



**HAL**  
open science

# Yang–Baxter $\sigma$ -model with WZNW term as E -model

Ctirad Klimcik

► **To cite this version:**

| Ctirad Klimcik. Yang–Baxter  $\sigma$  -model with WZNW term as E -model. 2025. hal-02907024

**HAL Id: hal-02907024**

**<https://hal.science/hal-02907024v1>**

Preprint submitted on 10 Feb 2025

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# YANG-BAXTER $\sigma$ -MODEL WITH WZNW TERM AS $\mathcal{E}$ -MODEL

CTIRAD KLIMČÍK

ABSTRACT. It turns out that many integrable  $\sigma$ -models on group manifolds belong to the class of the so-called  $\mathcal{E}$ -models which are relevant in the context of the Poisson-Lie T-duality. We show that this is the case also for the Yang-Baxter  $\sigma$ -model with WZNW term introduced by Delduc, Magro and Vicedo in [5].

## 1. INTRODUCTION

Integrability and T-dualisability are both rare properties of non-linear  $\sigma$ -models and it is not known what necessary conditions must be imposed on a target space metric  $G$  and on closed 3-form field  $H$  in such a way that the corresponding  $\sigma$ -model enjoys either one or another. It turns out, however, that those two properties share something in common, in the sense that the majority of integrable sigma models [1, 13, 14, 15] recently constructed fall in the class of the so-called  $\mathcal{E}$ -models [2], which are the building-blocks for the construction of T-dualisable models [3, 4]. It is not yet clear why it is so, but we find that trying to put all integrable  $\sigma$ -models under the common roof of the  $\mathcal{E}$ -models is a nice guiding principle which may eventually lead to a deeper understanding of this phenomenon. In particular, the purpose of the present note is to give the  $\mathcal{E}$ -model interpretation of one of a few integrable  $\sigma$ -models for which this was not yet done, namely the Yang-Baxter  $\sigma$ -model with WZNW term. This model was introduced in [5] (see also [6] for previous work) and it is a two parameter<sup>1</sup> deformation of the principal chiral model defined by the action

$$S[g] = -\frac{K}{4} \left( \int d\tau \oint (g^{-1}\partial_-g, (1 + \eta^2 + AR + \eta^2R^2)g^{-1}\partial_+g) + k \int d^{-1} \oint (g^{-1}dg, [g^{-1}\partial_\sigma g, g^{-1}dg]) \right). \quad (1.1)$$

Here  $(.,.)$  is the Killing-Cartan form on the Lie algebra  $\mathcal{G}$  of a simple compact Lie group  $G$ ,  $R : \mathcal{G} \rightarrow \mathcal{G}$  is the so-called Yang-Baxter operator (see Section 3 for its definition),  $\tau$  and  $\sigma$  stand respectively for the world-sheet time and space variables (the time and space integrals are notationally separated as  $\int$  and  $\oint$  expressing<sup>2</sup> the angular character of the variable  $\sigma$ ),  $g \equiv g(\sigma, \tau)$  is a dynamical field configuration with values in  $G$ , we set also  $\partial_\pm := \partial_\tau \pm \partial_\sigma$  and, finally, the parameters of the model are constrained by the relation

$$A = \eta \sqrt{1 - \frac{k^2}{1 + \eta^2}}. \quad (1.2)$$

<sup>1</sup>Since the constant  $K$  is just the overall normalisation constant and the relation (1.2) is supposed to hold, the action (1.1) is indeed a two-parameter deformation of the principal chiral model; the latter is recovered for the particular values  $\eta = A = k = 0$ .

<sup>2</sup>Note also our perhaps non-standard way of writing of the Wess-Zumino term which is however very useful for "travelling" between the first order and the second order description of the  $\sigma$ -model dynamics. Mathematically speaking, the object  $\oint (g^{-1}dg, [g^{-1}dg, g^{-1}\partial_\sigma g])$  should be viewed as a two-form on the infinite dimensional loop group manifold  $LG$ .

While Delduc, Magro and Vicedo showed in [5] that the model (1.1) is integrable only if the relation (1.2) holds, we show in the present paper that (1.1) has the structure of the  $\mathcal{E}$ -model also only in the case when the constraint (1.2) is imposed. This fact illustrates again the mysterious relation between the  $\mathcal{E}$ -model formalism and the integrability which motivates our work.

The plan of this note is as follows : We first review how the concept of the  $\mathcal{E}$ -model is built up on that of the Drinfeld double, then we introduce a particular Drinfeld double and show that the  $\mathcal{E}$ -model which corresponds to it is just the Delduc, Magro and Vicedo model (1.1). We also express the Lax pair of the integrable model (1.1) in the formalism of the  $\mathcal{E}$ -model therefore our paper can be viewed also as an alternative demonstration of the integrability of the model (1.1).

## 2. $\mathcal{E}$ -MODELS : GENERALITIES

Consider thus a real Lie algebra  $\mathcal{D}$  equipped with a non-degenerate symmetric ad-invariant bilinear form  $(\cdot, \cdot)_{\mathcal{D}}$  and consider an infinite-dimensional manifold  $P$  the points of which are smooth maps from a circle  $S^1$  into  $\mathcal{D}$ . By parametrizing the circle by the angle  $\sigma \in [0, 2\pi]$  we can associate to each  $\sigma$  a  $\mathcal{D}$ -valued function  $j(\sigma)$  on  $P$  which to every map  $S^1 \rightarrow \mathcal{D}$  attributes its value in  $\sigma$ . By picking further a basis  $T^A$  of  $\mathcal{D}$  we can construct also numerical functions  $j^A(\sigma)$  on  $P$  by the prescription

$$j^A(\sigma) := (j(\sigma), T^A)_{\mathcal{D}}. \quad (2.1)$$

We interpret the numerical functions  $j^A(\sigma)$  on  $P$  as coordinates on  $P$  and we define a Poisson structure  $\{\cdot, \cdot\}_P$  on  $P$  by the following Poisson brackets of the coordinates :

$$\{j^A(\sigma_1), j^B(\sigma_2)\} = F^{AB}{}_C j^C(\sigma_1) \delta(\sigma_1 - \sigma_2) + D^{AB} \delta'(\sigma_1 - \sigma_2). \quad (2.2)$$

Here the tensors  $F^{AB}{}_C$  and  $D^{AB}$  are defined by means of the structure of the Lie algebra  $\mathcal{D}$

$$D^{AB} := (T^A, T^B)_{\mathcal{D}}, \quad [T^A, T^B] = F^{AB}{}_C T^C. \quad (2.3)$$

