots, I_m$ the connected components of $\hat{S}_n(\varepsilon_n) \cap r_{\theta,y}$. Order this sequence in such a way that $I_i = (a_i, b_i)$, with $a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m$.*
- *If for some consecutive intervals $I_i, I_{i+1}, \ldots, I_{i+l-1}$, for all $a_i < t < b_{i+l}$ and $t \in r_{\theta,y}$, $d(t, \mathcal{X}_n) < 4\varepsilon_n$, define $A_i = (a_i, b_{i+l-1})$.*

- *Let $j$ be the number of disjoint open intervals $A_1, \ldots, A_j$ that this process ended with. Then define $\hat{n}_{\varepsilon_n}(\theta, y) = 2j$.*

Our first proposed estimator is

$$\hat{I}_{d-1}(\partial S) = \frac{1}{\beta(d)} \int_{\theta \in (\mathbb{S}^+)^{d-1}} \int_{y \in \theta^\perp} \hat{n}_{\varepsilon_n}(\theta, y) d\mu_{d-1}(y) d\theta. \tag{6}$$

### 4.2 $\alpha$-convex hull based approach

The $\alpha$-convex hull of a finite set of points $\mathcal{X}_n$ (defined by (2) with $S = \mathcal{X}_n$), which is also a consistent estimator of $S$ under some regularity conditions (see for instance Rodríguez-Casal (2007)), has the advantage that the $(d-1)$-dimensional Lebesgue measure of its boundary converges to the $(d-1)$-dimensional Lebesgue measure of $\partial S$ (see Theorem 2 below). This, together with the fact that $\partial C_\alpha(\mathcal{X}_n)$ is a rectifiable set (see the comment before Remark 1), suggests using Crofton's formula to estimate $|\partial C_\alpha(\mathcal{X}_n)|_{d-1}$. Then our second proposal is

$$\begin{aligned} \check{n}_\alpha(\theta, y) &= n_{\partial C_\alpha(\mathcal{X}_n)}(\theta, y) \\ \check{I}_{d-1}(\partial S) &= \frac{1}{\beta(d)} \int_{\theta \in (\mathbb{S}^+)^{d-1}} \int_{y \in \theta^\perp} \check{n}_\alpha(\theta, y) d\mu_{d-1}(y) d\theta. \end{aligned}$$

In this case, the computation of the intersection of a line with $\partial C_\alpha(\mathcal{X}_n)$ is not as direct as in the Devroye–Wise estimator. However, only weaker regularity restrictions on $\partial S$ will be required (see Theorem 2) to get the consistency of $\check{I}_{d-1}(\partial S)$.

## 5 Computational aspects of the algorithms

The algorithms to compute $\check{n}_\alpha(\theta, y)$ and $\hat{n}_{\varepsilon_n}(\theta, y)$ work for any finite set $\mathcal{X}_n = \{X_1, \ldots, X_n\} \subset S$ (not necessarily random). In the first case we assume $\alpha$ known, and in the second one, $\varepsilon_n > 0$ is a fixed real value. We will also provide some insight on how to choose $\varepsilon_n$ when $\mathcal{X}_n$ is a random sample from a random vector supported on $S$, under some regularity restrictions on $\partial S$. Regarding the parameter $\alpha$, in practice this value is usually unknown, however, as is proven in Theorem 1, any value smaller than the real one can be used in the algorithm.

The general case for stochastic processes indexed by $T \in \mathbb{R}^+$ is obtained by replacing the set $\mathcal{X}_n$ in the algorithm by a discretization of a trajectory of the process observed in $[0, T]$.

The values $\hat{I}_{d-1}(\partial S)$ and $\check{I}_{d-1}(\partial S)$ are approximated by means of the Monte Carlo method, choosing lines $r_{\theta_1,y_1},\ldots,r_{\theta_k,y_k}$ at random (w.r.t. $d\mu_{d-1}(y)\times d\theta$). Then, the estimators are given by

$$\hat{\hat{I}}_{d-1}(\partial S)=\frac{1}{\beta(d)}\frac{1}{k}\sum_{i=1}^{k}\hat{n}_{\varepsilon_n}(\theta_i,y_i)\quad\text{and}\quad\check{\check{I}}_{d-1}(\partial S)=\frac{1}{\beta(d)}\frac{1}{k}\sum_{i=1}^{k}\check{n}_{r}(\theta_i,y_i).$$

### 5.1 Devroye–Wise based approach

To compute $\hat{n}_{\varepsilon_n}(\theta,y)$ for a given $r_{\theta,y}$ we proceed as follows.

1. Identify the centres $\mathcal{Y}_{n'}=\{Y_1,\ldots,Y_{n'}\}$ of the boundary balls of $\hat{S}_n(\varepsilon_n)$ (see Aaron, Cholaquidis and Cuevas (2017)), i.e., the points $X_i\in\mathcal{X}_n$ such that

$$\max\{||x-X_i||:x\in\mathrm{Vor}(X_i)\}\geq\varepsilon_n.$$

2. Compute $d_i=d(r_{\theta,y},Y_i)$.

3. Compute the connected components $I_i$, of $r_{\theta,y}\cap\hat{S}_n(\mathcal{X}_n)$, according to the following steps: Initialize the list of the extremes of these intervals by list= $\emptyset$, and then,

   For $i=1$ to $n'$:

   - If $d_i\leq\varepsilon_n$ then compute $\{z_1,z_2\}=\mathcal{B}(Y_i,\varepsilon_n)\cap r_{\theta,y}$.
     - For $j=1$ to 2: if $d(z_j,\mathcal{X}_n)\geq\varepsilon_n$ do list=list$\cup\{z_j\}$.

   The $a_i$ and $b_i$ (and so the $I_i$) introduced in Definition 5 are obtained by a sorting procedure applied to the points $z_j$.

4 Obtain the $a_i'$ and $b_i'$ such that $I_i'=(a_i',b_i')$ are the connected components of $\hat{S}(4\varepsilon_n)\cap r_{\theta,y}$ by using the same procedure.

5. Lastly, compute $\hat{n}(\theta,y)$, as follows:

   initialization $\hat{n}=m$. For $i=1$ to $m-1$

   - If there exists $k$ such that $(b_i,a_{i+1})\subset I_k'$ then: $\hat{n}=\hat{n}-1$

### 5.2 $\alpha$-convex hull based approach

It is much more involved to compute $\check{n}_\alpha(\theta, y)$: it requires the computation of the $\alpha$-convex hull, as well as the convex hull, of the set $\mathcal{X}_n$. Recall that the convex hull of a sample is equal to the intersection of a finite number of half-spaces. In Edelsbrunner et al. (1983) it is proved, for dimension 2, but mentioned that the generalization is not difficult, that $C_\alpha(\mathcal{X}_n)^c$ is the union of a finite number of balls and the aforementioned half-spaces. The centres $O_i$ of these balls, and their radii $r_i$, are obtained by computing the Delaunay complex of the points. Let us write $C_\alpha(\mathcal{X}_n)^c = \bigcup_i E_i$, where $E_i$ is either a half-space or a ball. Observe that if the line $r_{\theta,y}$ is chosen at random (w.r.t. $d\mu_{d-1} \times d\theta$), $r_{\theta,y} \cap E_i$ contains fewer than 2 points.

Initialize list=$\emptyset$. Then:
for all $i$,

- compute $r_{\theta,y} \cap \partial E_i$
- For all $z \in r_{\theta,y} \cap \partial E_i$
    1. If for all $j$ $z \notin \mathring{E}_j$ do list=list$\cup\{z\}$

then $\check{n} = \#$list.

## 6 Main results

In this section we will state our main results. All proofs are given in the Appendix.

