Stability estimates for reconstruction from the Fourier transform on the ball

Mikhail Isaev, Roman Novikov

Stability estimates for reconstruction from the Fourier transform on the ball*

Mikhail Isaev
School of Mathematics
Monash University
Clayton, VIC, Australia
mikhail.isaev@monash.edu

Roman G. Novikov
CMAP, CNRS, Ecole Polytechnique
Institut Polytechnique de Paris
Palaiseau, France
IEPT RAS, Moscow, Russia
novikov@cmap.polytechnique.fr

Abstract

We prove Hölder-logarithmic stability estimates for the problem of finding an integrable function v on $\mathbb{R}^d$ with a super-exponential decay at infinity from its Fourier transform $\mathcal{F}v$ given on the ball B_r. These estimates arise from a Hölder-stable extrapolation of $\mathcal{F}v$ from B_r to a larger ball. We also present instability examples showing an optimality of our results.

Keywords: ill-posed inverse problems, Hölder-logarithmic stability, exponential instability, Chebyshev extrapolation
AMS subject classification: 42A38, 35R30, 49K40

1 Introduction

We consider the classical Fourier transform $\mathcal{F}$ defined by

$$\mathcal{F}v(\xi) := \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} e^{i\xi x} v(x) dx, \quad \xi \in \mathbb{R}^d,$$

*The first author's research is supported by the Australian Research Council Discovery Early Career Researcher Award DE200101045.

where v is a test function on $\mathbb{R}^d$ and $d \geq 1$. Let

$$B_r := \left\{\xi \in \mathbb{R}^d : |\xi| < r\right\}, \qquad \text{where } r > 0.$$

Assume that v is integrable and, for some $N, \sigma > 0$ and $\nu \geq 1$, we have that

$$Q_v(\lambda) := \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} e^{\lambda|x|} |v(x)| dx \leq N \exp\left(\sigma \lambda^\nu\right), \qquad \text{for all } \lambda \geq 0. \tag{1.1}$$

Remark 1.1. In particular, for the case of $\nu = 1$, the class of functions satisfying (1.1) includes all functions v with $\operatorname{supp} v \subset B_\sigma$ and $\frac{1}{(2\pi)^d} \|v\|_{\mathcal{L}^1(\mathbb{R}^d)} \leq N$. Furthermore, if v is such that

$$|v(x)| \leq C \exp\left(-\mu |x|^\eta\right) \text{ for some } \mu, C > 0 \text{ and } \eta > 1,$$

then assumption (1.1) holds with $\nu := \frac{\eta}{\eta - 1}$ and with some positive constants $\sigma = \sigma(\mu, \eta)$ and $N = N(C, \mu, \eta, d)$.

Under assumption (1.1), we consider the following two problems:

Problem 1.1. Given $\mathcal{F}v$ on the ball B_r. Find v.

Problem 1.2. Given $\mathcal{F}v$ on the ball B_r. Find $\mathcal{F}v$ on B_R, where $R > r$.

Problems 1.1 and 1.2 are fundamental in the theory of inverse coefficient problems. For example, Problem 1.1 with $r = 2\sqrt{E}$ can be regarded as a linearized inverse scattering problem for the Schrödinger equation with potential v at fixed positive energy E, for $d \geq 2$, and on the the energy interval $[0, E]$, for $d \geq 1$. More details can be found in [12, Section 4]. Problem 1.1 with $r = \omega_0$ also arises in a multi-frequency inverse source problem for the homogeneous Helmholtz equation with frequencies $\omega \in [0, \omega_0]$; see Bao et al. [2, Section 3] for more details. In addition, in many cases, Problem 1.2 is an essential step for solving Problem 1.1. For more applications related to Problems 1.1 and 1.2 in the case of compactly supported v, see [8] and references therein.

The present work continues the studies of our recent article [8], which considers the case of compactly supported functions v. Besides, in [8], we deal with reconstructions of $\mathcal{F}v$ on $[-R, R]^d$ and v on $\mathbb{R}^d$ from $\mathcal{F}v$ given on the cube $[-r, r]^d$, in place of the balls B_R and B_r. Due to the equivalence of $\|\cdot\|_2$ -norm and $\|\cdot\|_\infty$-norm in $\mathbb{R}^d$, these formulations are essentially equivalent, but B_R and B_r are more natural in the context of inverse problems.

In the present work, under assumption (1.1), we give Hölder-logarithmic stability estimates for Problem 1.1 in the norm of $\mathcal{L}^\infty(\mathbb{R}^d)$ and of $\mathcal{H}^s(\mathbb{R}^d)$, for any real s; see Section 3. (Note that the stability estimates of [8] are given in the norm of $\mathcal{L}^2(\mathbb{R}^d)$

only.) In addition, we obtain Hölder stability estimates for Problem 1.2 in the norm of $\mathcal{L}^\infty(B_R)$; see Section 5. The related reconstruction procedures are also given; see Sections 2 and 3. Besides, we present examples showing an optimality of our stability estimates and reconstruction procedures; see Section 4.

2 Reconstruction procedures

Let $\mathcal{F}^{-1}$ be the classical inverse Fourier transform defined by

$$\mathcal{F}^{-1}[u](x) := \int_{\mathbb{R}^d} u(\xi) e^{-i\xi x} d\xi, \qquad x \in \mathbb{R}^d.$$

For a given $r > 0$, we consider the following family of extrapolations $\mathcal{C}_{R,n} : \mathcal{L}^\infty(B_r) \to \mathcal{L}^\infty(B_R)$, depending on two parameters $R \geq r$ and $n \in \mathbb{N} := \{0, 1, \ldots\}$. For a function w on B_r (for example, such that $w \approx \mathcal{F}v|_{B_r}$), we define

$$[\mathcal{C}_{R,n} w](\xi) := \begin{cases} w(\xi), & \xi \in B_r, \\ \displaystyle\sum_{k=0}^{n-1} a_k\left(\frac{\xi}{|\xi|}\right) T_k\left(\frac{|\xi|}{r}\right), & \xi \in B_R \setminus B_r, \\ 0, & \xi \in \mathbb{R}^d \setminus B_R, \end{cases} \tag{2.1}$$

where $\xi = |\xi|\theta$ and, for $\theta \in S^{d-1}$,

$$a_k(\theta) = a_k[w](\theta) := \begin{cases} \dfrac{1}{\pi} \displaystyle\int_{-r}^{r} \frac{w(t\theta)}{\sqrt{r^2 - t^2}} dt, & \text{if } k = 0, \\ \dfrac{2}{\pi} \displaystyle\int_{-r}^{r} \frac{w(t\theta) T_k\left(\frac{t}{r}\right)}{\sqrt{r^2 - t^2}} dt, & \text{otherwise.} \end{cases} \tag{2.2}$$

In the above, $(T_k)_{k\in\mathbb{N}}$ stand for the Chebyshev polynomials on $\mathbb{R}$, which can be defined by $T_k(t) := \cos(k \arccos(t))$ if $t \in [-1, 1]$ and extended to $|t| > 1$ in a natural way. For $n = 0$, the sum in (2.1) is taken to be 0. Note that formulas (2.1) and (2.2) are correctly defined for almost all ξ and θ under the assumption that $w \in \mathcal{L}^\infty(B_r)$.

