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A multi-modal competitive hub location pricing problem with customer loyalty and elastic demand

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Abstract

This paper develops a multi-modal competitive hub location pricing problem whose target is the design of a transportation system for a company that plans to enter into a market with elastic demand, in which an existing transportation company operates its hub-and-spoke network. The entrant company aims to attract customers in the market by convenient locations for its hubs and proper pricing of its transportation services, while customer loyalty is different in the nodes. Hence, mixed-integer programming based on a multi-nominal logit model is proposed. Thereafter, to solve the single allocation hub-and-spoke model, it is decomposed into a bi-level model. In the new structure, the master problem is associated with hub location and assignment decisions, and the sub-problem is associated with pricing decisions. Moreover, upper and lower bounds are calculated to determine the price of transportation routes. Finally, based on a nested approach, a scatter search algorithm is used to search the solution space of the master problem, and a matheuristic method is designed to solve the pricing problem interactively. The proposed approach is employed to solve a case study in the postal service industry of Iran.

Keywords: Pricing; Hub location; Multi-modal transportation; Elastic demand; Customer loyalty.

1. Introduction

The number of companies that offer similar services has been increasing in recent years. As a result, competition among companies has increased to earn maximum profit and market share. Based on these conditions, companies must pay attention to the proper pricing of their services and design appropriate transportation routes to fulfill their customers' demands. For example, until recently, postal services in Iran were provided by a monopoly operator (Iran Post Company). To improve the quality of service and decrease prices, the government decided to open the market for other operators. The government believes that this policy is necessary to

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ensure the development of effective competition. Based on this case, we assume in this study that a new transportation company intends to enter a monopoly market. The entrant tries to compete against the incumbent while having complete information about the location of the incumbent's hubs and spokes. The entrant applies optimal pricing and designs an appropriate network to obtain the maximum profit and market share. Therefore, to achieve this goal, the entrant has to consider the condition of a rival company (i.e., the incumbent) as well.

Hub location problems (HLPs) are an important research area of logistic problems. That is because a solution of these problems allows a decrease in the number of transportation links between origin and destination nodes. Moreover, it reduces overall transportation costs by consolidating traffic flows from various origins and transferring them to hubs with various destinations. HLPs are applied in many settings, such as airline industries (Aykin, 1994; Jaillet et al., 1996; Adler and Smilowitz, 2007), package delivery firms (Kuby and Gray, 1993), message delivery networks (Klincewicz, 1998), cargo industries (Taylor et al., 1995; Lumsden et al., 1999), telecom industries (Lee et al., 1996), emergency services and mobile post offices (Bashiri et al., 1996), and many other transportation systems.

Studies of various hub network models began in 1986. The first model of an HLP was introduced by O'Kelly (1986). Thereafter, O'Kelly (1987) introduced the first formulation of the HLP as an optimization problem. Generally, HLPs can be classified into four parts (Campbell, 1994): center, covering, fixed costs and median problems. Also, a hub median problem can be considered as a single allocation or multiple allocations (Ghaffarinasab et al., 2018). Alumur et al. (2012) presented a multi-modal hub location problem that jointly considered transportation costs and travel times. They studied decisions about how to design a hub network with various possible transportation modes. A complete literature review of HLPs can be seen in Alumur and Kara (2008), Campbell and O'Kelly (2012), Farahani et al. (2013), Contreras (2015).

Competition among firms that utilize hub networks is an interesting topic of research. The first model of a competitive hub location problem was introduced by Marianov, Serra and ReVelle (1999). Thereafter, related studies were carried out by Sasaki and Fukushima (2001), Adler and Smilowitz (2007), Eiselt and Marianov (2009), Gelareh, Nickel, and Pisinger (2010) and Sasaki et al. (2014). Lüer-Villagra and Marianov (2013) were the first to explicitly utilize a pricing policy in a competitive HLP. They studied a single-modal and multiple-allocation HLP in a competitive environment. They argued that the revenue of firms that are new entrants into competitive markets depend on network design and pricing strategy.

Competition in a market can be divided into three major types: static competition, dynamic competition, and competition with foresight (Farahani et al. 2014). The basic assumption in a static competition is that the existing rivals (i.e., incumbents) will not react to the entrance of a new competitor. However, the entrant should consider the effect of their rivals' activities. As in

the other types of competition, incumbents can react to the entrance of a new competitor by changes in their characteristics, such as pricing. In dynamic competition, competitors simultaneously determine their competitive factors. In competition with foresight, a competitor will react to an entrant's decisions sequentially. In the field of facility location, pricing has been studied for some time; however, to the best of our knowledge, there are a few studies in the case of hub location problems. Table 1 represents a brief overview of the most closely related studies.

Table 1. Brief overview of competitive hub location and pricing problems

Publications	Competition type			Research questions and main contributions
	Dynamic	With foresight	Static	
Lüer-Villagra and Marianov, (2013)			*	- Entrant's hub location and pricing problem - Proving an equation to determine the optimal price of each entrant's route - Developing a genetic algorithm to solve the hub location problem
Čvokić et al. (2016)		*		- Both entrant's and incumbent's hub locations under fixed markups - Reformulating the lower level problem (LLP) to develop a matheuristic to solve the hub location problem
Čvokić et al. (2017)	*	*		- Hub location and pricing decisions for both competitors with and without relaxing the pre-commitment in terms of pricing - Providing that in the leader-follower hub location and pricing competition, where competitors are allowed to change their prices, there is a profit-maximizing solution for the leader
Esmaeili, M., & Sedehzade, S. (2018)		*		- Modeling a CHLP problem based on Stackelberg-game - Demand of each firm depends on its price under the Bernard's model - A solution approach is proposed using an imperialist competitive algorithm (ICA) and a closed expression
Čvokić et al. (2019)		*		- Modeling an r/p hub-centroid problem under the price war - Finding a unique finite Bertrand-Nash price equilibrium for the follower
This paper			*	- Developing a mathematical formulation for a multi-modal CHLP problem. - Considering the effect of customer loyalty and elastic demand. - Decomposing the model and proposing two nested evolutionary algorithms based on scatter search and differential evolution to solve a single allocation hub-and-spoke network.

A specific amount of price change does not always have the same effect on differences in demand. That is because the demand for a product/service can be elastic or inelastic, depending on the rate of change in demand relative to the change in the price of a product/service. Demand is elastic in the market when the response of demand is greater than a small proportional change in the price. Inelastic demand occurs when there is relatively less change in demand with greater change in price. There are a few studies that consider the effect of elastic demand. Redondo et al. (2012) studied the effect of elastic demand on competitive location problems. They showed that the assumption of fixed demand significantly affects location decisions; therefore, the correct type of demand (elastic or inelastic) must be considered for modeling location problems. Kaveh et al. (2016) designed a hub network with elastic demand, in which the demand is dependent on a utility that is related to the location of the hubs. Later, Rahimi et al. (2019) and Kaveh et al. (2019) utilized this concept in their hub location models; however, they

did not separately consider the effect of the price on demand. Two different hub location models considering the price-sensitive demand were studied by O'Kelly et al. (2015) and Čvokić and Stanimirović (2020); however, they developed their models for a non-competitive market.

Another concept that can be useful for studying demand behavior in a market is customer loyalty. This occurs when customers prefer buying a specific product or using a specific shop, rather than buying products that are related to other brands or using other shops. Customers display such behavior if they purchase a specific brand or product consistently over an extended period. For example, many customers prefer to use a specific travel agent because of good experiences with their services (Tasci, 2017). To the best of our knowledge, customer loyalty has not yet been considered in location and pricing models.

In the present paper, we develop a multi-modal competitive hub location pricing problem (CHLP) considering customer loyalty and elastic demand. The main contributions of the paper are as follows:

- Developing a mathematical formulation for a multi-modal CHLP problem.
- Modeling elastic demand in a competitive market.
- Considering the effect of customer loyalty on the decision-making process.
- Deriving a closed-form expression for the pricing problem when demand is inelastic.
- Deriving an upper bound and a lower bound for the price of each route in the hub network when demand is elastic.
- Decomposing the proposed model to a hub location problem (i.e., master problem) and a pricing problem (i.e., sub-problem). Then, proposing two nested evolutionary algorithms to tackle the complexity of solving this structure.
- Developing a scatter search algorithm to solve the single allocation multi-modal HLP.
- Developing a matheuristic based on differential evolution algorithms to solve the pricing sub-problem.
- Investigating the characteristics of the model in a real-world example by exploring the sensitivity analyses.

The rest of this paper is organized as follows. Section 2 describes a generic mathematical model of the problem and two extended versions considering the effect of elastic demand and customer loyalty. In Section 3, the problem has been decomposed to a master and a sub-problem based on location and pricing decisions. Thereafter, two solution approaches have been proposed to tackle the problem with/without elastic demand and customer loyalty considerations. Computational experiments have been explained in Section 4. Finally, in Section 5 conclusions of the research have been provided.

2. Mathematical formulation

2.1. Problem definition

It is assumed that there is a situation where a new distribution company (entrant) intends to enter into a monopoly market. In this market, an existing company (incumbent) serves customers by a distribution network utilizing a hub-and-spoke topology. The entrant wants to design its service network and determine the related prices to compete with the incumbent's servicing system. Both the entrant and incumbent offer their own servicing time and price for each pair of origin and destination (OD).

The problem can be defined on a directed graph G= (N, A), where A is the set of admissible arcs and N is the set of nodes. Each node represents a customer zone (e.g., demand for a city or a region). To give a more clear representation of the model, we explain the developing process in three phases. First, we develop the proposed model by Lüer-Villagra and Marianov, (2013) adding the possibility of considering transportation modes. Moreover, to accommodate it with our case, a constraint is added to convert the model to a single-allocation HLP. This can be considered as a generic model of our case. Thereafter, we extend the generic model considering the effect of elastic demand and the effect of customer loyalty in subsections 2.3 and 2.4 respectively. To obtain insightful results, the following assumptions are made:

1) In this research, we focus on designing an uncapacitated single-allocation hub network to have more adapt with the conditions of our case study in postal services.
2) The hub nodes are not interconnected (Luer-Villagra and Marianov, 2013).
3) On each OD, the fraction of demands absorbed by each company can be predicted based on a utility function using a logit model. In the related literature, logit models are extensively used to accommodate several attributes to select some alternatives (Zambrano-Rey et al., 2019; Čvokić et al., 2019).
4) The customers have an overview of all service times and prices when they want to select their services. For example, the required information is presented on the website's online store of each company.

Notations for the model are defined as follows:

Sets:

N	Set of nodes ($i, j, k, l \in N$)
P	Set of the incumbent's hub nodes
M	Set of transportation modes ($m \in M$)

Parameters:

K_{ij}^{m}	Fixed cost of arc ij for transportation mode m.

d_{ij} Distance of arc ij

φ_{ij}^{m} Travel cost per unit of transportation mode m along arc ij

τ_{ij}^{m} Travel time per unit of transportation mode m along arc ij

σ_{kl} Discount factor associated with the hub-to-hub travel cost along arc kl

δ_{kl} discount factor associated with the hub-to-hub travel time along arc kl

F_k Cost of locating a hub at node k

D_{ij} Inelastic demand, the flow that should be sent from node i to node j

D'_{ij} Elastic demand, the flow that should be sent from node i to node j

D_{ij}^{max} Maximum possible demand, the maximum flow that can be sent from node i to node j

β_1 Sensitivity parameter of customers to service time

β_2 Sensitivity parameter of customers to service price

q_{ij} Incumbent's price to flow from node i to node j

t'_{ij} incumbent's time to flow from node i to node j

η_{ij} Total utility of the incumbent's servicing system from node i to node j

CLI_j Customer loyalty index for incumbent's services in node j $(0 \leq CLI_j \leq 1)$

Decision variables:

Y_k 1 if a hub at node k is located by the entrant; 0, otherwise

H_{ij}^{m} 1 if a direct connection by transportation mode m on arc ij $(i, j \in N)$ is established by the entrant; 0, otherwise

$P_{ij/kl}$ Entrant's price to service the demand along arc ij, using intermediate hubs k and l

$x_{ij/kl}$ Fraction of the demand flow along arc ij through the entrant's hubs that are located at $k, l \in N$

z_{ij} Fraction of the demand flow along arc ij through the incumbent's hubs

$c_{ij/kl}$ Variable cost of the flow between nodes i and j, using hubs $k, l \in N$

$T_{ij/kl}$ Response time for demand in node j from node i, using hubs $k, l \in N$

2.2. Generic model: Competitive Hub Location Pricing Problem with Transportation Modes

The competitive hub location pricing problem with transportation modes (CHLP-TM) can be formulated by:

$$Z = \text{Max} \sum_{i,j,k,l \in N} (P_{ij/kl} - c_{ij/kl})\, D_{ij}\, x_{ij/kl} - \sum_{(i,j) \in A, m \in M} K_{ij}^{m}\, H_{ij}^{m} - \sum_{k \in N} F_k Y_k \tag{1}$$

s.t.