In fact, the non-degeneracy of the bilinear form  $(\cdot, \cdot)_{\mathcal{D}}$  guarantees that the Poisson structure  $\{\cdot, \cdot\}_P$  on  $P$  is symplectic<sup>3</sup> and can be directly used for construction of dynamical systems on  $P$ . An important class of such dynamical systems was introduced in [3, 4] and the members of this class are called the  $\mathcal{E}$ -models. Each  $\mathcal{E}$ -model is by definition a dynamical system on the symplectic manifold  $P$  defined by the Hamiltonian  $H_{\mathcal{E}}$  :

$$H_{\mathcal{E}} := \frac{1}{2} \int d\sigma (j(\sigma), \mathcal{E}j(\sigma))_{\mathcal{D}}, \quad (2.4)$$

where  $\mathcal{E} : \mathcal{D} \rightarrow \mathcal{D}$  is a self-adjoint  $\mathbb{R}$ -linear involution, i.e.

$$(\mathcal{E}x, y)_{\mathcal{D}} = (x, \mathcal{E}y)_{\mathcal{D}}, \quad \forall x, y \in \mathcal{D}; \quad \mathcal{E}^2 x = x, \quad \forall x \in \mathcal{D}. \quad (2.5)$$

Of course the time evolution of the current  $j(\sigma)$  is obtained by calculating its Poisson bracket with the Hamiltonian which gives

$$\partial_{\tau} j = \partial_{\sigma}(\mathcal{E}j) + [\mathcal{E}j, j]. \quad (2.6)$$

It turns out that every  $\mathcal{E}$ -model encapsulates the first order Hamiltonian dynamics of some non-linear  $\sigma$ -model provided that it exists a Lie subalgebra  $\mathcal{H} \subset \mathcal{D}$  the dimension of which is one half of the dimension of  $\mathcal{D}$  and which satisfies the so called isotropy condition  $(x, x)_{\mathcal{D}} = 0, \forall x \in \mathcal{H}$ . Denoting respectively  $D$  and  $H$  the (simply connected) groups corresponding to the Lie algebras

<sup>3</sup>The Poisson structure given by Eq. (2.2) is usually referred to as the symplectic current algebra.

$\mathcal{D}$  and  $\mathcal{H}$ , the target space of the  $\sigma$ -model is the space of the right cosets  $D/H$  and its action is given by the formula [7, 8]

$$S_{\mathcal{E}}[f] = \frac{1}{4} \int d\tau \oint (f^{-1} \partial_+ f, f^{-1} \partial_- f)_{\mathcal{D}} + \frac{1}{4} \int d^{-1} \oint (f^{-1} df, [f^{-1} \partial_{\sigma} f, f^{-1} df])_{\mathcal{D}-} - \frac{1}{2} \int d\tau \oint (P_f(\mathcal{E}) f^{-1} \partial_+ f, f^{-1} \partial_- f)_{\mathcal{D}}. \quad (2.7)$$

Here the  $\sigma$ -model field  $f(\tau, \sigma) \in \text{Image}(\Gamma)$ , where  $\Gamma : D/H \rightarrow D$  is some fixed global section of the total fibration  $D$  over the base space  $D/H$  (the ideal choice of  $\Gamma$  is some global section but if there is no global section of this fibration we can choose several local sections covering the base space) and  $P_f(\mathcal{E})$  is a projection operator from  $\mathcal{D}$  to  $\mathcal{D}$  defined by its kernel and its image:

$$\text{Ker} P_f(\mathcal{E}) = (\text{Id} + \text{Ad}_{f^{-1}} \mathcal{E} \text{Ad}_f) \mathcal{D}, \quad \text{Image} P_f(\mathcal{E}) = \mathcal{H}. \quad (2.8)$$

The formula (2.7) for the  $\sigma$ -model action from the  $\mathcal{E}$ -model data  $(P_{\mathcal{D}}, H_{\mathcal{E}})$  was derived in [7, 8] and the symplectomorphism associating to every solution of the equation of motion of the  $\sigma$ -model (2.7) the solution of the first order equation of motion (2.6) was given explicitly in [2]. It reads

$$j = \partial_{\sigma} f f^{-1} - \frac{1}{2} f (P_f(\mathcal{E}) f^{-1} \partial_+ f - P_f(-\mathcal{E}) f^{-1} \partial_- f) f^{-1}. \quad (2.9)$$

If we parametrize the  $\mathcal{D}$ -valued current  $j(\sigma)$  in terms of a  $D$ -valued variable  $l(\sigma)$  as

$$j(\sigma) = \partial_{\sigma} l(\sigma) l(\sigma)^{-1} \quad (2.10)$$

then the equation of motion (2.6) can be rewritten as

$$\partial_{\tau} l l^{-1} = \mathcal{E} \partial_{\sigma} l l^{-1} \quad (2.11)$$

and we can therefore infer a useful formula

$$\partial_{\pm} l l^{-1} = (\mathcal{E} \pm \text{Id}) j. \quad (2.12)$$

### 3. $\mathcal{E}$ -MODEL UNDERLYING THE YANG-BAXTER $\sigma$ -MODEL WITH WZNW TERM

We start by recalling the important concept of the Yang-Baxter operator  $R : \mathcal{G} \rightarrow \mathcal{G}$ , where  $\mathcal{G}$  is the Lie algebra of a real simple compact group  $G$ . Considering the standard Cartan-Weyl basis  $H^{\mu}, E^{\alpha}$  of the complexified algebra  $\mathcal{G}^{\mathbb{C}}$ , we introduce the basis of the real algebra  $\mathcal{G}$  as

$$T^{\mu} = iH^{\mu}, \quad B^{\alpha} = \frac{i}{\sqrt{2}}(E^{\alpha} + E^{-\alpha}), \quad C^{\alpha} = \frac{1}{\sqrt{2}}(E^{\alpha} - E^{-\alpha}), \quad (3.1)$$

and the operator  $R$  is then given by

$$RT^{\mu} = 0, \quad RB^{\alpha} = C^{\alpha}, \quad RC^{\alpha} = -B^{\alpha}. \quad (3.2)$$

The Yang-Baxter operator verifies the following crucial identity

$$[Rx, Ry] = R([Rx, y] + [x, Ry]) + [x, y], \quad \forall x, y \in \mathcal{G}. \quad (3.3)$$

Let us now specify which  $\mathcal{E}$ -model underlies the Yang-Baxter  $\sigma$ -model with WZNW term (1.1) via the formula (2.7). The relevant Lie algebra  $\mathcal{D}$  turns out to be the complexification  $\mathcal{G}^{\mathbb{C}}$  of the compact simple Lie algebra  $\mathcal{G}$ . (It must be stressed that  $\mathcal{G}^{\mathbb{C}}$  itself has to be viewed as the *real* Lie algebra, e.g. for  $\mathcal{G} = su(2)$ ,  $\mathcal{G}^{\mathbb{C}} = sl(2, \mathbb{C})$ , the real Lie algebra  $sl(2, \mathbb{C})$  is just the six-dimensional real Lie algebra of the Lorentz group.) There is a convenient way to describe each element  $z \in \mathcal{G}^{\mathbb{C}}$  in terms of two elements  $x, y \in \mathcal{G}$  as follows

$$z = x + iy. \quad (3.4)$$

The Lie algebra commutator in  $\mathcal{G}^{\mathbb{C}}$  is then defined in terms of the commutator of  $\mathcal{G}$ :

$$[z_1, z_2] := ([x_1, x_2] - [y_1, y_2]) + i([x_1, y_2] + [y_1, x_2]), \quad z_1, z_2 \in \mathcal{G}^{\mathbb{C}}, \quad x_1, y_1, x_2, y_2 \in \mathcal{G}, \quad (3.5)$$

where  $z_j = x_j + iy_j$ ,  $j = 1, 2$ .