Suppose $w \approx \mathcal{F}v|_{B_r}$. The transforms $\mathcal{C}_{R,n} w$ on B_R can be considered as a family of reconstruction procedures for Problem 1.2. The transforms $\mathcal{F}^{-1}\mathcal{C}_{R,n} w$ on $\mathbb{R}^d$ can be considered as a family of reconstruction procedures for Problem 1.1.

In Section 3, we give stability estimates for Problem 1.1 arising from the reconstructions $\mathcal{F}^{-1}\mathcal{C}_{R,n}$; see Theorem 3.1 and Theorem 3.2. In Section 5, we give stability estimates

for Problem 1.2 arising from the extrapolations $\mathcal{C}_{R,n}$; see Lemma 5.1, Theorem 5.2, and Corollary 5.4.

3 Stability estimates for Problem 1.1

We will assume that the unknown function $v : \mathbb{R}^d \to \mathbb{C}$ satisfies (1.1) for some $N, \sigma > 0$ and $\nu \geq 1$ and the given data w is such that, for some $\delta, r > 0$,

$$\|w - \mathcal{F}v\|_{\mathcal{L}^\infty(B_r)} \leq \delta < N, \tag{3.1}$$

where $\mathcal{F}$ is the Fourier transform. Note that if (1.1) holds then, for any $\xi \in \mathbb{R}^d$,

$$|\mathcal{F}v(\xi)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |v(x)| dx = Q_v(0) \leq N. \tag{3.2}$$

This explains the condition $\delta < N$ in assumption (3.1). Indeed, if the noise level δ is greater than N then the given data w tells about v as little as the trivial function $w_0 \equiv 0$.

To achieve optimal stability bounds, the parameters R and n in the reconstruction $\mathcal{F}^{-1}\mathcal{C}_{R,n}$ have to be chosen carefully depending on N, δ, r, σ. For any $\tau \in [0, 1]$, let

$$L_\tau(\delta) = L_\tau(N, \delta, r, \sigma, \nu) := \max\left\{1,\ \frac{1}{2}\left(\frac{(1-\tau)\ln\frac{N}{\delta}}{\sigma r^\nu}\right)^\tau\right\}. \tag{3.3}$$

Here and thereafter, we assume $0 < \delta < N$. Using (2.1), define

$$\mathcal{C}^*_{\tau,\delta} := \mathcal{C}_{R_\tau(\delta), n_\tau(\delta)}, \tag{3.4}$$

where

$$\begin{aligned} R_\tau(\delta) &= R_\tau(N, \delta, r, \sigma, \nu) := rL_\tau(\delta), \\ n_\tau(\delta) &= n_\tau(N, \delta, r, \sigma, \nu) := \begin{cases} \left\lceil \dfrac{(2-\tau)\ln\frac{N}{\delta}}{\ln 2 + \frac{1}{\tau\nu}\ln(2L_\tau(\delta))} \right\rceil, & \text{if } \tau > 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.5}$$

and $\lceil \cdot \rceil$ denotes the ceiling of a real number. Let

$$c(d) := \int\limits_{\partial B_1} 1\, dx = \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}. \tag{3.6}$$

Our first result is a stability estimate for Problem 1.1 in the norm $\mathcal{L}^\infty(\mathbb{R}^d)$. In addition to (1.1), we assume also that $v \in \mathcal{W}^m(\mathbb{R}^d)$, where the space $\mathcal{W}^m(\mathbb{R}^d)$, $m \geq 0$, and its

norm are defined by

$$\mathcal{W}^m(\mathbb{R}^d) := \left\{u \in \mathcal{L}^1(\mathbb{R}^d) \,:\, (1+|\xi|^2)^{\frac{m}{2}} \mathcal{F}u \in \mathcal{L}^\infty(\mathbb{R}^d)\right\},$$
$$||u||_{\mathcal{W}^m(\mathbb{R}^d)} := \left\|(1+|\xi|^2)^{\frac{m}{2}} \mathcal{F}u\right\|_{\mathcal{L}^\infty(\mathbb{R}^d)}.$$

We note that for integer m the space $\mathcal{W}^m(\mathbb{R}^d)$ contains the standard Sobolev space $\mathcal{W}^{m,1}(\mathbb{R}^d)$ of m-times smooth functions in $\mathcal{L}^1$ on $\mathbb{R}^d$.

Theorem 3.1. *Let the assumptions of* (1.1) *and* (3.1) *hold for some* $N, \sigma, r, \delta > 0$ *and* $\nu \geq 1$. *Assume also that* $v \in \mathcal{W}^m(\mathbb{R}^d)$, *for some real* $m > d$, *and that* $\|v\|_{\mathcal{W}^m(\mathbb{R}^d)} \leq \gamma_1$. *Then, for any* α *such that* $0 \leq \alpha \leq 1$, *the following estimate holds:*

$$\begin{aligned}\left\|v - \mathcal{F}^{-1}\mathcal{C}^*_{\tau,\delta} w\right\|_{\mathcal{L}^\infty(\mathbb{R}^d)} \leq \frac{8c(d)}{d} N^{1-\alpha} r^d \left(L_\tau(\delta)\right)^{d+1} \delta^\alpha \\ + \frac{c(d)}{m-d}\gamma_1 r^{-m+d}\left(L_\tau(\delta)\right)^{-m+d},\end{aligned} \tag{3.7}$$

where $\tau = 1 - \sqrt{1-(1-\alpha)\nu^{-1}}$ *and* $L_\tau(\delta)$, $\mathcal{C}^*_{\tau,\delta}$, $c(d)$ *are defined by* (3.3), (3.4), (3.6). *In particular, for any* β_1 *such that* $0 < \frac{\beta_1}{m-d} < 1 - \sqrt{1-\nu^{-1}}$, *we have*

$$\left\|v - \mathcal{F}^{-1}\mathcal{C}^*_{\tau,\delta} w\right\|_{\mathcal{L}^\infty(\mathbb{R}^d)} \leq c_1 \left(\ln(3+\delta^{-1})\right)^{-\beta_1}, \tag{3.8}$$

where $\tau = \frac{\beta_1}{m-d}$ *and* $c_1 = c_1(N, \sigma, \nu, r, m, \gamma_1, d, \beta_1)$ *is a positive constant.*

Our second result is a stability estimate for Problem 1.1 in the norm $\mathcal{H}^s(\mathbb{R}^d)$. Recall that the Sobolev space $\mathcal{H}^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, and its norm can be defined by

$$\mathcal{H}^s(\mathbb{R}^d) := \left\{u \in \mathcal{L}^2(\mathbb{R}^d) :\ \mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}u \in \mathcal{L}^2(\mathbb{R}^d)\right\},$$
$$||u||_{\mathcal{H}^s(\mathbb{R}^d)} := \left\|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}u\right\|_{\mathcal{L}^2(\mathbb{R}^d)}.$$