$$x_{ij/kl} = \frac{Y_k . Y_l . \sum_{m_1 \in M} H_{ik}^{m_1} . \sum_{m_2 \in M} H_{kl}^{m_2} . \sum_{m_3 \in M} H_{lj}^{m_3} . \exp(-\beta_1 T_{ij/kl} - \beta_2 P_{ij/kl})}{\sum_{s,t \in N} Y_s . Y_t . \sum_{m_1 \in M} H_{ik}^{m_1} . \sum_{m_2 \in M} H_{kl}^{m_2} . \sum_{m_3 \in M} H_{lj}^{m_3} . \exp(-\beta_1 T_{ij/st} - \beta_2 P_{ij/st}) + \eta_{ij}} \quad \forall\, i, j, k, l \in N \tag{2}$$

$$\eta_{ij} = \exp\left(-\beta_1 t'_{ij} - \beta_2 q_{ij}\right) \quad \forall\, i, j \in N \tag{3}$$

$$c_{ij/kl} = \sum_{m \in M} \varphi_{ik}^{m}\, d_{ik}\, H_{ik}^{m} + \sum_{m \in M} \varphi_{kl}^{m}\, \sigma_{kl} d_{kl}\, H_{kl}^{m} + \sum_{m \in M} \varphi_{lj}^{m}\, d_{lj}\, H_{lj}^{m} \quad \forall\, i, j, k, l \in N \tag{4}$$

$$T_{ij/kl} = \sum_{m \in M} \tau_{ik}^{m}\, d_{ik}\, H_{ik}^{m} + \sum_{m \in M} \tau_{kl}^{m}\, \delta_{kl}\, d_{kl}\, H_{kl}^{m} + \sum_{m \in M} \tau_{lj}^{m}\, d_{lj}\, H_{lj}^{m} \quad \forall\, i, j, k, l \in N \tag{5}$$

$$\sum_{m \in M} H_{ij}^{m} \leq 1 \quad \forall\, i, j \in N \tag{6}$$

$$Y_k \in \{0, 1\} \quad \forall\, k \in N \tag{7}$$

$$H_{ij}^{m} \in \{0,1\} \qquad \forall\,(i,j) \in A, m \in M \qquad (8)$$

$$P_{ij/kl} \geq 0 \qquad \forall\, i,j,k,l \in N \qquad (9)$$

Objective function (1) maximizes the profit of the entrant's decisions. The first term of the objective function indicates the net profit of transportation services. The second and third terms indicate the fixed cost of transportation routes and the fixed cost of locating hubs, respectively. According to a logit model, Constraint (2) assigns the entrant's flows. Eq. (3) defines the utility of an incumbent's network for the demand that should be sent from origin i to destination j. Constraints (4) and (5) state the transportation costs and times over a route $i \to k \to l \to j$, respectively. Constraint (6) ensures that one transportation mode can be maximally assigned between every two nodes. Constraints (7) to (9) represent the domain of the decision variables.

Also, if we want to have a single-allocation hub-and-spoke network, Constraint (10) should be considered as well:

$$1 - Y_i \leq \sum_{\substack{j \in N, j \neq i \\ m \in M}} H_{ij}^{m} \leq 1 - Y_i + (|N| - 1).Y_i \qquad \forall\, i \in N \qquad (10)$$

2.3. Considering the effect of elastic demand

In the previous model (CHLP-TM), demand D_{ij} is assumed to be fixed at all demand points. Now, let us make a more realistic assumption that the demand at each node is affected by the price. There are different possible expressions for the expenditure function in the literature. As Berman and Krass (2002) proposed, elastic demand based on an exponential expenditure function is defined by:

$$W_i(U_i) = W_i^{min} + (W_i^{max} - W_i^{min}) \times (1 - \exp(-\rho_i.U_i)) \qquad (11)$$

where U_i is the utility of service i, and expenditure function $W_i(U_i)$ is a non-negative function of the utility vector, which is non-decreasing for all components of U_i. Also, W_i^{max} and W_i^{min} are the maximum and minimum demands of service i, respectively. $\rho_i > 0$ is a positive constant.

In Eq. (11), it is assumed that utility has an additive effect on demand. However, we assume that demand is dependent on price. Despite utility, the price has a decreasing effect on demand. Considering this point, we can customize Eq. (11) as follows:

$$D'_{ij} = D_{ij}^{min} + (D_{ij}^{max} - D_{ij}^{min}) \times (\exp(-\beta_2.\bar{P}_{ij})) \qquad (12)$$

where $\bar{P}_{ij}$ is the mean of the prices that can be proposed for the demand in node j from node i.

$$\bar{P}_{ij} = \frac{q_{ij} + \sum_{\substack{m_1,m_2,m_3 \in M \\ k,l \in N}} P_{ij/kl} Y_k Y_l H_{ik}^{m_1} H_{kl}^{m_2} H_{lj}^{m_3}}{1 + \sum_{\substack{m_1,m_2,m_3 \in M \\ k,l \in N}} Y_k Y_l H_{ik}^{m_1} H_{kl}^{m_2} H_{lj}^{m_3}} \qquad \forall\, i,j \in N \qquad (13)$$

Based on the assumption of single allocation of the hub network, there is just one unique route between each arbitrary pair of nodes *i* and *j*. Also, based on Eq. (10), the transportation mode for the mentioned route will be unique. So, the price for the paired nodes *i* and *j* will be unique.

$$\bar{P}_{ij} = \frac{q_{ij} + p_{ij}}{2} \qquad \forall\, i,j \in N \qquad (14)$$

Since the market was a monopoly before the presence of an entrant, we can assume that the inelastic demand (D_{ij}) in the previous model is related to the incumbent's price (q_{ij}). Therefore, the elastic demand (D'_{ij}) should consider the following conditions:

$$D'_{ij} = \begin{cases} D_{ij}^{max} & \text{if } \bar{P}_{ij} \to 0 \\ D_{ij} & \text{if } \bar{P}_{ij} = q_{ij} \\ D_{ij}^{min} & \text{if } \bar{P}_{ij} \to +\infty \end{cases} \qquad (15)$$

It is also assumed that $D_{ij}^{min} = 0$. Considering $\bar{P}_{ij} = q_{ij}$ and replacing the second condition of (15) in (12), we will have:

$$D_{ij} = D_{ij}^{max} \times \exp(-\beta_2 . q_{ij}) \quad \Rightarrow \quad D_{ij}^{max} = D_{ij} \times \exp(\beta_2 . q_{ij}) \qquad (16)$$

Finally, we can calculate the elastic demand by:

$$D'_{ij} = D_{ij} \,.\exp(\,\beta_2(q_{ij} - \bar{P}_{ij})) \qquad (17)$$

The competitive hub location and pricing problem with elastic demand and transportation modes (CHLP-ED/TM) problem can be formulated by:

$$Z = max \sum_{i,j,k,l \in N} (P_{ij/kl} - c_{ij/kl})\, D'_{ij}\, x_{ij/kl} - \sum_{(i,j)\in A, m\in M} K_{ij}^{m}\, H_{ij}^{m} - \sum_{k\in N} F_k Y_k \qquad (18)$$

s.t.

Constraints (2) – (9), (12) and (13).

Moreover, if we want to have a single-allocation network in the CHLP-ED/TM, Constraints (12) and (13) should respectively be replaced by Constraints (17) and (14), and Constraint (10) should be added to the recent model.

2.4. Considering the effect of customer loyalty

Customer loyalty is the result of an affirmative perceived value of an experience, physical attribute-based satisfaction, or a consistently positive emotional experience that contains

products or services. In this model, customer loyalty has been considered in determining the market share of the entrant and incumbent. Hence, we characterize CLI_i as the loyalty index of the customer situated in node i to the incumbent's services. Therefore, we can modify Eq. (2) to Eq. (19).

$$x_{ij/kl} = \frac{(1 - CLI_i).\left(\sum_{m_1,m_2,m_3 \in M} Y_k Y_l \, H_{ik}^{m_1} \, H_{kl}^{m_2} \, H_{lj}^{m_3}\right).\exp(-\beta_1 \, T_{ij/kl} - \beta_2 \, P_{ij/kl})}{(1 - CLI_i).\sum_{\substack{m_1,m_2,m_3 \in M \\ s,t \in N}} Y_s Y_t \, H_{is}^{m_1} \, H_{st}^{m_2} \, H_{tj}^{m_3} \, .\exp(-\beta_1 \, T_{ij/st} - \beta_2 \, P_{ij/st}) + CLI_i.\eta_{ij}} \tag{19}$$

Note that if $CLI_i = 0.5$, Eq. (19) will be returned to Eq. (2) again. That is while for $CLI_i < 0.5$, where the customer has low satisfaction to the incumbent services, more flows of demand can be absorbed by the entrant. On the other hand for $CLI_i > 0.5$, the entrant hardly can increase its market share because of the customer's loyalty to the incumbent services.

Finally, to model the competitive hub location and pricing problem considering elastic demand and customer loyalty (CHLP-ED/CL/TM), we can reformulate (18) subject to Eqs. (3) – (9), (13), (17) and (19).

3. Solution approach

The mathematical models proposed in the previous section are NP-hard. Even without pricing decisions, solution of a hub location problem is very complex. Therefore, considering pricing decisions and related nonlinear terms in the model increases problem complexity. The literature has proposed some hybrid approaches to solve such models. Lüer-Villagra and Marianov (2013) decomposed their proposed competitive hub location pricing model to a master problem and a sub-problem with a bi-level structure. The master problem was related to hub-and-spoke location decisions, and the sub-problem was related to pricing decisions based on a multinomial logit (MNL) function. Consequently, they proposed a hybrid solution approach based on a genetic algorithm (GA) to solve a single-level reduced model. Zhang (2015) presented a competitive location and pricing model for a retailer based on an MNL function, and proposed a two-phase solution framework based on decomposition that contains two major components, location and pricing problems. They examined the performance of three pricing heuristics, including a path-following approach (PF), a gradient descent method (GD), and a gradient descent method with multiple random starting points (GDM), and three location heuristics, including a tabu search procedure (TS), greedy search (GS), and GA. Based on the computational experiments, the two hybrid approaches (i.e., GA+PF and TS+PF) outperformed the other approaches.

Bi-level decomposition algorithms are extensively used to solve certain nonconvex large-scale optimization problems. Based on the literature, decomposing the location and pricing model to a bi-level structure and interactively solving these sub-problems are the main

alternative for tackling such problems. As a result, we can transform the proposed hub location and pricing models into a bi-level structure in which a pricing problem serves as a sub-routine for the location problem. This structure is represented in Fig. 1.

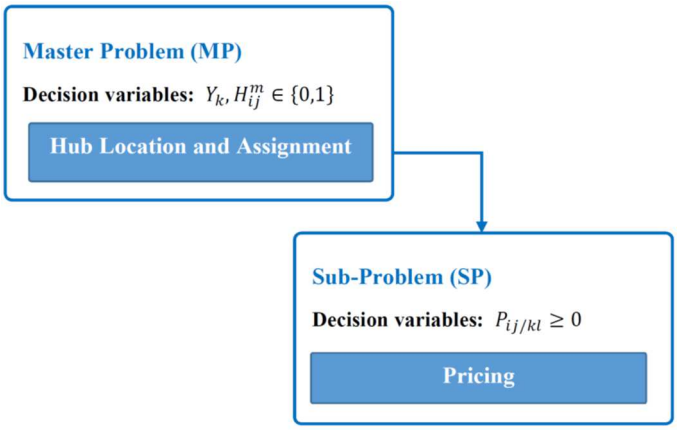


Fig. 1. Decomposition of the CHLP problem

Accordingly, we are faced with a bi-level model. There are various approaches to solving bi-level problems. For a complete literature review on bi-level optimization problems (BOPs), see Sinha et al. (2017). These authors reviewed the classical and evolutionary approaches to solving BOPs. They emphasized evolutionary methods because of a high degree of difficulty in real-world applications that usually lead to failure of the classical methods for solving BOPs. Hence, using heuristic/meta-heuristic and classical methods simultaneously (in a nested approach) is the main alternative for tackling BOPs (Talbi, 2013). Descriptions of some successful applications of the mentioned approach for solving multi-level problems can be found in (Parvasi et al., 2017; Akbari-Jafarabadi et al., 2017; Fard & Hajaghaei-Keshteli, 2018a; Fard & Hajaghaei-Keshteli, 2018b).

In the next section, a scatter search (SS) algorithm for the hub location problem is designed. Because of different concavity conditions in the pricing problems of the CHLP-TM and CHLP-ED/CL/TM, we discuss them separately, while the proposed approach to solve the hub location problem is same. Hence, an exact method for the pricing problem of the CHLP-TM and a matheuristic approach (named EGPSDE) for the pricing problem of CHLP-ED/TM and CHLP-ED/CL/TM are suggested. Table 2 presents a general view of the combination of the proposed approaches to solve the problem.