**Theorem 1.** *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside $\alpha$-rolling conditions. Assume also that $S$ is $(C, \varepsilon_0)$-regular for some positive constants $C$ and $\varepsilon_0$. Let $\mathcal{X}_n = \{X_1, \ldots, X_n\} \subset S$. Let $\varepsilon_n \to 0$ such that $d_H(\mathcal{X}_n, S) \leq \varepsilon_n$. Then*

$$\hat{I}_{d-1}(\partial S) = \frac{1}{\beta(d)} \int_{\theta \in (\mathbb{S}^+)^{d-1}} \int_{y \in \theta^\perp} \hat{n}_{\varepsilon_n}(\theta, y) d\mu_{d-1}(y) d\theta + \mathcal{O}(\sqrt{\varepsilon_n}).$$

*Moreover, for $n$ large enough,*

$$|\mathcal{O}(\sqrt{\varepsilon_n})| \leq \frac{5C\mathrm{diam}(S)}{6\sqrt{\alpha}} \sqrt{\varepsilon_n},$$

*$C$ being the constant of the $(C, \varepsilon_0)$-regularity of $S$.*

**Remark 2.** *Theorem 4 in Cuevas and Rodriguez-Casal (2004) gives some insight into how to choose the parameter $\varepsilon_n$ for the the case in which $\{X_1, \ldots, X_n\}$ is an iid sample of a random vector $X$ supported on $S$. It states that if $\varepsilon_n = C'(\log(n)/n)^{1/d}$, where $C'$ is a*

*large enough positive constant, then with probability one, for $n$ large enough, $S \subset \hat{S}_n$. In addition, $d_H(\partial S, \partial \hat{S}_n(\varepsilon_n)) \to 0$, $d_H(S, \hat{S}_n(\varepsilon_n)) \to 0$. Although this does not imply that $H^{d-1}(\partial \hat{S}_n)$ converges to $H^{d-1}(S)$, Theorem 1 states that we can consistently estimate the integralgeometric measure of $\partial S$ by means of* (5).

From Remark 2 and the previous theorem, we can obtain the rate of convergence for the iid case:

**Corollary 1.** *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the inside and outside $\alpha$-rolling conditions. Assume also that $S$ is $(C, \varepsilon_0)$-regular for some positive constants $C$ and $\varepsilon_0$. Let $X_1, \ldots, X_n$ be an iid sample of $X$ with distribution $P_X$ supported on $S$. Assume that $P_X$ has density $f$ (w.r.t. $\mu_d$) bounded from below by some $c > 0$. Let $\varepsilon_n = C'(\ln(n)/n)^{1/d}$ and $C' > (6/(c\omega_d))^{1/d}$. Then with probability one, for $n$ large enough,*

$$\hat{I}_{d-1}(\partial S) = \frac{1}{\beta(d)} \int_{\theta \in (\mathbb{S}^+)^{d-1}} \int_{y \in \theta^\perp} \hat{n}_{\varepsilon_n}(\theta, y) d\mu_{d-1}(y) d\theta + \mathcal{O}\Big( \Big( \frac{\ln(n)}{n} \Big)^{\frac{1}{2d}} \Big).$$

In a more general setting, the conclusion of Theorem 1 holds when the set of points $\mathcal{X}_n$ is replaced by the trajectory of any stochastic process $\{X_t\}_{t>0}$ included in $S$, observed in $[0, T]$, such that $d_H(\mathcal{X}_T, S) \to 0$ as $T \to \infty$. This is the case (for example) for some reflected diffusions and particularly for reflected Brownian motion (RBM). This has been recently proven in corollary 1 in Cholaquidis et al. (2016) for RBM without drift (see also Cholaquidis et al. (2020) for RBM with drift). RBM with drift is defined as follows: let $D$ be a bounded domain in $\mathbb{R}^d$ (i.e., a bounded, connected open set), such that $\partial D$ is $\mathcal{C}^2$. Given a $d$-dimensional Brownian motion $\{B_t\}_{t \geq 0}$, departing from $B_0 = 0$ and defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}_x)$, the RBM with drift is the (unique) solution to the following stochastic differential equation on $\overline{D}$:

$$X_t = X_0 + B_t + \int_0^t g(X_s) ds - \int_0^t \eta_{X_s} \xi(ds), \qquad \text{where } X_t \in \overline{D},\ \forall t \geq 0,$$

where the drift, $g(x)$, is assumed to be Lipschitz, and $\{\xi_t\}_{t \geq 0}$ is the corresponding *local time*: i.e., a one-dimensional continuous nondecreasing process with $\xi_0 = 0$ that satisfies $\xi_t = \int_0^t \mathbb{I}_{\{X_s \in \partial D\}} d\xi_s$.

From Corollary 1 together with Proposition 3 of Cholaquidis et al. (2016), we have the following result for RBM without drift:

**Corollary 2.** *Let $S \subset \mathbb{R}^d$ be a non-empty compact set with connected interior such that $S = \overline{int(S)}$, and suppose that $S$ fulfills the outside and inside $\alpha$-rolling conditions. Assume*

*also that $S$ is $(C, \varepsilon_0)$-regular for some positive constants $C$ and $\varepsilon_0$. Let $\{B_t\}_{t>0} \subset S$ be an RBM (without drift). Then, with probability one, for $T$ large enough,*

$$\hat{I}_{d-1}(\partial S) = \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \theta^\perp} \hat{n}_{\varepsilon_n}(\theta, y) d\mu_{d-1}(y) d\theta + o\Big(\Big(\frac{\ln(T)^2}{T}\Big)^{\frac{1}{2d}}\Big).$$

The following theorem states that the surface measure of the boundary of the $\alpha$-convex hull of an iid sample converges to the surface of the boundary of the set. Observe that in this case, with no need for the additional hypothesis of $(C, \varepsilon_0)$-regularity, the convergence rate is far better than the one obtained with the Devroye–Wise estimator.

**Theorem 2.** *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the inside and outside $\alpha$-rolling conditions. Assume also that $\partial S$ is of class $\mathcal{C}^3$ Let $X_1, \ldots, X_n$ be an iid sample of $X$ with distribution $P_X$ supported on $S$. Assume that $P_X$ has density $f$ (w.r.t. $\mu_d$) bounded from below by some $c > 0$. Suppose $\alpha' \leq \alpha$. Then with probability one, for $n$ large enough,*

1. $||\partial S|_{d-1} - |\partial C_{\alpha'}(\mathcal{X}_n)|_{d-1}| = \mathcal{O}((\ln(n)/n)^{1/(d+1)}$,
2. *as a consequence*

$$\check{I}_{d-1}(\partial S) = \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \theta^\perp} \check{n}_{\alpha'}(\theta, y) d\mu_{d-1}(y) d\theta + \mathcal{O}\Big(\Big(\frac{\ln(n)}{n}\Big)^{\frac{1}{d+1}}\Big).$$

# 7 Appendix

## 7.1 Proof of Theorem 1

**Sketch of the proof of Theorem 1**

The idea is to consider separately the set of lines that intersect $\partial S_n(\varepsilon_n)$:

1. If a line $r_{\theta,y} = y + \lambda\theta$ is 'far enough' (fulfilling condition $L(\varepsilon)$ for some $\varepsilon > 0$) from the tangent spaces, then our algorithm allows a perfect estimation of $n(y, \theta)$, see Lemma 4.
2. Considering the set of lines that are not 'far enough' from the tangent spaces (denoted by $\mathcal{A}_\varepsilon(\theta)$, see Definition 6), Corollary 3 states that the integral of $\hat{n}$ on $\mathcal{A}_\varepsilon(\theta)$ is bounded from above by $C'\varepsilon_n^{1/2}$, with $C'$ a positive constant.

### 7.1.1 Condition $L(\varepsilon)$

Now we define the two sets of lines to be treated separately: The lines that are 'far' from an affine tangent space, and the lines that are 'close to being tangent' to $\partial S$. More precisely,

assume that $\partial S$ is smooth enough so that for all $x \in \partial S$, the unit outer normal vector $\eta_x$ at $x$ is well defined. Now we define

$$\mathcal{T}_S = \{x + (\eta_x)^\perp : \ x \in \partial S\},$$

the collection of all the affine $(d-1)$-dimensional tangent spaces.