The second building block of an  $\mathcal{E}$ -model is the choice of the bilinear form  $(\cdot, \cdot)_{\mathcal{D}}$ . To recover the action (1.1), we choose for  $(\cdot, \cdot)_{\mathcal{D}}$  a bilinear form  $(\cdot, \cdot)_{C, \rho}$  depending on two real parameters  $C$  and  $\rho$ , which is defined in terms of the standard Killing-Cartan form  $(\cdot, \cdot)$  on  $\mathcal{G}^{\mathbb{C}}$  as follows:

$$(z_1, z_2)_{C, \rho} := C \mathcal{I}M(e^{i\rho} z_1 z_2) \equiv C \sin \rho((x_1, x_2) - (y_1, y_2)) + C \cos \rho((x_1, y_2) + (x_2, y_1)). \quad (3.6)$$

Here  $\mathcal{I}M$  stands for the imaginary part of a complex number.

The last thing to specify is the self-adjoint involution  $\mathcal{E} : \mathcal{D} \rightarrow \mathcal{D}$  leading to the model (1.1) via the formula (2.7). For that, we parametrize unambiguously any element  $z \in \mathcal{G}^{\mathbb{C}}$  in terms of two elements  $\mathcal{R}, \mathcal{J}$  of  $\mathcal{G}$  as follows

$$z = \frac{e^{-i\rho} \mathcal{R} + (\cosh p + e^{-i\rho} \sinh p) i \mathcal{J}}{2 \cosh p (\sinh p + \cos \rho \cosh p)}, \quad (3.7)$$

where  $p$  is some supplementary real parameter. Then the operator  $\mathcal{E}_{p, \rho} : \mathcal{D} \rightarrow \mathcal{D}$  is just defined by the flip  $\mathcal{R} \leftrightarrow \mathcal{J}$

$$\mathcal{E}_{p, \rho} z := \frac{e^{-i\rho} \mathcal{J} + (\cosh p + e^{-i\rho} \sinh p) i \mathcal{R}}{2 \cosh p (\sinh p + \cos \rho \cosh p)}, \quad (3.8)$$

which makes its involutivity manifest. Some work is needed to verify the self-adjointness of  $\mathcal{E}_{p, \rho}$ , i.e. the property

$$(\mathcal{E}_{p, \rho} z_1, z_2)_{C, \rho} = (z_1, \mathcal{E}_{p, \rho} z_2)_{C, \rho}. \quad (3.9)$$

(Note also that for  $p = \rho = 0$  the parametrization (3.7) becomes  $z = \mathcal{R} + i\mathcal{J}$  which explains the notation:  $\mathcal{R}$  and  $\mathcal{J}$  are, respectively, the "real" and the "imaginary" parts of  $z$ .)

To extract from our  $\mathcal{E}$ -model  $(\mathcal{E}_{p, \rho}, \mathcal{G}^{\mathbb{C}}, (\cdot, \cdot)_{C, \rho})$  the  $\sigma$ -model action (2.7) we need three more things: to identify the isotropic subalgebra  $\mathcal{H} \subset \mathcal{G}^{\mathbb{C}}$  and the corresponding subgroup  $H \subset G^{\mathbb{C}}$ , to choose the section  $\Gamma$  of the  $H$ -fibration over  $G^{\mathbb{C}}/H$  and to find explicit formula for the projection  $P_f(\mathcal{E}_{p, \rho})$ . Thus for the algebra  $\mathcal{H}$  we take the subspace of  $\mathcal{D} = \mathcal{G}^{\mathbb{C}}$  defined as the image of the  $\mathbb{R}$ -linear operator  $(R - \tan \frac{\rho}{2}(R^2 + 1) - i) : \mathcal{G} \rightarrow \mathcal{G}^{\mathbb{C}}$ :

$$\mathcal{H}_{\rho} := (R - \tan \frac{\rho}{2}(R^2 + 1) - i)\mathcal{G}, \quad (3.10)$$

The fact that the subspace  $\mathcal{H}_{\rho}$  is indeed a Lie subalgebra can be deduced from the identity

$$\begin{aligned} & \left[ \left( R - \tan \frac{\rho}{2}(R^2 + 1) - i \right) x, \left( R - \tan \frac{\rho}{2}(R^2 + 1) - i \right) y \right] = \\ & = \left( R - \tan \frac{\rho}{2}(R^2 + 1) - i \right) \left( \left[ \left( R - \tan \frac{\rho}{2}(R^2 + 1) \right) x, y \right] + \left[ x, \left( R - \tan \frac{\rho}{2}(R^2 + 1) \right) y \right] \right) \end{aligned} \quad (3.11)$$

which is the consequence of the identity (3.3). The crucial isotropy property of  $\mathcal{H}_{\rho}$  follows from the following anti-selfadjointness property of the operator  $R$  with respect to the Killing-Cartan form

$$(Rx, y) = -(x, Ry). \quad (3.12)$$

How the group  $H_{\rho}$  looks like? In fact, it is a semi-direct product of some real form of the complex Cartan torus  $T^{\mathbb{C}} \subset G^{\mathbb{C}}$  with the nilpotent group  $N$  featuring the Iwasawa decomposition of  $G^{\mathbb{C}} = GAN$ . To see it, it is convenient first to use the explicit form (3.2) of the Yang-Baxter

operator  $R$  to figure out that the subalgebra  $\mathcal{H}_\rho \subset \mathcal{D}$  can be written as a semi-direct sum of two Lie subalgebras of  $\mathcal{G}^\mathbb{C}$  (the ideal is on the right)

$$\mathcal{H}_\rho = \text{Span}_{\mathbb{R}}(e^{-\frac{i\rho}{2}} H^\mu) \ltimes \text{Span}_{\mathbb{C}}(E^\alpha) \equiv e^{-\frac{i\rho}{2}} \text{Lie}(A) \ltimes \text{Lie}(N). \quad (3.13)$$

Thus the subgroup  $H_\rho$  of  $G^\mathbb{C}$  is the semidirect product  $H_\rho := A_\rho \ltimes N$ , where  $A_\rho$  is the Abelian subgroup of  $G^\mathbb{C}$  the Lie algebra of which is  $\text{Lie}(A_\rho) = e^{-\frac{i\rho}{2}} \text{Lie}(A)$ . In the case of the group  $G^\mathbb{C} = SL(n, \mathbb{C})$ , the elements of  $A_\rho$  are appropriate diagonal matrices and the elements of  $N$  are complex upper-triangular matrices with units on the diagonal.