Table 2. Combination of the proposed solving approaches

Model types	Master problem (Hub location and assignment)	Sub-problem (Pricing)
CHLP-TM	Scatter search	Extracted closed-form expression
CHLP-ED/TM	Scatter search	Metaheuristic (EGPSDE)

3.1. Solution approach for the CHLP-TM model

3.1.1. Theoretical discussion

For each solution generated in the master problem (i.e., the values $\{\hat{Y}_k\}_{k\in N}$ and $\{\hat{H}_{ij}^m\}_{(i,j)\in A}$), we can define S_{ij} as the set of feasible pairs of hubs (k,l) that can connect the origin-destination (O-D) pair (i,j), that is:

$$S_{ij} = \left\{(k,l) \in N^2 \middle| \exists\, m_1, m_2, m_3 \in M, \hat{Y}_k = \hat{Y}_l = \hat{H}_{ik}^{m_1} = \hat{H}_{kl}^{m_2} = \hat{H}_{lj}^{m_3} = 1\right\} \tag{20}$$

Substituting Eq. (2) in Eq. (1), and using Eq. (20), the objective function of the entrant's pricing problem in the sub-problem is:

$$\text{Max}_P\, f(\hat{Y}, \hat{H}, P) = \sum_{i,j\in N} \frac{D_{ij} \sum_{(k,l)\in S_{ij}} (P_{ij/kl} - c_{ij/kl}) \exp(-\beta_1 T_{ij/kl} - \beta_2 P_{ij/kl})}{\sum_{(s,t)\in S_{ij}} \exp(-\beta_1 T_{ij/st} - \beta_2 P_{ij/st}) + \eta_{ij}} - \omega \tag{21}$$

where ω is a constant value that can be obtained from Eq. (22).

$$\omega = \sum_{(i,j)\in A, m\in M} K_{ij}^m \hat{H}_{ij}^m + \sum_{k\in N} F_k \hat{Y}_k \tag{22}$$

The entrant's optimal prices can be derived based on the first-order conditions that are represented in the following proposition.

Proposition 1. The optimal price of the entrant's service for each route $i \to k \to l \to j$ can be obtained from Eq. (23).

$$P_{ij/kl}^* = c_{ij/kl} + \frac{1}{\beta_2}\left\{1 + W_0\left[\frac{1}{\eta_{ij}} \sum_{(s,t)\in S_{ij}} \exp(-\beta_1 T_{ij/st} - \beta_2 c_{ij/st} - 1)\right]\right\} \tag{23}$$

where W_0 is the principal branch of the Lambert W function, which is defined as the inverse function of $f(\zeta) = \zeta e^{\zeta}$.

Proof. Lüer-Villagra and Marianov (2013) derived a closed-form expression for optimal pricing in the case of multiple bundles (i.e., hub pairs) with a single attribute (i.e., price) in the MNL utility function. Our proof and formula are an extension of the case of a multi-attribute utility function considering an extra attribute (i.e., non-price option). The objective function (21) can be separated in some independent expressions for each origin-destination pair (i,j). Hence, the

expression (24) for a special route $i \to s \to t \to j$ can be obtained based on using the first-order conditions $\frac{\partial \hat{Z}}{\partial P_{ij/kl}} = 0;\ \forall\, i,j,k,l \in N$.

$$\left[\sum_{(k,l)\in S_{ij}} \exp(-\beta_1\, T_{ij/kl} - \beta_2\, P_{ij/kl}) + \eta_{ij}\right].(1 - \beta_2(P_{ij/st} - c_{ij/st})) + \beta_2\left[\sum_{(k,l)\in S_{ij}} (P_{ij/kl} - c_{ij/kl}) \exp(-\beta_1\, T_{ij/kl} - \beta_2\, P_{ij/kl})\right] = 0 \quad (24)$$

Thereafter, we can consider another equivalent expression for another arbitrary route between origin node i and destination node j, such as $i \to u \to v \to j$. Then, we divide both expressions by β_2 and subtract them to obtain Eq. (25).

$$(P_{ij/uv} - c_{ij/uv} - P_{ij/kl} + c_{ij/kl})\,\eta_{ij} = 0 \quad (25)$$

Since η_{ij} in Eq. (25) is non-negative, the expression in parenthesis should be zero. In another way, if there are multiple optimal routes for the O-D pair (i,j), the margins $P_{ij/**} - c_{ij/**}$ will be equal. Let us define r_{ij} and Q_{ij} as:

$$r_{ij} = P_{ij/kl} - c_{ij/kl} \quad (26)$$

$$Q_{ij} = \sum_{(k,l)\in S_{ij}} \exp(-\beta_1\, T_{ij/kl} - \beta_2\, c_{ij/kl}) \quad (27)$$

Substituting r_{ij} and Q_{ij} in Eq. (24), we obtain:

$$\frac{Q_{ij}\exp(-1)}{\eta_{ij}} = (-1 + \beta_2\, r_{ij})\exp(-1 + \beta_2\, r_{ij}) \quad (28)$$

The Lambert function $W(z)$ can be defined such that $z = W(z)\exp(W(z))$. Let $z_{ij} = \frac{Q_{ij}\exp(-1)}{\eta_{ij}}$ and $(z_{ij}) = -1 + \beta_2\, r_{ij}$. Therefore, we have $-1 + \beta_2\, r_{ij} = W_0\left(\frac{Q_{ij}\exp(-1)}{\eta_{ij}}\right)$ and:

$$r_{ij} = \frac{1}{\beta_2}\left[1 + W_0\left(\frac{Q_{ij}\exp(-1)}{\eta_{ij}}\right)\right] \quad (29)$$

Substituting r_{ij} in Eq. (26), we can obtain the optimal value of $P_{ij/kl}$ as Eq. (23). □

Eventually, based on Proposition 1, the optimum value of the prices can be replaced by the sub-problem. Hence, reducing the bi-level structure to a single-level model, the CHLP-TM problem can be reformulated by:

$$Z = max_{Y,H} \sum_{i,j,k,l\in N} (P^*_{ij/kl} - c_{ij/kl})\, D_{ij}\, x_{ij/kl} - \sum_{(i,j)\in A, m\in M} K^m_{ij}\, H^m_{ij} - \sum_{k\in N} F_k Y_k \quad (30)$$

s.t.

Constraints (2) – (9) and (23).

Therefore, considering the binary variables of the resulting model, an appropriate binary search can be utilized to solve CHLP-TM.

3.1.2. Developing scatter search for the hub location problem

In this section, scatter search (SS) is presented to solve the CHLP-TM. Also, the presented algorithm can be utilized to solve the upper level of the CHLP-ED/CL/TM. Scatter search is a population-based meta-heuristic method. It has shown high-quality outcomes for combinatorial optimization problems to date. The search strategy of SS is based on combining the solution vectors that have proven effective in a variety of problem settings. It is worth noting that in the literature of meta-heuristics, obtaining good solutions is significantly dependent on designing the algorithm based on the problem at hand. In this regard, SS is specifically proposed by Marti et al. (2015) for an uncapacitated *p*-hub median problem. Based on the algorithm proposed by these authors, we design an SS algorithm for our model. Since there are some differences between the two models, we needed to improve some steps of the algorithm for matching. The general scheme of the proposed scatter search algorithm is presented in Fig. 2. Moreover, the details of main steps of the algorithm are represented as follows.

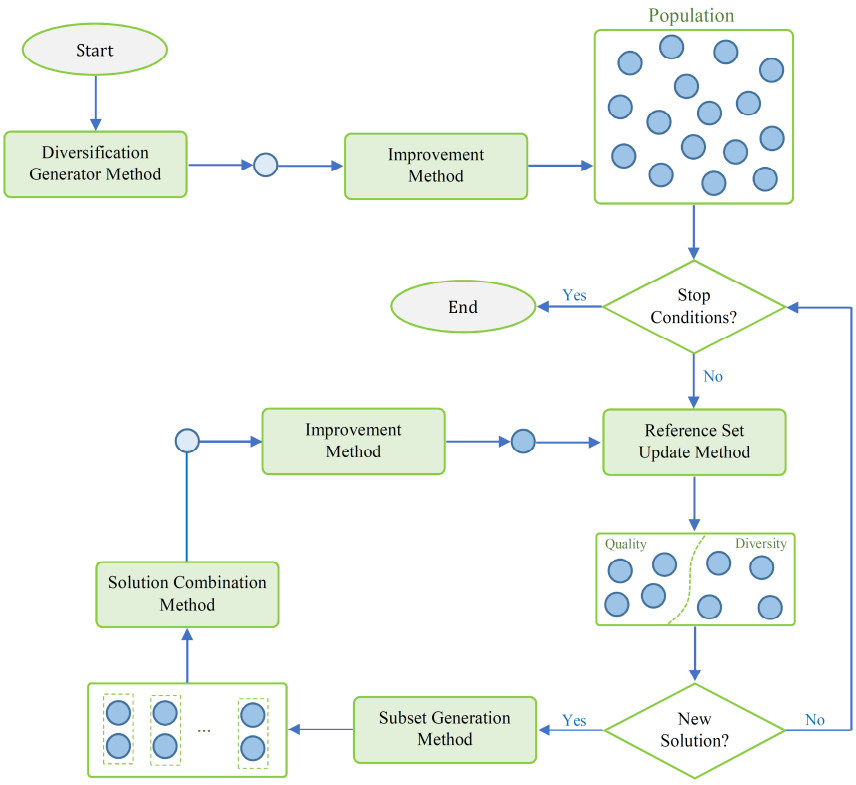


Fig. 2. General scheme of the proposed scatter search algorithm

Step 1. Diversification generator method (DGM): It generates an initial population (i.e., π feasible solutions) for the problem. Indeed, this method is a mechanism to generate the first generation of solutions that consider a balance between quality and diversity. An appropriate diversification generator for this problem is described in Marti et al. (2015); however, we needed to consider the differences between the two problems. These included multi-modal transportation, a single-allocation hub network, fixed cost of arcs, cost of locating a hub, and uncertainty regarding the number of hubs.

Step 1.1. Hub location methodology: To create an initial population, we develop seven different DGMs for the CHLP-TM. Six are based on greedy randomized adaptive search procedures (GRASP), and the seventh one is a random construction to provide diversity in an initial population. Indeed, each solution is a binary string ($Y_{1\times|N|}$) including p-hub nodes such that p is a random integer parameter in an interval $[\lceil\psi_{min}\times|N|\rceil,\lceil\psi_{max}\times|N|\rceil]$ where $0\leq\psi_{min}<\psi_{max}\leq 1$ are constant coefficients. These bounds help to accelerate the search process.

Construction of a solution in the greedy approaches is based on a greedy function g that evaluates the attractiveness of solutions. Let $h\in N$ be a candidate node for the hub location. If h is a hub, it will be used for transportation of loads among the other nodes, possibly the r terminals $i_1, i_2, \dots, i_r$ with a lower allocation cost to h, where r is a given parameter. Then, we compute $g(h)$ by:

$$g(h)=\sum_{s=1}^{r} Cost(i_s,h) \qquad \forall\, h\in N \qquad (31)$$

where $Cost(i,h)$ indicates the assignment cost of node i to hub h. Three different approaches are proposed to calculate $Cost(i,h)$. The three types of proposed cost functions, combined with the two GRASP designs (the sampled greedy and the semi-greedy), are utilized to generate the initial population (Marti et al., 2015).

$$Cost_1(i,h)=\overrightarrow{flow}_i\times\bar{\varphi}_{ih}+\bar{K}_{ih}+F_h \qquad (32)$$

$$Cost_2(i,h)=\overrightarrow{flow}_i\times\bar{\varphi}_{ih}+\overleftarrow{flow}_i\times\bar{\varphi}_{hi}+\bar{K}_{ih}+F_h \qquad (33)$$

$$Cost_3(i,h)=\overrightarrow{flow}_i\times\bar{\varphi}_{ih}+\left(\rho_1\left(1+\frac{\bar{\delta}_i}{2}\right)+\rho_2\left(1+\frac{\bar{\sigma}_i}{2}\right)\right)\overleftarrow{flow}_i\,\bar{\varphi}_{hi}+\bar{K}_{ih}+F_h \qquad (34)$$

where $\overrightarrow{flow}_i=\sum_{j\in N,j\neq i}d_{ij}$, is the sum of all the flows from node i to all terminals j. Also, $\overleftarrow{flow}_i=\sum_{j\in N,j\neq i}d_{ji}$ is the sum of all the flows from all nodes j to terminal i. Moreover, $\bar{K}_{ij}$, $\bar{\varphi}_{ij}$ and $\bar{\tau}_{ij}$ are the mean value of the various transportation parameters represented in Eqs. (35) – (37), respectively. Furthermore, the ot her parameters used in Eqs. (35) – (37) are represented in Eqs. (38) – (41).