**Definition 6.** *Let $\varepsilon \geq 0$. A line $r_{\theta,y} = y + \lambda\theta$ fulfills* ***condition*** *$L(\varepsilon)$ if $y$ is at a distance larger than $4\varepsilon$ from all the affine hyper-planes $w + \eta^\perp \in \mathcal{T}_S$ with $\langle \eta, \theta \rangle = 0$.*

*For a given $\theta$, we define*

$$\mathcal{A}_\varepsilon(\theta) = \Big\{ y \in \theta^\perp : ||y|| \leq diam(S) \text{ and } r_{\theta,y} \text{ does not satisfy } L(\varepsilon) \Big\}.$$

### 7.1.2 Some useful lemmas

**Lemma 1.** *Let $S$ be a compact set fulfilling the outside and inside $\alpha$-rolling conditions. Let $r_{\theta,y}$ be a line that fulfills condition $L(0)$ and $r_{\theta,y} \cap \partial S \neq \emptyset$. Then $r_{\theta,y}$ intersects $\partial S$ in a finite number of points.*

*Proof.* Because $S$ fulfills the outside and inside $\alpha$-rolling conditions, Theorem 1 in Walther, G. (1999) implies that for any $x \in \partial S$, the affine $(d-1)$-dimensional tangent space $T_x\partial S$ exists. If $r_{\theta,y}$ fulfills $L(0)$, then $r_{\theta,y}$ is not included in any hyper-plane tangent to $S$. Suppose that $\partial S \cap r_{\theta,y}$ is not finite. Then, by compactness, one can extract a subsequence $t'_n$ that converges to $y' \in \partial S$. Note that for all $(n,p) \in \mathbb{N}^2$ $(t'_n - t'_{n+p})/||t'_n - t'_{n+p}|| = \pm\theta$, which implies that $(t'_n - y')/||t'_n - y'|| = \pm\theta$. Lastly, if $n \to \infty$, then $\theta \in T_{y'}\partial S$. Considering $y'$, we have $y' \in \partial S$, $\theta \in T_{y'}\partial S$ and $y' \in r_{\theta,y}$, which contradicts the assumption that $r_{\theta,y}$ is not included in any hyper-plane tangent to $S$. □

**Lemma 2.** *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside $\alpha$-rolling conditions. Let $\varepsilon > 0$ such that $\varepsilon < 4\alpha$ and $\nu = 2\sqrt{2\varepsilon(\alpha - 2\varepsilon)}$. For any line $r_{\theta,y}$ fulfilling condition $L(\varepsilon)$ and $r_{\theta,y} \cap \partial S \neq \emptyset$, we have that $r_{\theta,y}$ meets $\partial S$ at a finite number of points $t_1, \ldots, t_k$, where $t_{i+1} - t_i > 2\nu$ for all $i = 1, \ldots, k-1$. Consequently, if $\varepsilon < \alpha/4$, then $k \leq \mathrm{diam}(S)\varepsilon^{-1/2}/(4\sqrt{\alpha})$.*

*Proof.* Note that if a line fulfills condition $L(\varepsilon)$, then it fulfills condition $L(0)$. Consequently, the fact that $r_{\theta,y}$ intersects $\partial S$ in a finite number of points follows from Lemma 1. Let us denote by $t_1 < \cdots < t_k$ the intersection of $r_{\theta,y}$ with $\partial S$. Proceeding by contradiction, assume that for some $i$, $t_{i+1} - t_i < 2\nu$. Let us denote by $\eta_{t_i}$ and $\eta_{t_{i+1}}$ the outer normal vectors at $t_i$ and $t_{i+1}$, respectively. We have two cases: the open interval $(t_i, t_{i+1}) \subset S^c$ or $(t_i, t_{i+1}) \subset int(S)$. Let us consider the first case (the proof for the second one is similar).

Because $(t_i, t_{i+1}) \subset \overline{S^c}$ and $S$ fulfills the inside $\alpha$-rolling condition on $t_i$, there exists $z \in S$ such that $t_i \in \partial\mathcal{B}(z,\alpha)$ and $\mathcal{B}(z,\alpha) \subset S$. In particular, $\mathcal{B}(z,\alpha) \cap (t_i, t_{i+1}) = \emptyset$, which implies $\langle \eta_{t_i}, \theta \rangle \geq 0$.

Reasoning in the same way but with $t_{i+1}$, we get $\langle \eta_{t_{i+1}} \theta \rangle \leq 0$. Given that $r_{\theta,y}$ is not included in any tangent hyperplane, we have that $\langle \eta_{t_i}, \theta \rangle > 0$ and $\langle \eta_{t_{i+1}}, \theta \rangle < 0$. Because $S$ fulfills the inside and outside $\alpha$-rolling conditions, $\partial S$ is a $(d-1)$-dimensional $\mathcal{C}^1$ manifold whose normal vector is Lipschitz (see Theorem 1 in Walther, G. (1999)). By Theorem 3.8 in Colesanti and Manselli (2010), there exists a curve $\gamma : [0,1] \to \partial S$ such that $\gamma(0) = t_i$, $\gamma(1) = t_{i+1}$ and $d(\gamma(t), r_{\theta,y}) < 4\varepsilon$ for all $t$. From $\langle \eta_{t_i}, \theta \rangle > 0$ and $\langle \eta_{t_{i+1}}, \theta \rangle < 0$, it follows that there exists an $s_0 \in (0,1)$ such that $\langle \eta_{\gamma(s_0)}, \theta \rangle = 0$, which contradicts the hypothesis that $y$ is at a distance larger than $4\varepsilon$ from all the $(d-1)$-dimensional hyperplanes tangent to $S$. This proves that $t_{i+1} - t_i > 2\nu$ for all $i = 1, \ldots, k-1$. □

**Lemma 3.** *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside $\alpha$-rolling ball conditions and with a $(C, \varepsilon_0)$-regular boundary. Then for all $\varepsilon \leq \varepsilon_0$,*

$$\int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_\varepsilon(\theta)} n(\theta, y) d\mu_{d-1}(y) d\theta \leq C \frac{\operatorname{diam}(S)}{2\sqrt{\alpha}} \sqrt{\varepsilon}.$$

*Proof.* From the proof of the previous lemma, it follows that for any $y \in E_\theta$ with $d(y, F_\theta) = l$, $n(\theta, y) \leq \operatorname{diam}(S) l^{-1/2}/(4\sqrt{\alpha})$. Hence,

$$\begin{aligned}
&\int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_\varepsilon(\theta)} n(\theta, y) d\mu_{d-1}(y) d\theta \\
&\quad = \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{l=0}^{\varepsilon} \int_{\{y \in \theta^\perp : d(y, F_\theta) = l\}} n(\theta, y) d\mu_{d-2}(y) dl d\theta \\
&\quad \leq \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{l=0}^{\varepsilon} \int_{\{y \in \theta^\perp : d(y, F_\theta) = l\}} \frac{1}{4} \operatorname{diam}(S) (\alpha l)^{-1/2} d\mu_{d-2}(y) dl d\theta \\
&\quad \leq \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{l=0}^{\varepsilon} \frac{1}{4} \operatorname{diam}(S) (\alpha l)^{-1/2} \int_{\{y \in \theta^\perp d(y, F_\theta) = l\}} d\mu_{d-2}(y) dl d\theta \\
&\quad \leq \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{l=0}^{\varepsilon} \frac{1}{4} \operatorname{diam}(S) (\alpha l)^{-1/2} \mu_{d-2}\Big(\big\{y \in \theta^\perp : d(y, F_\theta) = l\big\}\Big) dl d\theta.
\end{aligned}$$