If  $e^{\frac{i\rho}{2}} \neq \pm i$  (which we are going to suppose for the rest of this paper), there exists a global section of the  $H_\rho$ -fibration over  $G/H_\rho$ . Indeed, this can be deduced from the validity of the standard Iwasawa decomposition  $G^\mathbb{C} = GAN$  which implies in turn the validity of the modified Iwasawa decomposition  $G^\mathbb{C} = GA_\rho N$ . This means that any right  $H_\rho$ -coset in  $G^\mathbb{C}$  can be uniquely represented by some element of the compact group  $G$  which is, by definition, the image of that coset by the section  $\Gamma$ . The section  $\Gamma$  defined in this way has for the image the submanifold  $G$  of  $G^\mathbb{C}$  therefore the field  $f(\tau, \sigma)$  in the  $\sigma$ -model action (2.7) can be taken as  $G$ -valued and denoted as  $g(\tau, \sigma)$ .

In order to evaluate the  $\sigma$ -model action (2.7) corresponding to the  $\mathcal{E}$ -model  $(\mathcal{E}_{p,\rho}, \mathcal{G}^\mathbb{C}, (\cdot, \cdot)_{C,\rho})$ , it remains to make explicit the expression  $P_f(\mathcal{E}_{p,\rho})g^{-1}\partial_+g$ . To do that we first check that

$$g^{-1}\partial_+g = (e^{-i\rho} + i \cosh p + ie^{-i\rho} \sinh p)u + (R - \tan \frac{\rho}{2}(R^2 + 1) - i)v, \quad (3.14)$$

where  $u, v$  are the elements of  $\mathcal{G}$  given by the formulae

$$u = \left( \cos \rho + \sin \rho \sinh p + \left( R - \tan \frac{\rho}{2}(R^2 + 1) \right) (\cosh p + \cos \rho \sinh p - \sin \rho) \right)^{-1} g^{-1}\partial_+g; \quad (3.15)$$

$$v = \left( R - \tan \frac{\rho}{2}(R^2 + 1) - \frac{e^p \tan \frac{\rho}{2} + 1}{\tan \frac{\rho}{2} - e^p} \right)^{-1} g^{-1}\partial_+g. \quad (3.16)$$

We note that the  $u$ - and  $v$ -terms on the right hand side of Eq. (3.14) are respectively the elements of the subspaces  $\text{Ker}P_f(\mathcal{E}) = (\text{Id} + \text{Ad}_{f^{-1}}\mathcal{E}\text{Ad}_f)\mathcal{D}$  and  $\text{Image}P_f(\mathcal{E}) = \mathcal{H}_\rho$ . This means that it holds

$$P_f(\mathcal{E}_{p,\rho})g^{-1}\partial_+g = \left( R - \tan \frac{\rho}{2}(R^2 + 1) - i \right) \left( R - \tan \frac{\rho}{2}(R^2 + 1) - \frac{e^p \tan \frac{\rho}{2} + 1}{\tan \frac{\rho}{2} - e^p} \right)^{-1} g^{-1}\partial_+g. \quad (3.17)$$

Inserting the expression (3.17) in the general formula (2.7), we obtain the second order  $\sigma$ -model action corresponding to the  $\mathcal{E}$ -model  $(\mathcal{E}_{p,\rho}, \mathcal{G}^\mathbb{C}, (\cdot, \cdot)_{C,\rho})$ :

$$\begin{aligned} S_{\mathcal{E}_{p,\rho}}[g] &= \frac{C \sin \rho}{4} \oint \left( \int d\tau (g^{-1}\partial_+g, g^{-1}\partial_-g) + \int d^{-1}(g^{-1}dg, [g^{-1}\partial_\sigma g, g^{-1}dg]) \right) + \\ &+ \frac{C}{2} \int d\tau \oint \left( \left( \cos \rho - \sin \rho \left( R - \tan \frac{\rho}{2}(R^2 + 1) \right) \right) \left( R - \tan \frac{\rho}{2}(R^2 + 1) - \frac{e^p \tan \frac{\rho}{2} + 1}{\tan \frac{\rho}{2} - e^p} \right)^{-1} g^{-1}\partial_+g, g^{-1}\partial_-g \right). \end{aligned} \quad (3.18)$$

The  $\sigma$ -model (3.18) does not look like the Yang-Baxter  $\sigma$ -model with WZNW term as given by Eq.(1.1), however, it can be rewritten in that form by using the identity

$$R^3 = -R \quad (3.19)$$

which permits to invert easily the operator  $\left(R - \tan \frac{\rho}{2}(R^2 + 1) - \frac{e^\rho \tan \frac{\rho}{2} + 1}{\tan \frac{\rho}{2} - e^\rho}\right)$ . The resulting expression reads

$$S_{\mathcal{E}_{p,\rho}}[g] = \frac{C}{4 \cosh p} \oint \left( \int d\tau (g^{-1} \partial_+ g, g^{-1} \partial_- g) + \sin \rho \cosh p \int d^{-1} (dgg^{-1} \wedge [\partial_\sigma gg^{-1}, dgg^{-1}]) \right) + \frac{C}{4 \cosh p} \left( e^{2p} \cos^2 \frac{\rho}{2} - \sin^2 \frac{\rho}{2} \right) \int d\tau \oint ((-e^{-p} R + R^2 + 1) g^{-1} \partial_+ g, g^{-1} \partial_- g) \quad (3.20)$$

and this does coincide with the original action (1.1) of Delduc, Magro and Vicedo upon the identification

$$K = -\frac{C}{\cosh p}, \quad k = \sin \rho \cosh p, \quad \eta^2 = e^{2p} \cos^2 \frac{\rho}{2} - \sin^2 \frac{\rho}{2}, \quad A = -e^p \cos^2 \frac{\rho}{2} + e^{-p} \sin^2 \frac{\rho}{2}. \quad (3.21)$$

It can be also checked that the constraint (1.2) is satisfied by the values of the parameters  $K, k, \eta, A$  given by Eqs. (3.21), which means that the crucial property of integrability of the Yang-Baxter  $\sigma$ -model with WZNW term comes about automatically from our  $\mathcal{E}$ -model formalism.