$$\bar{K}_{ij} = \frac{\sum_{m\in M} K_{ij}^{m}}{|M|} \tag{35}$$

$$\bar{\varphi}_{ij} = \frac{\sum_{m\in M} \varphi_{ij}^{m}}{|M|} \tag{36}$$

$$\bar{\tau}_{ij} = \frac{\sum_{m\in M} \tau_{ij}^{m}}{|M|} \tag{37}$$

$$\bar{\delta}_{i} = \frac{\sum_{j\in N} \delta_{ij}}{|N| - 1} \tag{38}$$

$$\bar{\sigma}_{i} = \frac{\sum_{j\in N} \sigma_{ij}}{|N| - 1} \tag{39}$$

$$\rho_1 = \frac{\beta_1}{\beta_1 + \beta_2} \tag{40}$$

$$\rho_2 = \frac{\beta_2}{\beta_1 + \beta_2} \tag{41}$$

Step 1.2. Spoke allocation methodology: Once p hubs are selected for a solution, the allocation phase can begin. Unlike Marti et al. (2015), we propose a procedure that operates just one time, at the beginning of the algorithm execution. In this procedure, called the random weighted matrix (RWM) method, each node is assigned to a hub based on a random weighted matrix. In RWM, first, a weight matrix should be calculated based on the problem parameters. Then, the weight matrix ($H_{weights}$) multiplied by a same-dimension random matrix (H_{rand}) generates a random weighted matrix (H_{real}) for each solution.

$$H_{real}(i,h) = H_{rand}(i,h) \times H_{weights}(i,h) \qquad \forall\, i,h \in N \tag{42}$$

The procedure for calculating the weight matrix is represented in Eqs. (43) – (47).

$$Alloc_{cost}(i,h) = \bar{\varphi}_{ih} \times \overrightarrow{flow}_i + \sum_{j\in N} d_{ij}\, \bar{\varphi}_{hj} + \bar{K}_{ih} \qquad \forall\, i,h \in N \tag{43}$$

$$Alloc_{time}(i,h) = \bar{\tau}_{ih} \times \overrightarrow{flow}_i + \sum_{j\in N} d_{ij}\, \bar{\tau}_{hj} \qquad \forall\, i,h \in N \tag{44}$$

$$norm_{cost}(i,h) = \frac{\min_{h'\in N} Alloc_{cost}(i,h')}{Alloc_{cost}(i,h)} \qquad \forall\, i,h \in N \tag{45}$$

$$norm_{time}(i,h) = \frac{\min_{h'\in N} Alloc_{time}(i,h')}{Alloc_{time}(i,h)} \qquad \forall\, i,h \in N \tag{46}$$

$$H_{weights}(i,h) = \rho_1 \times norm_{time}(i,h) + \rho_2 \times norm_{cost}(i,h) \qquad \forall\, i,h \in N \tag{47}$$

Given a solution for the hub location, the allocation phase can be implemented based on H_{real}. Mohammadi et al. (2016) utilized such an approach, based on a random matrix instead of H_{real}. In this approach, the maximum value among the intersection arrays of each spoke row with the columns corresponding to the located hubs should be selected. An example is shown in Fig. 3. Eventually, allocation matrix (H) can be extracted as Fig. 4a. As shown in the extracted matrix, the allocation of the hub nodes to each other should also be considered.

		1	2	3	4	5	6
Location of hubs		0	1	0	0	0	1
Spoke	1	0	**0.362**	0.104	0.216	0.317	0.007
Hub	2	0.391	0	0.111	0.305	0.022	0.182
Spoke	3	0.153	**0.106**	0	0.192	0.304	0.016
Spoke	4	0.103	0.023	0.261	0	0.219	**0.117**
Spoke	5	0.311	**0.202**	0.228	0.153	0	0.196
Hub	6	0.251	0.103	0.018	0.114	0.231	0

Fig. 3. Allocation scheme (H_{real})

Step 1.3. Transportation mode selection: After allocation of spokes, a transportation mode on each generated link should be selected. To make this selection, a random matrix (H'_{real}) with the same dimension of H should be generated. Then, each bit of the matrix is multiplied by the number of transportation modes ($|M|$) and then rounded up. Finally, the transportation mode matrix (H') can be extracted by a bit-by-bit multiplication of two given matrices. For example, considering the mentioned instance in Fig. 3, a random selection of transportation modes is illustrated in Fig. 4.

Thereafter, the objective function value of each random solution should be calculated based on Eq. (30). It is worth noting that to calculate the value of $c_{ij/kl}$ and $T_{ij/kl}$, the situation where the following route $i \to k \to l \to j$ starts from/ends to a hub node should be considered because in such situations, the value of the origin-to-hub ($i \to k$) or hub-to-destination ($l \to j$) arc should be discounted. For example, based on the illustrated instance in Fig. 4a, the flow that should be sent from spoke node 1 to the hub node 6 has a route as $1 \to 2 \to 2 \to 6$. Hence, the hub-to-destination arc $2 \to 6$ should be discounted.

	1	2	3	4	5	6
1	0	1	0	0	0	0
2		0	1	0	1	**1**
3			0	0	0	0
4				0	0	1
5					0	0
6						0

a. Hub-and-spoke allocation matrix (H)

	1	2	3	4	5	6
1	0	**0.921**	0.271	0.381	0.302	0.754
2		0	**0.419**	0.850	**0.443**	**0.046**
3			0	0.736	0.159	0.227
4				0	0.106	**0.618**
5					0	0.901
6						0

b. A random matrix for transportation mode selection (H'_{real})

	1	2	3	4	5	6
1	0	2	0	0	0	0
2		0	1	0	1	1
3			0	0	0	0
4				0	0	2
5					0	0
6						0

c. Transportation mode matrix (H')

Fig. 4. Illustration of generating a random solution for transportation modes ($|N| = 6, p = 2, |M| = 2$)

Step 2. Reference set construction method: In this step, we need to construct a reference set ($RefSet$) by selecting only β solutions from the initial population. To construct the *RefSet*, the quality and diversity of the selected solutions should be considered (Laguna and Marti, 2012). Hence, at most $\beta/2$ of the *RefSet* members will be selected based on the quality criterion. For this purpose, we first order the population in descending order of their objective function values. Then, the ordered solutions are introduced one by one. If there is no other solution that has already been introduced with the same objective function value, a newly introduced solution can be added to the $RefSet$. The selection process continues until 50% of the members of the population have been examined, or $\beta/2$ qualified solutions are achieved. The rest of the members of *RefSet* will be selected from the rest of the initial population, based on the diversity criterion. This process tries to select those members of a population ($s \notin$ RefSet) that differ most from the selected solutions in the reference set ($t \in RefSet$). Therefore, the selected solution (s^*) should satisfy the following condition:

$$distance(s^*, \text{RefSet}) = Max_{s \notin \text{RefSet}} \{Min_{t \in \text{RefSet}}\, d_H(s,t)\} \tag{48}$$

where $d_H(s,t)$ is the number of non-common hubs in solutions s and t. In other words, $d_H(s,t) = p - |C|$ when $C = \{h: h \in H_s \cap H_t\}$. Also, if there is a possible tie when more than one solution takes the same maximum distance, we can use another distance criterion (d_A). This distance is based on the number of nodes allocated to the hubs in C. Let T_s^h be the set of arcs defined to connect the terminal nodes of solution s to the hubs in C. Therefore, the selected solution (s^*) should satisfy the following condition:

$$distance(s^*, \text{RefSet}) = Max_{s \notin \text{RefSet}} \{Min_{t \in \text{RefSet}}\, d_A(s,t)\} \tag{49}$$

where $d_A(s,t) = Min_{h \in C} \left|T_s^h \cap T_t^h\right|$ and $\left|T_s^h \cap T_t^h\right|$ is the number of assignments in common to hubs in both solutions. Finally, if there is still a tie, we can choose one of the solutions randomly.

Step 3. Subset generation method (SGM): Once the *RefSet* is constructed, it should be defined as a selection approach for choosing the current solutions, aiming at generating new ones. Therefore, we define an SGM to produce different subsets $X \subset RefSet$ that will be utilized for creating structured combinations in Step 4. The SGM is typically designed to generate all 2-element subsets from the *RefSet*. Then, for each subset, a fitness index should be calculated. The

fitness index is the mean fitness of the two individuals that are situated in each subset. This index will be used by a roulette wheel strategy to select the subsets.

Step 4. Solution combination method (SCM): It operates as an effective element of SS. In this paper, we borrow some GA operators (e.g., crossover and mutation) used by Lüer-Villagra and Marianov (2013), who considered two row-and-column-based 1-point crossover strategies. Moreover, a mutation strategy is utilized to improve the exploration aspect of their proposed algorithm. Here, we utilize their approaches with some minor changes. The operators should be applied on the binary vector of hub-and-spoke locations (Y) and the matrices with real values to assign the spokes and indicate the transportation modes (i.e. H_{real} and H'_{real}).

Step 5. Reference set update method: The new individuals constructed in the previous step are considered for membership in the reference set (i.e., *RefSet*). An individual may become a member of the *RefSet* if its fitness is better than the fitness of any of the individuals in the high-quality subset (i.e., $\beta/2$ of more qualified solutions in the *RefSet*). Alternatively, if a new individual improves the diversity of the *RefSet*, it can replace one that is currently in the diverse subset.

Step 6. Improvement method: Generally, there are four decisions that should be made in the problem. While the pricing decision is made in the sub-problem based on one of the presented approaches in Table 2, the rest three decisions are determined based on the methods presented in Step 1. To improve the achieved random solutions, three local search approaches are considered in the SS algorithm. For hub location and assignment decisions, we utilize the local search approaches proposed by Marti et al., 2015. Also, we propose a simple local search to seek the other kinds of transportation modes for each achieved solution. In this method, the transportation mode of a random number of routes are changed to seek for a possible improvement.

3.2. Solution approach for CHLP-ED/CL/TM

3.2.1. Theoretical discussions of CHLP-ED/CL/TM

Here, based on Propositions (2) and (3), upper and lower bounds of the pricing decision variables are presented. These values will facilitate the search process for the pricing problem.

Proposition 2. The lower bound of the entrant's prices for each route can be calculated by:

$$LB\left(P_{ijkl}\right) = c_{ijkl} + \frac{2}{3\,\beta_2} \qquad \forall\, i,j \in N\,,(k,l) \in S_{ij} \qquad (50)$$

Proof. Using the first-order conditions that are presented in Appendix A, the sign of the derivative is related to a term that is represented by:

$$\psi = (1 - CLI_i)\exp\left(-\beta_1 T_{ijkl} - \beta_2 P_{ijkl}\right)\left(1 - \frac{\beta_2}{2}\left(P_{ijst} - c_{ijst}\right)\right) + CLI_i\,\eta_{ij}\left(1 - \frac{3\beta_2}{2}\left(P_{ijst} - c_{ijst}\right)\right) \tag{51}$$

All of the terms in ψ are always non-negative except $\left(1 - \frac{\beta_2}{2}\left(P_{ijst} - c_{ijst}\right)\right)$ and $\left(1 - \frac{3\beta_2}{2}\left(P_{ijst} - c_{ijst}\right)\right)$. So, the sign of these terms affects the behavior of the first derivative. Therefore, the objective function is strictly increasing when $\psi > 0$.

$$\left.\begin{aligned} 1 - \frac{\beta_2}{2}\left(P_{ijst} - c_{ijst}\right) > 0 \;\Rightarrow\; P_{ijst} < c_{ijst} + \frac{2}{\beta_2} \\ 1 - \frac{3\beta_2}{2}\left(P_{ijst} - c_{ijst}\right) > 0 \;\Rightarrow\; P_{ijst} < c_{ijst} + \frac{2}{3\,\beta_2} \end{aligned}\right\} \Rightarrow\; P_{ijst} < c_{ijst} + \frac{2}{3\,\beta_2} \tag{52}$$

Until the condition of Eq. (52) is considered for all of the prices, the objective function is strictly increasing, which means that the entrant can increase its benefits by increasing the prices. □

Proposition 3. The upper bound of the entrant's prices for each route can be calculated by:

$$UB\left(P_{ijkl}\right) = c_{ijkl} + \frac{2}{\beta_2} \qquad \forall\, i,j \in N\,,(k,l) \in S_{ij} \tag{53}$$

Proof. Similarly, using the first-order conditions, the objective function is strictly decreasing when $\psi < 0$.

$$\left.\begin{aligned} 1 - \frac{\beta_2}{2}\left(P_{ijst} - c_{ijst}\right) < 0 \;\Rightarrow\; P_{ijst} > c_{ijst} + \frac{2}{\beta_2} \\ 1 - \frac{3\beta_2}{2}\left(P_{ijst} - c_{ijst}\right) < 0 \;\Rightarrow\; P_{ijst} > c_{ijst} + \frac{2}{3\,\beta_2} \end{aligned}\right\} \Rightarrow\; P_{ijst} > c_{ijst} + \frac{2}{\beta_2} \tag{54}$$

Until the condition of Eq. (54) is considered for all of the prices, the objective function is strictly decreasing, which means that if the entrant increases the prices, its benefits are sure to decrease. □

Also, when P_{ijst} is located in interval $\left[c_{ijst} + \frac{2}{3\,\beta_2}, c_{ijst} + \frac{2}{\beta_2}\right]$, the first and second terms of Eq. (51) are positive and negative, respectively. At any rate, the sign of ψ is related to the values of the other factors. Therefore, to search the optimal pricing strategy, it is sufficient that the solution space be limited based on the mentioned intervals.