By the definition of $\varphi_\theta$,

$$\Big|\big\{y \in \theta^\perp : l \leq d(y, F_\theta) \leq l + dl\big\}\Big|_{d-1} = \varphi_\theta(l + dl) - \varphi_\theta(l).$$

From the $(C, \varepsilon_0)$-regularity of $\partial S$ and the mean value theorem we obtain

$$\mu_{d-2}\Big(\big\{y \in \theta^\perp : d(y, F_\theta) = l\big\}\Big) \leq \sup_{\varepsilon \in (0, \varepsilon_0)} \varphi'_\theta(\varepsilon) \leq C,$$

which implies

$$\int_{\theta\in(\mathbb{S}^+)^{d-1}}\int_{y\in\mathcal{A}_\varepsilon(\theta)} n(\theta,y)d\mu_{d-1}(y)d\theta \leq \int_{\theta\in(\mathbb{S}^+)^{d-1}}\int_{l=0}^{\varepsilon} C\frac{1}{4}\mathrm{diam}(S)(\alpha l)^{-1/2}dld\theta \leq C\frac{\mathrm{diam}(S)}{2\sqrt{\alpha}}\sqrt{\varepsilon}.$$

□

**Lemma 4.** *Let $S$ be a compact set fulfilling the outside and inside $\alpha$-rolling conditions. Let $\mathcal{X}_n = \{X_1,\ldots,X_n\}\subset S$. Let $\varepsilon_n\to 0$ be such that $d_H(\mathcal{X}_n,S)\leq\varepsilon_n$. Let $r_{\theta,y}=y+\lambda\theta$ be any line fulfilling condition $L(\varepsilon_n)$. Then, for $n$ large enough, $n(y,\theta)=\hat{n}(y,\theta)$.*

*Proof.* Note that the choice of $\varepsilon_n$ ensures that $S\subset\hat{S}_n(\varepsilon_n)$, thus

$$r_{\theta,y}\cap S\subset\ r_{\theta,y}\cap\hat{S}_n(\varepsilon_n). \tag{7}$$

First, we will prove that

$$\hat{n}(\theta,y)\geq n(\theta,y). \tag{8}$$

Because $\hat{n}$ is not the number of connected components of $r_{\theta,y}\cap\hat{S}_n(\varepsilon_n)$, (8) does not follow directly from (7). If $r_{\theta,y}\cap\partial S=\emptyset$ inequality (8) holds. Assume $r_{\theta,y}\cap\partial S\neq\emptyset$. Let $t_1<\ldots<t_k$ be the intersection of $r_{\theta,y}$ with $\partial S$ (this set is finite due to Lemma 1). Let us prove that

$$\text{if } (t_i,t_{i+1})\subset S^c\text{, then: } \exists s\in(t_i,t_{i+1}) \text{ such that } d(s,S)>4\varepsilon_n. \tag{9}$$

Because $S$ fulfills the inside $\alpha$-rolling condition on $t_i$, there exists a $z_i\in S$ such that $t_i\in\partial\mathcal{B}(z,\alpha)$ and $\mathcal{B}(z,\alpha)\subset S$. Since $\mathcal{B}(z,\alpha)\cap(t_i,t_{i+1})=\emptyset$, it follows that $\langle\eta_{t_i},\theta\rangle\geq 0$ (recall that $\eta_{t_i}=(t_i-z_i)/\alpha$ and $t_{i+1}-t_i=||t_{i+1}-t_i||\theta$). Reasoning in the same way but with $t_{i+1}$, $\langle\eta_{t_{i+1}},\theta\rangle\leq 0$. By condition $L(\varepsilon_n)$ we obtain

$$\langle\eta_{t_i},\theta\rangle>0 \text{ and } \langle\eta_{t_{i+1}},\theta\rangle<0. \tag{10}$$

Suppose that for all $t\in(t_i,t_{i+1})$ we have $d(t,\partial S)\leq 4\varepsilon_n$. Take $n$ large enough such that $4\varepsilon_n<\alpha$. Because $\partial S$ fulfills the outside and inside $\alpha$-rolling conditions, by Lemma 2.3 in Pateiro-López and Rodríguez-Casal (2009) it has positive reach. Then, by Theorem 4.8 in Federer (1956), $\gamma=\{\gamma(t)=\pi_{\partial S}(t), t\in(t_i,t_i+1)\}$, the orthogonal projection onto $\partial S$ of the interval $(t_i,t_{i+1})$ is well defined and is a continuous curve in $\partial S$. By Theorem 1 in Walther, G. (1999), the map from $\partial S$ to $\mathbb{R}^d$ that sends $\eta_x\in\partial\mathcal{B}(0,1)$ to $x\in\partial S$ is Lipschitz. Thus, $t\to\langle\eta_{\gamma(t)},\theta\rangle$ is a continuous function of $t$ for all $t\in(t_i,t_{i+1})$, which, together with (10), ensures the existence of an $s\in(t_i,t_{i+1})$ such that $d(s,\gamma(s))\leq 4\varepsilon_n$ and

$\theta \in \eta^{\perp}_{\gamma(s)}$, which contradicts the assumption that $r_{\theta,y}$ fulfills condition $L(\varepsilon_n)$. This proves (9), which implies that

$$\text{if } (t_i, t_{i+1}) \subset S^c, \text{ then: } \exists s \in (t_i, t_{i+1}) \text{ such that } d(s, \mathcal{X}_n) > 4\varepsilon_n$$

and now (8) follows from (7).

Next we will prove the opposite inequality,

$$\hat{n}(\theta, y) \leq n(\theta, y). \tag{11}$$

Assume first $r_{\theta,y} \cap \partial S \neq \emptyset$. Let $\{t_1, \ldots, t_k\}$ be the intersection of $r_{\theta,y}$ with $\partial S$ (this set is finite due to Lemma 1).

Consider $t^* \in (t_i, t_{i+1}) \subset S^c$ and $t^* \in \hat{S}_n(\varepsilon_n)$. Equation (11) will be derived from the fact that $(t^*, t_{i+1}] \subset \hat{S}_n(\varepsilon_n) \cap r_{\theta,y}$ or $[t_i, t^*) \subset \hat{S}_n(\varepsilon_n) \cap r_{\theta,y}$.

Introduce $\psi(t) : (t_i, t_{i+1}) \to \mathbb{R}$ defined by $\psi(t) = d(t, \partial S)$. Consider points $t$ such that $d(t, \partial S) < \alpha$, and let $p_t \in \partial S$ such that $||p_t - t|| = d(t, \partial S)$. By item (3) in Theorem 4.8 in Federer (1956), $\psi'(t) = \langle \eta_{p_t}, \theta \rangle$.

Let $X_j$ be the closest observation to $t^*$ (recall that because $t^* \in \hat{S}_n(\varepsilon_n)$, we have $||X_j - t^*|| \leq \varepsilon_n$). Now, because there exists a point $p^* \in [t, X_j] \cap \partial S$, we obtain that $\psi(t^*) \leq \varepsilon_n$ and, because $r_{\theta,y}$ fulfills $L(\varepsilon_n)$, $\langle \eta_{p_{t^*}}, \theta \rangle \neq 0$.