#### 4. INTEGRABILITY

The first order field equations (2.6) of the  $\mathcal{E}$ -model defined by the data  $(\mathcal{E}_{p,\rho}, \mathcal{G}^{\mathbb{C}}, (\cdot, \cdot)_{C,\rho})$  acquire a particularly simple form if we parametrize the current  $j(\tau, \sigma)$  as in Eq. (3.7)

$$j = \frac{e^{-i\rho} \mathcal{R} + (\cosh p + e^{-i\rho} \sinh p) i \mathcal{J}}{2 \cosh p (\sinh p + \cos \rho \cosh p)}. \quad (4.1)$$

We then have

$$\mathcal{E}_{p,\rho} j = \frac{e^{-i\rho} \mathcal{J} + (\cosh p + e^{-i\rho} \sinh p) i \mathcal{R}}{2 \cosh p (\sinh p + \cos \rho \cosh p)} \quad (4.2)$$

therefore Eq. (2.6) for the data  $(\mathcal{E}_{p,\rho}, \mathcal{G}^{\mathbb{C}}, (\cdot, \cdot)_{C,\rho})$  can be equivalently rewritten in terms of two  $\mathcal{G}$ -valued equations :

$$\partial_\tau \mathcal{R} = \partial_\sigma \mathcal{J} + [\mathcal{J}, \mathcal{R}]; \quad (4.3a)$$

$$\partial_\tau \mathcal{J} = \partial_\sigma \mathcal{R}. \quad (4.3b)$$

The form (4.3) of the first order field equations is of course very familiar since it is identical with the Zakharov-Mikhailov field equations of the principal chiral model [9] allowing the so called Lax pair with spectral parameter. This means that given a solution  $\mathcal{R}(\tau, \sigma), \mathcal{J}(\tau, \sigma)$  of the equations of motion (4.3) and a complex number  $\lambda \neq \pm 1$  the following  $\mathcal{G}^{\mathbb{C}}$ -valued zero curvature identity takes place

$$\partial_+ A_-(\lambda) - \partial_- A_+(\lambda) + [A_-(\lambda), A_+(\lambda)] = 0, \quad (4.4)$$

where

$$A_\pm(\lambda) := \frac{\mathcal{J} \pm \mathcal{R}}{1 \pm \lambda}. \quad (4.5)$$

We thus conclude that our  $\mathcal{E}$ -model  $(\mathcal{E}_{p,\rho}, \mathcal{G}^{\mathbb{C}}, (\cdot, \cdot)_{C,\rho})$  is integrable, in the sense of admitting an infinite number of conserved observables extracted from the Lax pair  $A_\pm(\lambda)$  in the standard way [10]. We have thus given an alternative proof of integrability of the model (1.1) than that presented in [5].

We remark that the Yang-Baxter  $\sigma$ -model with WZNW term was shown to be integrable even in the stronger sense in [5] where it was demonstrated via the so called  $r/s$ -formalism that the conserved observables extracted from the Lax pair Poisson-commute. The matrix Poisson brackets of the components of the Lax connection, which intervene in this computation do depend on the

parameters  $C, p$  and  $\rho$ , which is not surprising since the Poisson brackets of the currents  $\mathcal{R}$  and  $\mathcal{J}$  depend on them. Indeed we have

$$\{\mathcal{J}^a(\sigma_1), \mathcal{J}^b(\sigma_2)\} = \frac{2 \cosh p}{C} f^{ab}{}_c \mathcal{J}^c(\sigma_1) \delta(\sigma_1 - \sigma_2) - \frac{4(\cosh p)^2 \sin \rho}{C} \delta^{ab} \delta'(\sigma_1 - \sigma_2); \quad (4.6a)$$

$$\{\mathcal{J}^a(\sigma_1), \mathcal{R}^b(\sigma_2)\} = \frac{2 \cosh p}{C} f^{ab}{}_c \mathcal{R}^c(\sigma_1) \delta(\sigma_1 - \sigma_2) - \frac{4(\cosh p)^2 (\cosh p + \cos \rho \sinh p)}{C} \delta^{ab} \delta'(\sigma_1 - \sigma_2); \quad (4.6b)$$

$$\{\mathcal{R}^a(\sigma_1), \mathcal{R}^b(\sigma_2)\} = -\frac{2 \cosh p ((\sinh p)^2 + (\cosh p)^2 + 2 \cos \rho \sinh p \cosh p)}{C} \mathcal{J}^c(\sigma_1) f^{ab}{}_c \delta(\sigma_1 - \sigma_2) + \quad (4.6c)$$

$$+ \frac{4 \sin \rho (\cosh p)^2}{C} \mathcal{R}^c(\sigma_1) f^{ab}{}_c \delta(\sigma_1 - \sigma_2) - \frac{4(\cosh p)^2 \sin \rho}{C} \delta^{ab} \delta'(\sigma_1 - \sigma_2). \quad (4.6d)$$

The brackets (4.6) were obtained from the symplectic current algebra (2.2) in the following way: we chose a basis  $T^A := (\tau^a, i\tau^a)$  of  $\mathcal{G}^C$  where  $\tau^a$  is an orthonormal basis of  $\mathcal{G}$  satisfying

$$(\tau^a, \tau^b) = -\delta^{ab}, \quad [\tau^a, \tau^b] = f^{ab}{}_c \tau^c, \quad (4.7)$$

we set

$$\mathcal{R}^a := (\mathcal{R}, \tau^a), \quad \mathcal{J}^a := (\mathcal{J}, \tau^b) \quad (4.8)$$

and, using the parametrization (4.1) of the  $\mathcal{D}$ -current  $j$ , we evaluated

$$(j, \tau^a)_{C, \rho} = \frac{C \mathcal{J}^a}{2 \cosh p}, \quad (j, (\sinh p + e^{-i\rho} \cosh p) i\tau^a)_{C, \rho} = \frac{C \mathcal{R}^a}{2 \cosh p}. \quad (4.9)$$

Finally we plugged the expressions (4.9) into the symplectic current algebra (2.2).