Theorem 3.2. *Let the assumptions of* (1.1) *and* (3.1) *hold for some* $N, \sigma, r, \delta > 0$ *and* $\nu \geq 1$. *Assume also that* $v \in \mathcal{H}^m(\mathbb{R}^d)$, *for some real* $m \geq -\frac{d}{2}$, *and that* $\|v\|_{\mathcal{H}^m(\mathbb{R}^d)} \leq \gamma_2$. *Then, for any* $\alpha \in [0,1]$ *and any* $s < m$, *the following estimate holds:*

$$\begin{aligned}\|v - \mathcal{F}^{-1}\mathcal{C}^*_{\tau,\delta} w\|_{\mathcal{H}^s(\mathbb{R}^d)} \leq 8(2\pi)^{d/2} c(d) N^{1-\alpha}\left(\int_0^{rL_\tau(\delta)} (1+t^2)^s t^{d-1} dt\right)^{1/2} L_\tau(\delta)\,\delta^\alpha \\ + \gamma_2 r^{-m+s}\left(L_\tau(\delta)\right)^{-m+s},\end{aligned} \tag{3.9}$$

where $\tau := 1 - \sqrt{1-(1-\alpha)\nu^{-1}}$ *and* $L_\tau(\delta)$, $\mathcal{C}^*_{\tau,\delta}$, $c(d)$ *are defined by* (3.3), (3.4), (3.6). *In particular, for any* β_2 *such that* $0 < \frac{\beta_2}{m-s} < 1 - \sqrt{1-\nu^{-1}}$, *we have*

$$\left\|v - \mathcal{F}^{-1}\mathcal{C}^*_{\tau,\delta} w\right\|_{\mathcal{H}^s(\mathbb{R}^d)} \leq c_2 \left(\ln(3+\delta^{-1})\right)^{-\beta_2}, \tag{3.10}$$

where $\tau = \frac{\beta_2}{m-s}$ *and* $c_2 = c_2(N, \sigma, \nu, r, m, s, \gamma_2, d, \beta_2)$ *is a positive constant.*

The proofs of Theorems 3.1 and 3.2 are given in Section 6. The first terms of the right-hand side in estimates (3.7) and (3.9) correspond to the error caused by the Hölder stable extrapolation of the noisy data w from B_r to $B_{R_\tau(\delta)}$ and the second (logarithmic) terms correspond to the error caused by ignoring the values of $\mathcal{F}v$ outside $B_{R_\tau(\delta)}$; see Section 6 for more details.

Let N, σ, ν, r, m, γ_1, γ_2, d be fixed. Then estimates (3.8) and (3.10) used for $v := v_1 - v_2$ and $w := w_0 \equiv 0$ yield the following corollary.

Corollary 3.3. *Let v_1 and v_2 be such that $v := v_1 - v_2$ satisfies (1.1) for some $N, \sigma > 0$ and $\nu \geq 1$. Let τ be such that $0 < \tau < 1 - \sqrt{1-\nu^{-1}}$. Then the following bounds hold.*

(a) If $v_1 - v_2 \in \mathcal{W}^m(\mathbb{R}^d)$, for some real $m < d$, and $\|v_1 - v_2\|_{\mathcal{W}^m(\mathbb{R}^d)} \leq \gamma_1$, then

$$\|v_1 - v_2\|_{\mathcal{L}^\infty(\mathbb{R}^d)} \leq c_1 \left(\ln \left(3 + \frac{1}{\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{\mathcal{L}^\infty(B_r)}} \right) \right)^{-\beta_1}, \tag{3.11}$$

where $\beta_1 = \tau(m-d)$ and c_1 is the constant of (3.8).

(b) If $v_1 - v_2 \in \mathcal{H}^m(\mathbb{R}^d)$ for some real $m \geq -\frac{d}{2}$, $\|v_1 - v_2\|_{\mathcal{H}^m(\mathbb{R}^d)} \leq \gamma_2$, and $s < m$, then

$$\|v_1 - v_2\|_{\mathcal{H}^s(\mathbb{R}^d)} \leq c_2 \left(\ln \left(3 + \frac{1}{\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{\mathcal{L}^\infty(B_r)}} \right) \right)^{-\beta_2}, \tag{3.12}$$

where $\beta_2 = \tau(m-s)$ and c_2 is the constant of (3.10).

One can see that the estimates of Theorem 3.1, Theorem 3.2, and Corollary 3.3 are available for any β_1, β_2 such that:

$$\begin{aligned} 0 < \beta_1 < \beta_1^{\max} &:= \left(1 - \sqrt{1-\nu^{-1}}\right)(m-d), \\ 0 < \beta_2 < \beta_2^{\max} &:= \left(1 - \sqrt{1-\nu^{-1}}\right)(m-s). \end{aligned} \tag{3.13}$$

In particular, for $\nu = 1$, we have that $\beta_1^{\max} = m - d$ and $\beta_2^{\max} = m - s$. In Section 4, we present instability examples showing that

- if $\nu = 1$ then (3.11) is impossible for $\beta_1 > m$ and (3.12) is impossible for $\beta_2 > m$ (when $d \geq 2$ and $s = 0$) and for $\beta_2 > m + \frac{1}{2}$ (when $d = 1$ and $s = 0$);
- if $\nu = 2$ then (3.11) and (3.12) (with $s = 0$) are impossible for $\beta_1, \beta_2 > m/2$.

These examples show that the logarithmic bounds (3.11) and (3.12) are rather optimal with respect to the values of the exponents β_1 and β_2. Consequently, in this respect, it

is impossible to essentially improve the stability bounds of Theorems 3.1 and 3.2, even using any other reconstruction procedure (for example, based on a more advanced basis instead of Chebyshev polynomials).

Moreover, observe that, for the case $\nu = 1$, the values of $\beta_1^{\max}$ and $\beta_2^{\max}$ are very close to the best possible: for example, we determined that $\beta_2 = m$ is indeed the threshold value for (3.12) when $d \geq 2$ and $s = 0$. However, we do not know whether the claimed exponents $\beta_1^{\max}, \beta_2^{\max} = \left(1 - \sqrt{1 - \nu^{-1}}\right) m + O(1)$ are also that close to optimal for $\nu > 1$. Our instability examples for $\nu = 2$ imply that they can not exceed $m/2$, but there is still a gap from $m/2$ down to $\left(1 - \frac{1}{\sqrt{2}}\right) m$.

Theorem 3.1, Theorem 3.2, and Corollary 3.3 illustrate similar stability behaviour in more complicated non-linear inverse problems. In fact, the relationship is closer than a mere illustration taking into account that the monochromatic reconstruction from the scattering amplitude in the Born approximation is reduced to Problem 1.1. In particular, estimates (3.10), (3.11) and (3.12) (with $\nu = 1$) should be compared with the results on the monochromatic inverse scattering problem obtained by Hähner, Hohage [4], Isaev, Novikov [7, Theorem 1.2] and Hohage, Weidling [5] under the assumption that v is a compactly supported sufficiently regular function on $\mathbb{R}^3$. More precisely, for this case, estimate (3.11) with $m > 3$ and $\beta_1 = \frac{m-3}{3}$ is similar to [7, Theorem 1.2]; estimate (3.12) with $s = 0$, $m > \frac{3}{2}$, and $\beta_2 = \frac{m}{m+3}$ is similar to [4, Theorem 1.2]; estimates (3.10), (3.12) with $m > 7/2$ and some appropriate $\beta_2 \in (0, 1)$ are similar to [5, Corollary 1.4]. For other known results on logarithmic and Hölder-logarithmic stability in inverse problems, see also Alessandrini [1], Bao et al. [2], Isaev [6], Isakov [9], Novikov [11], Santacesaria [13] and references therein.