3.2.2. Developing a matheuristic-based differential evolution algorithm for the pricing problem

The pricing sub-problem in the CHLP-ED/CL/TM model is non-linear, and we cannot ensure the concavity or convexity of the objective function. For this reason, and because of the high degree of complexity of the model, finding an optimal solution of the pricing problem using commercial software packages, even in instances with small or medium size, cannot be guaranteed. Therefore, we propose a matheuristic-based differential evolution (DE) algorithm and the mathematical characteristics of the pricing problem.

A matheuristic algorithm, which is a combination of mathematical programming and meta-heuristic approaches, is a growing field in operations research. For some recent studies, see (e.g., Aksen and Aras, 2013; Grangier et al., 2017; and Ghaffarinasab, 2018; Jelodari-Mamaghani et al., 2020). Fig. 5 represents a classification of matheuristics proposed by Sinha et al. (2017). Here, we propose an integrative combination that incorporates an exact algorithm (i.e., gradient search) in a popular and efficient variant of DE.

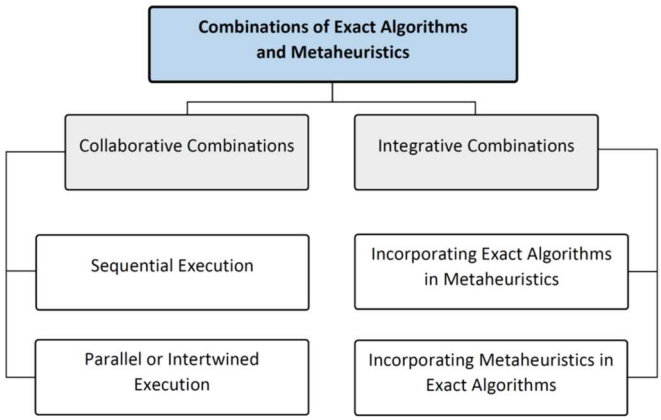


Fig. 5. Major structural classification of matheuristics/exact combinations

Differential evolution is a powerful metaheuristic algorithm, especially in the field of continuous optimization problems, as proposed by Storn and Price (1997). The satisfactory performance of DE in terms of convergence speed, accuracy, and robustness makes its application attractive in various real-world optimization problems (Das and Suganthan, 2011). Moreover, we can point out some other reasons to select DE in this case. The DE-based approach has also been successfully applied in the field of bi-level optimization problems. Zhu et al. (2006) utilized a nested approach based on DE to solve a nonlinear bi-level programming model with linear constraints. They used DE to search the feasible region of ULP and the interior point algorithm to LLP. Similarly, Angelo et al. (2013) combined two different specialized DE to handle both levels of a bi-level model. Besides, Angelo et al. (2015) studied a bi-level transportation routing problem and proposed a nested approach based on two intelligent

heuristics, namely ant colony optimization (ACO) to solve a routing problem in the upper-level problem (ULP), and DE to solve a transportation problem in the lower-level problem (LLP).

Wang et al. (2017) proposed a hierarchical modified differential evolution (HMDE) algorithm to make pricing decisions in a multi-product vendor-buyer supply chain. They applied a bi-level mathematical model to specify optimal wholesale and retail prices, advertising expenditures, and buyer's and vendor's ordering policies, considering environmental improvement. They utilized the HMDE to handle both levels of the proposed bi-level model.

Also, to enhance DE performance, several variants have been proposed in recent years. These variants are reviewed by Wu et al. (2018). The approach proposed in the present paper is based on EPSDE, a combination of mutation strategies and parameter values in differential evolution, which is a very popular and efficient variant of DE that is presented by Mallipeddi et al. (2011). Based on experimental studies, EPSDE has shown acceptable performance in comparison with other DE variants in solving some highly complex problems (Fan et al., 2017, Wu et al., 2018, and Mahmoodjanloo et al., 2020).

The procedure for EGPSDE, a variant of the DE algorithm with an ensemble of gradient-based mutation strategies and parameters, is presented below:

Step 1. Set the generation number $G = 0$ and randomly generate the initial population: Initialize a random population of NP individuals (i.e., $Pop_G = \{X_{1,G}, X_{2,G}, \dots, X_{NP,G}\}$). In the pricing problem, each individual is related to the entrant's price of the flow between each pair of nodes, where P_{ijkl} $(\forall\, i,j \in N, (k,l) \in S_{ij})$ is uniformly distributed in the range $\left[c_{ijkl} + \frac{2}{3\,\beta_2}, c_{ijkl} + \frac{2}{\beta_2}\right]$.

Step 2. Create three pools of strategies, including mutation strategies ($Pool_m$), crossover rate values ($Pool_{CR}$), and scaling factor values ($Pool_F$).

Step 2.1. Crossover strategies ($Pool_{CR}$): The different values of crossover rates for $Pool_{CR}$ will be selected from the range [0.1, 0.9] in steps of 0.1.

Step 2.2. Mutation strategies ($Pool_m$): The different mutation strategies that can be used in $Pool_m$ contain "DE/current-to-rand/1/gradient-base/bin", "DE/rand/1/gradient-base/bin" and "DE/current-to-best/1/gradient-base/bin".

DE/current-to-rand/1/gradient-base/bin:

$$V_{i,G} = X_{i,G} + (1 - \alpha_G) \times F_1 \times \left(X_{r_1,G} - X_{r_2,G}\right) + \alpha_G \times F_{2,G} \times \vec{\nabla} f\left(X_{i,G}\right) \tag{55}$$

DE/rand/1/gradient-base/bin:

$$V_{i,G} = X_{r_1,G} + (1 - \alpha_G) \times F_1 \times \left(X_{r_2,G} - X_{r_3,G}\right) + \alpha_G \times F_{2,G} \times \vec{\nabla} f\left(X_{i,G}\right) \tag{56}$$

DE/current-to-best/1/gradient-base/bin:

$$V_{i,G} = X_{i,G} + (1-\alpha_G).F_1.\left(X_{best,G} - X_{r_1,G}\right) + (1-\alpha_G).F_1.\left(X_{r_2,G} - X_{r_3,G}\right) + \alpha_G.F_{2,G}.\vec{\nabla}f\left(X_{i,G}\right) \quad (57)$$

where corresponding to a target vector $X_{i,G}$ (i.e., current solution), a mutation vector (i.e., $V_{i,G}$) is generated using one of the mentioned mutation strategies. In the above equations, F_1 and $F_{2,G}$, the scaling factors, are the parameters that control the magnitude of the random vectors and gradient vector respectively. r_1, r_2 and r_3 are exclusive integer numbers (i.e., from 1 to NP) that are different from i. $X_{best,G}$ is the best solution in the G-th generation. Moreover, α_G is a coefficient in the range 0-1 that controls the efficiency of the gradient vector. Indeed, the greater the value of α_G, the more the focus on exploitation. That is because the mutation strategy operates based on the gradient vector, and it is converted to a local search strategy. In contrast, the lower the value of α_G, the more the focus on exploration. Therefore, we consider a special value for α_G in each generation. The value of α_G starts from a very small value (i.e., α_{min}) at the beginning of the search procedure. Thereafter, it increases based on Eq. (58), which is borrowed from Beasley and Chu (1996).

$$\alpha_G = \alpha_{min} + \left((m_f - \alpha_{min})/(1 + \exp\left(-4 \times m_g \times (G - m_c)/(m_f - \alpha_{min})\right)\right)\right) \quad (58)$$

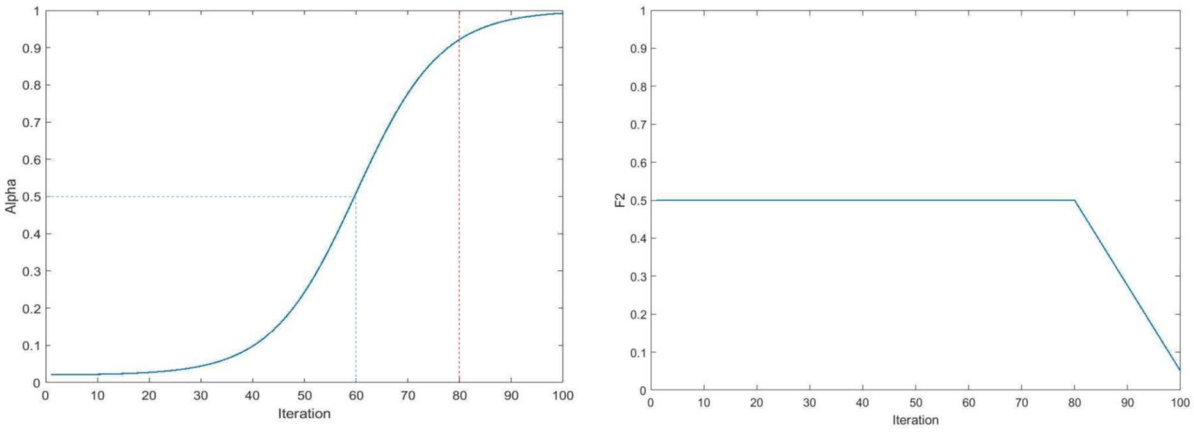
where G is the number of generations. α_G starts at α_{min} and tends to a final stable rate ($m_f - \alpha_{min}$). m_c specifies the number of generations required for α_G to reach $m_f/2$. Also, m_g specifies the increasing rate of α_G (i.e., a slope of the diagram). The changing procedure for α_G is illustrated in Fig. 6a.

Step 2.3. The values of scaling factors: In the mutation strategies, we have two types of scaling factors (i.e., F_1 and $F_{2,G}$). Like EPSDE, F_1 will be selected randomly from $Pool_F$, which includes different values that are taken in the range 0.4-0.9 in steps of 0.1. But we propose a different strategy for $F_{2,G}$, to control the process of exploration and exploitation more efficiently. Based on the behavior of α_G, the effect of a gradient vector will be enhanced in the last iterations of the algorithm (i.e., the exploitation phase). Therefore, we can consider a constant value for $F_{2,G}$ before the exploitation phase (e.g., for $G = 1$ to $0.8 \times G_{max}$). Thereafter, while converting the mutation strategies to gradient search, the value of $F_{2,G}$ decreases linearly for a more accurate search process. The changing procedure for $F_{2,G}$ is illustrated in Fig. 6b.

Step 3. Select a hybrid strategy randomly for each individual: Each hybrid strategy contains a mutation strategy from $Pool_m$, a crossover rate from $Pool_{CR}$, and a scaling factor from $Pool_F$.

Step 4. If the stopping criteria are not satisfied (e.g., $G \leq G_{max}$), carry out the following steps.

Step 4.1. Mutation step: For each individual ($X_{i,G}$), generate a mutation vector ($V_{i,G}$) using the related mutation strategy and scaling factor.



(a) Changing procedure of α_G (b) Changing procedure of $F_{2,G}$

Fig. 6. Changing procedure for the control parameters of the proposed mutation strategies (e.g., $m_g = 0.03, m_c = 60, G_{max} = 100, m_f = 1, \alpha_{min} = 0.02$)

Step 4.2. Crossover step: For each individual $X_{i,G} = \{x_{i,G}^1, x_{i,G}^2, \dots, x_{i,G}^D\}$ and its mutation vector $V_{i,G} = \{v_{i,G}^1, v_{i,G}^2, \dots, v_{i,G}^D\}$, generate a trial vector $U_{i,G} = \{u_{i,G}^1, u_{i,G}^2, \dots, u_{i,G}^D\}$ using the related crossover rate (CR_i).

$$u_{i,G}^j = \begin{cases} v_{i,G}^j & \text{if } rand_j \leq CR_i \text{ or } j = j_{rand} \\ x_{i,G}^j & \text{otherwise} \end{cases} \quad (59)$$

where D denotes the dimensionality of the problem ($j = 1, 2, \dots, D$). Also, $rand_j$ is a random value that is distributed in the range $[0, 1]$ uniformly, and j_{rand} is a random integer value that will be selected from $\{1, 2, \dots, D\}$. It guarantees that at least one dimension of $X_{i,G}$ and $U_{i,G}$ will be different.

Step 4.3. Selection step: Once $U_{i,G}$ is produced, it should be compared with $X_{i,G}$. If the trial vector $U_{i,G}$ is better than the target vector $X_{i,G}$, it should be replaced. So $X_{i,G+1}$ is found by:

$$X_{i,G+1} = \begin{cases} U_{i,G} & \text{if } f(U_{i,G}) \geq f(X_{i,G}) \\ X_{i,G} & \text{otherwise} \end{cases} \quad (60)$$

Step 4.4. Updating step: For each individual whose related strategies have not been successful ($f(U_{i,G}) < f(X_{i,G})$), a new parameter value from the stored successful combinations and a new

mutation strategy from the pools should be selected. Eventually, the generation number increases one step ($G = G + 1$).