Assume that, for instance, $\langle \eta_{p_{t^*}}, \theta \rangle < 0$. Then $\psi(t^*) \leq \varepsilon_n$ and $\psi'(t^*) < 0$. Suppose that there exists a $t' \in (t^*, t_{i+1})$ such that $\psi(t') \geq \varepsilon_n$ and consider $t'' = \inf\{t > t^*, \psi(t') \geq \varepsilon_n\}$. Then for all $t \in (t^*, t'')$ we have $\psi(t) \leq \varepsilon_n$, and thus $\psi$ is differentiable on this interval. From the fact that $\psi(t'') \geq \psi(t^*)$ and $\psi'(t^*) < 0$ we deduce that there exists a $\tilde{t} \in (t^*, t'')$ such that $\psi'(\tilde{t}) = 0$, which contradicts $L(\varepsilon_n)$ because $\psi(\tilde{t}) \leq \varepsilon_n$. To summarize, we have shown that if $\langle \eta_{p_{t^*}}, \theta \rangle < 0$, then $(t^*, t_{i+1}) \subset \hat{S}_n(\varepsilon_n)$. Symmetrically, if $\langle \eta_{p_{t^*}}, \theta \rangle > 0$, then $(t_i, t^*) \subset \hat{S}_n(\varepsilon_n)$, which concludes the proof.
Reasoning in the same way, if $r_{\theta,y} \cap \partial S = \emptyset$ and $\hat{n}(\theta, y) > 0$, a contradiction with condition $L(\epsilon_n)$ is obtained. □

**Lemma 5.** *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside $\alpha$-rolling conditions. Let $\varepsilon_n \to 0$ be a sequence such that $d_H(\mathcal{X}_n, S) \leq \varepsilon_n$, while $r_{\theta,y} = y + \lambda\theta$ and $A_1, \ldots, A_k$ are the sets in Definition 5. Put $A_i = (a_i, b_i)$ for $i = 1, \ldots, k$, and suppose that the sets are indexed in such a way that $a_1 < b_1 < a_2 < \ldots < b_k$. Then for all $i = 2, \ldots, k$, we have that $||a_i - b_{i-1}|| > 3\sqrt{\varepsilon_n \alpha}$ for $n$ large enough such that $3\sqrt{\alpha\varepsilon_n} < \alpha/2$ and $\varepsilon_n < \alpha/2$, which implies*

$$\hat{n}(\theta, y) \leq \frac{\operatorname{diam}(S)}{3\sqrt{\alpha}} \varepsilon_n^{-1/2}.$$

*Proof.* Assume by contradiction that for some $i$, $||a_i - b_{i-1}|| \leq 3\sqrt{\varepsilon_n \alpha}$. By construction, $[b_{i-1}, a_i] \subset \hat{S}_n(\varepsilon_n)^c \subset S^c$. Because $a_i$ and $b_i$ are on $\partial \hat{S}_n(\varepsilon_n)$, we have $d(a_i, \mathcal{X}_n) = d(b_{i-1}, \mathcal{X}_n) = \varepsilon_n$.

The projection $\pi_S : [b_{i-1}, a_i] \to \partial S$ is uniquely defined because $\partial S$ has reached at least $\alpha$ and $d(t, \partial S) \leq d(t, a_i) + d(a_i, \partial S) \leq ||a_i - b_{i-1}|| + d(a_i, \mathcal{X}_n)$ for all $t \in (b_{i-1}, a_i)$, $||a_i - b_{i-1}|| \leq 3\sqrt{\varepsilon_n \alpha} < \alpha/2$ and $d(a_i, \partial S) \leq \varepsilon_n \leq \alpha/2$. Moreover, $\pi$ is a continuous function. Hence $\max_{x \in [b_{i-1}, a_i]} ||x - \pi_S(x)|| \geq \varepsilon_n$, and the maximum is attained at some $x_0 \in [b_{i-1}, a_i]$. We will prove that $||x_0 - \pi_S(x_0)|| \geq 3\varepsilon_n$, which guarantees that $x_0 \in (b_{i-1}, a_i)$ and that $\eta_0$, the outward unit normal vector to $\partial S$ at $\pi_S(x_0)$, is normal to $\theta$. Indeed, suppose by contradiction that for all $t \in (b_{i-1}, a_i)$, $d(t, \partial S) \leq 3\varepsilon_n$. Then $d(t, \mathcal{X}_n) \leq 4\varepsilon_n$, which contradicts the definition of the points $a_i$ and $b_i$. Put $z_0 = \pi_S(x_0) + \eta_0 \alpha$. Observe that $d(a_i, S) \leq \varepsilon_n$ and $d(b_{i-1}, S) \leq \varepsilon_n$. From the outside $\alpha$-rolling condition at $\pi_S(x_0)$, and using the fact that $\eta_0$ is normal to $\theta$, we have (see Figure 7)

$$\overset{r}{\theta,y} \cap \mathcal{B}(z_0, \alpha - \varepsilon_n) \subset [b_{i-1}, a_i],$$

which implies, see Figure 7, that $||a_i - b_{i-1}|| \geq 2\sqrt{(\alpha - \varepsilon_n)^2 - (\alpha - l)^2}$, where $l = d(x_0, \pi_S(x_0))$. Therefore,

$$||a_i - b_{i-1}|| \geq 2\sqrt{(l - \varepsilon_n)(2\alpha - l - \varepsilon_n)}. \tag{12}$$

If we bound $l \geq 3\varepsilon_n$ and use the fact that $l = o(1)$, which follows from $l \leq ||b_{i-1} - a_i|| + \varepsilon_n \leq 3\sqrt{\varepsilon_n \alpha} + \varepsilon_n$, then we get, from (12),

$$||a_i - b_{i-1}|| \geq 2\sqrt{2\varepsilon_n(2\alpha - l - \varepsilon_n)} = 2\sqrt{4\varepsilon_n \alpha(1 + o(1))} = 4\sqrt{\alpha \varepsilon_n}(1 + o(1)),$$

and for $n$ large enough this contradicts $||a_i - b_{i-1}|| \leq 3\sqrt{\alpha \varepsilon_n}$.

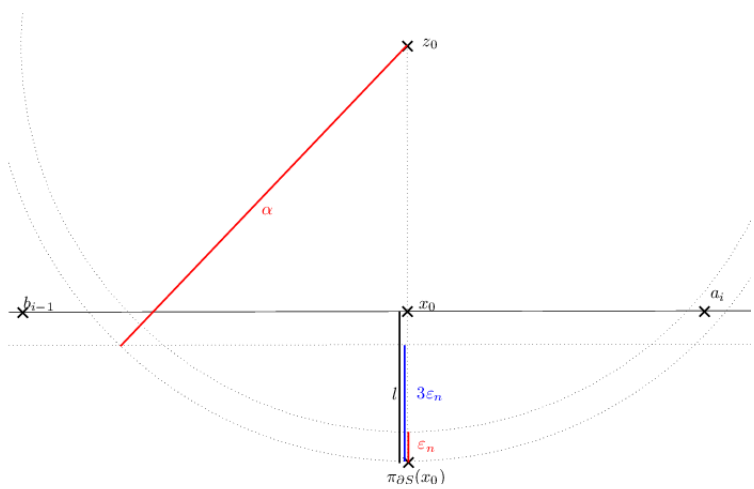


Figure 7

Lastly, the number of disjoint intervals $A_i$ is bounded from above by $\text{diam}(S)/(3\sqrt{\varepsilon_n \alpha})$. Thus, $\hat{n}(\theta, y) \leq \text{diam}(S)/(3\sqrt{\varepsilon_n \alpha})$. $\square$

**Corollary 3.** *Let $S \subset \mathbb{R}^d$ be a compact set fulfilling the outside and inside $\alpha$-rolling conditions and with a $(C, \varepsilon_0)$-regular boundary. For $n$ large enough such that $3\sqrt{\alpha\varepsilon_n} < \min(\alpha/2, \varepsilon_0)$, we have*

$$\int_\theta \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} \hat{n}(\theta, y) d\mu_{d-1}(y) d\theta \leq C \frac{\mathrm{diam}(S)}{3\sqrt{\alpha}} \sqrt{\varepsilon_n}.$$