As observed above, logarithmic and Hölder-logarithmic stability was established for many different inverse problems. However, to our knowledge, even for the compactly supported case, the estimates of Theorem 3.1, Theorem 3.2 (with $\alpha < 1$) and Corollary 3.3 are implied by none of results given in the literature before the recent work [8]. The related results of [8] are essentially equivalent to the special case of (3.9), (3.10), (3.12) when $v \in \mathcal{H}^m(\mathbb{R}^d)$ is compactly supported, m is a positive integer, and $s = 0$.

4 Examples of exponential instability for Problem 1.1

First, we recall the results from [8, Section 6]. Let A and B be open bounded domains in $\mathbb{R}^d$, $d \geq 1$. Then, for any fixed positive integer m and positive γ, we give examples of

real-valued functions $v_n \in C^m(\mathbb{R}^d)$ such that

$$\operatorname{supp}(v_n) \subseteq A, \qquad \|v_n\|_{C^m(\mathbb{R}^d)} \leq \gamma, \tag{4.1}$$

and the following asymptotics hold as $n \to +\infty$:

$$\|\mathcal{F}v_n\|_{\mathcal{L}^\infty(B)} = O(e^{-n}), \qquad \|v_n\|_{\mathcal{L}^\infty(\mathbb{R}^d)} = \Omega(n^{-m}),$$
$$\|v_n\|_{\mathcal{L}^2(\mathbb{R}^d)} = \begin{cases} \Omega(n^{-m}), & \text{for } d \geq 2, \\ \Omega(n^{-m-\frac{1}{2}}), & \text{for } d = 1. \end{cases}$$

Recall that for two sequences of real numbers a_n and b_n, we say $a_n = \Omega(b_n)$ if $a_n > 0$ always and $b_n = O(a_n)$. It follows from the above that, for any $\beta_1 > m$ and any constant $c_1 > 0$,

$$\|v_n\|_{\mathcal{L}^\infty(\mathbb{R}^d)} > c_1 \left(\ln\left(3 + \|\mathcal{F}v_n\|^{-1}_{\mathcal{L}^\infty(B_r)}\right)\right)^{-\beta_1}, \tag{4.2}$$

when n is sufficiently large. Similarly, for any β_2, where $\beta_2 > m$ if $d \geq 2$ and $\beta_2 > m + \frac{1}{2}$ if $d = 1$, and for any constant $c_2 > 0$, we have that

$$\|v_n\|_{\mathcal{L}^2(\mathbb{R}^d)} > c_2 \left(\ln\left(3 + \|\mathcal{F}v_n\|^{-1}_{\mathcal{L}^\infty(B_r)}\right)\right)^{-\beta_2}, \tag{4.3}$$

when n is sufficiently large.

Condition (4.1) and instability estimates (4.2) and (4.3) show an optimality (or nearly optimality) of the exponent β_1 in stability estimates (3.7), (3.8), (3.11) and of the exponent β_2 in stability estimates (3.9), (3.10), (3.12) with $s = 0$. Recall that Theorem 3.1, Theorem 3.2, and Corollary 3.3 require $\beta_1 < \beta_1^{\max}$, $\beta_2 < \beta_2^{\max}$, where $\beta_1^{\max}$ and $\beta_2^{\max}$ are defined in (3.13). In particular, for $\nu = 1$ (which includes the compactly supported case), we have that $\beta_1^{\max} = m - d$ and $\beta_2^{\max} = m$ (for $s = 0$), which are close to the infima of the exponents β_1 and β_2 in (4.2) and (4.3).

However, $\beta_1^{\max}(\nu)$ and $\beta_2^{\max}(\nu)$ decrease to 0 as $\nu \to +\infty$. In particular, for ν noticeable greater than 1, the instability behaviour exhibiting by the functions v_n recalled above become much less tight with respect to $\beta_1^{\max}(\nu)$ and $\beta_2^{\max}(\nu)$. This motivates us to construct other explicit examples of exponential instability for Problem 1.1, which are non-compactly supported and provide considerably smaller exponents β_1 and β_2 in the instability estimates in comparison with v_n.

For $k \in \mathbb{R}^d$, integer $m > 0$, and real $\varepsilon > 0$, consider the functions $v_{k,m,\varepsilon}$ defined by

$$v_{k,m,\varepsilon}(x) := \varepsilon |k|^{-m} e^{-x^2/2} \cos(kx), \qquad x \in \mathbb{R}^d. \tag{4.4}$$

Note that

$$\mathcal{F}v_{k,m,\varepsilon}(\xi) = \tfrac{1}{2}(2\pi)^{d/2}\varepsilon|k|^{-m}\left(e^{-(\xi-k)^2/2} + e^{-(\xi+k)^2/2}\right), \qquad \xi \in \mathbb{R}^d.$$

Similarly to Remark 1.1, we find that the functions $v_{k,m,\varepsilon}$ satisfy (1.1) for $\nu = 2$, any $\sigma > \frac{1}{2}$, and $N = \varepsilon|k|^{-m}N'(d,\sigma)$. Then, for any fixed $\sigma > \frac{1}{2}$, $r > 0$, integer $m > 0$, and real $\gamma_0, \gamma_1, \gamma_2 > 0$, we have that

$$N \le \gamma_0, \qquad \|v_{k,m,\varepsilon}\|_{\mathcal{W}^m(\mathbb{R}^d)} \le \gamma_1, \qquad \|v_{k,m,\varepsilon}\|_{\mathcal{H}^m(\mathbb{R}^d)} \le \gamma_2; \tag{4.5}$$

for all sufficiently small $\varepsilon > 0$ and $|k| > 1$; and, for fixed $\varepsilon > 0$, the following formulas hold as $|k| \to +\infty$:

$$\begin{aligned} \|\mathcal{F}v_{k,m,\varepsilon}\|_{\mathcal{L}^\infty(B_r)} &= O\left(\exp(-\alpha|k|^2)\right), \text{ for any } \alpha \in (0, \tfrac{1}{2}), \\ \|v_{k,m,\varepsilon}\|_{\mathcal{L}^\infty(\mathbb{R}^d)} &= \varepsilon|k|^{-m}, \qquad \|v_{k,m,\varepsilon}\|_{\mathcal{L}^2(\mathbb{R}^d)} = \Omega(|k|^{-m}). \end{aligned} \tag{4.6}$$

It follows from (4.6) that, for fixed scaling parameter ε, exponents $\beta_1, \beta_2 > m/2$, and constants $c_1, c_2 > 0$,

$$\|v_{k,m,\varepsilon}\|_{\mathcal{L}^\infty(\mathbb{R}^d)} > c_1\left(\ln\left(3 + \|\mathcal{F}v_{k,m,\varepsilon}\|^{-1}_{\mathcal{L}^\infty(B_r)}\right)\right)^{-\beta_1}, \tag{4.7}$$

$$\|v_{k,m,\varepsilon}\|_{\mathcal{L}^2(\mathbb{R}^d)} > c_2\left(\ln\left(3 + \|\mathcal{F}v_{k,m,\varepsilon}\|^{-1}_{\mathcal{L}^\infty(B_r)}\right)\right)^{-\beta_2}, \tag{4.8}$$

when $|k|$ is sufficiently large. Condition (4.5) and instability estimates (4.7), (4.8) show nearly optimality of the exponent β_1 in stability estimates (3.7), (3.8), (3.11) and of the exponent β_2 in stability estimates (3.9), (3.10), (3.12) with $s = 0$, for the case when $\nu = 2$. Namely, for this case,

$$\beta_1^{\max}(2) = \left(1 - \tfrac{1}{\sqrt{2}}\right)(m-d) \qquad \text{and} \qquad \beta_2^{\max}(2) = \left(1 - \tfrac{1}{\sqrt{2}}\right)m \text{ (for } s = 0).$$

One can see that $m/2$, which is the infima of the exponents β_1 and β_2 in (4.7) and (4.8), is substantially closer to $\beta_1^{\max}(2)$ and $\beta_2^{\max}(2)$ than the infima of the exponents β_1 and β_2 in (4.2) and (4.3), respectively.