4. Computational experiments

Here, the results and analyses of experiments are presented to evaluate the validity of the proposed mathematical model and the performance of the suggested algorithms. For the experiments, we utilize the approach of generating random instances, which was used by Lüer-Villagra and Marianov (2013). Moreover, some parameters are specific to the proposed model. The travel costs per unit (φ_{ij}^m) are randomly selected in the range $[0.5, 1]$ in step 0.1. Also, for each arc ij, the travel time per unit $\tau_{ij}^m = rand \times \frac{1}{\varphi_{ij}^m}$ where $rand$ is uniformly distributed in the range $[0.3, 0.6]$. To obtain the scale of β_1 and β_2, a discussion is presented in Subsection 4.1. Thereafter, to evaluate the validity of the CHLP-TM model and the performance of the proposed scatter search for the hub location sub-problem, the computational results are presented in Subsection 4.2. Also, to evaluate the validity of the pricing sub-problem of the CHLP-ED/CL/TM model and the performance of the related matheuristic method, the computational results are presented in Subsection 4.3. Finally, to validate the correctness of the CHLP-ED/CL/TM model and the performance of the joint solution approaches, an illustrative example is represented in Subsection 4.4.

4.1. Discussion of the scale of β_1 and β_2

Based on the theoretical discussion presented in Subsection 3.2.1, the scale of the sensitivity parameter of customers to service price (β_2) is highly important. That is because the optimal solution of the pricing sub-problem is situated in the interval $\left[c_{ijst} + \frac{2}{3\,\beta_2}, c_{ijst} + \frac{2}{\beta_2}\right]$, which depends on β_2. On the other hand, β_1 and β_2 are two conceptual coefficients that reflect customer tendencies. The ratio of these two factors can be extracted based on customer behavior; however, the scale of β_1 and β_2 are related to the scale of variable costs (c_{ijkl}) and response times (T_{ijkl}) respectively. Since the scales of τ_{ij}^m and φ_{ij}^m are considered the same, it is sufficient to discuss the scale of one of these two parameters (e.g., β_2). Based on Eqs. (50) and (53), the profit margin for each route ($P - C$) is situated in interval $\left[\frac{\kappa^L}{\beta_2}, \frac{\kappa^U}{\beta_2}\right]$. Hence, the scale of β_2 can be determined by:

$$\frac{\kappa^L}{P - C} \leq \beta_2 \leq \frac{\kappa^U}{P - C} \tag{61}$$

To estimate the scale of β_2, we utilize the concept of the profit margin ratio, also called the return-on-sales ratio, which can be estimated for each market sector. Therefore, we can replace $P - C$ by $[I_{min}, I_{max}] \times C$, where $[I_{min}, I_{max}]$ is a common range for the profit margin ratio in the market. A conservative interval for β_2 can be calculated as Eq. (62). The mean of this interval can be an acceptable estimation for β_2. Therefore, β_2 can be calculated from Eq. (63), where $\bar{C}$ is the mean of transportation costs (i.e., $\bar{C} = \bar{\varphi}_{ij}^{m} \times \bar{d}_{ij}$). For example, if $\bar{C} = 50$, $I_{min} = 5\%$ and $I_{max} = 15\%$, then $\hat{\beta}_2 = 0.444$. Eventually, based on customer behavior, the relative effect of time rather than cost can be extracted to estimate β_1. For example, if the effect of cost is twice the effect of time, then $\hat{\beta}_1 = 0.222$.

$$\frac{\kappa^L}{I_{max}.C} \leq \beta_2 \leq \frac{\kappa^U}{I_{min}.C} \tag{62}$$

$$\hat{\beta}_2 = \frac{I_{min}.\kappa^L + I_{max}.\kappa^U}{2.\bar{C}.I_{min}.I_{max}} \tag{63}$$

4.2. Computational results of the CHLP-TM model

This section investigates the validity of the CHLP-TM model and the approaches to its solution. To evaluate the performance of SS, the results are compared with the results of the GA proposed by Lüer-Villagra and Marianov (2013). Additionally, to solve the CHLP-TM model with the GA, we utilize the transportation mode selection approach presented in Section 3.1.2. In all instances, two types of transportation modes are considered. To tune the parameters of the SS and GA, we try to consider the values proposed in Marti et al. (2015) and Lüer-Villagra and Marianov (2013) respectively. For the SS, we consider $\pi = 100, \beta = 20, Iteration = 100$, $[\psi_{min}, \psi_{max}] = [0.1, 0.3]$ and a mutation probability of 1%. For the GA, we consider $nPop = 100, Iteration = 100$ and a mutation probability of 1%. The meta-heuristics are coded in MATLAB 2017 and tested on an Intel Core i7 processor with 2.5 GHz CPU and 8 GB of RAM. Each of the instances is run 20 times. The mean value of the level objective function, the standard deviation of each instance, and the CPU times are shown in Table 3. Moreover, the last column shows the percentage of differences between outputs of the SS and GA, i.e., $\Delta Z_{SS/GA} = \frac{Z_{SS} - Z_{GA}}{Z_{GA}} \times 100$.

The computational results of 27 random instances shown in Table 2 show that the proposed SS performs better in 20 instances. To more accurately evaluate the results, we utilize the relative percentage deviation (RPD) index to neutralize the effect of variant measures of instance problems. The index for a maximization problem can be computed by:

$$RPD = \frac{OF_{max} - OF_i}{OF_{max}} \tag{66}$$

where OF_{max} is the maximum value (i.e., best value) of the objective functions of 20 runs for each instance. Also, OF_i is the objective value of the i-th run for each instance problem. So, we have 540 data points with a similar scale for each algorithm. The performance of the two algorithms is illustrated in the boxplot of Fig. 7. The boxplot shows that the SS performed better in terms of both the mean value and dispersion of outputs. The computational results show the superiority of the proposed SS compared to the GA. The percentage of differences in the RPD of each instance is presented in Fig. 8.

Table 3. Computational results of CHLP-TM

ID	No. of nodes	Genetic algorithm			Scatter search			$\Delta Z_{SS/GA}(\%)$
		Z_{GA}	St. Dev.	CPU time (s)	Z_{SS}	St. Dev.	CPU time (s)	
HL-01	7	3,007,755	96,372	56	3,099,774	33,904	64	3/06
HL-02	7	3,422,759	109,240	34	3,394,766	43,954	33	**-0/82**
HL-03	7	3,444,960	96,978	51	3,337,114	43,490	43	**-3/13**
HL-04	8	3,601,163	106,292	63	3,731,214	38,535	72	3/61
HL-05	8	3,420,465	95,466	79	3,380,896	46,909	71	**-1/16**
HL-06	8	3,568,251	102,376	59	3,829,773	40,507	65	7/33
HL-07	9	4,707,494	100,236	76	4,903,573	59,261	81	4/17
HL-08	9	4,266,664	131,706	81	4,403,657	48,400	73	3/21
HL-09	9	4,171,062	131,464	86	4,358,149	57,047	98	4/49
HL-10	10	4,915,033	161,207	146	5,209,019	58,324	139	5/98
HL-11	10	4,939,191	143,598	127	4,915,610	57,361	148	**-0/48**
HL-12	10	4,508,556	148,522	135	4,413,085	53,408	151	**-2/12**
HL-13	11	4,958,490	108,735	184	5,372,238	65,642	189	8/34
HL-14	11	4,682,396	128,130	212	5,139,130	47,942	205	9/75
HL-15	11	5,426,456	156,445	201	5,948,956	64,410	220	9/63
HL-16	13	5,677,692	133,456	243	5,772,833	71,109	276	1/68
HL-17	13	6,106,429	168,599	237	5,896,537	70,479	228	**-3/44**
HL-18	13	6,284,311	192,533	256	6,321,120	78,114	283	0/59
HL-19	15	7,537,221	203,388	384	7,512,084	95,221	363	**-0/33**
HL-20	15	6,880,412	164,269	479	6,947,225	79,983	518	0/97
HL-21	15	7,067,451	219,806	466	7,391,838	86,428	478	4/59
HL-22	17	7,086,327	219,756	514	7,416,782	89,795	512	4/66
HL-23	17	8,386,967	262,656	601	8,742,379	107,443	631	4/24
HL-24	17	8,583,669	244,931	526	8,904,276	99,170	495	3/74
HL-25	20	9,037,651	328,299	757	9,550,987	93,383	716	5/68
HL-26	20	9,336,825	271,465	613	9,848,408	104,439	627	5/48
HL-27	20	8,944,959	259,662	675	9,131,655	100,324	669	2/09

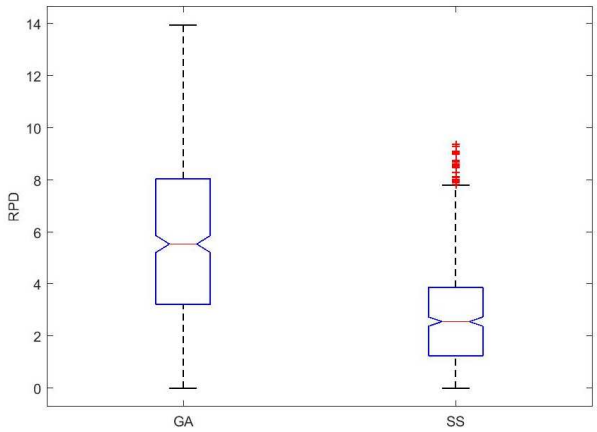


Fig. 7. Performance comparison of SS and GA for the CHLP-TM problem.

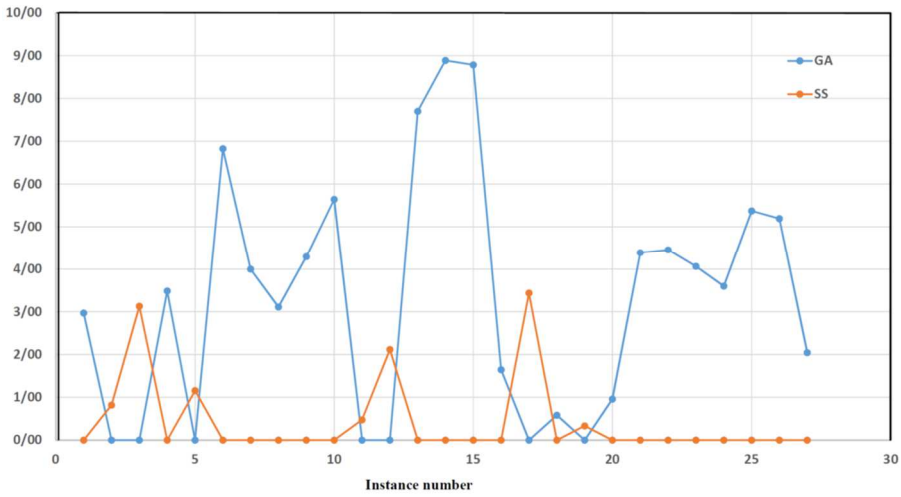


Fig. 8. Performance comparison of SS and GA in 27 instance problems.

4.3. Computational results of the CHLP-ED/CL/TM model

4.3.1. Performance evaluation of the EGPSDE algorithm

In this section, first, the validity of the proposed approach to solving the pricing sub-problem in the CHLP-ED/CL/TM model is presented. To evaluate the performance of EGPSDE, the results are compared with the results of two efficient and highly popular DE variants, EPSDE (Mallipeddi et al., 2011) and JADE (Zhang and Sanderson, 2009). Hence, we utilize the 27 random instances generated in the previous section. For each instance, a random solution of a hub-location problem is considered, and based on these fixed variables, the pricing sub-problem is solved by the suggested algorithms. Also, the customer loyalty index for the incumbent's services is considered to be 0.5 for each node ($CLI_j = 0.5$). The parameter values of the compared algorithms are presented in Table 4. Finally, the 27 instances are run 20 times each. Table 5 shows the mean value and the standard deviation of the obtained objective function for each instance over these runs.

The experiments show the superiority of the suggested algorithm compared to the EPSDE and JADE algorithms. As mentioned above, to more accurately evaluate the results, we utilize the RPD index. So, we will have 540 data points on a similar scale for each algorithm. The characteristics of RPD results are presented in Table 6. Moreover, the boxplots shown in Fig. 9 illustrate the performance of the three algorithms. The results confirm the acceptable performance of EGPSDE in terms of both the mean value and the dispersion of outputs.