### 7.1.3 Proof of Theorem 1

Without loss of generality, we can assume that $0 \in S$. Recall that for $\theta \in (\mathcal{S}^+)^{d-1}$, $\mathcal{A}_{\varepsilon_n}(\theta)$ is the set of all $y \in \theta^\perp$ such that $||y|| \leq \mathrm{diam}(S)$ and $r_{\theta,y}$ does not fulfill $L(\varepsilon_n)$. First, from Lemma 4, we have

$$|I_{d-1}(\partial S) - \hat{I}_{d-1}(\partial S)| \leq \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} |\hat{n}_{\varepsilon_n}(\theta, y) - n(\theta, y)| d\mu_{d-1}(y) d\theta.$$

So, by the triangle inequality we can bound the difference between the integralgeometric and its estimation by

$$\frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} \hat{n}_{\varepsilon_n}(\theta, y) d\mu_{d-1}(y) d\theta + \frac{1}{\beta(d)} \int_{\theta \in (\mathcal{S}^+)^{d-1}} \int_{y \in \mathcal{A}_{\varepsilon_n}(\theta)} n(\theta, y) d\mu_{d-1}(y) d\theta.$$

Now, by applying Corollaries 3 and Lemma 3, we get that

$$|I_{d-1}(\partial S) - \hat{I}_{d-1}(\partial S)| \leq \frac{5C\mathrm{diam}(S)}{6\beta(d)\sqrt{\alpha}} \sqrt{\varepsilon_n},$$

for $n$ large enough.

## 7.2 Proof of Theorem 2

Theorem 2 will be obtained from the two following lemmas. The first one states that eventually almost surely, the boundary of the $\alpha'$-convex hull of an iid sample drawn on a $\alpha$-convex support has some good geometrical properties.

The second one, which is purely geometric, bounds the difference between the measures of two sets, the first one having a positive reach $\alpha$ (as $\partial S$) and the second one having the same good geometrical properties as the boundary of $C_\alpha(\mathcal{X}_n)$.

We will introduce some notation. Let $A$ and $B$ be two sub-spaces of $\mathbb{R}^d$. We denote by $\angle(A, B)$ the operator norm of the difference between the orthogonal projection onto $A$, $\pi_A$, and the projection onto $B$, $\pi_B$, i.e., $\angle(A, B) = ||\pi_A - \pi_B||_{op}$. If $f$ is a function, then $\nabla f$ is its gradient and $\mathcal{H}_f$ is its Hessian matrix. Given a point $x$ in a $d-1$ dimensional manifold $E$, $N_x E = \{v \in \mathbb{R}^d : \langle v, u \rangle = 0, \forall v \in T_x E\}$ is the 1-dimensional orthogonal subspace. If $A = (a_{i,j})_{i,j}$ is a matrix, $||A||_\infty = \max_{i,j} |a_{i,j}|$.

**Lemma 6.** *Let $\{X_1, \ldots, X_n\}$ be an iid sample drawn according to a distribution $\mathbb{P}_X$ supported by $S$ a support satisfying the $\alpha$ inner and outer rolling ball properties. Also suppose that the density of the distribution is bounded below by a positive constant $f_0$. For each $\alpha' \leq \alpha$, there exists an $a = a(\alpha, \alpha')$ and a $c = c(\alpha, \alpha')$ such that with probability one, for $n$ large enough,*

1. $\partial C_{\alpha'}(\mathcal{X}_n) \cap \partial S = \emptyset$
2. $\partial C_{\alpha'}(\mathcal{X}_n) = \bigcup_{i=1}^m F_i$, *where $F_i$ is a compact $(d-1)$-dimensional $\mathcal{C}^2$ manifold, for all $i = 1, \ldots, m$.*
3. $d_H(\partial C_{\alpha'}(\mathcal{X}_n), S) \leq \varepsilon_n^2 < reach(E)$, *with* $\varepsilon_n = a(\ln(n)/n)^{1/(d+1)}$.
4. $\pi_{\partial S} : \partial C_{\alpha'}(\mathcal{X}_n) \to \partial S$ *the orthogonal projection onto $\partial S$ is one to one.*
5. *For all $i = 1, \ldots, m$ and all $x \in F_i$,* $\angle(N_x F_i, N_{\pi_{\partial S}(x)} \partial S) \leq c\varepsilon_n$.

*Proof.* 1. Note that $\partial S \cap \partial C_{\alpha'}(\mathcal{X}_n) \neq \emptyset$ implies that $\mathcal{X}_n \cap \partial S \neq \emptyset$, which is an event with null probability, and so

$$\mathbb{P}(\partial S \cap \partial C_{\alpha'}(\mathcal{X}_n) \neq \emptyset) = 0,$$

which proves that condition 1 is fulfilled.

2. Observe that $\partial C_{\alpha'}(\mathcal{X}_n)$ is a finite union of subsets of hyper-spheres of radius $\alpha'$ (this is proven in Edelsbrunner et al. (1983) for dimension 2, and the generalization to any dimension is easy). This proves condition 2.

3. Recall that in Rodríguez-Casal (2007) it is proven that for any $\alpha' \leq \alpha$ there exists an $a$ such that, with probability one for $n$ large enough,

$$d_H(\partial C_{\alpha'}(\mathcal{X}_n), \partial S) \leq a^2(\ln(n)/n)^{2/(d+1)}. \tag{13}$$

Hence, $d_H(\partial C_{\alpha'}(\mathcal{X}_n), \partial S) < reach(S) = \alpha$, with probability one for $n$ large enough. This proves conclusion 3.

4. To prove conclusions 4 and 5 let $x \in \partial C_{\alpha'}(\mathcal{X}_n)$, and put $x^* = \pi_{\partial S}(x)$, with $\hat{\eta}_x$ the outward unit normal vector of $\partial C_{\alpha'}(\mathcal{X}_n)$ at $x$ and $\eta_{x^*}$ the outward unit normal vector of $\partial S$ at $x^*$. We are going to prove that if Equation (13) holds and $a^2 \left(\ln(n)/n\right)^{\frac{2}{d+1}} \leq \alpha/2$, then

$$1 - \langle \hat{\eta}_x, \eta_{x^*} \rangle \leq \frac{2(\alpha + \alpha')}{\alpha\alpha'} a^2 \left(\frac{\ln(n)}{n}\right)^{\frac{2}{d+1}}. \tag{14}$$

Put $O = x + \alpha'\hat{\eta}_x$ and $O^* = x^* - \alpha\eta_{x^*}$ (see Figure 8), we will prove that

$$\mathcal{B}(O, \alpha') \subset C_{\alpha'}(\mathcal{X}_n)^c \text{ and } \mathcal{B}(O^*, \alpha) \subset S. \tag{15}$$

To prove the first inclusion, observe that $\partial C_r(\mathcal{X}_n)$ is a union of a finite number of subsets of $\partial B(O_i, \alpha')$ for some centres $O_i$, such that $B(O_i, \alpha') \subset C_{\alpha'}(\mathcal{X}_n)^c$. Now, if $x \in \partial\mathcal{B}(O, \alpha')$ (with $O$ one of these centres), it follows that $(O - x)/\alpha'$ is the outward unit normal vector of $\partial C_{\alpha'}(\mathcal{X}_n)$ at $x$, which concludes the proof. The second inclusion is a direct consequence the inner rolling ball condition.