For completeness, we give also the formula for the Hamiltonian (2.4) of our  $\mathcal{E}_{p, \rho}$ -model in the  $\mathcal{R}, \mathcal{J}$ -parametrization (4.1)

$$H_{\mathcal{E}_{p, \rho}} := \frac{C}{2} \int d\sigma \frac{(\cosh p + \cos \rho \sinh p)((\mathcal{R}, \mathcal{R}) + (\mathcal{J}, \mathcal{J})) - 2 \sin \rho (\mathcal{R}, \mathcal{J})}{4(\cosh p)^2 (\sinh p + \cos \rho \cosh p)^2}. \quad (4.10)$$

Combining the formulae (4.6) and (4.10) we find for the Poisson brackets  $\{\mathcal{R}, H_{\mathcal{E}_{p, \rho}}\}$  and  $\{\mathcal{J}, H_{\mathcal{E}_{p, \rho}}\}$

$$\{\mathcal{R}, H_{\mathcal{E}_{p, \rho}}\} = \partial_\sigma \mathcal{J} + [\mathcal{J}, \mathcal{R}]; \quad \{\mathcal{J}, H_{\mathcal{E}_{p, \rho}}\} = \partial_\sigma \mathcal{R} \quad (4.11)$$

in full accord with the equations of motion (4.3).

## 5. ANOTHER FORM OF THE FIRST ORDER FORMALISM

The reader might wish to understand better how the second order action (3.20) can be derived from the symplectic current algebra (4.6) and the Hamiltonian (4.10). In a general case, they may wish to consult the original papers [3, 7, 8], but the case of the Yang-Baxter  $\sigma$ -model with the WZNW term given by (3.20) can be understood quite simply by suitably parametrizing the symplectic current algebra in terms of the coordinates of another symplectic manifold which features in the first order description of the dynamics of nonlinear  $\sigma$ -models with the WZNW terms. Consider thus the spaces  $L\mathcal{G}$  and  $LG$  consisting of smooth maps from the circle  $S^1$  respectively into the Lie algebra  $\mathcal{G}$  and into the Lie group  $G$ . The manifold  $LG \times L\mathcal{G}$  is referred to as the (co)tangent bundle



of the loop group  $LG$  and it is equipped with a symplectic structure<sup>4</sup>  $\Omega$  given by two equivalent expressions:

$$\Omega = \frac{k}{2} \oint (dgg^{-1} \wedge \partial_\sigma(dgg^{-1})) + d \oint (dgg^{-1}, J_L), \quad (g, J_L) \in LG \times LG; \quad (5.1a)$$

$$\Omega = -\frac{k}{2} \oint (g^{-1}dg \wedge \partial_\sigma(g^{-1}dg)) - d \oint (g^{-1}dg, J_R), \quad J_R = -g^{-1}J_Lg + kg^{-1}\partial_\sigma g. \quad (5.1b)$$

Here the symbol  $\oint$  stands for the circular integral around the loop parameter  $\sigma$  and  $(\cdot, \cdot)$  is the Killing-Cartan form on  $\mathcal{G}$ .

It is useful to recall the Poisson brackets of the variables  $g, J_L, J_R$  induced by the symplectic form (5.1) that were computed e.g. in [11, 12]:

$$\{g(\sigma_1) \otimes g(\sigma_2)\} = 0; \quad (5.2a)$$

$$\{g(\sigma_1), (J_L(\sigma_2), \tau^a)\} = \tau^a g(\sigma_1) \delta(\sigma_1 - \sigma_2); \quad (5.2b)$$

$$\{g(\sigma_1), (J_R(\sigma_2), \tau^a)\} = -g(\sigma_1) \tau^a \delta(\sigma_1 - \sigma_2); \quad (5.2c)$$

$$\{(J_L(\sigma_1), \tau^a), (J_L(\sigma_2), \tau^b)\} = (J_L(\sigma_1), [\tau^a, \tau^b]) \delta(\sigma_1 - \sigma_2) + k(\tau^a, \tau^b) \delta'(\sigma_1 - \sigma_2); \quad (5.2d)$$

$$\{(J_R(\sigma_1), \tau^a), (J_R(\sigma_2), \tau^b)\} = (J_R(\sigma_1), [\tau^a, \tau^b]) \delta(\sigma_1 - \sigma_2) - k(\tau^a, \tau^b) \delta'(\sigma_1 - \sigma_2); \quad (5.2e)$$

$$\{(J_L(\sigma_1), \tau^a), (J_R(\sigma_2), \tau^b)\} = 0. \quad (5.2f)$$

Consider now an injective map  $\Upsilon : LG \times LG \rightarrow P$  from the cotangent bundle of the loop group  $LG$  into the symplectic current algebra  $P$  (based on the choice  $\mathcal{D} = \mathcal{G}^c$ ) given explicitly by

$$j(\sigma) \equiv \Upsilon(g, J_L) := \partial_\sigma g g^{-1} + \frac{e^{-i\rho}}{C} (R_{g^{-1}} + \tan \frac{\rho}{2} (R_{g^{-1}}^2 + 1) - i) (-J_L + C \sin \rho \partial_\sigma g g^{-1}), \quad (5.3)$$

where

$$R_{g^{-1}} := \text{Ad}g R \text{Ad}g^{-1}. \quad (5.4)$$

The crux of the story is the fact that, for the choice  $k = C \sin \rho$  and for the choice of the bilinear form  $(\cdot, \cdot)_{\mathcal{D}} = (\cdot, \cdot)_{C, \rho}$  given by (3.6), the map  $\Upsilon$  preserves the Poisson brackets (5.2) on  $LG \times LG$  and (2.2) on  $P$  (as it can be verified by somewhat tedious calculation employing crucially the Yang-Baxter identity (3.3)). The procedure of the extraction of a non-linear  $\sigma$ -model (3.20) from the  $\mathcal{E}_{p, \rho}$ -model based on the involution (3.8) then starts by pull-backing the symplecting form  $\omega$  and the Hamiltonian  $H_{\mathcal{E}_{p, \rho}}$  from  $P$  to  $LG \times LG$  where they become, respectively,  $\Upsilon^* \omega = \Omega$  and  $\Upsilon^* H_{\mathcal{E}_{p, \rho}} = H_{p, \rho}$  :

$$H_{p, \rho}(g, J_L) = \frac{1}{2} \oint (\Upsilon(g, J_L), \mathcal{E}_{p, \rho} \Upsilon(g, J_L))_{C, \rho}. \quad (5.5)$$

It is well-known how to obtain the first order action of a general dynamical system  $(P, \omega, H)$  if the symplectic form  $\omega$  were a coboundary, that is, if it existed a 1-form  $\theta$  such that  $\omega = d\theta$ . In this case the action  $S(\gamma)$  of a trajectory  $\gamma(\tau)$  in the phase space  $P$  would be given by

$$S(\gamma) = \int_{\tau_i}^{\tau_f} d\tau (\gamma^* \theta - (\gamma^* H) d\tau). \quad (5.6)$$

If  $\omega$  is not coboundary, we must write instead

$$S(\gamma) = \int_{\tau_i}^{\tau_f} d\tau (\gamma^* d^{-1} \omega - (\gamma^* H) d\tau). \quad (5.7)$$