Table 4. Parameter configuration of tested algorithms for the pricing sub-problem of CHLP-ED/CL/TM

Algorithm	Parameter settings
EPSDE	CR range [0.1, 0.9] and F range [0.4, 0.9], NP=100, Max Iteration=500
JADE	P=0.5, c=0.1, NP=100, Max Iteration=500
EGPSDE	CR range [0.1, 0.9] and F range [0.4, 0.9], NP=100, Max Iteration=500

Table 5. Computational results of the pricing sub-problem of CHLP-ED/CL/TM

	EGPSDE		EPSDE		JADE	
	Mean	St. Dev.	Mean	St. Dev.	Mean	St. Dev.
HL-01	**2,741,879**	22,475	2,517,438	23,030	2,325,178	38,372
HL-02	**2,918,176**	16,622	2,508,688	24,350	2,509,609	28,972
HL-03	2,903,098	23,649	**2,923,658**	27,837	2,851,759	85,357
HL-04	**2,962,347**	22,160	2,790,570	19,277	2,651,284	22,062
HL-05	**2,759,827**	18,797	2,561,641	21,563	2,338,709	35,215
HL-06	**4,264,153**	25,888	3,882,410	39,808	3,979,973	43,256
HL-07	**5,199,542**	54,921	4,809,435	42,114	4,481,471	37,596
HL-08	**3,872,636**	24,308	3,587,225	28,984	3,460,221	31,655
HL-09	**3,526,592**	26,674	3,256,957	28,150	3,161,938	41,224
HL-10	**4,868,947**	44,862	4,602,960	42,004	4,407,252	54,727
HL-11	**4,851,190**	34,694	4,222,342	43,218	4,539,650	36,763
HL-12	**4,288,237**	38,446	3,940,808	38,317	3,991,535	32,174
HL-13	**5,749,990**	40,161	5,442,322	48,831	4,946,738	40,843
HL-14	**5,441,531**	42,127	4,716,803	48,134	5,035,425	53,047
HL-15	**6,348,039**	51,986	5,661,624	61,985	5,536,895	89,406
HL-16	**6,505,784**	57,353	6,121,704	56,678	5,594,945	69,716
HL-17	**4,990,035**	38,757	4,678,431	40,738	4,611,149	68,385
HL-18	**6,147,391**	43,607	5,385,232	49,293	5,464,067	60,621
HL-19	**8,306,818**	73,614	8,038,362	71,408	7,688,339	85,263
HL-20	**6,258,580**	51,318	5,829,616	66,557	6,074,171	76,224
HL-21	**6,159,861**	54,536	5,801,960	50,800	5,821,553	57,769
HL-22	**6,181,584**	41,385	6,098,263	50,573	5,584,075	74,047
HL-23	**7,355,568**	76,470	7,069,777	63,621	6,518,722	83,430
HL-24	**7,088,710**	55,666	6,195,456	58,811	6,687,638	91,853
HL-25	**10,003,891**	91,897	9,524,973	91,248	9,936,625	79,241
HL-26	**7,533,688**	63,531	7,124,206	60,428	7,214,584	93,797
HL-27	**7,812,759**	58,996	7,320,473	78,745	7,403,805	101,476

Table 6. Specifications obtained from the RPD results to solve the pricing sub-problem.

	EGPSDE	EPSDE	JADE
Mean	2.712	9.827	11.419

Median	2.089	9.365	11.154
Standard Dev.	2.187	3.077	3.395
GAP	9.458	13.457	13.842

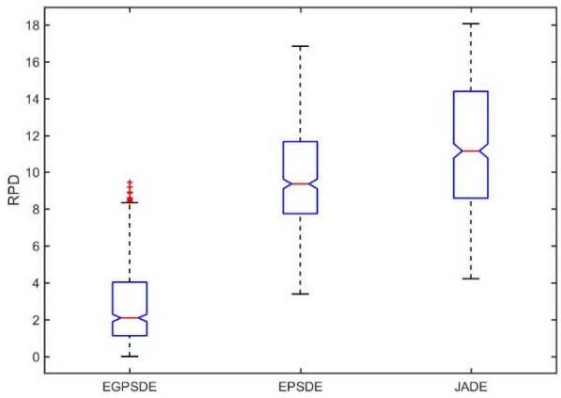
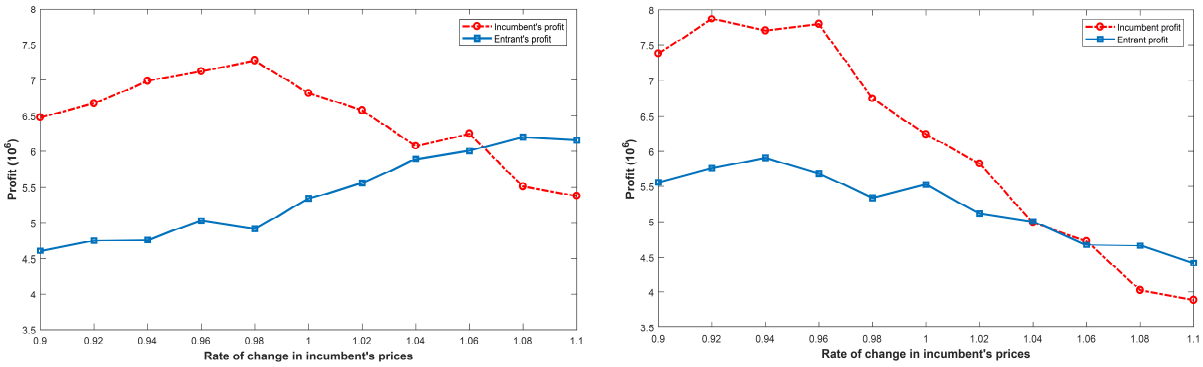


Fig. 9. Performance comparison of EGPSDE, EPSDE and JADE to solve the pricing sub-problem.

4.3.2. Sensitivity analysis of the pricing problem

Due to the effective performance of the proposed algorithms, the SS for a hub location problem and EGPSDE for a pricing problem, we use them in a nested approach to solving the CHLP-ED/CL/TM model. Here, to verify the behavior of the model, a sensitivity analysis is carried out for some instances. An effective sensitivity analysis illustrates the effect of the input parameters of the model. Thus, we consider some key parameters to assess the performance of the model.

Consequently, to study the behavior of the model related to market price changes, the incumbent's prices are changed by applying ± 10%. The behavior of the model, based on the range of profit gained by the entrant and incumbent under both elastic and inelastic demand in the market, is presented in Fig. 10. Fig. 10a shows the increase of the incumbent's prices in an inelastic market; there is always an uptrend in the entrant's profit; however, this process cannot be observed in an elastic market (Fig. 10b). In an elastic demand market, a rise in the incumbent's prices and a competitive pricing policy for the entrant consecutively lead to a reduction in market demand, which causes the profits of both rivals to be reduced. Also, it is worth noting that in both types of markets, the incumbent's profit changes in the form of a nearly concave function. Note that the function's maximum in both types of markets occurred before the current situation, with the coefficient of variation equal to 1 in the horizontal axis. It can be concluded that the incumbent is not in the best margin of its profit.



a. Inelastic demand

b. Elastic demand

Fig. 10. Entrant's and incumbent's profit as a function of incumbent's price, with $\Delta = 15\%$ (Ins-10).

In the following, to analyze two markets better according to elastic and inelastic demand, the trend of changes in the entrant's market share based on variations in market prices is studied. These trends are shown in Fig. 11. The first point that can be seen from these graphs is that the elastic demand curve is generally higher than that for inelastic demand. This indicates that the entrant's pricing policies are more effective in a competitive market for more elastic demand. Moreover, given the tangency of the two curves at higher price levels, it can be noted that the effectiveness of the entrant's pricing policies is decreased by increases in the incumbent's prices. This can be related to a reduction in the total quantity of the market's elastic demand if there is a rise in prices.

In general, what follows from the analysis of trends is that a rise in the incumbent's prices in an inelastic market is in a favor of the entrant. This is because the entrant will either attract more market share or gain more revenue by the possibility of increasing the profit margin. In an elastic demand market, an increase in the incumbent's prices leads to an increase in the market share of the entrant (Fig. 11); however, the entrant's demand will be also decreased due to a decline in overall market demand. These changes will eventually cause a reduction in the entrant's profit (Fig. 10b).

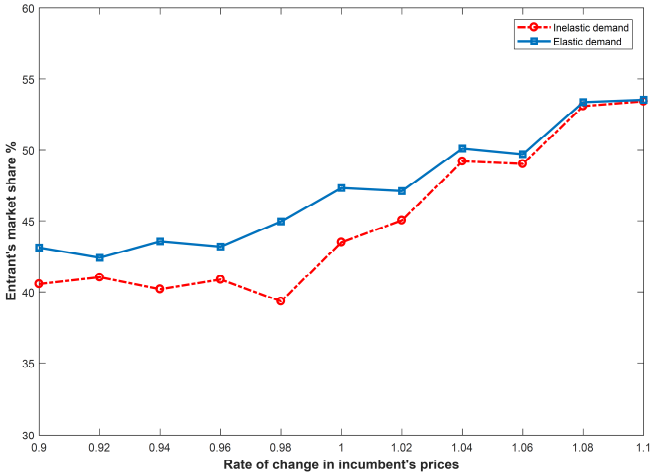


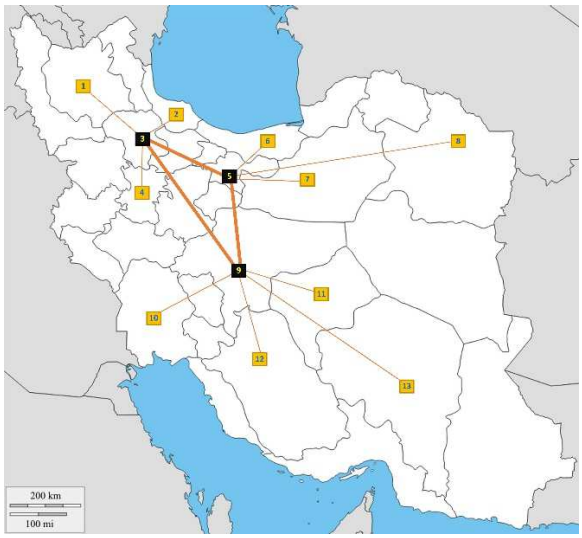
Fig. 11. Entrant's market share, for different values of the incumbent's price, with Δ= 15% (Ins-10).

4.3.3. Illustrative example

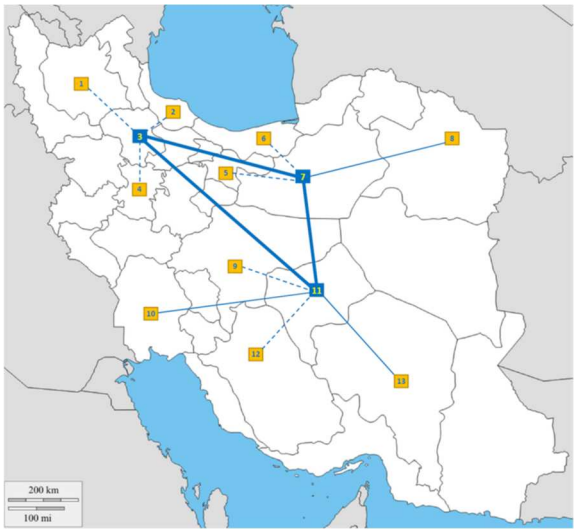
As previously mentioned, until recently, postal services in Iran were provided by a monopoly operator (Iran Post Company). In 2017, the Ministry of Information and Communications Technology of Iran decided to utilize a new operator from the private sector. Hence, this real case can be considered as an appropriate application of the proposed model.

In this section, we design an example based on Iran's postal industry. In this case, Iran National Post Company is considered the incumbent, and the new operator is considered the entrant. To facilitate and provide appropriate analyses, we consider a network with 13 nodes that are the most populated and industrialized cities in Iran. Fig. 12a depicts the incumbent's transportation network. Thereafter, the entrant's network with two transportation modes is designed by the proposed model (see Figs. 12b and 12c). In these figures, two transportation modes (i.e., road and air) are presented by dashed and solid lines, respectively.

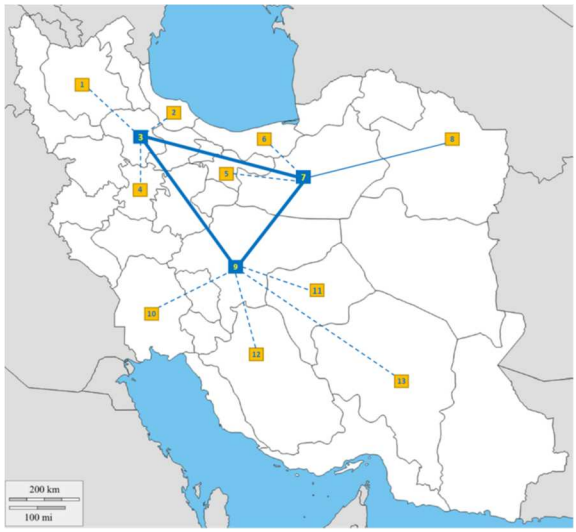
To analyze the effect of the customer loyalty index, the value of this index for all network nodes is set to 0.5 at first. As was seen in Section 2.4, in this case, by simplification of the CLI_i coefficients from the numerator and denominator, the impact of customer loyalty is considered to be ignored. Fig. 12b shows a solution obtained under these conditions. Then, by considering $CLI_{12} = CLI_{13} = 0.8$, we solve the problem one more time. Fig. 12c shows the solution obtained after increasing these two indices. As can be observed, the proposed network structure changes as these indices vary. This can also be important in the real world. Customers receive their postal services from the incumbent for many years. This may give customers a subjective sense of reliance on the system. Customer uncertainty about a new system or their normal resistance to change can justify the consideration of this index with values higher than 0.5. On the other hand, it may be that in some situations, due to inadequate performance by the previous operator in some parts of the network, the coefficient should be lower than 0.5.



a. Incumbent's hub network



b. Entrant's hub network with $CLI_i = 0.5$ ($Z = 559 \times 10^9$)



c. Entrant's network with $CLI_{12} = CLI_{13} = 0.8$ and $CLI_i = 0.5$ for $i \neq 12, 13$ ($Z = 488 \times 10^9$)

Fig. 12. Structure of hub networks for the incumbent (a) and the entrant (b, c), with $\beta_1 = 0.222, \beta_2 = 0.444$ and different values of the customer loyalty index.