Write $y^* = [O^*, O] \cap \partial\mathcal{B}(O^*, \alpha)$ and $y = [O^*, O] \cap \partial\mathcal{B}(O, \alpha')$. Then, from the second inclusion in (15), we get $y \in S$, and from the first inclusion in (15) we get $d(y, C_{\alpha'}(\mathcal{X}_n)) \geq ||y - y^*||$. This fact, combined with ((13)), implies that $||y - y^*|| \leq a^2(\ln(n)/n)^{2/(d+1)}$, which in turn implies

$$\alpha + \alpha' - ||O - O^*|| \leq a^2\Big(\frac{\ln(n)}{n}\Big)^{\frac{2}{d+1}}. \tag{16}$$

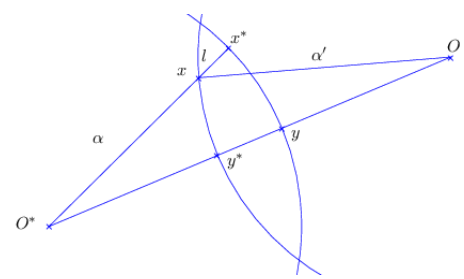


Figure 8

From $x^* = \pi_{\partial S}(x)$ we get that $x^* = x + l\eta_{x^*}$ where $l = ||x - x^*|| \leq a^2(\ln(n)/n)^{2/(d+1)}$. Then $O = O^* + (\alpha - l)\eta_{x^*} + \alpha'\hat{\eta}_x$ and

$$\begin{aligned}
\alpha + \alpha' - ||O - O^*|| &= \alpha + \alpha' - \sqrt{(\alpha')^2 + (\alpha - l)^2 + 2\alpha'(\alpha - l)\langle\hat{\eta}_x, \eta_{x^*}\rangle} \\
&= \alpha + \alpha' - \sqrt{(\alpha' + \alpha - l)^2 - 2\alpha'(\alpha - l)(1 - \langle\hat{\eta}_x, \eta_{x^*}\rangle)} \\
&= \alpha + \alpha' - (\alpha' + \alpha - l)\sqrt{1 - \frac{2\alpha'(\alpha - l)(1 - \langle\hat{\eta}_x, \eta_{x^*}\rangle)}{(\alpha' + \alpha - l)^2}} \\
&\geq l + \frac{\alpha'(\alpha - l)(1 - \langle\hat{\eta}_x, \eta_{x^*}\rangle)}{\alpha + \alpha' - l} \geq \frac{\alpha'\alpha(1 - \langle\hat{\eta}_x, \eta_{x^*}\rangle)}{2(\alpha + \alpha')}.
\end{aligned}$$

where in the first inequality of the last line we bounded $\sqrt{1 - 2B/A^2} \leq A(1 - B/A^2) = A - B/A$, and in the last inequality $\alpha - l \geq \alpha/2$.

This combined with ( 16) proves (14).

Next we show that from (14) it follows that the hypotheses 4) and 5) in Lemma 7 are fulfilled (with probability one for $n$ large enough). The proof of the bijectivity of $\pi_{\partial S}$ restricted to $\partial C_{\alpha'}(\mathcal{X}_n)$ follows the same ideas as those used to prove Theorem 3 in Aaron and Bodart (2016). The surjectivity follows from (13) and the rolling ball conditions, while the injectivity is a consequence of $\langle \hat{\eta}_x, \eta_{x^*} \rangle > 0$. To prove this last assertion, observe that if injectivity is not true, there exists a $y \in \partial S$ such that the half-line $\{y + t\eta_y, t \geq 0\}$ intersects $\partial C_{\alpha'}(\mathcal{X}_n)$ a first time pointing inside $C_{\alpha'}(\mathcal{X}_n)$ and then a second time 'pointing outside $C_{\alpha'}(\mathcal{X}_n)$' and at this second point we have $\langle \hat{\eta}_x, \eta_{x^*} \rangle \leq 0$.

Finally, Equation ( 14) implies that

$$\cos(\measuredangle(\hat{\eta}_x, \eta_{x^*})) \geq 1 - \frac{2(\alpha + \alpha')}{\alpha\alpha'} a^2 \left(\frac{\ln(n)}{n}\right)^{\frac{2}{d+1}},$$

and so

$$\measuredangle \hat{\eta}_x, \eta_{x^*} = \angle\Big(N_x \partial C_{\alpha'}(\mathcal{X}_n), N_{\pi_{\partial S}(x)} \partial S\Big) \leq 2a\sqrt{\frac{\alpha + \alpha'}{\alpha\alpha'}} \left(\frac{\ln(n)}{n}\right)^{\frac{1}{d+1}}.$$

□

**Lemma 7.** *Let $E \subset \mathbb{R}^d$ be a compact $(d-1)$-dimensional $\mathcal{C}^3$ manifold with positive reach $\alpha$. Let $\hat{E} \subset \mathbb{R}^d$ be a set such that*

1. $\hat{E} \cap E = \emptyset$
2. $\hat{E} = \bigcup_{i=1}^m F_i$, *where $F_i$ is a compact $(d-1)$-dimensional $\mathcal{C}^2$ manifold, for all $i = 1, \ldots, m$.*
3. $d_H(\hat{E}, E) \leq \varepsilon^2 < reach(E)$.
4. $\pi_E : \hat{E} \to E$ *the orthogonal projection onto $E$ is one to one.*
5. *For all $i = 1, \ldots, m$ and all $x \in F_i$, $\angle(N_x F_i, N_{\pi_E(x)} E) \leq c\varepsilon$.*

*Then,*

$$\Big(1 - 2(d-1)c\varepsilon - \mathcal{O}(\varepsilon^2)\Big)^{\frac{d-1}{2}} \leq \frac{|\hat{E}|_{d-1}}{|E|_{d-1}} \leq \Big(1 + 2c(d-1)\varepsilon + \mathcal{O}(\varepsilon^2)\Big)^{\frac{d-1}{2}}.$$

*Proof.* Fix $t > 0$. We will prove first that $E$ can be partitioned into $m$ connected sets $G_1, \ldots, G_m$ such that:

1. $|G_i \cap G_j|_{d-1} = 0$ for all $i \neq j$.
2. there exist $I(i) \in \mathbb{N}$ such that $\pi_E^{-1}(G_i) \subset F_{I(i)}$, for each $i = 1, \ldots, m$.
3. for each $i = 1, \ldots, m$ there exists an orthonormal basis $(e_1, \ldots, e_d)$ of $\mathbb{R}^d$, $H_i \subset \mathbb{R}^{d-1}$, and functions $f_i : H_i \to \mathbb{R}$, $\mathcal{C}^2$ such that:

$$G_i = \Big\{(x, f_i(x_1, \ldots, x_{d-1})) : x = \sum_{i=1}^{d-1} x_i e_i \in H_i\Big\}.$$

4. $\max_i(\max_{x \in G_i} ||\nabla f_i(x)||_\infty) \leq t$ and $\max_i(\max_{x \in G_i} ||\mathcal{H}_{f_i}(x)||_{op}) \leq \alpha + t$.

We provide a sketch of the proof, leaving the details to the reader. For any $x \in E$, consider the parametrization $\varphi_x : T_x E \cap \mathcal{B}(x, r_x) \to E$ such that $\nabla \varphi_x(x) = 0$ and $\mathcal{H}_f(x)$ is the second fundamental form, which is bounded by $\alpha$ in all directions (see Proposition 6.1 in Niyogi et al. (2008)). The regularity conditions on $E$ allow finding a radius $r_x$ such that for all $y \in B(x, r_x)$, $||\nabla \varphi_x(y)|| < t$, and $||\mathcal{H}_f(y)||_{op} < \alpha + t$. By compactness there exists a finite covering of $E$ by balls $B(x_1, r_1), \ldots, B(x_m, r_m)$, from which we extract only the Voronoi cells of $\{x_1, \ldots, x_m\}$. Let us denote by $V_i$ the Voronoi cell of $x_i$. Lastly, the family of sets $\{V_i \cap \pi_E(F_j)\}_{i,j}$ is the required partition.