It is also important to note that the instability behaviour exhibiting by the functions $v_{k,m,\varepsilon}$ defined in (4.4) is impossible for the compactly supported case, at least for sufficiently large m. This is because $\beta_1^{\max}(1)$ and $\beta_2^{\max}(1)$ (for $s = 0$) get bigger than $m/2$ so (4.7) and (4.8) would contradict to Corollary 3.3.

5 Stability estimates for Problem 1.2

Lemma 5.1. *Let $v \in \mathcal{L}^1(\mathbb{R}^d)$ and $\lambda, r, R > 0$ be such that $r \le R \le \lambda/2$, and $Q_v(\lambda) < +\infty$. Then, for any $w \in \mathcal{L}^\infty(B_r)$ and $n \in \mathbb{N}$, the following estimate holds:*

$$\|\mathcal{F}v - \mathcal{C}_{R,n}[w]\|_{\mathcal{L}^\infty(B_R)} \le 2\left(\frac{2R}{r}\right)^n \|w - \mathcal{F}v\|_{\mathcal{L}^\infty(B_r)} + 4\,Q_v(\lambda)\left(\frac{R}{\lambda}\right)^n.$$

Lemma 5.1 is proved in Section 7. Optimising the parameter n in Lemma 5.1, we obtain the following Hölder stability estimate for Problem 1.2.

Theorem 5.2. *Let $v \in \mathcal{L}^1(\mathbb{R}^d)$ and $\lambda, r, R > 0$ be such that $r \le R \le \lambda/2$, and $Q_v(\lambda) < +\infty$. Suppose that $\|w - \mathcal{F}v\|_{\mathcal{L}^\infty(B_r)} \le \delta$ for some function w and $0 < \delta < Q_v(\lambda)$. Then the following estimate holds:*

$$\|\mathcal{F}v - \mathcal{C}_{R,n^*}w\|_{\mathcal{L}^\infty(B_R)} \le \frac{8R}{r}\left(\frac{Q_v(\lambda)}{\delta}\right)^{\tau(\lambda)}\delta, \tag{5.1}$$

where

$$n^* := \left\lceil \frac{\ln\left(\frac{Q_v(\lambda)}{\delta}\right)}{\ln(2\lambda/r)} \right\rceil \qquad \text{and} \qquad \tau(\lambda) := \frac{\ln(2R/r)}{\ln(2\lambda/r)}. \tag{5.2}$$

Remark 5.3. Problem 1.2 is a particular case of the problem of stable analytic continuation; see, for example, Demanet, Townsend [3], Lavrent'ev et al. [10, Chapter 3], Tuan [14], and Vessella [15]. In particular, [3, Theorem 1.2] or [15, Theorem 1] lead to a Hölder stability estimate similar to (5.1). In the present work, we independently establish estimate (5.1) mainly for the purpose to give a simple explicit expression for the factor in front of the Hölder term $\delta^{1-\tau(\lambda)}$. Besides, we derive our estimates for specific analytic functions which are the Fourier transforms of functions satisfying (1.1).

Proof of Theorem 5.2. By the assumptions, we have that

$$\eta := \frac{\ln\left(\frac{Q_v(\lambda)}{\delta}\right)}{\ln(2\lambda/r)} > 0.$$

Note that η is the solution of the equation

$$\left(\frac{2R}{r}\right)^\eta \delta = Q_v(\lambda)\left(\frac{R}{\lambda}\right)^\eta.$$

Using also that $R \ge r$, we get

$$\left(\frac{2R}{r}\right)^{\eta+1}\delta = \frac{2R}{r}Q_v(\lambda)\left(\frac{R}{\lambda}\right)^\eta \ge 2\,Q_v(\lambda)\left(\frac{R}{\lambda}\right)^\eta.$$

By definition of n^* and η, we find that $\eta \le n^* < \eta + 1$. Then, applying Lemma 5.1, we obtain that

$$\begin{aligned}\|\mathcal{F}v - \mathcal{C}_{R,n^*}[w]\|_{\mathcal{L}^\infty(B_R)} &\le 2\left(\frac{2R}{r}\right)^{n^*}\delta + 4\,Q_v(\lambda)\left(\frac{R}{\lambda}\right)^{n^*} \\ &\le 2\left(\frac{2R}{r}\right)^{\eta+1}\delta + 4\,Q_v(\lambda)\left(\frac{R}{\lambda}\right)^{\eta} \\ &\le 4\left(\frac{2R}{r}\right)^{\eta+1}\delta.\end{aligned}$$

Then, by the definitions of $\tau(\lambda)$ and η, we get

$$\left(\frac{2R}{r}\right)^{\eta} = \exp\left(\eta \ln\frac{2R}{r}\right) = \exp\left(\eta\,\tau(\lambda)\ln\frac{2\lambda}{r}\right) = \left(\frac{Q_v(\lambda)}{\delta}\right)^{\tau(\lambda)}.$$

Combining the above formulas completes the proof. □

Theorem 5.2 leads to the following stability estimate for the extrapolation $\mathcal{C}^*_{\tau,\delta}$ used in Theorem 3.2.

Corollary 5.4. *Let the assumptions of* (1.1) *and* (3.1) *hold for some* $N, \sigma, r, \delta > 0$ *and* $\nu \ge 1$. *Then, for any* $\tau \in [0, \nu^{-1}]$, *we have*

$$\|\mathcal{F}v - \mathcal{C}^*_{\tau,\delta}[w]\|_{\mathcal{L}^\infty\left(B_{R_\tau(\delta)}\right)} \le 8L_\tau(\delta)\left(\frac{N}{\delta}\right)^{\nu\tau(2-\tau)}\delta = 8L_\tau(\delta)N^{1-\alpha}\delta^{\alpha},$$

where $L_\tau(\delta)$ *and* $R_\tau(\delta)$ *are defined in* (3.3) *and* (3.5) *and* $\alpha = \alpha(\tau) := 1 - \nu\tau(2-\tau)$. *In addition, the exponent* α *is positive if and only if* $0 \le \tau < 1 - \sqrt{1-\nu^{-1}} \le \nu^{-1}$.