Our proposed model includes three subjective parameters, β_1, β_2 and *CLI*. Normally, there is not a firm basis for calculating subjective parameters. Therefore, we are interested in assessing and comparing the sensitivity of the model to these parameters. Fig. 13 shows the effect of these parameters on a tornado diagram. These changes are obtained by applying a ±10% change in the parameters. As shown in Fig. 13, the *CLI* parameter has a greater impact on the target function value. This reveals the necessity of estimating this parameter properly in decision-making models.

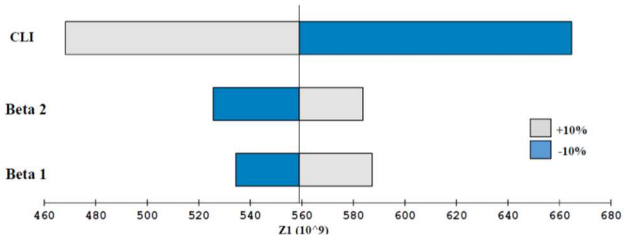


Fig. 13. Tornado diagram of the sensitivity analysis of the subjective parameters β_1, β_2 and CLI

As can be seen in this example, customer loyalty has a significant effect on the results. However, we attempt to show these effects based on a sensitivity analysis utilizing a simple equation presented in Eq. (19) while in the marketing literature, there are more deeply definitions in this field. For example, there are four distinctly different types of loyalty (i.e., inertia loyalty, mercenary loyalty, true loyalty and cult loyalty). Each level of loyalty requires different strategies to be achieved. Hence, it seems that entering into a market which the incumbent companies have own special strategies needs different approaches to competition. On this topic, the interested readers can refer to related literature (Benati and Hansen, 2002; Irani-kermani, 2017; Elshiewy et al., 2017).

5. Conclusions

In this paper, we developed a framework to design a hub-and-spoke network for a firm that plans to enter into a competitive market with elastic demand, in which the existing transportation company operates its network, and applied mill pricing. For this purpose, we developed a mixed-integer non-linear programming model and used a multi-nominal logit function to estimate the market share of each firm. Our approach is different from that of Lüer-Villagra and Marianov (2013), who just considered customer sensitivity to prices. In addition to prices, we consider customer sensitivity to service time duration. Moreover, for the first time, to the best of our knowledge, we modeled elastic demand in a competitive market and considered the effect of customer loyalty in a mathematical framework. To solve the proposed model, we decomposed it into a hub location problem (i.e., master problem) and a pricing problem (i.e., sub-problem). We then derived a closed-form expression for the pricing problem when demand is inelastic. Also, we derived upper and lower bounds for the price of each route in the hub network when demand is elastic. Eventually, in a nested structure, a scatter search (SS) algorithm and a matheuristic based on a differential evolution (DE) algorithm were proposed to solve the multi-modal hub location problem and the pricing sub-problem, respectively. The computational experiments confirmed the performance of the proposed algorithms.

Several sensitivity analyses were carried out to show the effect of elastic demand and customer loyalty in a competitive market. These analyses can be important from a management point of view because they showed that the rival's pricing policies in the market along with

elastic and inelastic demands could have different outcomes. Therefore, the type of market had to be considered in decision-making models. It was also important to consider customer loyalty in competitive markets. This could make the behavior of competitors in such models more realistic. The subjective nature of the customer loyalty index and the high sensitivity of the related model to its value highlights the necessity of carrying out further research on quantifying methods. The other subjective parameters of the model were β_1 and β_2. Although we proposed an approach to determining their scale in Section 4.1, there is a need for future research on the role of these parameters in making appropriate decisions and to determine their quantities more precisely based on data mining techniques.

Further, in the presented model, we tried to develop a method to estimate the behavior of customers based on the multi-nominal logit model. However, like any other estimation techniques, some errors could occur. For example, utilizing some other center measures (e.g., median or mode) instead of the mean value can be studied to explore the performance of the estimation model in some real-world cases. Hence, studying such methods can be interesting for future researches, in which some special approaches (e.g., statistical techniques) can be utilized. Moreover, although brand loyalty is extensively used in the marketing literature, it seems that there is a lack of such studies in the field of the logistics distribution network design. Hence, this paper can motivate interested research to make the related models much more applicable utilizing the concepts of brand loyalty. Furthermore, the model was developed in the point of view of an entrant company into a competitive distribution market. However, it can be revised for the point of view of the incumbent companies in the market or be utilized in other applications such as maritime transportation, industrial distribution, urban transportation and city logistics.

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Appendix A:

To describe the behavior of the objective function, the derivative sign patterns should be extracted by using the first-order conditions.

$$Z = \sum_{i,j \in N} \frac{D_{ij}(1 - CLI_i) \sum_{(k,l) \in S_{ij}} (P_{ijkl} - c_{ijkl}) \exp(-\beta_1 T_{ijkl} + \frac{\beta_2}{2} q_{ij} - \frac{3\beta_2}{2} P_{ijkl})}{(1 - CLI_i) \sum_{(s,t) \in S_{ij}} \exp(-\beta_1 T_{ijst} - \beta_2 P_{ijst}) + CLI_i \eta_{ij}} \quad \text{(A1)}$$

Let us introduce the following variables to facilitate the presentation of the process.

$$\gamma_{ijst} = \exp(-\beta_1 T_{ijst} - \beta_2 P_{ijst}) \quad \text{(A2)}$$

$$\gamma'_{ijst} = \exp(-\beta_1 T_{ijkl} + \frac{\beta_2}{2} q_{ij} - \frac{3\beta_2}{2} P_{ijkl}) \quad \text{(A3)}$$

$$\gamma''_{ijst} = \exp\left(\frac{\beta_2}{2}(q_{ij} - P_{ijst})\right) \quad \text{(A4)}$$

Hence, we have:

$$\gamma'_{ijst} = \exp(-\beta_1 T_{ijst} - \beta_2 P_{ijst}) \times \exp\left(\frac{\beta_2}{2} q_{ij} - \frac{\beta_2}{2} P_{ijst}\right)$$

$$\Rightarrow \gamma'_{ijst} = \gamma_{ijst} \times \gamma''_{ijst} \quad \text{(A5)}$$

Accordingly, the derivation of γ_{ijst}, γ'_{ijst} and γ'_{ijst} can respectively be calculated by:

$$\frac{\partial \gamma_{ijst}}{\partial P_{ijst}} = -\beta_2 \exp(-\beta_1 T_{ijst} - \beta_2 P_{ijst}) = -\beta_2 \gamma_{ijst} \quad \text{(A6)}$$

$$\frac{\partial \gamma'_{ijst}}{\partial P_{ijst}} = -\frac{3\beta_2}{2} \times \exp\left(-\beta_1 T_{ijkl} + \frac{\beta_2}{2} q_{ij} - \frac{3\beta_2}{2} P_{ijkl}\right) = -\frac{3\beta_2}{2} \gamma'_{ijst} \quad \text{(A7)}$$

$$\frac{\partial \gamma''_{ijst}}{\partial P_{ijst}} = -\frac{\beta_2}{2} \exp\left(\frac{\beta_2}{2}(q_{ij} - P_{ijst})\right) = -\frac{\beta_2}{2} \gamma''_{ijst} \quad \text{(A8)}$$

Moreover, let us to define the variable φ_{ij} as follows.

$$\varphi_{ij} = (1 - CLI_i) \sum_{(s,t) \in S_{ij}} \exp(-\beta_1 T_{ijst} - \beta_2 P_{ijst}) + CLI_i \eta_{ij} \quad \text{(A9)}$$

Therefore, we have:

$$\frac{\partial \hat{Z}}{\partial P_{ijkl}} = \left\{ \left[(1 - CLI_i) \sum_{(k,l)\in S_{ij}} \gamma_{ijkl} + CLI_i\, \eta_{ij} \right] \times \left(\gamma'_{ijst} \left(1 - \frac{3\beta_2}{2} \left(P_{ijst} - c_{ijst} \right) \right) \right) + \beta_2 (1 - CLI_i)\, \gamma_{ijst} \sum_{(k,l)\in S_{ij}} \left(P_{ijkl} - c_{ijkl} \right) \gamma'_{ijkl} \right\} / \varphi_{ij}{}^2 \tag{A10}$$

Also, consider the numerator of the obtained fraction in Eq. (A10) as follows.

$$\psi = \left[(1 - CLI_i) \sum_{(k,l)\in S_{ij}} \gamma_{ijkl} + CLI_i\, \eta_{ij} \right] \times \left(\gamma''_{ijst} \left(1 - \frac{3\beta_2}{2} \left(P_{ijst} - c_{ijst} \right) \right) \right) + \beta_2 (1 - CLI_i) \sum_{(k,l)\in S_{ij}} \left(P_{ijkl} - c_{ijkl} \right) \gamma'_{ijkl} \tag{A11}$$

Considering a single-assignment assumption, there is only one unique route between each arbitrary pair of nodes i and j such as $i \to s \to t \to j$. Hence, S_{ij} includes only the set (s, t). Therefore, substituting Equations (A2) – (A4) in Eq. A11, we can extract the following equations.

$$\psi = \left[(1 - CLI_i) \exp\left(-\beta_1\, T_{ijkl} + \frac{\beta_2}{2} q_{ij} - \frac{3\beta_2}{2} P_{ijkl} \right) + CLI_i\, \eta_{ij} \exp\left(\frac{\beta_2}{2} \left(q_{ij} - P_{ijst} \right) \right) \right] \times \left(1 - \frac{3\beta_2}{2} \left(P_{ijst} - c_{ijst} \right) \right) + \beta_2 (1 - CLI_i) \left(P_{ijkl} - c_{ijkl} \right) \exp(-\beta_1\, T_{ijkl} + \frac{\beta_2}{2} q_{ij} - \frac{3\beta_2}{2} P_{ijkl}) \tag{A12}$$

$$\psi = (1 - CLI_i) \exp\left(-\beta_1\, T_{ijkl} + \frac{\beta_2}{2} q_{ij} - \frac{3\beta_2}{2} P_{ijkl} \right) + CLI_i\, \eta_{ij} \exp\left(\frac{\beta_2}{2} \left(q_{ij} - P_{ijst} \right) \right) - \frac{3\beta_2}{2} (1 - CLI_i) \left(P_{ijst} - c_{ijst} \right) \exp\left(-\beta_1\, T_{ijkl} + \frac{\beta_2}{2} q_{ij} - \frac{3\beta_2}{2} P_{ijkl} \right) - \frac{3\beta_2}{2} CLI_i\, \eta_{ij} \left(P_{ijst} - c_{ijst} \right) \exp\left(\frac{\beta_2}{2} \left(q_{ij} - P_{ijst} \right) \right) + \beta_2 (1 - CLI_i) \left(P_{ijkl} - c_{ijkl} \right) \exp(-\beta_1\, T_{ijkl} + \frac{\beta_2}{2} q_{ij} - \frac{3\beta_2}{2} P_{ijkl}) \tag{A13}$$

$$\psi = (1 - CLI_i) \exp\left(-\beta_1\, T_{ijkl} + \frac{\beta_2}{2} q_{ij} - \frac{3\beta_2}{2} P_{ijkl} \right) + CLI_i\, \eta_{ij} \exp\left(\frac{\beta_2}{2} \left(q_{ij} - P_{ijst} \right) \right) - \frac{\beta_2}{2} (1 - CLI_i) \left(P_{ijst} - c_{ijst} \right) \exp\left(-\beta_1\, T_{ijkl} + \frac{\beta_2}{2} q_{ij} - \frac{3\beta_2}{2} P_{ijkl} \right) - \frac{3\beta_2}{2} CLI_i\, \eta_{ij} \left(P_{ijst} - c_{ijst} \right) \exp\left(\frac{\beta_2}{2} \left(q_{ij} - P_{ijst} \right) \right) \tag{A14}$$

$$\psi = (1 - CLI_i) \exp\left(-\beta_1\, T_{ijkl} + \frac{\beta_2}{2} q_{ij} - \frac{3\beta_2}{2} P_{ijkl} \right) \left(1 - \frac{\beta_2}{2} \left(P_{ijst} - c_{ijst} \right) \right) + CLI_i\, \eta_{ij} \exp\left(\frac{\beta_2}{2} \left(q_{ij} - P_{ijst} \right) \right) \left(1 - \frac{3\beta_2}{2} \left(P_{ijst} - c_{ijst} \right) \right) \tag{A15}$$

All of the terms in A15 are always non-negative except $\left(1 - \frac{\beta_2}{2}\left(P_{ijst} - c_{ijst}\right)\right)$ and $\left(1 - \frac{3\beta_2}{2}\left(P_{ijst} - c_{ijst}\right)\right)$. So, the sign of these terms has an effect on the behavior of the first derivative.