We will now introduce, for $x \in H_i$, $J_i(x)$ the block matrix $J_i(x) = (I_{d-1}, \nabla f_i(x))'$. Observe that this is the Jacobian matrix of the parametrization $\varphi_x$. Also $J_i(x)' J_i(x) = I_{d-1} + \nabla f_i(x) \nabla f_i(x)'$. Now if $v$ is any vector orthogonal to $\nabla f_i(x)$, $J_i'(x) J(x) v = v$, and it follows that 1 is an eigenvalue of $J_i'(x) J_i(x)$ with multiplicity $d-1$. On the other hand, $J_i'(x) J_i(x) \nabla f_i(x) = (1 + ||\nabla f_i(x)||^2) \nabla f_i(x) = ||n_x||^2$, where $n_x = (-\nabla f_i(x), 1) \in N_{(x, f(x))} G_i$. Then,

$$|G_i|_{d-1} = \int_{H_i} \sqrt{\det J_i(x)' J_i(x)} dx = \int_{H_i} ||n_x||_2 dx,$$

from which it follows that

$$|H_i|_{d-1} \leq |G_i|_{d-1} \leq (1+t)|H_i|_{d-1}. \tag{17}$$

Because $d_H(\hat{E}, E) < reach(E)$, by item (3) in Theorem 4.8 in Federer (1956) there exists a function $l$ such that for all $(x, f_i(x)) \in G_i$ and $y = \pi_E^{-1}((x, f_i(x))) \in \hat{E}$, we have that $y = x + f_i(x) e_d + l(x) n_x$ with $|l(x)|/||n_x||_2 = d(y, E) > 0$, because $\hat{E} \cap E = \emptyset$. Then $l(x) = ||n_x||_2 d(y, E)$ or $l(x) = -||n_x||_2 d(y, E)$. Since the sets $F_j$ are of class $\mathcal{C}^2$, again by item (3) in Theorem 4.8 in Federer (1956) $l(x)$ is of class at least $\mathcal{C}^1$.

By differentiation, for $j \in \{1, \ldots, d-1\}$ let $\hat{t}_j = dy/dx_j$ be the following vector of $T_y \hat{E}$.

$$\hat{t}_j = e_j + \frac{\partial f_i}{\partial x_j}(x) e_d + \frac{\partial l}{\partial x_j}(x) n_x - l(x) \left( \sum_{k=1}^{d-1} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(x) e_k \right) \tag{18}$$

Since $\hat{t}_j \in T_y\hat{E}$, $\pi_{N_y F_{I(i)}}(\hat{t}_j) = 0$, while

$$\pi_{N_x E}(\hat{t}_j) = \langle \hat{t}_j, n_x \rangle ||n_x||^{-1} = \frac{\partial l}{\partial x_j}(x)||n_x||_2 + \frac{l(x)}{||n_x||_2}\left(\sum_{k=1}^{d-1} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(x)\frac{\partial f_i}{\partial x_k}(x)\right)$$

by Hypothesis 5 and the fact that $||n_x||_2 \geq 1$,

$$\frac{\partial l}{\partial x_j}(x)||n_x||_2 + l(x)\left(\sum_{k=1}^{d-1} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(x)\frac{\partial f_i}{\partial x_k}(x)\right) \leq c\varepsilon.$$

If we bound $l(x)/||n_x|| \leq \varepsilon^2$, $||n_x|| > 1$, and use the fact that

$$\max_i(\max_{x \in G_i} ||\nabla f_i(x)||_\infty) \leq t,$$

and $\max_i(\max_{x \in G_i} ||\mathcal{H}_{f_i}(x)||_{op}) \leq \alpha + t$, then

$$\frac{\partial l}{\partial x_j}(x) \leq t(\alpha + t)\varepsilon^2 + c\varepsilon,$$

from which it follows that

$$\sup_x ||\nabla l(x)||_\infty \leq t(\alpha + t)\varepsilon^2 + c\varepsilon. \tag{19}$$

Using a change of variables, it turns out that

$$|\pi_E^{-1}(G_i)|_{d-1} = \int_{H_i} \sqrt{\det\left(\hat{J}_i(x)'\hat{J}_i(x)\right)}dx, \tag{20}$$

where, from (18),

$$\hat{J}_i(x) = \begin{pmatrix} I_{d-1} - l(x)\mathcal{H}_{f_i}(x) \\ \nabla f_i(x) \end{pmatrix} + n_x'\nabla l(x).$$

Next, we bound $||\hat{J}_i'(x)\hat{J}_i(x) - I_{d-1}||_\infty$ from above. First observe that $||l(x)\mathcal{H}_{f_i}(x)||_{op} \leq \varepsilon^2(\alpha + t)$. On the other hand, for every column $\nu$ of the matrix $||\nu||_2 \leq ||l(x)\mathcal{H}_{f_i}(x)||_{op}$. Also $||\nu||_\infty \leq ||\nu||_2$, and so $||\nu||_\infty \leq \epsilon^2(\alpha + t)$. We now can bound

$$\left\| \begin{matrix} l(x)\mathcal{H}_{f_i}(x) \\ \nabla f_i(x) \end{matrix} \right\|_\infty \leq \max\{\varepsilon^2(\alpha + t), t\}.$$

Lastly, if we use (19), $||n_x'\nabla l(x)||_\infty \leq \sqrt{1 + t^2}(t(\alpha + t))\varepsilon^2 + c\varepsilon$.

Introduce $\beta(t, \alpha, \varepsilon) = \max(\varepsilon^2(\alpha + t), t) + \sqrt{1 + t^2}(t(\alpha + t)\varepsilon^2 + c\varepsilon)$. We can decompose $\hat{J}_i'(x)\hat{J}_i(x) = I_{d\times(d-1)} + E$, where $I_{d\times(d-1)}$ is the $d \times (d-1)$ block matrix where the last

row is the $d-1$-dimensional null vector and the other $(d-1)\times(d-1)$ block is the identity, while $E$ is a matrix that fulfills $||E||_\infty < \beta$. Then

$||\hat{J}_i'(x)\hat{J}_i(x) - I_{d-1}||_\infty \leq 2\beta(t,\alpha,\varepsilon) + d\beta(t,\alpha,\varepsilon)^2$. Using that $||A||_{op} \leq (d-1)||A||_\infty$ for any $(d-1)\times(d-1)$ matrix $A$,

$$||\hat{J}_i'(x)\hat{J}_i(x) - I_{d-1}||_{op} \leq 2(d-1)\beta(t,\alpha,\varepsilon) + (d-1)^2\beta(t,\alpha,\varepsilon)^2 = B(t,\alpha,\varepsilon).$$

Now, using the fact that the operator norm is equal to the absolute value of the largest eigenvalue, and (20),

$$(1-B(t,\alpha,\varepsilon))^{\frac{d-1}{2}}|H_i|_{d-1} \leq |\pi_E^{-1}(G_i)|_{d-1} \leq (1+B(t,\alpha,\varepsilon))^{\frac{d-1}{2}}|H_i|_{d-1}$$

Considering also Equation (17) and summing over $i$, we obtain

$$\frac{(1-B(t,\alpha,\varepsilon))^{\frac{d-1}{2}}}{1+t}|E|_{d-1} \leq |\hat{E}|_{d-1} \leq (1+B(t,\alpha,\varepsilon))^{\frac{d-1}{2}}|E|_{d-1}$$

To conclude the proof it is then sufficient to make $t \to 0$.

□

#### 7.2.1 Proof of Theorem 2

Lemma 6 proves that all the hypotheses of Lemma 7 are fulfilled (with probability one for $n$ large enough) with $E = \partial S$, $\hat{E} = \partial C_{\alpha'}(\mathcal{X}_n)$, $\varepsilon_n = a(\ln(n)/n)^{1/(d+1)}$ and $c > 2\sqrt{(\alpha+\alpha')/(\alpha\alpha')}$. Lastly, we obtain that, with probability one, for $n$ large enough,

$$\Big||\partial S|_{d-1} - |\partial C_{\alpha'}(\mathcal{X}_n)|_{d-1}\Big| = O\Big(\Big(\frac{\ln(n)}{n}\Big)^{\frac{1}{d+1}}\Big).$$

Conclusion 2 of the theorem is a consequence of Theorem 3.2.26 in Federer (1969), (see page 261).

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