Strictly speaking, the object  $d^{-1} \omega$  is not well-defined but its variation is, and that is all what is needed to extract field equations out from (5.7). We do not enter the discussion how precisely the

<sup>4</sup>For  $k = 0$  the expressions (5.1) describe the canonical symplectic structure on the (co)tangent bundle.

term  $d^{-1}\omega$  should be handled in general (this is a well-known story for everyone who is familiar with the dynamics of the standard WZNW model) since what matters for us is that the symplectic form  $\Omega$  given by (5.1a) is the sum of two 2-forms from which the one containing explicitly the variable  $J_L$  is the coboundary. This permits us to write the first order action of a trajectory  $\gamma(\tau) := (g(\tau), J_L(\tau))$  of the dynamical system  $(LG \times L\mathcal{G}, \Omega, H_{p,\rho})$  as follows

$$\begin{aligned}
 S_{p,\rho} &= \int d\tau \oint \left( (\partial_\tau g g^{-1}, J_L) - \frac{1}{2}(\Upsilon(g, J_L), \mathcal{E}_{p,\rho} \Upsilon(g, J_L))_{C,\rho} \right) + \frac{C \sin \rho}{2} \int \gamma^* d^{-1} \oint (dgg^{-1} \frown \partial_\sigma(dgg^{-1})) = \\
 &= \int d\tau \oint \left( (\partial_\tau g g^{-1}, J_L) - \frac{e^{-p}}{C}(J_L, J_L) - \frac{\sinh p + \cos \rho \cosh p}{C(1 + \cos \rho)}(R_{g^{-1}} J_L, R_{g^{-1}} J_L) \right) + \\
 &+ \int d\tau \oint \left( \partial_\sigma g g^{-1}, \left( \sin \rho \cosh p + (\sinh p + \cos \rho \cosh p) \left( R_{g^{-1}} - \tan \frac{\rho}{2} (R_{g^{-1}}^2 + 1) \right) \right) J_L \right) \\
 &- \frac{C(\cosh p + \cos \rho \sinh p)}{2} \int d\tau \oint (\partial_\sigma g g^{-1}, \partial_\sigma g g^{-1}) + \frac{C \sin \rho}{2} \int \gamma^* d^{-1} \oint (dgg^{-1} \frown \partial_\sigma(dgg^{-1})).
 \end{aligned} \tag{5.8}$$

We note that the  $\mathcal{G}$ -valued current  $J_L$  plays the role of auxiliary field in (5.8) and, moreover, it appears there quadratically. It can be therefore easily eliminated giving a second order action

$$\begin{aligned}
 S_{p,\rho} &= \frac{C e^p (1 + \cos \rho)}{4} \int d\tau \oint (g^{-1} \partial_+ g, g^{-1} \partial_- g) + \\
 &+ \frac{C}{4} \left( \frac{1}{\cosh p} - e^p (1 + \cos \rho) \right) \int d\tau \oint (R g^{-1} \partial_+ g, R g^{-1} \partial_- g) + \\
 &+ \frac{C(\sinh p + \cos \rho \cosh p)}{2 \cosh p} \int d\tau \oint (R g^{-1} \partial_\tau g, g^{-1} \partial_\sigma g) + \frac{C \sin \rho}{4} \int d^{-1} \oint (dgg^{-1} \frown [\partial_\sigma g g^{-1}, dgg^{-1}]).
 \end{aligned} \tag{5.9}$$

Of course, the action (5.9) can be easily rewritten in the equivalent form (3.20) of the Yang-Baxter  $\sigma$ -model with WZNW terms, as it should.

## 6. OUTLOOK

The ordinary Yang-Baxter  $\sigma$ -model [13, 14] obtained from (3.20) by setting  $\rho = 0$  can be alternatively deformed to the two-parametric bi-Yang-Baxter  $\sigma$ -model [15] which is also integrable. There is an obvious open problem of "switching on" a non-vanishing  $\rho$  in the bi-Yang-Baxter case and to obtain in this way a three-parametric integrable deformation of the principal chiral model. This problem does not seem to be an easy one, however.

## REFERENCES

- [1] G. Arutyunov, R. Borsato and S. Frolov, *S-matrix for strings on  $\eta$ -deformed  $AdS_5 \times S^5$* , JHEP **1404** (2014) 002, arXiv:1312.3542 [hep-th];  
 J. Balog, P. Forgács, Z. Horváth and L. Palla, *A new family of  $SU(2)$  symmetric integrable  $\sigma$ -models*, Phys. Lett. **B324** (1994) 403, hep-th/9307030;  
 V.V. Bazhanov, G.A. Kotousov and S.L. Lukyanov, *Winding vacuum energies in a deformed  $O(4)$  sigma model*, Nucl. Phys. **B889** (2014) 817, arXiv:1409.0449 [hep-th];  
 I. V. Cherednik; *Relativistically invariant quasiclassical limits of integrable two-dimensional quantum models*, Theor. Math. Phys. **47** (1981) 422;  
 F. Delduc, M. Magro and B. Vicedo, *On classical  $q$ -deformations of integrable sigma-models*, JHEP **1311** (2013) 192, arXiv:1308.3581 [hep-th];  
 F. Delduc, T. Kameyama, M. Magro and B. Vicedo, *Affine  $q$ -deformed symmetry and the classical Yang-Baxter  $\sigma$ -model*, JHEP **1703** (2017) 126 doi:10.1007/JHEP03(2017)126, arXiv:1701.03691 [hep-th];  
 F. Delduc, M. Magro and B. Vicedo, *An integrable deformation of the  $AdS_5 \times S^5$  superstring action*, Phys. Rev. Lett. **112** (2014) no.5, 051601, arXiv:1309.5850 [hep-th];