Proof. First, we consider the case $L_\tau(\delta) = 1$. Then, (3.5) and (2.1) imply that $R_\tau(\delta) = r$ and $\mathcal{C}^*_{\tau,\delta}[w] \equiv w$. Recalling from (3.1) that $\delta < N$, we find that

$$\|\mathcal{F}v - \mathcal{C}^*_{\tau,\delta}[w]\|_{\mathcal{L}^\infty\left(B_{R_\tau(\delta)}\right)} = \|\mathcal{F}v - w\|_{\mathcal{L}^\infty(B_r)} \le \delta \le 8L_\tau(\delta)\left(\frac{N}{\delta}\right)^{\nu\tau(2-\tau)}\delta.$$

Next, suppose that

$$L_\tau(\delta) = \frac{1}{2}\left(\frac{(1-\tau)\ln\frac{N}{\delta}}{\sigma r^\nu}\right)^{\tau} > 1.$$

This is only possible when $\tau \ne 0$. Let

$$\lambda := r(2L_\tau(\delta))^{\frac{1}{\nu\tau}}.$$

Then, from (1.1), we get

$$Q_v(\lambda) \le N\exp(\sigma\lambda^\nu) = \delta\left(\frac{N}{\delta}\right)^{2-\tau}.$$

In addition, by the assumptions,

$$R_\tau(\delta) \geq r \qquad \text{and} \qquad \lambda \geq r\,(2L_\tau(\delta)) = 2R_\tau(\delta).$$

Observe that n^* defined in (5.2) for $\lambda = r(2L_\tau(\delta))^{\frac{1}{\nu\tau}}$ coincides with $n_\tau(\delta)$ defined in (3.5). Then, applying Theorem 5.2, we get that

$$\|\mathcal{F}v - \mathcal{C}^*_{\tau,\delta}[w]\|_{\mathcal{L}^\infty\left(B_{R_\tau(\delta)}\right)} \leq 8L_\tau(\delta)\left(\left(\frac{N}{\delta}\right)^{2-\tau}\right)^{\tau(\lambda)}\delta,$$

where $\tau(\lambda)$ is defined in (5.2). Note that $\tau(\lambda)$ is different from τ. However, we can replace $\tau(\lambda)$ by $\nu\tau$ in the estimate above since $\delta < N$ and

$$\tau(\lambda) = \frac{\ln(2R_\tau(\delta)/r)}{\ln(2\lambda/r)} = \frac{\ln(2L_\tau(\delta))}{\ln 2 + \frac{1}{\nu\tau}\ln(2L_\tau(\delta))} \leq \nu\tau.$$

The required bound follows. □

6 Proofs of Theorem 3.1 and Theorem 3.2

In this section, we prove Theorem 3.1 and Theorem 3.2. Their proofs are very similar. Starting from the inverse Fourier transform formula

$$v(x) = \int_{\mathbb{R}^d} e^{-i\xi x}\mathcal{F}v(\xi)d\xi, \quad x \in \mathbb{R}^d,$$

we analyse the contributions of the two regions $B_{R_\tau(\delta)}$ and $\mathbb{R}^d \setminus B_{R_\tau(\delta)}$. For the first region, we apply Corollary 5.4. For the second region we use the smoothness assumptions $\|v\|_{\mathcal{W}^m(\mathbb{R}^d)} \leq \gamma_1$ or $\|v\|_{\mathcal{H}^m(\mathbb{R}^d)} \leq \gamma_2$.

Note that α in estimates (3.7), (3.9) is the same as in Corollary 5.4. Indeed, as stated in Theorem 3.1 and Theorem 3.2, for a given $\alpha \in [0,1]$, we define

$$\tau = \tau(\alpha) := 1 - \sqrt{1-(1-\alpha)\nu^{-1}}.$$

Then, observe that $0 \leq \tau \leq 1 - \sqrt{1-\nu^{-1}} \leq \nu^{-1}$ and

$$1 - \nu\tau(2-\tau) = \alpha,$$

as in Corollary 5.4.

Recall also the definitions of $L_\tau(\delta)$, $R_\tau(\delta)$ and $c(d)$ from (3.3), (3.5), and (3.6), respectively. It is straightforward to check that (3.8) and (3.10) follow from (3.7) and (3.9), respectively. Indeed, for any $N, \sigma, r > 0$, $\nu \geq 1$, $\tau \in (0, 1 - \sqrt{1-\nu^{-1}})$, we have that

$$C_1 \left(\ln(3+\delta^{-1})\right)^\tau \leq L_\tau(\delta) \leq C_2 \left(\ln(3+\delta^{-1})\right)^\tau, \qquad \text{for } 0 < \delta < N,$$

where $C_1 = C_1(N, \sigma, \nu, r, \tau) > 0$ and $C_1 = C_1(N, \sigma, \nu, r, \tau) > 0$. Then, the second terms of the right-hand side on the estimates (3.7) and (3.9) dominates the first terms as $\delta \to 0$. Observing also that $\ln(3+\delta^{-1}) \geq 1$ for all $\delta > 0$, we can find some suitable constants c_1 and c_2 such that estimates (3.8) and (3.10) always hold (given the assumptions) with $\beta_1 = (m-d)\tau$ and $\beta_2 = (m-s)\tau$. Thus, to complete the proofs of Theorems 3.1 and 3.2, it remains to establish stability estimates (3.7) and (3.9).

6.1 Proof of estimate (3.7)

Observe that

$$\|v - \mathcal{F}^{-1}\mathcal{C}^*_{\tau,\delta} w\|_{\mathcal{L}^\infty(\mathbb{R}^d)} \leq \sup_{x \in \mathbb{R}^d} \int\limits_{\mathbb{R}^d} \left| e^{-i\xi x} \left(\mathcal{F}v(\xi) - \mathcal{C}^*_{\tau,\delta} w(\xi) \right) \right| d\xi = I_1 + I_2.$$

where

$$I_1 := \int\limits_{B_{R_\tau(\delta)}} |\mathcal{F}v(\xi) - \mathcal{C}^*_{\tau,\delta} w(\xi)| d\xi,$$

$$I_2 := \int\limits_{\mathbb{R}^d \setminus B_{R_\tau(\delta)}} |\mathcal{F}v(\xi) - \mathcal{C}^*_{\tau,\delta} w(\xi)| d\xi.$$

Using Corollary 5.4, we get that

$$I_1 \leq \int\limits_{B_{R_\tau(\delta)}} \left\| \mathcal{F}v - \mathcal{C}^*_{\tau,\delta} w \right\|_{\mathcal{L}^\infty(B_{R_\tau(\delta)})} d\xi \leq \int\limits_{B_{R_\tau(\delta)}} 8 L_\tau(\delta) N^{1-\alpha} \delta^\alpha \, d\xi$$
$$= \frac{8c(d)}{d} N^{1-\alpha} r^d \left(L_\tau(\delta)\right)^{d+1} \delta^\alpha.$$

Next, since $v \in \mathcal{W}^m(\mathbb{R}^d)$, we have that

$$|\xi|^m |\mathcal{F}v(\xi)| \leq (1+|\xi|^2)^{m/2} |\mathcal{F}v(\xi)| \leq \|v\|_{\mathcal{W}^m(\mathbb{R}^d)}.$$

Thus, we can bound

$$\begin{aligned} I_2 &= \int\limits_{\mathbb{R}^d \setminus B_{R_\tau(\delta)}} |\mathcal{F}v(\xi)| d\xi \leq c(d) \int\limits_{R_\tau(\delta)}^{+\infty} \frac{\|v\|_{\mathcal{W}^m(\mathbb{R}^d)}}{t^{m-d-1}} dt \\ &= \frac{c(d)}{m-d} \|v\|_{\mathcal{W}^m(\mathbb{R}^d)} (R_\tau(\delta))^{-m+d} \\ &= \frac{c(d)}{m-d} \|v\|_{\mathcal{W}^m(\mathbb{R}^d)} r^{-m+d} \left(L_\tau(\delta)\right)^{-m+d}. \end{aligned}$$

Combining the above bounds for I_1 and I_2 completes the proof of (3.7).