- S. Demulder, K. Sfetsos and D. C. Thompson, *Integrable  $\lambda$ -deformations: Squashing Coset CFTs and  $AdS_5 \times S^5$* , JHEP **1507** (2015) 019, arXiv:1504.02781 [hep-th];
- J. M. Evans and T. J. Hollowood, *Integrable theories that are asymptotically CFT*, Nucl. Phys. B **438** (1995) 469, [hep-th/9407113];
- A. Fateev, *The sigma model (dual) representation for a two-parameter family of integrable quantum field theories*, Nucl. Phys. **B473** (1996) 509;
- V.A. Fateev, E. Onofri and A.B. Zamolodchikov, *The Sausage model (integrable deformations of  $O(3)$   $\sigma$ -model)*, Nucl. Phys. **B406** (1993) 521;
- G. Georgiou and K. Sfetsos, *A new class of integrable deformations of CFTs*, JHEP **1703** (2017) 083, arXiv:1612.05012 [hep-th];
- G. Georgiou, K. Sfetsos and K. Siampos, *Double and cyclic  $\lambda$ -deformations and their canonical equivalents*, Phys. Lett. B **771** (2017) 576, arXiv:1704.07834 [hep-th];
- B. Hoare, *Towards a two-parameter  $q$ -deformation of  $AdS_3 \times S^3 \times M^4$  superstrings*, Nucl. Phys. B **891** (2015) 259, arXiv:1411.1266 [hep-th];
- B. Hoare, R. Roiban and A. A. Tseytlin, *On deformations of  $AdS_n \times S^n$  supercosets*, JHEP **1406** (2014) 002, arXiv:1403.5517 [hep-th];
- B. Hoare and A. A. Tseytlin, *On integrable deformations of superstring sigma models related to  $AdS_n \times S^n$  supercosets*, Nucl. Phys. **B897** (2015) 448, arXiv:1504.07213[hep-th];
- T. J. Hollowood, J. L. Miramontes and D. M. Schmidt, *Integrable Deformations of Strings on Symmetric Spaces*, JHEP **1411** (2014) 009, [arXiv:1407.2840 [hep-th];
- T.J.Hollowood, J.L. Miramontes and D. Schmidt, *An integrable deformation of the  $AdS_5 \times S^5$  superstring*, J. Phys. **A47** (2014) 49, 495402, arXiv:1409.1538 [hep-th];
- I. Kawaguchi, T. Matsumoto and K. Yoshida, *Jordanian deformations of the  $AdS_5 \times S^5$  superstring*, JHEP **1404** (2014) 153, arXiv:1401.4855 [hep-th];
- I. Kawaguchi and K. Yoshida, *Hybrid classical integrability in squashed sigma models*, Phys.Lett. **B705** (2011) 251, arXiv:1107.3662 [hep-th];
- S. L. Lukyanov, *The integrable harmonic map problem versus Ricci flow*, Nucl. Phys. **B865** (2012) 308, arXiv:1205.3201 [hep-th];
- T. Matsumoto and K. Yoshida, *Yang-Baxter sigma models based on the CYBE*, Nucl. Phys. B **893** (2015) 287 doi:10.1016/j.nuclphysb.2015.02.009 [arXiv:1501.03665 [hep-th]
- N. Mohammedi, *On the geometry of classical integrable two-dimensional nonlinear  $\sigma$ -models*, Nucl.Phys. **B839** (2010) 420, arXiv:0806.0550 [hep-th];
- K. Sfetsos, *Integrable interpolations: From exact CFTs to non-Abelian T-duals*, Nucl. Phys. B **880** (2014) 225, arXiv:1312.4560 [hep-th];
- K. Sfetsos and K. Siampos, *The anisotropic  $\lambda$ -deformed  $SU(2)$  model is integrable*, Phys. Lett. **B743** (2015) 160, arXiv:1412.5181 [hep-th].
- K. Sfetsos, K. Siampos and D. Thompson, *Generalised integrable  $\lambda$ - and  $\eta$ -deformations and their relation*, Nucl. Phys. **B899** (2015) 489, arXiv:1506.05784 [hep-th];
- S. J. van Tongeren, *On classical Yang-Baxter based deformations of the  $AdS_5 \times S^5$  superstring*, JHEP **1506** (2015) 048, arXiv:1504.05516 [hep-th];
- [2] C. Klimčık,  *$\eta$  and  $\lambda$  deformations as  $\mathcal{E}$ -models*, Nucl. Phys. **B900** (2015) 259, arXiv:1508.05832 [hep-th];
- [3] C. Klimčık and P. Ševera, *Dual non-Abelian duality and the Drinfeld double*, Phys. Lett. **B351** (1995) 455, hep-th/9502122; C. Klimčık, *Poisson-Lie T-duality*, Nucl. Phys. (Proc. Suppl.) **B46** (1996) 116, hep-th/9509095; P. Ševera, *Minimálne plochy a dualita*, Diploma thesis, 1995, in Slovak;
- [4] C. Klimčık and P. Ševera, *Poisson-Lie T-duality and loop groups of Drinfeld doubles*, Phys. Lett. **B372** (1996), 65-71, hep-th/9512040;
- [5] F. Delduc, M. Magro and B. Vicedo, *Integrable double deformation of the principal chiral model*, Nucl. Phys. **B891** (2015) 312-321, arXiv:1410.8066 [hep-th];
- [6] I. Kawaguchi, D. Orlando and K. Yoshida, *Yangian symmetry in deformed WZNW models on squashed spheres*, Phys. Lett. B **701** (2011) 475, arXiv:1104.0738 [hep-th];
- I. Kawaguchi and K. Yoshida, *A deformation of quantum affine algebra in squashed Wess-Zumino-Novikov-Witten models*, J. Math. Phys. **55** (2014) 062302, arXiv:1311.4696 [hep-th];
- [7] C. Klimčık and P. Ševera, *Non-Abelian momentum-winding exchange*, Phys.Lett. **B383** (1996) 281-286, hep-th/9605212;

- [8] C. Klimčík and P. Ševera, *Open strings and D-branes in WZNW model*, Nucl.Phys. **B488** (1997) 653, hep-th/9609112;
- [9] V.E. Zakharov and A.V. Mikhailov, *Relativistically invariant two-dimensional model of field theory which is integrable by means of the inverse scattering method*, Sov. Phys. JETP **47** (1978) 1017;
- [10] O. Babelon, D. Bernard and M. Talon, *Introduction to Classical Integrable Systems*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (2003);
- [11] C. Klimčík, *Quasitriangular WZW model*, Rev. Math. Phys. **16** (2004) 679-808, hep-th/0103118
- [12] B. Vicedo, *Deformed integrable  $\sigma$ -models, classical R-matrices and classical exchange algebra on Drinfel'd doubles*, J.Phys. **A48** (2015) 35, 355203, arXiv:1504.06303;
- [13] C. Klimčík, *Yang-Baxter  $\sigma$ -model and dS/AdS T-duality*, JHEP **0212** (2002) 051, hep-th/0210095;
- [14] C. Klimčík, *Integrability of the Yang-Baxter  $\sigma$ -model*, J. Math. Phys. **50** (2009) 043508, arXiv:0802.3518 [hep-th];
- [15] C. Klimčík, *Integrability of the bi-Yang-Baxter  $\sigma$ -model*, Lett. Math. Phys. **104** (2014) 1095, arXiv:1402.2105 [math-ph].

(C Klimčík) INSTITUT DE MATHÉMATIQUES DE LUMINY, AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE I2M, UMR 7373, 13453 MARSEILLE, FRANCE

*E-mail address:* ctirad.klimcik@univ-amu.fr