6.2 Proof of estimate (3.9)

The Parseval-Plancherel identity states that

$$\|u\|_{\mathcal{L}^2(\mathbb{R}^d)} = (2\pi)^{\frac{d}{2}} \|\mathcal{F}u\|_{\mathcal{L}^2(\mathbb{R}^d)} = (2\pi)^{-\frac{d}{2}} \|\mathcal{F}^{-1}u\|_{\mathcal{L}^2(\mathbb{R}^d)}. \tag{6.1}$$

Thus, we get that

$$\|v - \mathcal{F}^{-1}\mathcal{C}^*_{\tau,\delta} w\|_{\mathcal{H}^s(\mathbb{R}^d)} = (2\pi)^{\frac{d}{2}} \left\|(1+|\xi|^2)^{\frac{s}{2}} (\mathcal{F}v - \mathcal{C}^*_{\tau,\delta} w)\right\|_{\mathcal{L}^2(\mathbb{R}^d)} \leq (2\pi)^{\frac{d}{2}} (\tilde{I}_1 + \tilde{I}_2),$$

where

$$\begin{aligned} \tilde{I}_1 &:= \left(\int\limits_{B_{R_\tau(\delta)}} (1+|\xi|^2)^s |\mathcal{F}v(\xi) - \mathcal{C}^*_{\tau,\delta} w(\xi)|^2 d\xi \right)^{1/2}, \\ \tilde{I}_2 &:= \left(\int\limits_{\mathbb{R}^d \setminus B_{R_\tau(\delta)}} (1+|\xi|^2)^s |\mathcal{F}v(\xi) - \mathcal{C}^*_{\tau,\delta} w(\xi)|^2 d\xi \right)^{1/2}. \end{aligned}$$

Using Corollary 5.4, we get that

$$\begin{aligned} \tilde{I}_1 &\leq \left(\int\limits_{B_{R_\tau(\delta)}} (1+|\xi|^2)^s \left\|\mathcal{F}v - \mathcal{C}^*_{\tau,\delta} w\right\|^2_{\mathcal{L}^\infty(B_{R_\tau(\delta)})} d\xi \right)^{1/2} \\ &\leq 8N^{1-\alpha} \left(c(d) \int\limits_0^{R_\tau(\delta)} (1+t^2)^s t^{d-1} dt \right)^{1/2} L_\tau(\delta) \delta^\alpha. \end{aligned}$$

Applying (6.1) and recalling that $v \in \mathcal{H}^m(\mathbb{R}^d)$, we find that

$$\int\limits_{\mathbb{R}^d \setminus B_{R_\tau(\delta)}} (1+|\xi|^2)^s |\mathcal{F}v(\xi)|^2 d\xi \leq \left\| \frac{(1+|\xi|^2)^{\frac{m}{2}} \mathcal{F}v}{(R_\tau(\delta))^{m-s}} \right\|^2_{L^2(\mathbb{R}^d \setminus B_{R(\tau,\delta)})} \leq \frac{(2\pi)^{-d} \|v\|^2_{\mathcal{H}^m(\mathbb{R}^d)}}{(R_\tau(\delta))^{2(m-s)}}.$$

Thus, we can bound

$$\tilde{I}_2 \le \left(\int\limits_{\mathbb{R}^d \setminus B_{R_\tau(\delta)}} (1+|\xi|^2)^s |\mathcal{F}v(\xi)|^2 d\xi \right)^{1/2} \le \left\| \frac{(1+|\xi|^2)^{\frac{m}{2}} \mathcal{F}v}{(R_\tau(\delta))^{m-s}} \right\|_{L^2(\mathbb{R}^d \setminus B_{R_\tau(\delta)})}$$
$$\le \frac{(2\pi)^{-\frac{d}{2}} \|v\|_{\mathcal{H}^m(\mathbb{R}^d)}}{(R_\tau(\delta))^{m-s}} = (2\pi)^{-\frac{d}{2}} \|v\|_{\mathcal{H}^m(\mathbb{R}^d)} r^{-m+s} \left(L_\tau(\delta)\right)^{-m+s}.$$

Combining the above bounds for $\tilde{I}_1$ and $\tilde{I}_2$ completes the proof of (3.9).

7 Proof of Lemma 5.1

To prove Lemma 5.1, we need a bound for the error term in approximations of holomorphic functions by truncated series of Chebyshev polynomials stated in the following lemma. For completeness purposes, we include a proof of this bound.

Lemma 7.1. *Suppose that $f(z)$ is a holomorphic function in the ellipse*

$$D(\rho) := \{\cos z \,:\, z \in \mathbb{C} \text{ and } |\Im z| < \ln \rho\} \tag{7.1}$$

for some $\rho > 2$ and $\sup\limits_{z \in D(\rho)} |f(z)| \le M_\rho < +\infty$ for some $M_\rho > 0$. Then,

$$\left\| f - \sum_{k=0}^{n-1} b_k T_k \right\|_{L^\infty([-\rho', \rho'])} \le 2M_\rho \left(1 - \frac{2\rho'}{\rho}\right)^{-1} \left(\frac{2\rho'}{\rho}\right)^n,$$

for any $n \in \mathbb{N}$ and any $\rho' \in [1, \rho/2)$, where $(T_k)_{k\in\mathbb{N}}$ are the Chebyshev polynomials and

$$b_k := \begin{cases} \frac{1}{\pi} \int\limits_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt, & \text{if } k = 0, \\ \frac{2}{\pi} \int\limits_{-1}^{1} \frac{f(t) T_k(t)}{\sqrt{1-t^2}} dt, & \text{otherwise.} \end{cases} \tag{7.2}$$

Proof. First of all, we note that the condition $\rho' \in [1, \rho/2)$ ensures that interval $[-\rho', \rho']$ (of the real axe) lies in the ellipse $D(\rho)$. Indeed,

$$D(\rho) \cap \mathbb{R} = \left(-\tfrac{\rho+\rho^{-1}}{2}, \tfrac{\rho+\rho^{-1}}{2} \right).$$

Note also that

$$|\Im \zeta| < \rho/2 \qquad \text{for } \zeta \in D(\rho). \tag{7.3}$$

Let $g(z) := f(\cos z)$. Note that $g(z)$ is an even 2π-periodic holomorphic function in the stripe $|\Im z| < \ln \rho$ and, for all $k \in \mathbb{N}$,

$$\int_0^{2\pi} e^{ik\varphi} g(\varphi) d\varphi = \int_0^{2\pi} e^{-ik\varphi} g(\varphi) d\varphi = 2 \int_{-1}^{1} \frac{f(t) T_k(t)}{\sqrt{1-t^2}} dt,$$

Hence, by the Cauchy integral theorem, we get that

$$g(z) = \sum_{k=0}^{\infty} b_k \cos kz, \quad \text{for } |\Im z| < \ln \rho, \tag{7.4}$$

where

$$|b_k| = \left| \frac{1}{\pi} \int_0^{2\pi} e^{ik\varphi} g(\varphi) d\varphi \right| = \left| \frac{1}{\pi} \int_{0+i\ln\rho}^{2\pi+i\ln\rho} e^{ikz} g(z) dz \right| \leq$$

$$\leq \frac{1}{\pi} \int_0^{2\pi} e^{-k\ln\rho} |g(t)| dt \leq 2M_\rho \rho^{-k}, \quad k \in \mathbb{N}.$$

Using (7.4) and recalling that $T_k(t) := \cos(k \arccos(t))$ for $|t| \leq 1$, we get that

$$f(z) = \sum_{k=0}^{\infty} b_k T_k(z), \qquad z \in D(\rho).$$

Observe that if $|t| \leq 1$ then $|T_k(t)| \leq 1$, otherwise

$$|T_k(t)| = |\cosh(k \operatorname{arccosh}(t))| = \tfrac{1}{2} |(t - \sqrt{t^2-1})^k + (t + \sqrt{t^2-1})^k| \leq (2|t|)^k. \tag{7.5}$$

Combining the estimates above, we get that, for any $t \in [-\rho', \rho']$ and $n \in \mathbb{N}$,

$$\left| f(t) - \sum_{k=0}^{n-1} b_k T_k(t) \right| \leq \sum_{k=n}^{\infty} |b_k T_k(t)|$$

$$\leq 2M_\rho \sum_{k=n}^{\infty} \left(\frac{2\rho'}{\rho} \right)^k = 2M_\rho \left(1 - \frac{2\rho'}{\rho} \right)^{-1} \left(\frac{2\rho'}{\rho} \right)^n.$$

This completes the proof of Lemma 7.1. □

Now we are ready to prove Lemma 5.1. For $\theta \in S^{d-1}$, consider functions $f_{r,\theta} : \mathbb{R} \to \mathbb{C}$ defined by

$$f_{r,\theta}(s) := \mathcal{F}v\,(sr\,\theta) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{isr\theta x} v(x) dx, \quad s \in \mathbb{R}.$$

Provided $Q_v(\lambda) < +\infty$, we have that $f_{r,\theta}$ admits a holomorphic extension to the ellipse $D(\rho)$ defined by (7.1) with $\rho := 2\lambda/r$. Furthermore, using (7.3), we get that

$$|f_{r,\theta}(\zeta)| \leq \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} e^{r|\Im\zeta|\cdot|x|} |v(x)| dx \leq Q_v(\lambda), \qquad \text{for } \zeta \in D(\rho).$$

Applying Lemma 7.1 and taking into account that $\rho = 2\lambda/r$ and $r \leq R \leq \lambda/2$, we find that

$$\left\| f_{r,\theta} - \sum_{k=0}^{n-1} a_k(\theta) T_k \right\|_{\mathcal{L}^\infty([-R/r,R/r])} \leq 2\, Q_v(\lambda) \left(1 - \frac{R}{\lambda}\right)^{-1} \left(\frac{R}{\lambda}\right)^n.$$

It follows that

$$\|\mathcal{F}v - \mathcal{C}_{R,n}[\mathcal{F}v]\|_{\mathcal{L}^\infty(B_R)} \leq 4\, Q_v(\lambda) \left(\frac{R}{\lambda}\right)^n. \tag{7.6}$$

We note that

$$\mathcal{C}_{R,n}[w] - \mathcal{C}_{R,n}[\mathcal{F}v] = \mathcal{C}_{R,n}[w - \mathcal{F}v].$$

Observe that

$$\int\limits_{-r}^{r} \frac{|T_k(t/r)|}{\sqrt{r^2 - t^2}} dt \leq \int\limits_{-r}^{r} \frac{dt}{\sqrt{r^2 - t^2}} = \pi.$$

Recalling the definition of $\mathcal{C}_{R,n}$ and using the above two formulas and (7.5), we get that

$$\begin{aligned} &\|\mathcal{C}_{R,n}[w] - \mathcal{C}_{R,n}[\mathcal{F}v]\|_{\mathcal{L}^\infty(B_R)} \\ &\qquad \leq \frac{2}{\pi} \sum_{k=0}^{n-1} \|T_k\|_{\mathcal{L}^\infty([-R/r,R/r])} \|w - \mathcal{F}v\|_{\mathcal{L}^\infty(B_r)} \int\limits_{-r}^{r} \frac{|T_k(t/r)|}{\sqrt{r^2 - t^2}} dt \\ &\qquad \leq 2 \sum_{k=0}^{n-1} \left(\frac{2R}{r}\right)^k \delta \leq 2 \left(\frac{2R}{r}\right)^n \delta. \end{aligned}$$

This bound together with (7.6) implies Lemma 5.1.

References

[1] G. Alessandrini, Stable determination of conductivity by boundary measurements, *Applicable Analysis*, **27** (1988), 153–172.

[2] G. Bao, J. Lin, F. Triki, A multi-frequency inverse source problem, *Journal of Differential Equations*, **249** (2010), 3443–3465.

[3] L. Demanet, A. Townsend, Stable Extrapolation of Analytic Functions, *Foundations of Computational Mathematics*, **19** (2019), 297–331.

[4] P. Hähner, T. Hohage, New stability estimates for the inverse acoustic inhomogeneous medium problem and applications, *SIAM Journal on Mathematical Analysis*, **33**(3) (2001), 670–685.

[5] T. Hohage, F. Weidling, Variational source conditions and stability estimates for inverse electromagnetic medium scattering problems, *Inverse Problems and Imaging*, **11**(1) (2017), 203–220.

[6] M. Isaev, Energy and regularity dependent stability estimates for near-field inverse scattering in multidimensions, *Journal of Mathematics*, (2013), Article ID 318154, 10 p.; DOI:10.1155/2013/318154.

[7] M. Isaev, R.G. Novikov, New global stability estimates for monochromatic inverse acoustic scattering, *SIAM Journal on Mathematical Analysis*, **45**(3) (2013), 1495–1504.

[8] M. Isaev, R.G. Novikov, Hölder-logarithmic stability in the Fourier analysis, e-preprint arXiv:2005.01414.

[9] V. Isakov, Increasing stability for the Schrödinger potential from the Dirichlet-to-Neumann map, *Discrete and Continuous Dynamical Systems*, **4**(3) (2011), 631–640.

[10] M.M. Lavrent'ev, V.G. Romanov, S.P. Shishatskii, Ill-posed problems of mathematical physics and analysis, Translated from the Russian by J. R. Schulenberger. *Translations of Mathematical Monographs*, **64** (1986). American Mathematical Society, Providence, R.I., vi+290 pp.

[11] R.G. Novikov, New global stability estimates for the Gel'fand-Calderon inverse problem, *Inverse Problems*, **27** (2011), 015001, 21pp.

[12] R.G. Novikov, Multidimensional inverse scattering for the Schrödinger equation, e-preprint: https://hal.archives-ouvertes.fr/hal-02465839v1.

[13] M. Santacesaria, A Holder-logarithmic stability estimate for an inverse problem in two dimensions, *Journal of Inverse and Ill-posed Problems*, **23**(1) (2015), 51–73.

[14] V.K. Tuan, Stable analytic continuation using hypergeometric summation, *Inverse Problems*, **16**(1) (2000), 75–87.

[15] S. Vessella, A continuous dependence result in the analytic continuation problem, Forum Mathematicum, **11**(6) (1999), 695–703.